Lecture 11
Tue. 9.30.2014

Quadratic form geometry and development of mechanics of Lagrange and Hamilton

(Ch. 12 of Unit 1 and Ch. 4-5 of Unit 7)

Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 10)
Introducing 1st Lagrange and Hamilton differential equations of mechanics (Review Of Lecture 10)

Introducing the Poincare′ and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

A general contact transformation from sophomore physics

Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"
Intuitive-geometric development of " " " and " " "

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

Three ways to express energy: *Consider kinetic energy (KE) first*

1. ***Lagrangian*** *is explicit function of* ***velocity****:* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$L(v_k\ldots) = \tfrac{1}{2}(m_1 v_1^2 + m_2 v_2^2 + \ldots) = L(\mathbf{v}\ldots) = \tfrac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \ldots = \tfrac{1}{2}\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ldots$$

2. ***"Estrangian"*** *is explicit function of* **R***-rescaled velocity:*

(or l'Estrangian

or: ***"speedinum"*** *V* $\mathbf{V} = \mathbf{R}\bullet\mathbf{v}$ or: $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$E(V_k\ldots) = \tfrac{1}{2}(V_1^2 + V_2^2 + \ldots) = E(\mathbf{V}\ldots) = \tfrac{1}{2}\mathbf{V}\bullet\mathbf{1}\bullet\mathbf{V} + \ldots = \tfrac{1}{2}\begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \ldots$$

3. ***Hamiltonian*** *is explicit function of* $\mathbf{M}=\mathbf{R}^2$*-rescaled velocity:*

or: ***momentum p*** $\mathbf{p} = \mathbf{M}\bullet\mathbf{v}$ or: $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 v_1 \\ m_2 v_2 \end{pmatrix}$

$$H(p_k\ldots) = \tfrac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \ldots\right) = H(\mathbf{p}\ldots) = \tfrac{1}{2}\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p} + \ldots = \tfrac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \ldots$$

Scaling transformation between Lagrangian and Hamiltonian views of KE (Review of Lecture 9)

Introducing the (partial) differential equations of mechanics (Review Of Lecture 9)

→ *1st equations of Lagrange and Hamilton*

Introducing the (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have no explicit dependence on* ***momentum*** $\mathbf{p}=\mathbf{M}\bullet\mathbf{v}$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have no explicit dependence on* ***velocity*** $\mathbf{v}=\mathbf{M}^{-1}\bullet\mathbf{p}$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have no explicit dependence on* ***speedinum*** $\mathbf{V}=\mathbf{M}^{1/2}\bullet\mathbf{v}$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2} = \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2} = \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

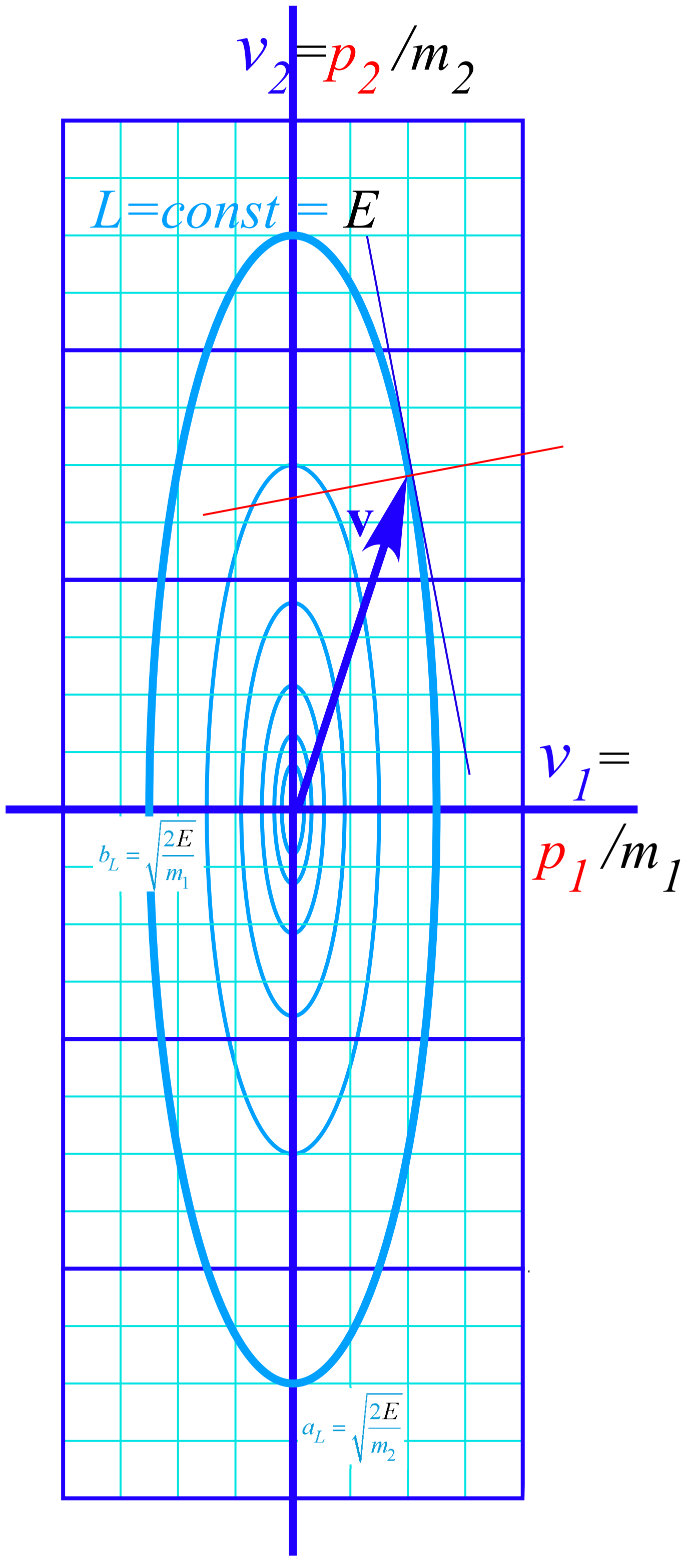
Hamilton's 1st equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

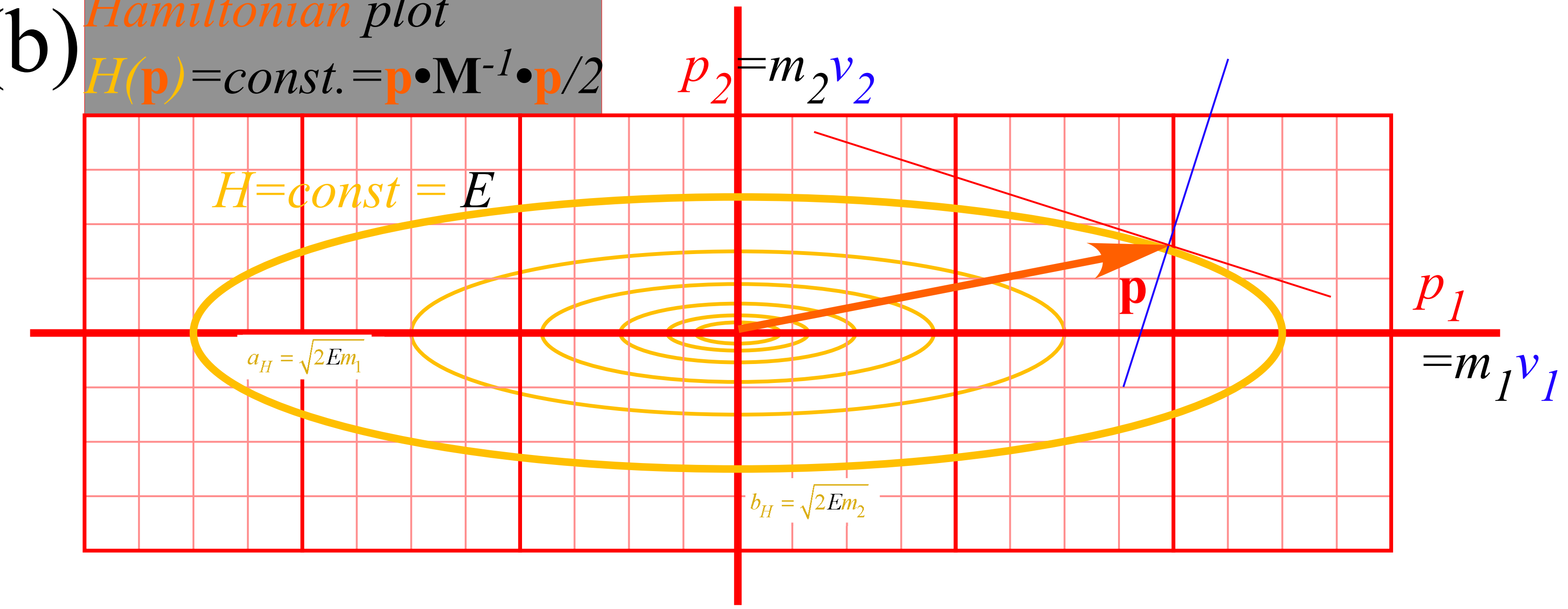
Forget Estrangian for now
(But, recall dual ellipse geometry in Lecture 10 p. 44-55)

Unit 1
Fig. 12.2

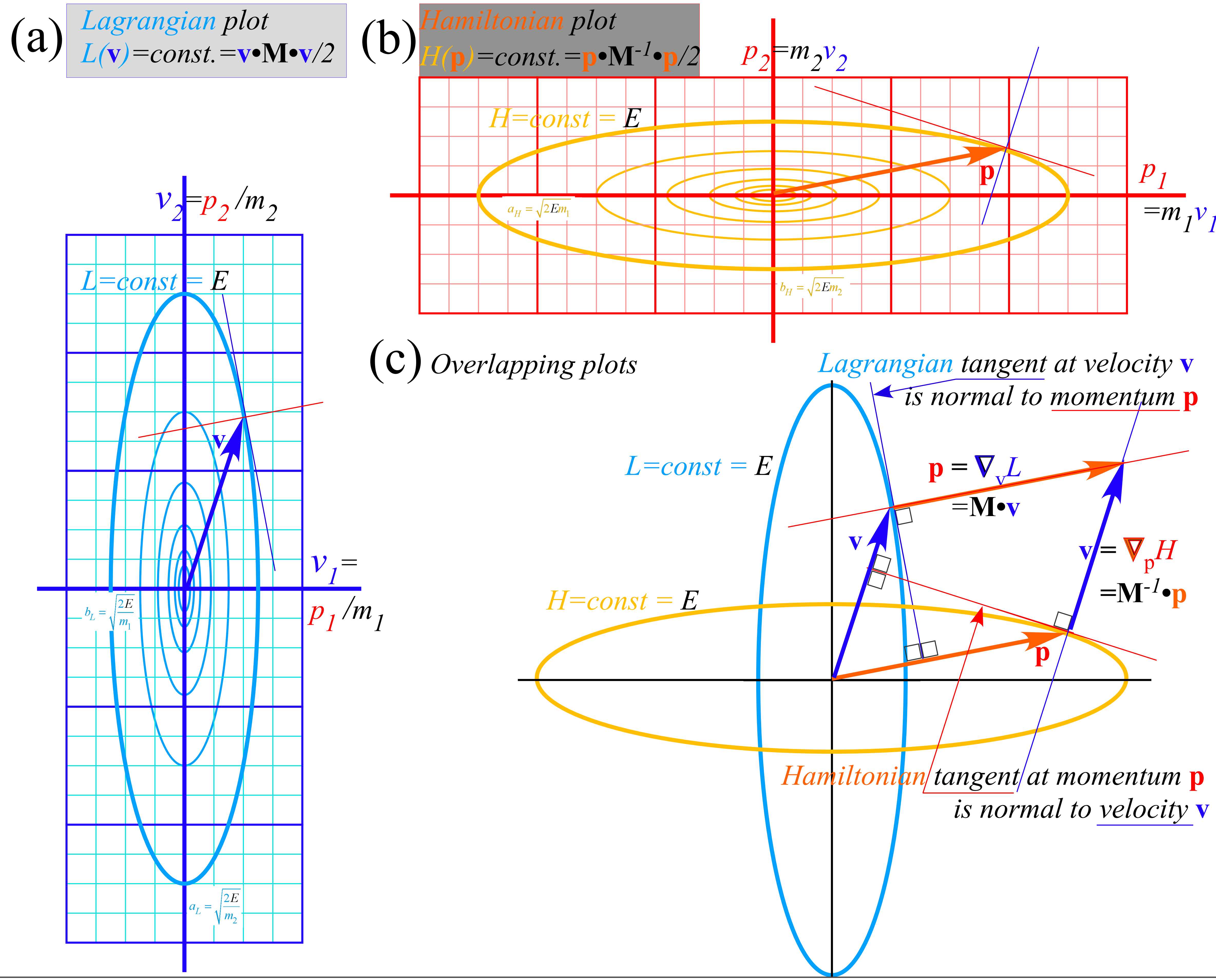
(a) *Lagrangian plot* $L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$



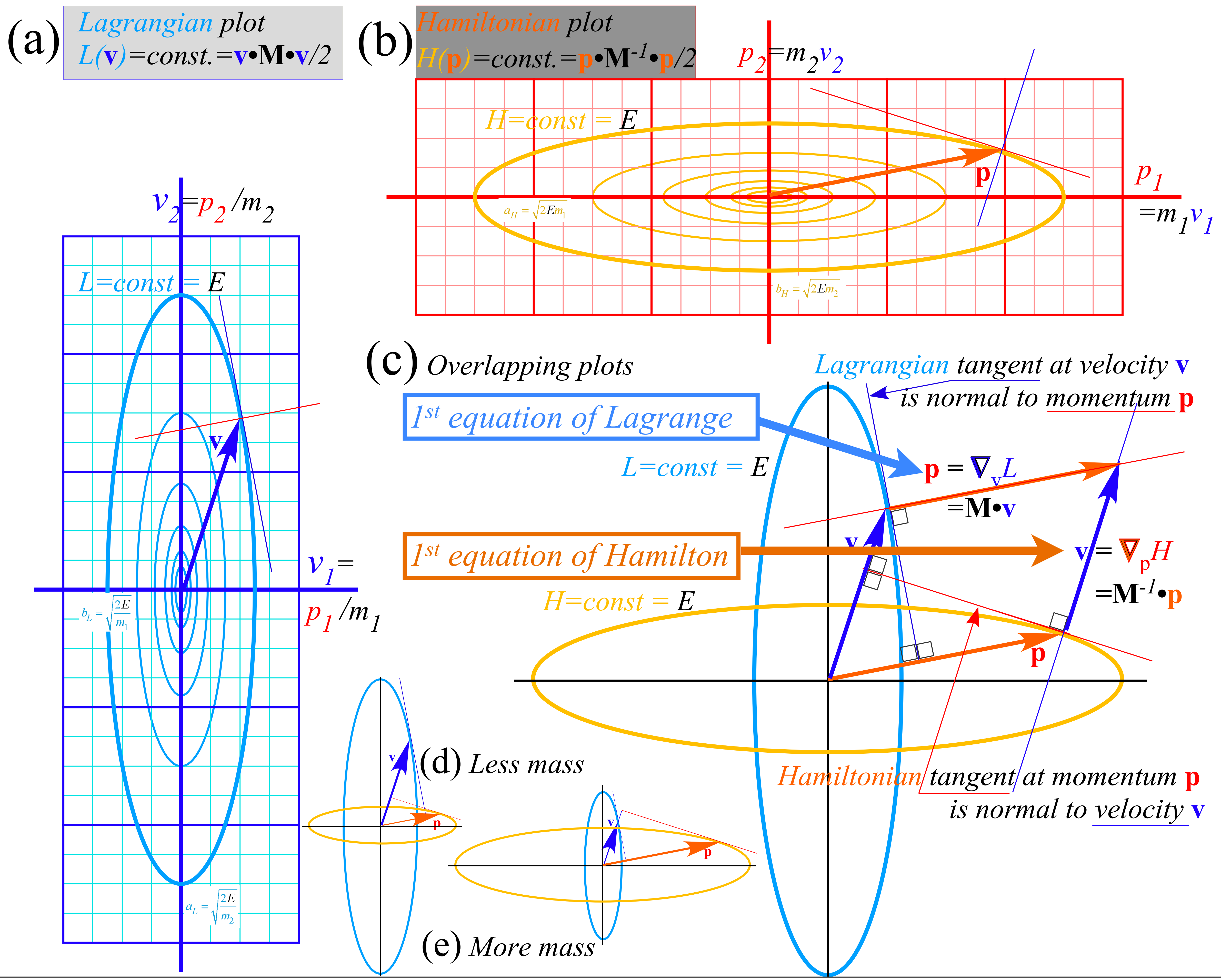
(b) *Hamiltonian plot* $H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$



Unit 1
Fig. 12.2



Unit 1
Fig. 12.2



Introducing the Poincare′ and Legendre contact transformations

Geometry of Legendre contact transformation

Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

Introducing the Poincare′ and Legendre contact transformations

Given matrix relation: $\mathbf{p}=\mathbf{M}\cdot\mathbf{v}$ *or its inverse:* $\mathbf{v}=\mathbf{M}^{-1}\cdot\mathbf{p}$ *you might be tempted to rewrite*

Q-forms $L(\mathbf{v}..)=(1/2)\mathbf{v}\cdot\mathbf{M}\cdot\mathbf{v}$ *or* $H(\mathbf{p}..)=(1/2)\mathbf{p}\cdot\mathbf{M}^{-1}\cdot\mathbf{p}$ *to be* $H=(1/2)\mathbf{p}\cdot\mathbf{v}$ *or equivalently* $L=(1/2)\mathbf{v}\cdot\mathbf{p}$.

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Numerically-CORRECT, but Differentially-WRONG!

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Numerically-CORRECT, but Differentially-WRONG! *(In classical physics* $\mathbf{p}\cdot\mathbf{v}$ *and* $\mathbf{v}\cdot\mathbf{p}$ *are identical)*

Instead try: $H(\mathbf{p}..)=\mathbf{p}\cdot\mathbf{v}-(1/2)\mathbf{v}\cdot\mathbf{p}=\mathbf{p}\cdot\mathbf{v}-L(\mathbf{v}..)$ *or else:* $L(\mathbf{v}..)=\mathbf{p}\cdot\mathbf{v}-H(\mathbf{p}..)$

Introducing the Poincare′ and Legendre contact transformations

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That is ... the ***Legendre contact transformation***

$$L(\mathbf{v})=\mathbf{p}\bullet\mathbf{v}-H(\mathbf{p}) \qquad \text{or:} \qquad H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$$

Introducing the Poincare′ and Legendre contact transformations

Given matrix relation: $\mathbf{p}=\mathbf{M}\bullet\mathbf{v}$ *or its inverse:* $\mathbf{v}=\mathbf{M}^{-1}\bullet\mathbf{p}$ *you might be tempted to rewrite*

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That is ... the ***Legendre contact transformation***

$$L(\mathbf{v})=\mathbf{p}\bullet\mathbf{v}-H(\mathbf{p}) \qquad \text{or:} \qquad H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$$

Now explicit dependency (non)-relations give the right derivatives

$$\frac{\partial L(\mathbf{v})}{\partial \mathbf{p}}=\frac{\partial}{\partial \mathbf{p}}\mathbf{p}\bullet\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad \frac{\partial H(\mathbf{p})}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\mathbf{p}\bullet\mathbf{v}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

$$0=\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad 0=\mathbf{p}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

Introducing the Poincare′ and Legendre contact transformations

Given matrix relation: $\mathbf{p}=\mathbf{M}\cdot\mathbf{v}$ *or its inverse:* $\mathbf{v}=\mathbf{M}^{-1}\cdot\mathbf{p}$ *you might be tempted to rewrite*

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$$0=\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad\qquad 0=\mathbf{p}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

