Quantum entanglement has attract much research interest in recent years since it is the central resource in applications such as quantum communication and computation. Multipartite continuous-variable (CV) entangled beams with different frequencies are necessary to connect different physical systems at the nodes of quantum networks [1] since they can be separated easily in applications [2]. It was predicted that two-color CV entanglement can be produced by a nondegenerate optical parametric oscillator (OPO) [3], which has been experimentally demonstrated below [4] and above [5] the oscillation threshold. Then, Villar et al. predicted that three-color CV entanglement can be generated using an OPO operating above the threshold [6], which has been verified by a recent experiment [2]. When one pump photon is annihilated, exactly two photons are created; that is, when $N$ pump photons are annihilated, $2N$ longer-wavelength photons are created. So the intensities of the three beams (i.e., pump, signal, and idler) will be strongly correlated when losses and uncertainties are minimized [7]. There are also many proposed schemes to produce multicolor CV entanglements, such as with cascaded nonlinearities inside an optical cavity [8–11], vector four-wave mixing (FWM) in a fiber [12], etc.

Alternatively, entanglement can also be produced in coherent atomic systems, which has attracted much attention since the entangled beams from atomic systems have narrower linewidths and longer coherence times. Moreover, the wavelengths of beams at about 800 nm allow them to be used directly in the quantum memory [13] and free-space (i.e., from a low-Earth-orbit satellite to a ground station) quantum teleportation [14]. Balić et al. reported that strong biphoton correlation can be produced by a FWM process in a cold atomic ensemble [15], which is similar to the entangled biphoton obtained by the process of spontaneous parametric down-conversion in nonlinear crystals. Duan et al. proposed a scheme to use bipartite CV entanglement generated in atomic ensembles for long-distance quantum communication [16], which has been experimentally demonstrated by Kuzmich et al. [17]. In addition, van der Wal et al. experimentally demonstrated the emission of two quantum-mechanically correlated light pulses produced by an ensemble of rubidium atoms [18], and Josse et al. also experimentally verified that the bipartite CV entanglement can be generated by cold atoms in a high-finesse optical cavity [19]. These bipartite CV entanglements are similar to the bipartite CV entanglement generated from an OPO below threshold [4]. Wu and Xiao obtained bright nonclassical correlated anti-Stokes and Stokes beams with Doppler-broadened four-level atoms inside an above-threshold optical ring cavity [20], which is similar to the bright two-color CV entanglement produced by a nondegenerate OPO operating above threshold, as in Ref. [5]. The Stokes and anti-Stokes beams are generated by the interaction among the pump and coupling beams and the multilevel atomic medium. When two photons (a pump photon and a coupling photon) are absorbed, two new photons (i.e., an anti-Stokes photon and a Stokes photon) are generated simultaneously. Therefore, the intensities of the pump, coupling, anti-Stokes, and Stokes beams will also be quantum correlated for energy conservation ($\omega_p + \omega_c = \omega_h + \omega_m$) [15], which is like the case for an OPO operating above threshold, as in Refs. [2] and [6].

Based on the above analysis and the previous experimental demonstration of two-color quantum correlation [20], we propose a scheme in this article to generate bright quadricolor CV entangled beams by adjusting the pump and coupling beams as well as the generated anti-Stokes and Stokes beams simultaneously on resonance in the optical cavities. We theoretically demonstrate entanglements among the four beams according to the inseparability criterion for multipartite CV entanglement proposed by van Loock and Furusawa [21]. Recently, a triply-resonant OPO using FWM with rubidium vapor inside an optical cavity has been experimentally achieved [22], so it is experimentally feasible to achieve a quadruply-resonant OPO and obtain quadricolor CV entangled beams by appropriately adjusting the previous experimental system used in Ref. [20].

We propose an experimentally feasible scheme to produce bright quadricolor continuous-variable (CV) entanglement by a four-wave-mixing process (FWM) with four-level atoms inside the optical ring cavities operating above threshold. The Stokes and anti-Stokes beams are generated via the pump beam (tuned close to one atomic transition) and the coupling beam (tuned to the resonance of another atomic transition), respectively. The quadruply resonant and narrowed linewidth of the cavity fields with different frequencies are achieved and quadricolor CV entanglement among the four cavity fields is demonstrated according to the criterion proposed by van Loock and Furusawa [Phys. Rev. A 67, 052315 (2003)]. This scheme provides a way to generate bright quadricolor CV entanglement and will be significant for applications in quantum information processing and quantum networks.

