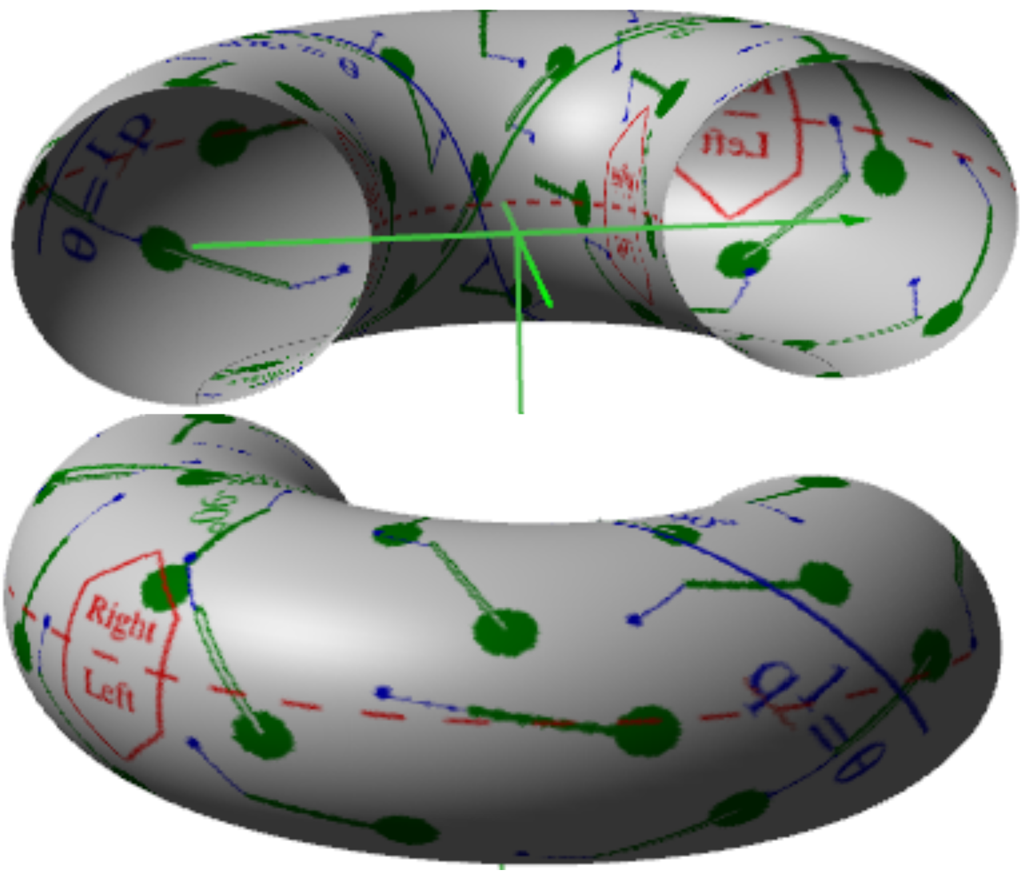
Unit 3 Coordinates and Transformations



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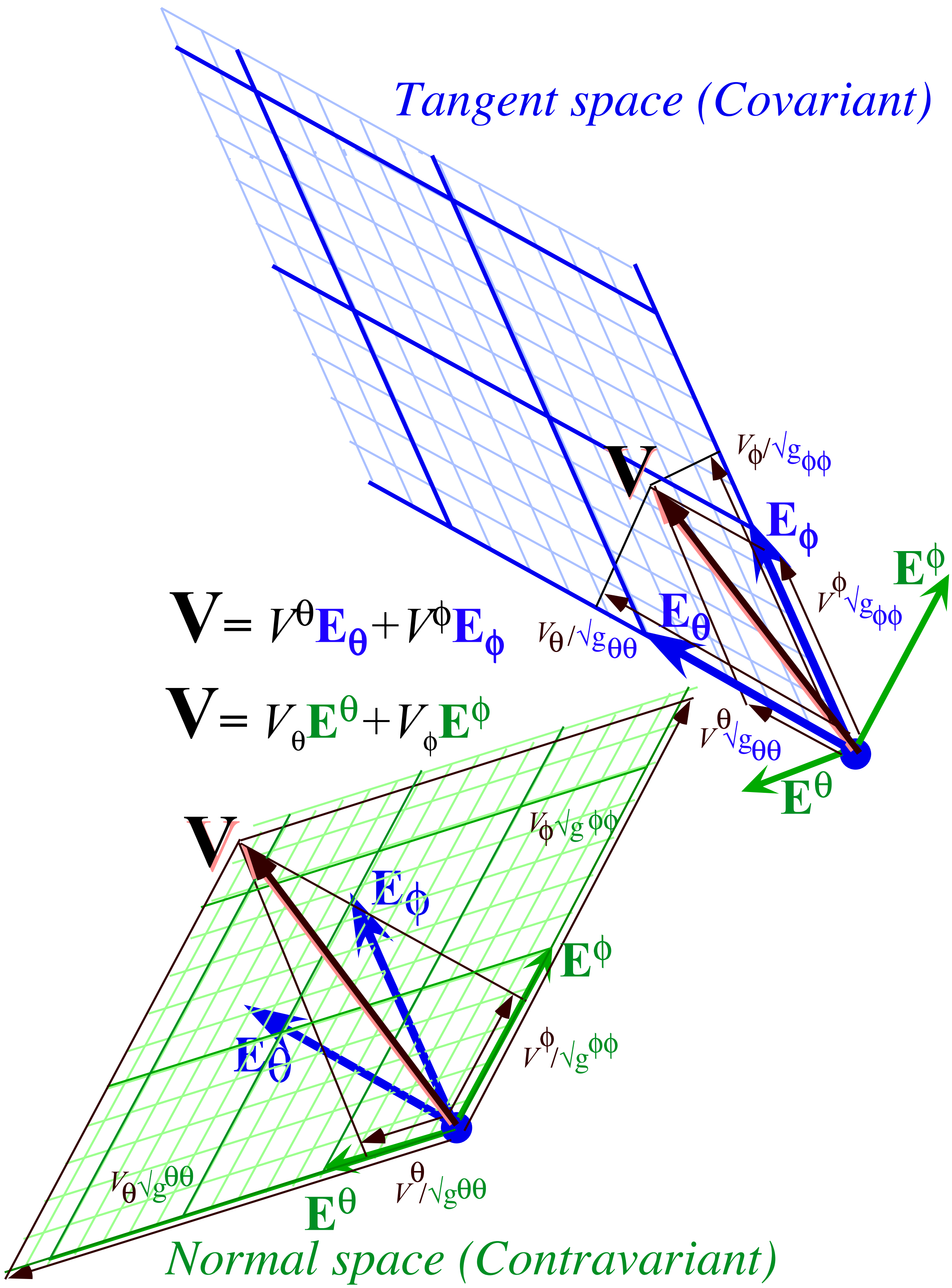
Lagrangian and Hamiltonian functions of generalized curvilinear coordinates (GCC) introduced in Units 1 and 2 are redeveloped using differential geometry of Riemann. The covariant metric tensor form of kinetic energy and Jacobian transformations are used to give an elegant approach to mechanics that is in the form used in general relativity. Christoffel expressions give Coriolis and related generalized inertial forces. GCC Hamiltonians with various symmetries provide effective potentials for analyzing motion on cylindrical, conical, spherical, toriodal, and hyperspherical surfaces.


Tangent space (Covariant)
V
$V_\phi/\sqrt{g_{\phi\phi}}$
$\mathbf{E}_\phi$
$\mathbf{E}^\phi$
$V^\phi\sqrt{g_{\phi\phi}}$
$\mathbf{E}_\theta$
$V_\theta/\sqrt{g_{\theta\theta}}$
$\mathbf{V} = V^\theta \mathbf{E}_\theta + V^\phi \mathbf{E}_\phi$
$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$
$V^\theta\sqrt{g_{\theta\theta}}$
$\mathbf{E}^\theta$
V
$V_\phi\sqrt{g^{\phi\phi}}$
$\mathbf{E}_\phi$
$\mathbf{E}^\phi$
$\mathbf{E}_\theta$
$V^\phi/\sqrt{g^{\phi\phi}}$
$V_\theta\sqrt{g^{\theta\theta}}$
$V^\theta/\sqrt{g^{\theta\theta}}$
$\mathbf{E}^\theta$
Normal space (Contravariant)

Unit 3. Coordinates and Transformations

Big BANG! Our universe begins with a cataclysmic creation of a space-time manifold. At least that is what Einstein's adaptation of Riemann's geometry to curved space-time relativity leads us to believe. Unit 3 reintroduces, in Riemann form, *generalized curvilinear coordinates (GCC).* GCC are used in Unit 2 to derive trebuchet and pendulum equations and general *nonlinear transformations* to solve them. (Recall transformation (2.9.13) that solves Hamiltonian equations for a g-free trebuchet.) Much of mechanics involves transforming to GCC to set up equations of motion and then transforming to other GCC to solve them. Fourier normal mode wave transformation is an example that occupies much of the following Unit 4.

GCC theory is also one of the foundations of *tensor analysis* and *differential geometry*, both of which have applications hydrodynamics, electrodynamics, and the mechanics of special and general relativity. (One of its first uses was for stress and tension analysis where the term *tensor* arose.)

Chapter 1. GCC manifolds and global properties

Our first examples of GCC are the trebuchet coordinates (q^1=θ, q^2=ϕ) introduced in Unit 2. Other examples are well known 2D-polar (q^1=r, q^2=θ), cylindrical (q^1=ρ,q^2=ϕ ,q^3=z), and 3D-spherical polar (q^1=r q^2=θ , q^3=ϕ) coordinates that decouple orbital equations and are *orthogonal curvilinear coordinates (OCC)*, a special case of GCC. More exotic OCC are parabolic (q^1=ρ, q^2=ν) and elliptic-hyperbolic (q^1=ζ, q^2=ξ, q^3=χ) OCC introduced in Ch. 10 of Unit 1 and Ch. 9 of this Unit 3. Unit 5 uses such OCC to treat equations for atomic orbits in electric fields (Stark orbits) and two-force-center molecular orbits.

It is tempting to just focus on OCC and ignore GCC. Many modern treatments of mechanics cheat the student by doing this and miss the great power and utility of GCC that is well worth a little extra work it takes to learn them. Also, most of the generalized coordinates found in mechanics, like the trebuchet angles, are in fact GCC systems and not OCC at all. Finally, GCC theory prepares one for modern tools such as exterior calculus and higher geometry and topology in n-dimensional phase space and space-time.

Generalized Curvilinear Coordinates (GCC): Global properties

Perhaps, the trebuchet is the oldest mechanics problem in human history, particularly considering analogous human throw and chop motions described in Unit 2 Ch. 1. The coordinate angles are a classic example of *non*orthogonal *generalized curvilinear coordinate (GCC)* system. To show this we plot the coordinate lines (q^1=θ =*const.*) on top of (q^2=ϕ =*const.*) in Fig. 3.1.1. The resulting curved lines (Actually, they are circles.) form a coordinate grid or *manifold* with a interesting topology.

Coordinate topology

Two overlapping manifolds are needed to describe the trebuchet. The upper drawing (Fig. 3.1.1a) applies when the trebuchet is shaped like your left hand and is pulling to throw its projectile to the left. The lower drawing (Fig. 3.1.1b) applies when the trebuchet is shaped like your right hand as if pulling to throw its projectile to the right. Either drawing appears to be part of a 3-dimensional *torus*, an optical illusion due to circles drawn by θ =*const.* or ϕ =*const.* compasses. But, this illusion shows some topological properties.

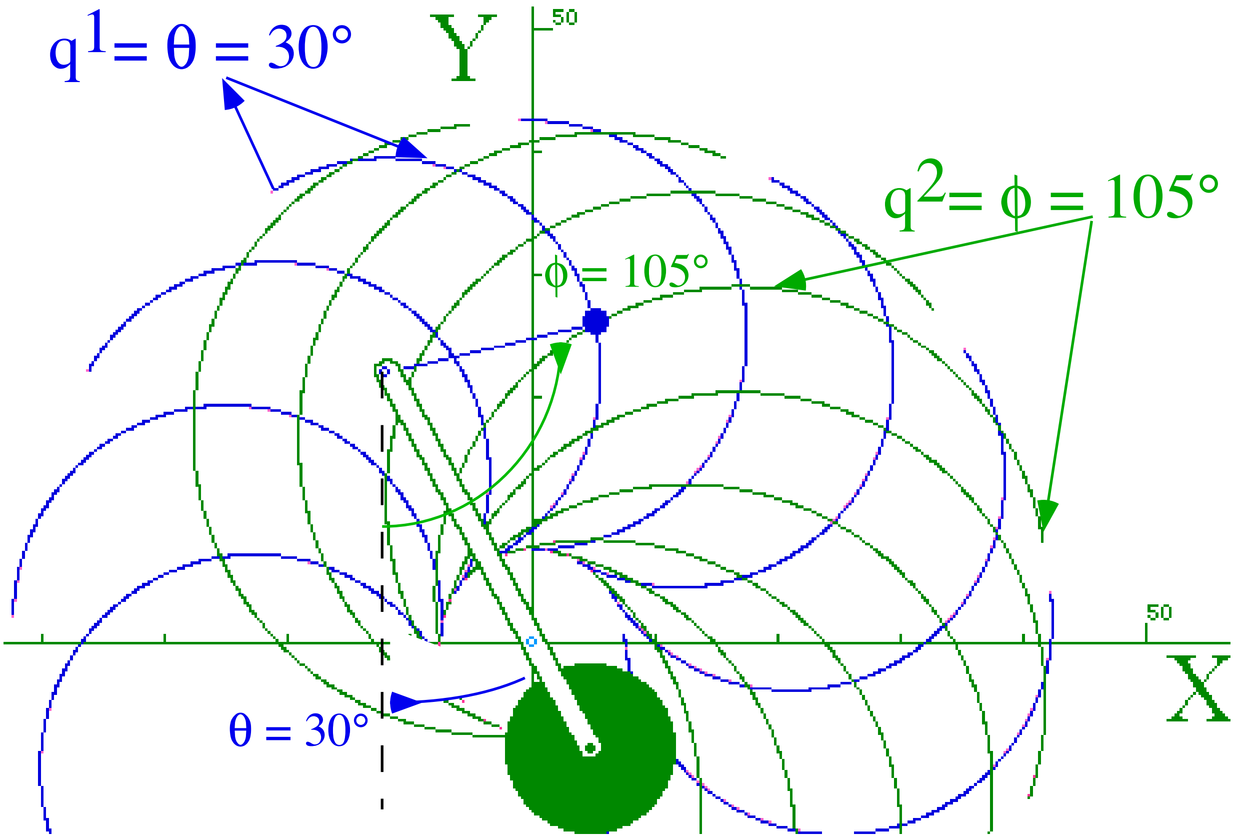


Fig. 3.1.1a (q^1=θ, q^2=φ)*Coordinate manifold for trebuchet (Left handed sheet.)*

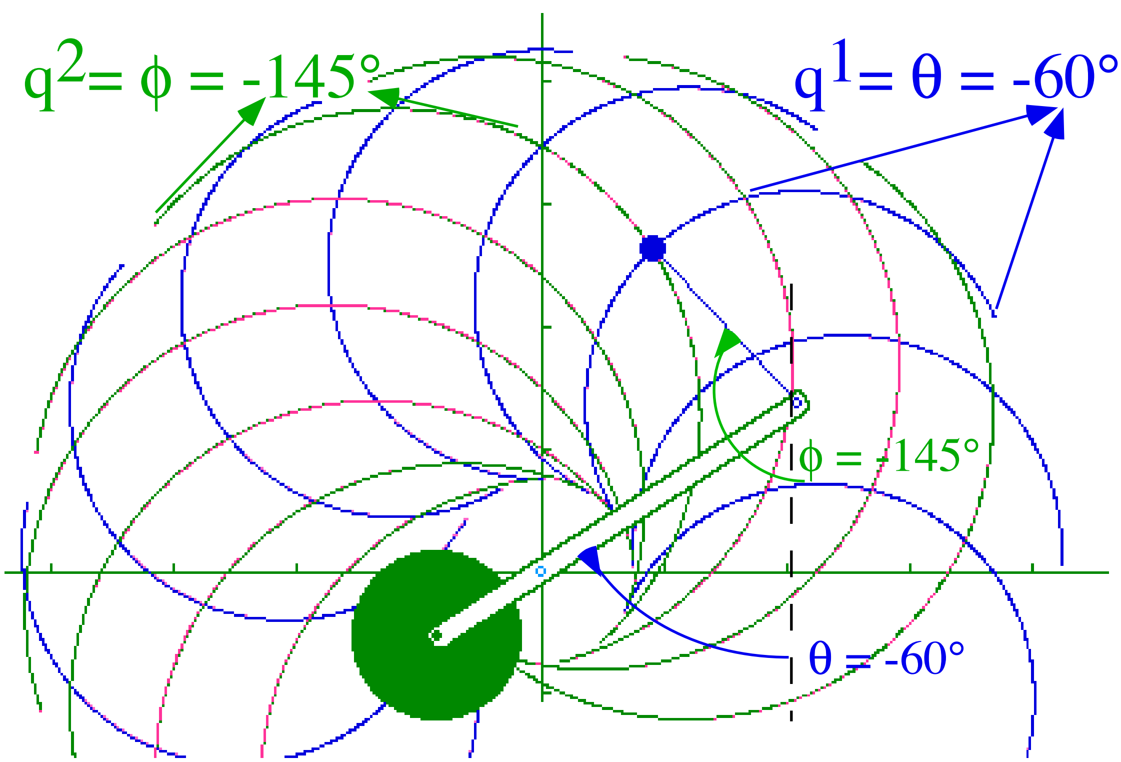


Fig. 3.1.1b (q^1=θ, q^2=φ)*Coordinate manifold for trebuchet (Right handed sheet.)*

The upper and lower drawing can be viewed as a torus "top" and "bottom" of a toroidal surface rendered in Fig. 3.1.2a. Trebuchet positions on the Fig. 3.1.2b torus are laid out in Fig. 3.1.3.

The θ–lines are not orthogonal to ϕ–lines in Fig. 3.1.1. We may "enforce" orthogonality by plotting (θ,ϕ) on an orthogonal Cartesian graph of its own as is done in Fig. 3.1.3 with tiny trebuchet images at $\pm 45°$ intervals to show the topology. Going along a supposedly straight path (say $\phi=0$ or the "X-axis") one runs into the same point with each passing of a 2π interval. Such a manifold may be mapped or wrapped onto a torus like Fig. 3.1.3 (below) so it can be non-redundant and finite-continuous.

Red dashed lines $(\theta=\phi\pm\pi)$ in Fig. 3.1.3 (upper left and lower right) map onto the *outer equator* of the torus in Fig 3.2.2b. It is a boundary between left and right handed "stretched-out" trebuchets. Central red dashed line $(\theta=\phi)$ in Fig. 3.1.3 is the torus *inner equator* between left and right "tucked-in" trebuchets.

As points rise above the equators in Fig. 3.1.3, the trebuchet becomes more right-handed until the coordinate difference $\theta-\phi$ at the top is $\theta-\phi=+\pi/2$, the most right-handed position. Below the equators the trebuchet becomes most left-handed at the bottom where $\theta-\phi=-\pi/2$.

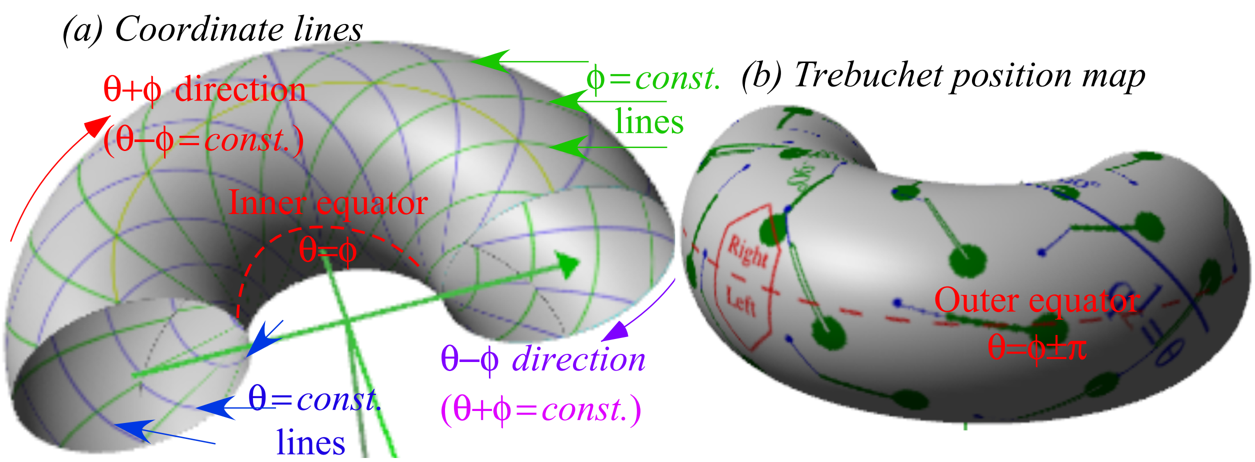


Fig. 3.1.2 Trebuchet torus. (a) $(q^1=\theta,\ q^2=\phi)$ *coordinate lines.(b)Trebuchet position map and equators.*

This example begins an analysis of *global* or *topological* properties of a coordinate system that underlie the actual mechanical coordinates and their *local* or *differential* properties. Topology characterizes connections and classifies equivalent closed paths in a space. Path A and B are *equivalent* if A can be deformed into B without cutting A. Imagine stretchy rubber bands wrapping the torus and only able to slide on it. Equivalent toroidal paths have the same *winding numbers* (N_{major},N_{minor}) . These integers count the number of times a path encircles the major or minor circumference, respectively.

The coordinate θ–lines and ϕ–lines both wrap once around each major or minor circumference so, apart from sign, one may classify them by $(N_{major}=1,N_{minor}=1)$. The $\theta+\phi$–lines wrap the doughnut hole or major circumference so their classification is $(N_{major}=1,N_{minor}=0)$, while the $\theta-\phi$–lines are in the $(0,1)$ class.

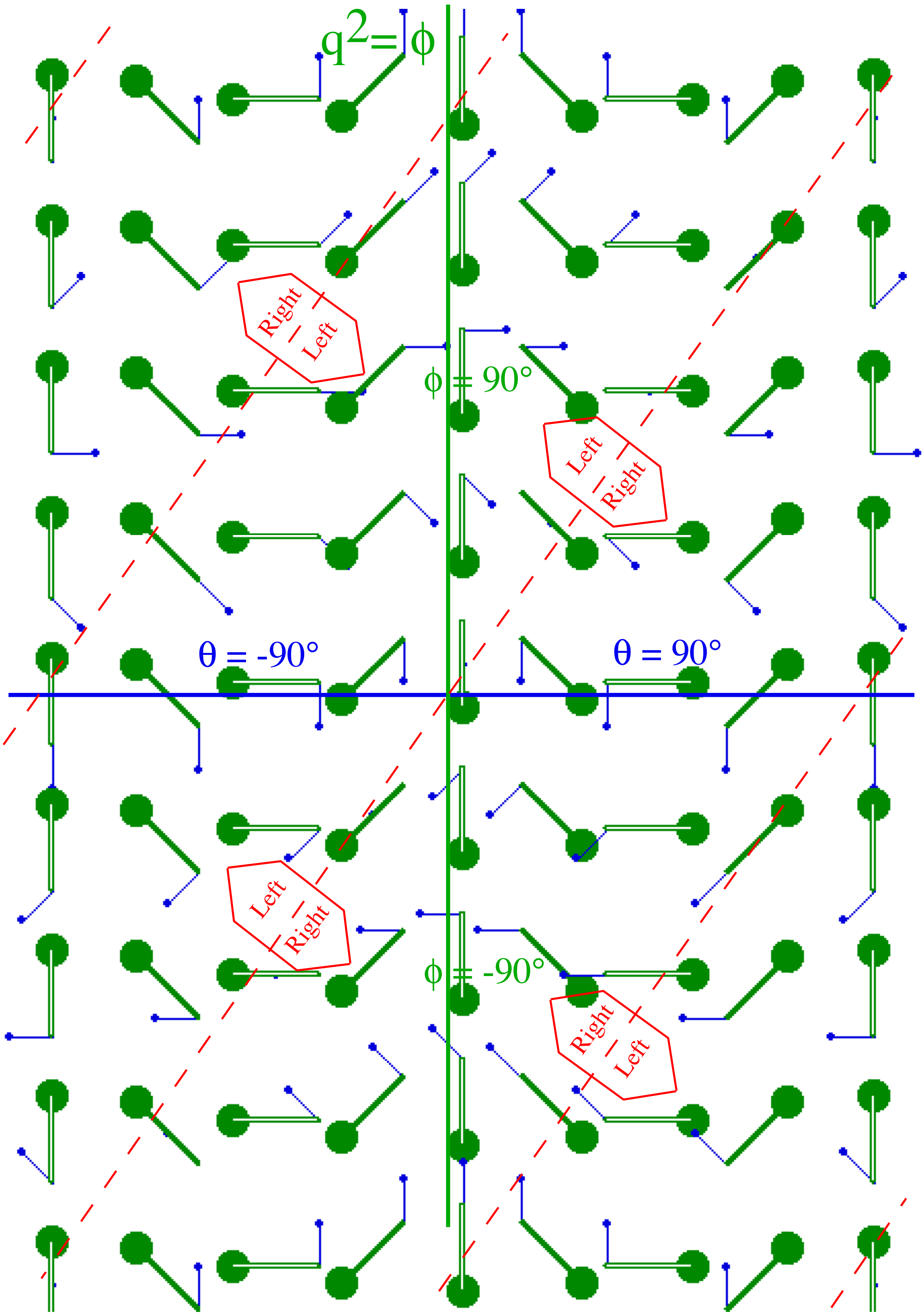


Fig. 3.1.3 "Flattened" ($q^1=\theta$, $q^2=\phi$) coordinate manifold for trebuchet

Coordinates are like temporary marks or "signposts" on a space to distinguish one point or region from another. Useful generalized coordinates will be ones that "respect" the topology of the system and provide the greatest continuity or connection to various parts of the space. Equations of motion are *local* differential recipes. Their *global* solutions are the cuisine we seek. Without good ingredients and ambiance, the outcome may not be palatable. As Dirac once said, “Nature is a stickler for good form!”

