Cavity-QED-based unconventional geometric phase gates with bichromatic field modes

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Abstract

Realization of two-qubit quantum phase gate is demonstrated using unconventional geometric phase in a cavity sustaining bichromatic field modes which are highly detuned from the atomic transition frequency. The two cavity modes are displaced simultaneously and thus acquire a geometric phase which can be used for realization of approximate phase gate operation.

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1. Introduction

Quantum computers which could be exponentially faster [1] than their classical counterparts essentially work on the fundamental principles of quantum mechanics. To realize such a device on large scale operation with high fidelity, suppression of decoherence is of prime concern. The architect of built-in-fault-tolerant quantum gates requires avoiding of dynamical phases and utilization of decoherence free subspace [2] or the concept of geometric phases [3–11]. It is possible to realize universal quantum gates [3–9] through geometric phases when Hamiltonian describing qubits undergoes changes along suitable cycles in a control space. The quantum computation is called a geometric quantum computation if the phase associated with the gate operation is pure geometric one. When the logical gates in quantum computing are implemented through geometric phases then these gates provide fault-tolerant characteristics because the geometric phases depend only on some global geometric features and hence should be robust against dephasing. A geometric gate obtained by driving the qubits to undergo appropriate adiabatic evolution is called conventional geometric phase-gates [10,12]. On the other hand, evolution of the state generator (e.g., displacement operator for the coherent state) along the closed path conditional on the state of the qubits provides the so-called unconventional geometric phase gates [10,12]. Several proposals using NMR [5], superconducting circuits [7], trapped ions [9], and semiconductor nano structures [11], for the adiabatic geometric quantum computation were given. This was also generalized for the nonadiabatic case both theoretically [10,12] and experimentally [13]. The universal quantum gates were proposed using nonadiabatic geometric phase [10] for Josephson junctions and for NMR systems. The gates categorized under conventional geometric quantum gates require certain extra operations in order to avoid dynamical phases that sometimes can bring additional errors. On the other hand, unconventional geometric quantum gates are independent of initial state of the system and can incorporate all advantages of conventional geometric gates and provide high fidelity. More specifically this is due to the fact that the unconventional geometric phase is a type of phase factor where the dynamic component is nonzero but proportional to geometric component so the total phase is dependent only on global geometric features [10]. The experimental realization of such gates has been reported in trapped ion system [13]. In a cavity QED
system where a single-mode field interacts with atoms and cavity field is displaced along a closed path depending on the state of the atoms, the system acquires a phase which can be used to implement the unconventional geometric phase gates [14]. During the gate operation the atoms do not undergo any transitions so the atomic decay does not affect the operation. Also, the cavity mode is disentangled from the atomic system throughout its operation under certain parametric conditions, so the gate operation is independent of cavity decay also. The interesting objective at this stage is when two modes interacts with the atoms in cavity-QED mode is disentangled from the atomic system throughout its operation under certain parametric conditions, so the gate operation the atoms do not undergo any transitions so the atomic decay does not affect the operation. Also, the cavity

M

For a path in parametric space consisting of

(7)

where \( \delta \xi \) is a complex parameter corresponding to the displacement related to TPCS. This operator satisfies the following relation for very small displacements \( |\delta \xi| \) and \( |\delta \xi_2| \) (when \( |\delta \xi| \) approaching zero, we neglect higher order terms in the expansion as they are quite small)

\[
B(\delta \xi_{1})B(\delta \xi_{2}) \cong B(\delta \xi_{1} + \delta \xi_{2}) \exp(i \Im(\delta \xi_{1}^{*} \delta \xi_{2}[a_{1\alpha}, a_{1\alpha}^{\dagger}]).
\]

(4)

For a path in parametric space consisting of \( M \) such very short straight displacements, it is possible to express total operation by

\[
B_{\text{Total}} = B(\delta \xi_{M})B(\delta \xi_{M-1}) \cdots B(\delta \xi_{2})B(\delta \xi_{1}) \cong B\left(\sum_{i=1}^{M} \delta \xi_{i}\right) \exp\left(i \Im\left(\sum_{m=2}^{M} \delta \xi_{m} \sum_{j=1}^{m-1} \delta \xi_{j}^{*}\right)[a_{1\alpha}, a_{1\alpha}^{\dagger}]\right).
\]

(5)

For infinitesimal displacement \( d\xi \) this arbitrary path \( C \) will be obtained when \( M \to \infty \)

\[
B_{\text{Total}} \cong B\left(\int_{C} d\xi\right) \exp(i \Phi)
\]

