LETTER TO THE EDITOR

Fock representation for two-mode quantum phase operators’ eigenstates

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Abstract. The quantum phase description of Noh, Fougères and Mandel (NFM) leads to a two-mode theory of phase in the strong local oscillator limit. We show that the explicit form of the eigenstates of the sine and the cosine phase operators in two-mode Fock space can be constructed. This brings great convenience to the NFM quantum phase approach.

Recently, phase operators and phase measurements in quantum optics have attracted the attention of physicists [1]. In [2], NFM proposed an operational quantum phase description, which means that the Hermitian phase operators can be defined in an operational way. Their theory is based on an eight-port homodyne-detection scheme. Then, in [3], Freyberger, Heni and Schleich (FHS) further showed that in the limit of a strong local oscillator, the NFM description contains an essential two-mode basis which leads to the simultaneously measurable operator pair

$$\hat{C} = \frac{\hat{x}}{\sqrt{\hat{x}^2 + \hat{p}^2}}$$

and

$$\hat{S} = \frac{\hat{p}}{\sqrt{\hat{x}^2 + \hat{p}^2}}$$

where $$\hat{x} = \hat{x}_1 + \hat{x}_{10}$$ and $$\hat{p} = \hat{p}_1 - \hat{p}_{10}$$ commute with each other and

$$\hat{x}_i = \frac{1}{\sqrt{2}}(\hat{a}_i + \hat{a}_i^\dagger) \quad \hat{p}_i = \frac{1}{\sqrt{2i}}(\hat{a}_i - \hat{a}_i^\dagger) \quad i = 1, 10.$$ (3)

where $$(\hat{a}_1, \hat{a}_{10})$$ are the two input mode operators defined in [3] for the eight-port interferometer in the operator description by NFM and they can be any arbitrary states. In this paper, we adopt the definitions used in [3] for convenience and use them as a starting point for our calculation.

The simultaneous eigenstates of $$\hat{x}$$ and $$\hat{p}$$ and their phase space are also considered in [3] as

$$\hat{x}|x, p\rangle = x|x, p\rangle \quad \hat{p}|x, p\rangle = p|x, p\rangle.$$ (4)
These eigenstates compose a complete set
\[ \int \int dx \ dp \langle x, p \rangle |x, p\rangle = 1 \ . \] (5)

With this condition, the phase operator pair can be described in the phase space as
\[ \hat{C} = \int \int dx \ dp \frac{x}{\sqrt{x^2 + p^2}} |x, p\rangle \langle x, p| \]
\[ \hat{S} = \int \int dx \ dp \frac{p}{\sqrt{x^2 + p^2}} |x, p\rangle \langle x, p| \] (6) (7)

with \([\hat{C}, \hat{S}] = 0\).

Although the representation of \(|x, p\rangle\) in terms of the quadrature states and of the number states of modes 1 and 10 is discussed in [3], we still think that it is necessary to derive the explicit form of \(|x, p\rangle\) in the two-mode Fock space. With an explicit form of \(|x, p\rangle\), many calculations can be made much easier and apparent, as will be seen in later sections. The paper is arranged as follows. In section 2, we derive the explicit form of \(|\xi\rangle\) with \(\xi = (1 + i\xi_2)x + ip\) and, then, employ the newly developed technique of integration within an ordered product (IWOP) of operators [4, 5] to prove the completeness relation of such a constructed eigenstate (5). As one can see, our approach is much simpler and more direct than the previous one adopted in [3]. In section 3, we deduce the normally ordered form of \(\hat{C}\) and \(\hat{S}\) by virtue of the IWOP technique. To compare our results with the single-mode phase description proposed by Paul in [6], we further analyse our state \(|\xi\rangle\) in section 4. We show that the normally ordered forms of \(\hat{C}\) and \(\hat{S}\) reduce to normally ordered cosine and sine Paul phase operators, respectively.

We now show that the explicit form of \(|x, p\rangle\) in a two-mode Fock space takes the form of
\[ |x, p\rangle = \exp \left[ -\frac{1}{4}|\xi|^2 + \xi a_1^\dagger + \xi^* a_{10}^\dagger - a_1 a_{10}^\dagger \right] |00\rangle \equiv |\xi\rangle \] (8)

where \(|00\rangle\) is the two-mode vacuum state, \(\xi = \xi_1 + i\xi_2 = 1/(\sqrt{2}) (x + ip)\) is a complex number.

