Three-qubit quantum-gate operation in a cavity QED system

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A scheme is proposed to obtain three-qubit gates, such as quantum-phase gate and \( C^3 \)-NOT gate operations, in a cavity QED system where highly detuned cavity-field modes interact with a four-level system in an inverted-Y configuration. The influence of the Stark shift is also included in such proposed gate operations. Since only the metastable lower levels are involved in the gate operations, the gates are not affected by the atomic decay rates. The potential application of such gates to realize Grover’s algorithm is also discussed.

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I. INTRODUCTION

In the last decade, considerable progress has been made in quantum computing, which relies on the quantum-information processing, quantum-computing networking, etc., using various physical systems [1]. It is well known that a quantum-computing network can be partitioned into a sequence of one-qubit rotation and two-qubit gates [2]. There are a large number of theoretical proposals for two-qubit gate operations and some experiments have been carried out to implement CNOT (controlled-NOT) gates or controlled phase gates in ion trap [3], cavity-QED system [4], NMR [5], quantum dots [6], and superconducting charge qubits [7].

Multiqubit-controlled quantum gates are very useful in the construction of quantum-computing networks, implementing quantum-error-correction protocols and quantum algorithms. Important steps in the direction of realizing multiqubit gates have been recently made in several physical schemes. There are proposals to implement three-qubit Toffoli gates with neutral atoms in an optical lattice [8] and hybrid atom-photon qubit via a cavity-QED system [9]. In another interesting work the implementation of a multiqubit unitary gate using adiabatic passage with a single-mode optical cavity has been proposed [10]. Experimental realization of a controlled phase gate in a three-qubit NMR system was reported [11]. Very recently, a \( n \)-qubit-controlled phase gate with superconducting quantum-interference devices coupled to a resonator was also proposed [12].

In this work we propose a way to realize a three-qubit-controlled phase gate and a \( C^3 \)-NOT (controlled-NOT gates with two controlling bits) gate in a cavity-QED system where a four-level atom in inverted-Y configuration interacts with three cavity-field modes. The cavity modes are considered to be highly detuned from the corresponding one-photon transitions. The Stark shifts are incorporated in this model. This system can be used to realize a number of three-qubit logic operations such as controlled quantum-phase gate and \( C^3 \)-NOT gate. In this scheme, the gate operation takes place in the lower states of the system and hence the effect of decoherence due to radiative damping is minimized in this system.

The rest of the paper is organized as follows. In Sec. II, we give the physical model with a theoretical description where a four-level system in inverted-Y configuration interacts with the three quantized cavity-field modes. Section III is devoted to a description of the implementations of a three-qubit quantum-phase gate and \( C^3 \)-NOT gate. The application of these three-qubit quantum gates to Grover’s algorithm is discussed in Sec. IV. Some concluding remarks are given in Sec. V.

II. MODEL

We consider a closed four-level atomic system in inverted-Y configuration, as shown in Fig. 1, which can be easily realized in a rubidium atom [13]. Essentially, this is a double-electromagnetically induced transparency (double-EIT) system [14] and many interesting results have been predicted for this system, such as vacuum Rabi splitting [15] and generalized dark-state polariton [16], etc. Levels \( |g\rangle \), \( |e\rangle \), and \( |d\rangle \) form a three-level ladder-type configuration and levels \( |l\rangle \), \( |e\rangle \), and \( |g\rangle \) form a three-level \( \Lambda \)-type configuration. So, the composite system consists of two subsystems, i.e., one ladder-type three-level system, and another \( \Lambda \)-type three-level system and hence exhibits double-EIT characteristics [14]. The transitions \( |g\rangle \) to \( |e\rangle \), \( |l\rangle \) to \( |e\rangle \), and \( |e\rangle \) to \( |d\rangle \) interact