Global properties are important for even the simplest curvilinear coordinates such as polar coordinates (r, θ). For polar coordinates the embedding surface that respects continuity at the origin is a cone as shown in Fig. 3.1.4. That way the radial coordinate r can go through zero into negative values without causing a discontinuous change in the polar angle θ.

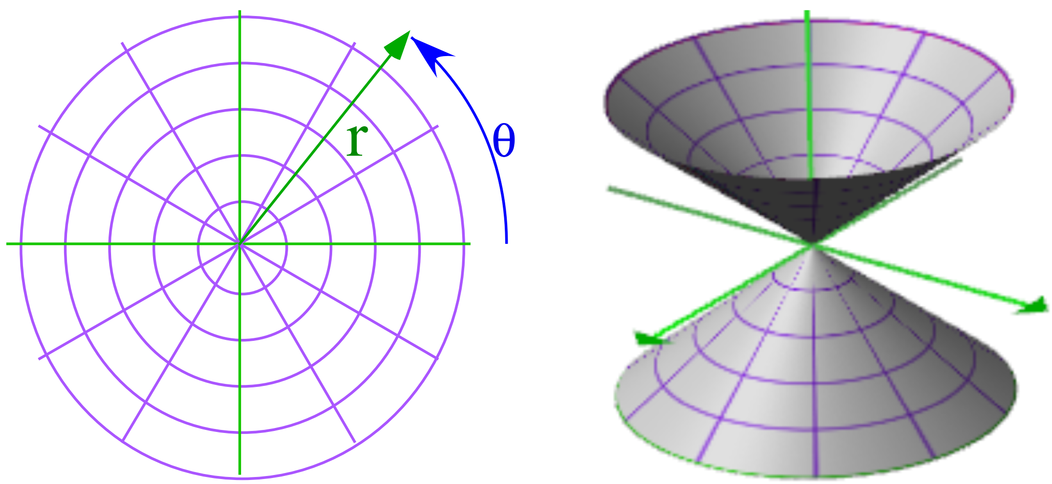


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

Exercise 3.1.1 Do a ruler & compass construction of trebuchet manifold assuming levers ℓ and r are equal.

Chapter 2. GCC differentials, Jacobians, and local properties

If you "cut and paste" the flat (θ,ϕ) graph in Fig. 3.1.3 onto the curved toroidal manifold surface in Fig. 3.1.2, you would crumple and tear the paper. The same applies to Fig. 3.1.4. Crumpling is related to the *differential* or *metric* properties of the space. Metric properties are determined by derivatives of *GCC definition functions* $q^m = q^m(x^j)$ or of *inverse definition functions* $x^j = x^j(q^m)$. Mechanics often deals with the inverse relations such as the following $x^j(\theta\ ,\ \phi\ )$ for the trebuchet. (Recall (2.2.1).)

$$x^1(\theta\ ,\ \phi\ ) = x = -\,r\ sin\ \theta + \ell\ \ sin\ \phi \tag{3.2.1a}$$

$$x^2(\theta\ ,\ \phi\ ) = y = \ r\ cos\ \theta\ -\ \ell\ \ cos\ \phi \tag{3.2.1b}$$

From now on we will be using an old standard notation q^m for generalized coordinates as well as for Cartesian coordinates x^j where the index is stuck "up" as a superscript where exponents normally go. This can be annoying for writing squares of a coordinate like $(q^m)^2$, but old diehard conventions die hard!

Most that we define in this section is based upon the *first differential* of an inverse coordinate definition function, which by the chain rule is the following. This is our most local definition.

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m\ , \qquad \text{or:}\quad \mathbf{dr} = \frac{\partial \mathbf{r}}{\partial q^m} dq^m \tag{3.2.2}$$

Tensor notation with an index-sum and Gibb's (•)-notation is used above and for its inverse below.

$$dq^m = \frac{\partial q^m}{\partial x^j} dx^j = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \mathbf{dr} \tag{3.2.3}$$

An important part of tensor notation is a so-called *dummy index rule* in which the sum over all independent variables is denoted by indices of any term being *repeated* on the *same* side of an equation. (3.2.2) sums m and (3.2.3) sums j. Sums replace the "dot" • product in Cartesian-Gibb's vector analysis.

The coefficients of the first differentials are called *Jacobian* matrix components. For the trebuchet the following matrix and its inverse *Kajobian* matrix are as follows. (Recall (2.2.7) and (2.2.8).)

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}, \qquad \left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} \tag{3.2.4a}$$

$$= \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}, \qquad = \frac{\begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{pmatrix}}{r\ell\sin(\theta-\phi)} \tag{3.2.4b}$$

(We referred to the first as the *Jacobian* matrix and its inverse as the *"Kajobian"* matrix, a little silly and quite arbitrary.) Existence of an inverse is essential so that partial derivative *chain rule relations* hold.

$$\frac{\partial x^j}{\partial q^m}\frac{\partial q^m}{\partial x^k} = \delta^j_k = \begin{Bmatrix} 0 \text{ if: } j \neq k \\ 1 \text{ if: } j = k \end{Bmatrix}, \qquad \frac{\partial x^j}{\partial q^n}\frac{\partial q^m}{\partial x^j} = \delta^m_n = \begin{Bmatrix} 0 \text{ if: } m \neq n \\ 1 \text{ if: } m = n \end{Bmatrix} \tag{3.2.5}$$

Inverses exist at a given point if and only if a Jacobian (or Kajobian) determinant is not zero (or infinity) there.

$$0 \neq \det\left|\frac{\partial x^j}{\partial q^m}\right| = \det\begin{vmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{vmatrix} = r\ell\sin(\theta-\phi)\,, \qquad 0 \neq \det\left|\frac{\partial q^m}{\partial x^j}\right| = \det\begin{vmatrix}\frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y}\end{vmatrix} = \frac{1}{r\ell\sin(\theta-\phi)} \tag{3.2.6}$$

Otherwise the Jacobian matrix is singular, and the point is called a *coordinate singularity*. For the trebuchet, singularities occur when $sin(\theta - \phi)=0$ which happens at the toris equators ($\theta = \phi$ and $\theta = \phi \pm \pi$). as "tuck" or "stretch" positions of projectile mass m at *6-o'clock* and *12-o'clock* relative to M-beam-line r in Fig. 2.2.2 or "start" or "release" points at *9-o'clock* or *3-o'clock* relative to beam-normal in Fig. 3.2.1.

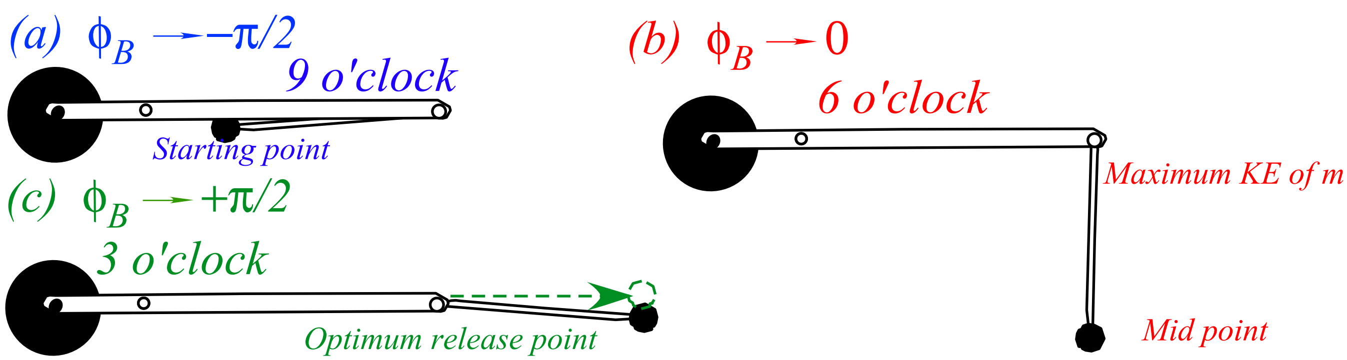


Fig. 3.2.1 Positions of a throwing sequence. (After Fig. 2.9.7)

GCC Unitary vectors E_k and E^k

The first differential (3.2.2) defines "unit-cells" for all combinations of its coordinate differentials. We rewrite it as follows while introducing a kind of generalized "quasi-unit" vectors.

$$\mathbf{dr} = \mathbf{E_m} dq^m \;, \quad \text{where:} \quad \mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} \tag{3.2.7}$$

The newly defined $\mathbf{E}_m$ is called a *GCC covariant unitary vector*. Note the word "unitary" rather than "unit"; these vectors are generally *not* unit-vectors in the Cartesian sense. Instead they indicate unit distances in the GCC system. If all the differentials were set equal to unity ($1=dq^m$) then the resulting coordinate vector differential would be the sum $\mathbf{dr}= \mathbf{E}_1+\mathbf{E}_2...+\mathbf{E}_N$ of all the covariant vectors. If only the first differential is set equal to unity ($1=dq^1$, $0=dq^2$,...) then $\mathbf{dr}= \mathbf{E}_1$, and so forth.

The problem is that GCC systems are curved and so straight fixed scale unit vector "rulers" become inaccurate for finite intervals. Equation (3.2.7) is a differential relation so the dq^m are assumed *infinitesimal*. Nevertheless, we can imagine constructing *tangent vectors* $\mathbf{E}_1$, $\mathbf{E}_2$... at each point on the curved manifold. These tangent vectors $\mathbf{E}_m$ are the covariant unitary vectors that span a *tangent space*.

A sketch of a typical tangent space is shown in Fig. 3.2.2. If the surface was flat then the vectors ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$) would exactly correspond to advancing coordinates (θ,ϕ) by incremental angle $d\theta$=1 radian and

$d\phi$=1 radian, respectively. Note each major coordinate grid in Fig. 3.2.2 represents $\pi/6$ or about half a radian, and the unitary vectors each span about two of their grid spaces. This is true even though the coordinate lines curve away significantly in one radian.

For small increments like $d\theta = 0.01$ or $d\phi = 0.01$, the tangent space is a more precise representation of the surface geometry. The vector $0.01\mathbf{E}_\theta$ shown in Fig. 3.2.1 is very nearly equal to the $d\theta = 0.01$ interval and pointing in the direction of *increasing* θ. The partial derivative with respect to GCC coordinate q^m in (3.2.7) gives **dr** when *only* q^m is allowed to vary.

Rewriting inverse differential (3.2.3) gives another kind of generalized "unit" vector.

$$dq^m = \frac{\partial q^m}{\partial x^j} dx^j = \mathbf{E}^\mathbf{m} \bullet \mathbf{dr} , \quad \text{where:} \quad \mathbf{E}^\mathbf{m} = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m \qquad (3.2.8)$$

The newly defined $\mathbf{E}^\mathrm{m}$ are called *GCC contravariant unitary vectors* and point along the gradient **grad**q^m *normal* to the q^m=const. line (or surface in 3D) in the direction of *increasing* q^m as in Fig. 3.2.3.

The two kinds of vectors make it much easier to deal with non-orthogonal coordinates and vectors. While neither the covariants nor the contravariants are orthogonal unit vectors by themselves, they are *mutually orthonormal* according to chain rule (3.2.5).

$$\frac{\partial q^m}{\partial x^j}\frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial \mathbf{r}}{\partial q^n} = \mathbf{E}^\mathbf{m} \bullet \mathbf{E}_\mathbf{n} = \delta^m_n = \begin{cases} 0 & \text{if: } m \neq n \\ 1 & \text{if: } m = n \end{cases} \qquad (3.2.9)$$

The two kinds of vectors occupy the columns and rows of the Jacobian and Kajobian matrices, respectively, as shown by the trebuchet examples below.

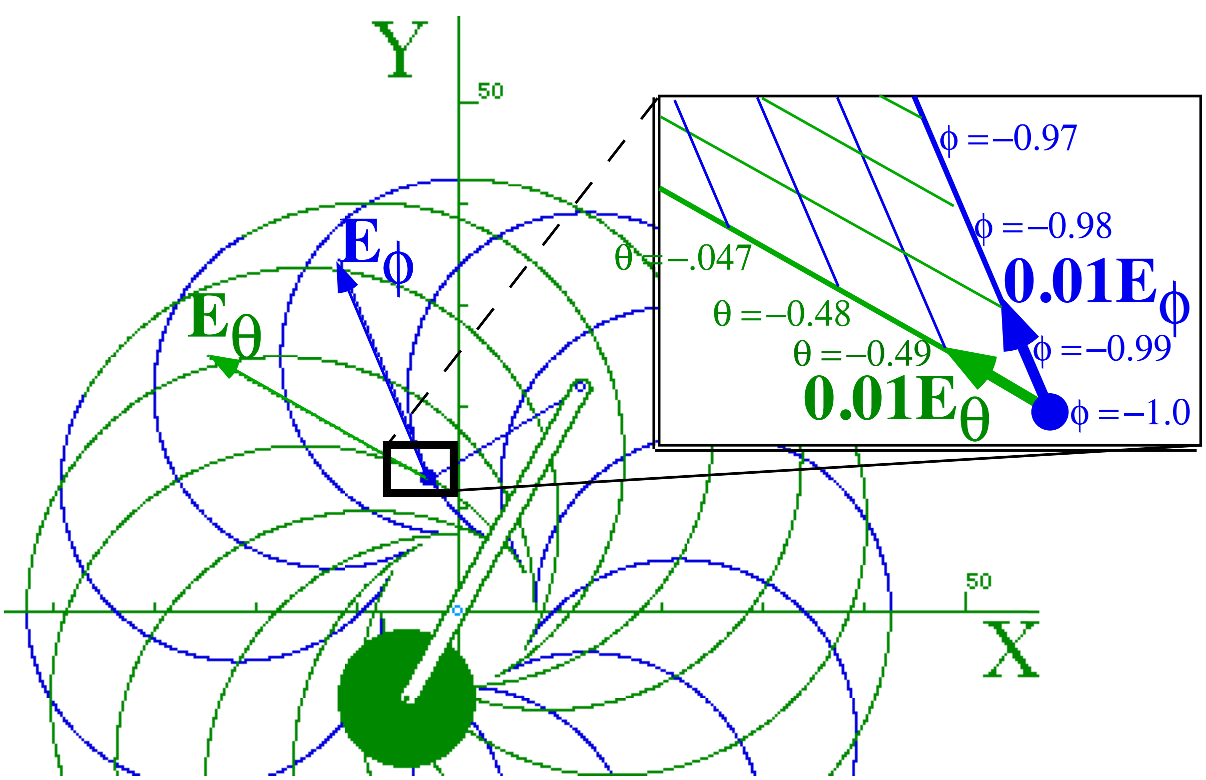


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad (3.2.10a) \qquad\qquad \left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \qquad (3.2.10b)$$

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}, \qquad \begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

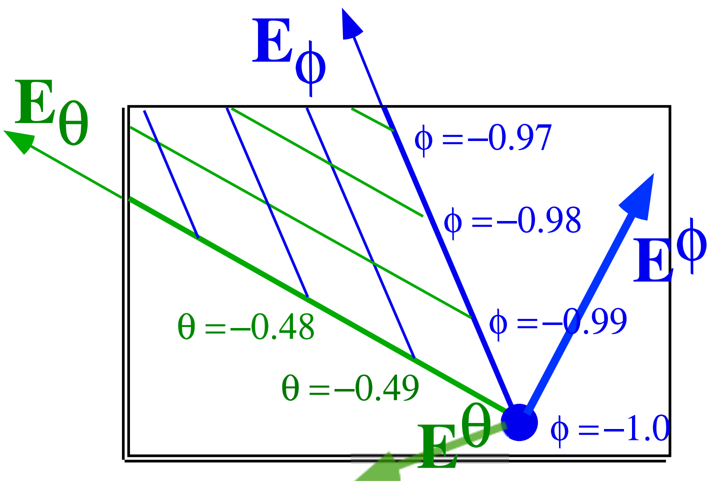


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

For the trebuchet coordinates the unitary vectors in Figs. 3.2.2 and Fig. 3.2.3 are as follows.

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}, \qquad \begin{array}{l} \mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi) \\ \mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi) \end{array} \qquad \begin{array}{c} (3.2.10a) \\ \textit{Example} \\ (3.2.10b) \end{array}$$

Mutual orthonormality $\mathbf{E}^m \bullet \mathbf{E}_n = \delta^m_n$ of $\mathbf{E}_m$'s with $\mathbf{E}^m$'s is analogous to that of lattice and reciprocal lattice or between bras and kets ($\langle m | n \rangle = \delta^m_n$) in quantum theory. *Any* vector **U**,**V**,**...** is expressed using *either* set,

$$\mathbf{U} = U^m\,\mathbf{E}_m = U_n\,\mathbf{E}^n, \qquad \mathbf{V} = V^m\,\mathbf{E}_m = V_n\,\mathbf{E}^n, \quad \ldots \qquad (3.2.11a)$$

where the *U^m, V^m,...are contravariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E}^m, \; V^m = \mathbf{V}\bullet\mathbf{E}^m \quad , \ldots \qquad (3.2.11b)$$

and the *U_n , V_n ,..are covariant components*

$$U_n = \mathbf{U}\bullet\mathbf{E}_n, \; V_n = \mathbf{V}\bullet\mathbf{E}_n \quad , \ldots \qquad (3.2.11c)$$

of the respective vectors. Because of the mutual orthonormality the scalar products of two vectors are as simple as they are for Cartesian vectors.

$$\mathbf{U}\bullet\mathbf{V} = U_n\,V^n = U^m\,V_m \qquad \ldots \qquad (3.2.11d)$$

To use this formalism one must make sure that indices "balance". Summations must always be between an up (contra) and a down (covariant) index, and an equal number of each kind of un-summed (unrepeated) index must exist on either side of an equation to balance it.

GCC Jacobian Transformations

Transformation between different GCC or to and from Cartesian coordinates is done using the chain rule and Jacobian (or Kajobian) matrices. The first step is to settle the transformation behavior of the unitary vectors. Definition (3.2.7) of vector $\mathbf{E_m}$ for coordinate system $\left\{q^1, q^2, \cdots\right\}$ is written in terms of new vectors $\mathbf{\bar{E}_{\bar{m}}}$ for a new "barred" coordinate system $\left\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\right\}$ using a chain-saw-sum rule.

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \text{ , or: } \mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}} \tag{3.2.12}$$

The result is the covariant transformation of unitary vectors. The "contras" transform inversely.

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}} \text{ , or: } \mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}} \tag{3.2.13}$$

To get the inverse of these transformations just replace "barred" quantities by "unbarred" ones.

$$\mathbf{\bar{E}_{\bar{m}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{E_m} \text{ ,} \qquad \mathbf{\bar{E}^{\bar{m}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{E^m} \tag{3.2.14}$$

Having set the base vector transformations, it is then an easy step to derive transformation rules for components of any vector **U** using (3.2.11). The key thing to remember is that a vector is an <u>invariant</u> thing; *it doesn't care which coordinate system or "viewpoint" you use to view it*. The same vector **U** can be written as many different ways as you have base vector sets $\mathbf{E_m}$ or $\mathbf{\bar{E}_{\bar{m}}}$.

$$\mathbf{U} = U^m\, \mathbf{E_m} = U_n\, \mathbf{E^n} = \bar{U}^{\bar{m}}\, \mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\, \mathbf{\bar{E}^{\bar{n}}} \tag{3.2.15}$$

The components do change from one viewpoint to the next according to (3.2.11) and (3.2.14).

$$\begin{aligned} \bar{U}_{\bar{m}} &= \mathbf{U} \bullet \mathbf{\bar{E}_{\bar{m}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{U} \bullet \mathbf{E_m} \text{ ,} & \bar{U}^{\bar{m}} &= \mathbf{U} \bullet \mathbf{\bar{E}^{\bar{m}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{U} \bullet \mathbf{E^m} \\ &= \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}U_m \text{ ,} & &= \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}U^m. \end{aligned} \tag{3.2.16}$$

Component transformation rules (3.2.16) exactly mimic those of the unitary vectors in (3.2.14). Covariant components transform inversely to contras so the scalar product is viewpoint-invariant.

$$\mathbf{U} \bullet \mathbf{V} = U^m\, V_m = U_n\, V^n = \bar{U}^{\bar{m}}\, \bar{V}_{\bar{m}} = \bar{U}_{\bar{n}}\, \bar{V}^{\bar{n}} \tag{3.2.17}$$

The same rules apply to tensors and tensor components. (A tensor is analogous to ket-bra operators $\mathbf{T} = T_{ij}\left\langle i \middle| j \right\rangle$ used in quantum theory.) A second rank tensor $\mathbf{\bar{\bar{T}}}$ may be written in any of the following ways.

$$\begin{aligned} \mathbf{\bar{\bar{T}}} &= T^{mn}\, \mathbf{E_m E_n} = T_{mn}\, \mathbf{E^m E^n} = T^m_n \mathbf{E_m E^n} = T^m_n\, \mathbf{E^n E_m} \\ &= \bar{T}^{\bar{m}\bar{n}}\, \mathbf{\bar{E}_{\bar{m}} \bar{E}_{\bar{n}}} = \bar{T}_{\bar{m}\bar{n}} \mathbf{\bar{E}^{\bar{n}} \bar{E}^{\bar{m}}} = \bar{T}^{\bar{m}}_{\bar{n}} \mathbf{\bar{E}_{\bar{m}} \bar{E}^{\bar{n}}} = \bar{T}^{\bar{m}}_{\bar{n}}\, \mathbf{\bar{E}^{\bar{n}} \bar{E}_{\bar{m}}} \end{aligned} \tag{3.2.18}$$

Tensor transformations are done using combinations of (3.2.16).

$$\bar{T}_{\bar{m}\bar{n}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T_{mn} \quad (3.2.19a) \qquad \bar{T}^{\bar{m}\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial \bar{q}^{\bar{n}}}{\partial q^n} T^{mn} \quad (3.2.19b) \qquad \bar{T}^{\bar{m}}_{\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n \quad (3.2.19c)$$

Exercise 3.2.1

Verify that metric tensor g_{mn} transforms to $\bar{g}_{\bar{m}\bar{n}}$ according to rules (3.2.19a).

Verify that metric tensor g^{mn} transforms to $\bar{g}^{\bar{m}\bar{n}}$ according to rules (3.2.19b).

Verify that metric tensor g^n_m transforms to $\bar{g}^{\bar{n}}_{\bar{m}}$ according to rules (3.2.19c). What is special about this case?