That is *Hamilton's 1st equation(s)* *and* *Lagrange's 1st equation(s)*

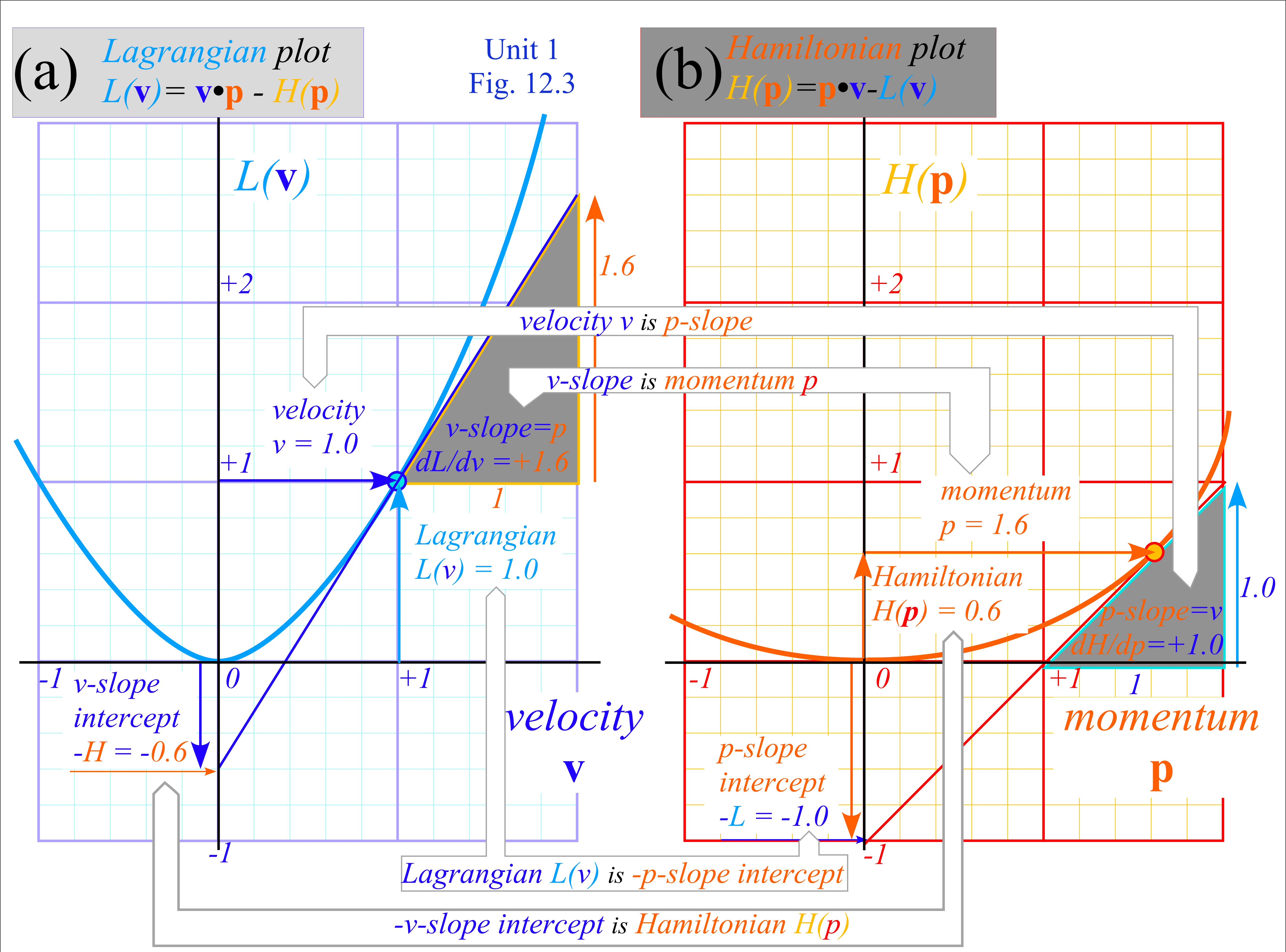
$$\mathbf{v}=\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad\qquad \mathbf{p}=\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

Introducing the Poincare´ and Legendre contact transformations

→ *Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)*

Example from thermodynamics

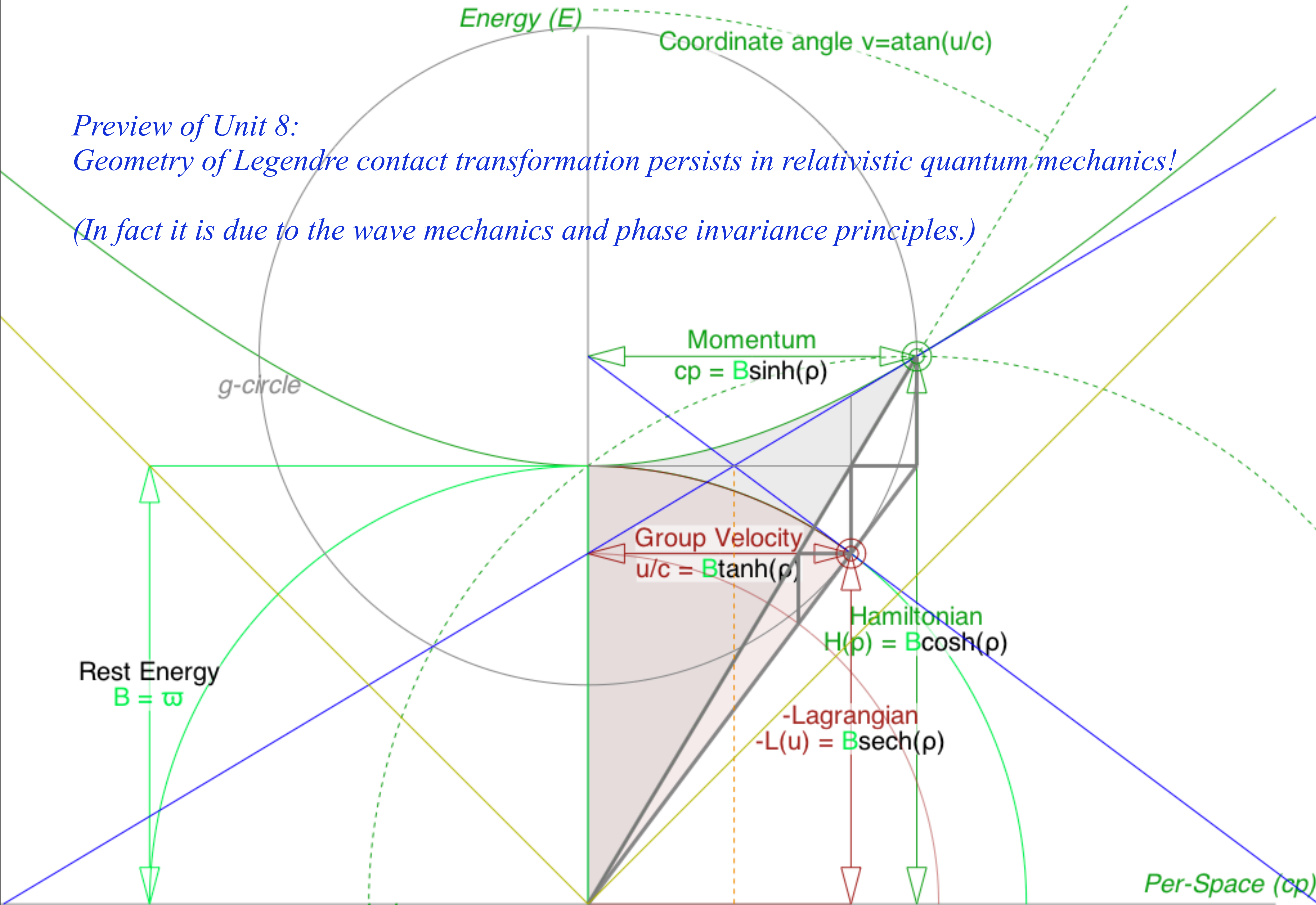
Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)



(a) Lagrangian plot
L(v)= v•p - H(p)
Unit 1
Fig. 12.3
(b) Hamiltonian plot
H(p)=p•v-L(v)
L(v)
H(p)
+2
1.6
+2
velocity v is p-slope
v-slope is momentum p
velocity
v = 1.0
v-slope=p
dL/dv =+1.6
+1
1
+1
momentum
p = 1.6
Lagrangian
L(v) = 1.0
Hamiltonian
H(p) = 0.6
1.0
p-slope=v
dH/dp=+1.0
-1
v-slope
intercept
-H = -0.6
0
+1
velocity
v
-1
0
+1
1
momentum
p
p-slope
intercept
-L = -1.0
-1
-1
Lagrangian L(v) is -p-slope intercept
-v-slope intercept is Hamiltonian H(p)

Preview of Unit 8:
Geometry of Legendre contact transformation persists in relativistic quantum mechanics!

(In fact it is due to the wave mechanics and phase invariance principles.)

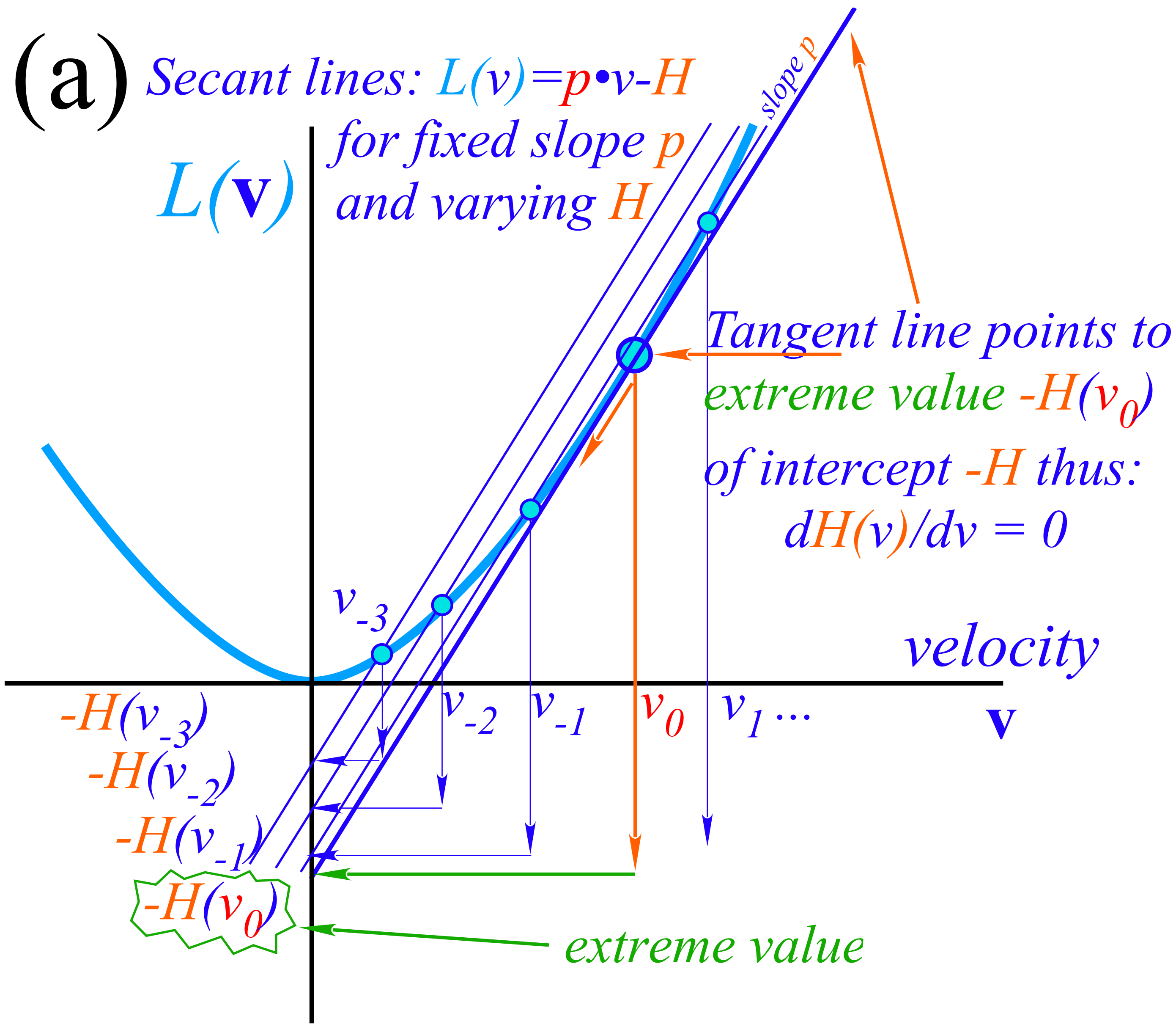


http://www.uark.edu/ua/modphys/testing/markup/RelaWavityWeb.html

How Legendre contact transformations work...(to make $\frac{\partial H}{\partial v}=0$ or $\frac{\partial L}{\partial p}=0$)

Secant lines $L(\mathbf{v}) = p \bullet v - H$ *of fixed slope* $p=\frac{\partial L}{\partial v}$
and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > \ldots$
for increasing velocity $v_{-2} > v_{-1} > \ldots > v_0$
lead to unique tangent to $L(\mathbf{v})$*-curve at the*
tangent contact point $v=v_0$ *that has* max $H(p\ v_0)$
Thus $\frac{\partial H}{\partial v}=0$

Unit 1
Fig. 12.4



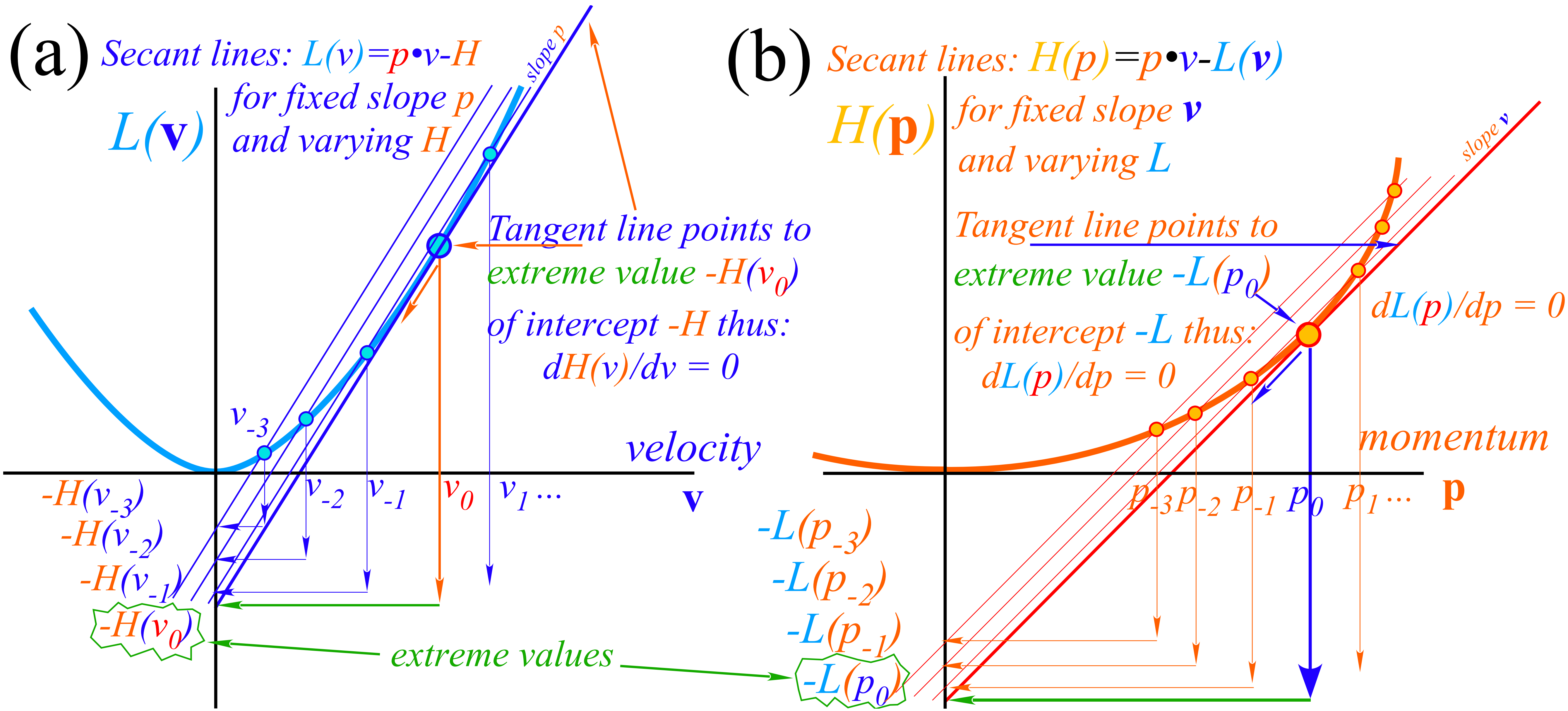
$\frac{\partial H}{\partial v}=0$ at each point $v=\frac{\partial H}{\partial p}$ of $L(v)$ with slope $p=\frac{\partial L}{\partial v}$

How Legendre contact transformations work... (to make $\frac{\partial H}{\partial v}=0$ or $\frac{\partial L}{\partial p}=0$)

Secant lines $L(\mathbf{v}) = p \bullet v - H$ *of fixed slope* $p=\frac{\partial L}{\partial v}$
and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > \ldots$
for increasing velocity $v_{-2} > v_{-1} > \ldots > v_0$
lead to unique tangent to $L(\mathbf{v})$*-curve at the tangent contact point* $v=v_0$ *that has* <u>*max*</u> $H(p\ v_0)$
Thus $\frac{\partial H}{\partial v}=0$

(Similarly...)

Unit 1
Fig. 12.4



$\frac{\partial H}{\partial v}=0$ at each point $v=\frac{\partial H}{\partial p}$ of $L(v)$ with slope $p=\frac{\partial L}{\partial v}$

$\frac{\partial L}{\partial p}=0$ at each point $p=\frac{\partial L}{\partial v}$ of $H(p)$ with slope $v=\frac{\partial H}{\partial p}$

Introducing the Poincare′ and Legendre contact transformations

Geometry of Legendre contact transformation *(Preview of Unit 8 relativistic quantum mechanics)*

→ *Example from thermodynamics*

Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

Example of Legendre contact transformation in thermodynamics

Internal energy U(S,V) is defined as a function of entropy S and volume V.

A new function *enthalpy H(S,P)* depends on entropy and *pressure P*.

It is a Legendre transform $H(S,P)=P\cdot V+U$ of energy $U(S,V)$ to new variable $P=-(\frac{\partial U}{\partial V})_S$.

Example of Legendre contact transformation in thermodynamics

Lagrangian $L(r,v)$ position r velocity v

Internal energy U(S,V) is defined as a function of entropy S and volume V.

Hamiltonian $H(r,p)$ position r *momentum* p

A new function *enthalpy H(S,P)* depends on entropy and *pressure P*.

$H(r,p) = p \cdot v - L$ Lagrangian $L(r,v)$ $p = \left(\frac{\partial L}{\partial v}\right)_r$

It is a Legendre transform $H(S,P) = P \cdot V + U$ of energy $U(S,V)$ to new variable $P = -\left(\frac{\partial U}{\partial V}\right)_S$.

Example of Legendre contact transformation in thermodynamics

Lagrangian $L(r,v)$ position r velocity v

Internal energy $U(S,V)$ is defined as a function of entropy S and volume V.

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$H(r,p)= p \cdot v - L$ Lagrangian $L(r,v)$ $p = \left(\frac{\partial L}{\partial v}\right)_r$

It is a Legendre transform $H(S,P)=P\cdot V+U$ of energy $U(S,V)$ to new variable $P = -\left(\frac{\partial U}{\partial V}\right)_S$.

Except for ±signs, it's our Hamiltonian $H(p)=p\cdot v-L(v)$ going from Lagrangian $L(v)$

to use new variable momentum $p = \left(\frac{\partial L}{\partial v}\right)_x$.

Introducing the Poincare′ and Legendre contact transformations

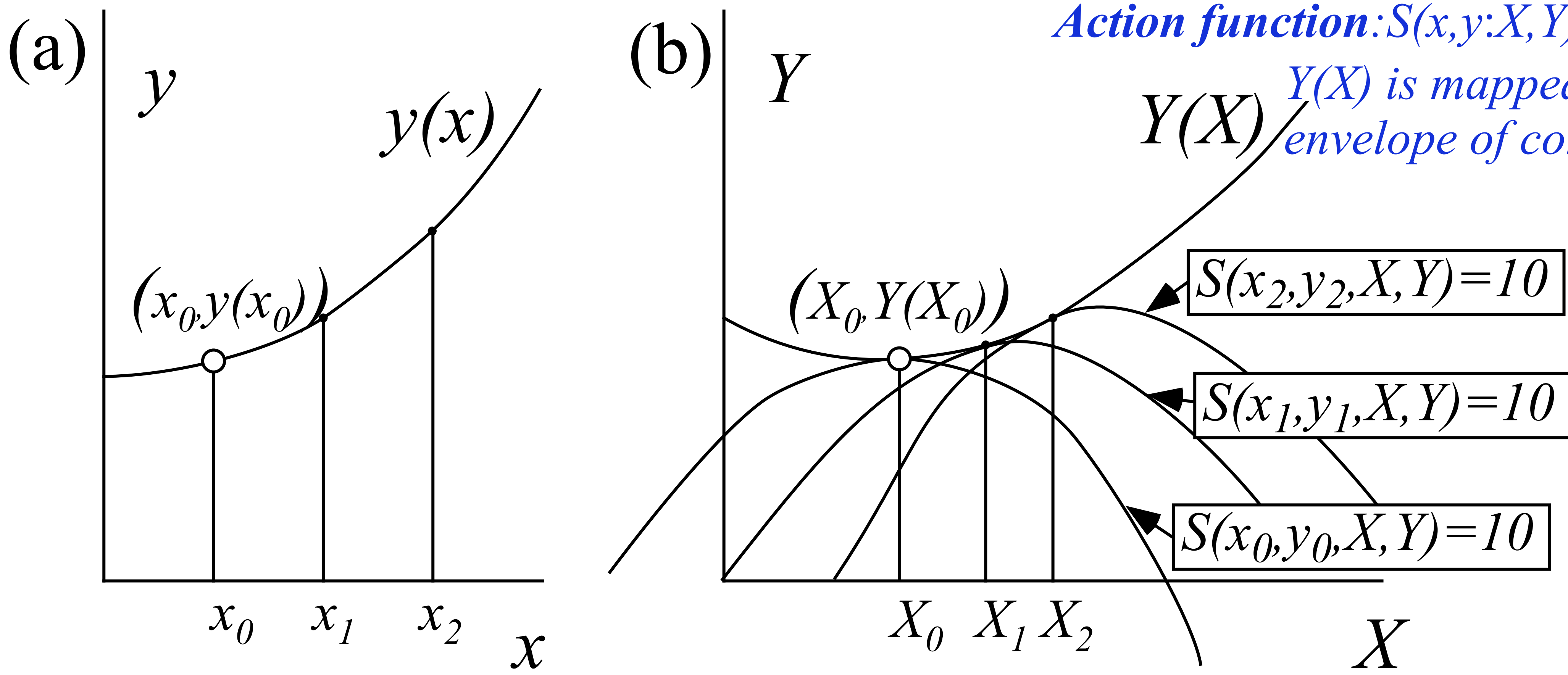
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)

Example from thermodynamics

→ *Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)*

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
Action function: *S(x,y:X,Y)=const. does mapping.*
Y(X) is mapped from y(x) as an envelope of contacting S=const. curves.