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FIG. 1. Schematic diagram of a four-level atomic system in inverted-Y configuration. Here, \( \omega_1 \), \( \omega_2 \), and \( \omega_3 \) are frequencies of probe, coupling, and pumping fields, respectively. The parameters \( \Delta_1 \), \( \Delta_2 \), and \( \Delta_3 \) are corresponding frequency detuning of these field frequencies from the respective atomic transition frequencies.
with the cavity modes having annihilation operators \( \hat{a} \), \( \hat{b} \), and \( \hat{c} \), respectively. The corresponding atomic transition frequencies and the cavity-field mode frequencies are \( \omega_{eg} \), \( \omega_{el} \), \( \omega_{de} \) and \( \omega_1 \), \( \omega_2 \), \( \omega_3 \), respectively. The Hamiltonian describing this system in the interaction picture can be written as

\[
H = \hbar [g_1 \hat{a} e^{i \Delta_1 t} |e\rangle \langle g| + g_2 \hat{b} e^{i \Delta_2 t} |e\rangle \langle l| + g_3 \hat{c} e^{i \Delta_3 t} |d\rangle \langle e| + \text{H.c.}],
\]

in which \( g_i \) (\( i = 1 - 3 \)) are atom-field coupling coefficients (which are assumed to be real) for the three transitions discussed above. The one-photon detunings of the cavity modes for these transitions are defined as \( \Delta_1 = \omega_{ek} - \omega_1 \), \( \Delta_2 = \omega_{el} - \omega_2 \), and \( \Delta_3 = \omega_{de} - \omega_3 \), respectively.

If we assume the initial photon occupation numbers of the three cavity modes to be \( n_1 \), \( n_2 \), \( n_3 \), respectively, then the state vector of the coupled atom-cavity system can be defined at any time in terms of the following basis vectors:

\[
\Psi(t) = a_1(t)|g\rangle n_1, n_2, n_3 \rangle + a_2(t)|e\rangle n_1 - 1, n_2, n_3 \rangle
+ a_3(t)|l\rangle n_1, n_2 + 1, n_3 \rangle + a_4(t)|d\rangle n_1 - 1, n_2, n_3 - 1 \rangle,
\]

where \( a_i(t) \) (\( i = 1 - 4 \)) are complex numbers representing occupation probability amplitudes of the corresponding basis vectors. The dynamical evolution of this coupled atom-cavity system can be obtained by the time-dependent Schrödinger equation

\[
\frac{i \hbar}{\partial t} \Psi = H \Psi,
\]

which gives equations of the motion for the probability amplitudes as

\[
i \dot{a}_1 = g_2 \sqrt{n_1} e^{-i \Delta_1 t} a_2,
\]

\[
i \dot{a}_2 = g_1 \sqrt{n_1} e^{i \Delta_1 t} a_1 + g_2 \sqrt{n_2 + 1} e^{i \Delta_2 t} a_3 + g_3 \sqrt{n_3} e^{-i \Delta_3 t} a_4,
\]

\[
i \dot{a}_3 = g_2 \sqrt{n_2 + 1} e^{-i \Delta_2 t} a_2,
\]

\[
i \dot{a}_4 = g_3 \sqrt{n_3} e^{i \Delta_3 t} a_2.
\]

To simplify the above equations further, we make use of the following transformations:

\[
a_1 \rightarrow p_1,
\]

\[
a_2 e^{-i \Delta_1 t} \rightarrow p_2,
\]

\[
a_3 e^{-i (\Delta_2 - \Delta_3) t} \rightarrow p_3,
\]

\[
a_4 e^{i (\Delta_3 + \Delta_2) t} \rightarrow p_4,
\]

and obtain the following set of equations:

\[
\dot{p}_1 = -i g_2 \sqrt{n_1} p_2,
\]

\[
\dot{p}_2 = -i \Delta_1 p_2 - i(g_1 \sqrt{n_1} p_1 + g_2 \sqrt{n_2 + 1} p_3 + g_3 \sqrt{n_3} p_4),
\]

\[
\dot{p}_3 = -i(\Delta_1 - \Delta_2) p_3 - i g_2 \sqrt{n_2 + 1} p_2,
\]

\[
\dot{p}_4 = -i(\Delta_1 + \Delta_3) p_4 - i g_3 \sqrt{n_3} p_2.
\]

