Number-difference–phase uncertainty relation for NFM operational quantum phase description

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Abstract

For the Noh, Fougers, and Mandel (NFM) operational quantum phase description, which is based on an eight-port homodyne-detection, we propose the number-difference–phase (ND-P) uncertainty relation and, then, discuss the mechanism of generation of ND-P squeezed states.

1. Introduction

It is well known that a state is defined to be photon-number squeezed if its photon number uncertainty falls below that of a coherent state, which happens at the expense of stretching the corresponding phase uncertainty [1,2]. The photon-number (or amplitude) squeezed state is often referred to as sub-Poissonian light because its standard deviation falls below that of the Poisson distribution that characterizes the coherent state. A photon-number squeezed state is an ideal minimum-uncertainty one when the condition $\Delta n \Delta \sin \phi = |\langle \cos \phi \rangle|$ is satisfied. Based on Susskind–Glogower phase operators [3], a number-phase interaction is proposed that is analogous to the usual single-mode quadrature-squeezing Hamiltonian [4]. However, the Susskind–Glogower phase operator is not unitary and cannot be a function of a Hermitian angle operator $\phi$. According to the definition

$$e^{i\phi} = \frac{1}{\sqrt{n+1}} \hat{a} \quad e^{-i\phi} = \frac{1}{\sqrt{n+1}} \hat{a}^\dagger,$$

$$\hat{n} = \hat{a}^\dagger \hat{a}, \quad (1)$$

and

$$\cos \phi = \frac{1}{2} (e^{i\phi} + e^{-i\phi}), \quad \sin \phi = -\frac{i}{2} (e^{i\phi} - e^{-i\phi}), \quad (2)$$

one gets the number-phase uncertainty relations

$$[\cos \phi, \hat{n}] = i \sin \phi, \quad [\sin \phi, \hat{n}] = -i \cos \phi, \quad (3)$$

and

$$[\cos \phi, \sin \phi] = \frac{i}{2} |0\rangle \langle 0|,$$

$$(\cos \phi)^2 + (\sin \phi)^2 = 1 - \frac{1}{2} |0\rangle \langle 0|. \quad (4)$$

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The vacuum state projector $|0\rangle\langle 0|$ in Eq. (4) "spoils" both the commutivity of operators $\cos \phi$ and $\sin \phi$ and prevents them from obeying a familiar trigonometric identity [5]. This suggests that in states of the field with large average number occupations, hence small vacuum component, the derivations in Eq. (4) can be treated as approximately correct. Therefore, the previous treatment is limited to the low intensity situation.

In Ref. [6], Noh, Fougers, and Mandel (NFM) proposed an operational quantum phase description, which means that the Hermitian phase operators can be defined as observables in an operational way. Their theory is based on an eight-port homodyne-detection scheme. Then, in Ref. [7], Freyberger, Heni, and Schleich (FHS) further showed that in the limit of a strong local oscillator, the NFM description contains an essential two-mode basis which leads to the simultaneously measurable operator pair (named NFM phase operators)

$$\hat{c} = \frac{\hat{x}}{\sqrt{\hat{x}^2 + \hat{p}^2}},$$

(5)

and

$$\hat{s} = \frac{\hat{p}}{\sqrt{\hat{x}^2 + \hat{p}^2}},$$

(6)

where $\hat{x} = \hat{x}_1 + \hat{x}_2$ and $\hat{p} = \hat{p}_1 - \hat{p}_2$ commute with each other and

$$\hat{x}_1 = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p}_1 = \frac{1}{\sqrt{2} i} (\hat{a} - \hat{a}^\dagger),$$

(7)

$$\hat{x}_2 = \frac{1}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger), \quad \hat{p}_2 = \frac{1}{\sqrt{2} i} (\hat{b} - \hat{b}^\dagger),$$

(8)

where $(\hat{a}, \hat{b})$ are the two input mode operators defined in Ref. [6] for the eight-port interferometer in the operator description by NFM.

The simultaneous eigenstates of $\hat{x}$ and $\hat{p}$ and their phase space are also considered in Refs. [6,7] as

$$|x, p\rangle = x |x, p\rangle, \quad \hat{p} |x, p\rangle = p |x, p\rangle.$$  

(9)

These eigenstates compose a complete set

$$\int \int dx \, dp \, |x, p\rangle \langle x, p| = 1.$$  

(10)

With this condition, the phase operator pair can be described in the phase space as

$$\hat{C} = \int \int dx \, dp \frac{x}{\sqrt{x^2 + p^2}} |x, p\rangle \langle x, p|,$$

(11)

$$\hat{S} = \int \int dx \, dp \frac{p}{\sqrt{x^2 + p^2}} |x, p\rangle \langle x, p|,$$

(12)

with $[\hat{C}, \hat{S}] = 0$. The explicit eigenvector $|x, p\rangle$ of cosine and sine phase operators in two-mode Fock space, which has been constructed in Refs. [8,9], takes the form of

$$|x, p\rangle = \exp \left[ -\frac{1}{2} |\xi|^2 \hat{a}^\dagger + \xi \hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger \right] |0, 0\rangle$$

$$\equiv |\xi\rangle,$$

(13)

where $|0, 0\rangle$ is the two-mode vacuum state. $\xi = \xi_1 + i \xi_2 = (1/\sqrt{2})(x + ip)$ is a complex number. $\xi_2$ is the parameter corresponding to the input mode $\hat{b}$. This $|\xi\rangle$ is orthonormal and complete, e.g.

$$\langle \xi' | \xi \rangle = \delta^{(2)}(\xi' - \xi),$$

(14)

and

$$\int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| = 1.$$  

(15)

Using this $|\xi\rangle$ representation and the technique of integration within an ordered product of operators [10], the normally ordered forms of $\hat{C}$ and $\hat{S}$ are shown in Ref. [8].