(6)

in which the phase factor is defined as

\[
\Phi \cong \Im\left(\int_{C} \xi^{*} d\xi\right)[a_{1\alpha}, a_{1\alpha}^{\dagger}].
\]

(7)

If the path \( C \) is a closed path then

\[
B_{\text{Total}} = B(0) \exp(i \Phi),
\]

(8)

and for the Fock state \( |n_{1}, n_{2}\rangle \) we get

\[
\Phi = \langle n_{1}, n_{2}| \hat{\Phi}|n_{1}, n_{2}\rangle = (n_{1} + n_{2} + 1) \Im\left(\hat{\Phi} \xi^{*} d\xi\right).
\]

(9)
Note that the phase operator $\Phi$ is determined by two important considerations. The first one is the area involved in the parametric phase space and the second one is the quantized states of field modes. In this way the generalized geometric phase shift is different from what has been discussed in the earlier work [10,12,14] where the geometric phase shift turns out to be independent of the field mode state. However, the generalized geometric phase reduces to the one discussed in Refs. [10,12] when the field is considered to be in the vacuum state and then it is given by

$$
\Phi = \text{Im}\left( \int \xi^* d\xi \right).
$$

(10)

Thus the generalized unconventional geometric phase (within the approximation discussed above) corresponding to the operator defined by Eq. (3) for a TPCC or TMSS is given by Eqs. (7), (9) and (10).

Next, we prove that all the phase factor accumulated in a closed path in a phase space leads to the generalized unconventional geometric phase. For any quantum system with initial wavefunction $|\psi(0)\rangle$, evolving to final wavefunction $|\psi(T)\rangle$, in a cyclic evolution, then $\langle\psi(0)|\psi(T)\rangle = e^{i\gamma(T)}|\langle\psi(0)|\psi(T)\rangle|$, where $\gamma(T)$ is the total phase acquired during the evolution. The wavefunction at anytime satisfy $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ and the unitary operator is approximately given by [10]

$$
U(t) = \hat{T}e^{-i\int H_{\text{int}}(t')dt'} \cong e^{i\chi(T)} B(\xi),
$$

(11)

where $\chi(T) \equiv -2\pi(K/(\eta_1 + \eta_2))^2(n_1 + n_2 + 1)$ for the cyclic evolution and $B(\xi)$ is defined in (3) above. Also, from the nature of expression in Eq. (11) one can show that $\gamma(T) \equiv \chi(T)$. The explicit calculation of geometric phase gives the expression

$$
\Phi(T) = 2\pi(K/(\eta_1 + \eta_2))^2(n_1 + n_2 + 1) = -\chi(T).
$$

(12)

So, the dynamic phase accumulated during cyclic evolution is $\gamma(T) = \gamma(T) - \Phi(T) = 2\chi(T)$ within the approximation. Hence the dynamic phase is proportional to the geometric component consistent with the definition of unconventional geometric phase factor in Ref. [10]. Thus the phase factor accumulated in a closed path in phase space is the generalized unconventional geometric phase. Above analysis is more justified when both $n_1$ and $n_2$ are near zero.

3. Two mode cavity QED system

The physical model considered here for the cavity QED system consists of two identical three-level atoms, having two ground states $|g_k\rangle$ and $|f_k\rangle$ and an excited state $|e_k\rangle$ (Fig. 1). The qubit operation is performed using two ground states $|g_k\rangle$ and $|f_k\rangle$, and the excited state $|e_k\rangle$ acts as an auxiliary state. The atoms undergoes bi-photonic transition from $|f_k\rangle$ to $|e_k\rangle$ by absorbing photons from the bichromatic field modes with coupling constant $g$ and detuning $\Delta = \omega_1 - 2\omega_c$. The atoms are also driven by a classical field characterized by Rabi frequency $\Omega_L$, frequency $\omega_L$. The other ground level $|g_k\rangle$ is not involved in the interaction, so one can describe this system by the following Hamiltonian

$$
H = \omega_1 \sum_{k=1,2} S_k^+ + (\omega_c - \epsilon) a_1^+ a_1 + (\omega_c + \epsilon) a_2^+ a_2 + g \sum_{k=1,2} (a_1^+ S_k^+ + a_1 S_k^-) + \Omega_L \sum_{k=1,2} (e^{-i\omega_L t} S_k^+ + e^{i\omega_L t} S_k^-),
$$