Next, we work in the limit of large one-photon detunings of the cavity modes with their respective atomic transitions, i.e., \( \Delta_i \gg g \), and for the sake of simplicity we assume \( g_1 = g_2 = g_3 = g \). Under this condition, the state \( |e\rangle \) can be adiabatically eliminated by formally integrating the equation \( p_2 \) under slowly varying amplitude approximation and substituting the result in the remaining equations [17]. Alternatively, the adiabatic elimination of \( |e\rangle \) can be carried out by neglecting the time derivative [18] of \( p_2 \), i.e., one can set \( p_2 = 0 \) and substitute the value in the remaining equations. The probability amplitude equations (6) reduce to

\[
\dot{p}_1 = \frac{g^2}{\Delta_1} (n_1 p_1 + \sqrt{n_1 (n_2 + 1)} p_3 + \sqrt{n_2} n_1 p_4),
\]

\[
\dot{p}_3 = -i(\Delta_1 - \Delta_2) p_3 + \frac{g^2}{\Delta_1} (\sqrt{n_1 (n_2 + 1)} p_1 + (n_2 + 1) p_3

+ \sqrt{n_2} n_1 p_4),
\]

\[
\dot{p}_4 = -i(\Delta_1 + \Delta_3) p_4 + \frac{g^2}{\Delta_1} (\sqrt{n_3} n_1 p_1 + \sqrt{n_2} (n_2 + 1) p_3 + n_3 p_4).
\]

The above equations for the probability amplitudes can be derived by using the following effective Hamiltonian [17,18]:

\[
H_{\text{eff}} = -\frac{\hbar g^2}{\Delta_1} [\hat{a} \hat{a}^\dagger |g\rangle \langle g| + \hat{b} \hat{b}^\dagger |l\rangle \langle l| + \hat{c} \hat{c}^\dagger |d\rangle \langle d|] - \frac{\hbar g^2}{\Delta_1} [\hat{a} \hat{b}^\dagger |g\rangle \langle l| + \hat{b} \hat{a}^\dagger |l\rangle \langle g| + \hat{c} \hat{c}^\dagger |d\rangle \langle d|] + \sum_{\langle d| \hat{b}^\dagger \hat{c}^\dagger |l\rangle \langle l| + h(\Delta_1 - \Delta_2) |l\rangle \langle l| + h(\Delta_1 + \Delta_3) |d\rangle \langle d|.\]

In this effective Hamiltonian the first square bracket represents the Stark shifts while the second square bracket gives the interactions leading to transitions from the initial state \( |g\rangle \) to the two final states \( |l\rangle \) and \( |d\rangle \) corresponding to the \( \Lambda \) subsystem and ladder subsystem of our four-level model. The last two terms are the consequences of the two-photon detunings of the two subsystems. It is interesting to note that the Stark-shift terms are of the same order of magnitude as the field-atom interaction terms [18]. The equations described in Eqs. (7) give interesting results under the following two limiting conditions on the parameters.

Case I. \( \Delta_1 > \Delta_2, \Delta_3 \). Though we have restricted \( \Delta_i \gg g \) to derive Eq. (7), we can further impose the condition \( \Delta_1 > \Delta_2, \Delta_3 \) on Eq. (7). We obtain the solution of Eq. (7) under the initial conditions: \( p_1(0) = 1 \), \( p_2(0) = 0 \), \( p_4(0) = 0 \); \( n_1 = 1 \), \( n_2 = 0 \), \( n_3 = 1 \) as

\[
p_1(t) = e^{-i t} \left[ \cos(\Omega t) + \frac{i}{2 \Omega} \left( \Delta_1 - \frac{g^2}{\Delta_1} \right) \sin(\Omega t) \right] p_1(0),
\]

\[
p_3(t) = -i(\Delta_1 - \Delta_2) p_3 - i g_2 \sqrt{n_2 + 1} p_2,
\]

\[
p_4(t) = -i(\Delta_1 + \Delta_3) p_4 - i g_3 \sqrt{n_3} p_2.
\]
Note that these equations depend on all \( H_9004 \) \( i/g \), i.e., further simplified under a more explicit assumption of equations of motion for the probability amplitudes can be obtained. We use initial conditions \( p_1(0) = 1 + g^2/2 + 9g^4/\Delta_1^2 \), goes as follows. Although we have used the condition \( \Delta_1 \gg g \) in deriving Eq. (7), however, the equations of motion for the probability amplitudes can be further simplified under a more explicit assumption of \( \Delta_1 + \Delta_3 \gg g \). Under such a condition, we can further set \( \dot{p}_4 = 0 \) and thus obtain the evolution of the system involving only two lower states as

\[
p_1 = i\frac{g^2}{\Delta_1 \Delta_3} \left[ p_1 - i\frac{g^2 n_3 (n_2 + 1)}{\Delta_1 - g^2 n_3} \right] p_3,
\]

\[
p_3 = i\frac{g^2 n_3 (n_2 + 1)}{\Delta_1 - g^2 n_3} p_1 = \left[ \frac{1}{2}(\Delta_1 - \Delta_2) - i\frac{g^2 (n_2 + 1)}{\Delta_1 - g^2 n_3} \right] p_3.
\]