Chapter 3. GCC Metric Tensors g_{mn} and g^{mn}

A very famous tensor is the *metric tensor* **g** that has the following *covariant (and contravariant) metric components g_{mn} (and g^{mn})* defined, respectively, as follows.

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ . \tag{3.3.1}$$

The "mixed" covariant-contravariant metric components receive less notoriety but are most important.

$$g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = \delta_m^n = \begin{cases} 0 \ \text{ if: } m \neq n \\ 1 \ \text{ if: } m = n \end{cases} \tag{3.3.2}$$

Because of (3.2.9), they are simply delta-tensors or unit operators. In fact, the abstract tensor **g** <u>is</u> the δ-tensor, though hardly anyone uses this notation! Caution: δ_{mn} is g_{mn} and <u>not</u> δ_n^m in GCC.

Covariant metric coefficients express covariant unitary vectors in terms of contras and vice-versa.

$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \ , \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \ . \tag{3.3.3}$$

Clearly, they are *matrix* inverses of each other. (Do not set $g_{mn} = 1/g^{mn}$ unless $g_{mn} = \delta_m^n \, g_m$ *is diagonal*.)

$$\langle g_{mn} \rangle = \langle g^{mn} \rangle^{-1} \ , \qquad \langle g^{mn} \rangle = \langle g_{mn} \rangle^{-1} \ . \tag{3.3.4}$$

Also, co-and-contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \quad V^m = g^{mn}V_n \ , \quad T^{mm'} = g^{mn}g^{m'n'}V_{nn'} \ , \ etc. \tag{3.3.5}$$

Metric coefficients measure off differential arc length given GCC coordinate differentials. From the fundamental definition (3.2.7) we have the arc length square using (3.3.1).

$$(ds)^2 = \mathbf{dr} \bullet \mathbf{dr} = \left(\mathbf{E_m} dq^m\right) \bullet \left(\mathbf{E_n} dq^n\right) \tag{3.3.6a}$$

$$(ds)^2 = g_{mn} dq^m dq^n \tag{3.3.6b}$$

(The metric relation is treated as an *axiom* in general relativity theory using *four* dimensions of space-time.) The g_{mn} are scale factors. Metric coefficients for trebuchet coordinates in (3.2.7) to (3.2.10) are as follows.

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\theta\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} r^2 & -r\ell\cos(\theta-\phi) \\ -r\ell\cos(\theta-\phi) & \ell^2 \end{pmatrix} \ . \tag{3.3.7}$$

The diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors.

$$\left|\mathbf{E_m}\right| = \sqrt{\mathbf{E_m} \bullet \mathbf{E_m}} = \sqrt{g_{mm}} \tag{3.3.8a}$$

(No sum here since index m appears on <u>both</u> sides of equation.) The geometric interpretation of a vector's components relative to the tangent space is sketched in Fig. 3.3.1. The diagonal square roots $\sqrt{g^{mm}}$ are the lengths of the contravariant unitary vectors.

$$\left|\mathbf{E^m}\right| = \sqrt{\mathbf{E^m} \bullet \mathbf{E^m}} = \sqrt{g^{mm}} \tag{3.3.8b}$$

A vector's components relative to the normal space is sketched in Fig. 3.3.2.

The area of a parallelogram 2-cell in a tangent space spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$ is the following.

$$Area\left(V^1 E_1, V^2 E_2\right) = V^1 V^2 \left|\mathbf{E_1}\times\mathbf{E_2}\right| = V^1 V^2 \sqrt{\left(\mathbf{E_1}\times\mathbf{E_2}\right)\bullet\left(\mathbf{E_1}\times\mathbf{E_2}\right)}$$

The Levi-Civita identity (1.A.7) reduces cell area to a 2-by-2 metric determinant. (Recall $g_{12} = g_{21}$.)

$$Area\left(V^1 E_1, V^2 E_2\right) = V^1 V^2 \sqrt{\left(\mathbf{E_1}\bullet\mathbf{E_1}\right)\left(\mathbf{E_2}\bullet\mathbf{E_2}\right) - \left(\mathbf{E_1}\bullet\mathbf{E_2}\right)\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)}$$

$$= V^1 V^2 \sqrt{g_{11}g_{22} - g_{12}g_{12}} = V^1 V^2 \sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}} \qquad (3.3.9)$$

The volume of a parallelepiped 3-cell in a 3D tangent space spanned by $V^1\mathbf{E}_1$, $V^2\mathbf{E}_2$ and $V^3\mathbf{E}_3$ is also expressed in terms of a metric determinant or by appealing directly to a Jacobian determinant *J*. Recall from (3.2.10) that *J*-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E_1}\times\mathbf{E_2}\bullet\mathbf{E_3}\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix} \qquad (3.3.10)$$

From (3.3.1) it follows that a metric matrix is a matrix product of a Jacobian *J* with its transpose J^{T}.

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J \qquad (3.3.11)$$

Using the determinant product ($det|A|\ det|B| = det|A\bullet B|$) and symmetry ($det|A^T| = det|A|$) gives

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3 \det|J| = V^1V^2V^3\sqrt{\det|g|} \qquad (3.3.12)$$

Length, area, and volume for contravariant vector cells are inverses of the ones for covariant vector cells. This is true since the Jacobian is inverse Kajobian and the determinant of a matrix inverse the inverse of the determinant. That is: $det|K^{-1}|=(det|K|)^{-1} =1/\ det|K|=det|J|$.

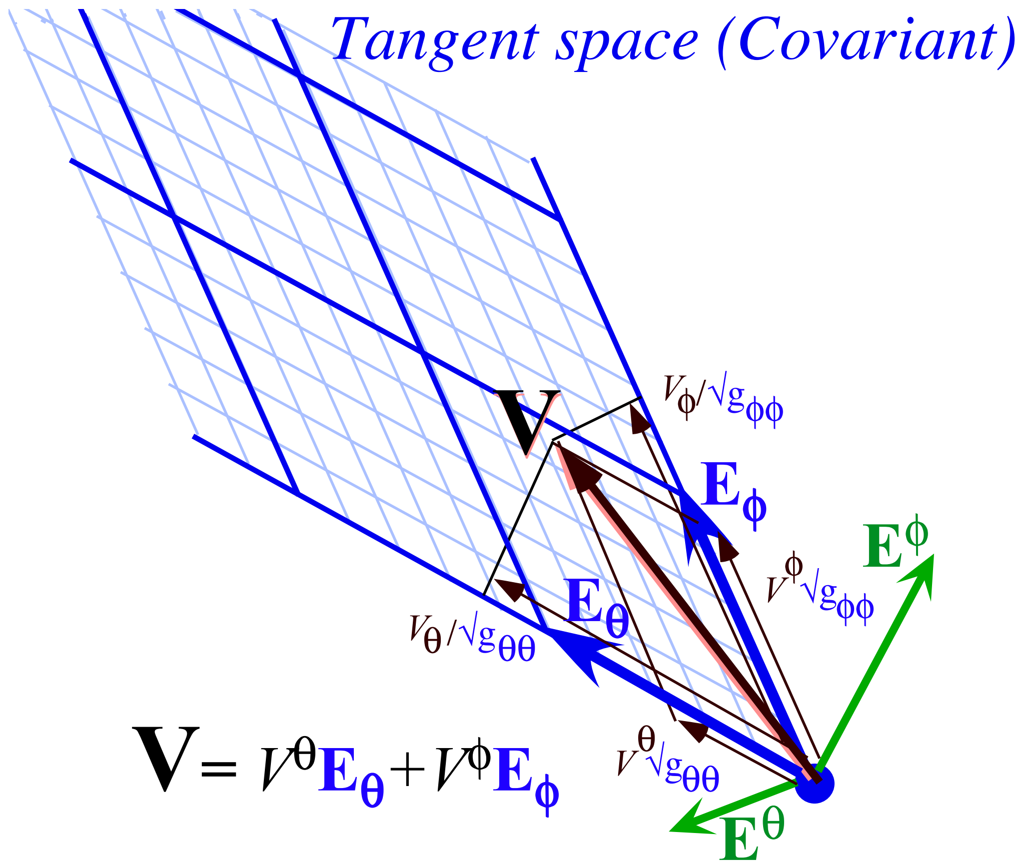


Fig. 3.3.1 Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

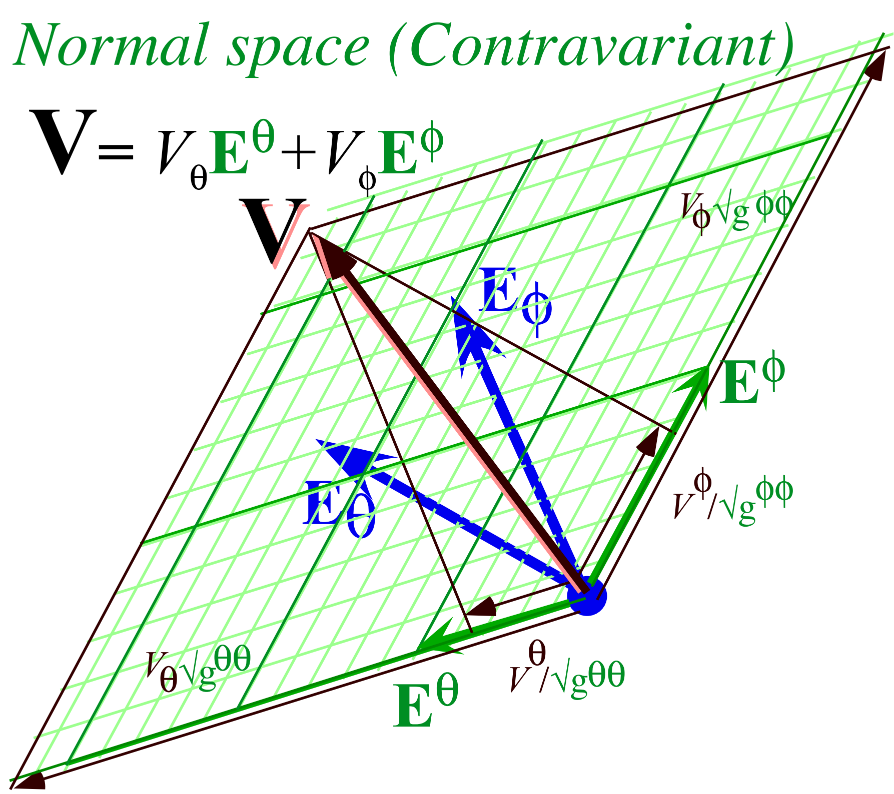


Fig. 3.3.2 Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Mass weighted metric tensors: GCC kinetic coefficients $\gamma_{\mu\upsilon}$

For mechanics, the metric sum (3.3.6) is easily made into the kinetic energy of any particle of mass m described by GCC coordinates q^υ and velocities $\dot{q}^\upsilon$ by scaling $g_{\mu\upsilon}$ by m, that is, let $\gamma_{\mu\upsilon} = m\cdot g_{\mu\upsilon}$.

$$T = \frac{mv^2}{2} = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 = \frac{m}{2} g_{\mu\upsilon}\frac{dq^\mu}{dt}\frac{dq^\upsilon}{dt} = \frac{m}{2} g_{\mu\upsilon}\dot{q}^\mu \dot{q}^\upsilon \qquad (3.3.13)$$

Metric coefficients for just the trebuchet projectile mass m are from (3.3.7).

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\theta\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} r^2 & -r\ell\cos(\theta-\phi) \\ -r\ell\cos(\theta-\phi) & \ell^2 \end{pmatrix}. \qquad (3.3.14)$$

Adding the kinetic energy $MR^2\dot{\theta}^2/2$ of the big mass M yields the total trebuchet kinetic energy.

$$\begin{aligned} T = \frac{mv^2}{2} + \frac{MR^2\dot{\theta}^2}{2} &= \frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi) + \frac{1}{2}m\ell^2\dot{\phi}^2 \\ &= \frac{1}{2}\gamma_{\theta\theta}\,\dot{\theta}^2 + \frac{1}{2}\gamma_{\theta\phi}\,\dot{\theta}\dot{\phi} + \frac{1}{2}\gamma_{\phi\theta}\,\dot{\phi}\dot{\theta} + \frac{1}{2}\gamma_{\phi\phi}\dot{\phi}^2 \end{aligned} \qquad (3.3.15)$$

A total kinetic energy expression is defined by *mass-weighted metric* or *kinetic coefficients* $\gamma_{\mu\nu}$.

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n \qquad (3.3.16)$$

The trebuchet kinetic coefficients agree with previously derived (2.3.11) and (2.6.10a).

$$\begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\theta\phi} & \gamma_{\phi\phi} \end{pmatrix} = \begin{pmatrix} mr^2 + MR^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}. \qquad (3.3.17)$$

These are the ones used in Unit 2 Ch. 3 for trebuchet equations and in (2.4.3) for GCC momentum.

Exercise 3.3.1 Understanding metric system

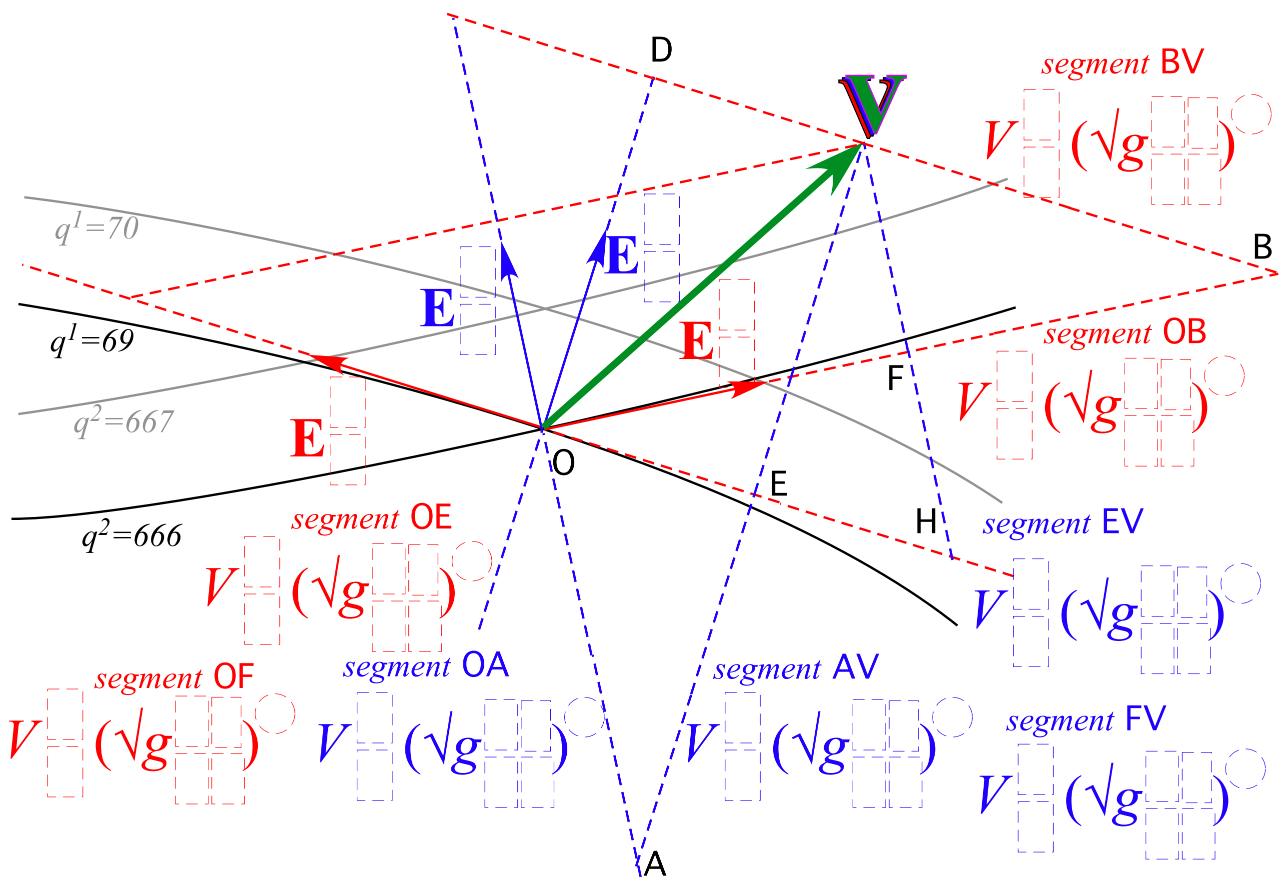
GCC components of a vector **V** in the figure below are realized by line segments OA, BV, etc. Give each segment length by single terms of the form V_m or V^m times $(\sqrt{g_{mm}})^{+1}$, $(\sqrt{g_{mm}})^{-1}$, $(\sqrt{g^{mm}})^{+1}$, or $(\sqrt{g^{mm}})^{-1}$ with the correct $m=1$ or 2. Also label each unitary vector as $\mathbf{E}_1$, $\mathbf{E}^1$, $\mathbf{E}_2$, or $\mathbf{E}^2$ whichever it is. You should be able to do this quickly without looking at the text figures.

Standard coordinate problems

Exercise 3.3.2 Compute Jacobian, Kajobian, $\mathbf{E}_m$, $\mathbf{E}^m$, metric tensors g_{mn} and g^{mn} for the following OCC.

Cylindrical coordinates $\{q^1=\rho, q^2=\phi\}$: $x=x^1=\rho\cos\phi$, $y=x^2=\rho\sin\phi$.

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$: $x=x^1=r\sin\theta\cos\phi$, $y=x^2=r\sin\theta\cos\phi$, $z=x^3=r\cos\theta$.



Exercise 3.3.3 "Plopped" Parabolic Coordinates

Consider the GCC(Cartesian) definition: $\bar{q}^{\,1} = (x)^2 + y \quad \bar{q}^{\,2} = (y)^2 - x$ (see following Figure)

(a) Does an analytic Cartesian coordinate definition $x^j = x^j(\bar{q}^{\,m})$ exist? If so find it.

(b) Derive the Jacobian, Kajobian, unitary vectors $\mathbf{E}_m$, $\mathbf{E}^m$, and metric tensors for this GCC.

(c) On the appropriate graph on the following page sketch the unitary vectors at the point $(x=1, y=1)$ (Arrow) and at the point $(x=1, y=0)$. Where, if anywhere, are they OCC?

(d) Find and indicate where, if anywhere, are the singularities of this GCC.

Exercise 3.3.4 "Sliding" Parabolic Coordinates

Consider the Cartesian(GCC) definition: $x = 0.4\,(q^1)^2 - q^2\,,\quad y = q^1 - 0.4\,(q^2)^2$ (see following Figure)

(a) Does an analytic GCC coordinate definition $q^m = q^m(x^j)$ exist? If so find it.

(b) Derive the Jacobian, Kajobian, unitary vectors $\mathbf{E}_m$, $\mathbf{E}^m$, and metric tensors for this GCC.

(c) On the appropriate graph on the following page sketch the unitary vectors at the point $(x=1, y=1)$ (Arrow) and at the point $(x=1, y=0)$. Where, if anywhere, are they OCC?

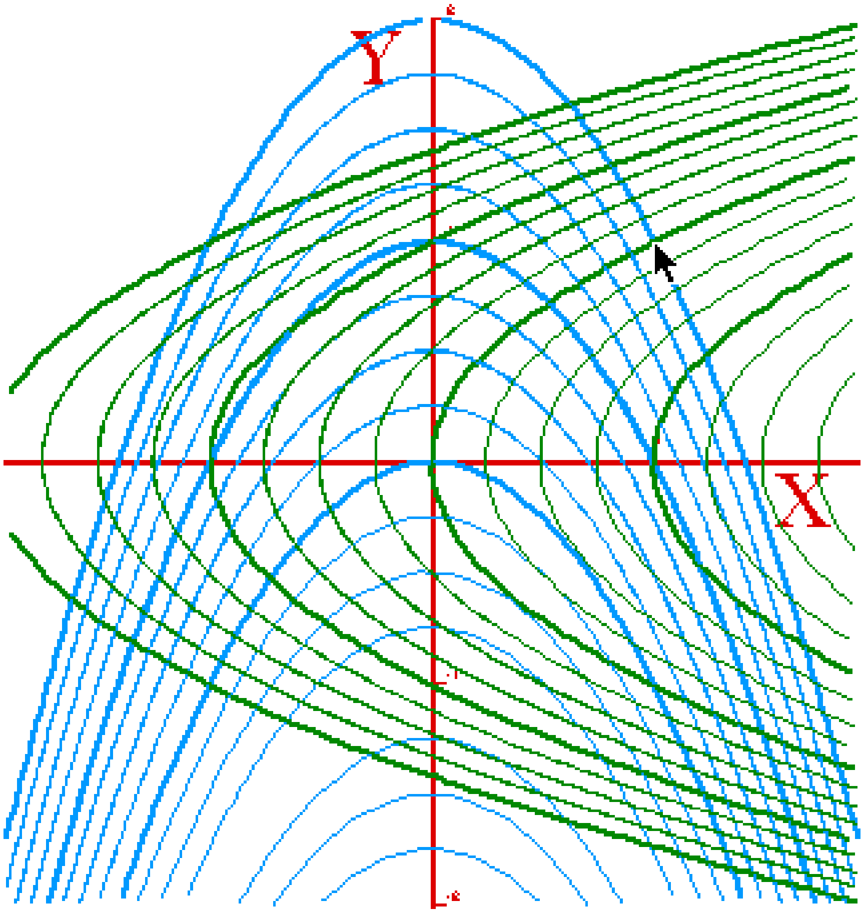
(d) Find and indicate where, if anywhere, are the singularities of this GCC.

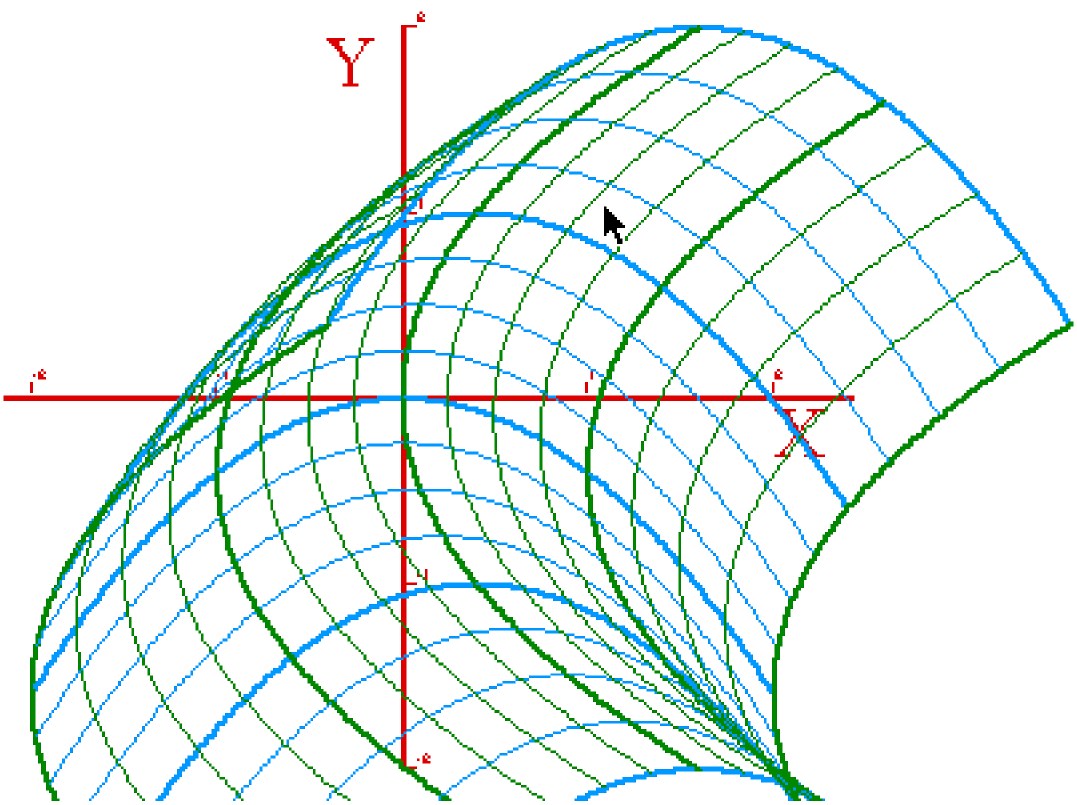
Exercise 3.3.5 "Professional" Parabolic (or Hyperbolic) Coordinates

Consider the GCC-Cartesian definition: $q^1 = (x^1)^2 - (x^2)^2\,,\ q^2 = 2(x^1)(x^2)$. Both $(x^1=x, x^2=y)$ and $(q^1=u, q^2=v)$ are Orthogonal Curvilinear Coordinates (OCC) related by an analytic function $w=z^2$ or $(u+iv)=(x+iy)^2$. For different purposes it may be convenient to treat either one as Cartesian. (Recall Fig. 10.7 in Unit 1.)

