Lecture 11
Wed. 9.26.2018

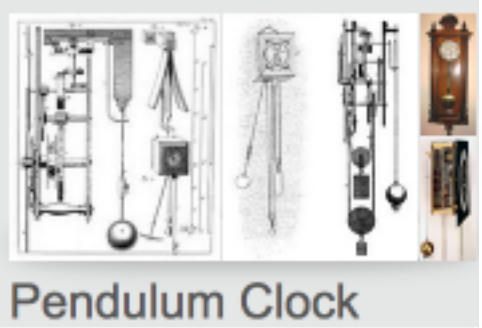
Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)
- *Clues for future assignments (Web Simulation: CoulIt)*

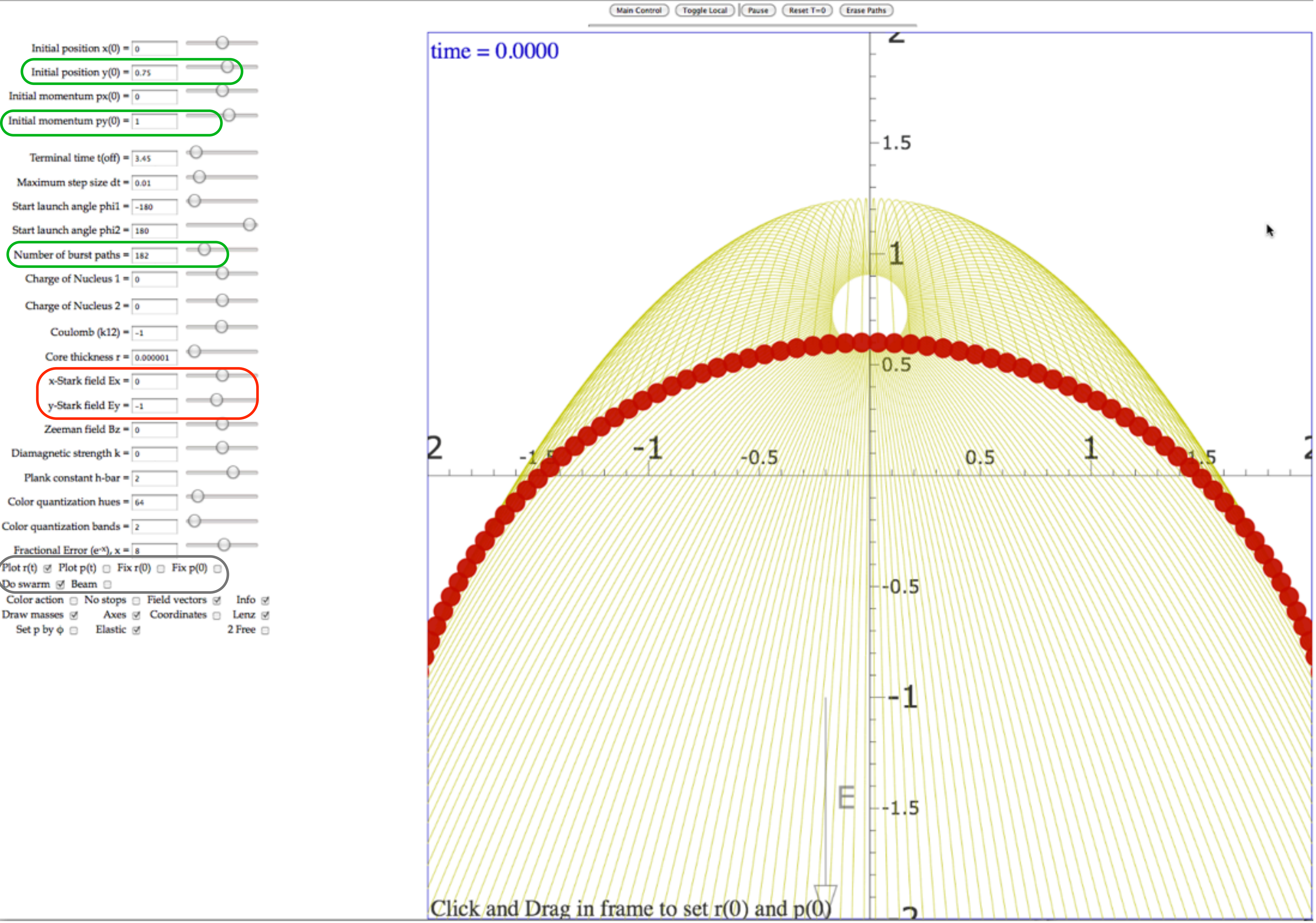
Examples of Hamiltonian mechanics in phase plots
- *1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*
- *1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")*

Exploring phase space and Lagrangian mechanics more deeply
- *A weird "derivation" of Lagrange's equations*
- *Poincare identity and Action,* Jacobi-Hamilton *equations*
- *How Classicists might have "derived" quantum equations*
- *Huygen's contact transformations enforce minimum action*
- *How to do quantum mechanics if you only know classical mechanics*
- *("Color-Quantization" simulations:* Davis-Heller "Color-Quantization" or "Classical Chromodynamics"*)*




Christaan Huygens (1629-1695)

➡ *Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)*
Some clues for future assignments *(Web Simulation: CouIIt)*

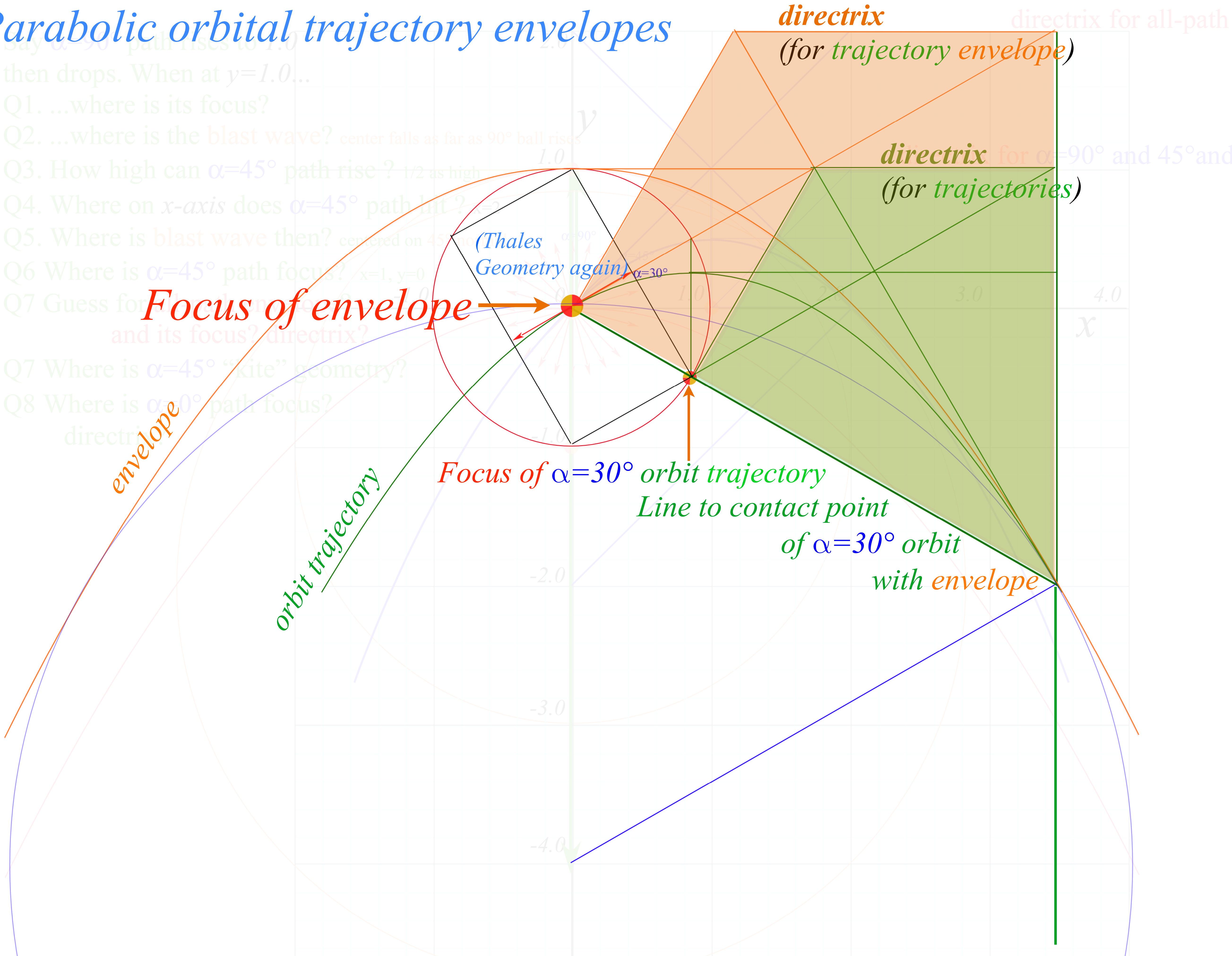


Initial position x(0) = 0
Initial position y(0) = 0.75
Initial momentum px(0) = 0
Initial momentum py(0) = 1
Terminal time t(off) = 3.45
Maximum step size dt = 0.01
Start launch angle phi1 = -180
Start launch angle phi2 = 180
Number of burst paths = 182
Charge of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 0.000001
x-Stark field Ex = 0
y-Stark field Ey = -1
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 64
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8

Plot r(t) ☑ Plot p(t) ☐ Fix r(0) ☐ Fix p(0) ☐
Do swarm ☑ Beam ☐
Color action ☐ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by φ ☐ Elastic ☑ 2 Free ☐

Link ⇒ CoulIt - Simulation of the *Volcanoes of Io*

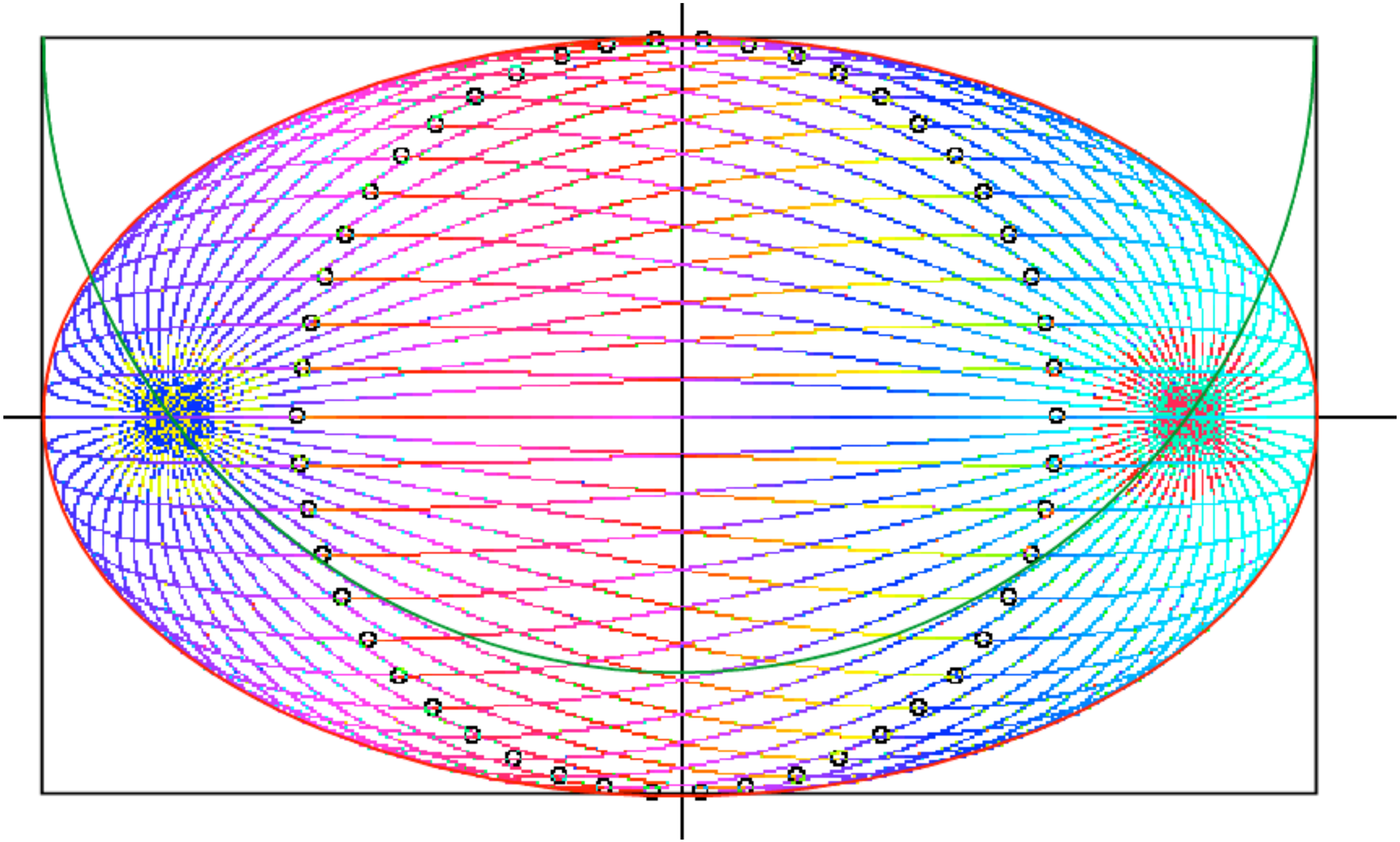
Parabolic orbital trajectory envelopes



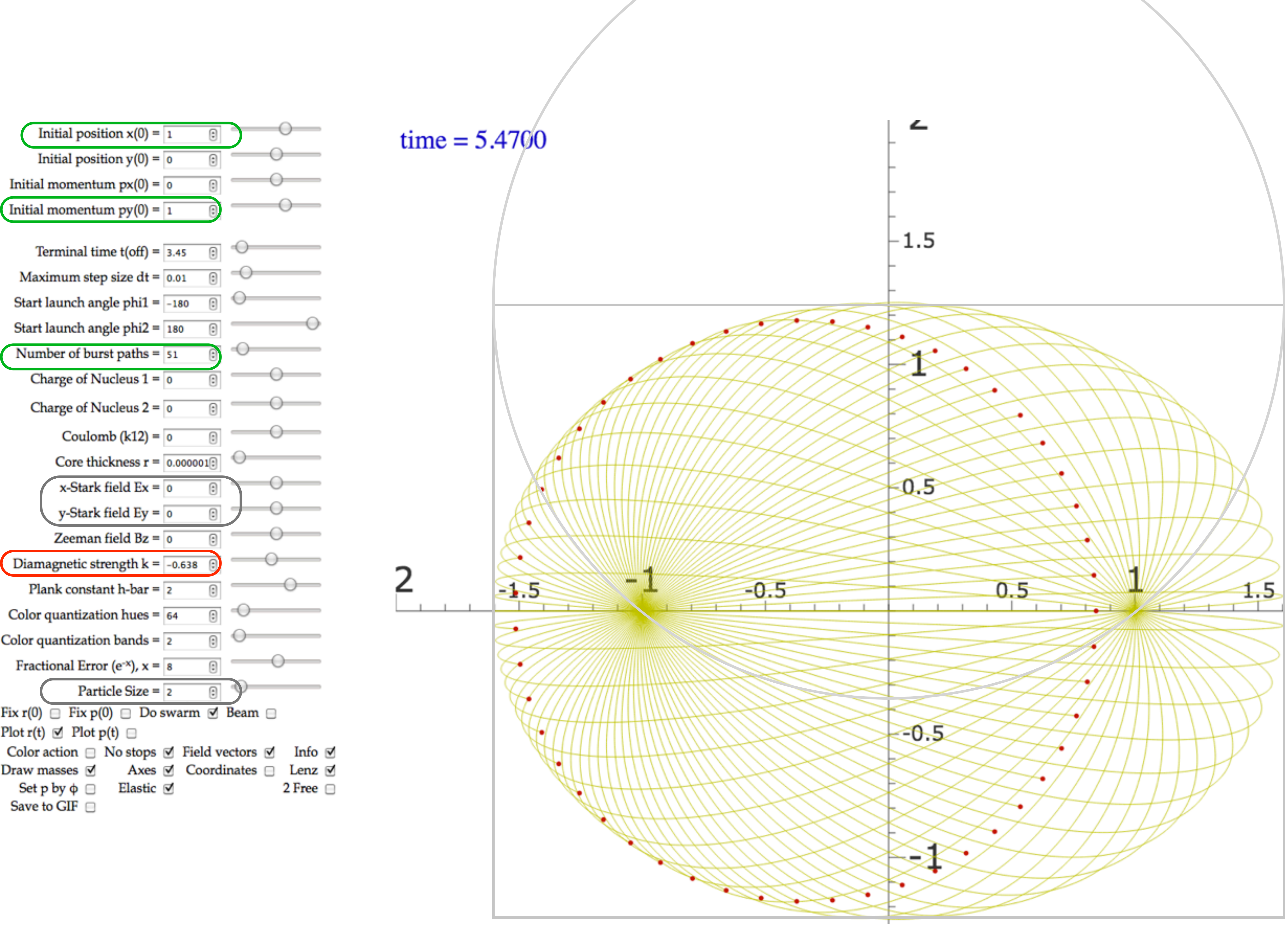
Parabolic and 2D-IHO orbital envelopes (Review of Lecture 9 p.56-81 and a generalization.)

Some clues for future assignments *(Web Simulation: CoulIt)*

Exploding-starlet elliptical envelope and contacting elliptical trajectories



*([Web Simulation: CouIIt - Exploding*Starlet]() {IHO Potential})*



Initial position x(0) = 1

Initial position y(0) = 0

Initial momentum px(0) = 0

Initial momentum py(0) = 1

Terminal time t(off) = 3.45

Maximum step size dt = 0.01

Start launch angle phi1 = -180

Start launch angle phi2 = 180

Number of burst paths = 51

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.000001

x-Stark field Ex = 0

y-Stark field Ey = 0

Zeeman field Bz = 0

Diamagnetic strength k = -0.638

Plank constant h-bar = 2

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 2

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☑ Beam ☐

Plot r(t) ☑ Plot p(t) ☐

Color action ☐ No stops ☑ Field vectors ☑ Info ☑

Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑

Set p by ϕ ☐ Elastic ☑ 2 Free ☐

Save to GIF ☐

(Web Simulation: CoulIt - Exploding*Starlet {IHO Potential})

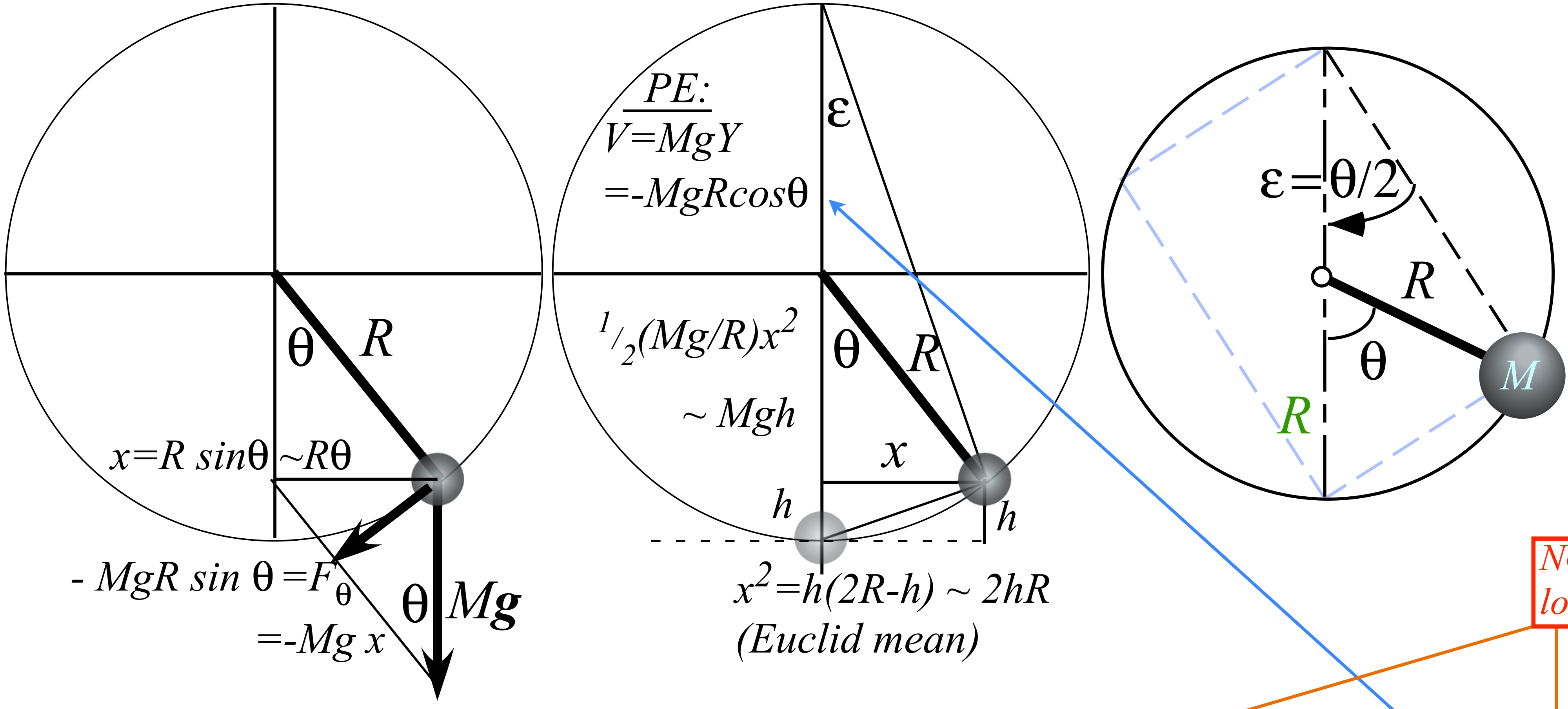
Examples of Hamiltonian mechanics in phase plots