(13)

where $\omega_1$, $\omega_c$, $\epsilon$, and $\omega_L$ are the frequencies of the resonant transition between $|f_k\rangle$ and $|e_k\rangle$, the mean frequency of the two cavity modes, half of the frequencies difference of the two cavity modes, and the frequency of classical driving field, respectively. The ladder operators are defined as $S_k^+ = |e_k\rangle \langle f_k|$, $S_k^- = |f_k\rangle \langle e_k|$ and $S_k^\leftrightarrow = \frac{1}{2}(|e_k\rangle \langle f_k| - |f_k\rangle \langle e_k|)$. The cavity field operators are defined as in Section 2. In the rotating frame with respect to cavity frequency $\omega_c$ and assuming $\epsilon \ll \Delta$, the Hamiltonian is transformed to

$$
H_r = \sum_{k=1,2} \Delta S_k^+ + (g a_1^+ a_2^+ e^{i\omega_c t} + \Omega_L e^{i\delta'}) S_k^- + (g a_1 a_2 e^{-i\omega_c t} + \Omega_L e^{-i\delta'}) S_k^+.
$$

(14)

Fig. 1. Schematics of the three-level atom interacting with two cavity modes ($a_1, a_2$) and classical field ($\Omega_L$).
in which $\delta = \omega_L - \omega_c$. Under the dispersive interaction limit, i.e., $\Delta \gg \Omega_L$, $g$, $\delta$, there is no exchange of energy between atoms and the fields, then the effective Hamiltonian describing such interaction is represented by

$$H_{\text{dis}} = \sum_{k=1,2} \frac{1}{\Delta} \left[ (n_1 + n_2 + 1) g^2 a_1^+ a_2^+ a_1 a_2 + \Omega_L^2 + \Omega_L g a_1 a_2 e^{i\delta' t} + \Omega_L g a_1^+ a_2^+ e^{-i\delta' t} \right] |e_k \rangle \langle e_k| - |f_k \rangle \langle f_k| + \frac{1}{\Delta} (n_1 + n_2 + 1) g^2 (S_1^+ S_2^- + S_1^- S_2^+) [a_1 a_2, a_1^+ a_2^+] \tag{15}$$

where $n_1, n_2$ correspond to occupation numbers of the cavity field modes and $\delta' = \omega_L - 2 \omega_c$. We can decompose the above Hamiltonian into two parts and express them as

$$H_{d1} = \sum_{k=1,2} \frac{1}{\Delta} \left[ (n_1 + n_2 + 1) g^2 a_1^+ a_2^+ a_1 a_2 + \Omega_L^2 \right] |e_k \rangle \langle e_k| - |f_k \rangle \langle f_k| + \frac{1}{\Delta} (n_1 + n_2 + 1) g^2 |e_k \rangle \langle e_k| [a_1 a_2, a_1^+ a_2^+]$$

$$H_{d2} = \sum_{k=1,2} \frac{1}{\Delta} \left[ \Omega_L g a_1 a_2 e^{i\delta' t} + \Omega_L g a_1^+ a_2^+ e^{-i\delta' t} \right] |e_k \rangle \langle e_k| - |f_k \rangle \langle f_k| + \frac{1}{\Delta} (n_1 + n_2 + 1) g^2 (S_1^+ S_2^- + S_1^- S_2^+) [a_1 a_2, a_1^+ a_2^+] \tag{16}$$

Next, we apply unitary transformation to the wavefunction defined by $\Psi(t) = U(t)\Psi'(t)$, where $U(t) = \exp(-iH_{d1}t)$ and the Hamiltonian transforms to $H'_d = U^* H_d U$. The expression for the transformed Hamiltonian is given by

$$H''_d = \sum_{k=1,2} \frac{\Omega_L g}{\Delta} \left[ (a_1^2 + a_2^2)(\frac{\Delta}{\Delta}) \right] |e_k \rangle \langle e_k| + \frac{1}{\Delta} \left[ (n_1 + n_2 + 1) g^2 (S_1^+ S_2^- + S_1^- S_2^+) [a_1 a_2, a_1^+ a_2^+] \right] \tag{17}$$

In the above Hamiltonian the term responsible for generating unconventional geometric phase during the evolution of qubit states $|g_k \rangle$ and $|f_k \rangle$ can be identified as

$$H''_{d2} = -\sum_{k=1,2} \frac{\Omega_L g}{\Delta} \left[ (a_1^2 + a_2^2)(\frac{\Delta}{\Delta}) \right] |e_k \rangle \langle e_k| + \frac{1}{\Delta} \left[ (n_1 + n_2 + 1) g^2 (S_1^+ S_2^- + S_1^- S_2^+) [a_1 a_2, a_1^+ a_2^+] \right] \tag{18}$$