Case II. \( \Delta_1 + \Delta_3 \gg g \). Another interesting situation, which further simplifies Eq. (7), goes as follows. Although we have used the condition \( \Delta_1 \gg g \) in deriving Eq. (7), however, the equations of motion for the probability amplitudes can be further simplified under a more explicit assumption of \( \Delta_1 + \Delta_3 \gg g \). Under such a condition, we can further set \( \dot{p}_4 = 0 \) and thus obtain the evolution of the system involving only two lower states as

\[
p_1 = i\frac{g^2}{\Delta_1 \Delta_3} \left[ p_1 - i\frac{g^2 n_3 (n_2 + 1)}{\Delta_1 - g^2 n_3} \right] p_3,
\]

\[
p_3 = i\frac{g^2 n_3 (n_2 + 1)}{\Delta_1 - g^2 n_3} p_1 = \left[ \frac{1}{2}(\Delta_1 - \Delta_2) - i\frac{g^2 (n_2 + 1)}{\Delta_1 - g^2 n_3} \right] p_3.
\]

Note that these equations depend on all \( \Delta_1 \) (\( i = 1-3 \)) and all \( n_i \) (\( i = 1-3 \)). The solution of these equations can easily be obtained. We use initial conditions \( n_1 = 1, n_2 = 0, \) and \( n_3 \) arbitrary, i.e.,

\[
p_1 = e^{i\Delta t} \left[ \cos(Qt) + i\frac{g^2}{2Q} \left[ (\Delta_1 - \Delta_2) \sin(Qt) \right] p_1(0) + i\frac{g^2}{\Delta_1 Q} \sin(Qt) p_3(0) \right],
\]

\[
p_3 = e^{i\Delta t} \left[ \cos(Qt) - i\frac{g^2}{2Q} \left[ (\Delta_1 - \Delta_2) \sin(Qt) \right] p_3(0) + i\frac{g^2}{\Delta_1 Q} \sin(Qt) p_1(0) \right],
\]

in which

\[
s = \frac{1}{2} \left[ \frac{2g^2}{\Delta_1 - g^2 n_3} - (\Delta_1 - \Delta_2) \right],
\]

\[
Q = \frac{1}{2} \left[ (\Delta_1 - \Delta_2)^2 + \left( \frac{2g^2}{\Delta_1 - g^2 n_3} \right)^2 \right]^{1/2},
\]

Under the two-photon resonance condition, i.e., \( \Delta_2 = \Delta_1 \), we get the solution

\[
p_1 = \frac{1}{2} [p_1(0) + p_3(0)] (e^{i\Phi} - 1) + p_1(0),
\]

\[
p_3 = \frac{1}{2} [p_1(0) + p_3(0)] (e^{i\Phi} - 1) + p_3(0),
\]

where \( \Phi = 2g^2/\Delta_1 \).

III. LOGIC GATE OPERATION

For case I, discussed above in Eq. (9), if we select the experimental parameters in such a way that \( r = 0 \rightarrow \Delta_1 = 2g^2 \) and \( W = g \), then under the condition \( W = \pi \), \( p_1(\pi) \) becomes \(-1\) for the initial conditions of \( p_1(0) = 1 \) and \( p_3(0) = p_4(0) = 0 \). Note that \( p_3(t) \) and \( p_4(t) \) do not evolve in this case. In this situation one can perform a three-qubit phase-gate operation, which will be discussed in detail in the text following case II.