- *1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*
  - *Circular pendulum dynamics and elliptic functions*
  - *Cycloid pendulum dynamics and "sawtooth" functions*
- *1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")*

1D Pendulum and phase plot

(Unit 2 Chapter 7 Fig. 2)

(a) Force geometry (b) Energy geometry (c) Time geometry

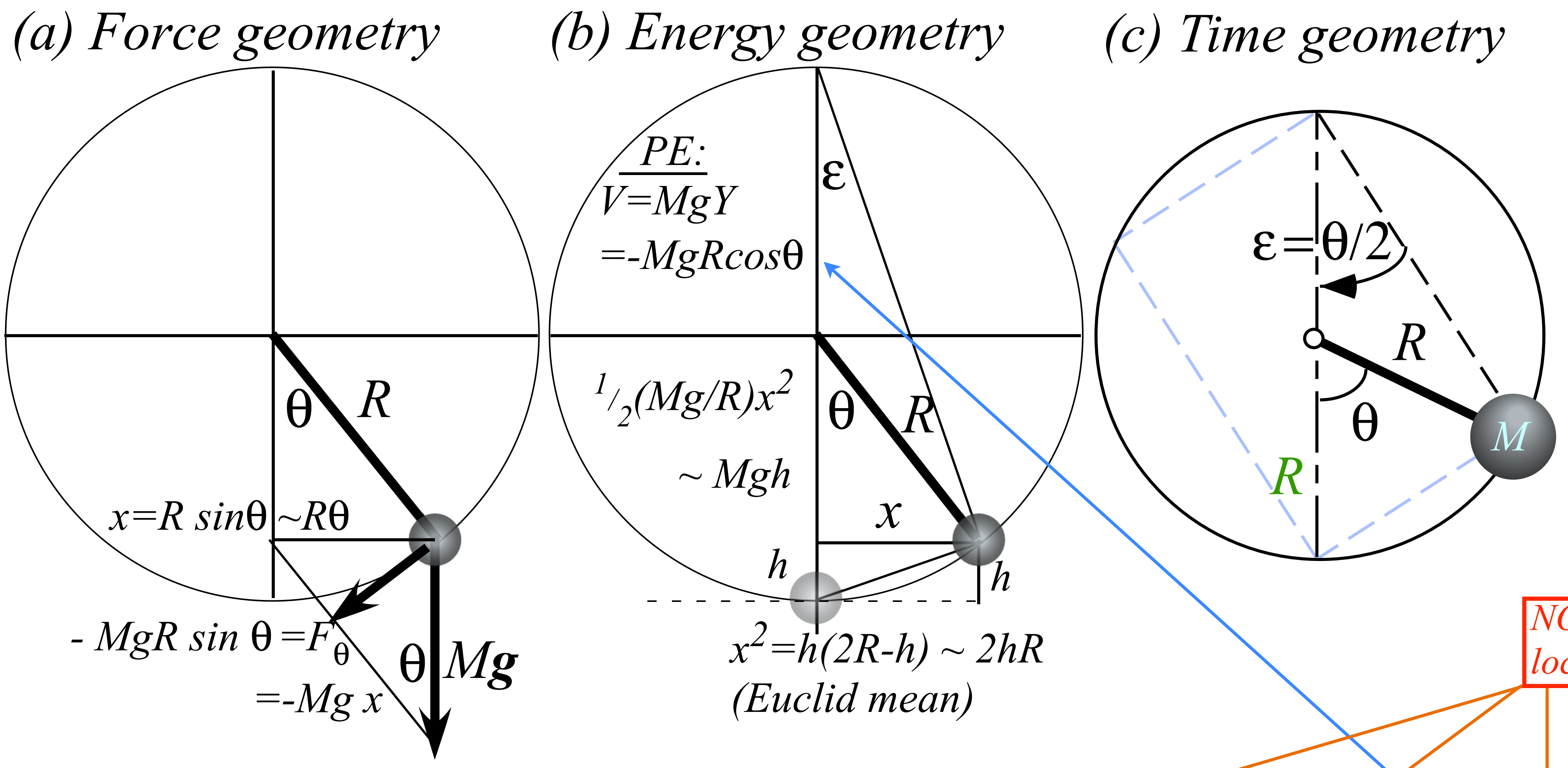


NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

1D Pendulum and phase plot



(a) Force geometry

(b) Energy geometry

(c) Time geometry

NOTE: Very common loci of ± sign blunders

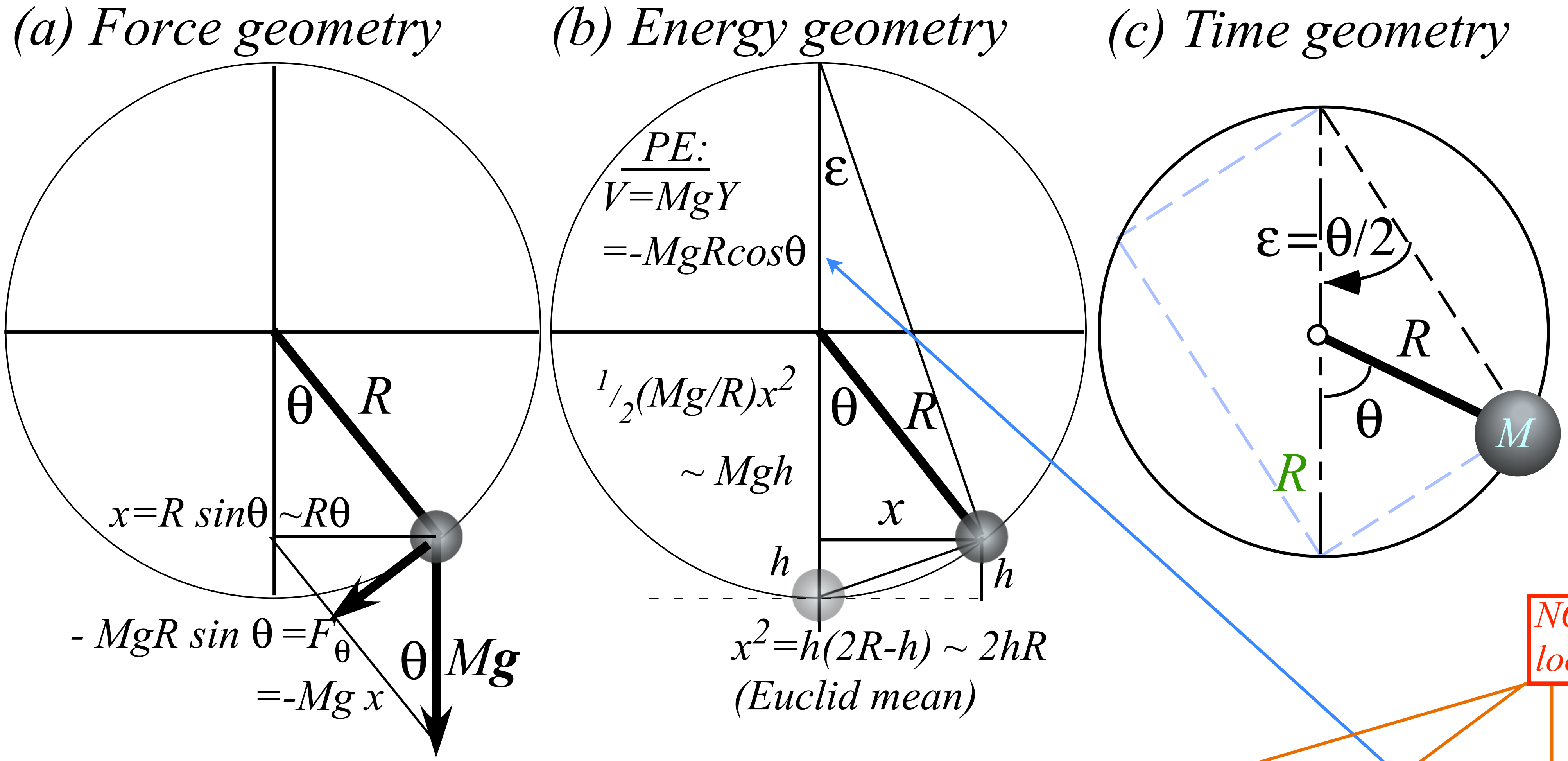
Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}{p_\theta}^2+U(\theta)=\frac{1}{2I}{p_\theta}^2-MgR\cos\theta\ =E\quad =\text{const.}$$

1D Pendulum and phase plot



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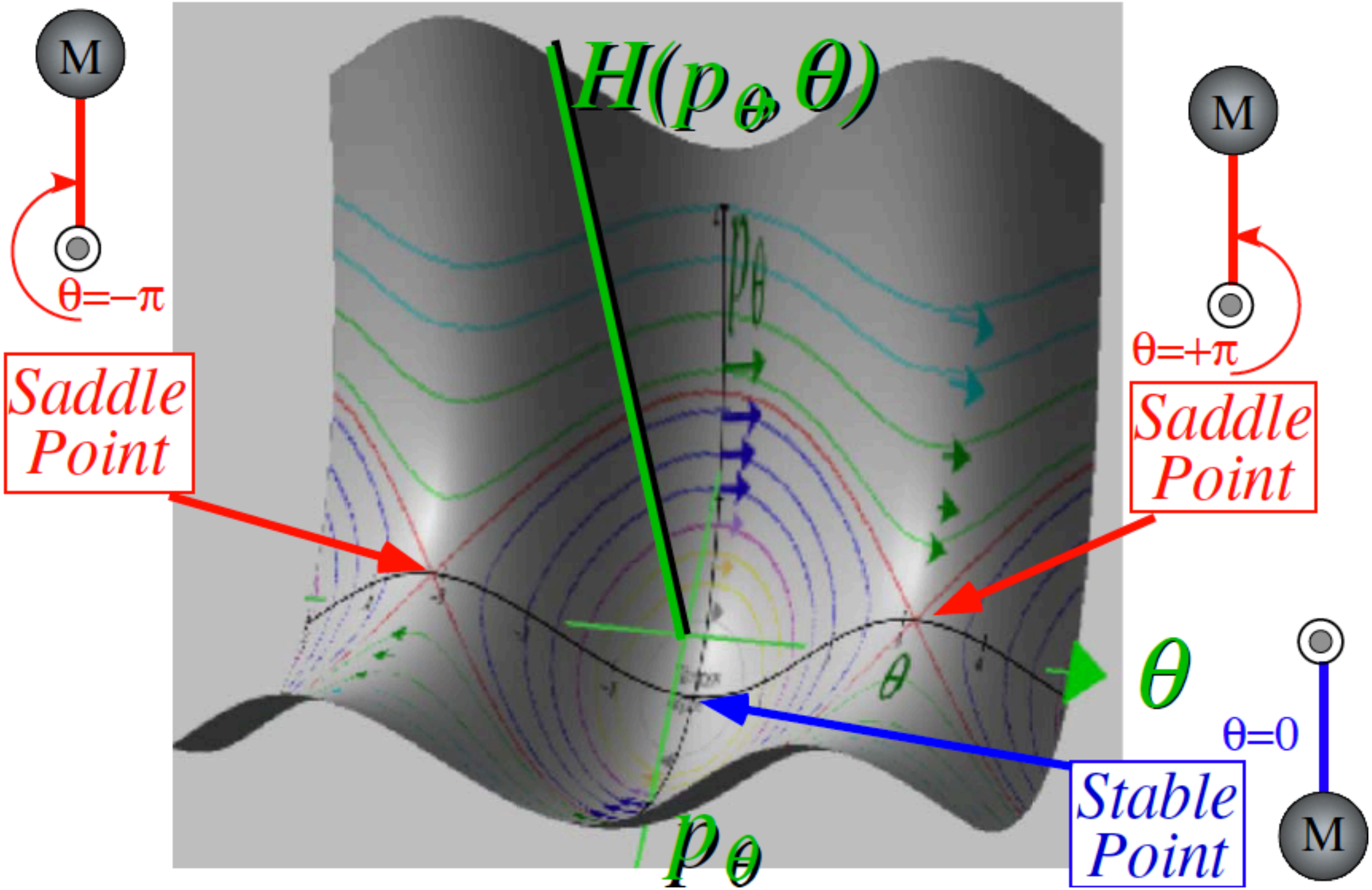
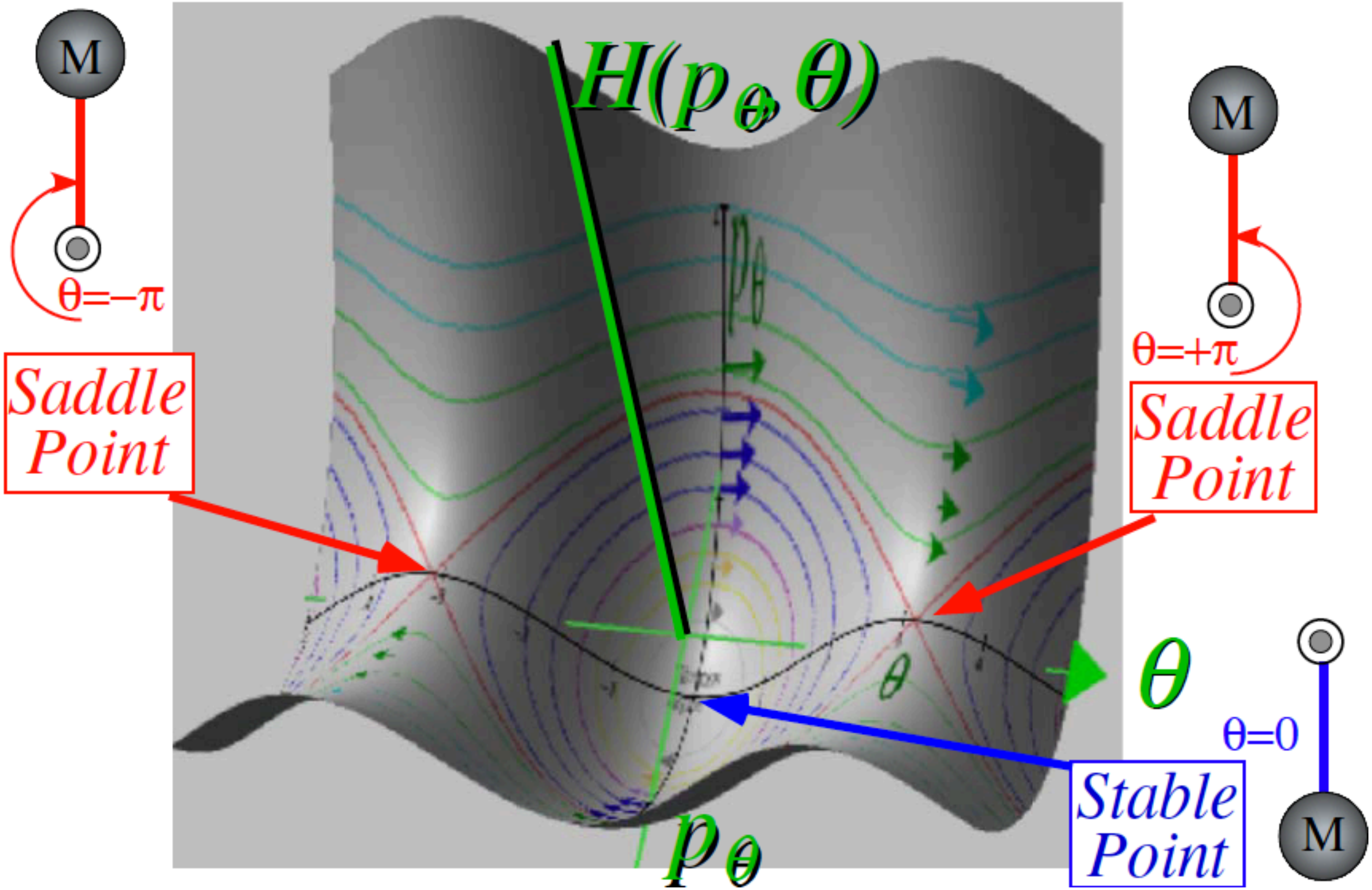
implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



(unstable
"balancing"
point)

Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta^{\,2} - MgR\cos\theta\ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$



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Funny way to look at Hamilton's equations:

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix} = \mathbf{e_H}\times\left(-\nabla H\right) = \overrightarrow{\text{(H-axis)}}\times\overrightarrow{\text{(fall line)}}, \text{ where:}\begin{cases}\overrightarrow{\text{(H-axis)}} = \mathbf{e_H} = \mathbf{e_q}\times\mathbf{e_p}\\ \overrightarrow{\text{(fall line)}} = -\nabla H\end{cases}$$

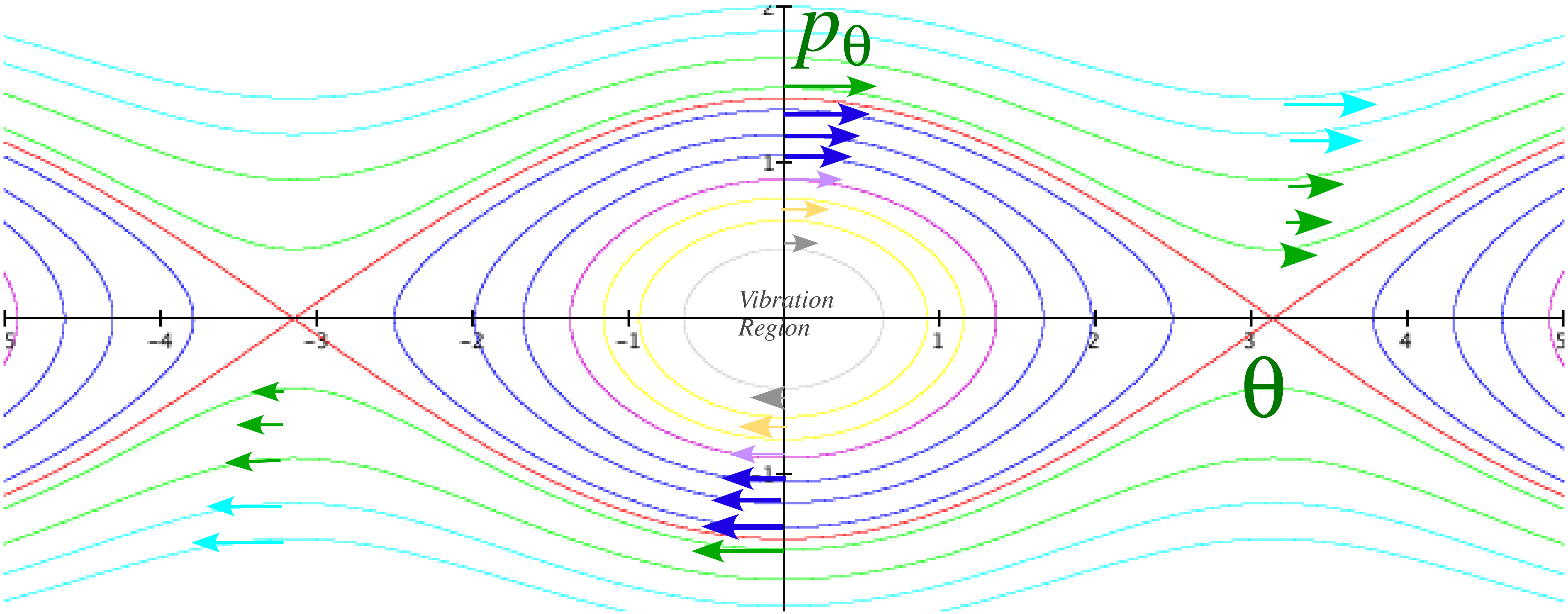

Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

Examples of Hamiltonian mechanics in phase plots

- *1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*
  - → *Circular pendulum dynamics and elliptic functions*
  - *Cycloid pendulum dynamics and “sawtooth” functions*
- *1D-HO phase-space control (Old Mac OS & Web Simulations of “Catcher in the Eye”)*

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$

Let $E = MgY = -MgR\cos\theta_0$ be potential energy where $KE = 0$ or $p_\theta = 0$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

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Let $E=MgY=-MgR\cos\theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I\left(E + MgR\cos\theta\right)} / I \quad \text{where: } I = MR^2$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H= KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta^{\,2}+U(\theta)=\frac{1}{2I}p_\theta^{\,2}-MgR\cos\theta \;= E \quad =\text{const.}$$

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Circular pendulum dynamics and elliptic functions

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Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\int_0^{\theta_0}dt=\left(\text{Travel time 0 to }\theta_0\right)=\tau_{1/4}$$