DOI: 10.1103/PhysRevA.83.012321 PACS number(s): 03.67.Bg, 03.67.Mn, 42.50.Dv, 42.50.Lc
Therefore, intensity anticorrelation is expected between the pump (coupling) beam and the anti-Stokes (Stokes) beam. From the frequency constraint translated into a constraint for the phase fluctuations of the four fields, one can see that the phase fluctuations between the anti-Stokes (Stokes) beam and the pump (coupling) beam should be correlated. Conversely, the phase fluctuations between the anti-Stokes and Stokes beams or the pump and coupling beams should be anticorrelated. Then, from the multiparticle CV entanglement criteria reported in Ref. [21], the satisfaction of the three inequalities

\begin{align}
V_{p-c} &= \Delta^2 \left( \frac{\hat{X}_p - \hat{X}_c}{\sqrt{2}} \right) + \Delta^2 \left( \frac{\hat{Y}_p + \hat{Y}_c}{\sqrt{2}} + g_s \hat{Y}_s + g_{as} \hat{Y}_{as} \right) < 1, \\
V_{c-s} &= \Delta^2 \left( \frac{\hat{X}_c + \hat{X}_s}{\sqrt{2}} \right) + \Delta^2 \left( \frac{\hat{Y}_c - \hat{Y}_s}{\sqrt{2}} + g_p \hat{Y}_p + g_{as} \hat{Y}_{as} \right) < 1, \\
V_{s-as} &= \Delta^2 \left( \frac{\hat{X}_a - \hat{X}_{as}}{\sqrt{2}} \right) + \Delta^2 \left( \frac{\hat{Y}_a + \hat{Y}_{as}}{\sqrt{2}} + g_p \hat{Y}_p + g_c \hat{Y}_c \right) < 1,
\end{align}

should be sufficient to verify the quadripartite entanglement, where \( \hat{X}_i = (\hat{a}_i^{\text{out}} + \hat{a}_i^{\text{out}})/(i = p, c, s, a) \) represent the amplitude quadratures of the fields, \( \hat{Y}_i = (\hat{a}_i^{\text{out}} - \hat{a}_i^{\text{out}})/2 \) stand for their phase quadratures, \( \hat{a}_i^{\text{out}} \) are the output fields for the four modes in cavities, and \( g_i \) are the adjustable parameters chosen to minimize the left-hand side of the inequalities.

Here, the energy levels of \(^{87}\text{Rb}\) atoms serve as the medium to produce a nonlinear FWM process. In present study, we only focus on the entanglement characteristics among the four optical fields rather than that between atoms and optical fields. So the interaction Hamiltonian in this nondegenerate FWM process can be simply written by optical field operators as [16,24]

\[
H_{\text{int}} = i\hbar \kappa \hat{a}_p \hat{a}_c \hat{a}_s ^\dagger \hat{a}_a ^\dagger + \text{H.c.},
\]

where \( \kappa \) is the dimensionless nonlinear coupling coefficient and is commonly taken to be a real number [4,9]. Considering the driving fields in the cavity, the Hamiltonian for the pump and coupling beams is given by

\[
H_{\text{ext}} = i\hbar \epsilon_p \hat{a}_p ^\dagger + i\hbar \epsilon_c \hat{a}_c ^\dagger + \text{H.c.},
\]

where \( \epsilon_i \) are the classical pump and coupling laser amplitudes. The loss of the \( i \)th mode in the cavity can be written as [25]

\[
L_i \hat{\rho} = \gamma_i (2 \hat{a}_i \hat{a}_i ^\dagger - \hat{a}_i ^\dagger \hat{a}_i - \hat{\rho} \hat{a}_i ^\dagger \hat{a}_i),
\]

where \( \gamma_i \) stand for the damping rates for the corresponding cavity modes which are related to the amplitude transmission coefficients [9]. The master equation for the four cavity modes is

\[
\frac{d \hat{\rho}}{dt} = -\frac{i}{\hbar} [H_{\text{int}} + H_{\text{ext}}, \hat{\rho}] + \sum_i L_i \hat{\rho}.
\]
representation as \[26\]

\[
\frac{\partial \alpha_p}{\partial t} = \epsilon_p - \gamma_p \alpha_p - \kappa \alpha_p \alpha_d - \sqrt{2} \alpha_p \alpha_d \eta(t),
\]