(a) Plot $(q^1=u, q^2=v)$ coordinate curves in a Cartesian $(x^1=x, x^2=y)$ graph. Derive the Jacobian, Kajobian, unitary vectors and metric tensors for this GCC.

(b) Plot $(x^1=x, x^2=y)$ coordinate curves in a Cartesian $(q^1=u, q^2=v)$ graph. Derive the Jacobian, Kajobian, unitary vectors and metric tensors for this GCC.


Y
X


Y
X

Chapter 4. Covariant derivative: Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

Dynamics involves GCC q^m derivatives of vectors **U** or tensors **T**. GCC are curves so derivatives have two contributions, one due to changing **U** components and another due to curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(U^j \mathbf{E_j}\right) = \frac{\partial U^m}{\partial q^i}\left(\mathbf{E_m}\right) + U^n \frac{\partial \mathbf{E_n}}{\partial q^i} \qquad (3.4.1)$$

The second term due to curving $\mathbf{E_n}$ needs special treatment. Partial derivative of $\mathbf{E_n}$ expands as follows.

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\mathbf{E}^{\ell} = \Gamma^m_{in}\mathbf{E}_m \qquad (3.4.2)$$

Christoffel coefficients $\Gamma_{ij;k}$ *of the first kind* are defined by: $\Gamma_{in;m} = \frac{\partial \mathbf{E_n}}{\partial q^i} \bullet \mathbf{E_m} = \Gamma_{ni;m}$

Christoffel coefficients $\Gamma_{ij}{}^{k}$ *of the second kind* are defined by: $\Gamma^m_{in} = \frac{\partial \mathbf{E_n}}{\partial q^i} \bullet \mathbf{E^m} = \Gamma^m_{ni}$

These Christoffel relations (3.4.1-2) give vector coordinate derivatives in terms of covariant $\mathbf{E}_{\mathrm{m}}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma^m_{in}\right)\mathbf{E_m}$$

Note the symmetry relation due to partial order invariance.

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E_i}}{\partial q^n}\ . \qquad (3.4.3)$$

This makes Christoffel symbols symmetric in two indices. ($\Gamma_{in;m} = \Gamma_{ni;m}$ and $\Gamma^m_{in} = \Gamma^m_{ni}$)

Writing derivatives using contra $\mathbf{E}^{\mathrm{m}}$ might appear to require a third kind of Γ-coefficient, say, a Λ.

$$\frac{\partial \mathbf{E^n}}{\partial q^i} = \Lambda^n_{im}\mathbf{E^m}\ \text{, where: } \Lambda^n_{im} = \frac{\partial \mathbf{E^n}}{\partial q^i} \bullet \mathbf{E_m} \qquad (3.4.4)$$

But, since orthonormality (3.2.9) is constant, the Λ-coefficients are just minus Γ-coefficients.

$$0 = \frac{\partial\left(\delta^n_m\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E^n} \bullet \mathbf{E_m}\right)}{\partial q^i} = \frac{\partial \mathbf{E^n}}{\partial q^i} \bullet \mathbf{E_m} + \mathbf{E^n} \bullet \frac{\partial \mathbf{E_m}}{\partial q^i}$$

$$0 = \quad \Lambda^n_{im} \quad + \quad \Gamma^n_{im}$$

So a vector derivative can be expressed using $\Gamma_{im}{}^{n}$ in terms of either $\mathbf{E}_{\mathrm{m}}$ or $\mathbf{E}^{\mathrm{m}}$ as follows

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma^m_{in}\right)\mathbf{E_m} = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma^n_{im}\right)\mathbf{E^m} \qquad (3.4.5a)$$

$$= \qquad U^m_{;i}\ \mathbf{E_m} \qquad = \qquad U_{m;i}\ \ \mathbf{E^m}$$

Here the *covariant derivative* $U^m{}_{;i}$ *of a contravariant component* U^m is defined by

$$U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma^m_{in}\ , \qquad (3.4.5b)$$

and the *covariant derivative* $U_{m;i}$ *of a covariant component* U_m is defined by

$$U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n \Gamma^n_{im} \,. \tag{3.4.5c}$$

Christoffel coefficients are combinations of 1st derivatives of metric coefficients according to (3.4.3).

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n} = \Gamma_{nm;i} + \Gamma_{in;m} \quad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad \text{(switched i} \leftrightarrow \text{m)}$$

Combining this gives:

$$\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} - \frac{\partial g_{mi}}{\partial q^n} + \frac{\partial g_{in}}{\partial q^m}\right) \tag{3.4.6}$$

What's a tensor? What's not?

Vector or tensor components must transform according to the rules derived in (3.2.16) and (3.2.19). Does $U^m{}_{;n}$ from (3.4.5b) qualify as a tensor? Let us test it using the following.

$$\frac{\partial \mathbf{U}}{\partial q^n} = U^m_{;n}\ \mathbf{E_m}\ \text{, or: } U^m_{;n} = \frac{\partial \mathbf{U}}{\partial q^n}\bullet\ \mathbf{E^m} \tag{3.4.7a}$$

Using chain-saw-sums and (3.2.14) one transforms the "bar" view of the above components.

$$\bar{U}^{\bar{m}}{}_{;\bar{n}} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar{q}^{\bar{n}}}\bullet\ \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar{q}^{\bar{n}}}\bullet\ \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n}\bullet\ \bar{\mathbf{E}}_{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n}\bullet\ \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{E}_m \tag{3.4.7b}$$

The result checks perfectly, that is, the transformation of $U^m{}_{;n}$ is like that of general $T^m{}_n$ in (3.2.19c).

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}U^m_{;n} \tag{3.4.8}$$

So the *covariant* derivative components satisfy the rules for a co-contra tensor. It had to be so since the original vector partial derivative is a valid vector under transformation.

$$\frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \mathbf{U}}{\partial q^n} \tag{3.4.9}$$

But, the simple derivative does *not* give a valid co-contra tensor. At first it looks like it might be OK.

$$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$$

But $\bar{U}^{\bar{m}}$ still needs to be expressed by (3.2.16) in terms of U^m and differentiated with respect to q^n.

$$\begin{aligned}\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} &= \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial}{\partial q^n}\left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}U^m\right)\\ &= \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}\frac{\partial U^m}{\partial q^n} + U^m\frac{\partial}{\partial q^n}\left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\right)\end{aligned} \tag{3.4.10}$$

The first term is just a $T^m{}_n$ transformation but “bad” second terms spoil it unless the Jacobian is *constant*.

That's an important difference between linear transformation theory and that of GCC. Curvilinear coordinates, orthogonal or not, have *variable* Jacobian transformation matrices.

$$\text{So: } U^m_{\;;n} \text{ are tensor components, but } \frac{\partial U^m}{\partial q^n} \text{ aren't.}$$

Also the Christoffel coefficients are not tensor components, either. In fact they have "bad" terms that just cancel the "bad" terms in (3.4.10) so the total covariant derivatives in (3.4.5) become the valid tensor generalization of the ordinary partial derivatives.

You should prove that the metric g_{mn}, $g^m{}_n$, and g^{mn} are, in fact, valid tensor components as are the delta $\delta^m{}_n$ unit tensors. In fact the δ and **g** tensors are really one and the same. (See exercises.)

Chapter 5. Lagrange equations of motion with explicit t-dependence

Classical mechanics is concerned with finding coordinates that simplify equations of motion. The next few chapters are devoted to deriving and applying several forms of GCC equations of motion.

The first GCC form is that of Lagrange's equations introduced in (2.3.8) and (2.6.5). We redo that derivation here for general GCC and with the extra complication of allowing *generalized coordinates with explicit time-dependence (GCWETD)*. This will include our own Earth coordinates as well as any accelerated reference frame whose position is preordained by N coordinate functions $q^m = q^m(x^1, x^2, \ldots, x^N, t)$ that may depend explicitly upon time t as well as N Cartesian coordinates x^j of particles relative to an inertial frame.

Also, we start with a modern form for Newton's Cartesian equations of motion that has a *tensorial inertial mass* M_{jk} in its Newton-2 ($f=Ma$)-equations. *(Recall that we sum over repeated indices!)*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k \qquad (3.5.1)$$

We will see this goes along with a revised kinetic energy. Instead of $T=(^1/_2)Mv^2$ we will have

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k \quad (\text{where:} M_{jk} = M_{kj}) \ . \qquad (3.5.2)$$

Why such strange force and energy forms? One example is anisotropic effective mass μ_{jk} of quasi-particles in solids. Another is inertia tensor I_{jk}. (If this sort of modern physics is uncomfortable just replace the M_{jk} matrix with diagonal masses $M_{jk} = \delta_{jk} M_k$, and you're back with Newton in the 17-th century!)

To generalize (3.5.1) to GCC or GCWETD we need those two lemmas about Jacobian derivatives. The first is the chain rule (3.3.2) converted to a velocity relation, as follows

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m + \left\{ \frac{\partial x^j}{\partial t} dt \right\} \quad \text{or:} \quad \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\}, \qquad (3.5.3)$$

Lemma 1 and Lemma 2 still works

Here terms due to explicit time dependence are surrounded by braces {}. For *generalized coordinates without time dependence (GCWOTD)* or fixed GCC, we simply drop braced terms. For either GCWETD (moving GCC) or GCWOTD (fixed GCC) the following lemma results. (Recall (2.2.14).)

Lemma 1.
$$\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}, \qquad (3.5.4)$$

In other words, coordinate Jacobians equal corresponding velocity Jacobians as they did in Ch. 2 of Unit 2.

The second lemma involves acceleration and 2nd-order total derivative. From (3.5.3)

$$\ddot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m + \frac{d}{dt}\left\{ \frac{\partial x^j}{\partial t} \right\}, \qquad (3.5.5a)$$

A second chain rule application gives

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right) = \frac{\partial^2 x^j}{\partial q^m \partial q^n} \dot{q}^m + \left\{ \frac{\partial^2 x^j}{\partial q^m \partial t} \right\} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right), \qquad (3.5.5b)$$

Then (3.5.3) gives the lemma. (Recall (2.2.16). Again the Unit 2 result holds here, too.))

Lemma 2. $$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)=\frac{\partial \dot{x}^j}{\partial q^m}, \tag{3.5.6}$$

…and Work still works

Now consider the incremental work $dW = F_j dx^j$ done by applied forces for arbitrary differential changes dq^m of coordinates and intervals dt of time.

$$dW = f_j dx^j = f_j\left(\frac{\partial x^j}{\partial q^m}dq^m+\left\{\frac{\partial x^j}{\partial t}dt\right\}\right)=M_{jk}\,\ddot{x}^k\left(\frac{\partial x^j}{\partial q^m}dq^m+\left\{\frac{\partial x^j}{\partial t}dt\right\}\right) \tag{3.5.7}$$

This is true for any choice of dq^m including all zero except for one. In other words the sum over m must be true term-by-term. Here, for example, is the m^{th} term.

$$F_m = f_j\frac{\partial x^j}{\partial q^m}=M_{jk}\,\ddot{x}^k\,\frac{\partial x^j}{\partial q^m} \tag{3.5.8}$$

The left hand side is a covariant vector transformation (See first of (3.2.16)) from Cartesian f_j to GCC *covariant force components* F_m.

$$F_m = f_j\frac{\partial x^j}{\partial q^m}$$

The right hand side can be rearranged using derivative identity $\ddot{A}B=\frac{d}{dt}\left(\dot{A}B\right)-\dot{A}\dot{B}$.

$$F_m = M_{jk}\frac{d}{dt}\left(\dot{x}^k\frac{\partial x^j}{\partial q^m}\right)-M_{jk}\dot{x}^k\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right), \tag{3.5.9a}$$

Then we use *Lemma 1* (3.5.4) and *Lemma 2* (3.5.6)

$$F_m = M_{jk}\frac{d}{dt}\left(\dot{x}^k\frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right)-M_{jk}\dot{x}^k\left(\frac{\partial \dot{x}^j}{\partial q^m}\right), \tag{3.5.9b}$$

Finally we use identities: $$M_{jk}\left(v^j\frac{\partial v^k}{\partial q}\right)=\frac{\partial}{\partial q}\left(\frac{1}{2}M_{jk}v^jv^k\right),\ (M_{jk}=M_{kj}=const.) \tag{3.5.9c}$$

The result is *Lagrange's covariant force equation*

$$F_m=\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m}-\frac{\partial T}{\partial q^m}, \tag{3.5.10a}$$

where

$$T=\frac{1}{2}M_{jk}\,v^jv^k=\frac{1}{2}M_{jk}\,\dot{x}^j\dot{x}^k \tag{3.5.10b}$$

is the *generalized kinetic energy* which in GCWETD (moving GCC) is

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\left\{\frac{\partial x^j}{\partial t}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\left\{\frac{\partial x^k}{\partial t}\right\}\right) \tag{3.5.10c}$$

or in GCWOTD (fixed GCC) has a basic metric form (Recall (3.3.16).) with {} terms all zero.

$$T = \frac{1}{2} M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \dot{q}^m \dot{q}^n = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n \tag{3.5.10d}$$

(3.5.10) is a "four-wheel-drive" equation of mechanics; it's supposed to go anywhere and not get stuck! All subsequent equations are derived from this one, and most of them have special requirements that may make them less general. When in doubt we start from (3.5.10a).

Chapter 6. Riemann equations of motion (No explicit t-dependence)

In GCWOTD (fixed GCC) kinetic energy T has basic metric form (3.5.10d) with explicit time dependencies (braced {}-terms) all zero. The kinetic metric γ_{mn} is a covariant tensor transform (3.2.19c) of the original Cartesian inertia tensor M_{ij} in (3.5.1).

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n} \qquad (3.6.1)$$

Now we convert the Lagrange equations for fixed GCC to a tensor form.

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}, \qquad (3.6.2)$$

First, note how a derivative of a metric sum is reduced. (Recall also (3.5.9c).)

$$\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{mn}\dot{q}^n\frac{\partial \dot{q}^m}{\partial \dot{q}^\ell} + \gamma_{mn}\dot{q}^m\frac{\partial \dot{q}^n}{\partial \dot{q}^\ell} = \gamma_{mn}\dot{q}^n\delta^m_\ell + \gamma_{mn}\dot{q}^m\delta^n_\ell = \left(\gamma_{\ell n} + \gamma_{n\ell}\right)\dot{q}^n$$

$$= 2\gamma_{\ell n}\dot{q}^n$$

This first term involves the *canonical momentum p_ℓ*.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n \qquad (3.6.3a)$$

p_ℓ is also called a *covariant momentum*. The inverse or *contravariant kinetic metric γ^{mn}* is defined to reverse the momentum definition. (In other words, contravariant momentum is just velocity.)

$$\dot{q}^n = p_\ell \gamma^{\ell n} = p^n \qquad (3.6.3b)$$

Each GCC velocity is just a *contravariant momentum p^m*. The canonical form of Lagrange's equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}, \qquad (3.6.4)$$

is valid for all GCC, fixed or otherwise, but the following is for GCWOTD only.

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$
$$= \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n,$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

This is rearranged into an equation involving Christoffel coefficients of the first kind like (3.4.6).

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$
$$= \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n \ , \qquad (3.6.2)$$
$$= \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

The result will be called *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n \ , \qquad (3.6.10a)$$

where the *kinetic Christoffel coefficients of the first kind* are defined analogously to (3.4.6).

$$\Gamma_{mn;\ell} \equiv \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right] \qquad (3.6.10b)$$

For numerical solutions we use the *contravariant Riemann equations*.

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n \qquad (3.6.10c)$$

where the *kinetic Christoffel coefficients of the second kind* are defined by

$$\Gamma^k_{mn} = \gamma^{k\ell}\Gamma_{mn;\ell} \ , \qquad (3.6.10d)$$

and the *contravariant generalized force* F^k is (from (3.5.8)) given in terms of Cartesian f_j.

$$F^k = \gamma^{k\ell}F_\ell = \gamma^{k\ell}\frac{\partial x^j}{\partial q^\ell} f_j \ , \qquad (3.6.10e)$$

This completes the derivation of the world's most complicated form of Newton's $f=ma$ equations!

Sometimes we call this the "Teutonic" approach to mechanics since it involves the mathematics of the great geometers Gauss and Riemann. It has advantages (easy numerical programming and explicit geometrical transformation properties) and disadvantages (not convenient in moving coordinates and complicated notations) when compared to the "French" or "Celtic" approach of Lagrange and Hamilton. Still, it has been the choice for general relativity. There the "moving coordinate disadvantage" vanishes immediately as time becomes the 4^{th} (or 0^{th}) coordinate dimension $t=x^4$ (or: $t=x^0$) and total t-derivatives are replaced by derivatives with respect to a proper invariant time τ or length λ.

Intrinsic derivatives

In an attempt to simplify the Riemann equations one defines a covariant derivative based construction called an *intrinsic derivative of contravariant vector components*.

$$\frac{\delta V^k}{\delta t} = \frac{dV^k}{dt} + \Gamma^k_{mn}V^m\dot{q}^n = \frac{\partial V^k}{\partial q^n}\dot{q}^n + \Gamma^k_{mn}V^m\dot{q}^n = V^k_{\ ;n}\dot{q}^n \qquad (3.6.11a)$$

Ther is also an equivalent *intrinsic derivative of covariant vector components*.

$$\frac{\delta V_k}{\delta t} = \frac{dV_k}{dt} - \Gamma^m_{kn}V_m\dot{q}^n = \frac{\partial V_k}{\partial q^n}\dot{q}^n - \Gamma^m_{kn}V_m\dot{q}^n = V_{k\,;\,n}\dot{q}^n \qquad (3.6.11b)$$

Then Newton's equations look very familiar! Force is the intrinsic time derivative of momentum.

$$F_k = \frac{\delta p_k}{\delta t} \qquad (3.6.12a) \qquad\qquad F^k = \frac{\delta p^k}{\delta t} \qquad (3.6.12b)$$

The intrinsic derivative takes account of all the "fictitious forces" due to the curving coordinates by using the covariant derivatives (3.4.5) instead of simple partial derivatives in *tensor chain rules*.

$$\frac{\delta V^k}{\delta t} = V^k_{;n}\dot{q}^n \text{ , replaces: } \frac{dV^k}{dt} = \frac{\partial V^k}{\partial q^n}\dot{q}^n \text{ where: } V^k_{;n} = \frac{\partial V^k}{\partial q^n} + \Gamma^k_{mn}V^m \qquad (3.6.13a)$$

$$\frac{\delta V_k}{\delta t} = V_{k\,;\,n}\dot{q}^n \text{ , replaces: } \frac{dV_k}{dt} = \frac{\partial V_k}{\partial q^n}\dot{q}^n \text{ where: } V_{k\,;\,n} = \frac{\partial V_k}{\partial q^n} - \Gamma^m_{kn} V_m \qquad (3.6.13b)$$

Exercise 3.6.1 Simplifying Riemann calculations

The Riemann formulas for Christoffel-Coriolis coefficients Γ^{ℓ}_{mn} or $\Gamma_{mn;k}$ given in (3.4.6) use definition (3.4.2). If equations (3.6.10) of Riemann match equations (3.5.10) of Lagrange, then it is easier to derive Christoffel coefficients using the latter.

(a) Derive all Γ^{ℓ}_{mn} or $\Gamma_{mn;k}$ for mass m in 3D cylindrical coordinates $\{q^1=\rho,\ q^2=\phi\}:\ x=x^1=\rho\cos\phi,\ y=x^2=\rho\sin\phi\}$ while finding co-and-contravariant velocity, momentum, and Riemann equations.

(b) Derive all Γ^{ℓ}_{mn} or $\Gamma_{mn;k}$ for mass m in 3D spherical coordinates $\{q^1=r,\ q^2=\theta,\ q^3=\phi\ \}:\ x=x^1=r\sin\theta\cos\phi,\ y=x^2=r\sin\theta\cos\phi,$ $z=x^3=r\cos\theta$ while finding co-and-contravariant velocity, momentum, and Riemann equations.

(Try this first without looking ahead for the answers.)

Chapter 7. Christoffel expansion of fictitious forces: Rotating frames

Rotational centrifugal and Coriolis effects are prime examples of so-called fictitious forces. They can be derived quite simply in cylindrical coordinates $(q^1 = \rho,\ q^2 = \phi,\ q^3 = z)$ system defined as follows.

$$x = \rho \cos \phi\,, \qquad y = \rho \sin \phi\,, \qquad z = z\,, \tag{3.7.1}$$

It should also help your physical understanding of metric and Christoffel coefficients and show how to calculate rotational forces easily. Then we will derive them another way using Gibb's vector analysis.

The first step in GCC analysis is to get the Jacobian and Kajobian matrices from (3.2.10).

$$\langle J \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \\ = \langle J^{-1} \rangle \end{array} \tag{3.11.2}$$

Whichever is easier is done first. Then matrix inversion gives the other, and then both the covariant and contravariant unitary vectors follow. They are sketched in Fig. 3.6.1.

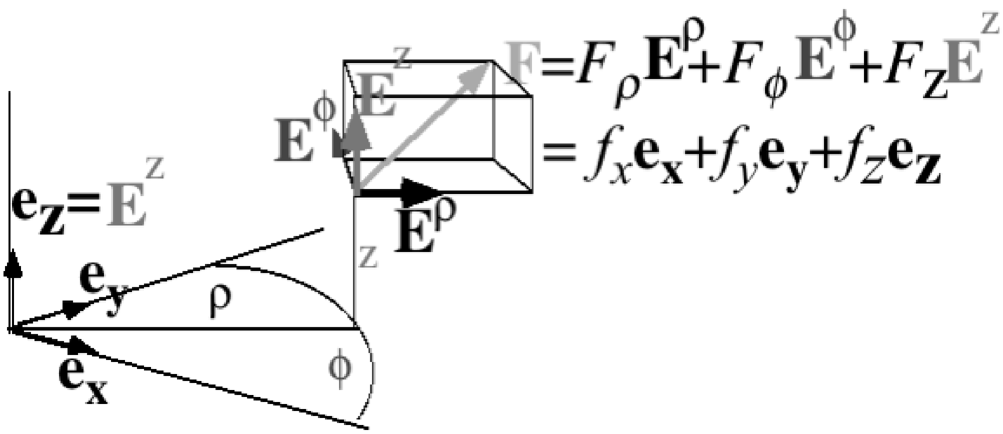


Fig. 3.6.1 Covariant force vector components in a cylindrical normal space ($\mathbf{E}^\rho, \mathbf{E}^\phi, \mathbf{E}^z$).

Cylindrical coordinates are an example of an *orthogonal curvilinear coordinate (OCC)* system, so covariant unitary vectors point in the same direction as contravariant unitary vectors. One could normalize these vectors, but this is mostly a waste of time. Instead we will see that *co*variant quantities are natural for Hamiltonian *physics* while *contra*variant quantities are natural for Lagrangian *computation*.

First, the covariant force components shown in Fig. 3.6.1 drive Lagrange's equations (3.5.10a) and (3.6.4). They are a Jacobian transformation (3.5.8) of the Cartesian components.

$$
\begin{aligned}
F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0 \\
F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0 \\
F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z
\end{aligned}
\tag{3.7.3}
$$

Note that F_ρ is a radial *force* with units of *Newton*, but F_ϕ is a *torque* with units of *N·m* or *Joule*.