Unit 1
Fig. 12.7

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or ***Action function****: $S(x,y:X,Y)$=const. does mapping.*

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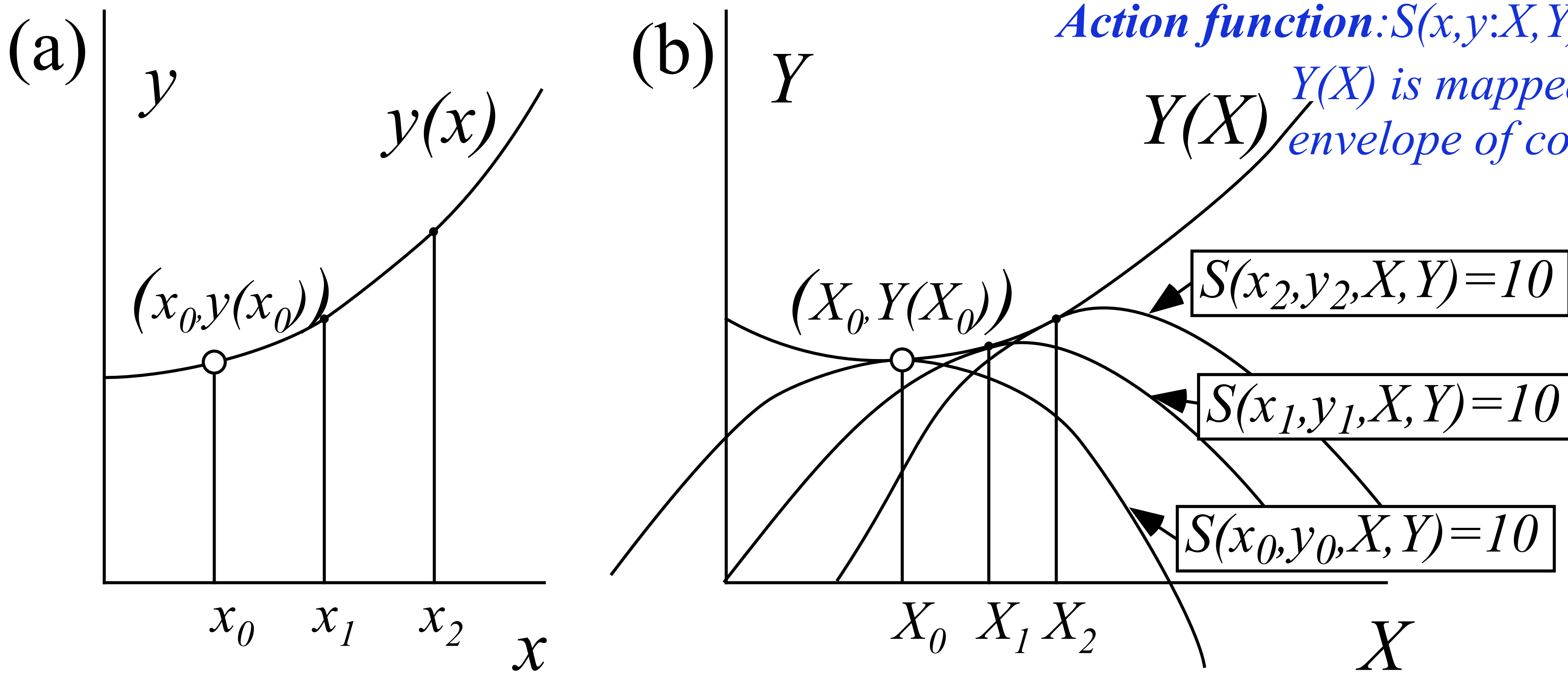
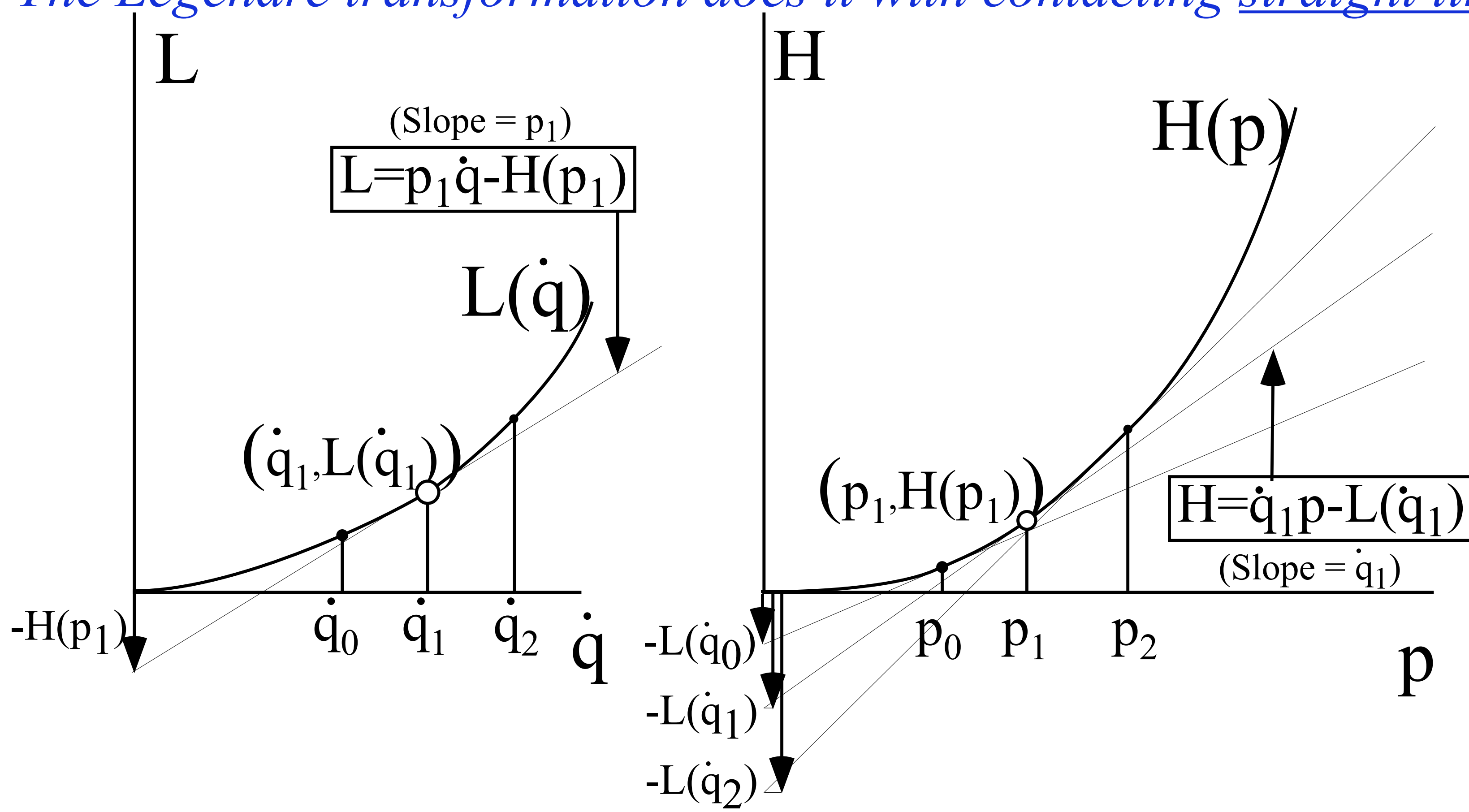


Unit 1
Fig. 12.7

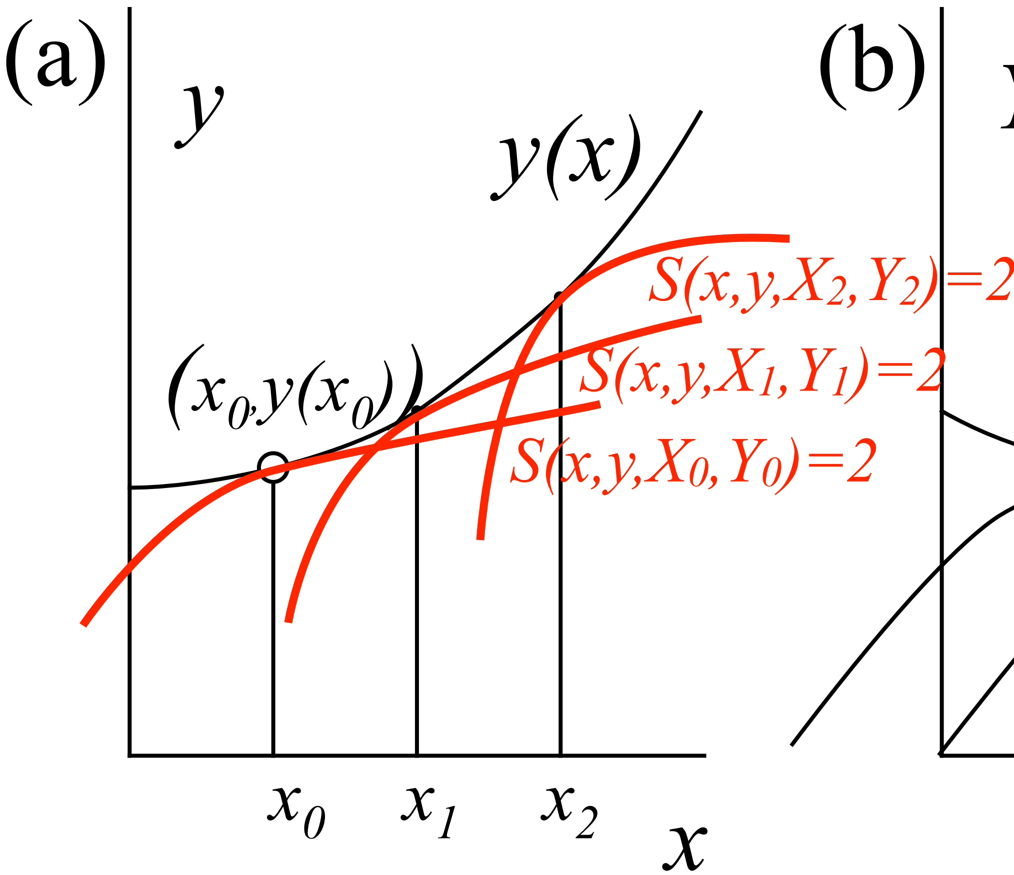
The Legendre transformation does it with contacting *<u>straight line</u>* *tangents.*

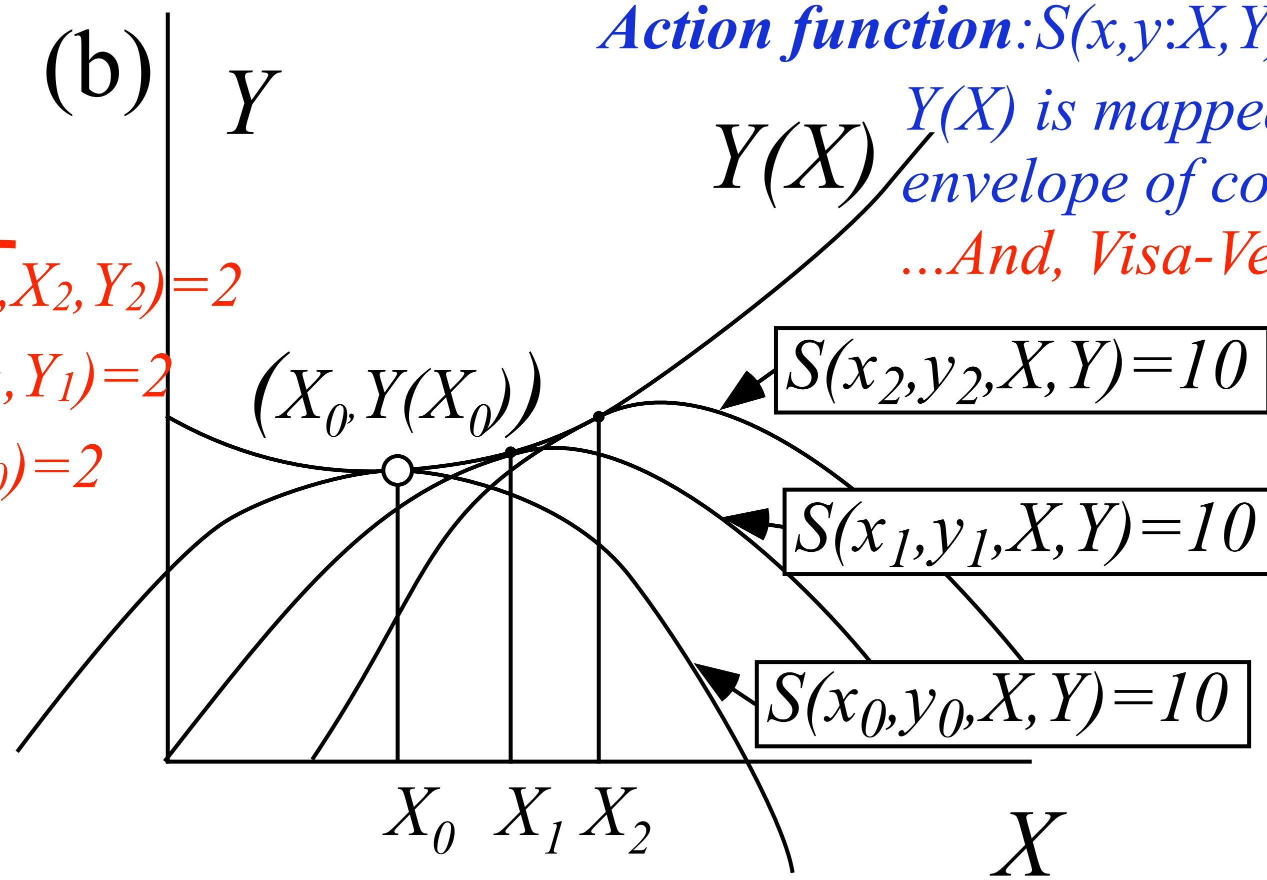


Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation

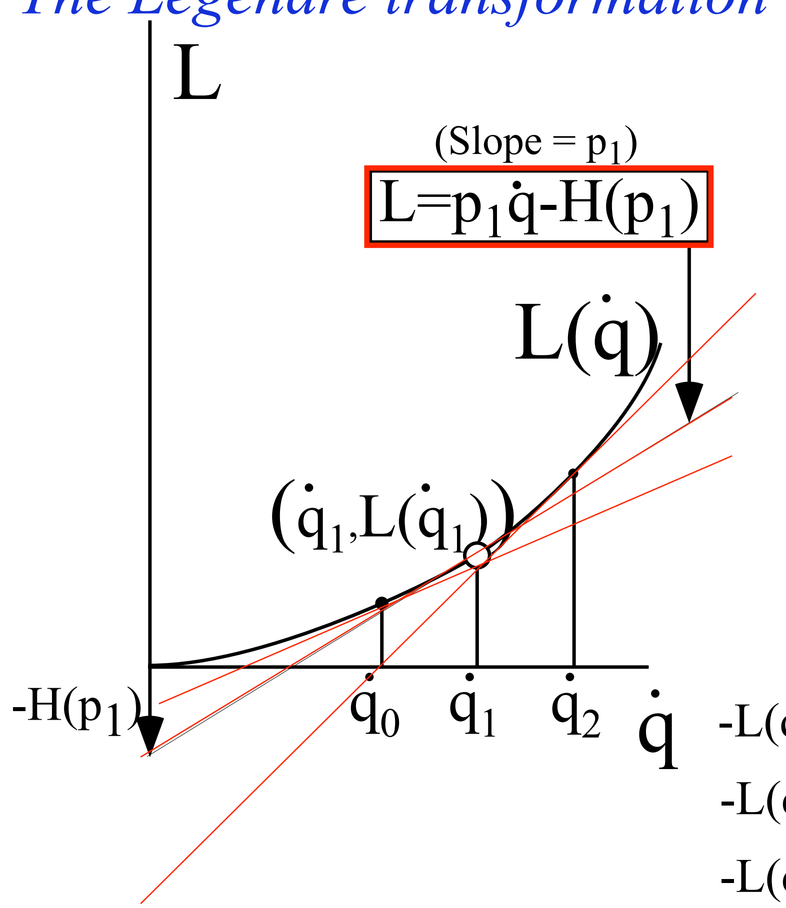
Active-Contact-Transformation Generator or
Action function: *$S(x,y:X,Y)$=const. does mapping.*
$Y(X)$ is mapped from $y(x)$ as an envelope of contacting S=const. curves.
...And, Visa-Versa !...

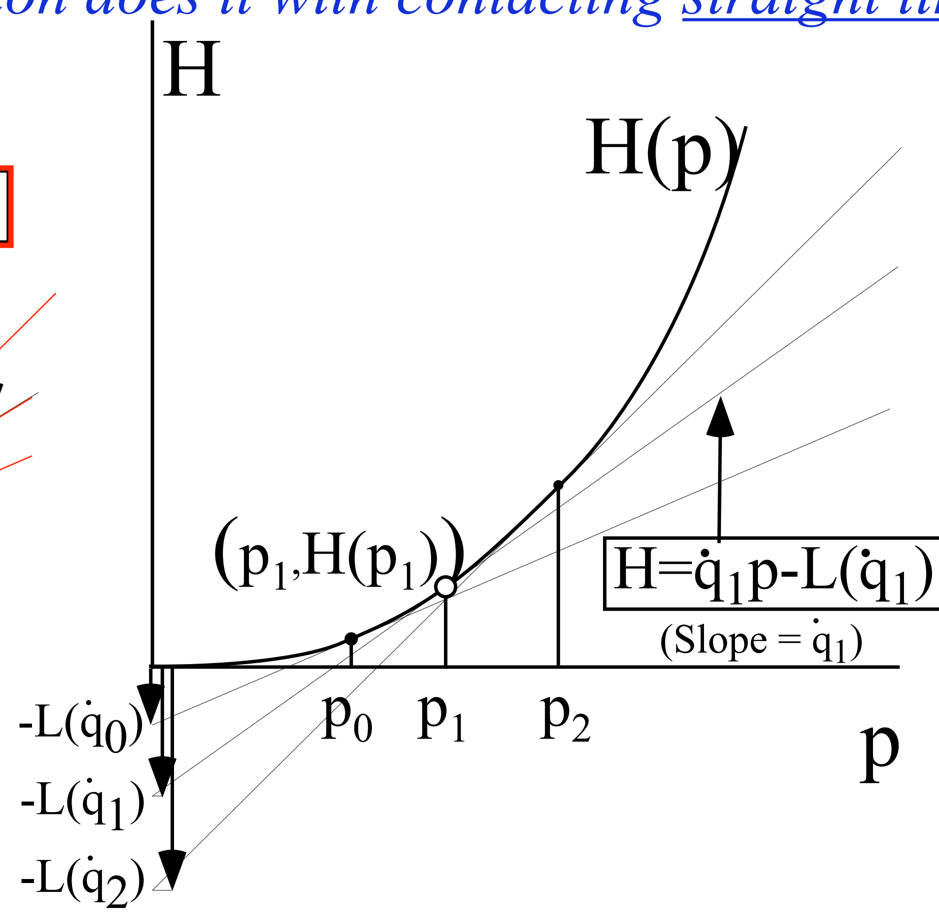




Unit 1
Fig. 12.7

The Legendre transformation does it with contacting <u>straight line</u> tangents.





Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation

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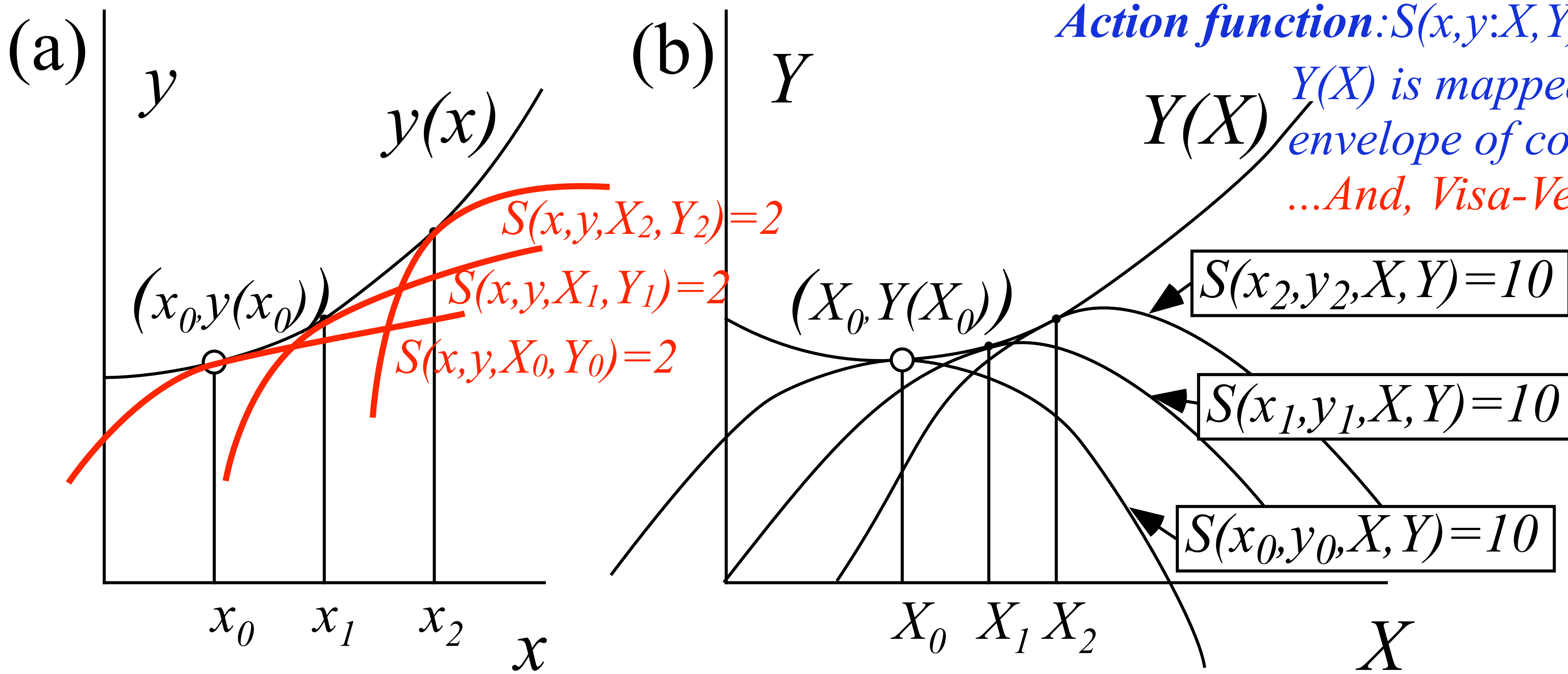
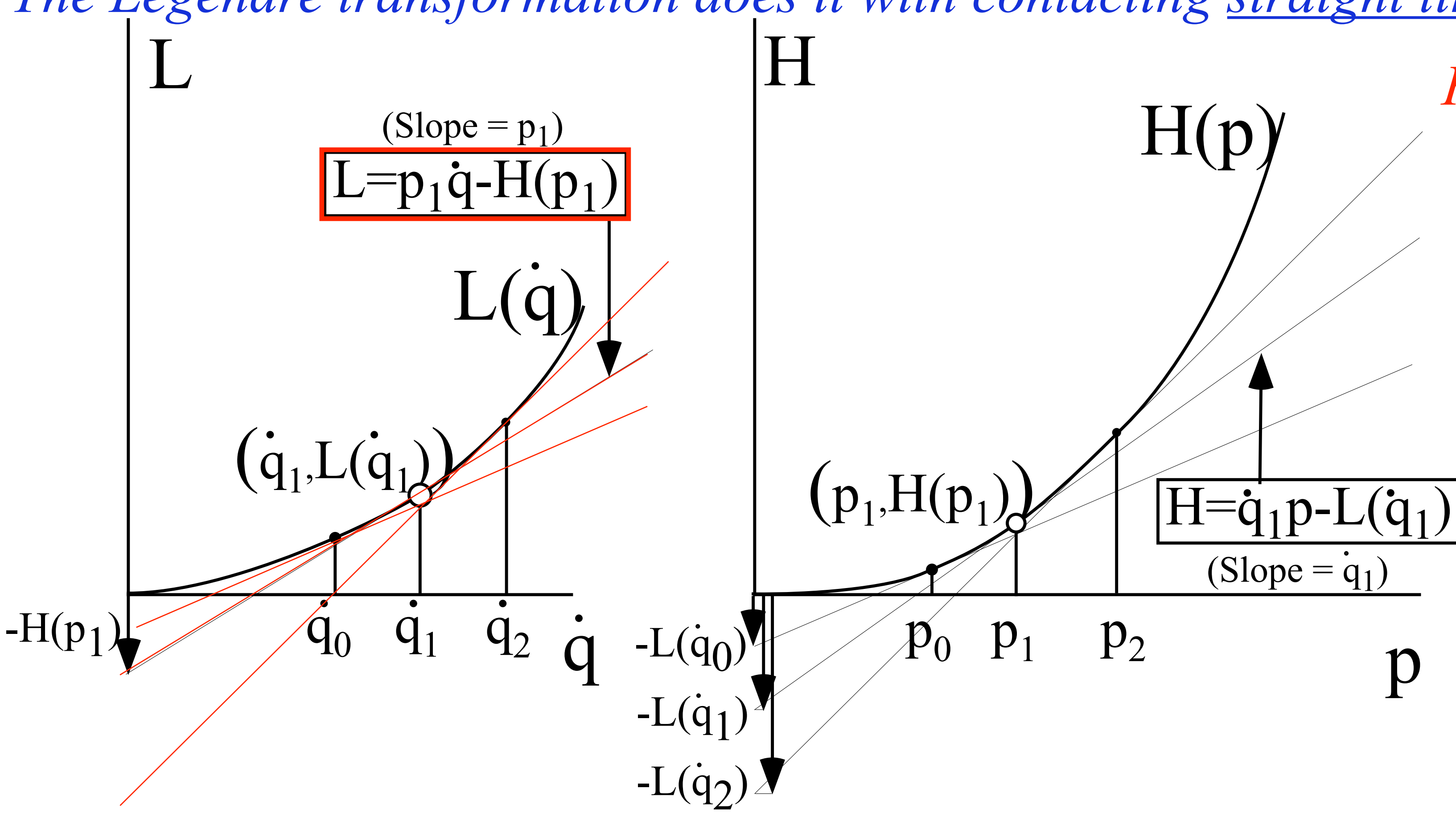


Unit 1
Fig. 12.7

The Legendre transformation does it with contacting straight line tangents.



Poincare's differential action

$$dS = L\,dt = p \cdot \dot{q}\,dt - H \cdot dt$$
$$= p \cdot dq - H \cdot dt$$

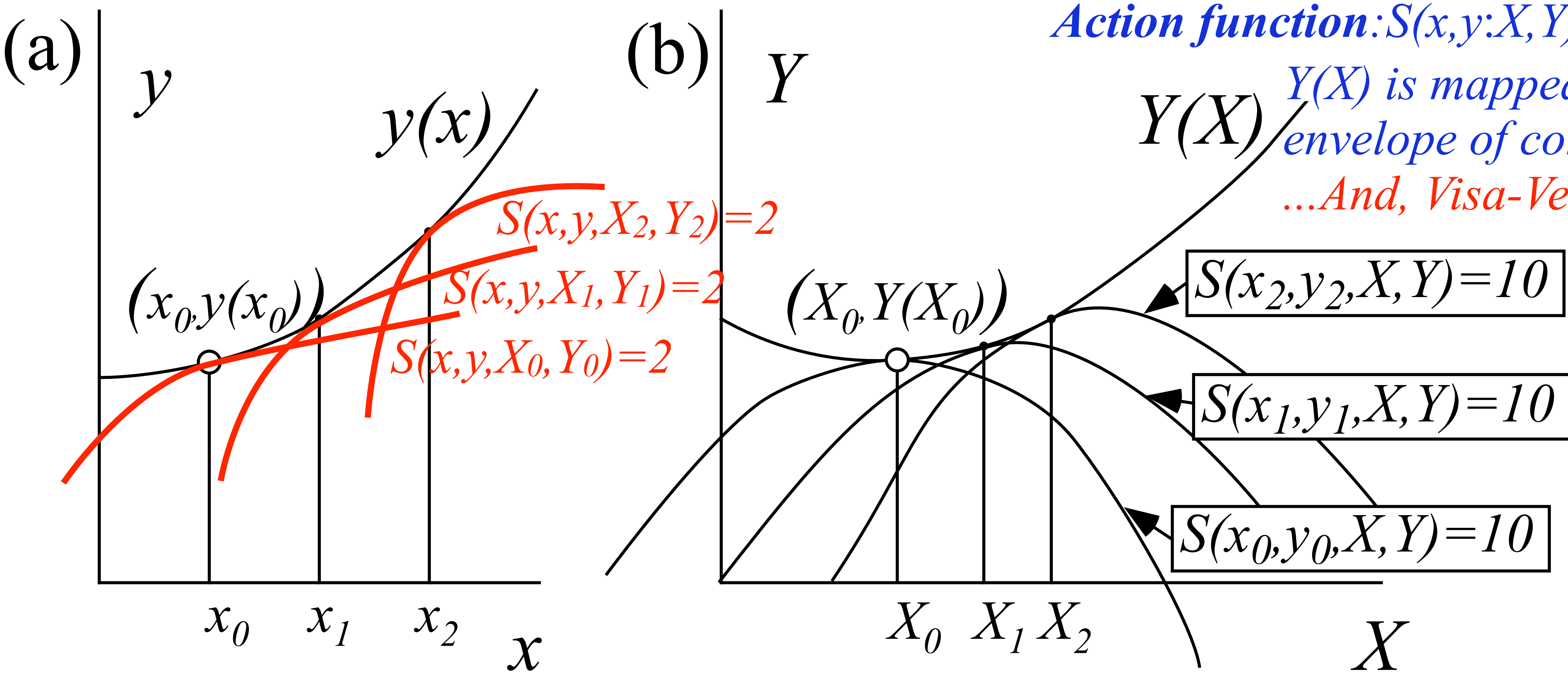
Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or ***Action function***: $S(x,y{:}X,Y)$=const. *does mapping.*

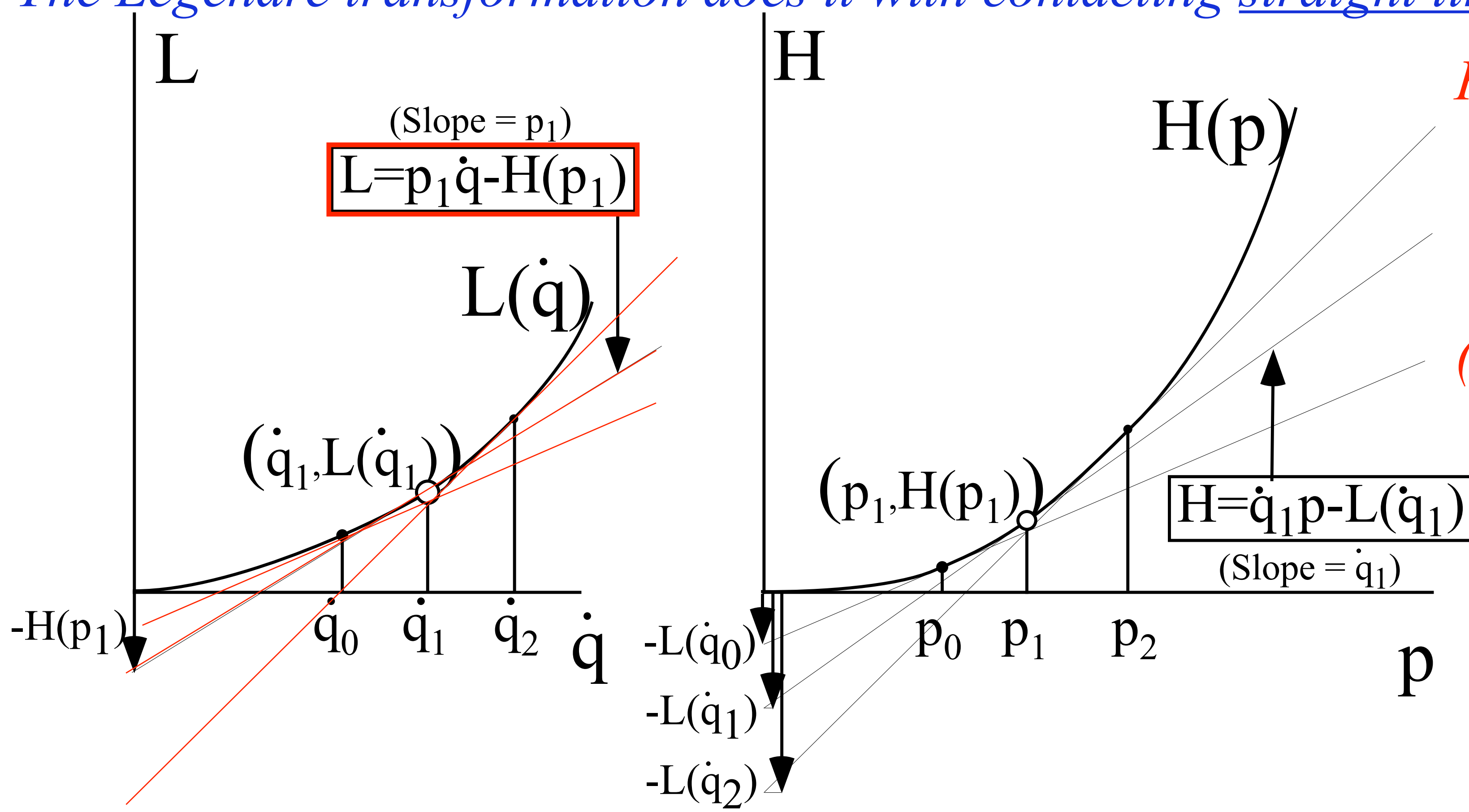
$Y(X)$ is mapped from $y(x)$ as an envelope of contacting S=const. curves.

...And, Visa-Versa !...



Unit 1
Fig. 12.7

The Legendre transformation does it with contacting straight line tangents.



Poincare's differential action

$$dS = L\,dt = p\cdot\dot{q}\,dt - H\cdot dt$$
$$= p\cdot dq - H\cdot dt$$

(Quantum phase differential)

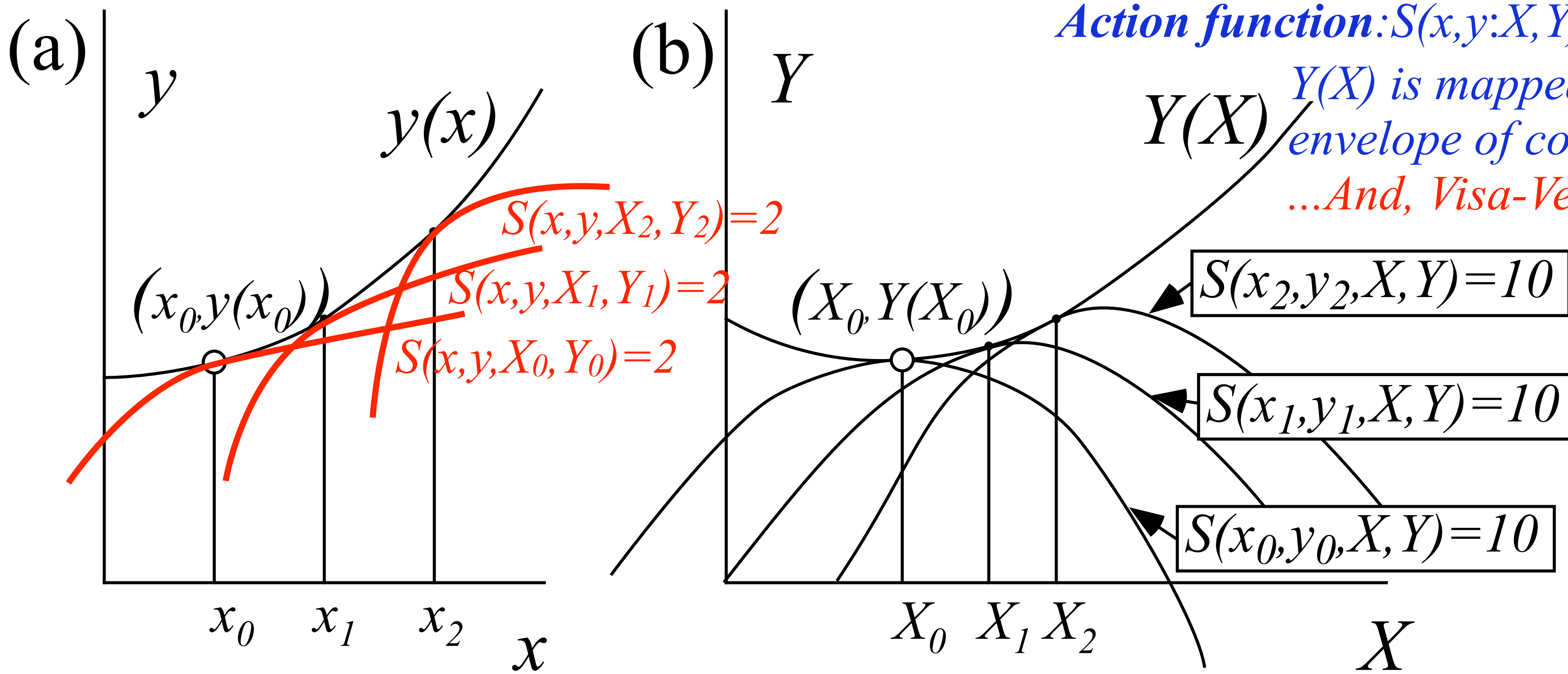
Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or ***Action function****: S(x,y:X,Y)=const. does mapping.*

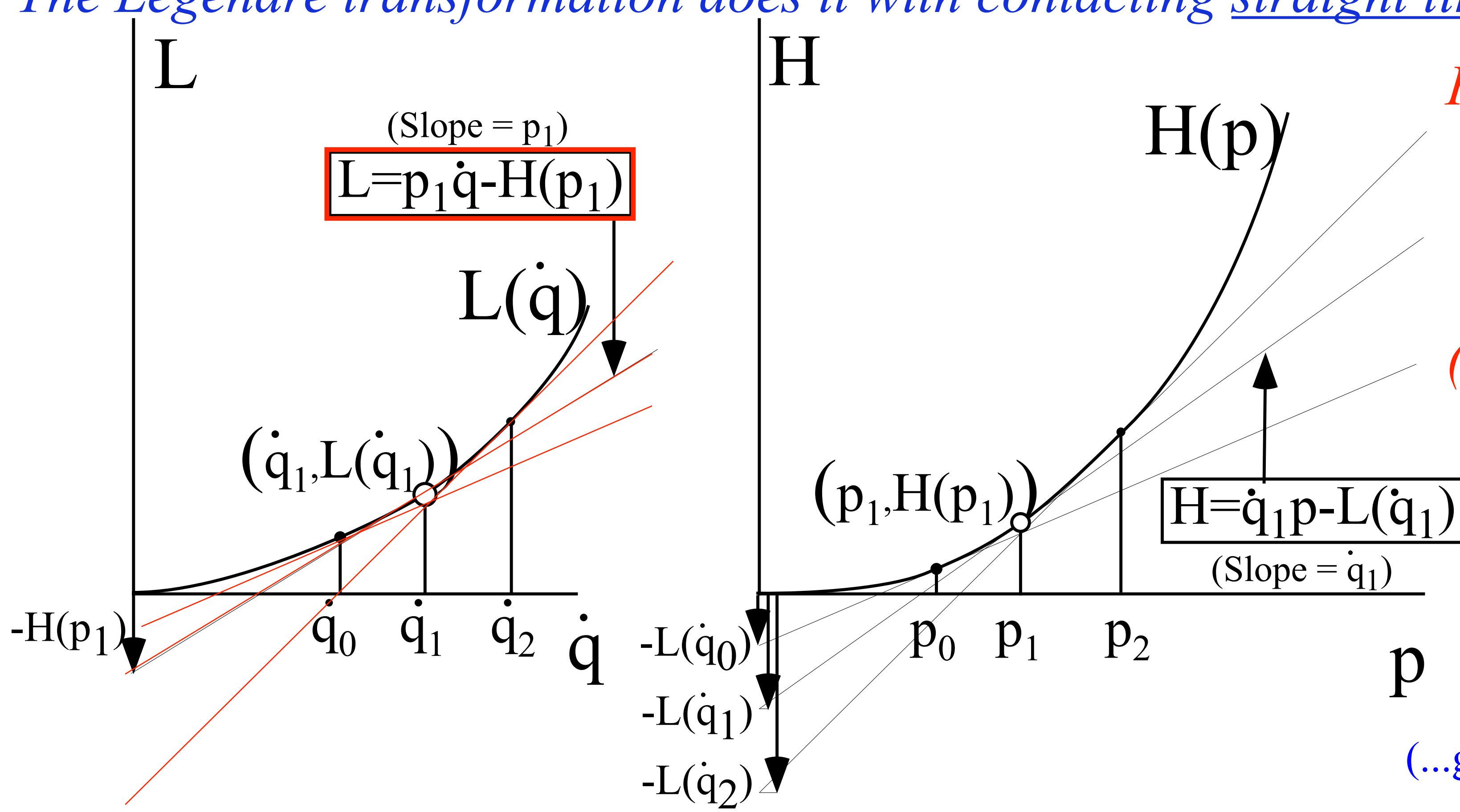
Y(X) is mapped from y(x) as an envelope of contacting S=const. curves.