Circular pendulum dynamics and elliptic functions

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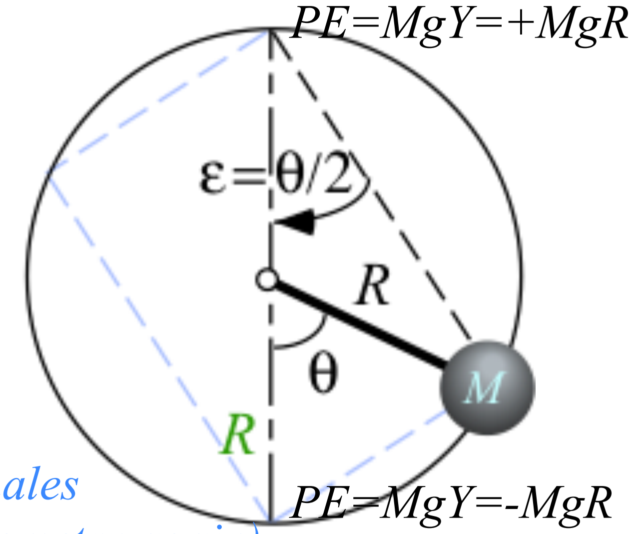
$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E+MgR\cos\theta)} / I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E+MgR\cos\theta)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\varepsilon,$$



(Thales Geometry again)

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) $= -MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}{p_\theta}^2+U(\theta)=\frac{1}{2I}{p_\theta}^2-MgR\cos\theta\ =E\quad =const.$$

implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$

Let $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

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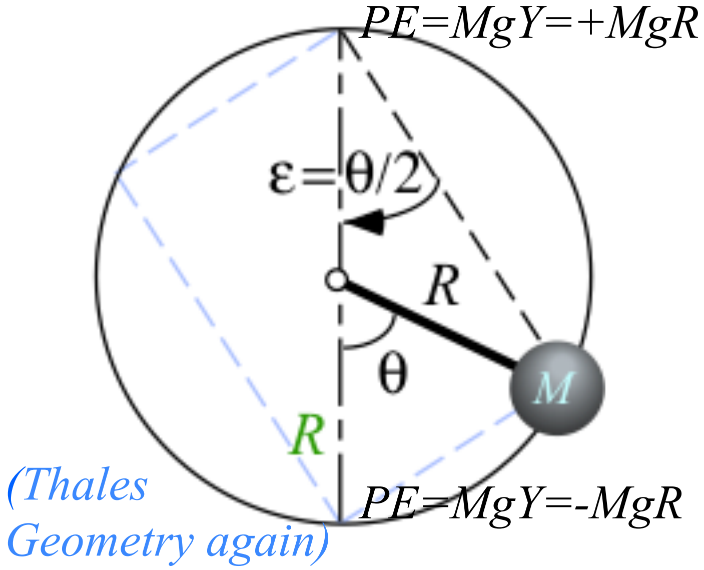
Quadrature integral gives quarter-period $\tau_{1/4}$*:*

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\int_0^{\theta_0}dt=\left(\text{Travel time 0 to }\theta_0\right)=\tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\sin^2\varepsilon,\qquad \cos\theta-\cos\theta_0=2\sin^2\varepsilon_0-2\sin^2\varepsilon$$



(Thales Geometry again)

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

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implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$

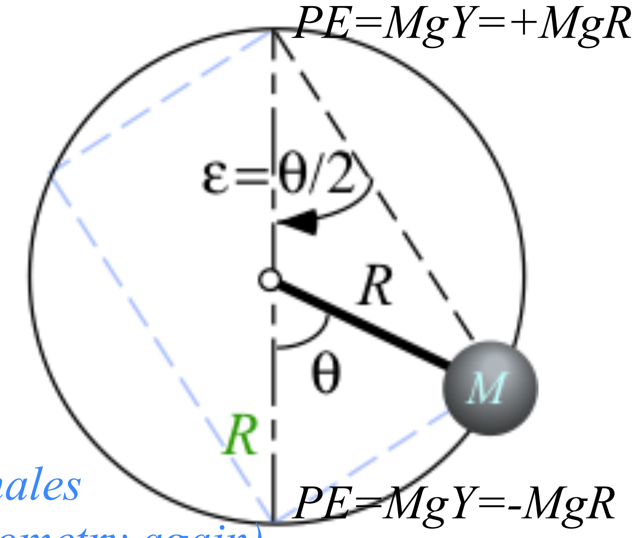
Let $E=MgY=-MgR\cos\theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

$E=MgY=-MgR\cos\theta_0$

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\frac{d\theta}{dt}=p_\theta/I=\sqrt{2I\left(E+MgR\cos\theta\right)}/I \quad \text{where: } I=MR^2 \quad \text{or}: dt=\frac{d\theta}{\sqrt{2\left(E+MgR\cos\theta\right)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\int_0^{\theta_0}dt=\left(\text{Travel time 0 to }\theta_0\right)=\tau_{1/4}$$



(Thales Geometry again)

Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\sin^2\varepsilon,\qquad \cos\theta-\cos\theta_0=2\sin^2\varepsilon_0-2\sin^2\varepsilon$$

$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta = E \quad =\text{const.}$$

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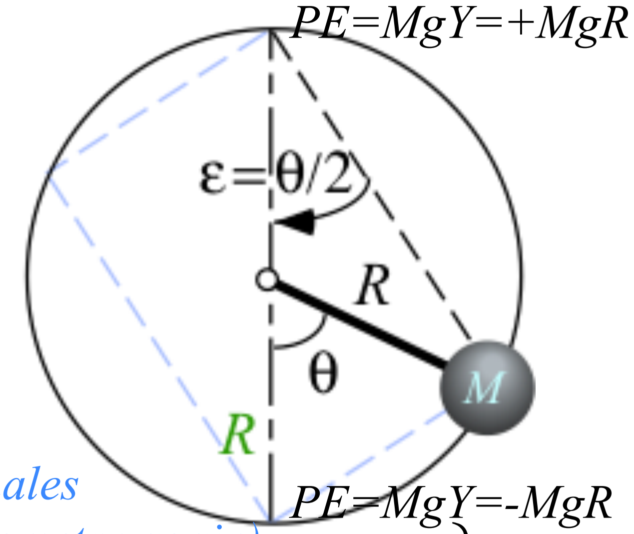

(Thales Geometry again)

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\int_0^{\theta_0}dt=\left(\text{Travel time 0 to }\theta_0\right)=\tau_{1/4}$$

Uses a half-angle coordinate $\varepsilon=\theta/2$

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$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{k\,d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\ \text{, where:}\left\{\begin{array}{c}1/k=\sin\varepsilon_0=\sin\dfrac{\theta_0}{2}\\ I=MR^2\end{array}\right\}$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta = E \quad =\text{const.}$$

implies: $p_\theta=\sqrt{2I(E+MgR\cos\theta)}$

Let $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

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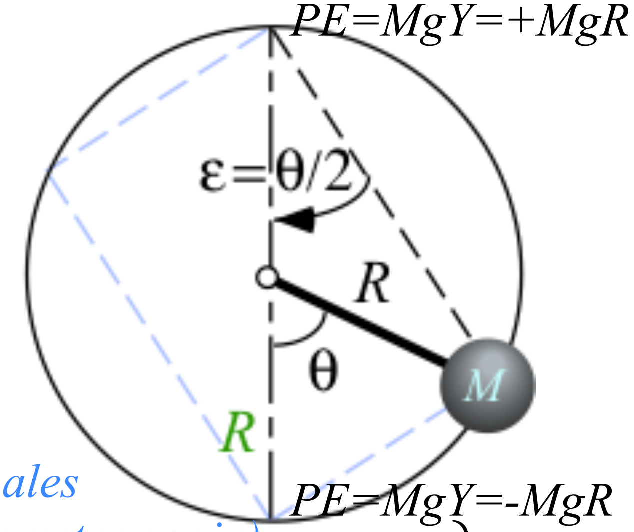

(Thales Geometry again)

$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{k\,d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}, \text{ where:}\left\{\begin{array}{c}1/k=\sin\varepsilon_0=\sin\dfrac{\theta_0}{2}\\ I=MR^2\end{array}\right\}$$

The integral is an *elliptic integral of the first kind:* $F(k,\varepsilon_0)=am^{-1}$ or the "inverse amu" function.

$$F(k,\varepsilon_0)\equiv\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\equiv am^{-1}(k,\varepsilon_0)$$

Circular pendulum dynamics and elliptic functions

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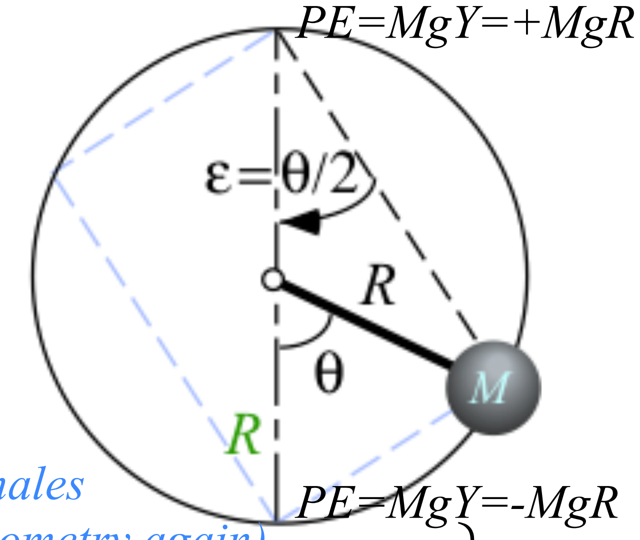

(Thales Geometry again)

$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{k\,d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\text{, where:}\left\{\begin{array}{c}1/k=\sin\varepsilon_0=\sin\dfrac{\theta_0}{2}\\ I=MR^2\end{array}\right\}$$

The integral is an *elliptic integral of the first kind:* $F(k,\varepsilon_0)=am^{-1}$ or the "inverse amu" function.

$$F(k,\varepsilon_0)\equiv\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\equiv am^{-1}(k,\varepsilon_0)$$

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For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ *where potential energy is* $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}{p_\theta}^2 + U(\theta) = \frac{1}{2I}{p_\theta}^2 - MgR\cos\theta = E \quad = const.$$

implies: $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

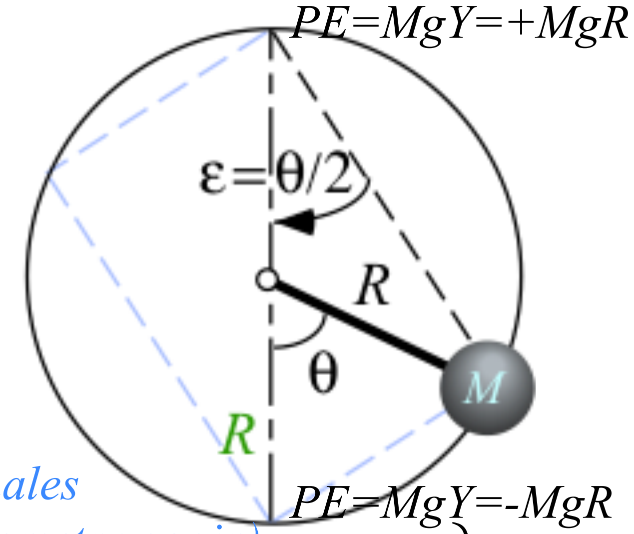
Let $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

$E=MgY=-MgR\cos\theta_0$

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E+MgR\cos\theta)}/I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E+MgR\cos\theta)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$



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The integral is an *elliptic integral of the first kind:* $F(k,\varepsilon_0)=am^{-1}$ or the "inverse amu" function.

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low $\varepsilon \ll 1$: $t = \sqrt{\frac{R}{g}}\int_0^{\varepsilon(t)}\frac{d\varepsilon}{\sqrt{\varepsilon_0^2-\varepsilon^2}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon}{\varepsilon_0}\Big|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon(t)}{\varepsilon_0}$

For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

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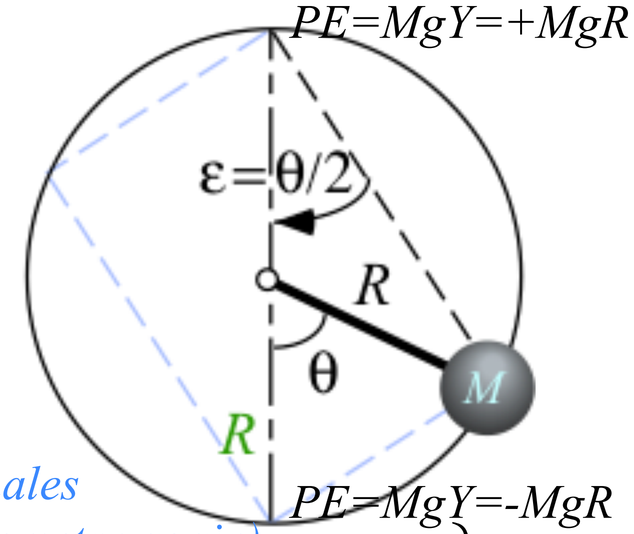

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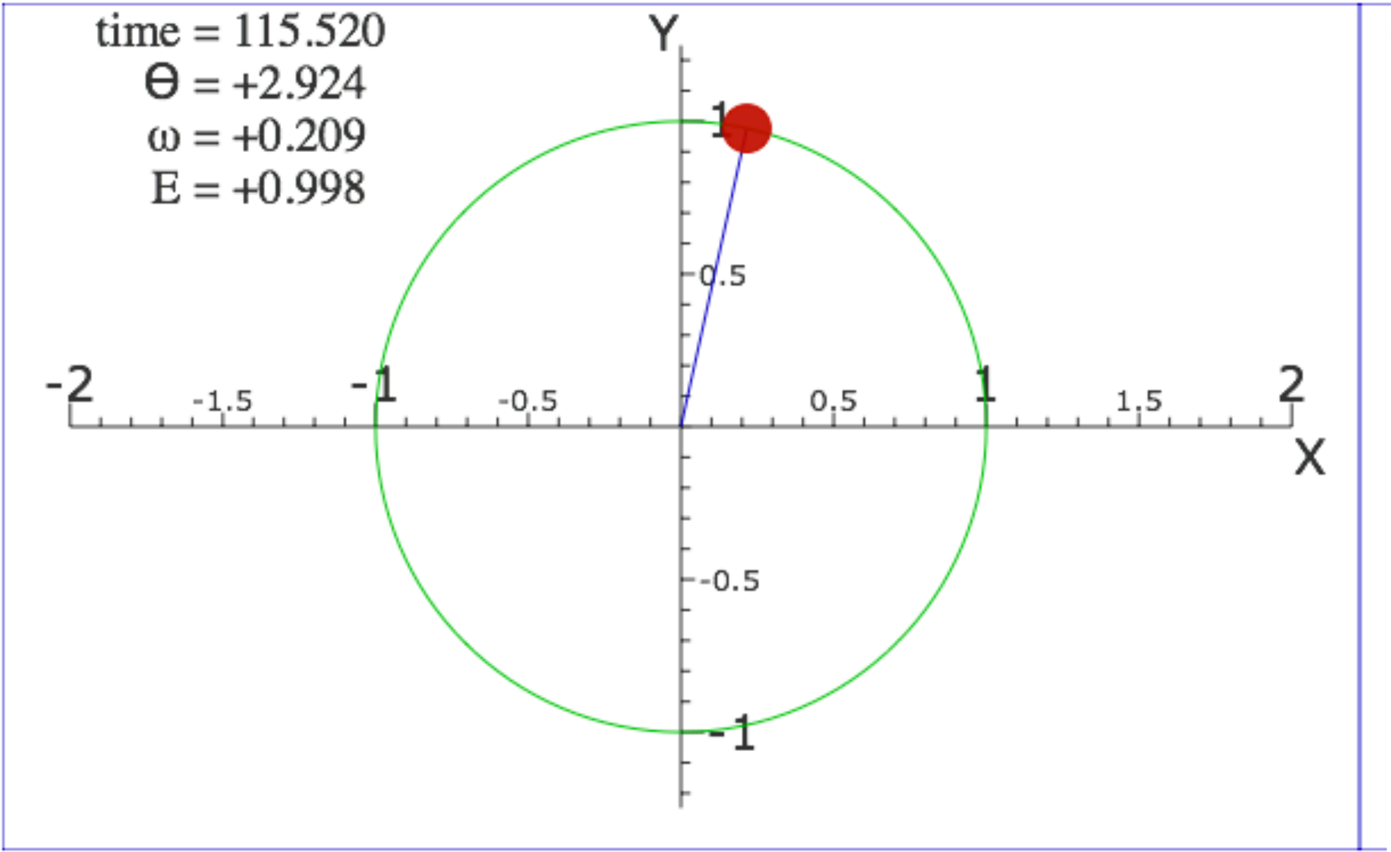
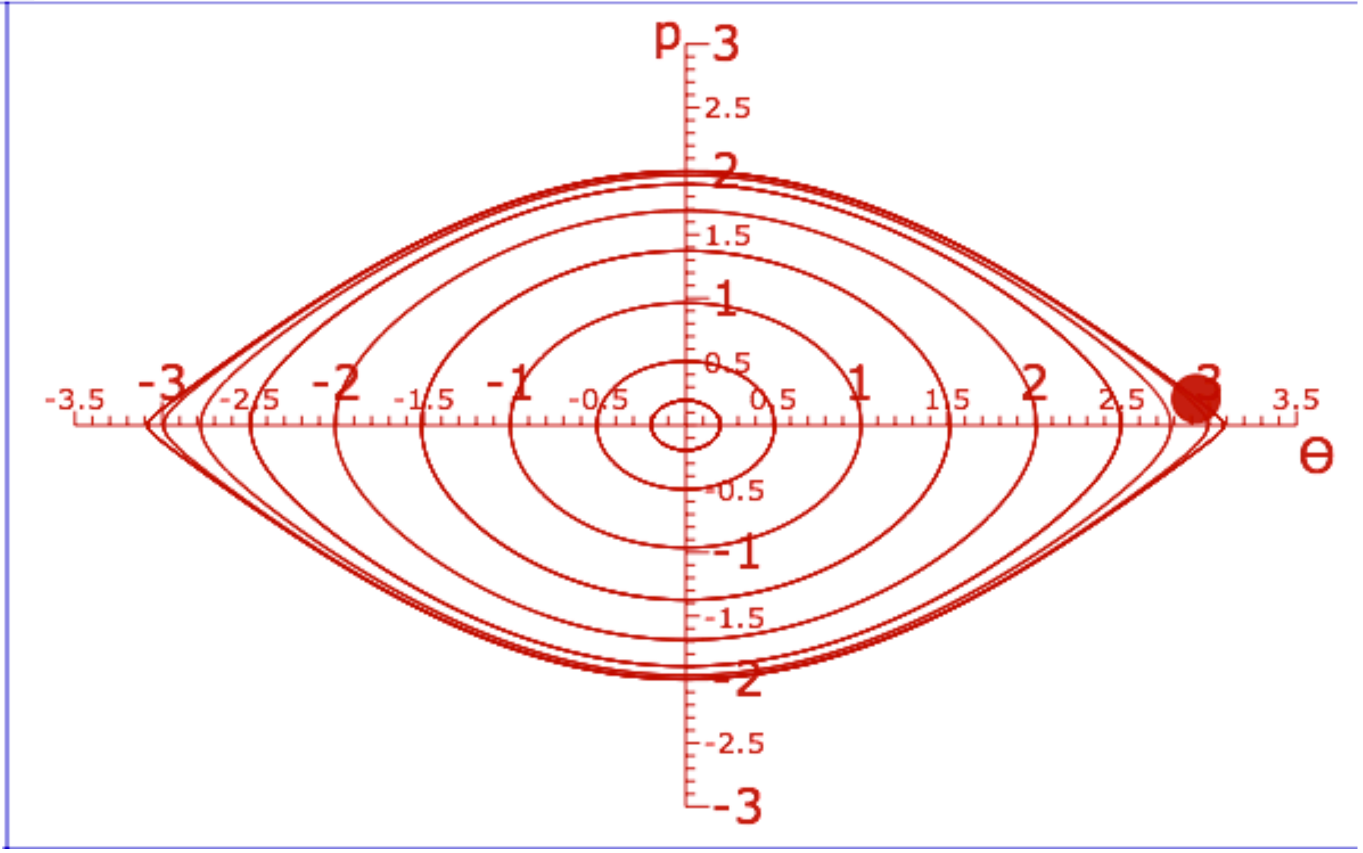
..reduces to sine...

$$\varepsilon(t)=\varepsilon_0\sin\sqrt{\frac{g}{R}}t=\varepsilon_0\sin\omega\,t\text{ , where: }\omega=\sqrt{\frac{g}{R}}$$

For low amplitude $\varepsilon\ll 1$: $\sin\varepsilon_0\simeq\varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions

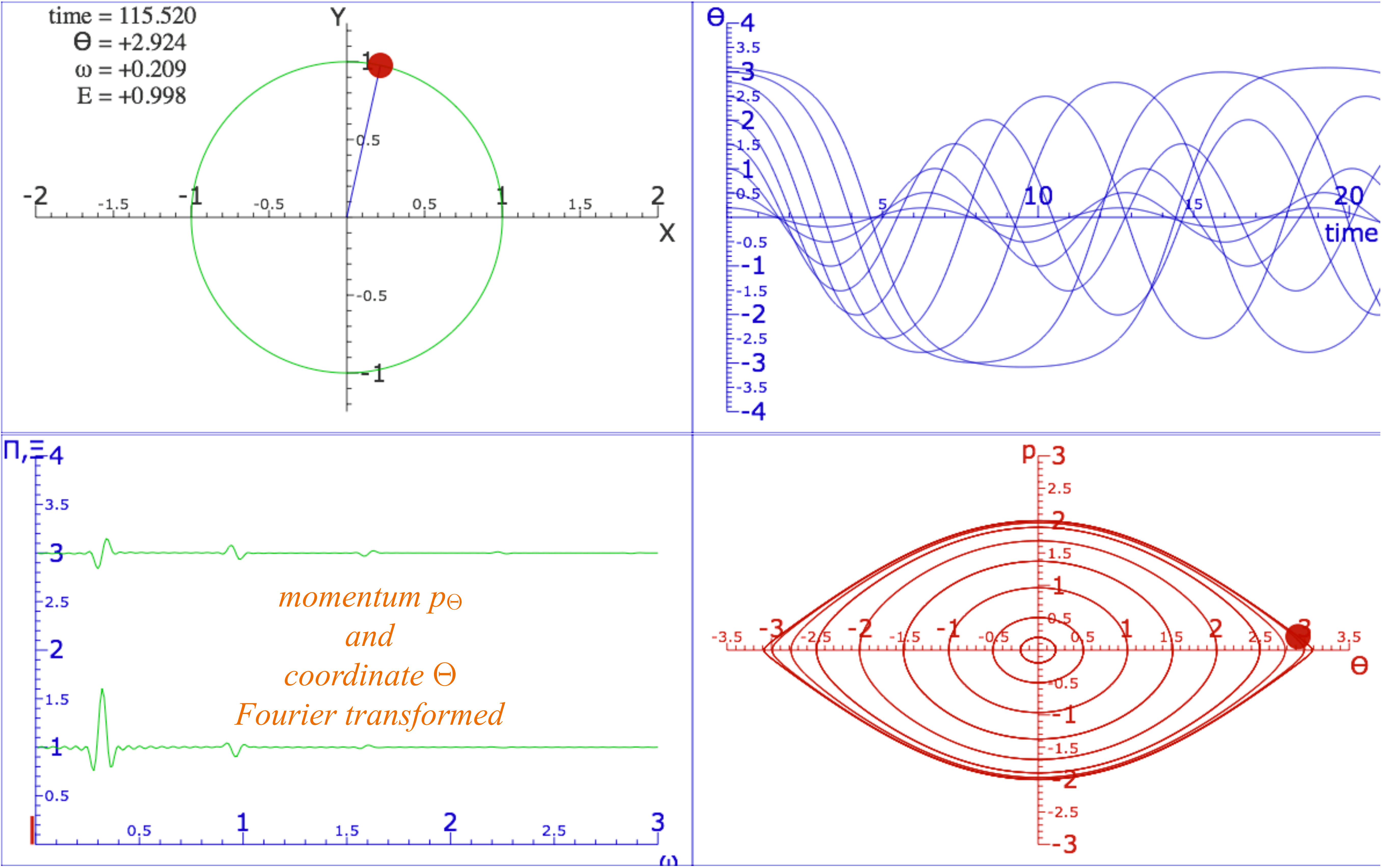




(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)

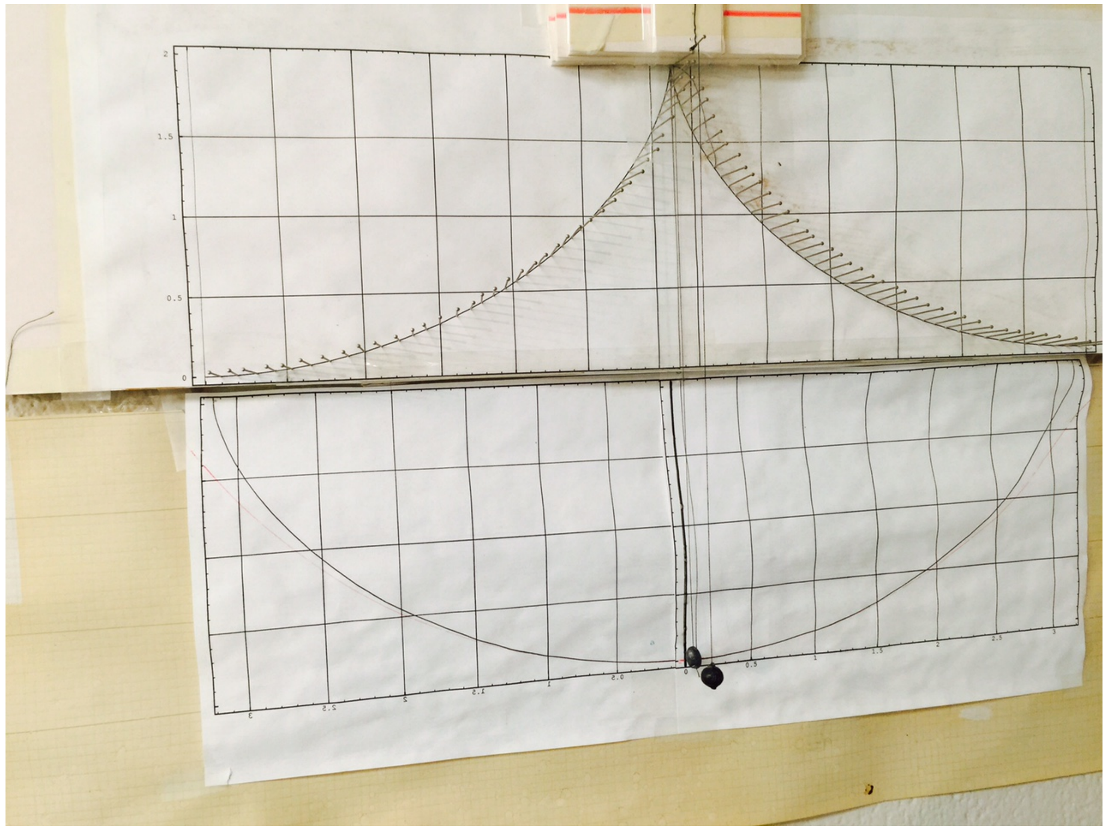
Circular pendulum dynamics and elliptic functions



momentum p_Θ
and
coordinate Θ
Fourier transformed

(Simulations of pendulum)

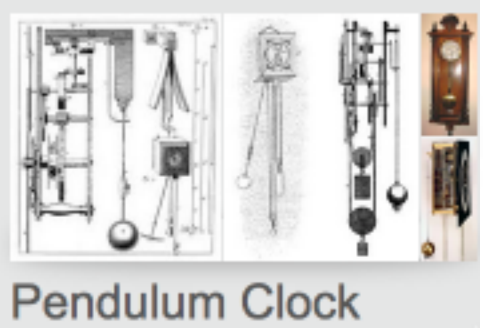
(See also: Simulation of cycloidally constrained pendulum)



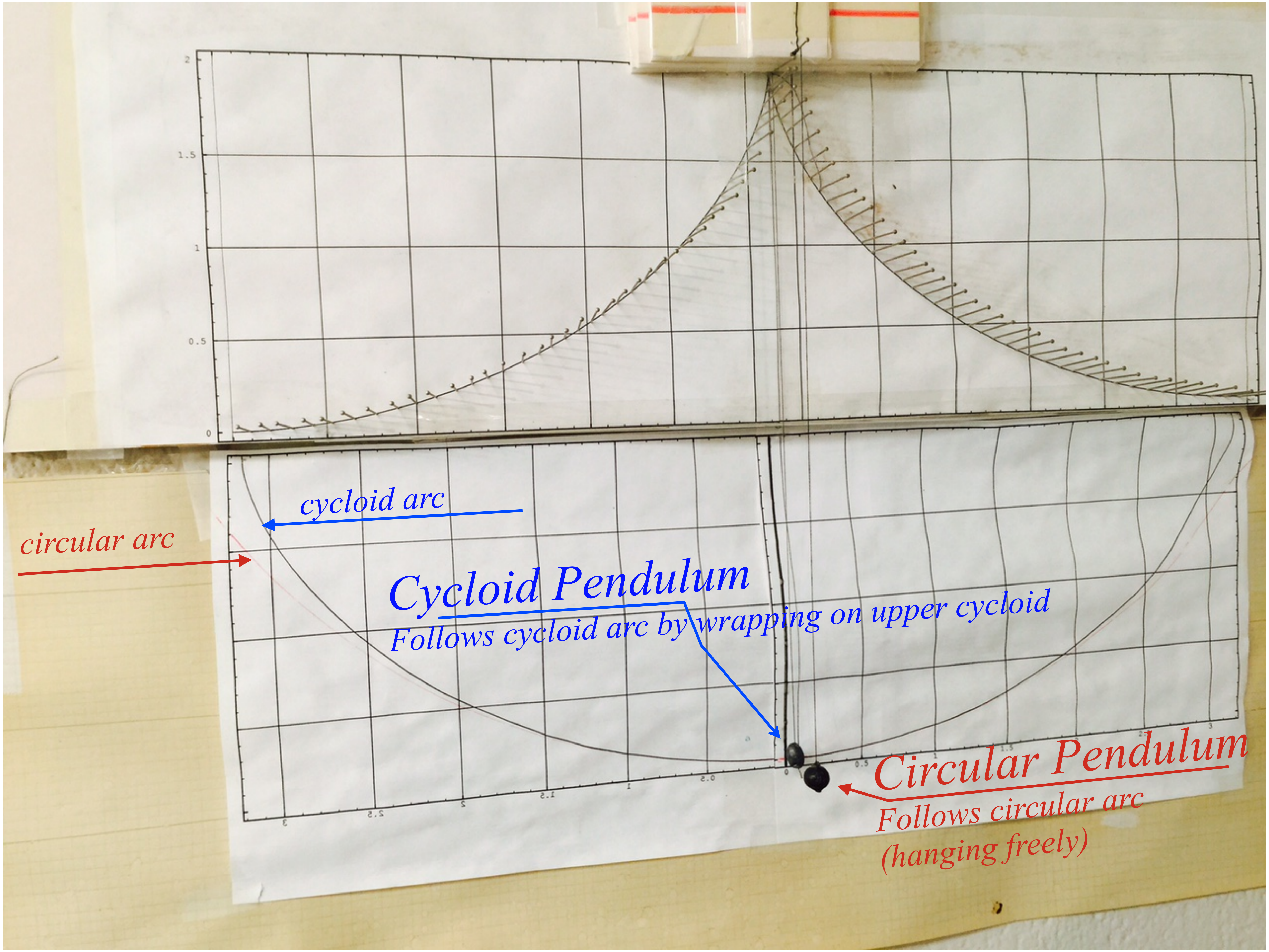
U of A (PHYS 241)
Cycloid pendulum

Examples of Hamiltonian mechanics in phase plots

- *1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))*
  - *Circular pendulum dynamics and elliptic functions*
  - *Cycloid pendulum dynamics and "sawtooth" functions*
- *1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")*



Christaan Huygens
(1629-1695)



cycloid arc

circular arc

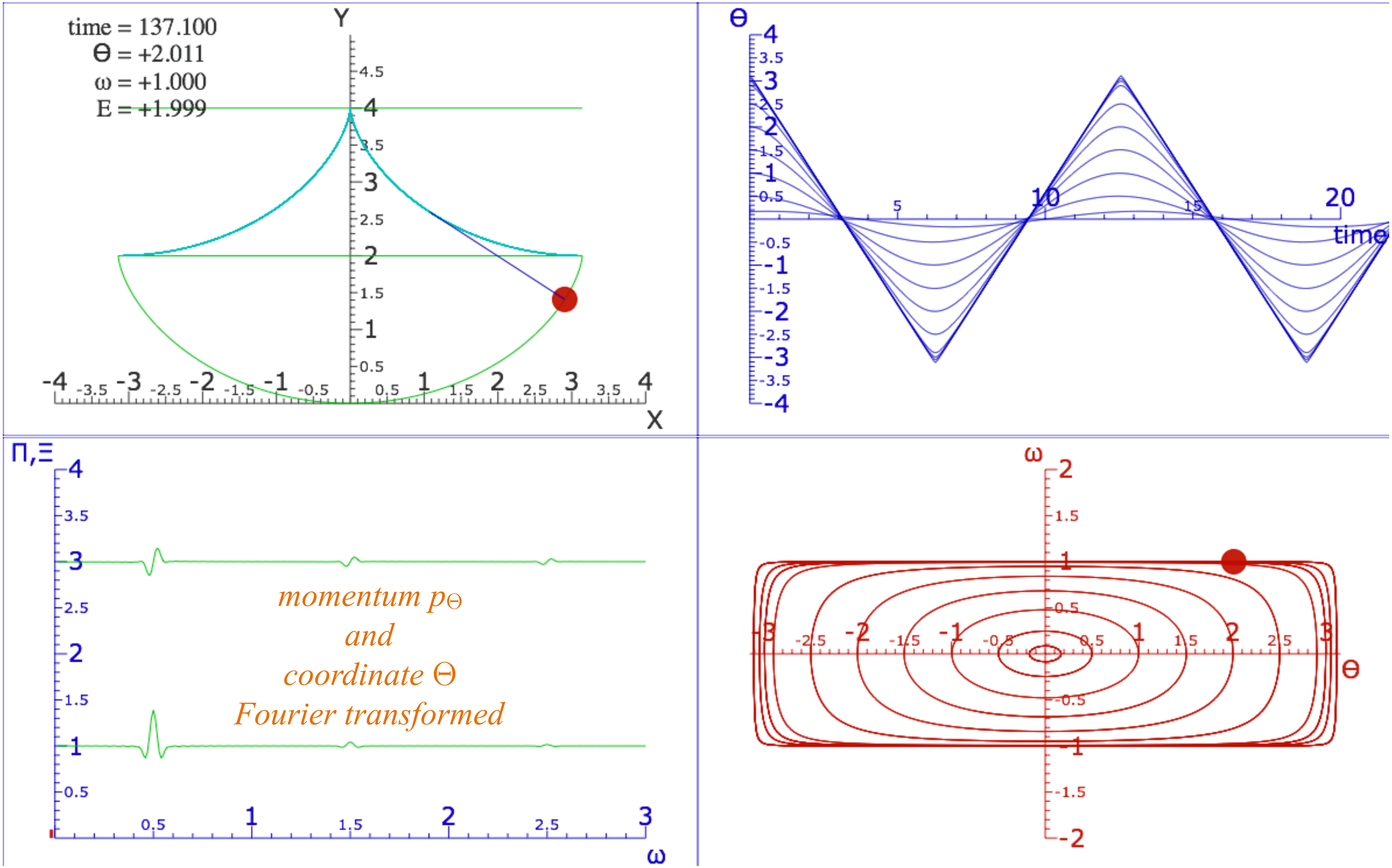
Cycloid Pendulum

Follows cycloid arc by wrapping on upper cycloid

Circular Pendulum

Follows circular arc (hanging freely)