Actually, by operating \(\hat{a}_1\) and \(\hat{a}_{10}\) on \(|x, p\rangle\), respectively, we have
\[ \hat{a}_1 |\xi\rangle = (\xi - \hat{a}_{10}^\dagger) |\xi\rangle \quad \hat{a}_{10} |\xi\rangle = (\xi^* - \hat{a}_1^\dagger) |\xi\rangle \ . \] (9)

After combining them, we obtain
\[ (\hat{a}_1 + \hat{a}_{10} + \hat{a}_1^\dagger + \hat{a}_{10}^\dagger) |\xi\rangle = \sqrt{2}x |\xi\rangle \quad \text{or} \quad (\hat{x}_1 + \hat{x}_{10}) |\xi\rangle = x |\xi\rangle \] (10)
and
\[ (\hat{a}_1 - \hat{a}_{10} - \hat{a}_1^\dagger + \hat{a}_{10}^\dagger) |\xi\rangle = i\sqrt{2}p |\xi\rangle \quad \text{or} \quad (\hat{p}_1 - \hat{p}_{10}) |\xi\rangle = p |\xi\rangle \ . \]

Thus, \(|\xi\rangle\) is a state such that the real and imaginary parts of \(\xi\) correspond to the eigenvalues of \(\hat{x}_1 + \hat{x}_{10}\) and \(\hat{p}_1 - \hat{p}_{10}\), respectively.

With the use of the normally ordered product form
\[ |00\rangle \langle 00| = : e^{-\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_{10}^\dagger \hat{a}_{10}} : \] (11)
and the mathematical formula
\[ \int \frac{d^2\xi}{\pi} e^{\lambda |\xi|^2 + f\xi + g\xi^*} = \frac{1}{\lambda} e^{-fg/\lambda} \]
as well as the IWOP technique [4, 5], we can immediately perform the following integration:

\[
\int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| = \int \frac{d^2 \xi}{\pi} : \exp\{-|\xi|^2 + \xi(\hat{a}_1^\dagger + \hat{a}_{10}) + \xi^*(\hat{a}_1 + \hat{a}_{10}^\dagger) \\
- \hat{a}_1\hat{a}_{10}^\dagger - \hat{a}_1^\dagger \hat{a}_{10} - \hat{a}_1\hat{a}_{10}^\dagger - \hat{a}_1^\dagger \hat{a}_{10}\} : =: \exp((\hat{a}_1^\dagger + \hat{a}_{10}) (\hat{a}_{10}^\dagger + \hat{a}_1) - (\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger)) : = 1.
\]

(12)

This approach is, as one may conclude, more concise than the derivation of equations (B.16) and (B.17) in [3]. Besides, we can easily see that

\[
\langle \xi'| (\hat{x}_1 + \hat{x}_{10}) |\xi\rangle = x' \langle \xi' | \xi\rangle = x \langle \xi' | \xi\rangle
\]

and

\[
\langle \xi'| (\hat{p}_1 + \hat{p}_{10}) |\xi\rangle = p' \langle \xi' | \xi\rangle = p \langle \xi' | \xi\rangle.
\]

(13)

Thus,

\[
\langle \xi' | \xi\rangle = \pi \delta(\xi_1 - \xi_1') \delta(\xi_2 - \xi_2')
\]

(14)

which is the orthogonal condition for the eigenstates. The overlap between the two-mode coherent state \(\langle z_1 z_2 | \xi\rangle\) is

\[
\langle z_1 z_2 | \xi\rangle = \exp\left[-\frac{|\xi|^2}{2} + \xi z_1^* + \xi^* z_2^* - z_1^* z_2^* - \frac{1}{2}(|z_1|^2 + |z_2|^2)\right].
\]

At this point, we state that the common eigenstates of the relative coordinate \((\hat{x}_1 - \hat{x}_{10})\) and total momentum \((\hat{p}_1 + \hat{p}_{10})\) of two particles (their commutative property was first pointed out by Einstein et al [7]) are obtained in [8].

Since we have got the explicit form of the eigenstate |\(\xi\rangle\), we are now able to perform the following integration with the IWOP:

\[
\int \frac{d^2 \xi}{\pi} \xi |\xi\rangle \langle \xi| = \int_{-\infty}^{\infty} \frac{d|\xi|^2}{2\pi} \int_{0}^{2\pi} d\theta \ e^{i\theta} : \exp\{-|\xi|^2 + \xi (\hat{a}_1^\dagger + \hat{a}_{10}) + \xi^* (\hat{a}_1 + \hat{a}_{10}^\dagger) - (\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger)\} :
\]

\[= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d|\xi|^2 e^{-|\xi|^2} \sum_{n=0}^{\infty} \frac{|\xi|^{2n+1}}{n! (n+1)!} e^{-(\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger)} : \]

\[= : \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n! (n+1)!} (\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger) : .
\]