...And, Visa-Versa !...



Unit 1
Fig. 12.7

The Legendre transformation does it with contacting *<u>straight line</u>* *tangents.*



Poincare's differential action

$$dS = L\,dt = p\cdot\dot{q}\,dt - H\cdot dt$$
$$= p\cdot dq - H\cdot dt$$

(Quantum phase differential)

Unit 1
Fig. 12.9

This extraordinary claim
needs extraordinary proof!

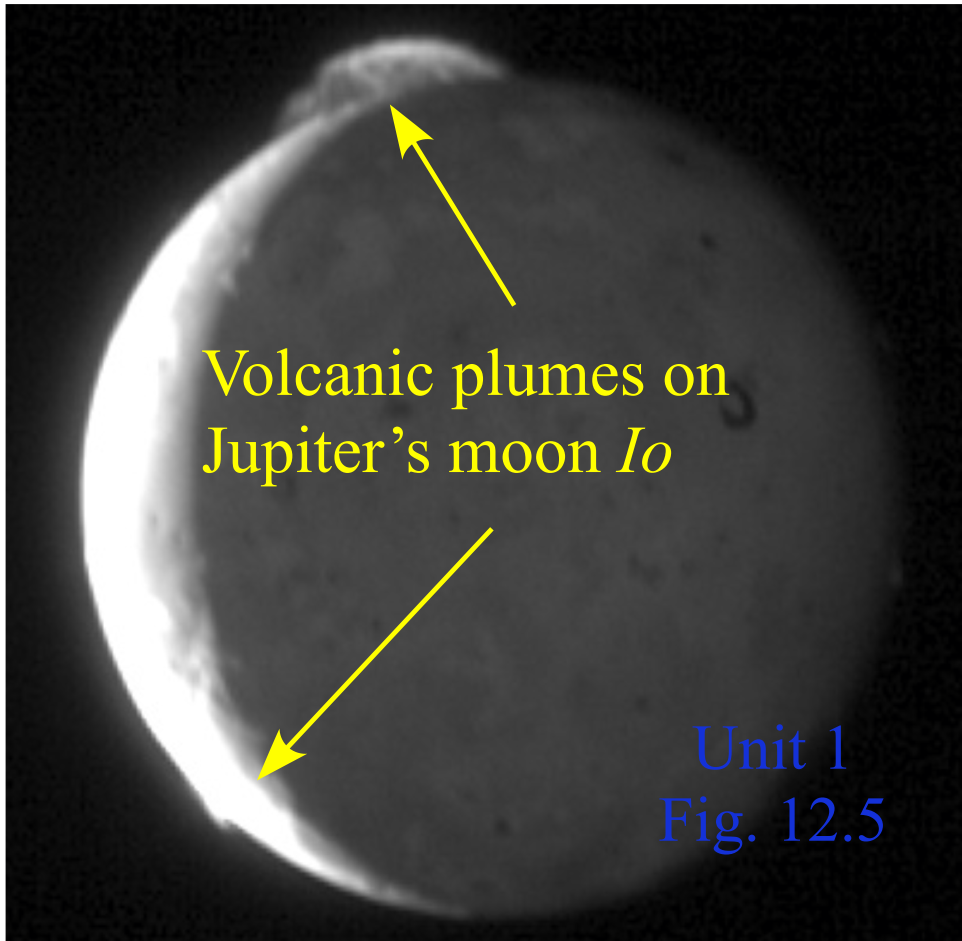
(...given in Ch. 12 Unit 1 and in Unit 8.)

A general contact transformation from sophomore physics

→ *Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"*

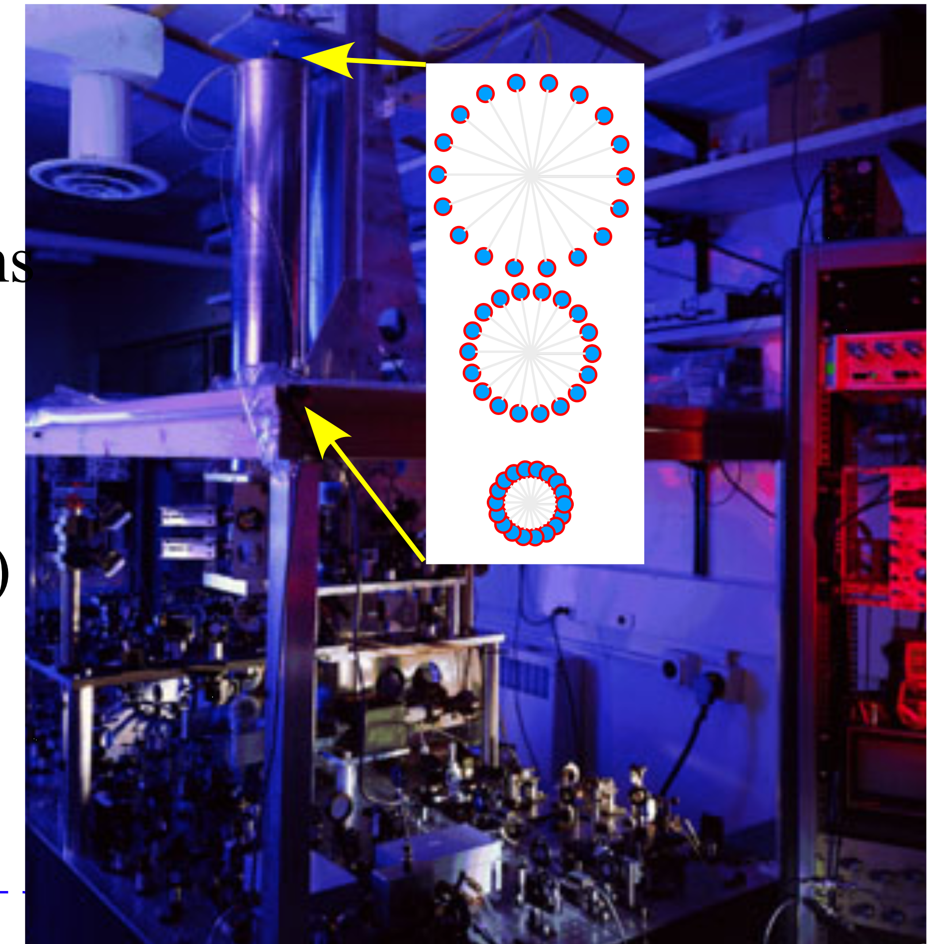
Intuitive-geometric development of " " " and " " "

(a)

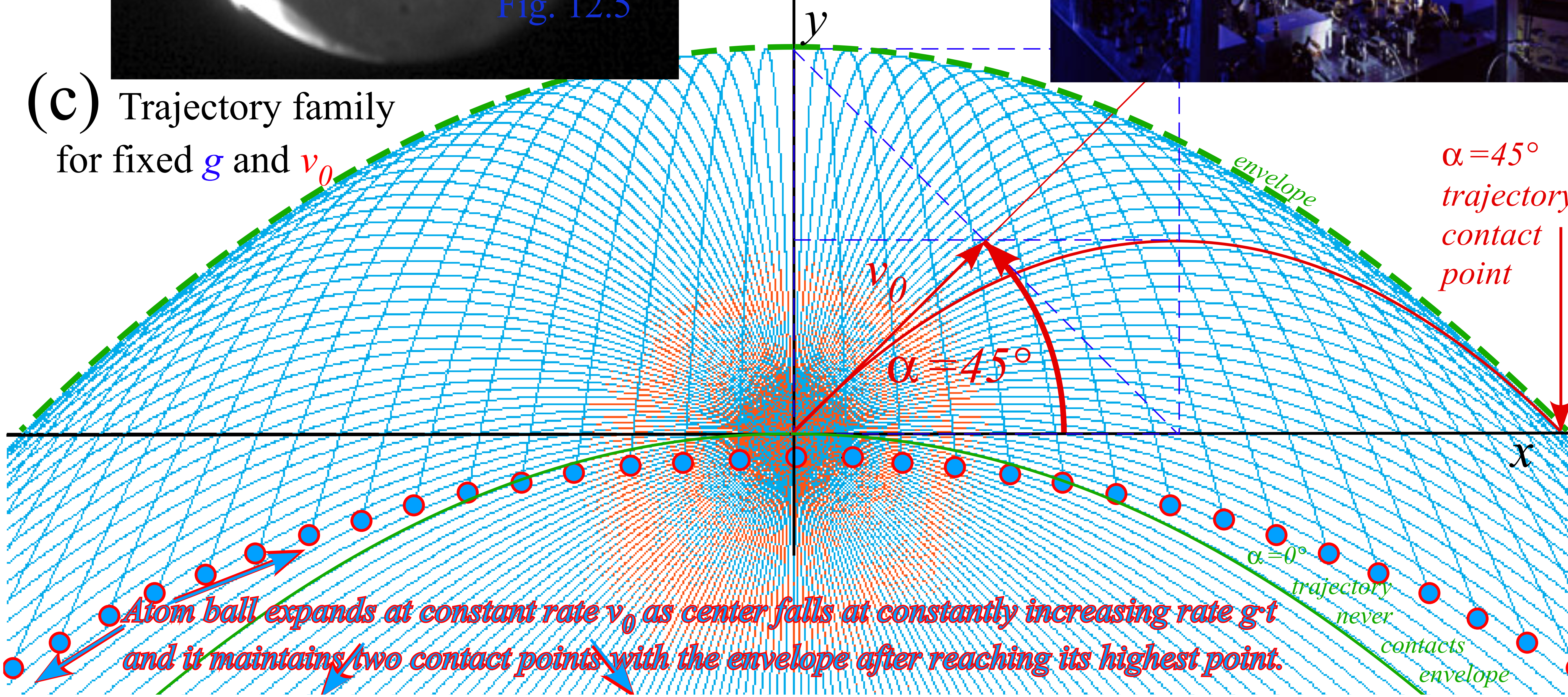


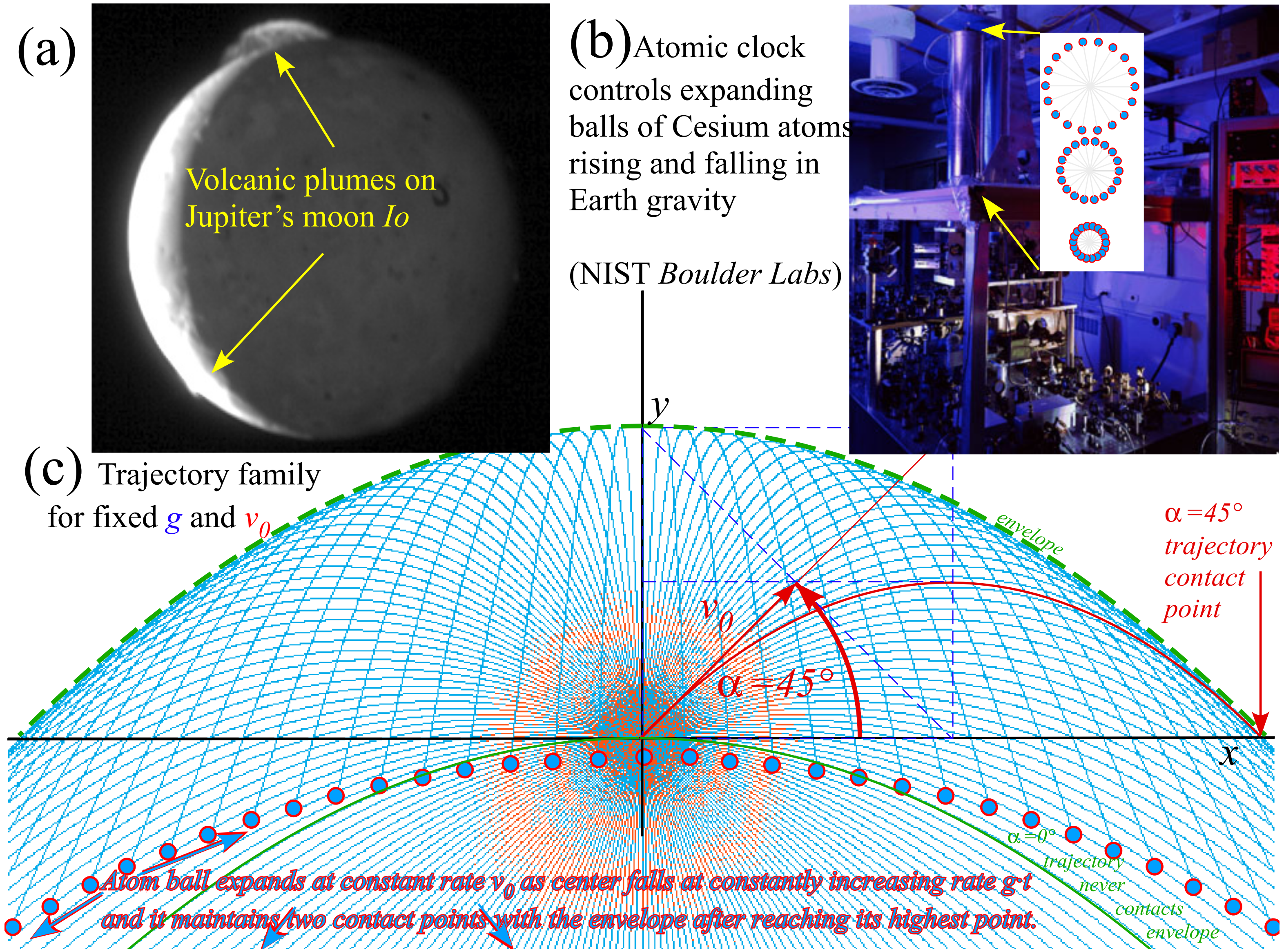
(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity

(NIST *Boulder Labs*)



(c) Trajectory family for fixed g and v_0





Unit 1
Fig. 12.5

UP-1 formulas for trajectories in constant gravity g

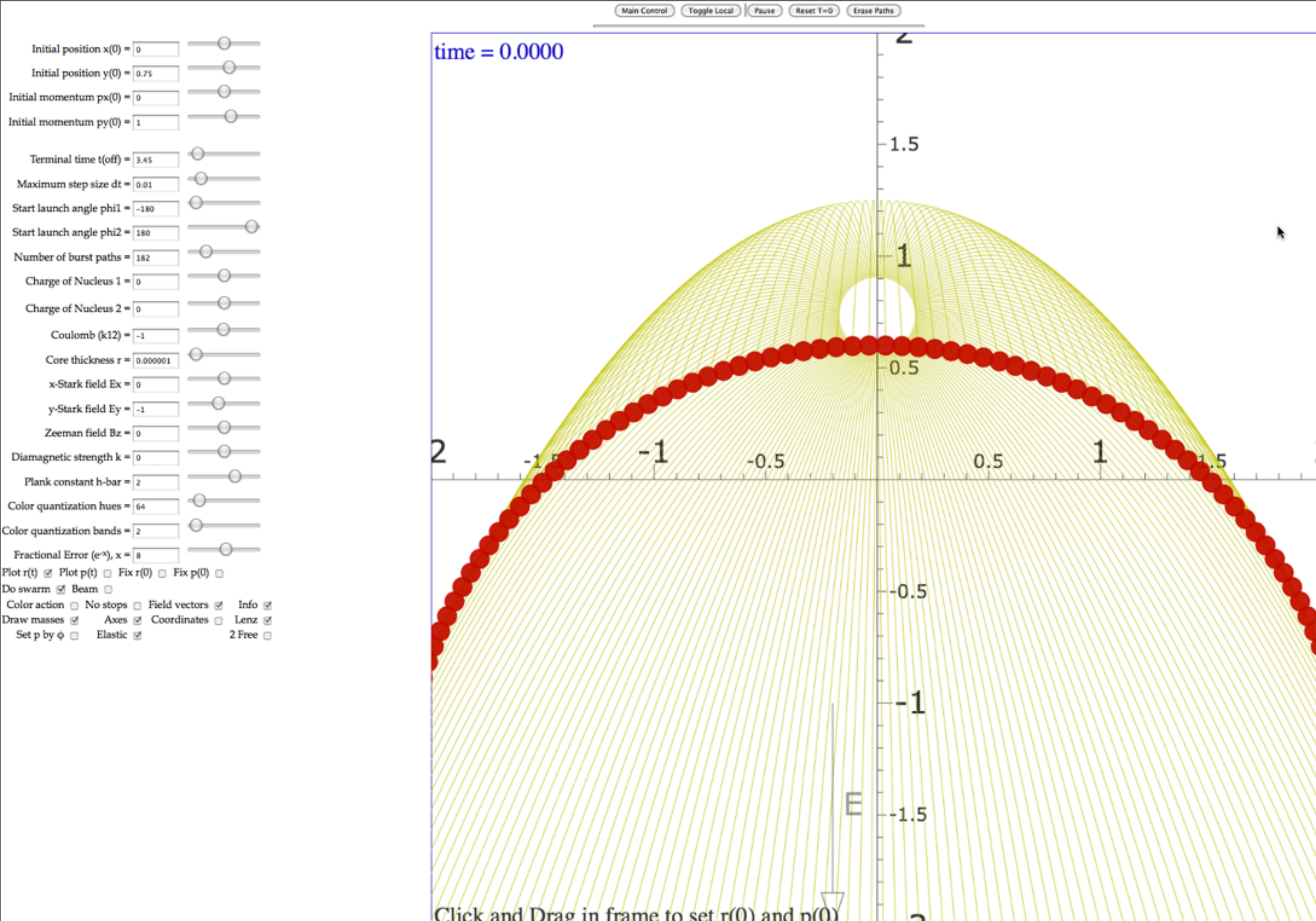
$$x(t) = (v_0 \cos\alpha)t \qquad y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$\dot{x}(0) = v_x(0) = v_0 \cos\alpha \qquad \dot{y}(0) = v_y(0) = v_0 \sin\alpha$$

Substitute time $t = x/(v_0 \cos\alpha)$ into $y(t)$

$$y(x) = \frac{v_0 \sin\alpha}{v_0 \cos\alpha} x - \frac{gx^2}{2v_0^2 \cos^2\alpha}$$

$$y(x) = x \tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha}$$



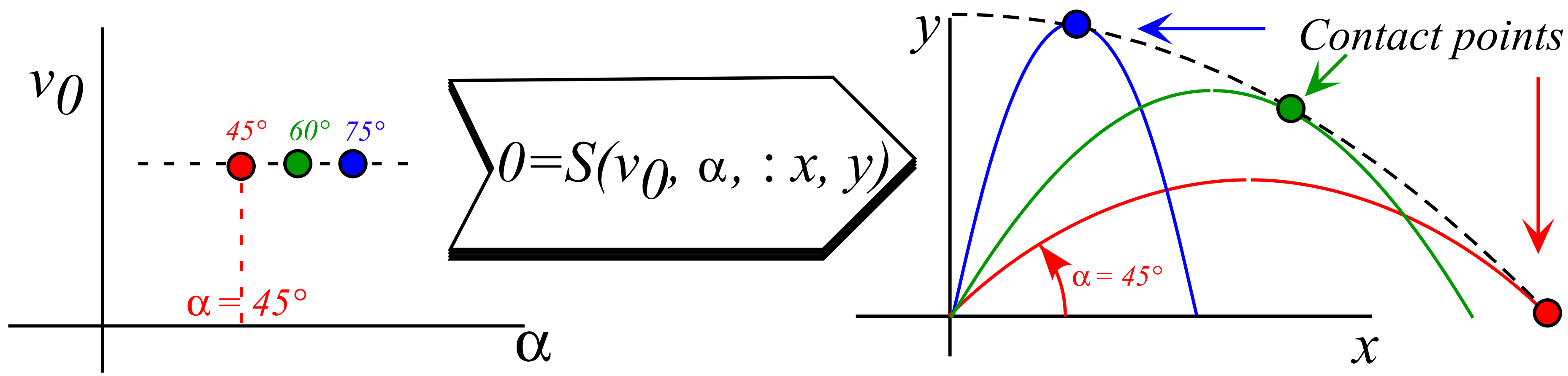
http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

becomes:

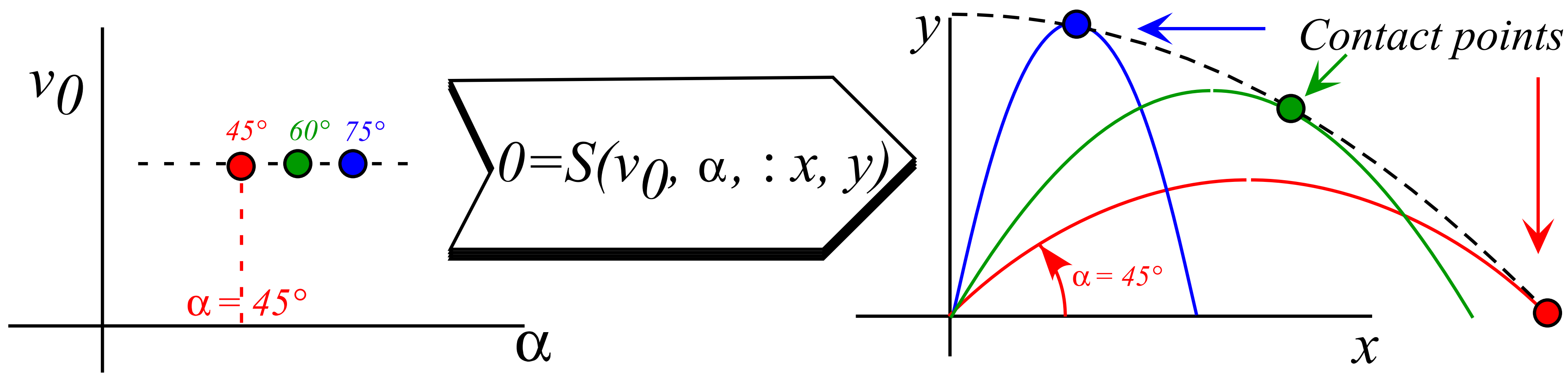
$$S(v_0, \alpha : x, y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Unit 1
Fig. 12.6

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha: x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$ *becomes:* $$S(v_0,\alpha : x, y) = -y + x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} = 0$$



Unit 1
Fig. 12.6

Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $$\frac{\partial S(v_0,\alpha : x, y)}{\partial\alpha} = 0$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

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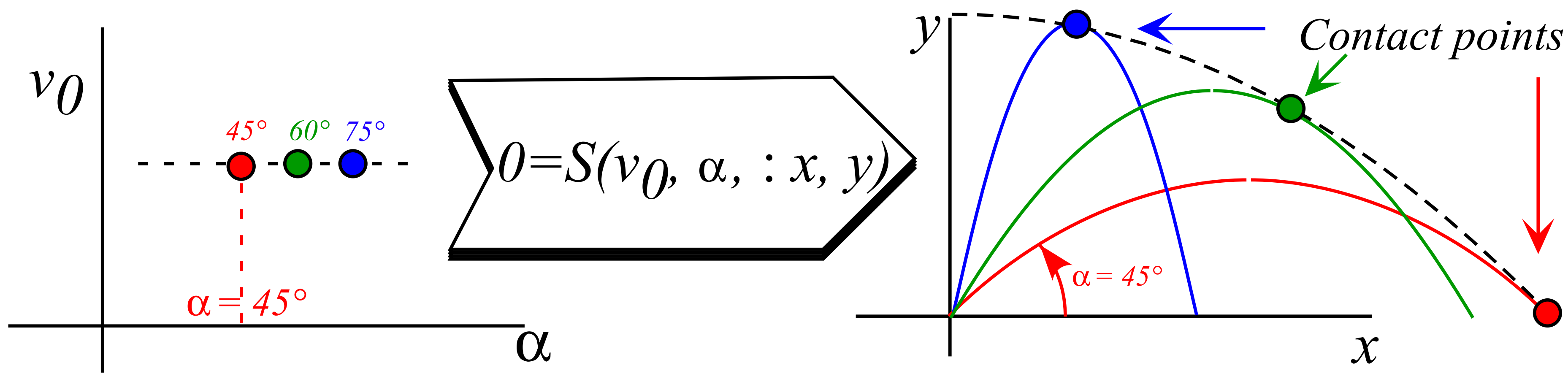


Unit 1
Fig. 12.6

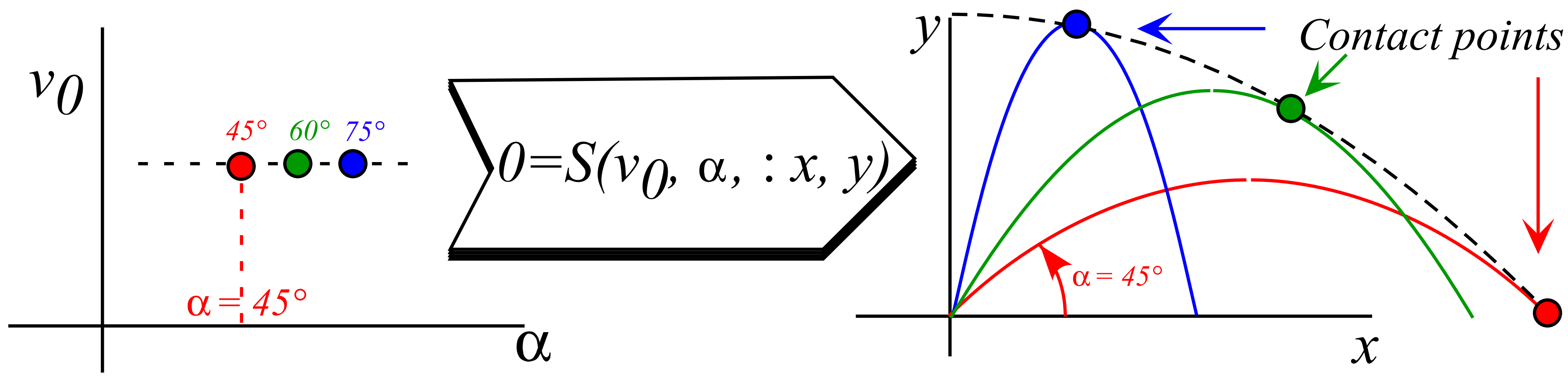
Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where:

$$\frac{\partial S(v_0,\alpha : x,y)}{\partial \alpha} = 0$$

$$x\frac{\partial \tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial \cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha}$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$ *becomes:* $$S(v_0,\alpha : x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



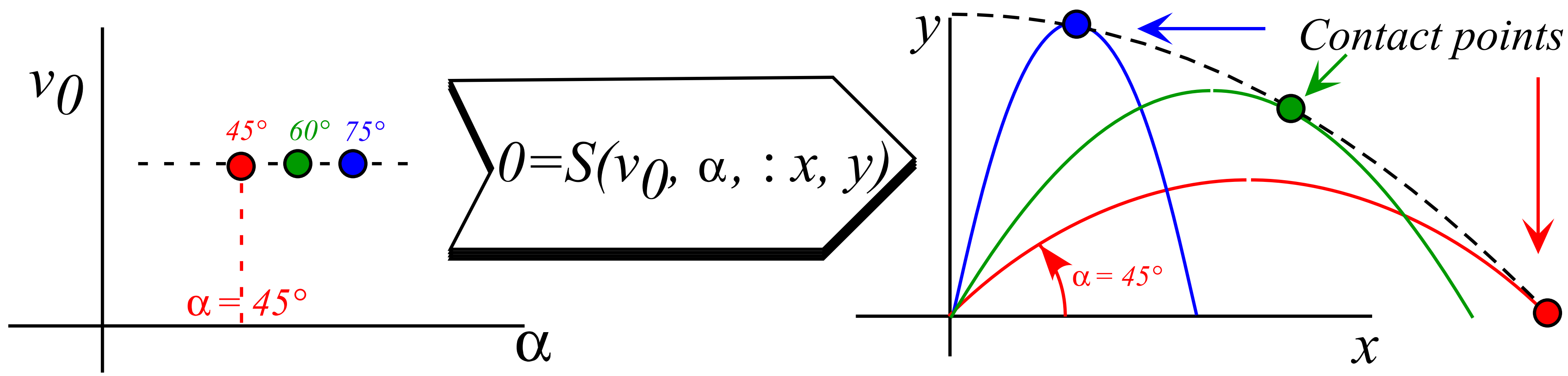
Unit 1
Fig. 12.6

Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where:
$$\frac{\partial S(v_0,\alpha : x,y)}{\partial\alpha} = 0$$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha}$$ *gives:* $$\tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}\ .$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \qquad \textit{becomes:} \qquad S(v_0,\alpha : x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Unit 1
Fig. 12.6

Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory

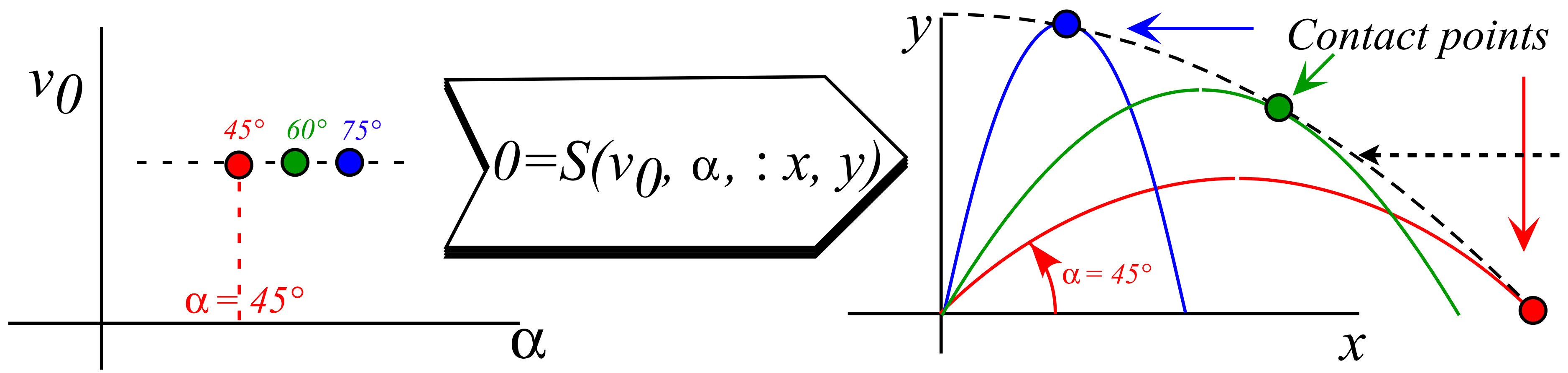
where: $\dfrac{\partial S(v_0,\alpha : x,y)}{\partial\alpha} = 0$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha} \qquad \tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}.$$

$$y_{env}(x) = x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1+\tan^2\alpha\right) \Rightarrow y_{env}(x) = x\frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2}\left(1+\frac{v_0^4}{g^2x^2}\right)$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$ *becomes:* $$S(v_0,\alpha:x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $\frac{\partial S(v_0,\alpha:x,y)}{\partial\alpha} = 0$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha}$$

$$\tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}.$$

$$y_{env}(x) = x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1+\tan^2\alpha\right) \Rightarrow y_{env}(x) = x\frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2}\left(1+\frac{v_0^4}{g^2x^2}\right)$$

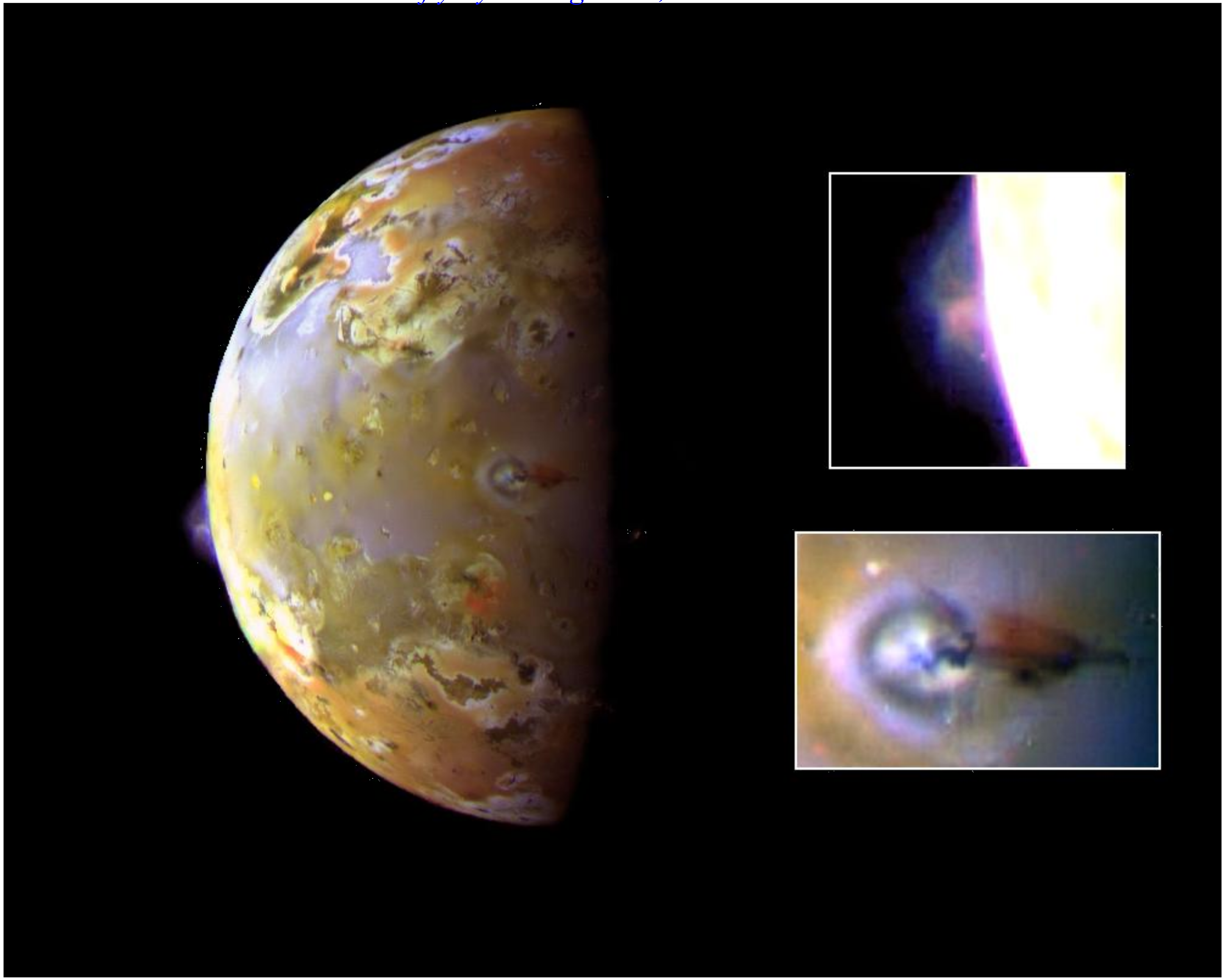
$$y_{env}(x) = \frac{v_0^2}{g} - \frac{gx^2}{2v_0^2} - \frac{g x^2}{2v_0^2}\frac{v_0^4}{g^2 x^2} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

Envelope function

The Plumes of Prometheus

NASA-Galileo Project

Io fly-by on August 18, 1997



http://antwrp.gsfc.nasa.gov/apod/image/9708/prometheus_gal_big.jpg

http://antwrp.gsfc.nasa.gov/apod/ap970818.html

http://science.nasa.gov/science-news/science-at-nasa/1999/ast04oct99_1/

Io's Alien Volcanoes

Science @NASA
Inform Inspire Involve
science.nasa.gov

Space Science News home

Pretty bad sketch of plumes
(LasVegas model of planetary ejecta?)

Do these guys need a geometry lesson?

Io's Alien Volcanoes

Go fly a kite?

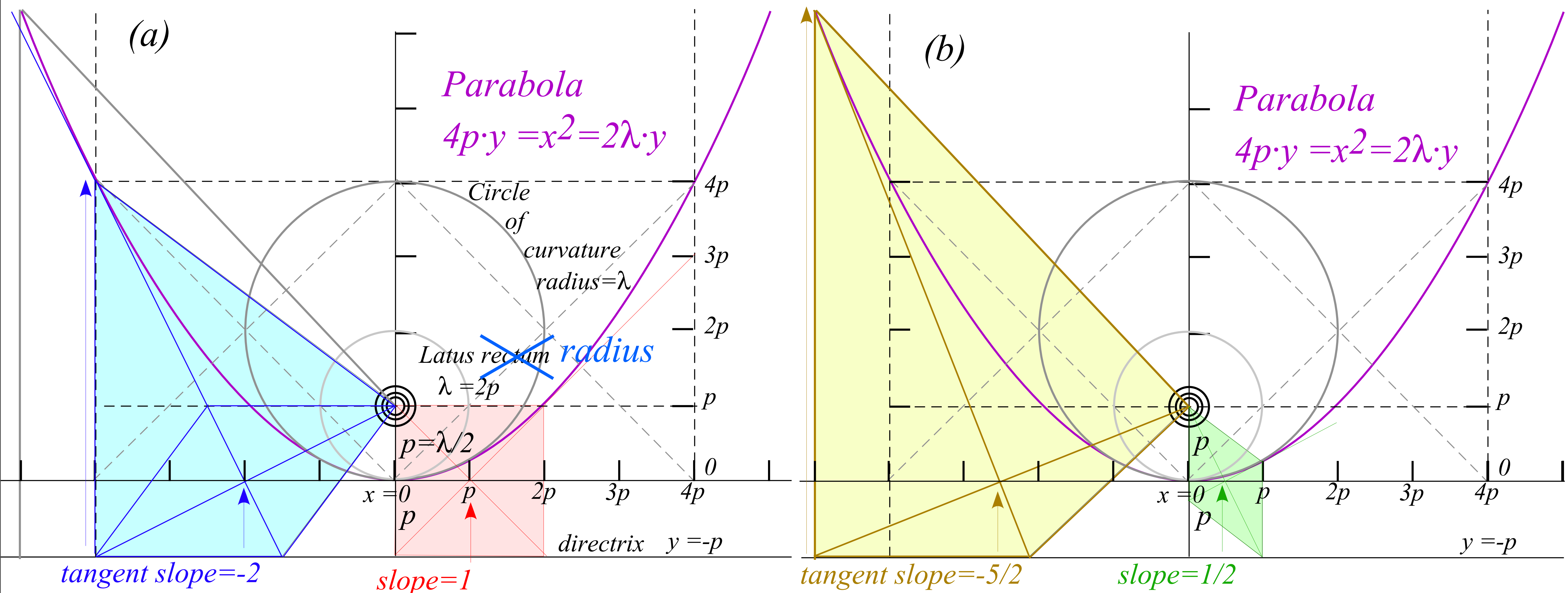
Scientists are eager for a closer look at the solar system's strangest and most active volcanoes when Galileo flies by Io on October 11.

October 4, 1999: Thirty years ago, before the Voyager probes visited Jupiter, if you had described Io to a literary critic it would have been declared overwrought science fiction. Jupiter's strange moon is literally bursting with volcanoes. Dozens of active vents pepper the landscape which also includes gigantic frosty plains, towering mountains and volcanic rings the size of California. The volcanoes themselves are the hottest spots in the solar system with temperatures exceeding 1800 K (1527 C). The plumes which rise 300 km into space are so large they can be seen from Earth by the Hubble Space Telescope. Confounding common sense, these high-rising ejecta seem to be made up of, not blisteringly hot lava, but frozen sulfur dioxide. And to top it all off, Io bears a striking resemblance to a pepperoni pizza. Simply unbelievable.

Right: Digital Radiance simulation of Pillan Patera just before the Galileo flyby. click for animation → .

...conventional parabolic geometry...carried to extremes...