\[
\frac{\partial \alpha_c}{\partial t} = \epsilon_c - \gamma_c \alpha_c - \kappa \alpha_c \alpha_d - \sqrt{2} \alpha_c \alpha_d \eta(t),
\]

\[
\frac{\partial \alpha_s}{\partial t} = \kappa \alpha_p \alpha_d^\dagger - \gamma_s \alpha_s + \sqrt{2} \kappa \alpha_c \eta(t),
\]

\[
\frac{\partial \alpha_m}{\partial t} = \kappa \alpha_p \alpha_m^\dagger - \gamma_m \alpha_m + \sqrt{2} \kappa \alpha_c \eta(t),
\]

where \(\eta\) are real noise terms and have \(\langle \eta(t) \rangle = 0\) and \(\langle \eta(t) \eta(t') \rangle = \delta(t-t')\). In this nonlinear FWM process, the pump and coupling beams are both coherent driving fields and have similar quantum characteristics. In order to simplify the calculation we assume they have the same damping rate (i.e., \(\gamma_p = \gamma_c = \gamma_s\) and \(\epsilon_p = \epsilon_c = \epsilon\)). Likewise, we assume the anti-Stokes and Stokes beams have the same damping rate with \(\gamma_r = \gamma_a = \gamma_b\). To obtain the output spectra one has to know the classical steady-state solutions \[11,25\]. By letting \(\partial \alpha_i / \partial t = 0\), it is easy to find the solutions when the noise terms are ignored \[11,25\]. We find the pump threshold at \(\epsilon_{th} = \gamma_a \sqrt{\gamma_b / \kappa}\). When \(\epsilon < \epsilon_{th}\), the steady-state outputs can be obtained as \(A_i = A_i = \epsilon / \gamma_a\) (i = p, c) and \(A_i = A_b = 0\) (i = s, a). When \(\epsilon > \epsilon_{th}\), the steady-state outputs can be obtained as \(A_i = A_a = \pm \sqrt{\gamma_a / \kappa}\) (i = p, c) and \(A_i = A_b = \pm \sqrt{\gamma_a (\epsilon - \epsilon_{th}) / (\epsilon \kappa)}\) (i = s, a), where \(A_a\) is the mean value of the pump or coupling field in the steady state, and \(A_b\) is the value of the anti-Stokes or Stokes field in the steady state. In the present scheme we only consider the situation for the field modes to oscillate above the threshold.

In the following, one can decompose the system variables into their steady-state values and small fluctuations around the steady-state values as \(\alpha_i = A_i + \delta \alpha_i\). Then one can use this linearized analysis as a method to calculate the spectra for the output cavity modes \[11,25\]. Equation (6) can be linearized as \[26\]

\[
\frac{d}{dt} \delta \alpha = -M \delta \alpha + B \eta,
\]

where \(\delta \alpha = [\delta \alpha_p, \delta \alpha_c, \delta \alpha_s, \delta \alpha_p, \delta \alpha_c, \delta \alpha_m]^T\), \(B\) contains the coefficients of the noise terms, \(\eta\) is the noise matrix, and \(M\) is the drift matrix with the steady-state values inserted and is given by

\[
M = \begin{pmatrix}
\gamma_a & 0 & 0 & \kappa A_p^2 A_p & 0 & \kappa A_p^2 A_b & 0 \\
0 & \gamma_a & \kappa A_c^2 & 0 & 0 & \kappa A_c^2 A_b & 0 \\
0 & \kappa A_p^2 & \gamma_a & 0 & \kappa A_p^2 A_p & 0 & \kappa A_p^2 A_b \\
\kappa A_p^2 & 0 & 0 & \gamma_a & 0 & \kappa A_p^2 A_p & 0 \\
-\kappa A_a A_b^* & 0 & -\kappa A_a A_b^* & 0 & \gamma_b & 0 & 0 \\
0 & -\kappa A_a A_b^* & 0 & -\kappa A_a A_b^* & 0 & \gamma_b & -\kappa A_a^2 \\
-\kappa A_a A_b^* & 0 & -\kappa A_a A_b^* & 0 & 0 & -\kappa A_a^2 & \gamma_b \\
0 & -\kappa A_a A_b^* & 0 & -\kappa A_a A_b^* & 0 & 0 & \gamma_b
\end{pmatrix}.
\]

Equation (7) can be solved in the frequency domain as \(\delta \alpha(\omega) = (M + i \omega I)^{-1} B \eta\), where \(I\) is the identity matrix and \(\omega\) is the analysis frequency. The output fields can be obtained along with the well-known input-output relations \[26,27\].