Since this is an OCC system the kinetic metric coefficients (3.6.1) are diagonal for both the covariant and contravariant components. From Jacobian (3.7.2) we derive the covariant γ_{mn}.

$$
\begin{aligned}
\gamma_{\rho\rho} &= m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m \\
\gamma_{\phi\phi} &= m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2 \\
\gamma_{zz} &= m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m
\end{aligned}
\tag{3.7.4a}
$$

The contra's are just the inverses since this is a diagonal-metric OCC system.

$$
\begin{aligned}
\gamma^{\rho\rho} &= 1/m \\
\gamma^{\phi\phi} &= 1/\left(m\rho^2\right) \\
\gamma^{zz} &= 1/m
\end{aligned}
\tag{3.7.4b}
$$

The covariant coefficients γ_{mn} are the ones in the kinetic energy expression (3.5.10d).

$$
T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 \tag{3.7.5a}
$$

They also give the canonical covariant momenta (3.6.3a).

$$
\begin{aligned}
p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad & p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad & p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} \\
&= m\dot{\rho} & &= m\rho^2\dot{\phi} & &= m\dot{z}
\end{aligned}
\tag{3.7.5b}
$$

Note that p_ρ is a *radial* momentum, but p_ϕ is an *angular* momentum with units of *Joule.sec*. The covariant metric coefficient $\gamma_{\phi\phi} = m\rho^2$ is a *moment* of inertia or *angular* inertia. The covariant quantities have physical significance but the contras are computationally simpler. Contra-momentum is just GCC *velocity*.

$$
p^\rho = \dot{\rho} \qquad p^\phi = \dot{\phi} \qquad p^z = \dot{z} \tag{3.7.5c}
$$

This comparison extends to the covariant Lagrange equations of motion (3.6.4). By combining the Lagrange and covariant Riemann equations (3.6.10c) one can easily derive and understand the physical significance of the Christoffel coefficients. Here, again are the two competing equations, the Lagrange and the Riemann *co*variant *force* equations of motion.

$$
F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n \tag{3.7.6a}
$$

We use (3.7.5) for cylindrical-polar coordinates.

$$
\begin{aligned}
F_\rho &= \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\,\dot{q}^m\dot{q}^n \\
&= \frac{d\left(m\dot{\rho}\right)}{dt} - \frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho
\end{aligned}
\tag{3.7.6b}
$$

$$
\begin{aligned}
F_\phi &= \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\,\dot{q}^m\dot{q}^n \\
&= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt} - 0 = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\rho} = m\rho = \Gamma_{\phi\rho;\rho}
\end{aligned}
\tag{3.7.6c}
$$

Only three non-zero Christoffel coefficients appear, and only two are independent. The term $-m\rho\dot{\phi}^2$ corresponds to *centrifugal* force while the term(s) $2m\rho\dot{\rho}\dot{\phi}$ correspond to (both) *Coriolis* force terms.

The *contra*variant equations are *acceleration* equations. The concept of mass and force drop out in this case. The general *contra*variant Riemann equations are repeated below from (3.6.10c).

$$F^k = \gamma^{jk}F_j = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n \tag{3.7.7a}$$

In cylindrical coordinates they are as follows.

$$
\begin{aligned}
F^\rho &= \gamma^{\rho\rho}F_\rho = \ddot{q}^\rho + \Gamma^\rho_{mn}\dot{q}^m\dot{q}^n \\
&= \ddot{\rho} - \rho\dot{\phi}^2 \quad \text{so: } \Gamma^\rho_{\phi\phi} = -\rho
\end{aligned}
\tag{3.7.7b}
$$

$$
\begin{aligned}
F^\phi &= \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho + \Gamma^\rho_{mn}\dot{q}^m\dot{q}^n \\
&= \ddot{\phi} + 2\dot{\rho}\dot{\phi}/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi} = 1/\rho = \Gamma^\phi_{\phi\rho}
\end{aligned}
\tag{3.7.7c}
$$

Coordinate acceleration is then given in terms of applied-F and fictitious acceleration components.

$$\ddot{\rho} = F^\rho + \rho\dot{\phi}^2 \quad (\textit{Centrifugal}) \qquad (3.7.8a) \qquad\qquad \ddot{\phi} = F^\phi - 2\dot{\rho}\dot{\phi}/\rho \quad (\textit{Coriolis}) \qquad (3.7.8a)$$

Coriolis and centrifugal forces

To visualize the two "fictitious" forces or accelerations in (3.7.6) and (3.7.7), consider a sketch in Fig. 3.6.1 of a low-pressure area with inward winds ($\dot{\rho} < 0$) on a counter clockwise ($\dot{\phi} > 0$) rotating portion of the Earth (Northern Hemisphere). With no outside force ($F_\rho = 0 = F_\phi$) (3.7.7c) gives Coriolis acceleration counter clockwise ($\ddot{\phi} = -2\dot{\rho}\dot{\phi}/\rho > 0$) This is rule for large weather systems including hurricanes. Since weather drifts to the East, a warm South wind generally precedes a low-pressure area in the Northern Hemisphere. Positive is the preferred rotation for smaller ones, too, like tornados and waterspouts. The 1/ρ factor assures intensity increases near the center of rotation.

Centrifugal and Coriolis forces are two sides of the same coin, indeed, the latter ($2m\rho\dot{\rho}\dot{\phi}$) is minus the derivative of the former ($-m\rho\dot{\phi}^2$) with respect to $\dot{\phi}$. For example, reducing $\dot{\phi}$ also reduces centrifugal acceleration for a East to West wind since it is going against Earth rotation. So it feels less North to South acceleration than Earth-fixed objects, that is, a Coriolis acceleration toward North. Vice-versa for the West wind which veers South. On the other hand a North to South wind travels toward a part of the Earth that is moving Eastward more rapidly since it is farther from the North polar rotation axis. This explains why it appears to veer Westerly as it tends to fall behind the increasingly rapid Eastward land rotation.

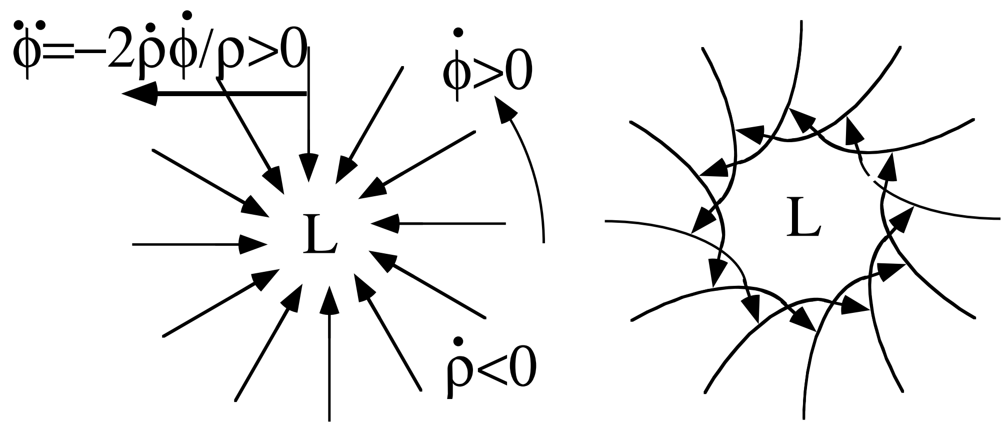


Fig. 3.6.2 Coriolis acceleration causing cyclonic winds on inflow trying to fill a low L.

A covariant constraint 'aint : $F_m^{constraint}=0$

If a mass is confined to a frictionless surface that happens to be a coordinate surface, say, $q^3=const.$, then the constraint force vector **F=N** is normal to the surface. That means that **F** lies along **grad** q^3 that by (3.2.8) is the *contra*variant unitary vector $\mathbf{E}^3$ that is orthogonal to covariant $\mathbf{E}_1$ and $\mathbf{E}_2$.

$$\mathbf{F}^{constraint}=\mathbf{N}=N_3\,\mathbf{E}^3 \qquad \mathbf{E}_1\bullet\mathbf{E}^3=0=\mathbf{E}_2\bullet\mathbf{E}^3 \tag{3.7.8}$$

That means all the covariant components of the constraint force are zero except the one N_3 that prevents motion normal to the surface and keeps it in the tangent space. So $\mathbf{F}^{constraint}$ has no tangent components.

$$F_1^{constraint}=N_1=0, \qquad F_2^{constraint}=N_2=0. \tag{3.7.9}$$

Therefore, the covariant force components F_1 or F_2 in the Lagrange or Riemann equations for motion on the surface do not contain any contribution by constraint forces and $\mathbf{F}^{constraint}$ can be ignored. The same applies to a mass confined to a frictionless line that is the intersection of two coordinate surface, say, $q^2=const.$and $q^3=const.$ so the constraint force vector $\mathbf{F}^{constraint}$ is a combination of normals to the surfaces.

$$\mathbf{F}^{constraint}=F_2^{constraint}\,\mathbf{E}^2+F_3^{constraint}\,\mathbf{E}^3\text{, but: }F_1^{constraint}=0. \tag{3.7.10}$$

So covariant force component F_1 for the remaining free coordinate equations contains no constraint effects.

Galilean relativity of centrifugal and Coriolis forces

The 3D whirl vector ω *and rotation matrix $R[\omega\cdot t]$*

The velocity field $\mathbf{v}(\mathbf{r})$ of radial $\mathbf{r}$-vectors rotating at *angular velocity* ω around an axis $\mathbf{e}_\omega$ uses Darboux's *whirl* or *omega vector* $\omega=\omega\,\mathbf{e}_\omega$. Each $\mathbf{v}(\mathbf{r})$ is a cross product of whirl ω with $\mathbf{r}$ and perpendicular to both.

$$\mathbf{v}(\mathbf{r})=\omega\times\mathbf{r} \quad (3.7.11a) \qquad |\mathbf{v}(\mathbf{r})|=\omega\cdot r\sin\angle_\omega^r \quad (3.7.11b)$$

Velocity $\dot{\mathbf{r}}=\mathbf{v}$ of $\mathbf{r}$ is ω times its perpendicular distance $r\sin\angle_\omega^r$ to whirl axisω. Rotation matrix$\mathbf{R}[\Theta]$ maps each lab-fixed unit vector $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ to $\{\bar{\mathbf{e}}_1(t),\bar{\mathbf{e}}_2(t),\bar{\mathbf{e}}_3(t)\}$. Derivative of *axis vector* $\boldsymbol{\Theta}=\boldsymbol{\omega}\cdot t$ is whirl $\boldsymbol{\omega}=\dot{\boldsymbol{\Theta}}$.

$$\bar{\mathbf{e}}_k(t)=\mathbf{R}[\boldsymbol{\Theta}]\bullet\mathbf{e}_k\,(=\mathbf{R}[\boldsymbol{\omega}\cdot t]\bullet\mathbf{e}_k\text{ if direction }\mathbf{e}_\Theta=\mathbf{e}_\omega\text{ of rotation axis }\boldsymbol{\Theta}\text{ is fixed.)} \tag{3.7.12}$$

A t-derivative $\dot{\mathbf{R}}[\boldsymbol{\omega}\cdot t]_{t=0}$ at $t=0$ of $\mathbf{R}[\boldsymbol{\omega}\cdot t]$ must give initial velocity $\dot{\bar{\mathbf{e}}}_k(0)=\boldsymbol{\omega}\times\mathbf{e}_k$ to each unit vector $\bar{\mathbf{e}}_k(t)$.

$$\dot{\mathbf{R}}[\boldsymbol{\omega}\cdot t]_{t=0}\bullet\mathbf{e}_1 = \tfrac{d}{dt}\,\bar{\mathbf{e}}_1 = \dot{\bar{\mathbf{e}}}_1(0) = \boldsymbol{\omega}\times\mathbf{e}_1 = (\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3)\times\mathbf{e}_1 = \quad 0 \quad +\omega_3\mathbf{e}_2 \; -\omega_2\mathbf{e}_3$$

$$\dot{\mathbf{R}}[\boldsymbol{\omega}\cdot t]_{t=0}\bullet\mathbf{e}_2 = \tfrac{d}{dt}\,\bar{\mathbf{e}}_2 = \dot{\bar{\mathbf{e}}}_2(0) = \boldsymbol{\omega}\times\mathbf{e}_2 = (\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3)\times\mathbf{e}_2 = -\omega_3\mathbf{e}_1 + \quad 0 \quad +\omega_1\mathbf{e}_3$$

$$\dot{\mathbf{R}}[\boldsymbol{\omega}\cdot t]_{t=0}\bullet\mathbf{e}_3 = \tfrac{d}{dt}\,\bar{\mathbf{e}}_3 = \dot{\bar{\mathbf{e}}}_3(0) = \boldsymbol{\omega}\times\mathbf{e}_3 = (\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3)\times\mathbf{e}_3 = +\omega_2\mathbf{e}_1 - \;\omega_1\mathbf{e}_2 + \quad 0$$

Initial t-derivative matrix $\dot{R}_{ik}[\boldsymbol{\omega}\cdot t]_{t=0} = \mathbf{e}_i\bullet\dot{\mathbf{R}}[\boldsymbol{\omega}\cdot t]\bullet\mathbf{e}_k = \mathbf{e}_i\bullet\dot{\bar{\mathbf{e}}}_k$ is a ε-matrix sum of Levi-Civita ε_{ijk} of (1.A.3a).

$$\begin{pmatrix} \dot{R}_{1,1} & \dot{R}_{1,2} & \dot{R}_{1,3} \\ \dot{R}_{2,1} & \dot{R}_{2,2} & \dot{R}_{2,3} \\ \dot{R}_{3,1} & \dot{R}_{3,2} & \dot{R}_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \omega_1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \omega_2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \omega_3\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dot{R}_{ik}[\boldsymbol{\omega}\cdot t]_{t=0} \quad = \quad \omega_j\varepsilon_{ijk} \quad = \omega_1 \quad \varepsilon_{i1k} \quad + \omega_2 \quad \varepsilon_{i2k} \quad + \omega_3 \quad \varepsilon_{i3k} \; = \boldsymbol{\omega}\bullet\ddot{\boldsymbol{\varepsilon}}_{ik} \tag{3.7.13}$$

Fig. 3.6.3 shows $\dot{\bar{\mathbf{e}}}_1(0)$, $\dot{\bar{\mathbf{e}}}_2(0)$, and $\dot{\bar{\mathbf{e}}}_3(0)$ due to whirls $\boldsymbol{\omega}_1 = \omega_1\mathbf{e}_1$, $\boldsymbol{\omega}_2 = \omega_2\mathbf{e}_2$, and $\boldsymbol{\omega}_3 = \omega_3\mathbf{e}_3$, respectively. If ω_1, ω_2, and ω_3, occur together the angular velocity is a single total whirl Ω_{TOTAL}. (The Darboux sum rule.)

$$\mathbf{V}_{TOTAL}(\mathbf{r}) = \mathbf{v}_1(\mathbf{r}) + \mathbf{v}_2(\mathbf{r}) + \mathbf{v}_3(\mathbf{r}) = \omega_1 \times \mathbf{r} + \omega_2 \times \mathbf{r} + \omega_3 \times \mathbf{r} = (\omega_1+\omega_2+\omega_3)\times\mathbf{r} = \Omega_{TOTAL}\times\mathbf{r} \tag{3.7.14}$$

Galilean addition of angular velocities ω_k is just as valid as the addition of their linear velocities $\mathbf{v}_k$. More generally, a rotation matrix $\mathbf{R}[\boldsymbol{\omega}\cdot t]$ satisfies exponential differential ε-matrix equations at all times.

$$\tfrac{d}{dt}R_{ik}[\boldsymbol{\omega}\cdot t] = \dot{R}_{ik}[\boldsymbol{\omega}\cdot t] = \boldsymbol{\omega}\bullet\ddot{\boldsymbol{\varepsilon}}_{ij}R_{jk}[\boldsymbol{\omega}\cdot t] \tag{3.7.15a}$$

$$\tfrac{d^n}{dt^n}\mathbf{R}[\boldsymbol{\omega}\cdot t] = (\boldsymbol{\omega}\bullet\ddot{\boldsymbol{\varepsilon}})^n\mathbf{R}[\boldsymbol{\omega}\cdot t] \tag{3.7.15b}$$

The latter n matrix products of $(\boldsymbol{\omega}\bullet\ddot{\boldsymbol{\varepsilon}})$ and $\mathbf{R}$ leads to a matrix exponential expression for $\mathbf{R}[\boldsymbol{\omega}\cdot t] = e^{(\boldsymbol{\omega}\bullet\ddot{\boldsymbol{\varepsilon}})t}$.

Rotating vs. lab coordinates

A radial position vector $\mathbf{r}$ may be expressed in lab $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ or rotating $\{\bar{\mathbf{e}}_1(t),\bar{\mathbf{e}}_2(t),\bar{\mathbf{e}}_3(t)\}$ bases. If $\mathbf{r} = \mathbf{r}(t)$ varies in time then so must its lab coordinates $x_i(t)$, but its "fixed-to-the-stars" lab unit vectors $\mathbf{e}_j$ *do not*.

$$\mathbf{r}(t) = x_i(t)\mathbf{e}_i = \bar{x}_k(t)\bar{\mathbf{e}}_k(t) = \bar{x}_k(t)\mathbf{R}[\boldsymbol{\omega}\cdot t]\bullet\mathbf{e}_k \tag{3.7.16}$$

(Sum-rules apply, but in OCC $x_j = x^j$.) Matrix $R_{ik}[\boldsymbol{\omega}\cdot t]$ finds x_i given $\bar{x}_k$, *vice-versa* for $R^{-1}_{ik}[\boldsymbol{\omega}\cdot t] = R_{ik}[-\boldsymbol{\omega}\cdot t]$.

$$\begin{aligned} x_i(t) &= \mathbf{e}_i\bullet\mathbf{r}(t) = \bar{x}_k(t)\mathbf{e}_i\bullet\mathbf{R}[\boldsymbol{\omega}\cdot t]\bullet\mathbf{e}_k \\ &= R_{ik}[\boldsymbol{\omega}\cdot t]\bar{x}_k(t) \end{aligned} \tag{3.7.17a}$$

$$\begin{aligned} \bar{x}_k(t) &= R^{-1}_{ki}[\boldsymbol{\omega}\cdot t]x_i(t) \\ &= R_{ki}[-\boldsymbol{\omega}\cdot t]x_i(t) \end{aligned} \tag{3.7.17b}$$

The 1st time derivative $\dot{R}_{ik}[\boldsymbol{\omega}\cdot t]$ of rotation matrix $R_{ik}[\boldsymbol{\omega}\cdot t]$ adds a term to each velocity transformation.

$$\dot{x}_i = R_{ik}[\boldsymbol{\omega}\cdot t]\dot{\bar{x}}_k + \dot{R}_{ik}[\boldsymbol{\omega}\cdot t]\bar{x}_k \tag{3.7.18a}$$

$$\dot{\bar{x}}_k = R_{ki}[-\boldsymbol{\omega}\cdot t]\dot{x}_i + \dot{R}_{ki}[-\boldsymbol{\omega}\cdot t]x_i \tag{3.7.18b}$$

At $t=0$ rotation $\mathbf{R}$ is unit matrix $R_{ik}[0] = \delta_{ik}$. Its 1st derivative is $\dot{R}_{ik}[0] = \omega_j\varepsilon_{ijk} = (\omega\bullet\varepsilon)_{ik}$ by (3.7.13).

$$\begin{aligned} \dot{x}_i(0) &= \dot{\bar{x}}_i(0) + \varepsilon_{ijk}\omega_j\bar{x}_k \\ \dot{\mathbf{x}}(0) &= \dot{\bar{\mathbf{x}}}(0) + \;\boldsymbol{\omega}\times\bar{\mathbf{x}} = \dot{\bar{\mathbf{x}}} + (\omega\bullet\varepsilon)\bar{\mathbf{x}} \end{aligned} \tag{3.7.19a}$$

$$\begin{aligned} \dot{\bar{x}}_k(0) &= \dot{x}_k(0) - \varepsilon_{kji}\omega_j x_i \\ \dot{\bar{\mathbf{x}}}(0) &= \dot{\mathbf{x}}(0) - \;\boldsymbol{\omega}\times\mathbf{x} = \dot{\mathbf{x}} - (\omega\bullet\varepsilon)\mathbf{x} \end{aligned} \tag{3.7.19b}$$

Acceleration transformations are t-derivatives of velocity transformations (3.7.18). Two new terms arise.

$$\begin{aligned} \ddot{x}_i &= \tfrac{d}{dt}\left(R_{ik}[\boldsymbol{\omega}\cdot t]\dot{\bar{x}}_k + \dot{R}_{ik}[\boldsymbol{\omega}\cdot t]\bar{x}_k\right) \\ &= R_{ik}[\boldsymbol{\omega}\cdot t]\ddot{\bar{x}}_k + 2\dot{R}_{ik}[\boldsymbol{\omega}\cdot t]\dot{\bar{x}}_k + \ddot{R}_{ik}[\boldsymbol{\omega}\cdot t]\bar{x}_k \end{aligned} \tag{3.7.20a}$$

$$\begin{aligned} \ddot{\bar{x}}_k &= \tfrac{d}{dt}\left(R_{ki}[-\boldsymbol{\omega}\cdot t]\dot{x}_i + \dot{R}_{ki}[-\boldsymbol{\omega}\cdot t]\dot{x}_i\right) \\ &= R_{ki}[-\boldsymbol{\omega}\cdot t]\ddot{x}_i + 2\dot{R}_{ki}[-\boldsymbol{\omega}\cdot t]\dot{x}_i + \ddot{R}_{ki}[-\boldsymbol{\omega}\cdot t]\dot{x}_i \end{aligned} \tag{3.7.20b}$$

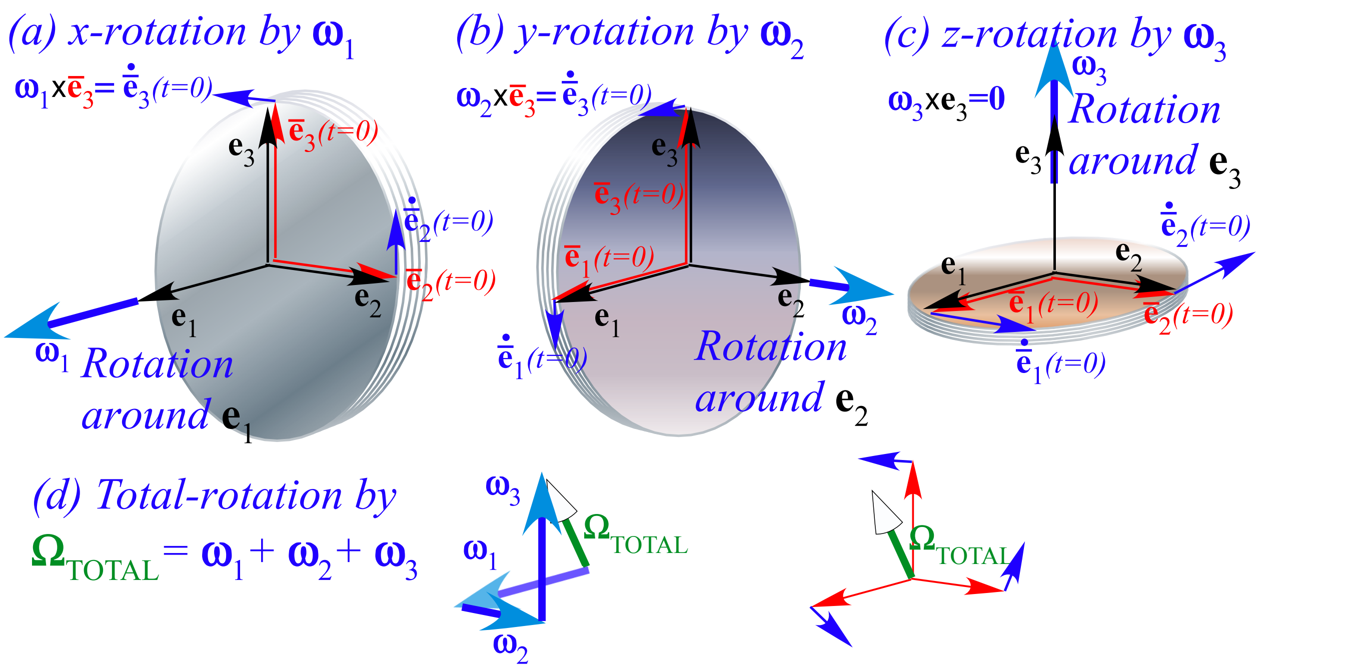


Fig. 3.6.3 Initial effects on unit vectors of angular velocity (a) ω_1, (b) ω_2, (c) ω_3, and (d) Ω_{TOTAL}.