(15)

By using the expression of the gamma function

\[
\int_{0}^{\infty} r^{\lambda-1} e^{-r} dr = \Gamma(\lambda)
\]

(16)

with

\[
\Gamma(n + \frac{1}{2}) = \frac{(2n - 1)!!}{2^n} \sqrt{\pi}
\]

we obtain

\[
\int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n! (n+1)!} (\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger) : e^{-(\hat{a}_1^\dagger + \hat{a}_{10})(\hat{a}_1 + \hat{a}_{10}^\dagger)} : .
\]

(17)
Therefore, we have
\[ \hat{C} = \int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| \]
and
\[ \hat{S} = \int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| \]

It is interesting to notice that the expectation values of \( \hat{C} \) and \( \hat{S} \) in the second mode (\( \hat{a}_{10} \) mode) vacuum state are
\[ \langle 0| \hat{C} |0\rangle = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{n!(n + 1)!} |\xi\rangle (\hat{a}^\dagger_{10} \hat{a}_{10})^n |\xi\rangle \]
(20)
and
\[ \langle 0| \hat{S} |0\rangle = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{n!(n + 1)!} \hat{p}_{10} (\hat{a}^\dagger_{10} \hat{a}_{10})^n |\xi\rangle \]
(21)

We can also use the IWOP technique to directly perform the following special integrations for the single-mode coherent state \( |z = re^{i\varphi}\rangle \), which gives
\[ \int \frac{d^2 z}{\pi} e^{i\varphi}|z\rangle \langle z| = \int_{0}^{\infty} d\varphi e^{i\varphi} - \int_{-\pi}^{\pi} d\varphi e^{i\varphi} e^{i\varphi} |\hat{a}_{10}^\dagger \hat{a}_{10}| e^{-\hat{a}_{10}^\dagger \hat{a}_{10}} : \]
(22)

Therefore, we have
\[ \int \frac{d^2 z}{\pi} \cos \varphi |z\rangle \langle z| = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{2})}{n!(n + 1)!} \hat{p}_{10} (\hat{a}^\dagger_{10} \hat{a}_{10})^n |\xi\rangle \]
(23)

which is the normally ordered form of the Paul cosine operator.

Comparing equations (23) and (20), we see that
\[ \int \frac{d^2 z}{\pi} \cos \varphi |z\rangle \langle z| = \langle 0| \hat{C} |0\rangle \]
(24)

This relation has been pointed out in [3]; however, the normally ordered Paul operator was not shown.

We emphasize that the explicit form (8) of \( |\xi\rangle \) can make things simpler and easier to calculate. For example, the overlap \( \langle mn|\xi\rangle \), where \( |mn\rangle \) is the two-mode number state, can be calculated as the following:
\[ \langle mn|\xi\rangle = \langle mn| \int \frac{d^2 z_1 d^2 z_2}{\pi^2} |z_1 z_2\rangle \langle z_1 z_2| \xi \rangle \]
\[ = e^{-|z|^2/2} \int \frac{d^2 z_1 d^2 z_2}{\pi^2} e^{-|z_1|^2 - |z_2|^2 + \xi z_1^\dagger + \xi^* z_2^\dagger} \sqrt{n!m!} \]
\[ = \sum_{l=0}^{\min(m,n)} (-1)^l \xi^{m-l} \xi^{n-l} \frac{n!m!}{l!(n-l)!(m-l)!} e^{-|z|^2/2}. \]
(25)
In particular, when \( n = 0 \), equation (25) reduces to

\[
\langle m | \xi \rangle = \frac{\xi^m}{\sqrt{m!}} e^{-\frac{1}{4}(x^2 + p^2)} = \langle m | \xi \rangle_c
\]  

where \( |\xi\rangle_c \) is a single-mode coherent state.

Actually, the result of equation (26) is closely related to the structure of \( |\xi\rangle \); it is easily seen that

\[
0 \langle 0 | \xi \rangle = e^{-\frac{|\xi|^2}{2}} \hat{a}^\dagger | 0 \rangle_1 = |\xi\rangle_c.
\]  

(27)

In summary, the explicit form of the common eigenstate of the operators \( \hat{C} \) and \( \hat{S} \) makes the NFM phase description much more convenient. Connection between the two-mode NFM phase description and the single-mode Paul phase description is easily established using this eigenstate \( |\xi\rangle \) and the IWOP technique.

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References