Cycloid pendulum dynamics and “sawtooth” functions



momentum p_Θ

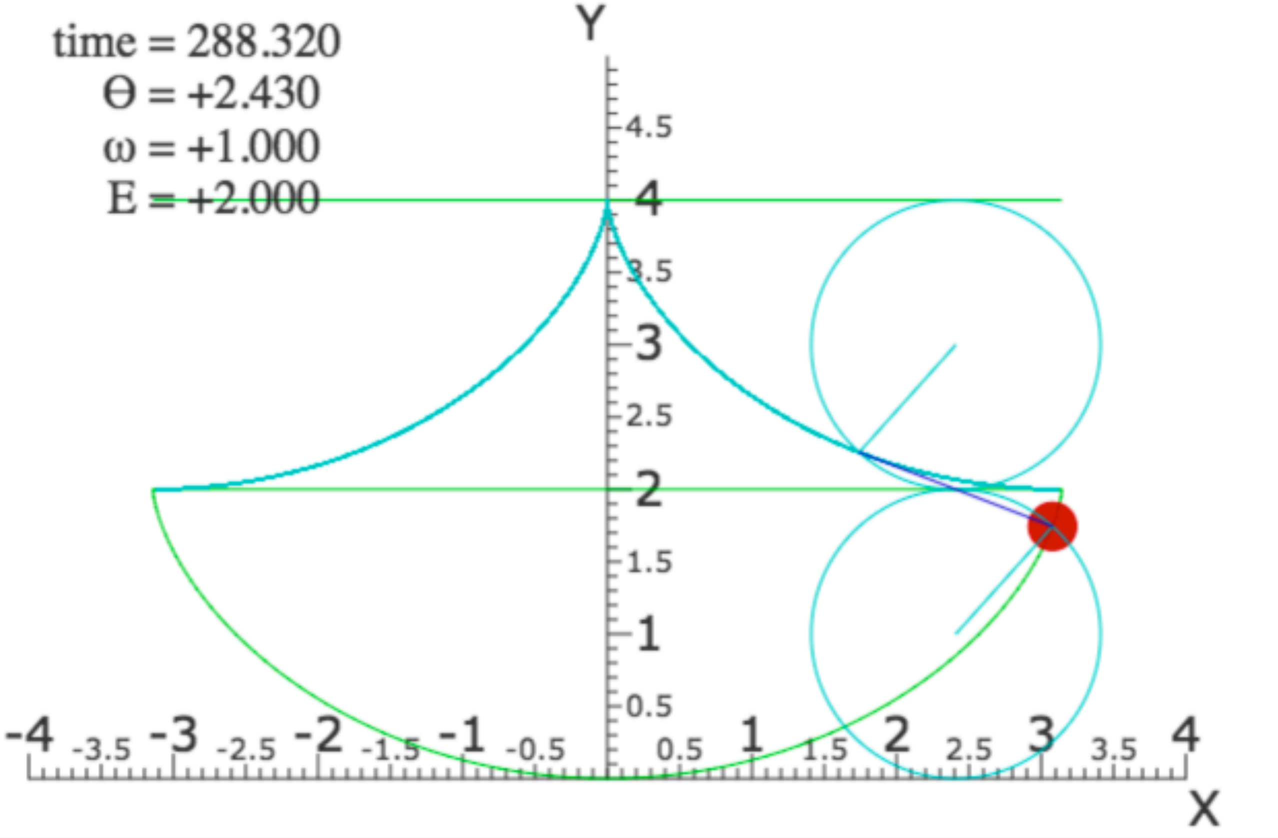
and

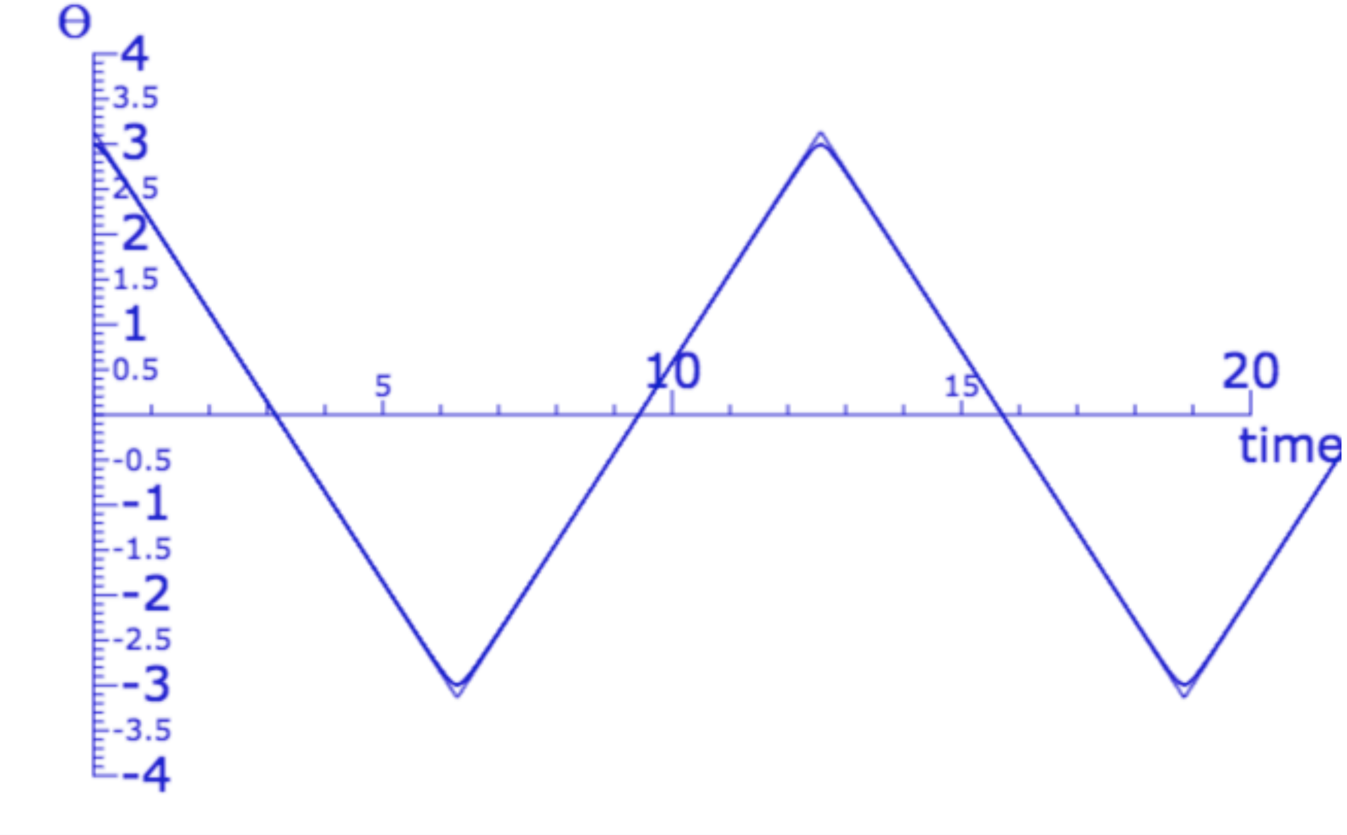
coordinate Θ

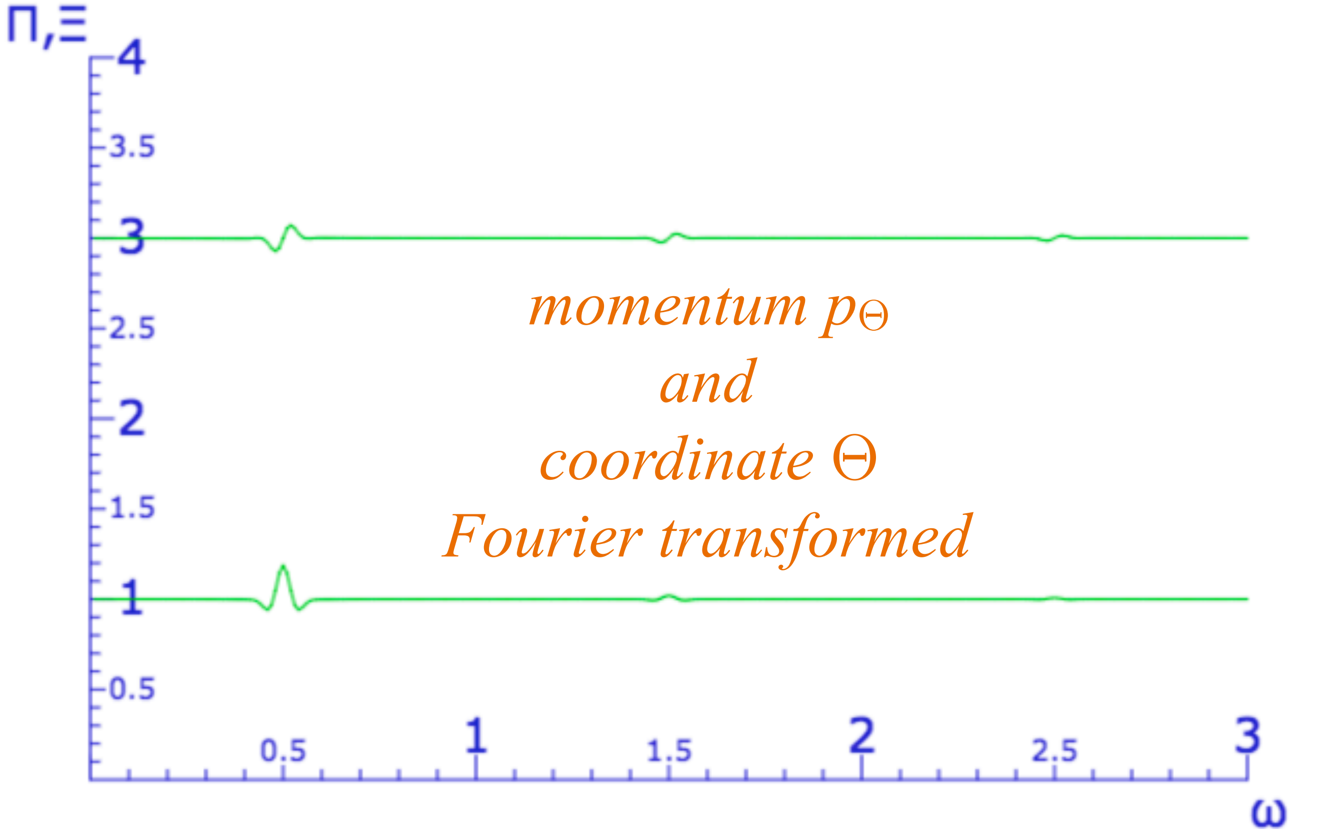
Fourier transformed

(Simulations of cycloidally constrained pendulum)

Cycloid pendulum dynamics and "sawtooth" functions







momentum p_Θ
and
coordinate Θ
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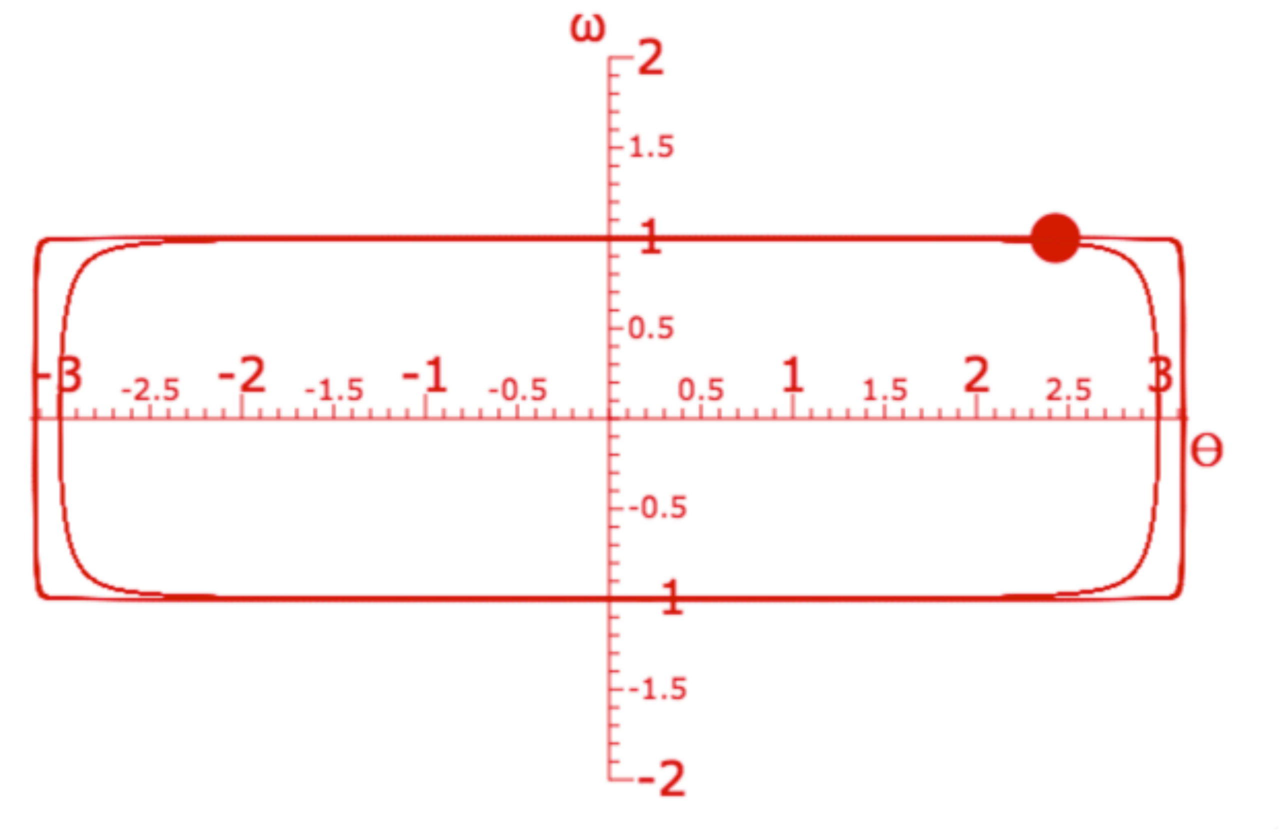

(Simulations of cycloidally constrained pendulum)

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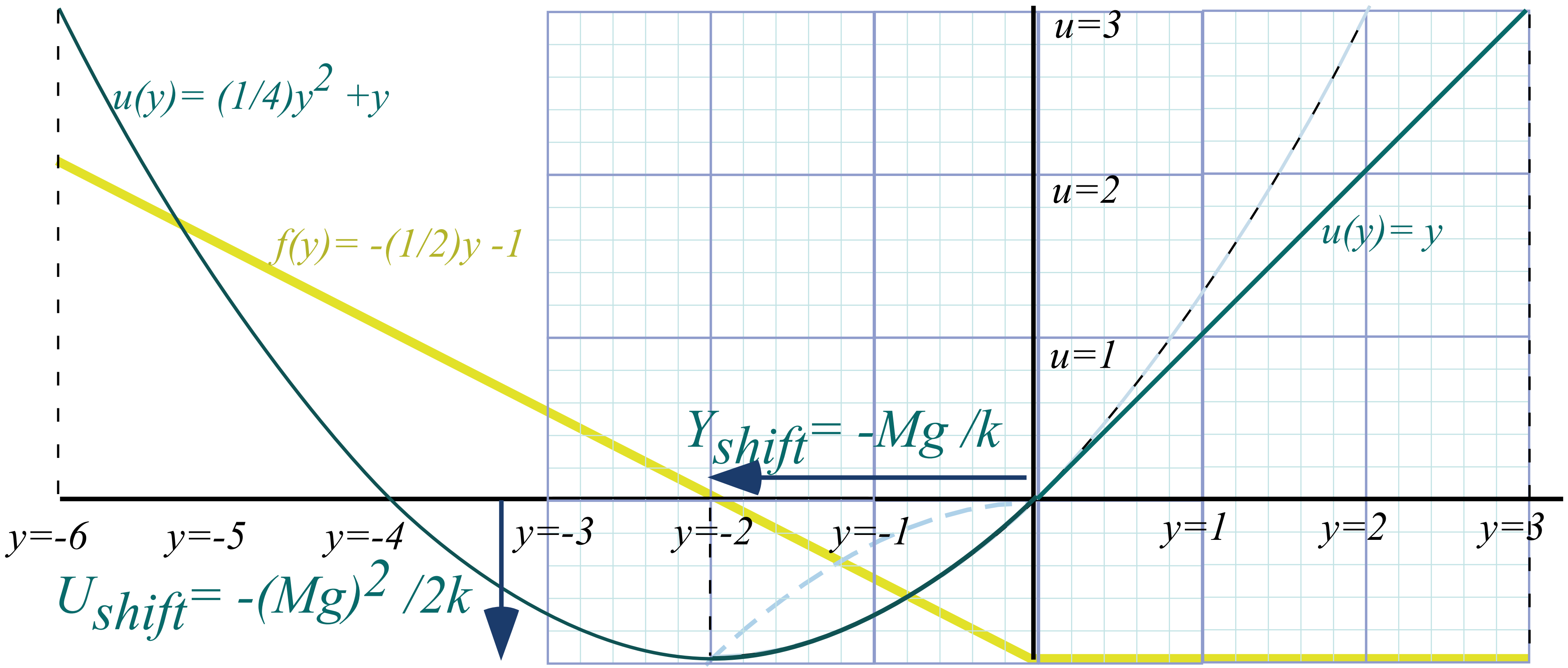
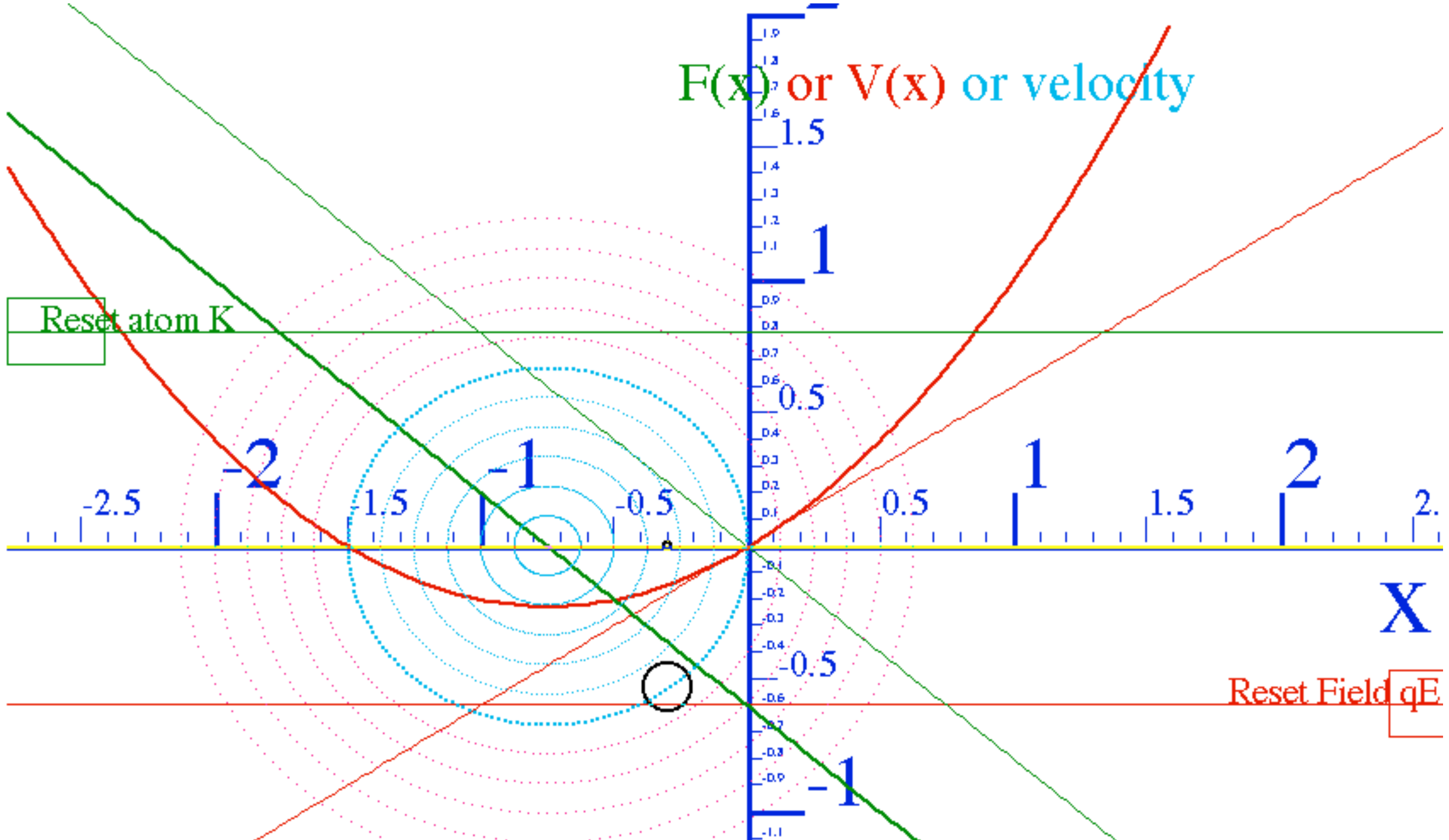
1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")

$F(Y)=-kY-Mg$ $\qquad$ $U(Y)=(1/2)kY^2+Mg\,Y$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action, Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

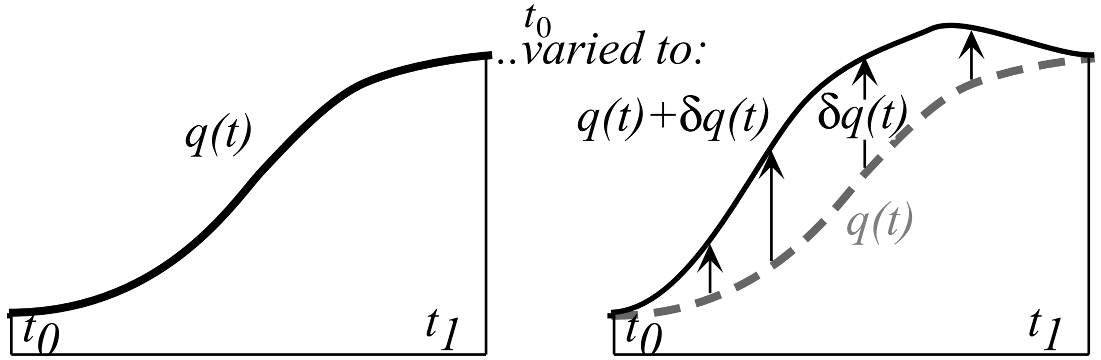
Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange "derivation" of Lagrange's equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q)=\int_{t_0}^{t_1} dt\ L(q(t),\dot{q}(t),t).$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1. There we demand it not vary at all.(1)

$$\delta q(t_0)=0=\delta q(t_1) \qquad (1)$$

1st order L(q+δq) approximate:

$$S(q+\delta q)=\int_{t_0}^{t_1} dt\left[L(q,\dot{q},t)+\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right] \text{ where: } \delta\dot{q}=\frac{d}{dt}\delta q$$

A weird "derivation" of Lagrange's equations

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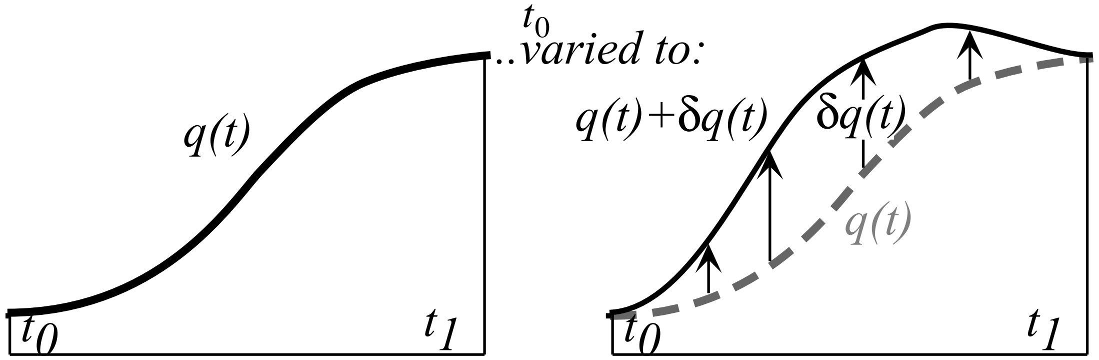

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Replace $\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

A weird "derivation" of Lagrange's equations

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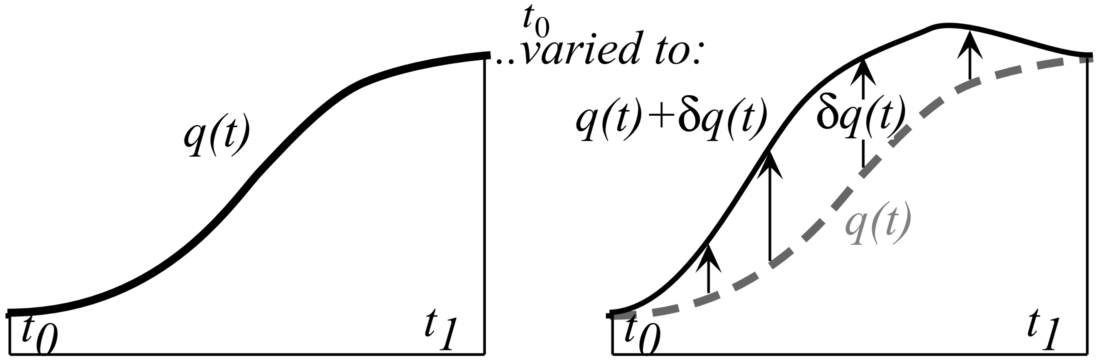

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A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

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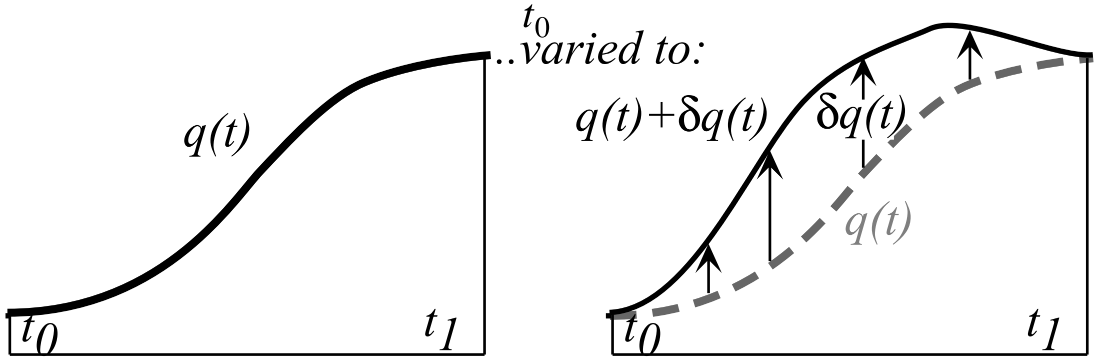

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$$=\int_{t_0}^{t_1} dt\ L(q,\dot q,t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right]\delta q+\left(\frac{\partial L}{\partial \dot q}\delta q\right)\Bigg|_{t_0}^{t_1}$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

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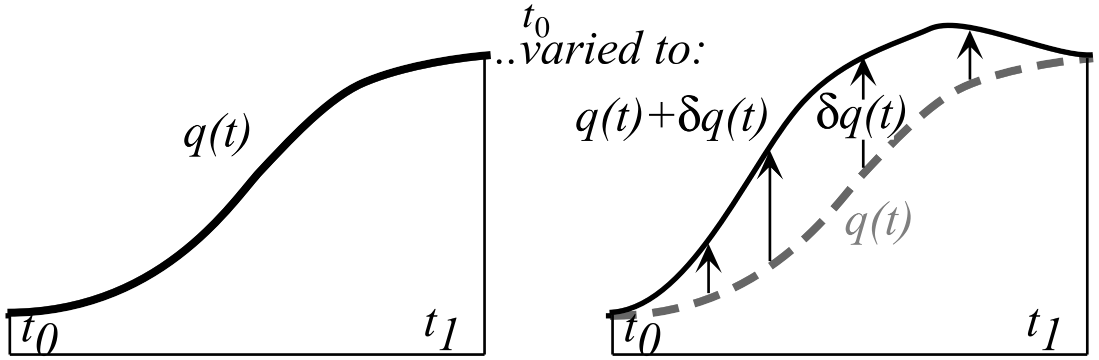

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due to requiring (1)

Third term vanishes by (1). This leaves first order variation: $\delta S=S(q+\delta q)-S(q)=\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

$$\delta S=0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Euler-Lagrange equation(s)

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

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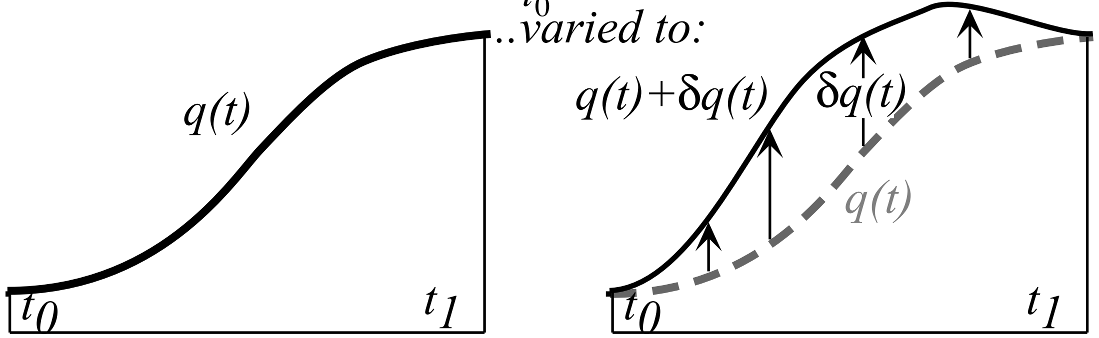

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$$=\int_{t_0}^{t_1} dt\ L(q,\dot{q},t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q+\cancel{\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)}\Bigg|_{t_0}^{t_1}$$

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$$\delta S=0\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Euler-Lagrange equation(s)

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian L = T - U???