By (3.7.15), matrix $R_{ik}[t=0]=\delta_{ik}$ has derivatives $\dot R_{ik}[0]=(\omega\bullet\varepsilon)_{ik}$ and $\ddot R_{ik}[0]=(\omega\bullet\varepsilon)_{ij}(\omega\bullet\varepsilon)_{jk}+(\dot\omega\bullet\varepsilon)_{ik}$.

$$\ddot{\mathbf{x}}(0)=\ddot{\bar{x}}+2(\omega\bullet\varepsilon)\dot{\bar{\mathbf{x}}}+(\omega\bullet\varepsilon)(\omega\bullet\varepsilon)\bar{\mathbf{x}}+(\dot\omega\bullet\varepsilon)\bar{\mathbf{x}} \qquad \ddot{\bar{\mathbf{x}}}(0)=\ddot{\mathbf{x}}-2(\omega\bullet\varepsilon)\dot{\mathbf{x}}+(\omega\bullet\varepsilon)(\omega\bullet\varepsilon)\mathbf{x}-(\dot\omega\bullet\varepsilon)\mathbf{x} \qquad (3.7.21)$$

At $t=0$ we let $\bar{\mathbf{x}}$ be $\mathbf{x}$ and $\dot{\bar{\mathbf{x}}}$ be $\dot{\mathbf{x}}-(\omega\bullet\varepsilon)\mathbf{x}$ by (3.7.19b) or, *vice-versa*, let $\dot{\mathbf{x}}$ be $\dot{\bar{\mathbf{x}}}+(\omega\bullet\varepsilon)\bar{\mathbf{x}}$ by (3.7.19a).

$$\ddot{\mathbf{x}}(0)=\ddot{\bar{x}}+2(\omega\bullet\varepsilon)\dot{\mathbf{x}}-(\omega\bullet\varepsilon)(\omega\bullet\varepsilon)\mathbf{x}+(\dot\omega\bullet\varepsilon)\mathbf{x} \qquad \ddot{\bar{\mathbf{x}}}(0)=\ddot{\mathbf{x}}-2(\omega\bullet\varepsilon)\dot{\bar{\mathbf{x}}}-(\omega\bullet\varepsilon)(\omega\bullet\varepsilon)\bar{\mathbf{x}}-(\dot\omega\bullet\varepsilon)\bar{\mathbf{x}} \qquad (3.7.22)$$

Levi-Civita rules $\varepsilon_{ijk}\varepsilon_{kab}=\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja}$ reduce vector triple cross products to dot and tensor products.

$$\begin{aligned}\ddot{\mathbf{x}}(0)&=\ddot{\bar{\mathbf{x}}}+2\boldsymbol{\omega}\times\dot{\mathbf{x}}-\ \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x})\quad+\dot{\boldsymbol{\omega}}\times\mathbf{x}\\&=\ddot{\bar{\mathbf{x}}}+2\boldsymbol{\omega}\times\dot{\mathbf{x}}-(\boldsymbol{\omega}\boldsymbol{\omega}-\omega^2\mathbf{1})\bullet\mathbf{x}+\dot{\boldsymbol{\omega}}\times\mathbf{x}\end{aligned} \quad (3.7.23a) \qquad \begin{aligned}\ddot{\bar{\mathbf{x}}}(0)&=\ddot{\mathbf{x}}-2\boldsymbol{\omega}\times\dot{\bar{\mathbf{x}}}-\ \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\bar{\mathbf{x}})\quad-\dot{\boldsymbol{\omega}}\times\bar{\mathbf{x}}\\&=\ddot{\mathbf{x}}-2\boldsymbol{\omega}\times\dot{\bar{\mathbf{x}}}-(\boldsymbol{\omega}\boldsymbol{\omega}-\omega^2\mathbf{1})\bullet\bar{\mathbf{x}}-\dot{\boldsymbol{\omega}}\times\bar{\mathbf{x}}\end{aligned} \quad (3.7.23b)$$

Let us compare cylindrical OCC formulae (3.7.8) to (3.7.23b) above with $\omega=\dot\phi$ and $\dot{\bar{x}}=\dot\rho$.

$$\ddot\rho=F^\rho+\rho\dot\phi^2 \quad (Centrifugal) \qquad (3.7.8a)_{repeat} \qquad\qquad \rho\ddot\phi=F^\phi\rho-2\dot\rho\dot\phi \quad (Coriolis) \qquad (3.7.8a)_{repeat}$$

The following assumed ω and $\bar{\mathbf{x}}$ vectors give terms in (3.7.23b) that correspond to OCC results *(3.7.8)*.

$$\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)=(0,0,\dot\phi) \qquad \bar{\mathbf{x}}(0)=(\bar{x}_1,\bar{x}_2,\bar{x}_3)=(\rho,0,0) \qquad \dot{\bar{\mathbf{x}}}(0)=(\dot\rho,0,0) \qquad \dot{\boldsymbol{\omega}}=(0,0,\ddot\phi) \qquad (3.7.24a)$$

$$-\ \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\bar{\mathbf{x}})=(\rho\dot\phi^2,0,0) \qquad -2\boldsymbol{\omega}\times\dot{\bar{\mathbf{x}}}=(0,-2\dot\rho\dot\phi,0) \qquad -\dot{\boldsymbol{\omega}}\times\bar{\mathbf{x}}=(0,-\rho\ddot\phi,0) \qquad \ddot{\bar{\mathbf{x}}}(0)=(\ddot\rho,0,0) \quad (3.7.24b)$$

Centrifugal ($\rho\dot\phi^2$), Coriolis ($-2\dot\rho\dot\phi$), trans-Coriolis ($-\rho\ddot\phi$), and radial acceleration ($\ddot\rho$) terms match lab F^μ.

$$(\ddot\rho,0,0)=(F^\rho,F^\phi\rho,0)+(0,-2\dot\rho\dot\phi,0)+(\rho\dot\phi^2,0,0)+(0,-\rho\ddot\phi,0) \qquad (3.7.25)$$

It helps to compare different approaches like this in order to have confidence in either of the results.

Chapter 8. GCC Lagrangian and Hamiltonian functions and equations

Here we redo the derivation in Unit 2 Ch. 6 of Lagrangian and Hamiltonian equations in tensor notation for fixed GCC (GCWOTD). Both require that external covariant forces F_m be derivatives of a *potential function* $V(q^m, t)$ of coordinates and time, only.

$$F_m = -\frac{\partial V}{\partial q^m} \tag{3.8.1a}$$

Then the *Lagrangian function L=T-V*

$$L{=}L(q\ ,\dot{q}\ ,t) = T(q\ ,\dot{q}\ ,t)\text{-}V(q\ ,t) \tag{3.8.1b}$$

simplifies Lagrange's equations (3.6.4).

$$\frac{dp_m}{dt} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} = \frac{\partial T}{\partial q^m} + F_m = \frac{\partial \left(T - V\right)}{\partial q^m}\ , \quad \text{or:}\ \ \dot{p}_m = \frac{\partial L}{\partial q^m} \tag{3.8.1c}$$

Canonical momentum is defined using T and L as in (2.4.1c) and (3.6.3).

$$p_m = \frac{\partial T}{\partial \dot{q}^m} = \frac{\partial L}{\partial \dot{q}^m} \tag{3.8.1d}$$

The idea of the *Hamiltonian formulation* is to treat generalized *momenta* and coordinates as independent variables rather than generalized *velocities* and coordinates. The total derivative of L is

$$\dot{L}\left(q\ ,\dot{q}\ ,t\right) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t} \tag{3.8.2}$$

Inserting Lagrange equations (3.8.1c-d) and using the identity: $\dot{p}\frac{dq}{dt} + p\frac{d\dot{q}}{dt} = \frac{d}{dt}\left(p\dot{q}\right)$ gives

$$\begin{aligned}\dot{L}\left(q\ ,\dot{q}\ ,t\right) &= \frac{dL}{dt} = \dot{p}_m\frac{dq^m}{dt} + p_m\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}\\ &= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot{q}^m\right) + \frac{\partial L}{\partial t}\end{aligned} \tag{3.8.3}$$

Reordering gives *Legendre-Poincare* forms (Recall (1.12.11) in Unit 1 and (2.6.9) in Unit 2.)

$$\frac{d}{dt}\left(L - p_m\dot{q}^m\right) = \frac{\partial L}{\partial t}\ , \tag{3.8.5a}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}\ , \tag{3.8.5b}$$

Here the *Hamiltonian function H* is defined as in (1.12.11b) and (2.6.9).

$$H = H\left(p_m\ ,q^m, t\right) = p_m\dot{q}^m - L \tag{3.8.5c}$$

Hamiltonian H satisfies *Hamilton's equations* as in (2.6.13) and (2.6.14).

$$\frac{\partial H}{\partial p^m} {=} \dot{q}_m \tag{3.8.5d}$$

$$-\frac{\partial H}{\partial q^m} {=} \dot{p}_m = \frac{\partial L}{\partial q^m} \tag{3.8.5e}$$

The Hamiltonian function has the extraordinary property that its *total* time derivative is equal to its partial time derivative as in (3.8.5b). If p's and q's obey the equations of motion then H is constant or *conserved* provided L (and therefore H) has no *explicit* time dependence.

This is a familiar energy conservation relation which follows from using the metric definition: $p_m = \Sigma\gamma_{mn}\,\dot{q}^n$ of covariant momentum from (3.6.3a) in the definition (3.8.5c) of H.

$$H = p_m \dot{q}^m - L = \left(\gamma_{mn}\dot{q}^n\right)\dot{q}^m - \left(T - V\right)$$
$$= \gamma_{mn}\dot{q}^m\dot{q}^n - \frac{1}{2}\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right) + V \tag{3.8.6a}$$

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = T + V \equiv E \qquad \text{(Numerically correct ONLY!)} \tag{3.8.6b}$$

$$H = \frac{1}{2}\gamma^{mn}p_m p_n + V = T + V \equiv E \qquad \text{(Formally and Numerically correct)} \tag{3.8.7}$$

So, the *Hamiltonian* is the <u>sum</u> of kinetic energy T and potential V which is the *total energy* $E=T+V$. Equations (3.8.5) amount to the *conservation of total energy* if L and H are *not* an *explicit* function of time.

One always writes a Hamiltonian $H=H(q, p, t)$ as an explicit function of coordinates and *momenta* like (3.8.7), not as coordinates and velocities, as in (3.8.6b) The corrected equation (3.8.7) uses the contravariant metric γ^{mn}, the inverse of the covariant γ_{mn}. On the other hand, Lagrangian independent variables are coordinates and *velocities*, that is $L = L\left(q,\ \dot{q},\ t\right)$.

Symmetry and conservation laws

The absence of explicit time dependence implies energy conservation; or more correctly, that the Hamiltonian H was a *constant of the motion*.

$$\frac{\partial L}{\partial t} = 0 = \frac{\partial H}{\partial t} \quad \text{implies: } \dot{H} = 0, \text{ or } H = E = \text{constant} \tag{3.8.8}$$

Similarly, the absence of explicit coordinate dependence leads to *momentum conservation*.

$$\frac{\partial L}{\partial q^m} = 0 = \frac{\partial H}{\partial q^m} \quad \text{implies: } \dot{p}_m = 0, \text{ or } p_m = M = \text{constant} \tag{3.8.9}$$

Hamilton's equations show the beautiful relation between *symmetry* in a generalized coordinate q^m and *conservation* of its *conjugate momentum* p_m. Symmetry, in this case, means that the system 'looks the same' for all values of the coordinate q^m or is *invariant* to changes of that coordinate. In other words, we find no 'lumps'; the Hamiltonian doesn't go up or down as q^m changes. Because of this symmetry or 'smoothness', the system cannot alter momentum corresponding or *conjugate* to this coordinate. In other words: "No lumps means no bumps!" Some texts call such a coordinate an *ignorable* coordinate. Bad terminology! We shall not ignore conserved coordinates. Ignoring symmetry is a big mistake.

An important quantity in theoretical mechanics is the following differential from (3.8.5c)

$$dS = L\,dt = p_m dq^m - H\,dt \tag{3.8.10}$$

It is known as *Poincare's invariant* or *differential of action S*. It is clearly invariant to a GCC transformation since it is a combination of an invariant covariant-contra sum and scalar functions.

Separation of GCC Equations: Effective Potentials

Now we show how a GCC equation of motion might be separated into independent "curvilinear normal modes", something that might seem oxymoronic. We will take as our first example the cylindrical polar coordinates that were worked out in Ch. 7. The kinetic energy (3.7.5) gives the following two forms, only the second of which is a formally correct function of covariant momentum p_m.

$$\begin{aligned} H &= \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V \quad \left(\text{Numerically correct ONLY!}\right) \\ &= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V \quad \left(\text{Formally **and** Numerically correct}\right) \end{aligned} \tag{3.8.11}$$

If the potential V is *isotropic*, that is, cylindrical symmetric, then it will be a function of radius ρ alone. ($V = V(\rho)$) Then H has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu \qquad \text{if: } V = V(\rho) \tag{3.8.12a}$$

Similarly, if H has no explicit z–dependence then the z–momenta is constant, too.

$$m\dot{z} = p_z = const. = k \quad \text{if: } V = V(\rho) \tag{3.8.12b}$$

This reduces the isotropic Hamiltonian to

$$H = \frac{1}{2m}p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = const. \tag{3.8.13}$$

The effect of symmetry here is to reduce the problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const. \tag{3.8.14a}$$

The cost is to have an *effective potential* $V^{eff}(\rho)$ given as follows. (Let $k=0$ to suppress z-motion.)

$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho) \tag{3.8.14b}$$

The $1/\rho^2$ term is a *centrifugal barrier*. It reduces the attraction of the "real" $V(\rho)$ near $\rho=0$. Imagine riding this $V^{eff}(\rho)$ spinning around with mass m at the polar angle velocity $\omega = \dot{\phi}$ given by (3.8.12a).

$$\dot{\phi} = \mu/\left(m\rho^2\right) \tag{3.8.14c}$$

The angular rate varies inversely with the square of the radius ρ, which, in turn, satisfies Hamilton's equation (3.8.5d). We use $H=E$ from (3.8.14a).

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)} \tag{3.8.15a}$$

The equation is solved by a *quadrature integral* for time versus radius. Recall pendulum case (2.7.10).

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0 \tag{3.8.15b}$$

For a bound oscillation a similar integral gives the time for one-half of an oscillation or orbit.

$$\int_0^{\tau_{1/2}} dt = \int_{\rho_{\text{perigee}}}^{\rho_{\text{apogee}}} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{\text{eff}}(\rho)\right)}} = \left(\text{Travel time } \rho_{\text{perigee}} \text{ to } \rho_{\text{apogee}}\right) = \tau_{1/2} \tag{3.8.15b}$$

This is the orbit time from the furthest "up" point (*apogee*) to the closest (*perigee*) point. These are the turning points for which radial velocity $\dot{\rho}$ and momentum $p_\rho = m\dot{\rho}$ are zero. The points can be found for a given energy E by solving the effective potential equation (3.8.14a) with $p_\rho = 0$

$$V^{\text{eff}}\left(\rho_{perigee}\right) = E = V^{\text{eff}}\left(\rho_{apogee}\right) \tag{3.8.16}$$

Kepler's law and orbital paths

Isotropic potentials conserve angular momentum μ and so the polar angular rate varies as inverse square of distance according to $\dot{\phi} = \mu / m\rho^2$ or

$$\rho^2 d\phi = \left(\mu / m\right) dt \tag{3.8.17}$$

Since $\rho^2 d\phi/2$ is area of an infinitesimal arc let's integrate to relate time and area swept by radius ρ.

$$A = \text{Area swept in time } \Delta t = \int_0^{\phi(\Delta t)} \frac{1}{2}\rho^2 d\phi = \int_0^{\Delta t} \left(\mu / 2m\right) dt = \left(\mu / 2m\right)\Delta t \tag{3.8.18a}$$

This is *Kepler's Law: Equal areas are swept in equal times*. Furthermore, the (reduced) mass m of a planet can be given in terms of observed orbit period τ, angular momentum μ, and total orbit area A.

$$m = \mu \cdot \tau / 2 \cdot A = \frac{\text{Angular momentum} \cdot \text{Orbital Period}}{2 \cdot \text{Orbital Area}} \tag{3.8.18b}$$

Kepler's law holds in all radial potentials, not just the Coulomb *GM/r* potential for which it was discovered or the inside-Earth oscillator potential discussed in Unit 1 Ch. 9. That is due to their spherical symmetry.

Momentum relation (3.8.14c) can be used to convert the orbital time integral to one for orbital path.

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{d\rho}{d\phi}\frac{d\phi}{dt} = \frac{d\rho}{d\phi}\frac{\mu}{m\rho^2} = \pm\sqrt{\frac{2}{m}\left(E - V^{\text{eff}}(\rho)\right)} \tag{3.8.19a}$$

$$\frac{m}{\mu}\left(\phi(t) - \phi(0)\right) = \frac{m}{\mu}\int_{\phi(0)}^{\phi(t)} d\phi = \int_{\rho(0)}^{\rho(t)} \frac{d\rho}{\rho^2\sqrt{\frac{2}{m}\left(E - V^{\text{eff}}(\rho)\right)}} \tag{3.8.19b}$$

The orbital path integral simplifies somewhat using inverse radius $u=1/\rho$.

$$\frac{m}{\mu}\left(\phi(\rho_1) - \phi(\rho_0)\right) = -\int_{u_0}^{u_1} \frac{du}{\sqrt{\frac{2}{m}\left(E - V^{\text{eff}}(1/u)\right)}} \quad \text{where: } u \equiv \frac{1}{\rho} \text{, and: } du = \frac{-d\rho}{\rho^2} \tag{3.8.20}$$

This can be used to describe orbits 2D HO potential $V(\rho)=\frac{1}{2}k\rho^2$ discussed in Ch. 9 of Unit 1 and will be taken to more detail in Unit 5.

Small radial oscillations

If there is a stable circular orbit, then its radius ρ_{stable} will be a minimum for the effective potential. This minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0\,, \quad \text{with: } \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0\,. \tag{3.8.21}$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \tag{3.8.22}$$

An effective "spring constant" may exist at the stable point.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \tag{3.8.23}$$

This provides an approximate radial frequency of oscillation.

$$\omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}} \tag{3.8.24}$$

Otherwise, the potential is quartic (4^{th} order) or higher degree such as the super-ball potential discussed in Section 1.7. Such a system has “soft-mode’ behavior with a zero-frequency low-amplitude limit.

Comparing radial ω_ρ to the polar angular frequency $\dot{\phi}$ determines orbit shape and closure. Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_\phi} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic} \tag{3.8.25}$$

Some generic shapes resulting from various ratios $n_\rho : n_\phi$ are sketched in Fig. 3.8.1. The patterns are the polar coordinate analogs of more familiar *Lissajous* patterns seen in Cartesian oscillations for similarly rational eigenfrequency ratios. (This is discussed in the following Unit 4.)

For most applications of (3.8.24-15) the rationality test is an approximation that depends upon having only small deviations from the stable radius of a purely circular orbit. The figures below have a width between turning radii that would be highly exaggerated for most potentials since (3.8.25) generally applies only over a limit range of ρ around ρ_{stable}.

The first two ratios $n_\rho : n_\phi = 1:1$ and $n_\rho : n_\phi = 2:1$ are the most commonly seen or widely known cases which include, respectively, the Coulomb potential $V(\rho) = k/\rho$ and the isotropic harmonic oscillator potential $V(\rho) = k\,\rho^2/2$ treated in Unit. 5. There is something special about the latter two. As noted already, they maintain their integral ratios for large amplitude motion, as well, and produce perfect ellipses.

The orbit paths in Fig. 3.8.1 were drawn for sinusoidal motion in both the radial and angular coordinates as occurs in optical Faraday polarization oscillation. Rarely is the time behavior so simple, so the precise form of orbital paths will depend on the detailed time behavior of each problem.

Often orbits appear near some near-rational ratio. For example *1.02:1* or *0.8:1* are near *1:1* and therefore would appear at first to be like the *1:1* shape in Fig. 3.8.1 but slowly move away from that. Such slow movement takes the form of either *retrograde precession* (for $\omega_\rho / \omega_\phi > n_r / n_\phi$) or else *prograde precession* (for $\omega_\rho / \omega_\phi < n_\rho / n_\phi$). The former case where, for example, ratio *1.0241..:1* is ~2% greater than 1/1 means the radial oscillation is 2% quicker than the angular rotation. So apogees and perigees rotate or precess in a direction opposite or *retrograde* to that of the orbit at 2% its angular rate or for the latter ratio of *0.8:1*, with a 2% *prograde* precession. See Fig. 3.8.1b. The same applies to ratios near 2:1 as shown in Fig. 3.8.1c or to any ratio such as in Fig. 3.8.1a. Any finite decimal is an integer ratio.

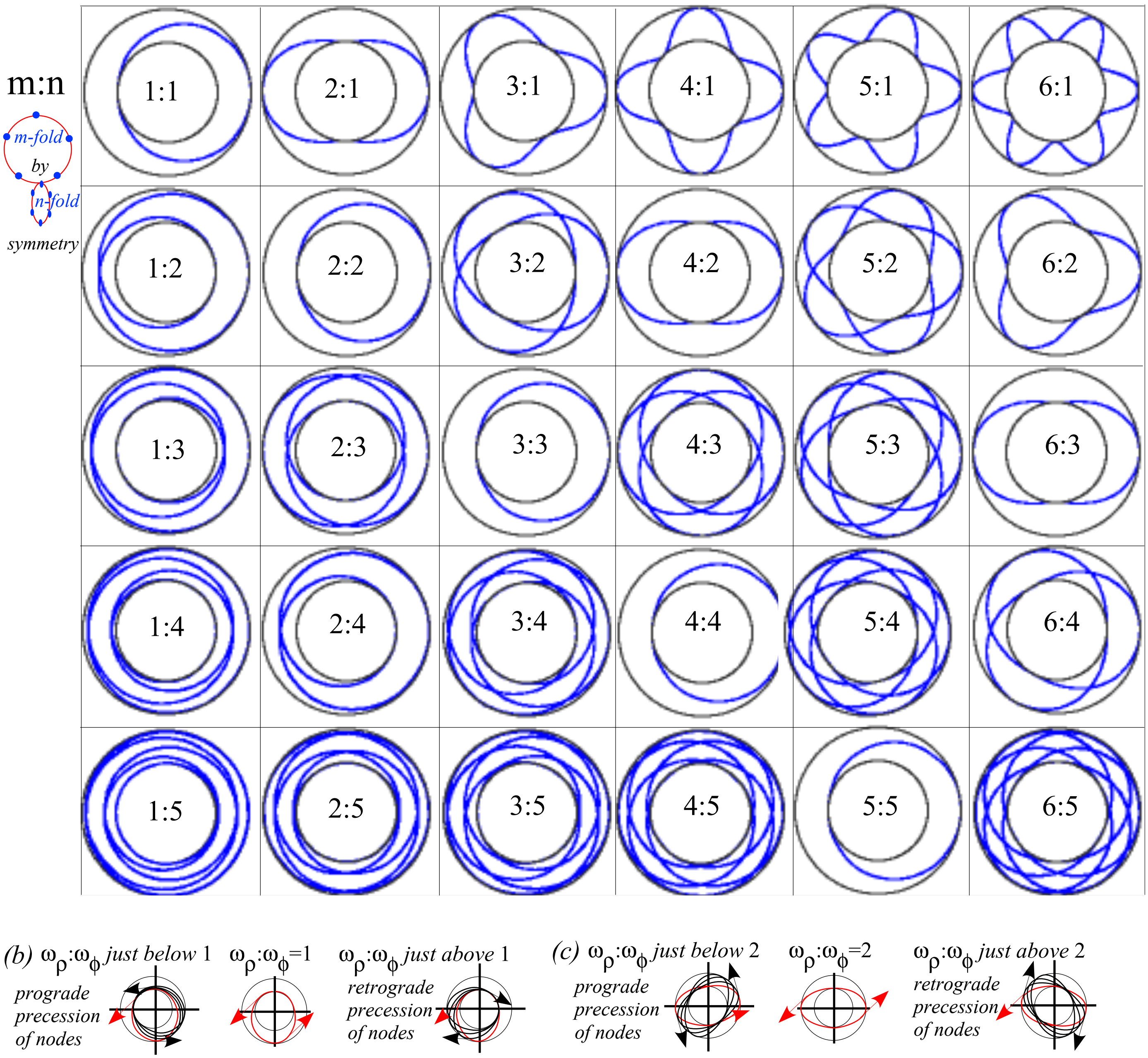
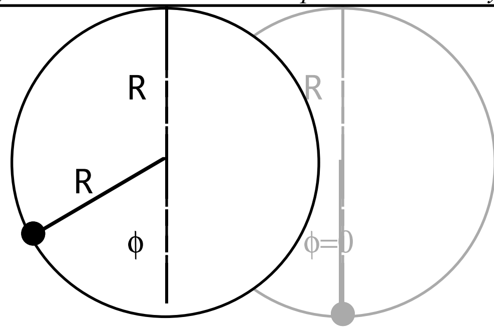


Fig. 3.8.1 Orbit patterns with radial:angular frequency ratios $n_\rho : n_\phi$. (b) Near 1:1 (c) Near 2:2

Huygen's problem. For over 40 years Christian Huygens worked to improve harmonicity of pendulums and only just before he died solved the problem you are about to do. Let's hope it doesn't take you that long!