Next, we focus our attention on case II, i.e., for the solution given in Eq. (12), which is obtained under the conditions of large single-photon detunings such as \( \Delta_1 \gg g \) and \( \Delta_1 + \Delta_3 \gg g \). In this solution we keep \( n_3 = 1 \) for the quantum-gate implementation. If we further set the condition

\[
\Delta_1 - \Delta_2 = \frac{2g^2}{\Delta_1 - g^2 n_3},
\]

and use the interaction time \( t = \pi \) such that \( Q = \pi \), or

\[
g = \pi \sqrt{2g/\Delta_1},
\]

then \( p_1(t) \) \( [p_3(t)] \) becomes \(-1\) for the initial condition \( p_1(0) \) \( [p_3(0)] \) = 1. This selection of interaction time allows one to perform the following three-qubit phase-gate operation (PHASE):

\[
|0\rangle_a |0\rangle_b |0\rangle_c \rightarrow |0\rangle_a |0\rangle_b |0\rangle_c,
\]

\[
|0\rangle_a |1\rangle_b |0\rangle_c \rightarrow |0\rangle_a |0\rangle_b |1\rangle_c,
\]

\[
|0\rangle_a |0\rangle_b |l \rightarrow |0\rangle_a |0\rangle_b |l,
\]

\[
|0\rangle_a |1\rangle_b |l \rightarrow |0\rangle_a |1\rangle_b |l,
\]

\[
|0\rangle_a |0\rangle_b |1 \rightarrow |0\rangle_a |1\rangle_b |0,
\]

\[
|0\rangle_a |0\rangle_b |l \rightarrow -|0\rangle_a |1\rangle_b |l.
\]

This gate operation clearly involves the atomic ground-state basis and the Fock-state basis in \( \hat{b} \) and \( \hat{c} \) modes. The two
field modes act as the controlling qubits and the two atomic states form the target qubit. For the phase-gate operation described above we have used the condition $\Delta_1 \neq \Delta_2 \neq \Delta_3$ and $\Delta_i > g$. We would like to emphasize the fact that a three-qubit quantum phase gate (QPG) operation is possible only under this condition of detuning in the current system. Once detunings are equal, then the two-photon resonance condition is satisfied and we no longer have a QPG operation. So, in order to have a successful QPG operation we need to avoid the two-photon resonance condition. In the QPG operation the cavity modes $b$, $c$ and the ground metastable states $|l\rangle$ and $|g\rangle$ of the atom are involved and transitions between these atomic ground states are dipole forbidden. This implies that decoherence mechanism due to radiative damping is minimal in this gate, which makes this QPG scheme advantageous. In order to perform gate operation faithfully one needs to do it within an interaction time $\tau$ such that the cavity damping constant is much less than $1/\tau$, i.e., using a very high-$Q$ cavity.

In order to achieve $C^\dagger$-NOT gate operation from the QPG operation we have to perform a rotation of the target qubit before and after the QPG operation by applying the Hadamard transformation in general on the atomic states. The Hadamard transformation $H$ on the states $|x\rangle$ and $|y\rangle$ is defined as

\[
H|x\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle),
\]
\[
H|y\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle).
\]  

(18)

By applying the above operations in the proper sequence we get $C^\dagger$-NOT gate operation as

\[
|0\rangle_a|0,0,0,0\rangle \rightarrow |0\rangle_a|0,0,0,0\rangle,
\]
\[
|0\rangle_a|1,0,0,0\rangle \rightarrow |0\rangle_a|1,0,0,0\rangle,
\]
\[
|0\rangle_a|0,0,0,0\rangle \rightarrow |0\rangle_a|0,0,0,0\rangle,
\]
\[
|0\rangle_a|1,0,0,0\rangle \rightarrow |0\rangle_a|1,0,0,0\rangle,
\]
\[
|0\rangle_a|0,1,0,0\rangle \rightarrow |0\rangle_a|0,1,0,0\rangle,
\]
\[
|0\rangle_a|1,1,0,0\rangle \rightarrow |0\rangle_a|1,1,0,0\rangle,
\]
\[
|0\rangle_a|0,0,1,0\rangle \rightarrow |0\rangle_a|0,0,1,0\rangle,
\]
\[
|0\rangle_a|1,0,1,0\rangle \rightarrow |0\rangle_a|1,0,1,0\rangle.
\]  

(19)

Clearly, we obtain a $C^\dagger$-NOT gate where field qubits of modes $b$ and $c$ are controlling qubits and the atomic qubit is the controlled or target qubit.