This is because atoms are in the ground states and under the dispersive condition atoms do not go to the excited state and hence the excited state operator and dipole–dipole interaction terms are ignored. For the infinitesimal interval $[t, t + dt]$, the evolution of states is governed by the operator

$$B(d\xi) = \exp(-iH''_{d2}t),$$

where

$$d\xi = -i \frac{\Omega_L g}{\Delta} e^{-i(\delta' + \frac{(n_1 + n_2 + 1)^2}{\Delta}) \frac{\Delta}{\Delta}} dt \tag{19}$$

The qubit state which contains a single [double] state $|f_k \rangle$ evolves by the operator $B(d\xi) [B(2d\xi)]$ while the other states do not change. For a preselected interaction time with cavity mode we have the two-mode squeezed state parameter

$$\xi = -i \frac{\Omega_L g}{\Delta} \int_0^t e^{-i(\delta' + \frac{(n_1 + n_2 + 1)^2}{\Delta}) \frac{\Delta}{\Delta}} d\tau = \frac{\Omega_L g}{\Delta(\delta' + (n_1 + n_2 + 1)^2 g^2)} \left[ e^{-i(\delta' + \frac{(n_1 + n_2 + 1)^2}{\Delta}) \frac{\Delta}{\Delta}} - 1 \right] \tag{20}$$

The geometric phase shifts for the qubit states $|g_k \rangle |f_k \rangle$ (|fk⟩gk⟩) and $|f_k \rangle |f_l \rangle$ are governed by $\Phi$ and $\Phi'$, respectively where

$$\Phi = (n_1 + n_2 + 1) \text{Im} \left( \int_C \xi' d\xi' \right) = \frac{(n_1 + n_2 + 1)(\Omega_L g)^2}{\Delta(\delta' + (n_1 + n_2 + 1)^2 g^2)} \left[ t - \frac{\sin(\delta' + \frac{(n_1 + n_2 + 1)^2}{\Delta}) \frac{\Delta}{\Delta}}{\delta' + \frac{(n_1 + n_2 + 1)^2}{\Delta} g^2} \right] \tag{21}$$

and

$$\Phi' = (n_1 + n_2 + 1) \text{Im} \left( \int_C 2\xi'^* d[2\xi'] \right) = 4\Phi.$$  

$$\tag{22}$$

$$\tag{23}$$
Chapter 4: Phase gate operation with unconventional geometric phase

Let us assume that the cavity field modes are initially in the vacuum state $|0,0\rangle$ and the displacement parameter of TPCS changes along a closed path. So, when it comes back to the original point in the phase space, a geometric phase is acquired by the state vector depending upon the condition of atomic states and cavity modes. To obtain a closed path in the parametric space of $\xi$, one has to restrict the interaction time of atoms with cavity field according to following condition

$$\left| \delta' + \frac{(n_1 + n_2 + 1)}{\Delta} \right| t = 2p\pi, \quad (24)$$

where $p$ is an integer. Interaction time in certain circumstances can be controlled by the time of flight of atoms crossing the cavity. Under the specific condition of interaction time, i.e., when $n_1 = n_2 = 0$, closed path evolution of constituting states of qubits under the operation of $H_{\text{dis}}$ are

$$|g_1,g_2\rangle|0,0\rangle \rightarrow |g_1,g_2\rangle|0,0\rangle,$$
$$|g_1,f_2\rangle|0,0\rangle \rightarrow e^{i\frac{\pi}{4}}e^{i\Phi}B(x)|g_1\rangle|f_2\rangle|0,0\rangle \rightarrow e^{i\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}|g_1\rangle|f_2\rangle|0,0\rangle,$$
$$|f_1,g_2\rangle|0,0\rangle \rightarrow e^{i\frac{\pi}{4}}e^{i\Phi}B(x)|f_1\rangle|g_2\rangle|0,0\rangle \rightarrow e^{i\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}|f_1\rangle|g_2\rangle|0,0\rangle,$$
$$|f_1,f_2\rangle|0,0\rangle \rightarrow e^{i\frac{\pi}{4}}e^{i\Phi}B(2\xi)|f_1\rangle|f_2\rangle|0,0\rangle \rightarrow e^{i\left(\frac{\pi}{4} + \frac{\pi}{4}\right)}|f_1\rangle|f_2\rangle|0,0\rangle,$$

$$\text{where} \quad \Phi = \frac{(\Omega Lg)^2}{\Delta(\Delta' + g^2)}t, \quad \Phi' = 4\Phi. \quad (26)$$