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

→ ***Poincare identity and Action,*** Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

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Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \text{ implies: } \mathbf{v}\cdot dt = d\mathbf{r} \right)$$

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This is the time differential dS of *action* $S = \int L\cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

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Unit 8 shows *DeBroglie law* $\mathbf{p} = \hbar \mathbf{k}$ and *Planck law* $H = \hbar \omega$ make *quantum plane wave phase* Φ*:*

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Q:When is the *Action*-differential dS integrable?

A: A differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* *∂-symmetry* *of W:*

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y\partial x} = \frac{\partial^2 W}{\partial x\partial y} = \frac{\partial f_y}{\partial x}$$

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dS is integrable if: $\boxed{\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}}$ and: $\boxed{\frac{\partial S}{\partial t} = -H}$

These conditions are known as Jacobi-Hamilton *equations*

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How Jacobi-Hamilton *could have “derived” Schrodinger equations*

(Given “quantum wave”)

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

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Try 1st **r**-*derivative of wave* ψ

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Momentum Operator
or **p***-op in* **r***-basis*

$$\mathbf{p} \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}$$

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

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Try 1st t-derivative of wave ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)}$$

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

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Schrodinger time equation

$$\mathrm{i}\hbar\dot{\psi}(\mathbf{r},t) = H\psi(\mathbf{r},t)$$

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How Classicists might have "derived" quantum equations

➡ ***Huygen's contact transformations enforce minimum action***

How to do quantum mechanics if you only know classical mechanics


Christaan Huygens
(1629-1695)

Huygen's contact transformations enforce minimum action

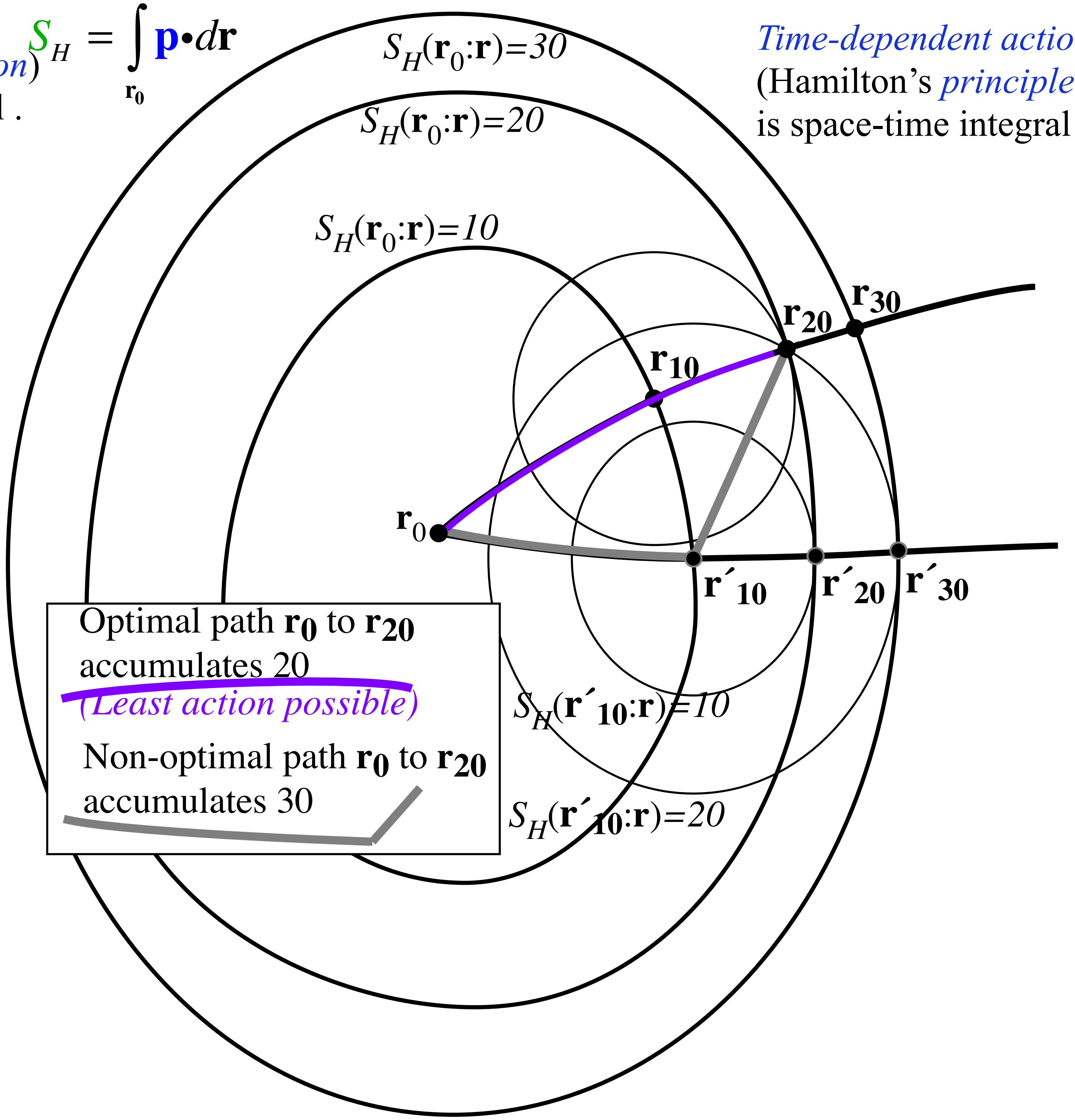
Each point $\mathbf{r}_k$ on a wavefront "broadcasts" in all directions.
*Only **minimum action** path interferes constructively*

Time-independent action
(Hamilton's *reduced action*)
is a purely spatial integral .

$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p}\bullet d\mathbf{r}$$

Time-dependent action
(Hamilton's *principle action*)
is space-time integral .

$$S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} \left(\mathbf{p}\bullet d\mathbf{r} - H\cdot dt\right)$$



Optimal path $\mathbf{r}_0$ to $\mathbf{r}_{20}$
accumulates 20
(Least action possible)

Non-optimal path $\mathbf{r}_0$ to $\mathbf{r}_{20}$
accumulates 30

Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

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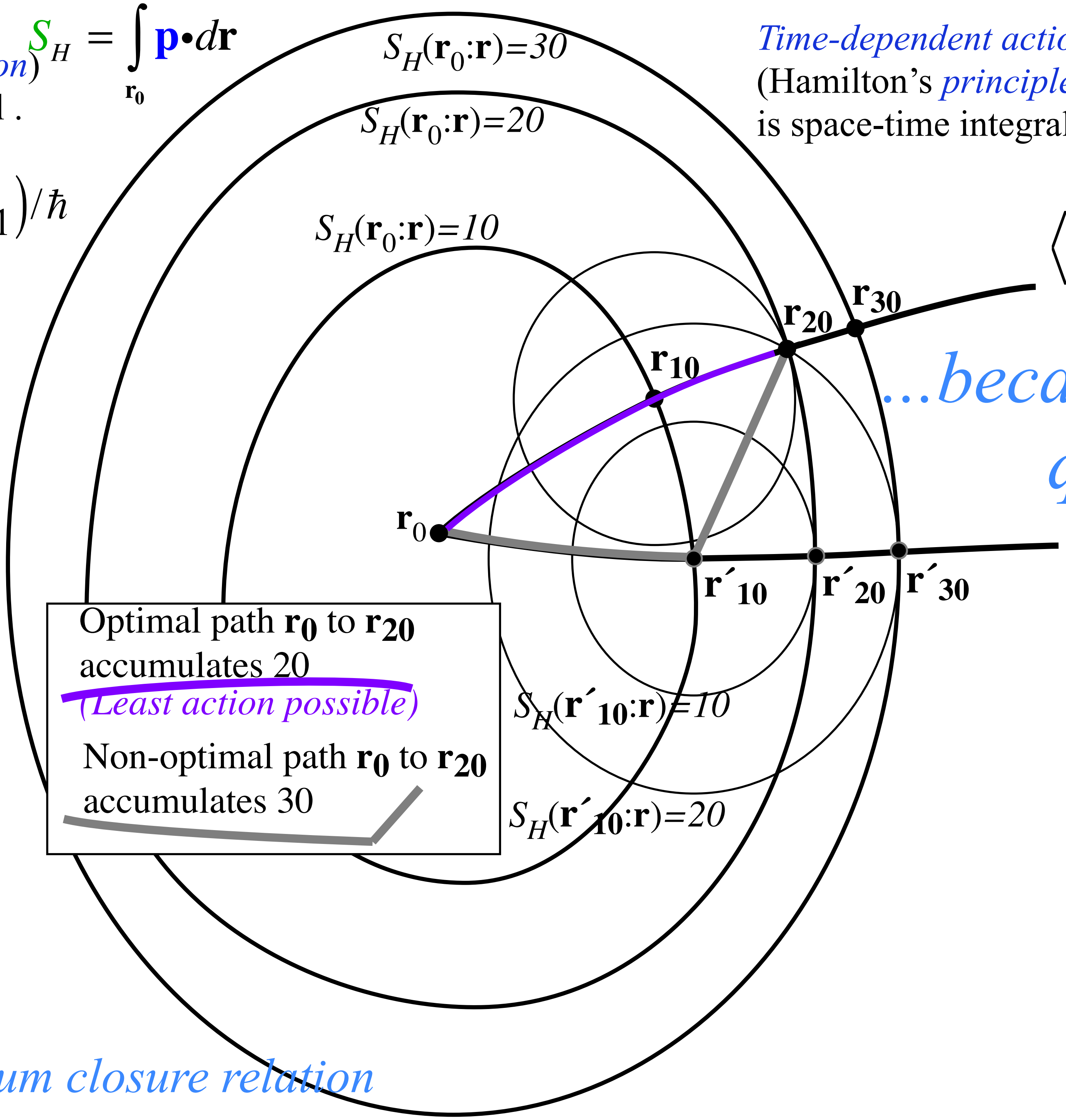
$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1)/\hbar}$$



Optimal path $\mathbf{r_0}$ to $\mathbf{r_{20}}$
accumulates 20
(Least action possible)

Non-optimal path $\mathbf{r_0}$ to $\mathbf{r_{20}}$
accumulates 30

...because action *<u>is</u>*
quantum wave phase

Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i\left(S_H(\mathbf{r}_0:\mathbf{r}') + S_H(\mathbf{r}':\mathbf{r}_1)\right)/\hbar} = e^{i S_H(\mathbf{r}_0:\mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

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→ ***How to do quantum mechanics if you only know classical mechanics***
Davis-Heller “Color-Quantization” or “Classical Chromodynamics”

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple *n* of 2π after a closed loop integral $S_H(\mathbf{r}_0 : \mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \bullet d\mathbf{r}$. The integer *n* (*n = 0, 1, 2,...*) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi\hbar n = hn$$

Numerically integrate Hamilton's equations and Lagrangian *L*. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L\, dt + Ht .$$

How to do quantum mechanics if you only know classical mechanics

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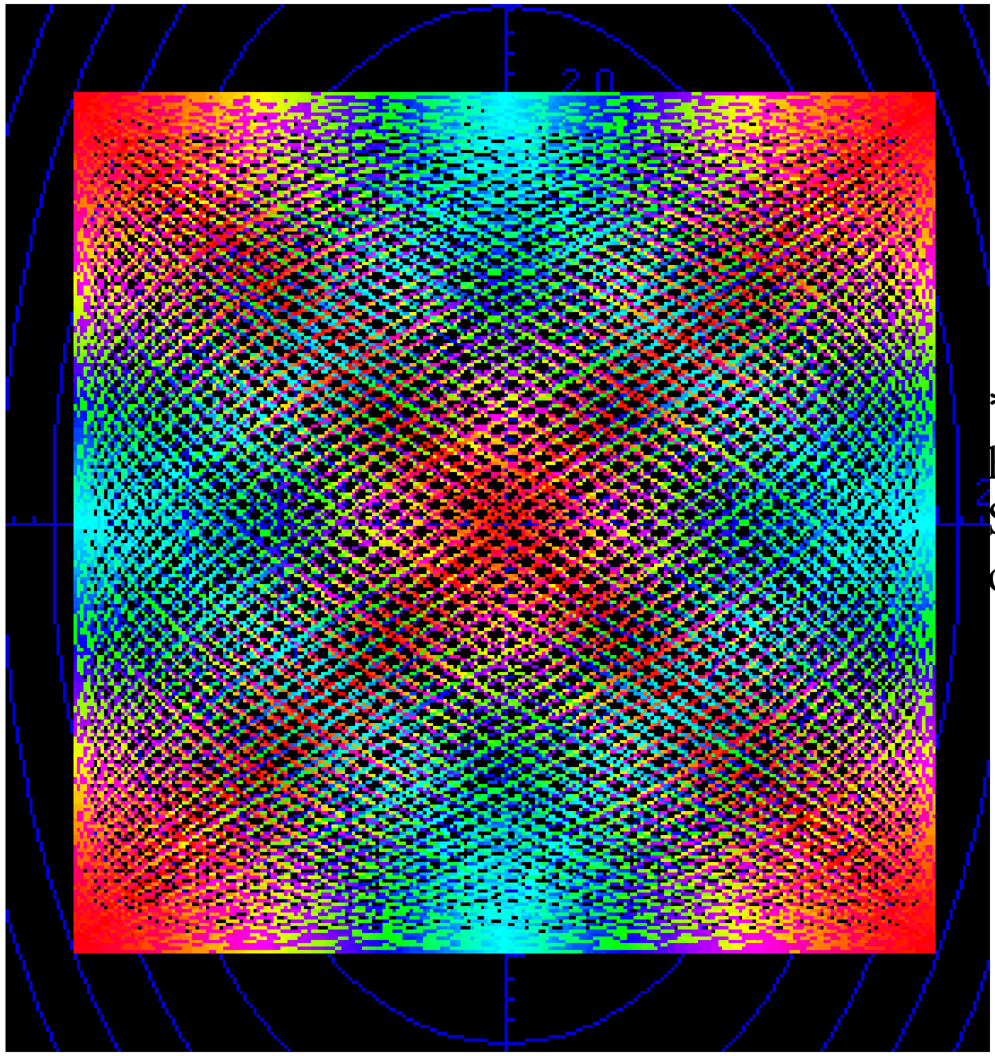
$$S_H(\mathbf{0}:\mathbf{r})=S_p(\mathbf{0},0:\mathbf{r},t)+Ht=\int_0^t L\,dt+Ht\,.$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* 2π as, for example,

0=red, π/4=orange, π/2=yellow, 3π/4=green, π=cyan (opposite of red), *5π/4=indigo, 3π/2=blue, 7π/4=purple, and 2π=red* (full color circle).

Interpolating action on a palette of *32* colors is enough precision for low quanta.



simulation
by
"Color U(2)"

Unit 1
Fig.
12.13

*closed system has quantized E. Standing wave has only two phases(±) *cyan* and *red*

[Quantum dynamical tunneling in bound states - Wavepacket and Color-quantization - M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)]

[The Semiclassical Way to Molecular Spectroscopy: Eric J. Heller, Acc. Chem. Res. 1981, 14, 368-375]

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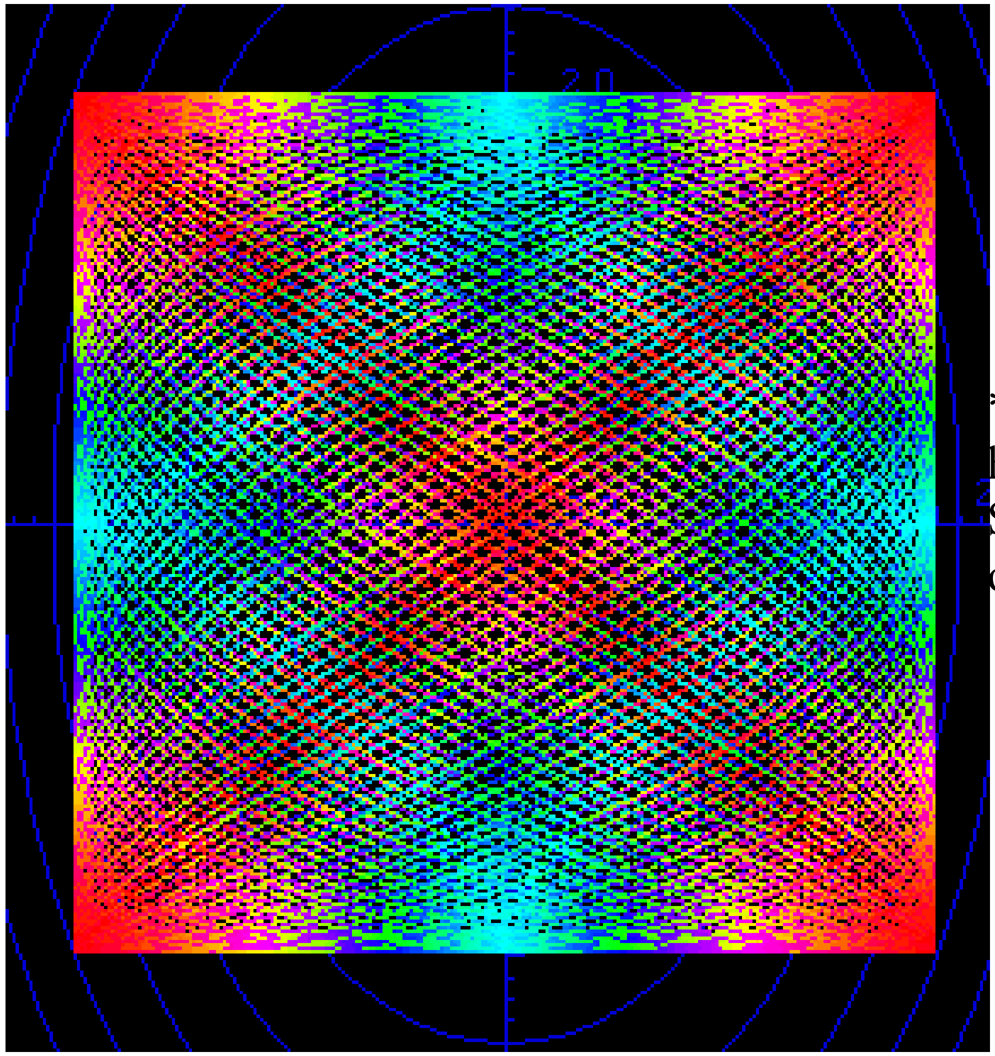
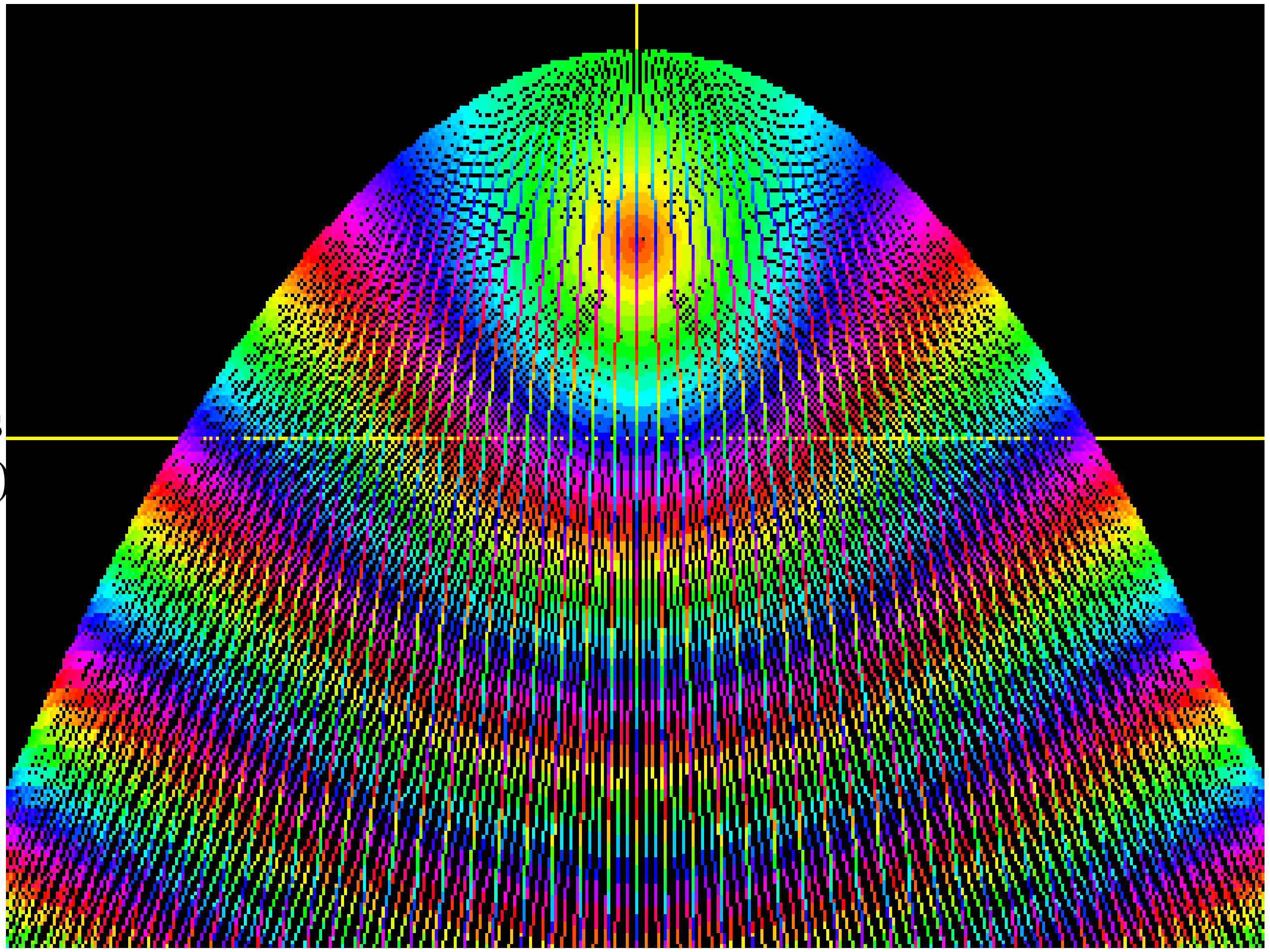