IV. APPLICATION TO GROVER’S ALGORITHM

Searching a database using quantum mechanics (quantum-mechanical superpositions and quantum entangle-
ments) can be much faster than its classical counterpart as predicted by Grover [17]. To find an object from an unsorted database having $N$ objects one usually needs $O(N)$ steps in classical searching algorithm, however, quantum mechanically only $O(\sqrt{N})$ steps [17] are required. Essentially, the Grover’s algorithm requires inversion of the phase of the desired basis state (out of equal amplitude $N=2^n$ states, $n =$ number of qubits) and then inverts all the basis states about the average amplitude of all the states. This can be achieved by making use of diffusion transform $D$, which increases the probability of identifying the desired state. The diffusion transform is given by [1]

\[
D_{ij} = \frac{2}{N} - \delta_{ij}.
\]  

(20)

This step is carried out $\pi\sqrt{N}/4$ times to maximize the probability for the desired state and finally a measurement of the whole system is done to get the desired state of interest [17].

Next, we discuss how in general the three-qubit $(N=8)$ phase gate can be used to implement the Grover’s algorithm. Then we use the three-qubit gate discussed above along with one-qubit gates to implement it. The quantum circuit is given in Fig. 2. Let the initial state of the three qubits be $|000\rangle$ and the ancillary qubit be in the state $|l\rangle$. The first Hadamard gates prepare the three qubits in equal superposition state and ancillary in the $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ state such that the wave function for the quantum computation becomes [1]

\[
|\zeta\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)(|0\rangle-|1\rangle).
\]  

(21)

The search problem can be recast as an oracle problem in which $n$ items are labeled as $|0,1,\ldots,N-1\rangle$ and $|\zeta_0\rangle$ is an unknown marked item. The $n$-bit binary function is computed by the oracle as

\[
f:|0,1|^n \rightarrow |0,1|,
\]  

(22)

and it has the following values:

\[
f(\zeta) = 1 \quad \text{if } \zeta = \zeta_0,
\]
\[
=0 \quad \text{otherwise}.
\]  

(23)

After the oracle query [1] $|\zeta\rangle\rightarrow|\zeta\rangle\oplus f(\zeta)$, the value of $f(\zeta)$ is loaded into ancillary qubit. The state of ancillary is so chosen that it remains the same for $f(\zeta)=0$ but changes its sign when $f(\zeta)=1$. For simplicity we set $f(\zeta)=1$ when

\[
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\]
\( \zeta = \zeta_0 = (110) \) and the circuit shown in Fig. 2 solves the searching problem for eight possible values of \( \zeta \). The wave function for quantum computation after oracle query is

\[
|\zeta \rangle |\xi \rangle = \frac{1}{2} \left( |000 \rangle + |001 \rangle + |010 \rangle + |011 \rangle + |100 \rangle + |101 \rangle - |110 \rangle - |111 \rangle \right) (|0 \rangle - |1 \rangle),
\]

(24)

which is different from the original wave function \([\text{Eq. (21)}]\) by a sign difference on the marked state. The change in sign of ancillary is kicked back in front of the register \( |\zeta \rangle \) and hence the ancillary register is unchanged and will be of no importance for further work. The important step in Grover’s algorithm is to convert the phase difference in front of \( |110 \rangle \) to amplitude difference using the unitary transformation \( D_{ij} \) of Eq. (20) with \( n = 3 \). The transformation \( D \) can be decomposed as

\[
D = H^{\otimes 3} D' H^{\otimes 3},
\]

(25)

where the diagonal matrix \( D' \) is given by

\[
D' = \delta_{ij} \quad (i = 1),
\]

\[
e = - \delta_{ij} \quad (i = 2 - 8),
\]

(26)

which gives the controlled shift (minus sign) in front of the basis element \( |000 \rangle \). The matrix \( D' \) can be decomposed (up to an overall phase) as

\[
D' = \sigma_x^{\otimes 3} (I^{\otimes 2} H) C^{\otimes 3} \text{NOT}(I^{\otimes 2} H) \sigma_x^{\otimes 3} = \sigma_x^{\otimes 3} \text{CPHASE}\sigma_x^{\otimes 3}.
\]

(27)

The operation involving \( \sigma_x \) and Hadamard transformation \( H \), etc., can be obtained by one-bit unitary gates. In a cavity-QED system, the passage of a resonant two-level atom through the cavity can give one-bit unitary gate operation. One may need a classical field pulse during the middle of atomic transit to complete the task. We briefly outline this procedure here just for the sake of completeness [19].