Recall Lecture 8 p.16 to 18



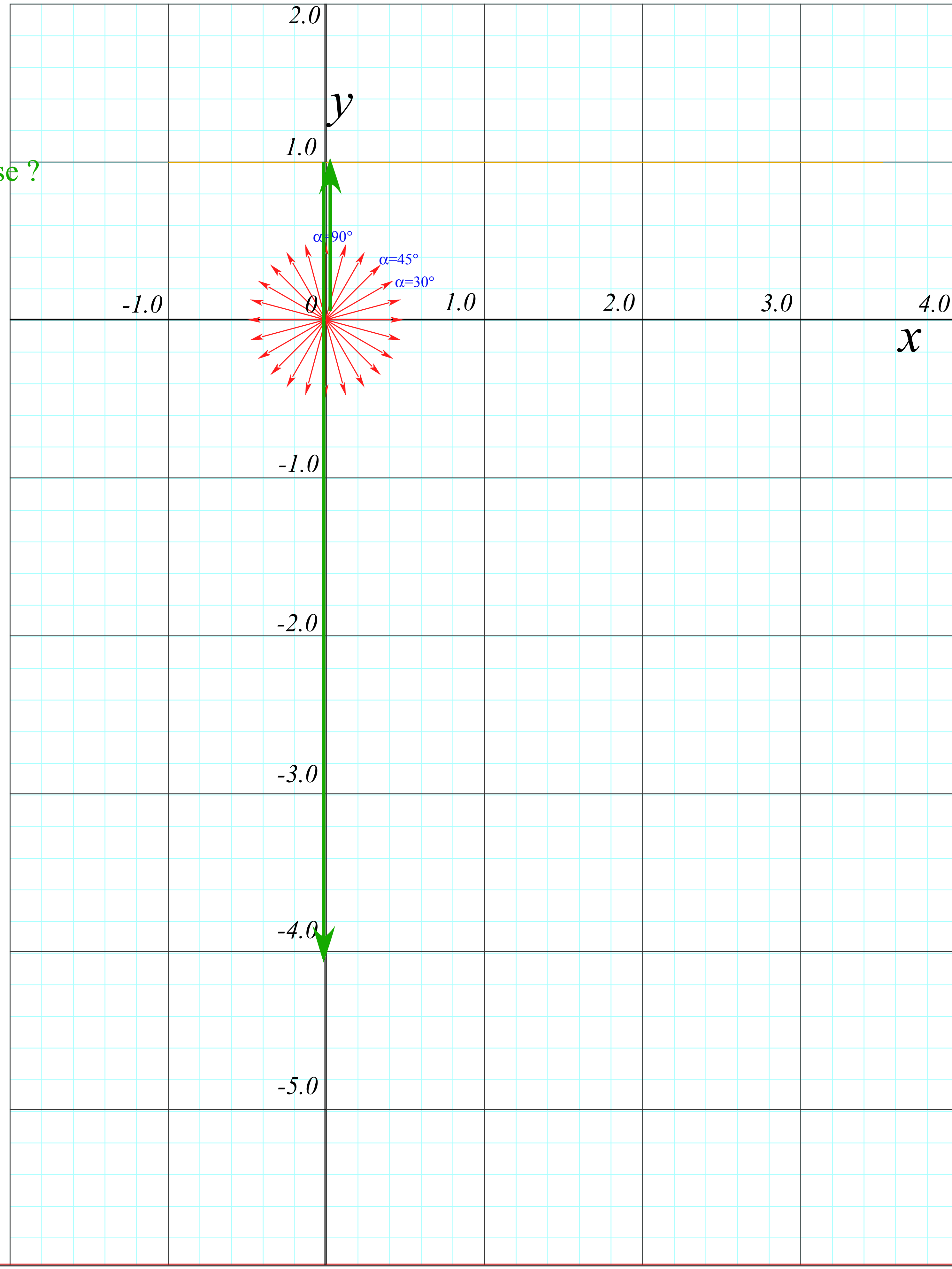
Unit 1
Fig. 9.4

A general contact transformation from sophomore physics

Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"

→ *Intuitive-geometric development of " " " and " " "*

Say $\alpha=90°$ path rises to *1.0*
then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45°$ path path rise ?

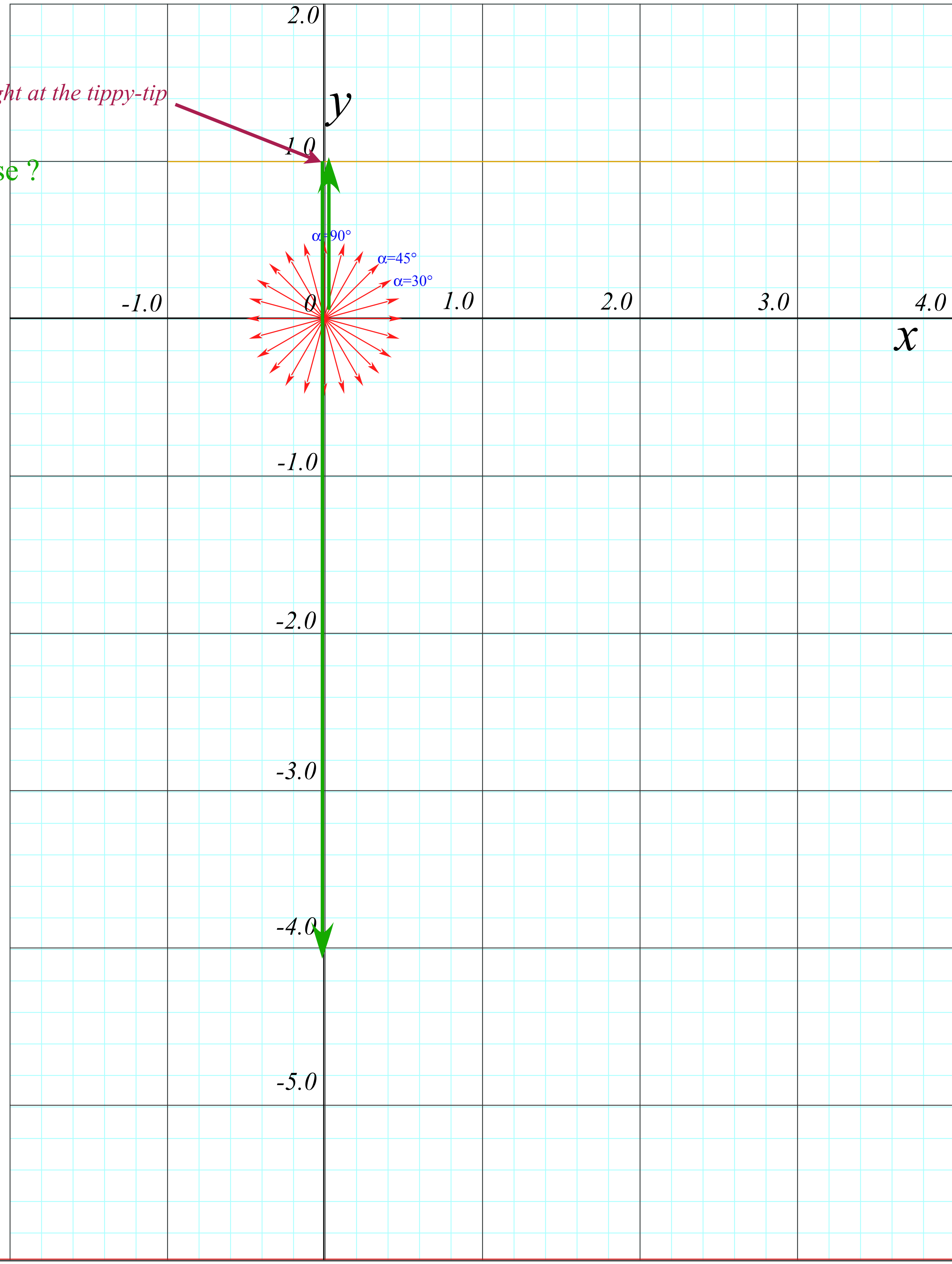


Say $\alpha=90°$ path rises to *1.0* then drops. When at $y=1.0$...

Q1. ...where is its focus? → *Right at the tippy-tip*

Q2. ...where is the blast wave?

Q3. ...how high can $\alpha=45°$ path path rise ?



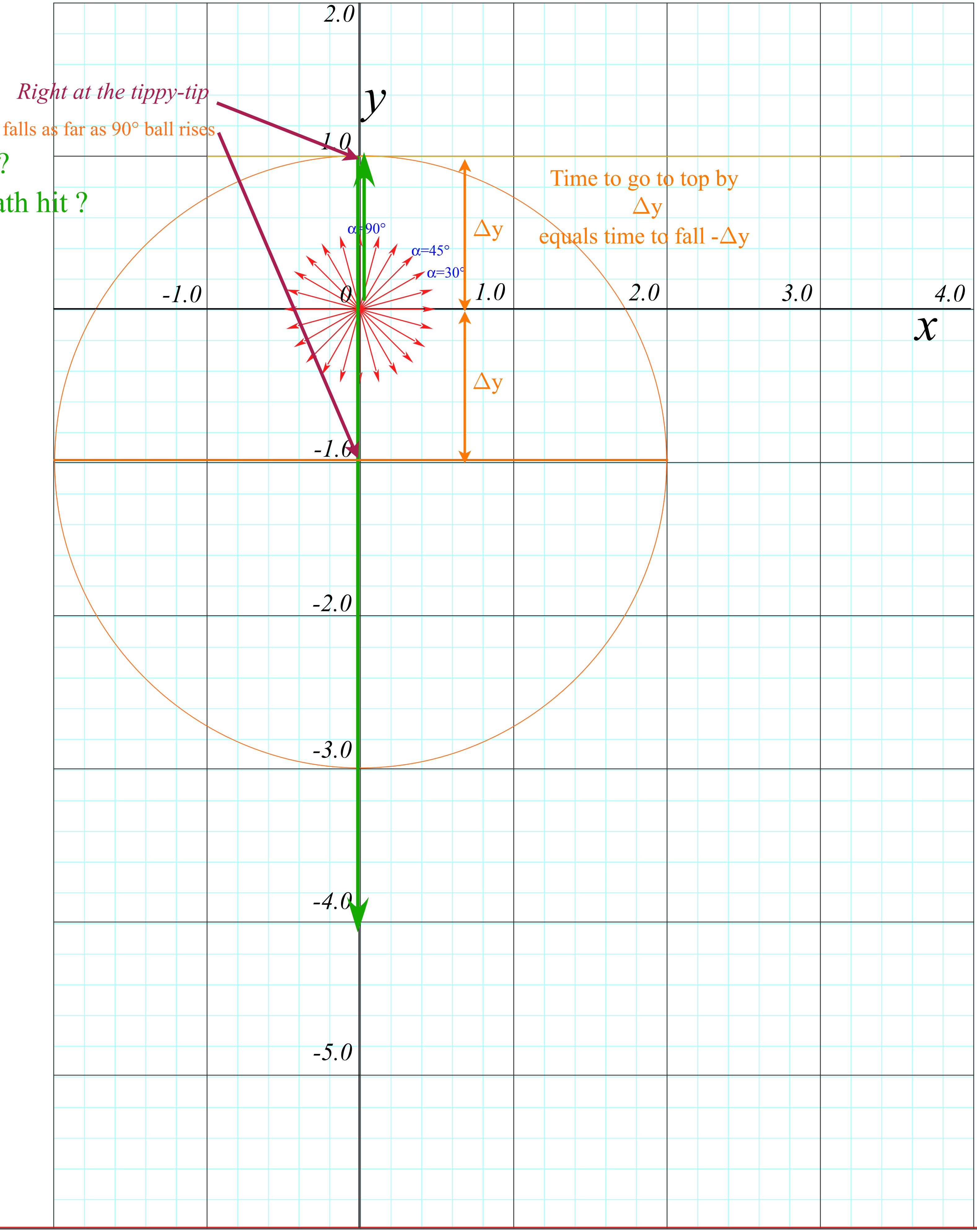
Say α=90° path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

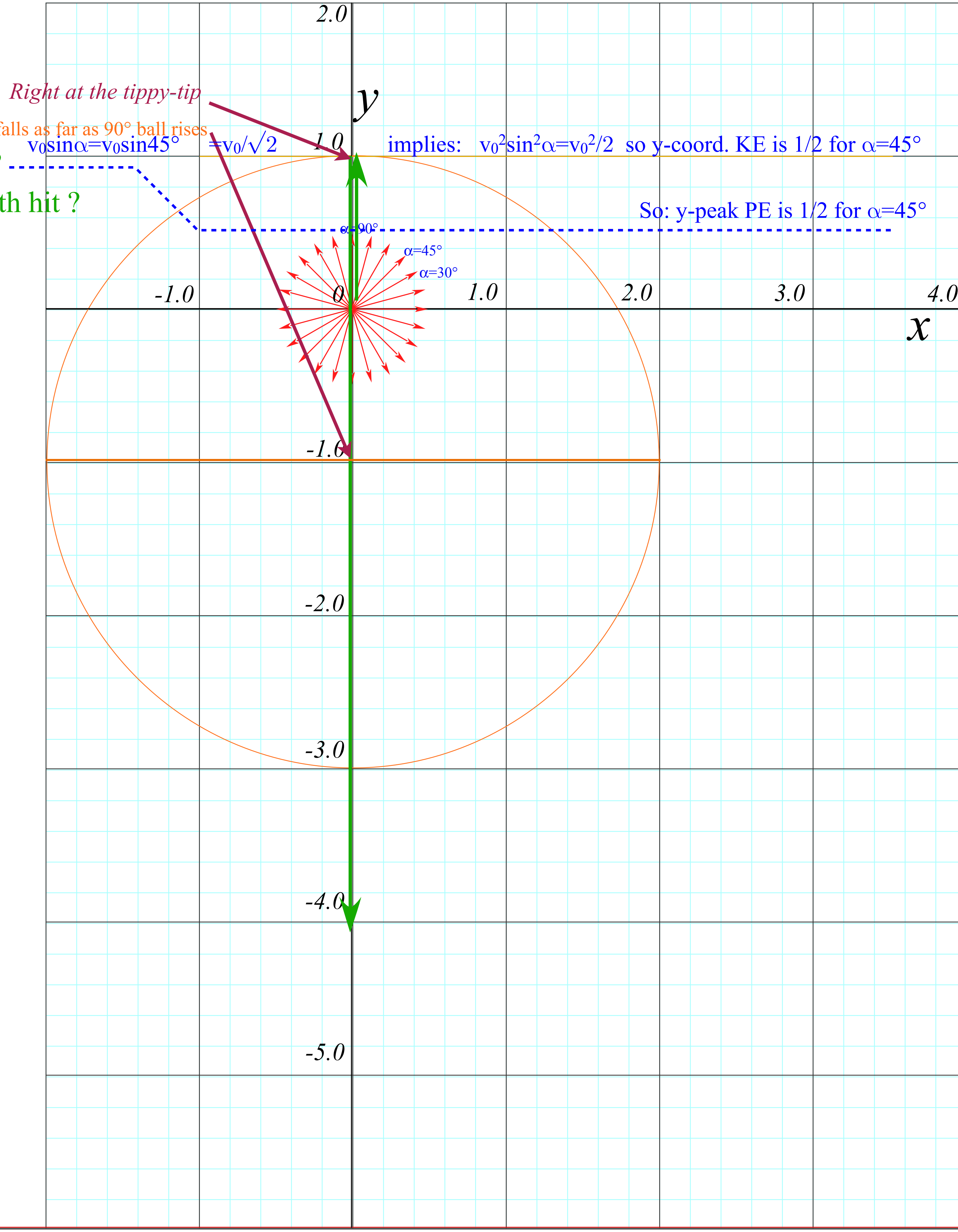
Q2. ...where is the blast wave?

Q3. How high can α=45° path rise ?

Q4. Where on *x-axis* does α=45° path hit ?



Say $\alpha=90°$ path rises to *1.0*
then drops. When at *y=1.0*...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can $\alpha=45°$ path rise ?
Q4. Where on *x-axis* does $\alpha=45°$ path hit ?



Say α=90° path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

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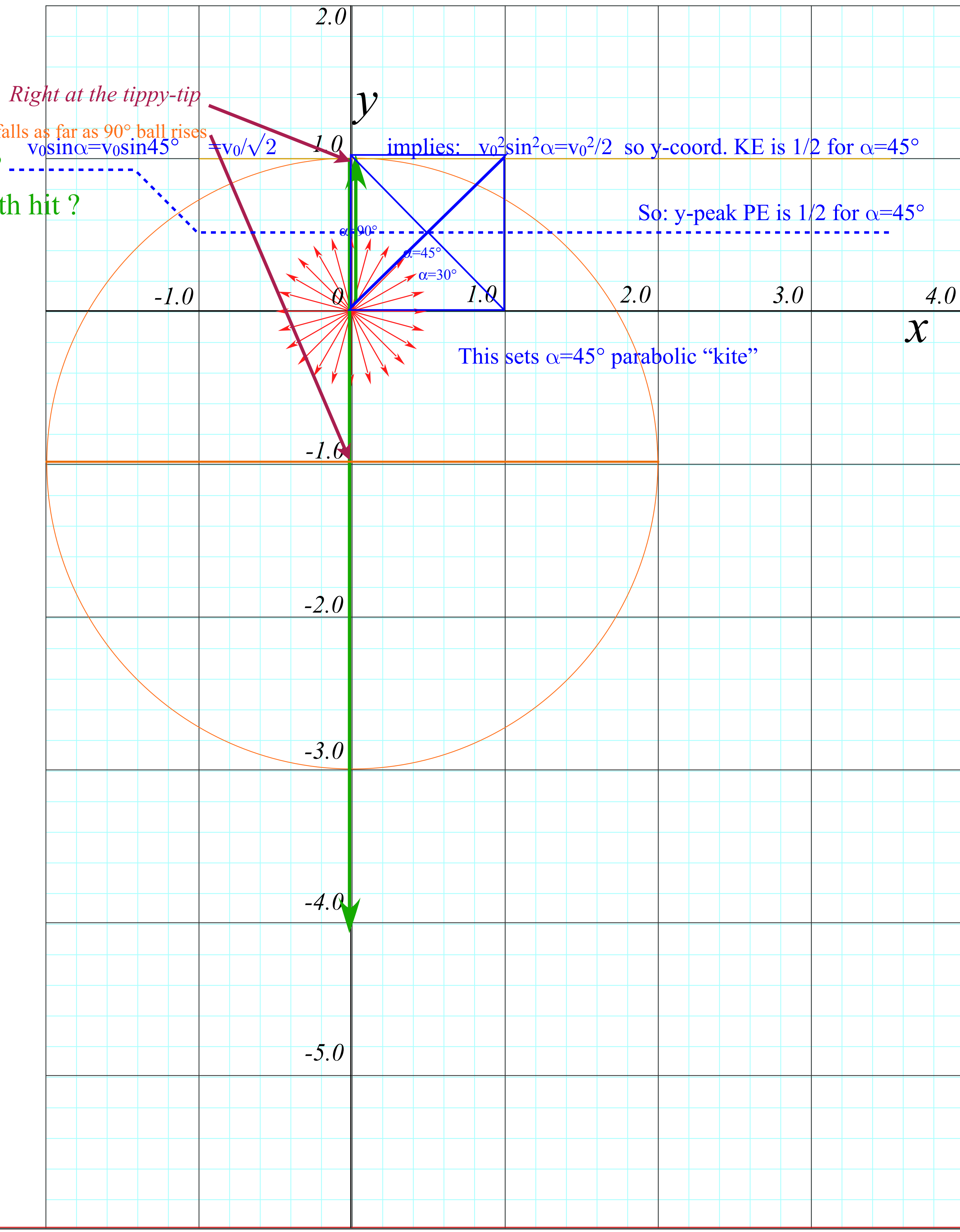
Q3. How high can α=45° path rise ?

Q4. Where on *x-axis* does α=45° path hit ?



Say α=90° path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as 90° ball rises

Q3. How high can α=45° path rise ? 1/2 as high

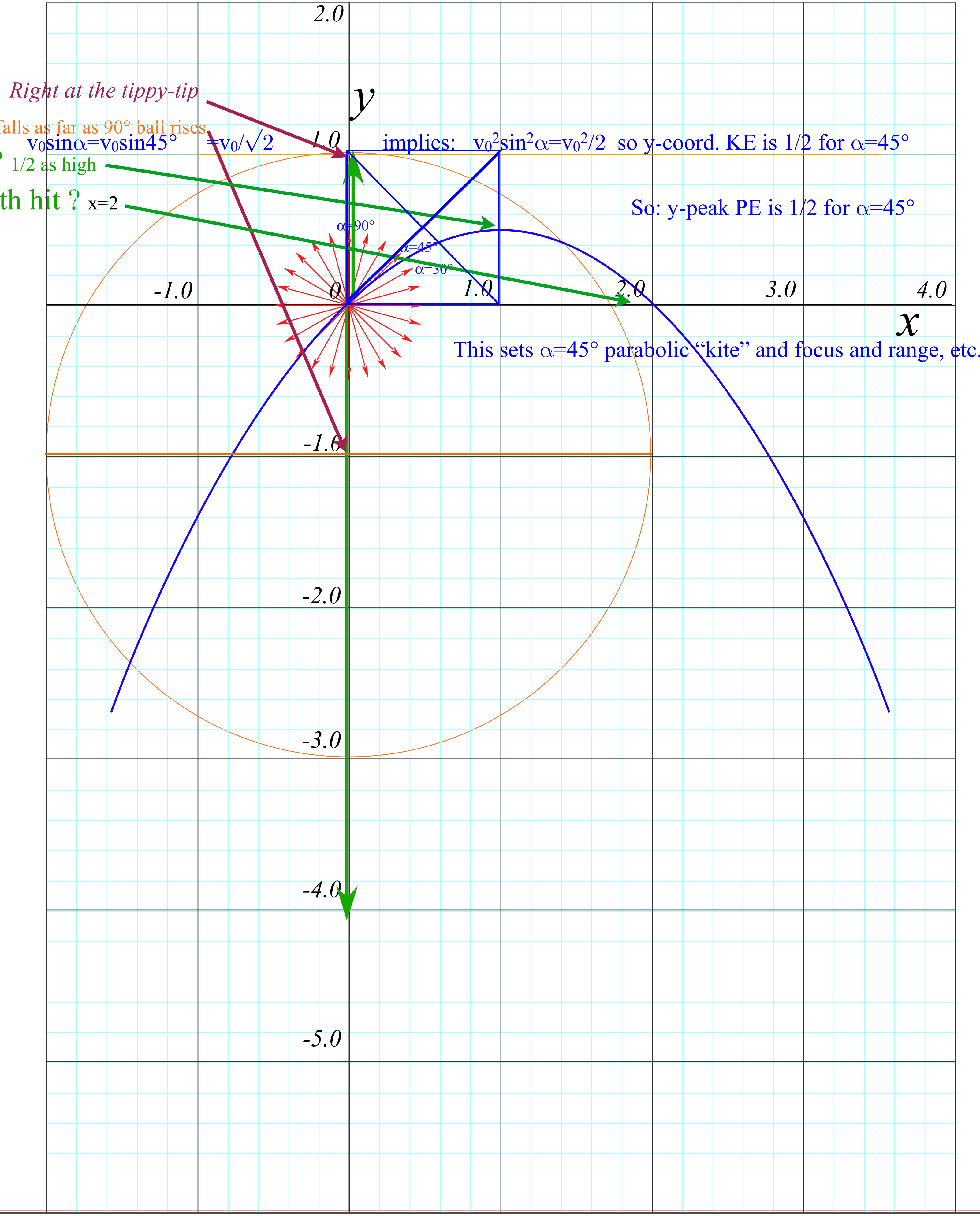
Q4. Where on *x-axis* does α=45° path hit ? x=2

Q5. Where is blast wave then?