Simulation by "Color U(2)"

Unit 1 Fig. 12.13

*closed system has quantized E
Standing wave has only two phases(±) *cyan* and *red*

*open system has continuous energy

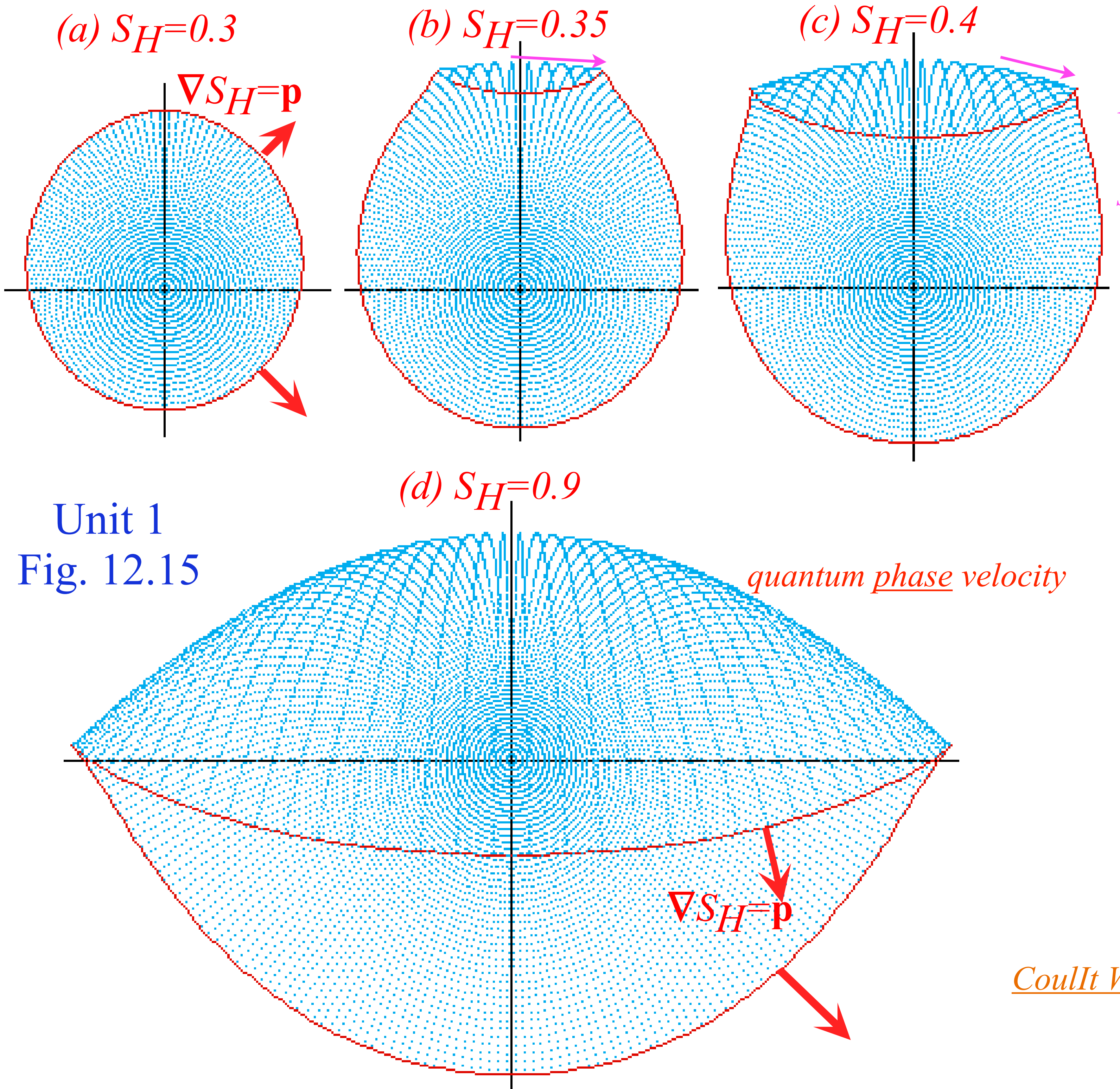


Unit 1 Fig. 12.14

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

Quantum “phase wavefronts”


(a) $S_H=0.3$
(b) $S_H=0.35$
(c) $S_H=0.4$
(d) $S_H=0.9$

wavefront
“cat ears”
scoot outward..

quantum phase velocity

Unit 1
Fig. 12.15

CoulIt Web Simulation with "Quantum phase front"

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

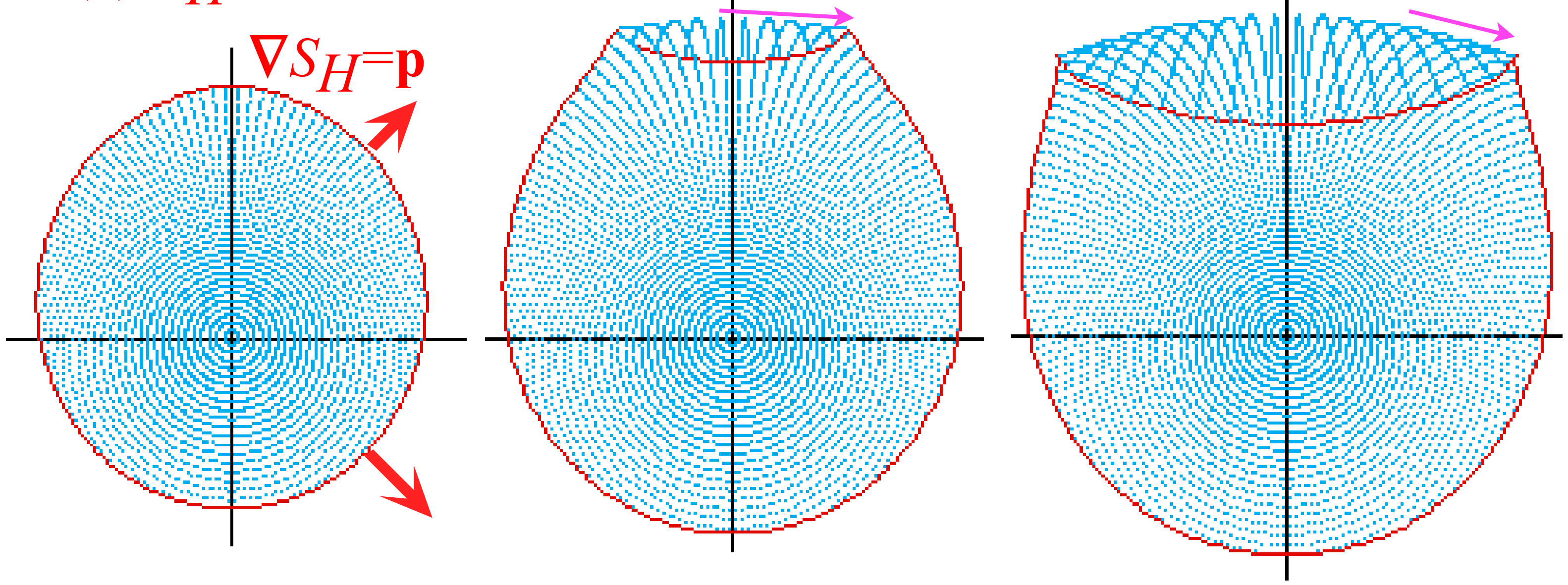
This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"

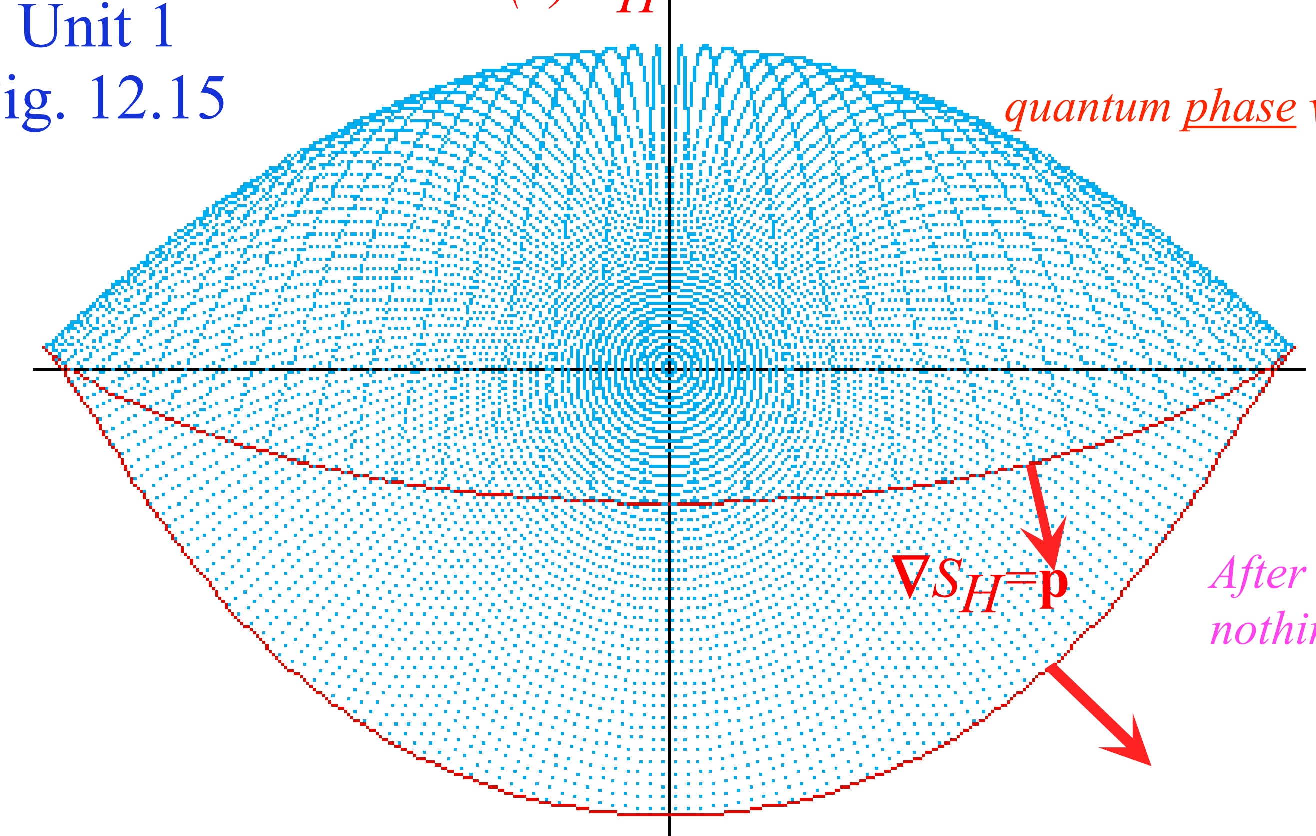
(a) $S_H=0.3$ *(b) $S_H=0.35$* *(c) $S_H=0.4$*



wavefront "cat ears" scoot outward..

Unit 1
Fig. 12.15

(d) $S_H=0.9$



quantum phase velocity

After a while …
nothing left but a smile!

CoulIt Web Simulation with "Quantum phase front"

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0{:}\mathbf{r},t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

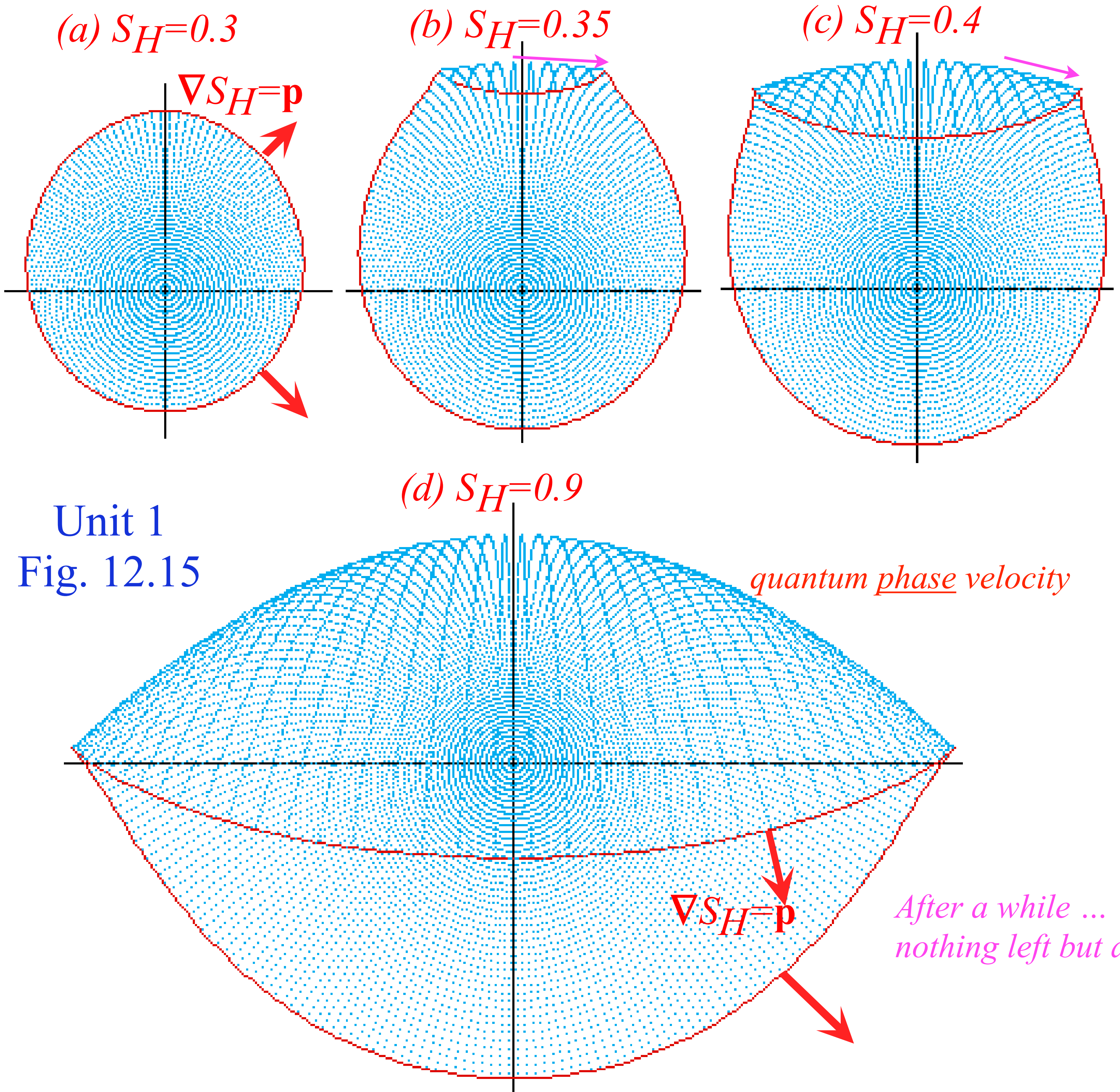
$$V_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$V_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



wavefront
"cat ears"
scoot outward..

quantum phase velocity

After a while ...
nothing left but a smile!