A question thus naturally arises: with respect to $\hat{C}$ and $\hat{S}$ of Eqs. (5) and (6), what is the observable operator which possesses the similar commutative relations as Eqs. (3)? In other words, can we find the partner (or conjugate) operator, say $\hat{D}$, of the NFM phase operator, that satisfies the following commutation relations.

$$[\hat{C}, \hat{D}] = i \hat{S}, \quad [\hat{S}, \hat{D}] = -i \hat{C}. $$

(16)

Because $\hat{C}$ and $\hat{S}$ include $\sqrt{\hat{x}^2 + \hat{p}^2}$, it seems difficult to find such a $\hat{D}$ operator, which might be the reason why FHS and NFM have not proposed the relation (16) and no such partner (or conjugate) operator $\hat{D}$ for the NFM phase operator has been reported so far.

In Section 2 we search for this operator $\hat{D}$, which turns out just to be the two-mode number-difference
operator. We then analyze the number-difference-phase (ND-P) uncertainty relation. In Section 3, we introduce the concept of ND-P squeezed states and suggest a ND-P interaction that can generates these states. Because the number-difference states were recently experimentally demonstrated in a feedback scheme in an optical parametric oscillator system [11] and the NFM phase also has a link with the Shapiro–Wagner phase operator formalism, as we will point out later, the discussion in our work seems interesting.

2. ND-P uncertainty relation

To find the $\hat{D}$ operator, we notice from Eqs. (7) and (8) that

\[
\hat{x} + i \hat{p} = \sqrt{2} (\hat{a} + \hat{b}^\dagger) \equiv \sqrt{2} \hat{\gamma},
\]

\[
\hat{x} - i \hat{p} = \sqrt{2} (\hat{a}^\dagger + \hat{b}) = \sqrt{2} \hat{\gamma}^\dagger.
\]

then

\[
\hat{x}^2 + \hat{p}^2 = 2 \hat{\gamma} \hat{\gamma}^\dagger = 2 \hat{\gamma} \hat{\gamma}^\dagger.
\]

Note $[\hat{\gamma}, \hat{\gamma}^\dagger] = 0$. As one can see later, Eq. (18) is essential for us to find the suitable operator $\hat{D}$ which obeys Eq. (16) and the ND-P uncertainty relation.

As a result of Eqs. (5), (61, (17), and (181, we can express the NFM phase operators as

\[
\hat{C} = \frac{\hat{x}}{\sqrt{\hat{x}^2 + \hat{p}^2}} = \frac{\hat{\gamma} + \hat{\gamma}^\dagger}{\sqrt{2} \hat{\gamma} + \hat{\gamma}^\dagger},
\]

and

\[
\hat{S} = \frac{\hat{p}}{\sqrt{\hat{x}^2 + \hat{p}^2}} = \frac{\hat{\gamma} - \hat{\gamma}^\dagger}{i \sqrt{2} \hat{\gamma} + \hat{\gamma}^\dagger}.
\]

At this point it is not difficult to see that the two-mode number-difference operator

\[
\hat{D} = \hat{a} ^\dagger \hat{a} - \hat{b} ^\dagger \hat{b}
\]

obeys the relations

\[
[\hat{D}, \hat{\gamma}] = -\hat{\gamma}, \quad [\hat{D}, \hat{\gamma}^\dagger] = \hat{\gamma}^\dagger, \quad [\hat{D}, \hat{\gamma} \hat{\gamma}^\dagger] = 0.
\]

Thus, $\hat{D}$ really satisfies the ND-P relation for NFM operational phase, e.g.

\[
[\hat{D}, \hat{C}] = -i \hat{S}, \quad [\hat{D}, \hat{S}] = -i \hat{C}.
\]

Obviously, the ND-P uncertainty relation is an important complement to the theoretical analysis by Freyberger et al. [7].