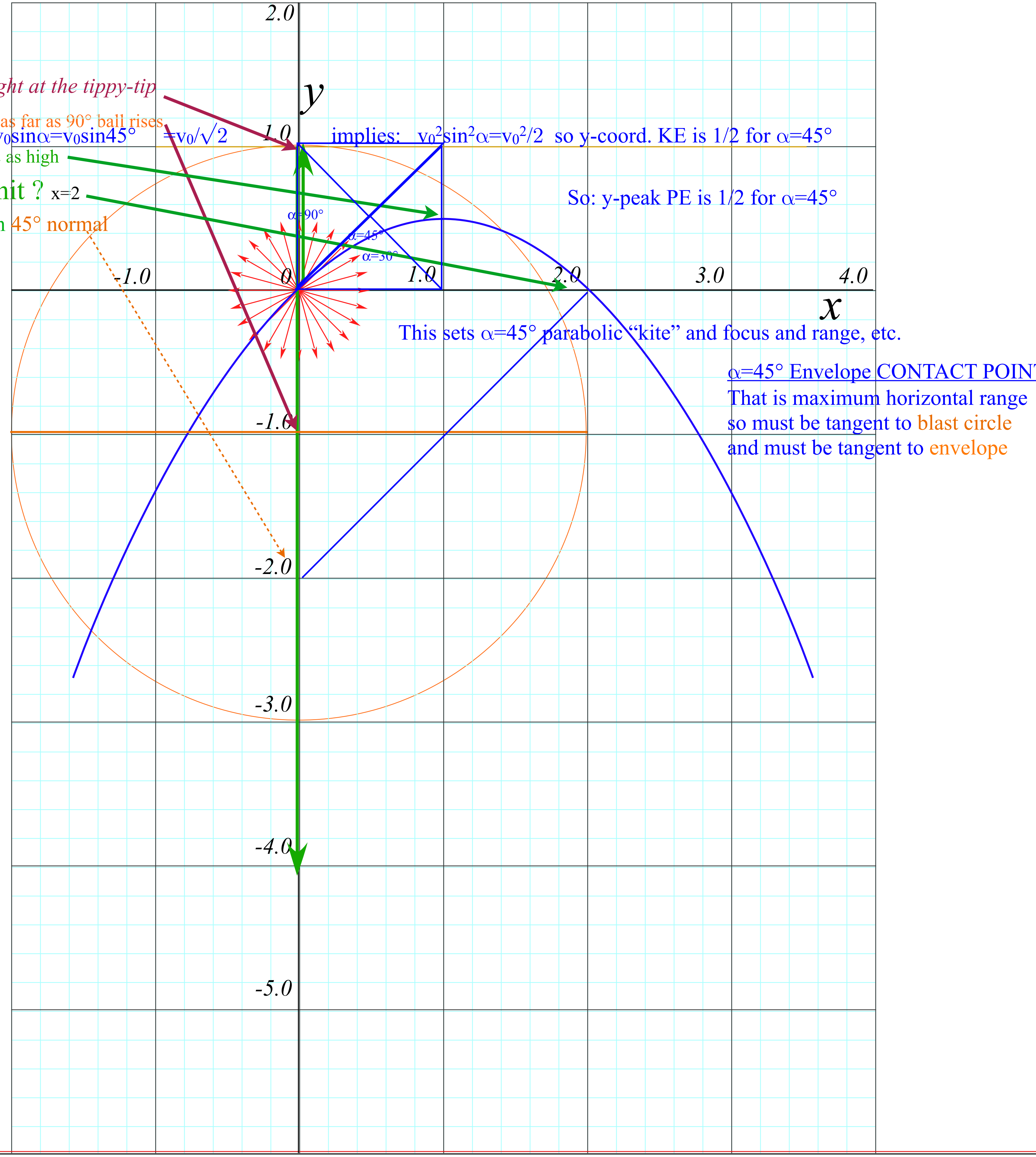
Q6 Where is α=45° path focus?

Q7 Guess for all-path envelope?

and its focus? directrix?



Say $\alpha=90°$ path rises to *1.0*
then drops. When at $y=1.0$...
Q1. ...where is its focus? *Right at the tippy-tip*
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can $\alpha=45°$ path rise ? 1/2 as high
Q4. Where on *x-axis* does $\alpha=45°$ path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is $\alpha=45°$ path focus?
Q7 Guess for all-path envelope?
and its focus? directrix?



Say α=90° path rises to *1.0*
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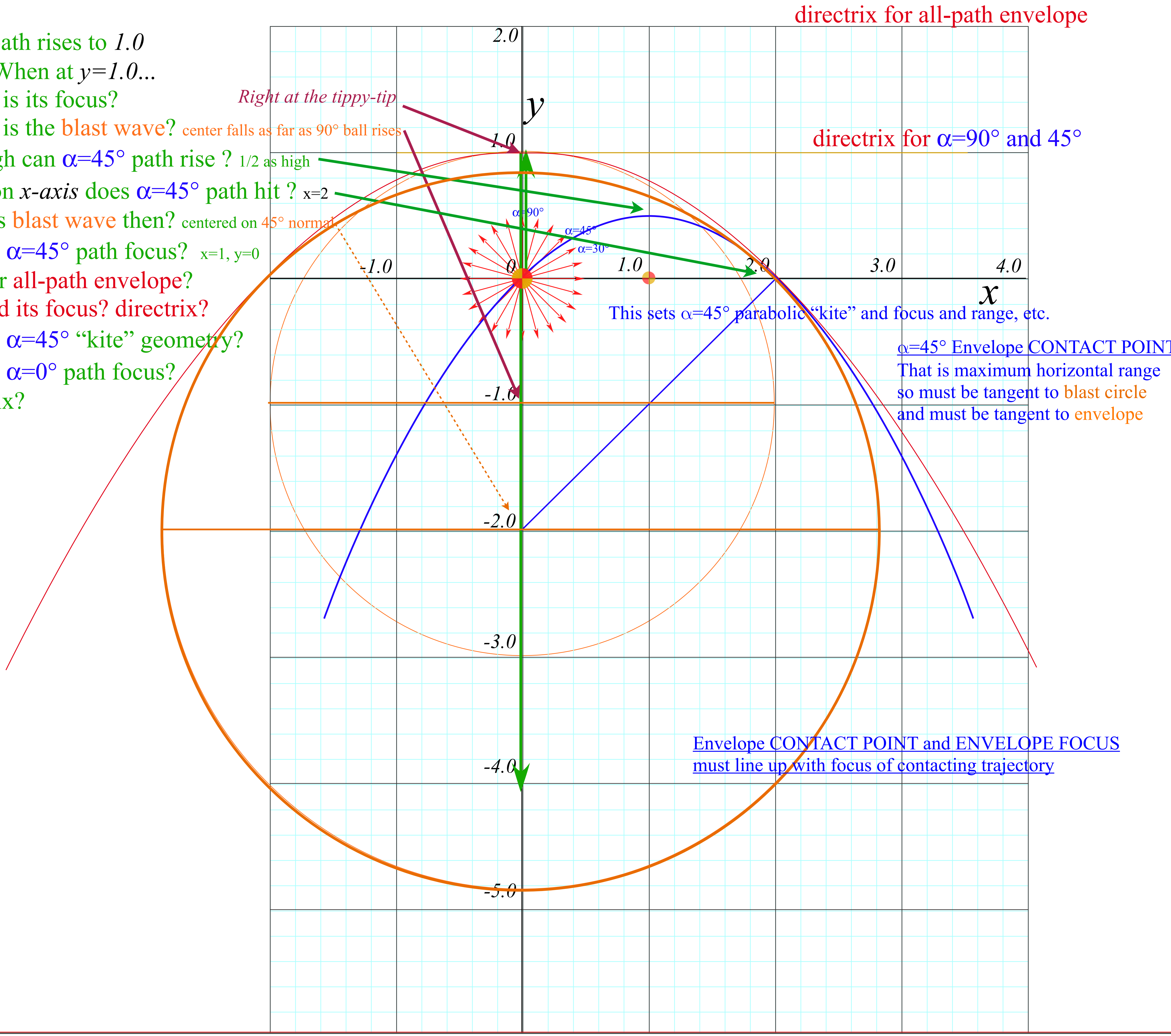
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Guess for all-path envelope?
and its focus? directrix?
Q7 Where is α=45° "kite" geometry?
Q8 Where is α=0° path focus?
directrix?



Say α=90° path rises to *1.0*
then drops. When at *y=1.0...*
Q1. ...where is its focus?
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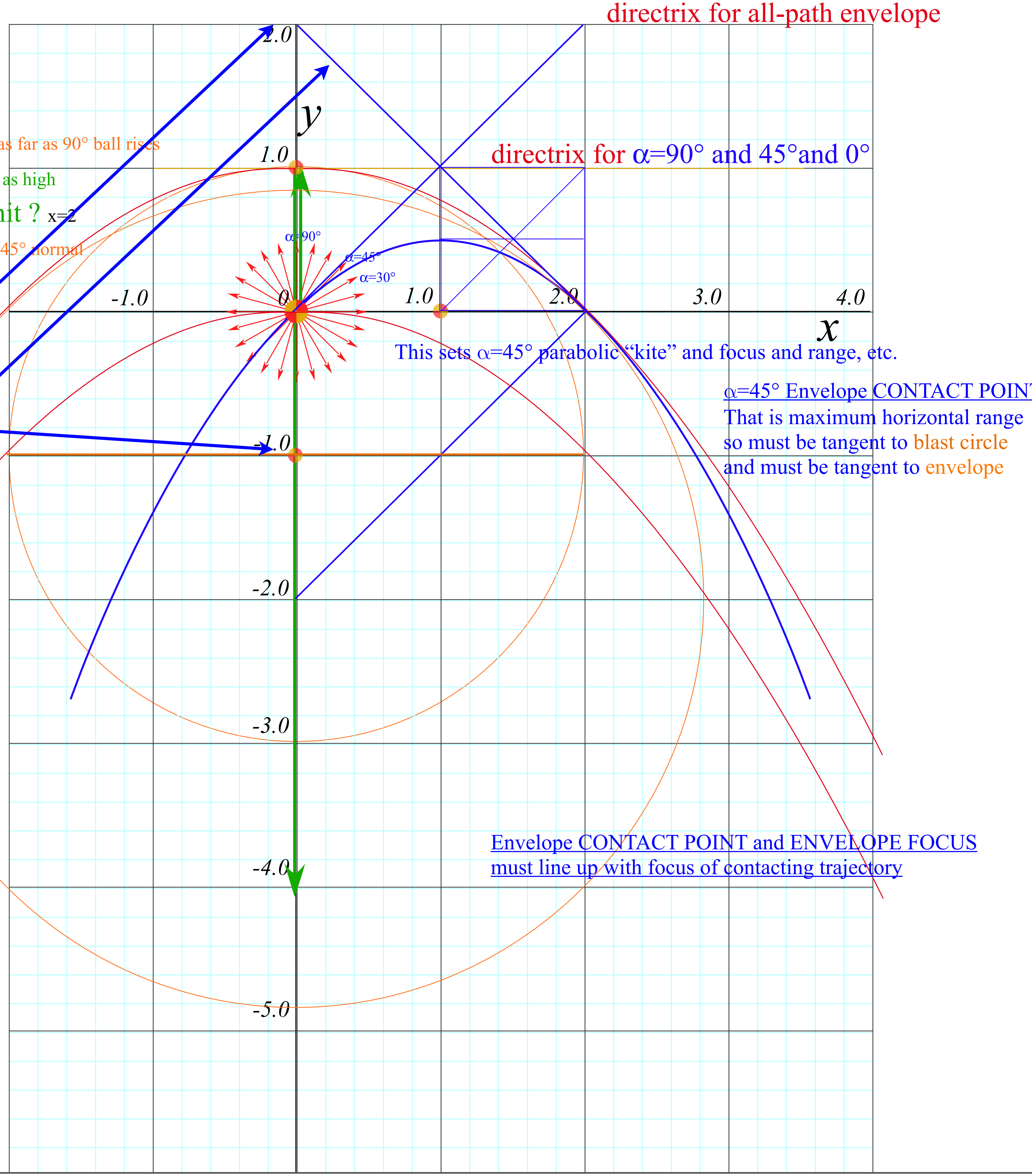
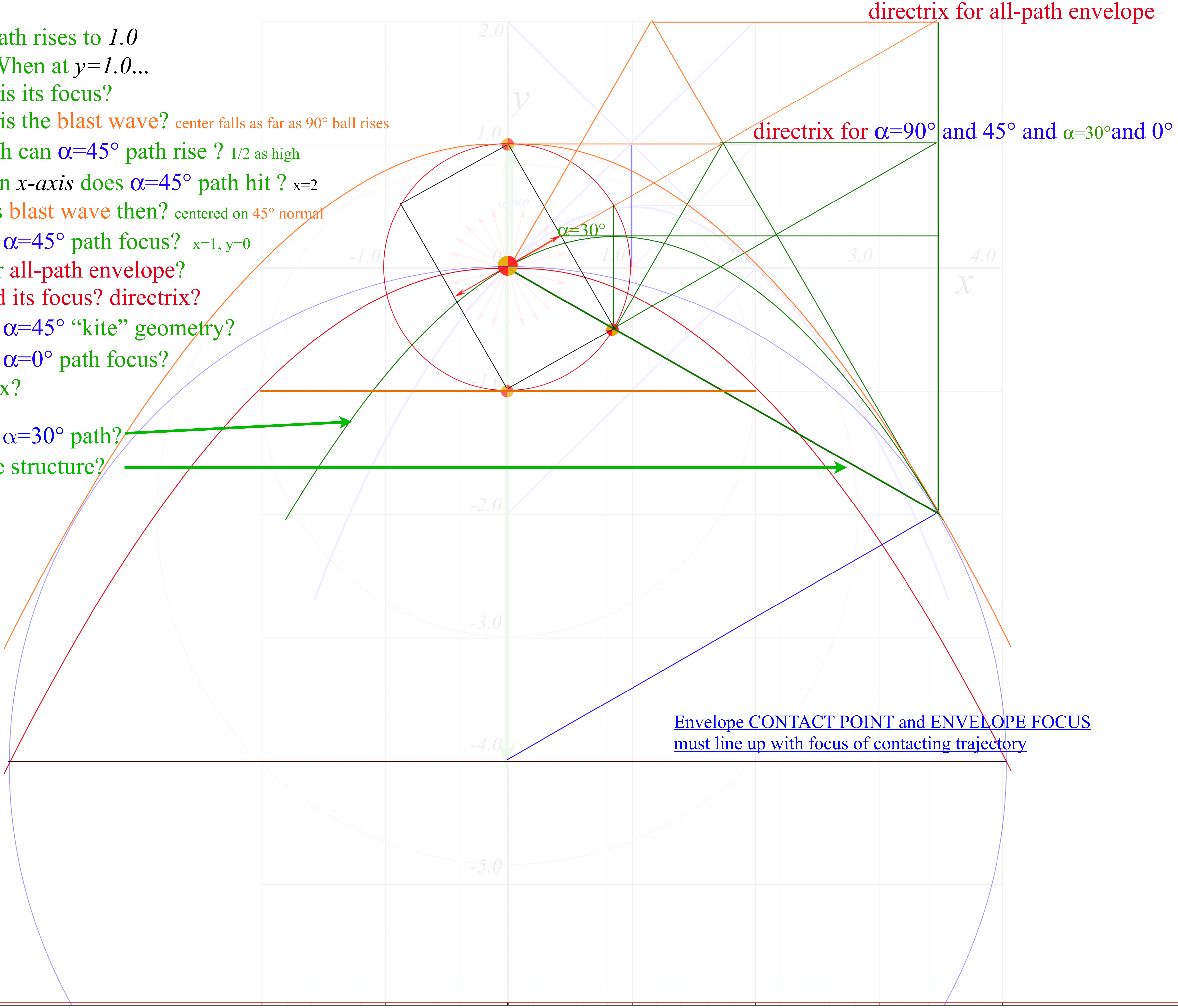
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Guess for all-path envelope?
and its focus? directrix?
Q7 Where is α=45° "kite" geometry?
Q8 Where is α=0° path focus?
directrix?

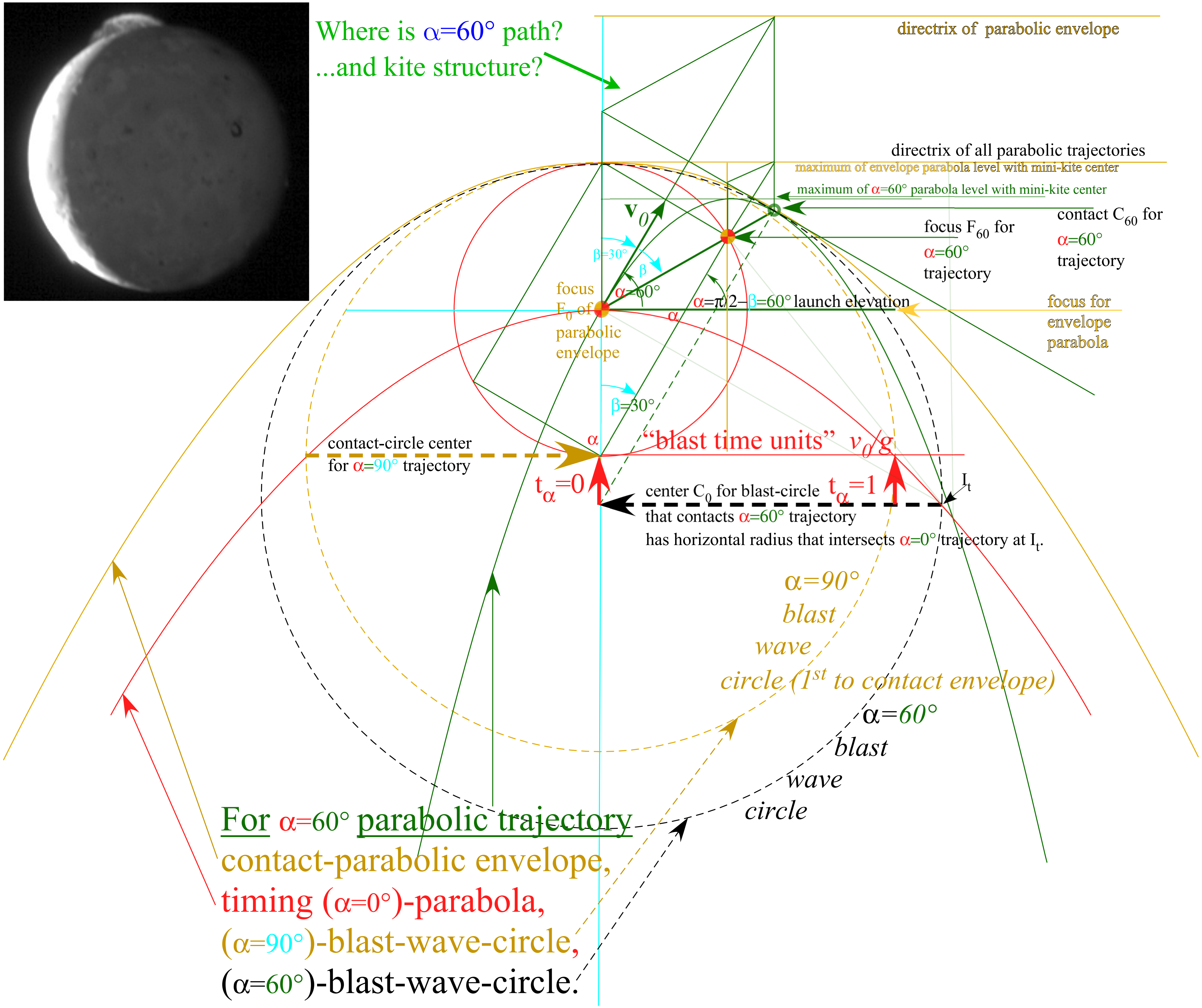
Where is a=30° path?



Say α=90° path rises to *1.0*
then drops. When at *y=1.0*...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Guess for all-path envelope?
and its focus? directrix?
Q7 Where is α=45° "kite" geometry?
Q8 Where is α=0° path focus?
directrix?

Where is α=30° path?
...and kite structure?





Lecture 11 ends here
Tue. 9.30.2014