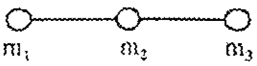
CANDIDACY EXAM 2000

MECHANICS

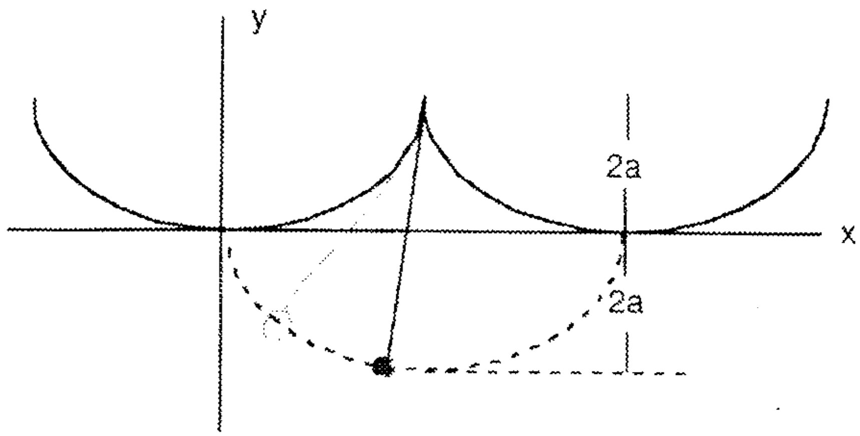
A1.

Three rigid spheres are connected by light, elastic rods, as shown in the figure. The relative masses are $m_1:m_2:m_3 = 1:2:1$.



Assuming small oscillations, so the longitudinal oscillations do not couple to the transverse oscillations, describe all the normal modes of the system, sketch them, and state all you can about their relative frequencies. Ignore translations of the center of mass and rotations of the molecule as a whole, that is, restrict yourself to motions with (total) ***P***=***L***=0.

A2.



A pendulum of mass m is suspended from the cusp of a cycloid cut in rigid support (see figure). The path described by the pendulum bob is a cycloid, given by the equations

$x = a(\phi - \sin\phi)$

$y = a(\cos\phi - 1)$

where the length of the pendulum is l=4a, and ϕ is the angle of rotation of the circle generating the cycloid. Find the Lagrangian, the Lagrange equations of motion, and the time period of the oscillations.

A3.

A mass m moves in a circular orbit of radius r_0 under the influence of a central force whose potential is $-km/r^n$.

(a) If the angular momentum of the particle is **L**, write down an expression for the effective radial potential seen by the particle, as a function of r.

(b) Use the result in (a) to find an expression for the value of r leading to a circular orbit, for the given **L**.

(c) Show that this circular orbit will be stable under small oscillations (that is, the mass will oscillate about the circular orbit) if $n < 2$.

E & M

B1.

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart. A resistor R is connected across the rails and a uniform magnetic field $\mathbf{B}$ pointing into the page, fills the entire region.

a.) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

b.) What is the magnetic force on the bar? In what direction?

c.) If the bar starts out with speed v_0 at time t=0, and is left to slide, what is its speed at a later time t ?

d.) The bar eventually stops moving. Where does its kinetic energy go? Prove that energy is conserved (that this other form of energy acquire all of the initial kinetic energy).

B2.

An electric dipole of moment p is placed at a distance d from a grounded conducting plane. The dipole makes an angle of 90° with the plane.

(a) Find the force experienced by the dipole

(b) How much work is required to move the dipole to infinity?

B3.

Consider a wire of length $2L$ and radius a carrying a current I that is uniformly distributed over the cross section of the wire. Find an expression for the external and internal vector potential $\mathbf{A}$ and magnetic field $\mathbf{B}$ at a point equidistant from the ends of the wire (assume $L \gg a$).

THERMO & STAT

C1. According to the kinetic theory of gases, the pressure of a gas is the result of the collisions between the gas molecules and the walls of their container. In the following, we will assume that: i) the gas is made of a large number of molecules that make elastic collisions with each other and with the walls of their container; ii) The molecules are separated, on the average, by large distances, and they exert no forces on each other except when they collide. In other words, the gas can be considered as an ideal gas.
Let us also assume that we have a rectangular container of volume V containing N molecules, each of mass m and moving with the same speed v_{av}. The cartesian x, y and z axes are chosen along the three vertices of the rectangular container.
(a) Demonstrate that the change in the x-component of the momentum of *all* molecules hitting the wall during a time Δt is given by:

$$\Delta p_x = \frac{N}{V} m v^2_{av,x} A\, \Delta t$$

where A is the container area perpendicular to the x-axis.