Note that, in contrast to Eq. (41, here we see that $[\cos \phi, \sin \phi] = 0, \cos^2 \phi + \sin^2 \phi = 1$ strictly hold. These commutation relations show that the number-difference and phase operators exactly satisfy

\[
\Delta D \Delta \cos \phi \geq \frac{1}{2} |\langle \sin \phi \rangle|, \\
\Delta D \Delta \sin \phi \geq \frac{1}{2} |\langle \cos \phi \rangle|.
\]

3. Time evolution for the number-difference and phase operators

The ND-P squeezed state is defined when its photon number-difference uncertainty falls below that of the two-mode coherent state. In the spirit of Ref. [4], we write the Hamiltonian for generating ND-P squeezed states as

\[
\hat{H} = \frac{\hbar}{2} \lambda \{ \hat{D}, \sin \phi \},
\]

where $\lambda$ is the coupling constant and $\{ \} \}$ means an anticommutator. We then search for the time evolution for the operators $\hat{D}$ and $\tan \frac{\phi}{2}$. According to the Heisenberg equation, we calculate

\[
\frac{d \hat{D}}{dt} = \frac{1}{i \hbar} [\hat{D}, \hat{H}] = \frac{i}{2} \lambda \{ \hat{D}, \cos \phi \},
\]

\[
\frac{d}{dt} \sin \phi = -\frac{\lambda}{2} \sin 2 \phi,
\]

\[
\frac{d}{dt} \cos \phi = \lambda \sin^2 \phi.
\]

As a result of Eqs. (27)–(29), we have

\[
\frac{d}{dt} \tan \frac{\phi}{2} = \frac{d}{dt} \left( 1 - \cos \phi \right) = -\lambda \tan \frac{\phi}{2}.
\]
The solution to this equation is
\[ \tan \frac{1}{2} \phi = e^{-\lambda t} \tan \frac{1}{2} \phi(0), \] (31)
which implies the phase's evolution behavior.

On the other hand, from Eqs. (23) and (24), we derive
\[ [\hat{D}, \{ \hat{D}, \sin \phi \}] = i \{ \hat{D}, \cos \phi \} \] (32)
and
\[ [\hat{D}, -\{ \hat{D}, \cos \phi \}] = i \{ \hat{D}, \sin \phi \}. \] (33)

Then, by noticing
\[ \sin \phi \hat{D}^2 \cos \phi = \frac{1}{2} \hat{D}^2 \sin 2\phi - i \{ \cos \phi, \hat{D} \} \cos \phi, \] (34)
\[ \cos \phi \hat{D}^2 \sin \phi = \frac{1}{2} \hat{D}^2 \sin 2\phi + i \{ \sin \phi, \hat{D} \} \sin \phi, \] (35)
and
\[ \sin \phi \hat{D} \sin \phi + \cos \phi \hat{D} \cos \phi = \hat{D}, \] (36)
we obtain
\[ [\{ \hat{D}, \cos \phi \}, \{ \hat{D}, \sin \phi \}] = 4i \hat{D}. \] (38)

Eqs. (32), (33), and (38) imply that \[ \hat{D} \equiv \hat{J}_z, \]
\[ \frac{1}{2} i [\hat{D}, \cos \phi] = \hat{J}_z, \] and \[- \frac{1}{2} i [\hat{D}, \sin \phi] = \hat{J}_z \] constitute a closed SU(2) Lie algebra. Then immediately follows the time evolution of the number-difference operator
\[ \hat{D}(t) = e^{-i\hat{H}t/\hbar} \hat{D}(0) e^{i\hat{H}t/\hbar} \]
\[ = \frac{i}{2} \left( e^{i\lambda t} \hat{J}_+ + e^{-i\lambda t} \hat{J}_- \right), \] (39)
where
\[ \hat{J}_+ = \hat{J}_+ + i \hat{J}_y = \hat{D} + \frac{1}{2} \{ \hat{D}, \cos \phi \} \]
\[ = \frac{1}{2} \{ \hat{D}, 1 + \cos \phi \}, \] (40)
\[ \hat{J}_- = \hat{J}_- - i \hat{J}_y = \frac{1}{2} \{ \hat{D}, 1 - \cos \phi \}. \] (41)

Comparing our Eqs. (31) and (39) with the results (Eqs. (7) and (9)) of Ref. [4], we conclude that the Hamiltonian (26) can really generate ND-P squeezed states for which phase squeezing increases the number-difference fluctuations. It is worth emphasizing that in this paper, Eqs. (31) and (39) rigorously hold because \[ \{ \cos \phi, \sin \phi \} = 0. \] It will be interesting to find the experimental realization of the interaction Hamiltonian defined in Eq. (26). The discussion of possible experiments will be very similar to the detailed consideration as given at the end of Ref. [4] and, therefore, will not be repeated here. On the other hand, from Eq. (17) it is interesting to realize that \[ \hat{Y} \] and \[ \hat{Y}^\dagger \] are just the Shapiro–Wagner (SW) heterodyne-measurable phases [12]. Hence, we are sure that the NFM experiment and SW phase measurement scheme have something intrinsically in common.

In summary, for the NFM operational quantum phase description we have introduced the corresponding ND-P uncertainty relation, which goes a step further from the original FHS theoretical analysis. The dynamics for generating such a kind of squeezed states has been analyzed. We believe that the expectation values of this number-difference operator \[ \hat{D}, \] which is the conjugate operator of the operationally defined quantum-phase operator by NFM using the eight-port homodyne detection scheme, can be experimentally measured in the existing experimental arrangement [6] and the uncertainty relation between the operationally defined quantum phase and the number-difference can be experimentally tested.

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References