From now on, we numerically calculate the values of inequalities in Eq. (1) according to the results obtained above. In Fig. 2 we plot the minimum of the inequalities versus the analysis frequency normalized to \(\gamma_a\) with \(\gamma_a = 0.03\), \(\gamma_b = 0.015\), \(\epsilon = 1.5 \epsilon_{th}\), and \(\kappa = 1\). It is obvious that the minimal values of inequalities are all less than 1 in a wide range of analysis frequencies, which is sufficient to demonstrate that the pump, coupling, anti-Stokes, and Stokes beams are CV entangled with each other. In this FWM process, the production of a pair of new photons (i.e., anti-Stokes and Stokes photons) must be accompanied by the annihilation of a pair of pump and coupling photons simultaneously due to energy conservation. Therefore, the sum of decreasing intensities of the pump and coupling beams is equal to that of the anti-Stokes and Stokes beams, and the four beams are quantum correlated, which is similar to the case of an OPO operating above

![FIG. 2. (Color online) Minima of the inequalities versus the analysis frequency normalized to \(\gamma_a\) with \(\gamma_a = 0.03\), \(\gamma_b = 0.015\), \(\epsilon = 1.5 \epsilon_{th}\), and \(g = 1\).](image-url)
γa

threshold [2]. The smaller the value of the inequality, the higher the degree of entanglement and, therefore, the stronger the degree of quantum correlation. In Fig. 2 one can see that the value of \( V_{\text{v-as}} \) is the smallest of the three inequalities, which indicates that the degree of entanglement between the anti-Stokes and Stokes beams is the biggest. This is similar to the case of twin beams produced by spontaneous parametric down-conversion.

In Fig. 3, we plot the minima of the inequalities versus the pump-power parameter \( \epsilon \) normalized to the threshold of the pump with \( \gamma_a = 0.03, \gamma_b = 0.015, \omega = \gamma_a, \) and \( \kappa = 1.0 \). One can see that the minima of the inequalities are less than 1 when the optical oscillators are operated above the threshold, and it verifies that the four field modes are entangled. Moreover, the value of \( V_{\text{v-as}} \) is also the smallest in the three inequalities and it demonstrates that the degree of quantum correlation between the anti-Stokes and Stokes beams is the strongest. In addition, the best quadricolor CV entanglement can be obtained at about \( \epsilon = 1.3 \epsilon_{\text{th}} \). When \( \epsilon < \epsilon_{\text{th}} \), the steady-state values of anti-Stokes and Stokes beams are both equal to zero, and the powers of the pump and coupling beams are higher than that of the anti-Stokes and Stokes beams. Thus, the quantum characteristics of the pump and coupling beams are not present in such a case and they are not entangled with the weak anti-Stokes and Stokes fields. With the increase of the pump above the threshold, the anti-Stokes and Stokes fields will be enhanced; however, the pump and coupling beams become weaker since their energies are transferred to the anti-Stokes and Stokes beams. Then, their quantum characteristics are present and bright quadricolor CV entanglement can be obtained. Upon further increasing the pump far above the threshold, the gain begins to saturate, the intensities of pump and coupling beams are higher than that of the signal and idler beams, and their quantum characteristics vanish when the inequalities exceed 1.

In summary, we proposed a scheme to produce bright quadricolor CV entanglement from an atomic ensemble in an optical cavity operating above threshold, which is experimentally feasible based on our previous experiment [20]. We have theoretically demonstrated that quadripartite entanglement is present in this system according to the criterion for multipartite CV entanglement. These bright quadricolor entangled beams with different frequencies and narrow linewidths are spatially separated after generation, which can be employed for the needs of quantum networks [2]. Moreover, their wavelengths at about 800 nm can be directly used in quantum memory and storage [13] and can be used as important entanglement resources in free-space quantum teleportation [14].

Y. B. Yu is supported by the National Natural Science Foundations of China (No. 10804059), the Zhejiang Provincial Natural Science Foundation (No. Y6090488), the Ningbo Natural Science Foundation (No. 2008A610006), and the K. C. Wong Education Foundation. M. Xiao is supported in part by the National Science Foundation (USA).