Exercise 3.8.1 A really scary roller coaster

Let particle m be constrained to move with no friction along a cycloid of radius R in gravity g. (Equivalently, lead weight m is stuck to a circular steel hoop that rolls without slip while hanging on a smooth magnetic ceiling as sketched above.)

(a) Write parametric equations $x(\phi)$, $y(\phi)$ for mass m as hoop rolls by angle ϕ from point where m has lowest PE$=mgy=0$.

(b) Derive Lagrangian $L(d\phi/dt,\phi)$ and find canonical momentum and equation of motion.

(c) Derive total energy and Hamiltonian function H. Is either L or H a constant of the motion?

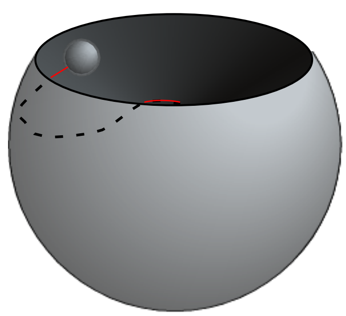
(d) Derive an expression for the arc length $s(\phi)$ that m travels from its lowest point.

(e) Show the period of oscillation of this mass kicked from $\phi=0$ is independent of initial velocity for velocity less than_______? (What?)

(f) Derive $\phi(t)$ for free oscillation. Does an initial $v_x(0)$ exist so hoop rolls across a ceiling at constant $v_x(t)=v_x(0)$? Discuss. If, instead the hoop rolls on a *floor*, does such a constant velocity state exist either exactly or approximately?

(g) *Evolution geometry*. Prove that a cycloid may be generated by a string unwrapping from a cycloid.

Involution geometry. Show that a cycloid's center-of-curvature points lie on another cycloid above it.



Exercise 3.8.2 The fishbowl orbiter

A mass is sliding frictionlessly in a spherical bowl with gravity $g \sim 10 m \cdot s^{-2}$.

(a) Set up a suitable coordinate description and Lagrangian equations.

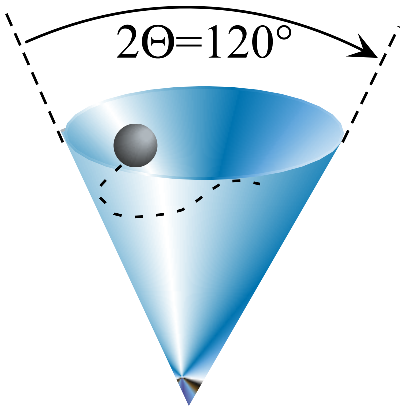
(b) Give both covariant and contravariant momenta for the problem.

(c) Write Hamilton's equations of motion. Indicate conserved quantities, if any.

(d) Write a quadrature integral formula using effective potential formalism.

(e) For the low energy $(0<\theta_{eq}<<1)$ case (particle around bottom of bowl) estimate radial and angular oscillation frequencies in terms of conserved quantities and equilibrium polar angle θ_{eq}. Which closed orbit (if any) in Fig. 3.8.1 is closest? If orbit is not perfectly closed, say if aphelion or perihelion precession is in direction of orbital velocity (prograde precession) or against it (retrograde precession).

(e) Answer (e) for the high energy case $(\theta_{eq} \sim \pi/2)$ (particle around equator of bowl).



Exercise 3.8.3 The funnel orbiter

A mass m is sliding frictionlessly in a circular cone of polar angle θ with gravity $g\sim 10m{\cdot}s^{-2}$.

a. Derive Lagrangian and Lagrange equations of motion for radius r and angle ϕ.

b. Derive Hamiltonian and Hamilton's equations of motion. Note any conserved quantities.

c. Derive 1D radial effective potential and corresponding equation of motion.

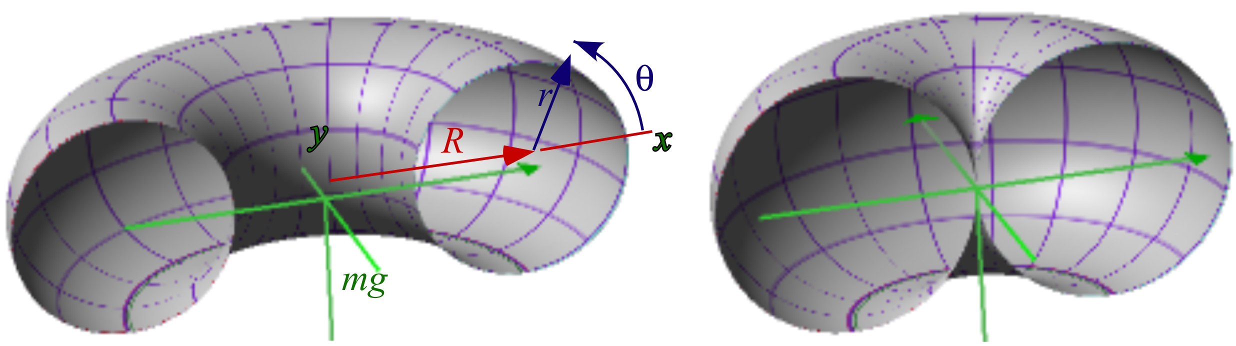
Consider nearly circular orbits and estimate radial and angular frequencies in terms of angle θ .

d. Find what if any angle $\theta_{1:1}$ gives a closed $\omega_r / \omega_\phi = 1/1$ orbit.

e. Find what if any angle $\theta_{2:1}$ gives a closed $\omega_r / \omega_\phi = 2/1$ orbit.

f. Is aphelion precession in direction of orbit (prograde) or against it (retrograde)…

…for a cone with angle θ slightly greater than $\theta_{1:1}$? …for a cone with angle θ slightly less than $\theta_{2:1}$?



Exercise 3.8.4 The toroidal orbiter

Consider a mass m is frictionlessly constrained to a torus which has a major radius of R and a minor radius of r. (See Figure.) Let the major azimuthal angle ϕ be measured from the x axis counter-clockwise around the center circle of radius R which lies in the xy-plane and supports the minor radius r. The angle of elevation θ of the minor radius above the xy-plane is measured from the outer equator of the torus. (Positive θ is in the $+z$ direction at the outer equator.) For some of the problems below let there also be a gravitational field with acceleration g along the negative z-axis.

a. Derive Lagrangian and Lagrange equations of motion for angle θ and angle ϕ.

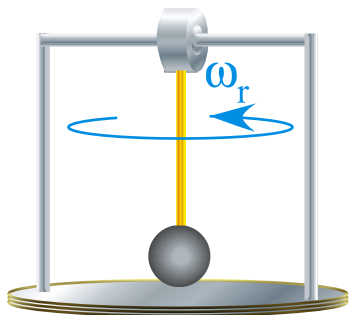
b. Derive Hamiltonian and Hamilton's equations of motion. Note any conserved quantities.

c. Derive 1D effective potential and corresponding equation of motion.

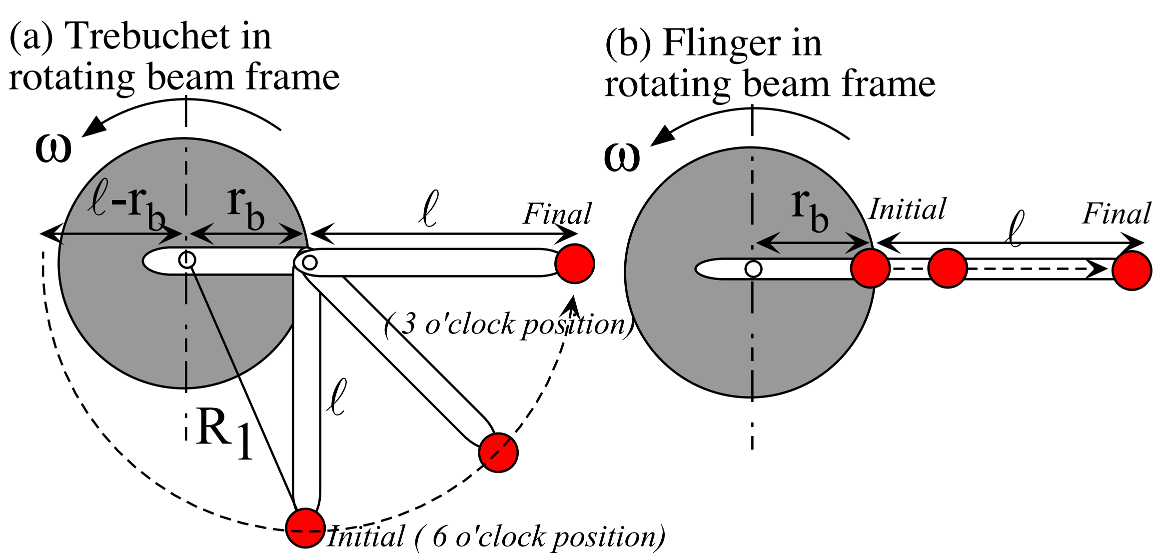
Extra problems involving generalized coordinates with explicit time dependence (GCWETD)

Problems so far have not involved the most general GCC that have time dependency. Study the following classic cases involving GCC with constant rotation. Try to do without looking at answers!

Exercise 3.8.5 GCWETD as world turns
Derive *GCWETD* Lagrange equations for a spherical coordinate rotating at constant angular velocity ω such as an Earth fixed system.
a. Show fictitious force terms as well as real ones and compare results to analysis leading to (3.7.7) or (3.7.25).
b. Extend analysis to include variable $\omega(t)$.



Exercise 3.8.6 Pendulum-on-turntable
Suppose a pendulum supported by a circular ball bearing may swing without friction in the vertical plane of the bearing. The bearing plane is secured to a turntable that rotates at a constant angular frequency ω_r. The pendulum consists of a mass m at the end of a rod of length ℓ and negligible mass with natural frequency of small θ-angle motion at zero-ω_r in gravity acceleration g given by $\omega_0(\omega_r{=}0)$.
a. First, what is $\omega_0(\omega_r{=}0)$?
b. Derive the Lagrangian and Hamiltonian using spherical coordinates in the rotating frame.
c. Derive the θ-equilibrium points and small-oscillation frequency $\omega_0(\omega_r)$ as a function of g and frequency ω_r. Overlay plots of an effective θ-potential for several key values of ω_r.



Exercise 3.8.6 Trebuchet-on-turntable (algebraic version)
Consider a rotating frame of constant angular frequency ω_r such as pictured in the models in Fig. 2.9.3 that compare trebuchet mechanics to flinger mechanics. The flinger just rotates a frictionless tube in which a mass m slides from radius r_b out to radius $r_b+\ell$. The trebuchet swings a mass m on rod ℓ around pivot point at radius r_b and lets m go at radius $r_b+\ell$. The idea is to compare the final velocities of the two devices.

Derive Lagrangian and Hamiltonian for each using appropriate coordinates in the rotating frame.

For the case of equal inner and outer radii ($r_b=\ell$) determine which device can give higher final velocity. Discuss effects of possible initial conditions on the results for the trebuchet model.

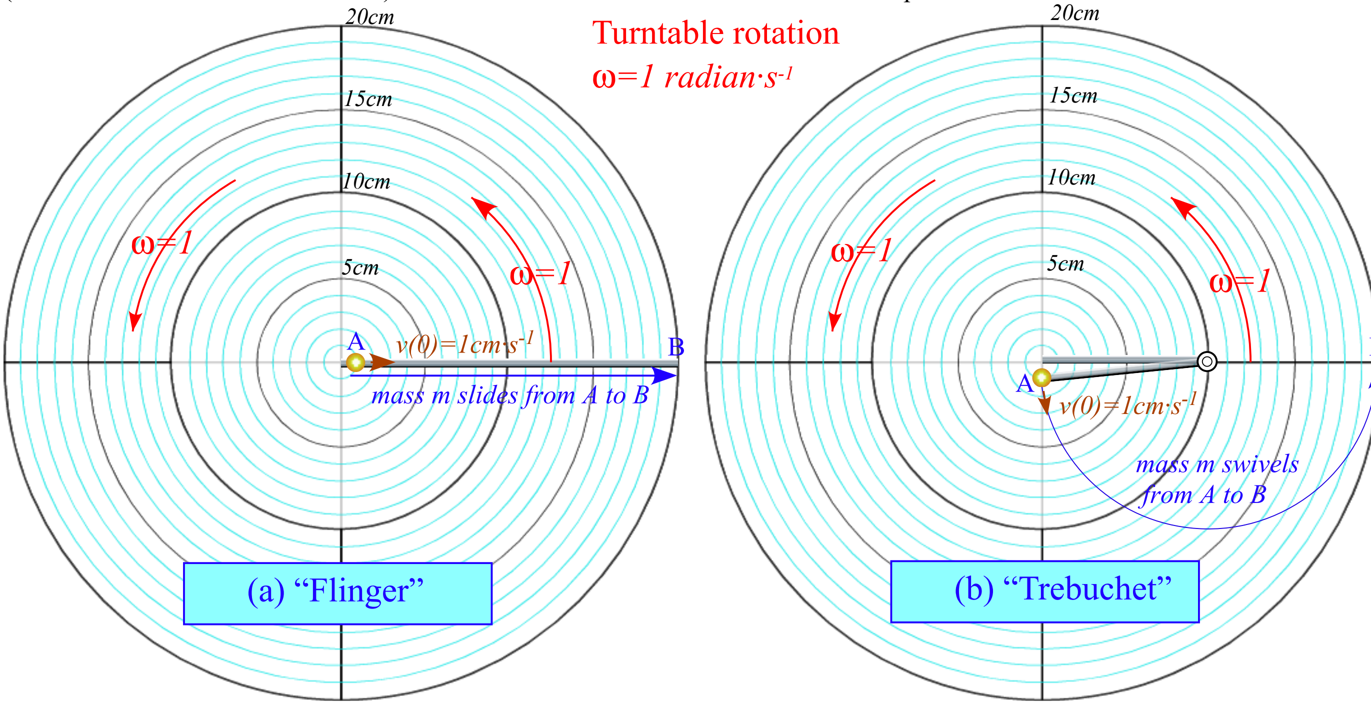
Exercise 3.8.7 Trebuchet-on-turntable (geometric version)

Compare dynamics of mass m on a "Flinger" (Fig. (a)) to what it does on a "Trebuchet" (Fig. (b)).

Both begin at point A of radius r(0)=1cm. from the center of a turntable rotating at ω=1(rad)s^{-1}. Both have an initial speed of v(0) =1cm·s^{-1} relative to turntable. Both masses move from that point A to a final point B having radius r(t_r)=20cm where we assume m is then released into the laboratory.

In Fig. (a) m slides 19cm along a rod of length ℓ =20cm. (The 20cm rod is fixed to turntable.)

In Fig. (b) m swivels on a rod of length ℓ =10cm around a point fixed to turntable at r=10cm radius.



Relative to turntable…

a. Find m release speed for "Flinger." __________

Relative to laboratory…

b. Find m release speed for "Flinger."__________

c. To scale†, sketch lab $\mathbf{v}(t_r)$ assuming release at B.

† Let 1cm be 1cm·s^{-1}.

Relative to turntable…

Find m release speed for "Trebuchet."__________

Relative to laboratory…

Find m release speed for "Trebuchet."__________

To scale†, sketch lab $\mathbf{v}(t_r)$ assuming release at B.

† Let 1cm be 1cm·s^{-1}.

d. How long does *m* take to go from A to release point B? _____sec.

e. Plot or (preferably) construct its orbit on a polar graph like Fig. (a) but in the lab frame.

Chapter 9. Constraint analysis: Comparing GCC and other approaches

Direct Lagrangian approach

Let us consider some ways to analyze a particle m constrained to a curve $y=\frac{1}{2}kx^2$ on (x,y)-plane of Fig. 3.9.1a with gravitational potential $V(\mathbf{r})=mgy$. One way is to insert the constraint into the Lagrangian.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy \qquad \text{where: } y=\frac{1}{2}kx^2 \quad \text{and: } \quad \dot{y}=kx\dot{x} \tag{3.9.1a}$$

The resulting Lagrangian involves just one degree of freedom x, one momentum p_x, and one force $f_x=\frac{\partial L}{\partial x}$.

$$\begin{aligned} L&=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 & p_x&=\frac{\partial L}{\partial \dot{x}} & f_x&=\frac{\partial L}{\partial x} \\ &=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2) & &=m(\dot{x}+k^2x^2\dot{x}) & &=m(k^2x\dot{x}^2-gkx) \end{aligned} \tag{3.9.1b}$$

Lagrange equation of motion $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ has the form $\ddot{x}=-K(x,\dot{x})x$ with a varying spring parameter K.

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx) \quad \text{, or:} \qquad \ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx \tag{3.9.2}$$

(A particle in a uniform-g parabolic well is a *harmonic* oscillator only for small x and v.) Another way to approach this problem lets us re-examine Lagrangian-Riemann GCC theory and its relation to constraints.

GCC approach: y-line parabolic coordinates

Another way to treat constraint curve $y=\frac{1}{2}kx^2$ is to incorporate it with x into GCC defined as follows.

$x=q^1=X$ (3.9.3a) $\qquad y=\frac{1}{2}kx^2+q^2=kX^2/2+Y$ (3.9.3b)

This choice is easily inverted, but that is not really needed for a GCC approach.

$q^1=X=x$ (3.9.3b) $\qquad q^2=Y=y-\frac{1}{2}kx^2$ (3.9.3c)

Fig. 3.9.1b shows grid lines for $q^1=X=const.$(vertical lines) and $q^2=Y=const.$(vertical family of parabolas). The notation (X,Y) is used to avoid confusing indices of (q^1,q^2) and exponents. Also shown are the covariant $\mathbf{E}_k$ in columns of Jacobian J matrix and the contravariant $\mathbf{E}^k$ in rows of Kajobian K below.

$$\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix}=J \quad \begin{matrix} \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix} \\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}\end{matrix} \quad (3.9.4a) \qquad \begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \quad \begin{matrix} \mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix} \\ \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}\end{matrix} \quad (3.9.4b)$$

This gives the 1st coordinate differentials and velocity relations with their inverses. *(Lemma 1)*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} \quad (3.9.4c) \qquad \begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} \quad (3.9.4d)$$

The kinetic energy uses kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}$.

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{m}{2}(g_{XX}\dot{X}^2+2g_{XY}\dot{X}\dot{Y}+g_{YY}\dot{Y}^2)=\frac{1}{2}(\gamma_{XX}\dot{X}^2+2\gamma_{XY}\dot{X}\dot{Y}+\gamma_{YY}\dot{Y}^2) \tag{3.9.5a}$$

Covariant γ_{AB} and contravariant (inverse) γ^{AB} coefficients relate momentum $p_A = \gamma_{AB}\dot{q}^B$ to velocity $\dot{q}^A = \gamma^{AB}p_B$.

$$\begin{pmatrix} \gamma_{XX} & \gamma_{XY} \\ \gamma_{YX} & \gamma_{YY} \end{pmatrix} = m\begin{pmatrix} g_{XX} & g_{XY} \\ g_{YX} & \gamma_{YY} \end{pmatrix} = m\begin{pmatrix} 1+k^2x^2 & kx \\ kx & 1 \end{pmatrix} \quad (3.9.5b) \qquad \begin{pmatrix} \gamma^{XX} & \gamma^{XY} \\ \gamma^{YX} & \gamma^{YY} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & -kx \\ -kx & 1+k^2x^2 \end{pmatrix} \quad (3.9.5c)$$

$$= m\begin{pmatrix} \mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y \\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y \end{pmatrix} \qquad = \frac{1}{m}\begin{pmatrix} \mathbf{E}^X\bullet\mathbf{E}^X & \mathbf{E}^X\bullet\mathbf{E}^Y \\ \mathbf{E}^Y\bullet\mathbf{E}^Y & \mathbf{E}^Y\bullet\mathbf{E}^Y \end{pmatrix}$$

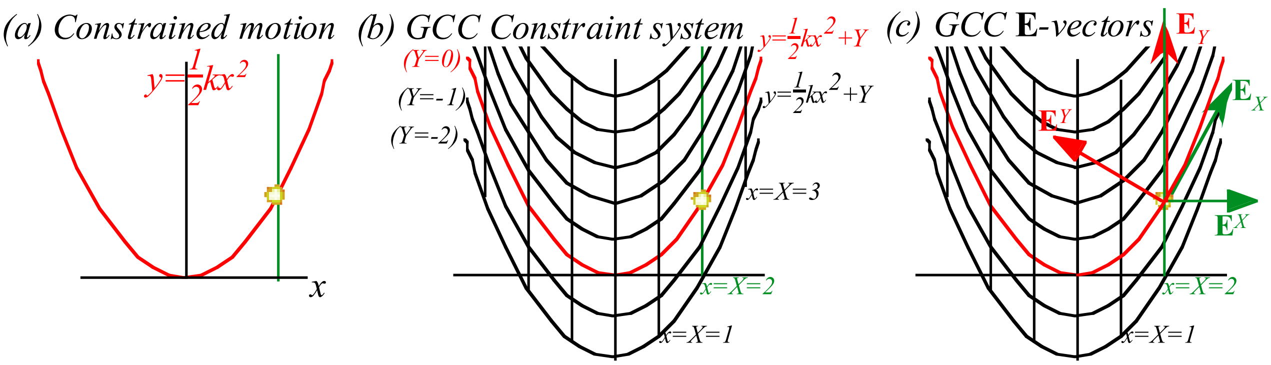


Fig. 3.9.1 Constraints (a) Constraining curve (b) GCC system (c) GCC unitary vectors

The Lagrangian L is the difference $L=T-V$ of kinetic (3.9.5a) and potential energy in (3.9.1b).

$$L = T - V = m\left[\tfrac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \tfrac{1}{2}\dot{Y}^2 - gY - \tfrac{gk}{2}X^2\right] \quad (3.9.6a)$$

Covariant momentum p_A is velocity $\dot{q}^A$ derivative $\frac{\partial L}{\partial \dot{q}^A}$ of L. Force f_A is a coordinate q^A-derivative $\frac{\partial L}{\partial q^A} = \dot{p}_A$.

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix} \quad (3.9.6a) \qquad \begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} \quad (3.9.6b)$$

The latter expands into equations of motion for $X=x$ and $Y= y-\frac{1}{2}kx^2$.

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} \; m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} \quad (3.9.7)$$

These covariant (Lagrange) equations have no constraint so covariant forces are zero. $(F_X^{cov} = 0 = F_Y^{cov})$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \quad m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix} \quad (3.9.8a)$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \quad m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix} \quad (3.9.8b)$$

Clearing $(\ddot{X},\ddot{Y})$ by γ^{AB} gives contravariant (Riemann) equations with zero contra-force $F^A_{con} = \gamma^{AB} F^{cov}_B = 0$.

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix} \quad (3.9.9a)$$

Contra-force has mass divided out to give Newtonian acceleration of (3.9.2) that now appears in (3.9.9).

$$\ddot{x} = 0 = \ddot{X} \quad (3.9.10a) \qquad -g = \ddot{y} = \frac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right) = k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\left(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0\right) \quad (3.9.9b)$$

Recall that acceleration relations appearing in contravariant equations (3.7.7) lose the mass *m*, too.