After the single qubit operation is performed, i.e., $|f_k\rangle \rightarrow e^{-i(\Phi + \frac{\pi}{4})}|f_k\rangle$, we obtain

$$|g_1,g_2\rangle|0,0\rangle \rightarrow |g_1\rangle|g_2\rangle|0,0\rangle,$$
$$|g_1,f_2\rangle|0,0\rangle \rightarrow |g_1\rangle|f_2\rangle|0,0\rangle,$$
$$|f_1,g_2\rangle|0,0\rangle \rightarrow |f_1\rangle|g_2\rangle|0,0\rangle,$$
$$|f_1,f_2\rangle|0,0\rangle \rightarrow e^{i\Phi}|f_1\rangle|f_2\rangle|0,0\rangle. \quad (27)$$

This completes the realization of an approximate unconventional geometric phase gate on the TPCS state operator in a physical system of cavity QED sustaining bichromatic field modes. By selecting the parametric condition in such a way that $2\Phi = -\pi$ then a $\pi$ phase gate from unconventional geometric phase can be implemented in this two-mode squeezed states system. This can be obtained when $\Omega L = g$, $\delta' = g^2/\Delta$, $(g^2/\Delta)\mu = \pi$ [12]. An interesting situation can arise when we consider the two-mode state as $|0,1\rangle$. This state produces similar kind of phase-gating as the state $|1,0\rangle$ does and there are degenerate phase gates under such a situation.

Since the excited state $|e\rangle$ is not populated in this model so we do not have any effect of radiation damping on the gate operation. In order to avoid cavity decay we need to have condition $\frac{\Omega Lg}{(\Delta' + g^2)} \ll 1$ which implies $B(x) \sim 1$ and gating operation is not affected by the cavity decay because atoms and fields are disentangled under this condition [12]. For the condition of parameter $\Delta = 15g$, $\delta' = 5g$, $\Omega L = 2g$ and $n_1 = n_2 = 0$, the time required to implement gate is $t' = \frac{\Delta(\Delta' + g^2)}{2(\Omega Lg)^2} = 150\pi/g$ and the probability of excitation to the excited state during gate operation is $6.6 \times 10^{-4}$ and efficient decay time is approximately $4 \times 10^4/g$ (assuming $\gamma = g/27$ [15]) and hence the gate error would be roughly $1.2 \times 10^{-2}$ which is smaller than the value reported in earlier experiments [16].

Recently, there are proposals to trap ions/neutral atoms inside the optical cavity. Instead of vibrational states of ions the optical-cavity mode plays that role. In order to avoid the effect of finite transit time of atoms/ions passing through the optical cavity, many avenues are pursued in cavity QED systems, e.g., far off-resonant trapping beams, near resonant light with $\tilde{n} \sim 1$ intracavity photons, and single trapped ions in high finesse optical cavities. Very recently for quantum information processing in cavity QED systems, extended trapping time of the order of $3s$ were achieved maintaining strong atom-field coupling constant. This kind of intracavity far-off resonant trap (FORT) has been used to trap single Cs atoms [17] located with $\tilde{n} \sim 1$ photon. The parametric value of $g$ was equal to $2\pi \times 47$ MHz. One can estimate the two-photon atom-field coupling constant to be $2\pi \times 5$ MHz which leads to time required for faithful gate operation to be less than $100\mu$s. For stronger driving fields this time could be less than $10\mu$s. In order to perform gate operation faithfully one needs to do it within the transit time of atoms. However in FORT the transit time does not cause any problem as the atoms are trapped for a longer time than the interaction time required to perform gate operation. Once gate operation is faithfully achieved atoms can be released from the trapping condition.
5. Conclusions

In summary, we have proposed a generalized unconventional geometric phase (under approximation) for the evolution of a two-photon coherent state or two-mode squeezed state generator on a closed path. This phase is found to be different from an ordinary coherent state displacement operator because of the bimodal nature of the operator. Using this phase we have given a scheme for realizing an approximate unconventional two qubit phase gates with classically driven three-level atoms interacting nonlinearly with bichromatic field modes in a cavity. The gate operation is performed using two lower states so the gate is not affected by the radiative damping of atoms. Under certain condition it is also not affected by the cavity decay. In this way using a TPCS (which is a nonclassical two-mode squeezed state also), we can implement an approximate unconventional geometric phase gate operation.

Note added in proof

An interesting work to implement unconventional geometric phase gate by highly squeezed operator with a cavity QED system has been reported [18] after submission of this manuscript.

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