Let the two-level atom with upper and lower states \( |\alpha \rangle \) and \( |\beta \rangle \), respectively, interact with a cavity mode \( \hat{a} \) (\( G = \) interaction strength) such that its interaction Hamiltonian is described by

\[
H_{int} = \hbar G (\hat{a} |\alpha \rangle \langle \beta| + \hat{a}^\dagger |\beta \rangle \langle \alpha|),
\]

(28)

and the corresponding unitary operator is

\[
U(t) = \cos(Gt \sqrt{\hat{a}^\dagger \hat{a} + 1}) |\alpha \rangle \langle \alpha| + \cos(Gt \sqrt{\hat{a}^\dagger \hat{a} + 1}) |\beta \rangle \langle \beta|
\]

\[
- i \sin(Gt \sqrt{\hat{a}^\dagger \hat{a} + 1}) \sqrt{\hat{a}^\dagger \hat{a} + 1} |\alpha \rangle \langle \beta|
\]

\[
- i \sqrt{\hat{a}^\dagger \hat{a} + 1} \sin(Gt \sqrt{\hat{a}^\dagger \hat{a} + 1}) |\beta \rangle \langle \alpha|.
\]

(29)

For a choice of interaction time \( T = \pi/2G \), the state \( |\beta \rangle |0 \rangle \) does not evolve but \( |\beta \rangle |1 \rangle \) evolves to \(-i|\alpha \rangle |0 \rangle \). Next, the atom interacts with a classical field pulse of amplitude \( E \) and duration \( \tau' \), which prepares the coherent superposition of the atomic states

\[
|\alpha \rangle \to \cos(R) |\alpha \rangle + i e^{-i\eta} \sin(R) |\beta \rangle,
\]

\[
|\beta \rangle \to i e^{i\eta} \sin(R) |\alpha \rangle + \cos(R) |\beta \rangle,
\]

(30)

in which angle \( R = \Omega \tau' / 2 \), Rabi frequency \( \Omega = |\mu| E / \hbar \), and the dipole moment \( \mu = |\mu| e^{i \eta} \). The cavity-field state remains in state \( |0 \rangle \) during this interaction. Again, the atom interacts with the cavity field for the interaction time \( T = \pi/2g \), comes out in state \( |\beta \rangle \), and consequently projects the cavity field (depending on its initial state \( |0 \rangle \) or \( |1 \rangle \)) to

\[
|0 \rangle \to \cos(R) |0 \rangle + i e^{-i\eta} \sin(R) |1 \rangle,
\]

\[
|1 \rangle \to e^{-i\eta} \sin(R) |0 \rangle - \cos(R) |1 \rangle.
\]

(31)

The expression for the one-bit operator is

\[
U_{R, \eta} = \left( \begin{array}{cc}
\cos(R) & e^{-i\eta} \sin(R) \\
-ie^{-i\eta} \sin(R) & -\cos(R)
\end{array} \right)
\]

\[
= \cos(R) \sigma_z + (\eta \sin(R)) \sigma_x - (\sin(\eta) \sin(R)) \sigma_y.
\]

(32)

Clearly, for \( R = \pi/2, \eta = 0 \); \( U_{R, \eta} = \sigma_z \) and for \( R = \pi/4, \eta = 0 \); \( U_{R, \eta} = (1/\sqrt{2})(\sigma_z + \sigma_y) = H. \) In this way we complete the cavity-QED implementation of Grover’s algorithm by using one-qubit gates and a three-qubit quantum-phase gate. Our scheme needs a high-Q cavity prepared appropriately in two modes and the passage of two- and three-level atoms with controlled interaction times will perform Grover’s algorithm.

V. CONCLUSIONS

In this work we have studied the proposal for realizing three-photon qubits in a cavity-QED system consisting of an interaction of a four-level system in inverted-Y configuration with three different cavity modes. Using such a system, the implementations of a quantum-phase gate, \( C^3 \text{-NOT} \) gate, and Grover’s algorithm are demonstrated.

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