Constraint force components are covariant F^{cov}_B

It is now shown that frictionless constraint forces must be *covariant* F^{cov}_B, while frictional or driving forces are *contravariant* F^A_{con} components as described below. Consider two geometries of any applied force **F**.

$$\mathbf{F} = F^{cov}_X \mathbf{E}^X + F^{cov}_Y \mathbf{E}^Y = F^{cov}_X \nabla X + F^{cov}_Y \nabla Y \quad (3.9.10a)$$

$$\mathbf{F} = F^X_{con}\mathbf{E}_X + F^Y_{con}\mathbf{E}_Y = F^X_{con}\frac{\partial \mathbf{r}}{\partial X} + F^Y_{con}\frac{\partial \mathbf{r}}{\partial Y} \quad (3.9.10b)$$

A frictionless constraint of mass *m* to the parabola *Y=const.* must be normal to it along its gradient ∇Y.

$$\mathbf{F}(Y = const.) = F^{cov}_X \mathbf{E}^X + F^{cov}_Y \mathbf{E}^Y = 0 \cdot \nabla X + F^{cov}_Y \nabla Y = 0 \cdot \mathbf{E}^X + F^{cov}_Y \mathbf{E}^Y \quad (3.9.11)$$

So constraint requirements in covariant equations (3.9.8b) are $\dot{Y} = 0 = \ddot{Y}$ and $F^{cov}_X = 0$.(but $F^{cov}_Y \neq 0$).

$$m\begin{pmatrix} (1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ F^{cov}_Y \end{pmatrix} \quad (3.9.12a)$$

This nicely gives an *X*-acceleration like (3.9.2) and a covariant constraint force **F** to cause it.

$$\ddot{X} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX \quad (3.9.12b)$$

$$\mathbf{F} = F^{cov}_Y \mathbf{E}^Y = m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix} -kX \\ 1 \end{pmatrix} = m(k\dot{X}^2 + g)\frac{1}{1+k^2X^2}\begin{pmatrix} -kX \\ 1 \end{pmatrix} \quad (3.9.12c)$$

It is reassuring that constraint **F** boils down to $\begin{pmatrix} 0 \\ mg \end{pmatrix}$ when *m* is at equilibrium . $(X = 0 = \dot{X})$ It also gives the extra centripetal constraint force $mk\dot{X}^2$ as the mass bottoms out while speeding by *X=0*.

Frictional force components are contravariant F^A_{con}

The contra equations (3.9.9b) can be just that. They describe contra-motion if tangential friction is at play.

$$\begin{pmatrix} \ddot{X}+0 \\ \ddot{Y}+k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

A contra-component F^A_{con} multiplies tangent co-vector $\mathbf{E}_A = \frac{\partial \mathbf{r}}{\partial A}$ along which mass m goes if only coordinate A advances. Fixed X has $F^X_{con} = 0$ so Y obeys $\ddot{Y} + g = F^Y_{con}$. Fixed Y makes $F^X_{con} = \ddot{X}$ so X obeys $k\dot{X}^2 + g = F^Y_{con}$.

Suppose a coefficient of friction μ_Y adds to F^Y_{con} a drag $F^Y_{drag} = -\mu_Y |\mathbf{F}|$ proportional to the normal force (3.9.12c). Then a drag term $F^{drag}_X = \gamma_{XY} F^Y_{drag} = -mkX\mu_Y |\mathbf{F}|$ is added to $F^{cov}_X = 0$ in (3.9.12a).

Parabolic OCC approach

Elegant treatments of parabolic systems use parabolic OCC (orthogonal curvilinear coordinates) based on the complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$. Two expansions, one of z and one of absolute square z^2 give relations between Cartesian (x,y) and OCC(u,v).

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \tag{3.9.14a}$$

$$r^2 = z^* z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2 \tag{3.9.14b}$$

These relations give simple equations for orthogonal intertwining parabolas shown in Fig. 3.9.2.

$$\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \\ r &= u^2 + v^2 \end{aligned} \quad (3.9.15a) \qquad \begin{aligned} 2u^2 &= \sqrt{x^2+y^2} + x = r + x \\ 2v^2 &= \sqrt{x^2+y^2} - x = r - x \end{aligned} \quad (3.9.15b) \qquad \begin{aligned} y^2 &= 4u^2v^2 = 4u^2(u^2 - x) \\ y^2 &= 4u^2v^2 = 4v^2(v^2 + x) \end{aligned} \quad (3.9.15c)$$

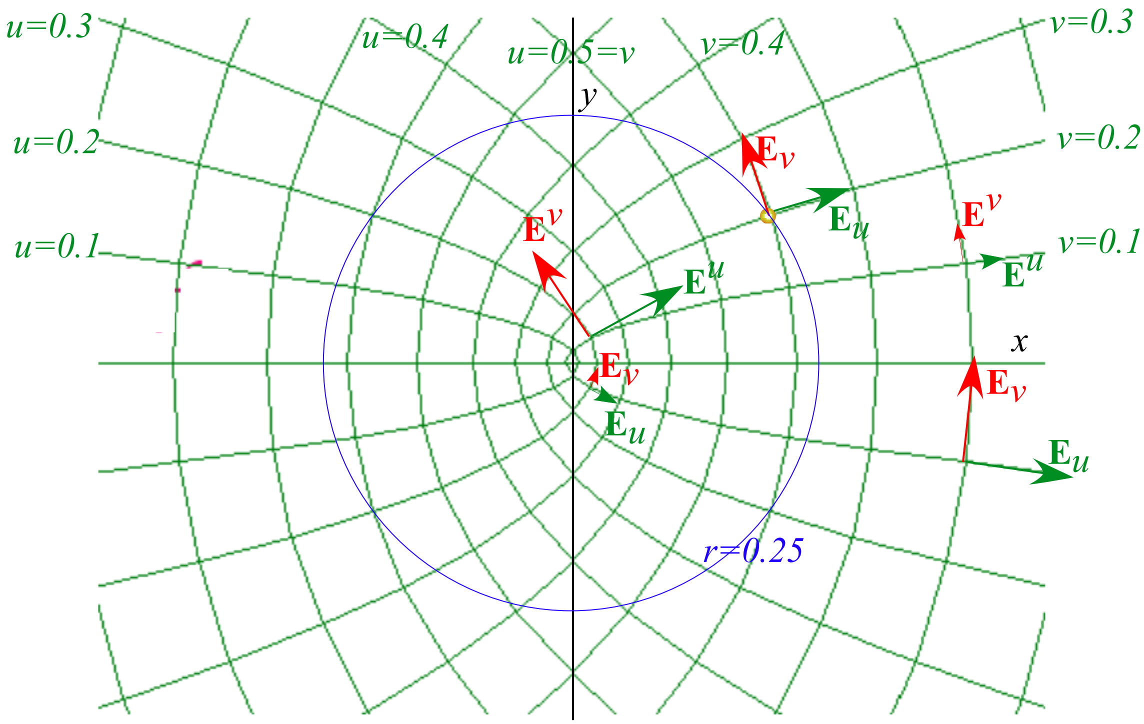


Fig. 3.9.2 Parabolic coordinates and OCC unitary vectors

Covariant *($\mathbf{E}_u$,$\mathbf{E}_v$)* and contra *($\mathbf{E}^u$,$\mathbf{E}^v$)* differ in length only, the former grow by $\sqrt{r}$ as contras shrink by $1/\sqrt{r}$.

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix} \text{(3.9.16a)} \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \tfrac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix} \text{(3.9.16b)}$$

Metric $g_{uv}=\mathbf{E}_u\bullet\mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V \tag{3.9.17a}$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V \tag{3.9.17b}$$

For a *Stark-Coulomb potential* $V=\varepsilon x + k/r$ Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \tfrac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \quad = \quad h_u \quad + \quad h_v \tag{3.9.18}$$

A zero Stark-field *(ε=0)* gives h_u or h_v harmonic oscillation if $E<0$. It is unstable or anharmonic otherwise.

$$\dot{p}_u = -\frac{\partial h_u}{\partial u} = -8Eu + 16\varepsilon u^3 \qquad \dot{u} = \frac{\partial h_u}{\partial p_u} = p_u/m \qquad \dot{p}_v = -\frac{\partial h_v}{\partial v} = -8Ev - 16\varepsilon v^3 \qquad \dot{v} = \frac{\partial h_v}{\partial p_v} = p_v/m$$

The resulting orbits and scattering trajectories will be studied in Unit 5.

To apply OCC-parabolic coordinates *(u,v)* to the parabolic constraint problem in the preceding section we set the Stark ε_x term to $mgx=mg(u^2-v^2)$ and drop the Coulomb $1/(u^2+v^2)$ term. (Cartesian x and y are switched.) To constrain the mass m to a parabola amounts to setting either $u=const.$ or else $v=const.$, depending on the direction of parabola. Equations of motion then follow (3.9.18) with $\dot{v}=0=\ddot{v}$ or else $\dot{u}=0=\ddot{u}$, respectively. It may seem that using OCC is a rather overblown attack for that problem.

Lagrange multiplier approaches

A usually convenient way to add constraints without introducing new GCC or OCC manifolds involves the *Lagrange multiplier* or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ in (3.9.1) is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient ∇c^1. ($\mathbf{F} \propto \nabla c^1$)

$$\mathbf{F} = \lambda\nabla c^1 = \lambda\nabla(\tfrac{1}{2}kx^2 - y) = \lambda\begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda\begin{pmatrix} kx \\ -1 \end{pmatrix} \tag{3.9.20}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*. It is like a covariant constraint component F_1^c of a contravariant $\mathbf{E}^1 = \nabla c^1$ vector that arises if $c^1(x,y)=const.$ was a coordinate line causing a force $\mathbf{F} = F_1^c\nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force $\mathbf{F}$ to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g}$.

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = -\begin{pmatrix} 0 \\ mg \end{pmatrix} \text{ with constraint } \mathbf{F} = F_1^c \nabla c^1 \text{ becomes: } \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \tag{3.9.21}$$

Constraint function $y=\frac{1}{2}kx^2$ gives derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$ from which we get multiplier λ.

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \text{ gives } \textit{Lagrange multiplier}: \lambda = m(-k\dot{x}^2 - kx\ddot{x} - g) \tag{3.9.22}$$

Then the λ function gives the new constrained x-equation of motion.

$$\begin{aligned} m\ddot{x} = \lambda kx &= -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx) \\ (1 + k^2x^2)\ddot{x} &= (-k\dot{x}^2 - g)kx \end{aligned} \tag{3.9.23}$$

This agrees with (3.9.2) and (3.9.12b) while (3.9.22) agrees with the constraint in (3.9.12c).

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines. It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using the chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \text{ where: } \frac{\partial c}{\partial q^k} = \frac{\partial}{\partial q^k}c = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k \bullet \nabla c \tag{3.9.24}$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad (3.9.25a) \qquad\qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k} \qquad (3.9.25b)$$

Two or more constraints $c^1(q^k) = const.,\ c^2(q^k) = const.,\cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad (3.9.25c) \qquad\qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda_\gamma\frac{\partial c^\gamma}{\partial q^k} \qquad (3.9.25d)$$

In three dimensions, two constraints would determine curve(s) by their intersection(s) thereby reducing three degrees of freedom to a single degree of freedom along the curve(s). Two constraints in a two-degree system determines point(s) of intersection so that no motion is possible. (That's a showstopper!)

Constraints may be determined by differential relations that are not integrable. Since the Lagrange method uses differentials anyway, lack of integral surface functions c^γ is no obstacle. A comparison of the integral constraint relations with the more general relations is shown below.

<u>*Integral constraint differentials*</u> <u>*General differential constraint relations*</u>

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1}dq^1 + \frac{\partial c^1}{\partial q^2}dq^2 + \ldots \qquad 0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1}dq^1 + \frac{\partial c^2}{\partial q^2}dq^2 + \ldots \qquad 0 = C^2_1 dq^1 + C^2_2 dq^2 + \ldots \tag{3.9.25e}$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots$$

Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots \qquad \dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 C^1_1 + \lambda_2 C^2_1 + \ldots$$

$$\dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 \frac{\partial c^1}{\partial q^2} + \lambda_2 \frac{\partial c^2}{\partial q^2} + \ldots \qquad \dot{p}_2 - \frac{\partial L}{\partial q^2} = \lambda_1 C^1_2 + \lambda_2 C^2_2 + \ldots \tag{3.9.25f}$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots$$

If a general differential cannot be integrated to give an actual constraint function it is called a *non-holonomic constraint*. I guess that means that integrable ones are holonomic, but it escapes me why we need the longer words. A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j} \tag{3.9.26}$$

The force components $F^\gamma_k = \frac{\partial c^\gamma}{\partial q^k} = C^\gamma_k$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

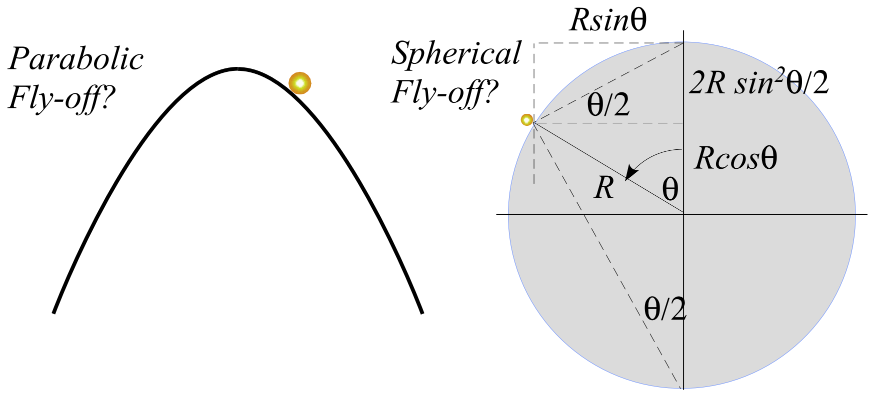
$$\frac{\partial F^\gamma_k}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F^\gamma_j}{\partial q^k} \qquad \frac{\partial C^\gamma_k}{\partial q^j} \;\begin{matrix}\textit{may or}\\ \textit{may not be}\end{matrix}\; \frac{\partial C^\gamma_j}{\partial q^k}$$

Recall that a conservative force $\mathbf{F} = -\nabla V$ is one of zero curl $\nabla \times \mathbf{F} = \mathbf{0}$ and satisfies a similar reciprocity.

$$\frac{\partial F_k}{\partial x^j} = -\frac{\partial^2 V}{\partial x^j \partial x^k} = \frac{\partial F_j}{\partial x^k}$$

Its closed path integrals $\oint \mathbf{F} \bullet d\mathbf{r}$ are zero or *conservative* and work integrals $\int_1^2 \mathbf{F} \bullet d\mathbf{r}$ are path independent.

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \bullet d\mathbf{r} = -V\Big|_{\mathbf{r}_1}^{\mathbf{r}_2} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \qquad \oint \mathbf{F} \bullet d\mathbf{r} = 0$$

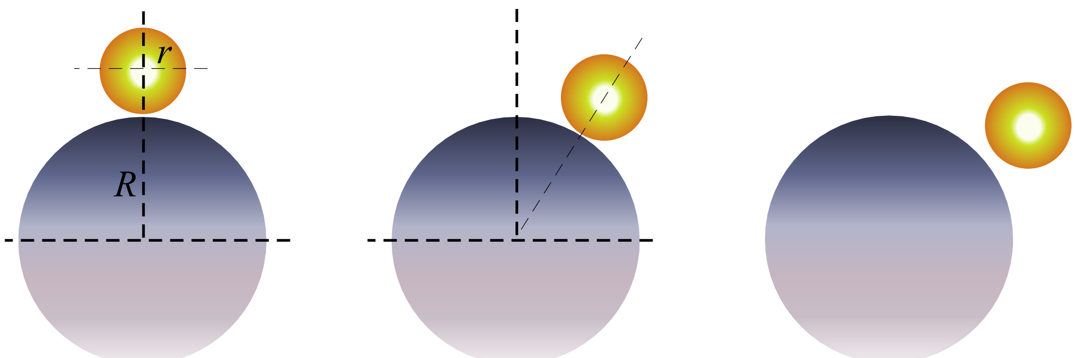


Exercise 3.9.1 Easy as sliding off a parabola

Elementary constraint problem of mass m sliding in a parabolic well is an anharmonic oscillator in Fig. 3.9.1. Consider how m sliding on inverted parabola might fall off under gravity $g=10m{\cdot}s^{-2}$.

(a) Suppose an inverted parabolic road $y=-\frac{1}{2}kx^2$ with m starting with near-zero $v(0)$ at $x=0$ on top. Show whether there are x_{fly}, y_{fly}, and v_{fly} values where the mass m would fly off the road. Analyze and discuss.

(b) Do a similar analysis for mass m sliding on a sphere of radius R. Compare to parabolic result of (a).



Exercise 3.9.1 Easy as rolling off a log

The mother-of-all-roll-off-the-log problems was on a 2004 qual-exam and worked correctly by a student but incorrectly by professor γ. (Fortunately, we caught γ 's error before scores were finalized.)

A ball of radius r and mass $m=1kg$ starting at the top of a fixed log of radius R and begins rolling down it.

Assuming the sphere rolls without slipping calculate the angle from vertical where it last contacts the log.

Let $R=20cm$ and $r=1cm$, and $g=10m{\cdot}s^{-2}$ but give algebraic answers first. Then try $R=1cm$ and $r=20cm$.

More difficult problem:

Assuming a coefficient of stiction is $\mu_S=2$ find the angle where it starts slipping.

Even more difficult problem:

Assuming the log of mass $M=10kg$ and length $L=5cm$ can rotate, too, answer each of the problems above.

References

Unit 3 Review Topics and Formulas

Deriving Lagrangian-Deriviative equations

Revised Cartesian Newton-(*f=Ma*)-equations of motion: (M_{jk} not function of x_j, $v_j = \dot{x}_j$, or time *t*.)

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

Revised kinetic energy $T=(^1/_2)Mv^2$:

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k \quad \text{(where } M_{jk}=M_{kj} \text{ are constants)}.$$

Chain rule (3.3.2) converted to a velocity relation: (Brace terms zero in fixed GCC)

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m + \left\{ \frac{\partial x^j}{\partial t} dt \right\} \quad \text{or:} \quad \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \qquad (3.5.3) \qquad \textit{Lemma 1.}\ \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}$$

Acceleration relations including moving-GCC-{terms}.

$$\ddot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m + \frac{d}{dt}\left\{ \frac{\partial x^j}{\partial t} \right\}, \qquad (3.5.5a)$$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right) = \frac{\partial^2 x^j}{\partial q^m \partial q^n} \dot{q}^m + \left\{ \frac{\partial^2 x^j}{\partial q^m \partial t} \right\} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) (3.5.5b) \qquad \textit{Lemma 2.}\ \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m}$$

Work $dW = F_j dx^j$ allows arbitrary coordinate changes dq^m and intervals *dt* of time.

$$dW = f_j dx^j = f_j \left(\frac{\partial x^j}{\partial q^m} dq^m + \left\{ \frac{\partial x^j}{\partial t} dt \right\} \right) = M_{jk}\, \ddot{x}^k \left(\frac{\partial x^j}{\partial q^m} dq^m + \left\{ \frac{\partial x^j}{\partial t} dt \right\} \right) \qquad (3.5.7)$$

So *m*-Sum is true term-by-term for covariant GCC force $F_m = f_j \frac{\partial x^j}{\partial q^m}$ with $A = M_{jk}\, \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$.

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\, \ddot{x}^k \frac{\partial x^j}{\partial q^m} \left[= \ddot{A}B = \frac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B} \right] \qquad (3.5.8)$$

$$F_m = M_{jk} \frac{d}{dt}\left(\dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right), \qquad (3.5.9a)$$

Use *Lemma 1* and *Lemma 2*: $F_m = M_{jk} \frac{d}{dt}\left(\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \right) - M_{jk} \dot{x}^k \left(\frac{\partial \dot{x}^j}{\partial q^m} \right)$, (3.5.9b)

Use: $M_{jk} v^j \frac{\partial v^k}{\partial q} = \frac{\partial}{\partial q}\left(\frac{M_{jk}}{2} v^j v^k \right) \Rightarrow F_m = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^m}\left(\frac{M_{jk}}{2} \dot{x}^k \dot{x}^j \right) - \frac{\partial}{\partial q^m}\left(\frac{M_{jk}}{2} \dot{x}^k \dot{x}^j \right) = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$ (3.5.10)

Relating Lagrangian-Deriviative equations to Reimann equations (fixed GCC only) Given *LD* eqations:

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial\left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial\left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell} \text{ (3.6.2) and GCC metric: } \gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad (3.6.1)$$

where: $$\frac{\partial\left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} = \gamma_{mn} \dot{q}^n \frac{\partial \dot{q}^m}{\partial \dot{q}^\ell} + \gamma_{mn} \dot{q}^m \frac{\partial \dot{q}^n}{\partial \dot{q}^\ell} = \gamma_{mn} \dot{q}^n \delta^m_\ell + \gamma_{mn} \dot{q}^m \delta^n_\ell = \left(\gamma_{\ell n} + \gamma_{n\ell} \right) \dot{q}^n = 2\gamma_{\ell n} \dot{q}^n$$

Results: *canonical momentum* $p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$ and *contravariant* $\dot{q}^n = p_\ell \gamma^{\ell n} = p^n$

LD equations *(for fixed GCC only)*:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$
$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n \quad , \qquad \text{where: } \frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n , \quad (3.6.2)$$

Results: *covariant Riemann equations* with *kinetic Christoffel coefficients* $\Gamma_{mn;\ell}$ *of 1st kind*.

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n \qquad \Gamma_{mn;\ell} \equiv \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right] \qquad (3.6.10ab)$$

Numerics use *contravariant Riemann equations* with *Christoffel coefficients* $\Gamma^k_{mn} = \gamma^{k\ell}\Gamma_{mn;\ell}$ *of 2nd kind*.

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n \qquad \Gamma^k_{mn} = \gamma^{k\ell}\Gamma_{mn;\ell}$$, with *contravariant force*: $F^k = \gamma^{k\ell}F_\ell = \gamma^{k\ell}\frac{\partial x^j}{\partial q^\ell}f_j$

Geometric basis of Christoffel coefficients and Covariant deivative

U-derivative has two parts: one for changing **U** components and another for curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(U^j\mathbf{E_j}\right) = \frac{\partial U^m}{\partial q^i}\left(\mathbf{E_m}\right) + U^n\frac{\partial \mathbf{E_n}}{\partial q^i} \qquad (3.4.1)$$

Curving $\mathbf{E_n}$ expansion: $\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\mathbf{E}^\ell = \Gamma^m_{in}\mathbf{E}_m$ involves *Christoffel coefficients* $\Gamma_{in;m}$ *or* $\Gamma_{in}{}^m$

1st kind: $\Gamma_{in;m} = \frac{\partial \mathbf{E_n}}{\partial q^i}\bullet\mathbf{E_m} = \Gamma_{ni;m}$ *or equivalently:* *2nd kind*: $\Gamma^m_{in} = \frac{\partial \mathbf{E_n}}{\partial q^i}\bullet\mathbf{E^m} = \Gamma^m_{ni}$

Curving $\mathbf{E}^m$ expansion: $\frac{\partial \mathbf{E^n}}{\partial q^i} = \Lambda^n_{im}\mathbf{E^m}$ involves $\Lambda^n_{im} = \frac{\partial \mathbf{E^n}}{\partial q^i}\bullet\mathbf{E_m}$ coefficients.

Orthonormality $\mathbf{E^n}\bullet\mathbf{E_m} = \delta^n_m$ is constant so Λ-coefficients are just minus Γ-coefficients.

$$0 = \frac{\partial\left(\delta^n_m\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E^n}\bullet\mathbf{E_m}\right)}{\partial q^i} = \frac{\partial \mathbf{E^n}}{\partial q^i}\bullet\mathbf{E_m} + \mathbf{E^n}\bullet\frac{\partial \mathbf{E_m}}{\partial q^i} = \Lambda^n_{im} + \Gamma^n_{im} \quad \Rightarrow \quad \Lambda^n_{im} = -\Gamma^n_{im}$$

Covariant derivative $U^m{}_{;i}$ *of a contravariant* U^m or equivalently $U_{m;i}$ *of a covariant* U_m is defined by

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n\Gamma^m_{in}\right)\mathbf{E_m} = \left(\frac{\partial U_m}{\partial q^i} - U_n\Gamma^n_{im}\right)\mathbf{E^m} = U^m_{;i}\,\mathbf{E_m} = U_{m;i}\,\mathbf{E^m} \qquad (3.4.5a)$$

where: $U^m_{;i} = \frac{\partial U^m}{\partial q^i} + U^n\Gamma^m_{in}$, *or equivalently:* $U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n\Gamma_{im}$ (3.4.5bc)