(b) Use Newton's second law and the ideal gas law to demonstrate that:

$$v^2_{av} = \frac{3kT}{m}$$

where k is Boltzmann's constant and T is the gas temperature.

(c) In fact, all the molecules in the gas does not have the same speed. We have:

$$dN = N f_1(v) dv$$

where dN is the number of molecules that have speeds ranging between v and v+dv. $f_1(v)$ is the Maxwell-Boltzmann (speed) distribution function, and is given by:

$$f_1(v) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{1.5} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

Find the analytical expression for the velocity v_{max} for which $f_1(v)$ is maximum. Is v_{max} smaller, equal to, or larger than v_{av}?

(d) By assuming that the energy E of one molecule is only a kinetic energy, demonstrate that the Maxwell-Boltzmann energy distribution function is given by:

$$f_2(E) = \frac{2}{\sqrt{\pi}} \left(\frac{1}{kT}\right)^{1.5} E^{0.5} \exp\left(-\frac{E}{kT}\right)$$

C2. Consider a gas of N massless identical non-interacting particles at temperature T and volume V. For massless particles, the relation between energy and momentum is $E = c|\mathbf{p}|$, where c = speed of light.
Find:
(a) the single particle partition function
(b) the partition function of the system
(c) the Helmholtz free energy of the system
(d) the specific heat at constant volume

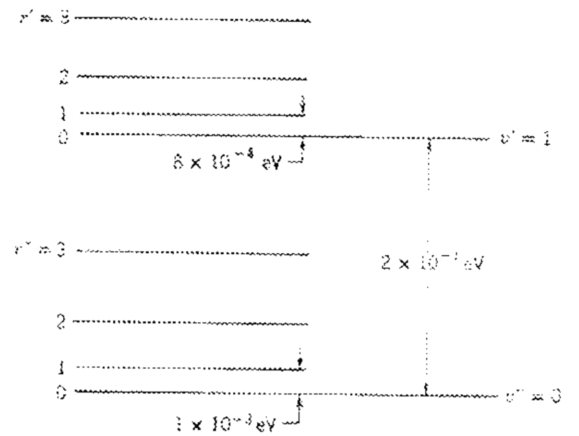
C3. Two identical perfect gases with the same pressure P and the same number of particles N, but with different temperatures T_1 and T_2, are confined in two vessels, of volume V_1 and V_2, which are then connected. Find the change in entropy after the system has reached equilibrium.

MODERN

D1. An atom obeys LS coupling scheme.

(a) Using the vector model, find an expression for the angle θ between the total angular momentum vector **J** and the total magnetic moment of the electron μ_T (sum of the orbital and intrinsic magnetic moments). Your answer should be in terms of the quantum numbers j, l, and s.
(b) What is the time-averaged value of the total magnetic moment?
(c) Now consider the specific case of the 3^2P state of Na atom. The two fine structure components of the $3^2P \rightarrow 3^2S$ transition have wavelengths λ_1 = 5896 Å and λ_2 = 5890 Å. Using this information, determine the precession frequency of μ_T in the 3^2P state of Na.

D2.



The energy-level diagram for the rotational levels in each of the two lowest vibrational states of the electronic ground state for a diatomic molecule is given in the above figure. (The index r labels rotational levels, the index v labels vibrational levels.)

(a) Find the energies of the transitions that give rise to allowed spectral lines (in the infrared spectrum) assuming that the molecule is made of two non-identical nuclei.
(b) If the nuclei are identical would any direct rotational or vibration-rotational emission lines be observed? Why? What is a way to probe experimentally the energy levels of molecules of identical atoms?
(c) Calculate the rotational inertia, i.e., the moment of inertia, of the molecule in each vibrational level.
(d) Calculate the zero-point (vibrational) energy

D3. In a nuclear reaction $^9Be+a \rightarrow {}^{12}C + n$, alpha particles from ^{210}Po, with energy of 5.30 MeV bombard a ^9Be target at rest.
(a) Find the Q value of the reaction
(b) Find the energy of the neutrons at angles 0°, 90°, 180° with the direction of the incident alpha particles in the non-relativistic approximation, assuming that the residual nucleus ^{12}C is left in its first excited state, 4.43 Mev above the ground state.
Mass of ^4He = 4u + 2.42 MeV/c^2
Mass of ^9Be = 9u + 11.35 MeV/c^2
Mass of n = u + 8.07 MeV/c^2
Mass of ^{12}C = 12u + 4.43 MeV/c^2
u = 931 MeV/c^2

D4. The concentration of charge carriers in a semiconductor is a critical parameter in device performance. Using the definition of current as being the rate of charge flowing past a surface ($I = \Delta Q/\Delta t$),

(a) Find an expression for the number of charge carriers per unit volume n, in terms of the amount of current flowing through the semiconductor, I, the fundamental unit of charge q, the average speed of the charge carrier, v, and the rectangular area of the surface the charge flows through, $A = w \times h$.