Unit 1
Fig. 12.15

16th Century carving on St. Wifred's in Grappenhall

...on St. Nicolas

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

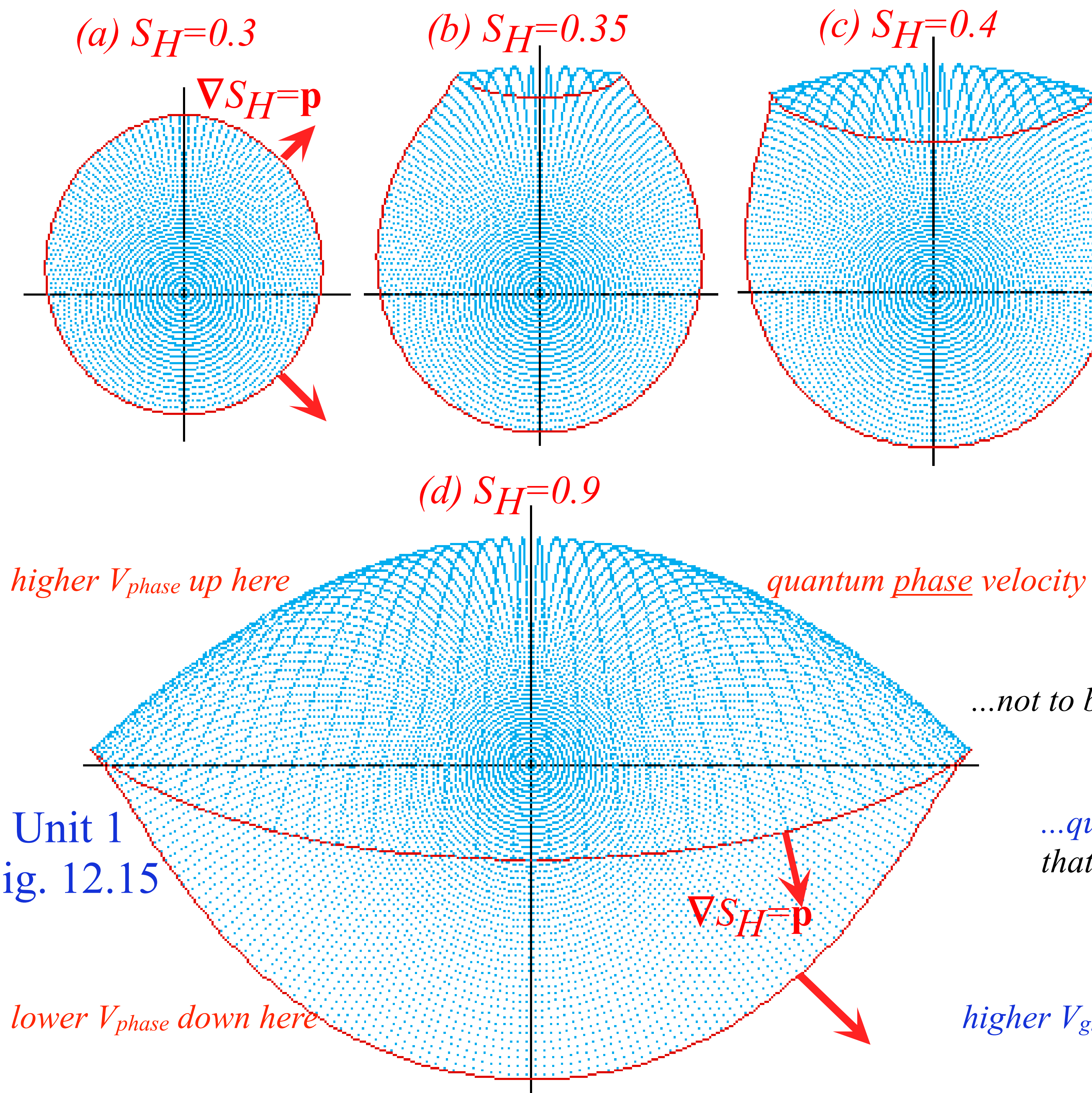
$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

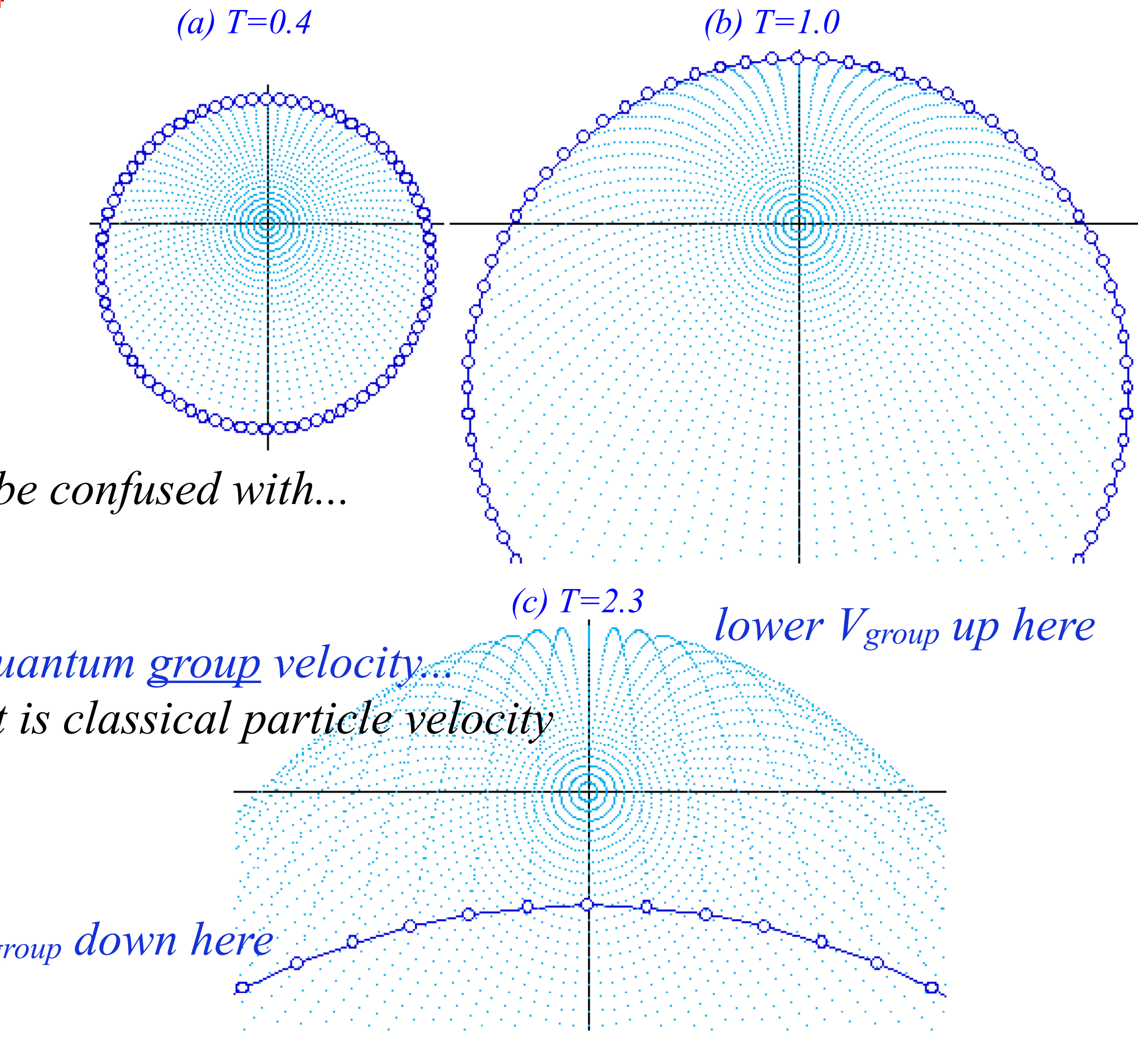
Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



...not to be confused with...

Classical "blast wavefronts"

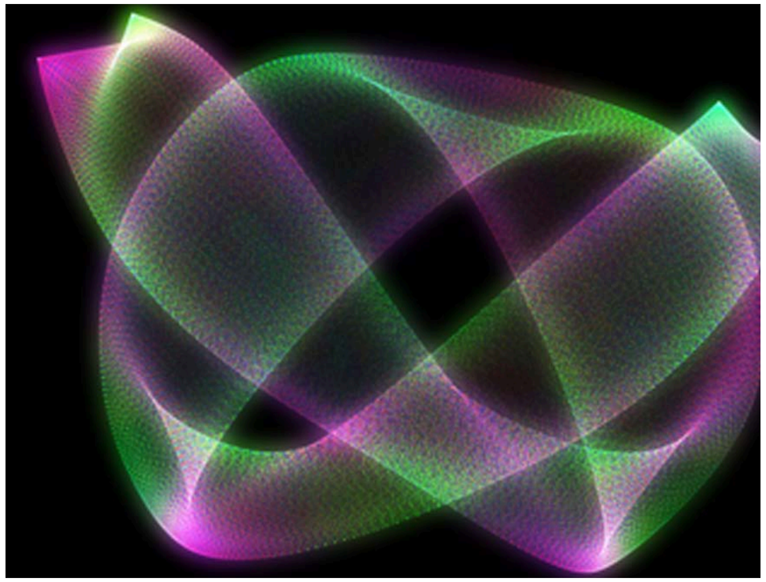


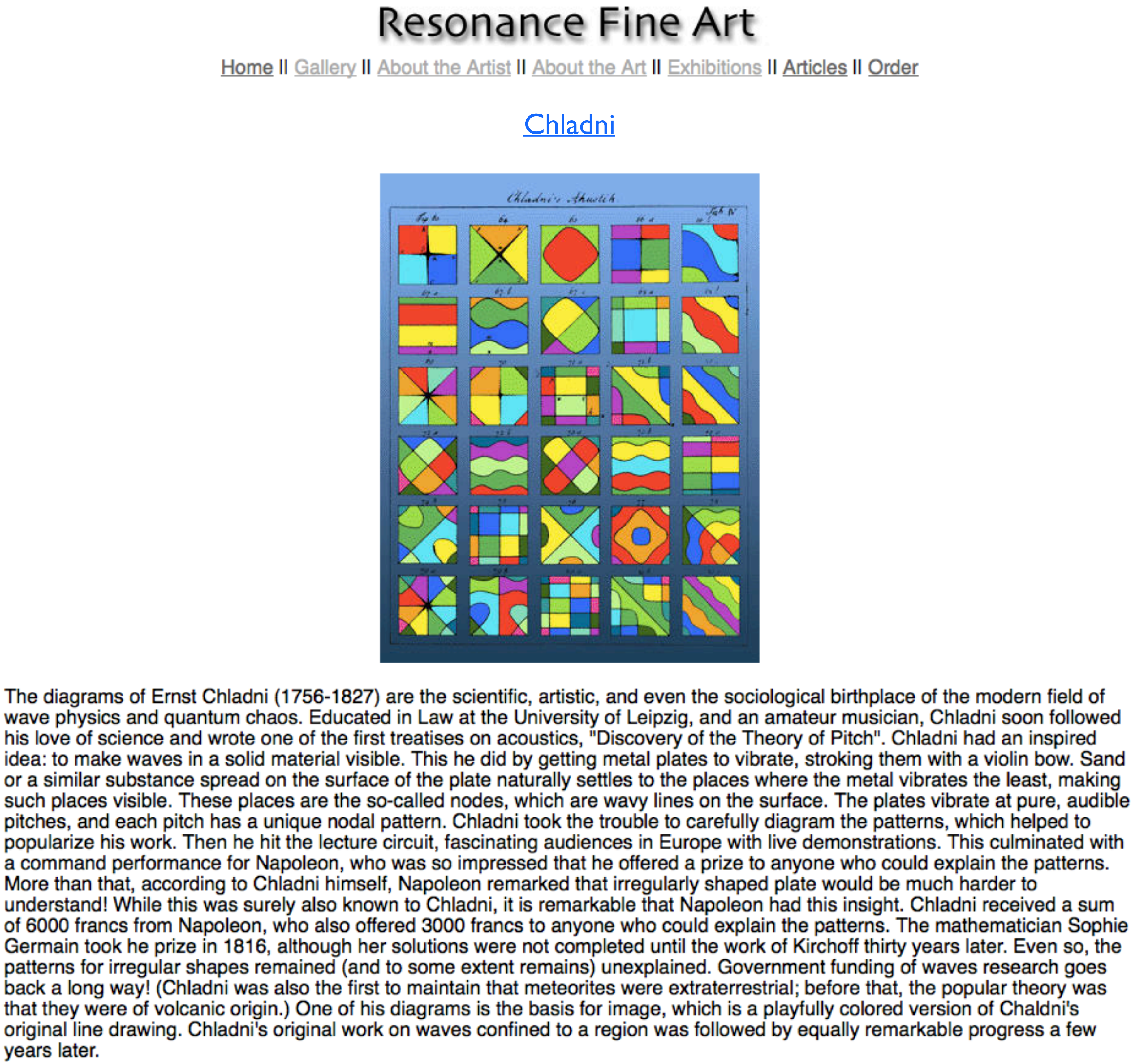
...quantum group velocity...
that is classical particle velocity

Unit 1
Fig. 12.15

Check out the Heller Galleries

http://jalbum.net/en/browse/user/album/1696720





Resonance Fine Art

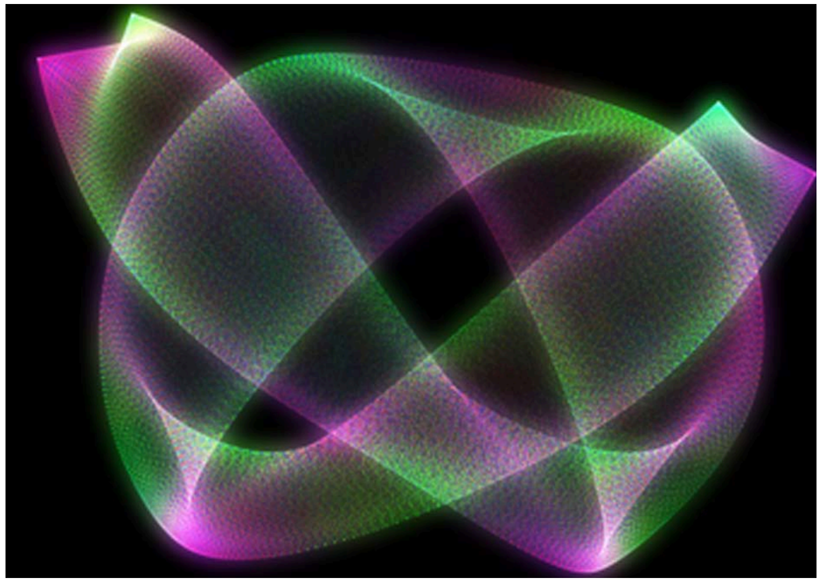
Home II Gallery II About the Artist II About the Art II Exhibitions II Articles II Order

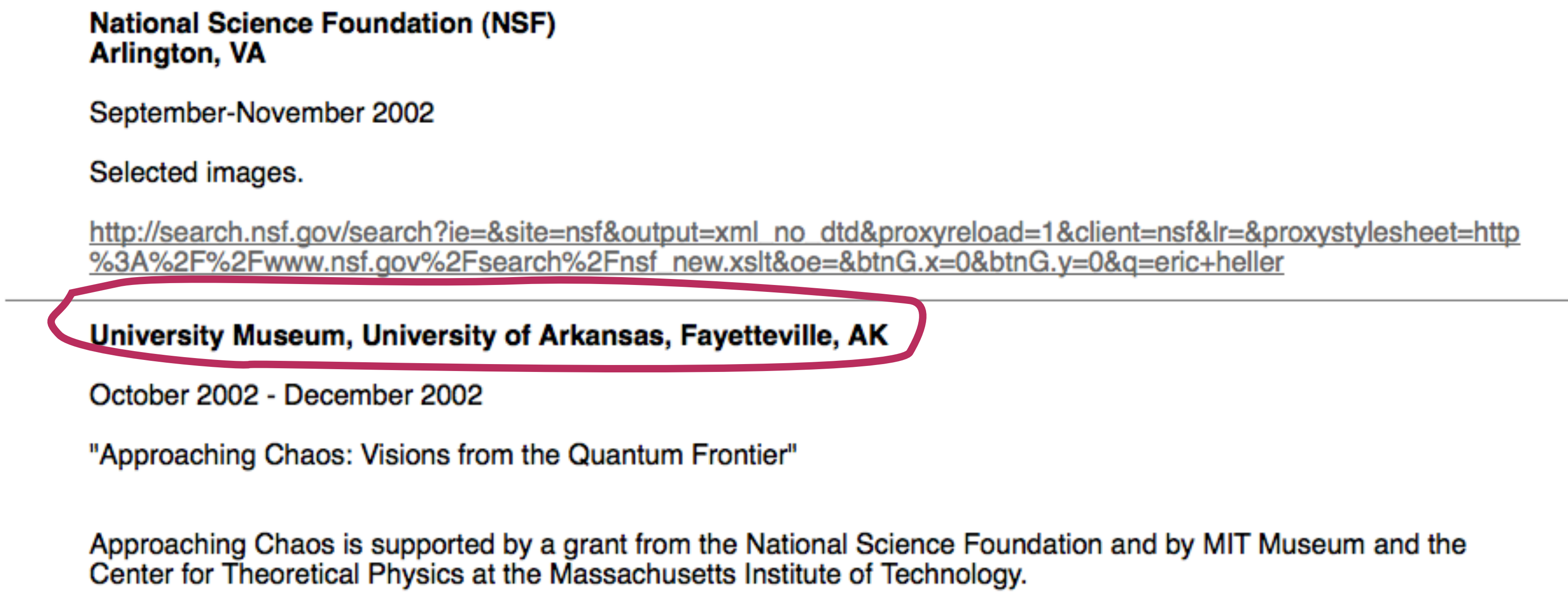
Chladni

The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took he prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chaldni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries

http://jalbum.net/en/browse/user/album/1696720





National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

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University Museum, University of Arkansas, Fayetteville, AK

October 2002 - December 2002

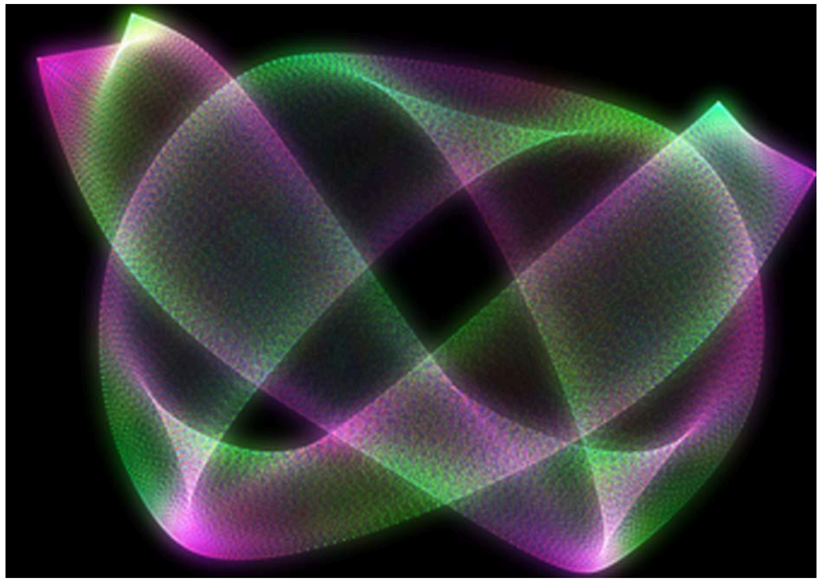
"Approaching Chaos: Visions from the Quantum Frontier"

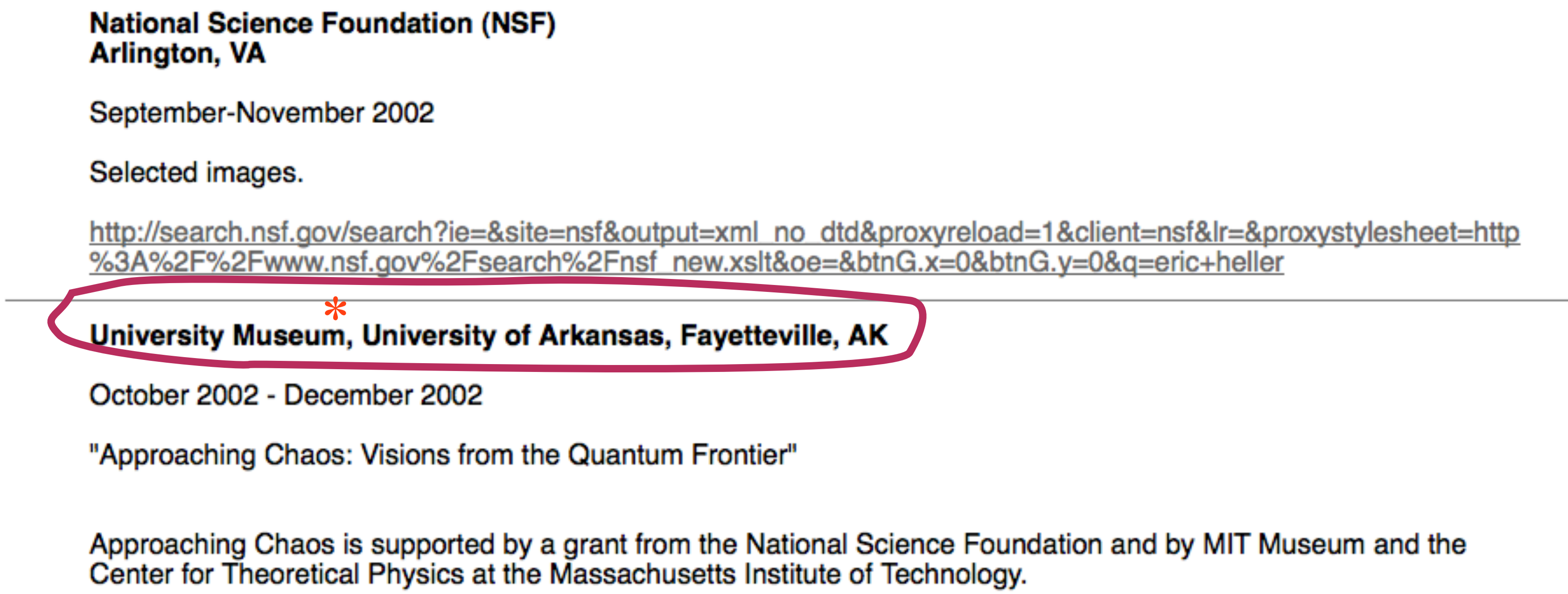
Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

Bessel 21

Check out the Heller Galleries

http://jalbum.net/en/browse/user/album/1696720





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University Museum, * **University of Arkansas, Fayetteville, AK**

October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

*UAF *Museum closed after this exhibit*

Bessel 21

Lecture 11 ends here
Wed. 9.26.2018

A running collection of links to course-relevant sites and articles

Physics Web Resources	***"Texts"***	***Classes***
Comprehensive Harter-Soft Resource Listing	Classical Mechanics with a Bang!	2014 AMOP
UAF Physics YouTube channel	Quantum Theory for the Computer Age	2017 Group Theory for QM
LearnIt Physics Web Applications	Principles of Symmetry, Dynamics, and Spectroscopy	2018 AMOP
	Modern Physics and its Classical Foundations	2018 Adv Mechanics

Neat external material to start the class:

- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

These *are* hot off the presses:

- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:

- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:

- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Unique URL Listing for 2018 Adv Mechanics Lecture 11

http://ejheller.jalbum.net/Eric%20J%20Heller%20Gallery/
http://ejheller.jalbum.net/Eric%20J%20Heller%20Gallery/#Chladni.jpg
http://ejheller.jalbum.net/Eric%2520J%2520Heller%2520Gallery/%23Bessel%252021.jpg
http://homepage.univie.ac.at/mario.barbatti/papers/heller/heller_acs_14_368_1981.pdf
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