(b) Why is it impossible to use this formula to measure the concentration of carriers?

(c) If a magnetic field is applied perpendicularly to the current flow, draw a diagram showing the forces that act on the charge carrier in a direction perpendicular to both the current flow and the magnetic field.

(d) Explain in words what happens to a negatively charged particle entering the region with the magnetic field. Repeat for a positive charge.

(e) Balance the forces found in part (c) to predict the voltage genrated, V. Combine this result with your result from part (a), to find a new expression for n. Explain why this formula makes it possible to determine n.

QUANTUM

E1. Imagine that the electron in a hydrogen atom is in the combined position-spin state

$$R_{12}\left(\sqrt{\frac{1}{3}}\,Y_1^0|+\rangle+\sqrt{\frac{2}{3}}\,Y_1^1|-\rangle\right)$$

Here R_{12} is the radial wavefunction $R_{12}=\frac{1}{\sqrt{24}}a_0^{-3/2}\frac{r}{a_0}\exp\left(-\frac{r}{2a_0}\right)$; Y_1^0 is the spherical harmonic $Y_1^0=\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$; $Y_1^1=-\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{i\phi}$; and $|+\rangle$ and $|-\rangle$ are "spin up" and "spin down" states (relative to the z axis), sometimes written as χ_+ and χ_- instead.

(a) If you measured the orbital angular momentum squared (L^2), what values might you get, and what is the probability of each?

(b) Same for the z-component of orbital angular momentum (L_z)

(c) Same for the spin angular momentum squared (S^2)

(d) Same for the z-component of spin angular momentum (S_z).

Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$ be the total angular momentum.

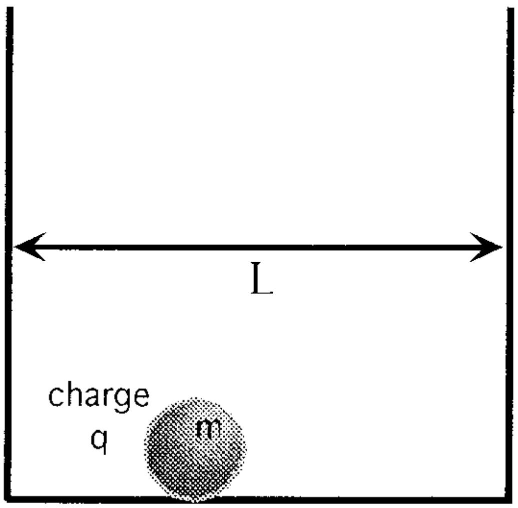
(e) If you measured J^2 what values might you get, and what is the probability of each?

(f) Same for J_z.

(g) If you measured the position of the particle, what is the probability density for finding it at r,θ,ϕ?

(h) If you measured both the z-component of the spin and the distance from the origin (note that these are compatible observables), what is the probability density for finding the particle with spin up and at a radius r?

E2.



Consider a particle of mass m in a one-dimensional box of length L, that is, let it be in an infinite square-well potential as indicated in the Figure.

(a) Calculate the energy of the two lowest stationary states according to standard quantum mechanics. Give energy values and normalized wavefunctions.

(b) Suppose the particle has probability $P1$ to be in the first or lowest state and probability $P2$ to be in the second or next lowest level where $1 = P1 + P2$. Calculate the expected energy of the particle for a 50-50 combination ($P1 = P2$).

(c) Calculate the expected position $\langle x(t) \rangle$ as a function of constants m, q, L, $\hbar$, and probabilities $P1$ and $P2$.

(d) Use your result from (c) to tell what $P1$ and $P2$ give the maximum $\langle x \rangle$ and what that value is as a fraction of length L. (Numerical answer only will be accepted for this part.)

E3. Calculate the probability current density $\vec{j} = -i(\hbar/2\mu)\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$ for the hydrogenic wavefunction $\psi_{nlm}(\vec{r}) = R_{nl}(r)\Theta_{lm}(\theta)e^{im\phi}e^{-iE_n t/\hbar}$, where μ is the reduced mass, and R_{nl} and Θ_{lm} are real functions of r and θ, respectively. Interpret your result.

Note: in spherical coordinates the gradient operator is

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$