Lecture 15
Tue.10.11.2016

GCC Lagrange and Riemann Equations for Trebuchet
(Ch. 1-5 of Unit 2 and Unit 3)

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

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Chapter 1. The Trebuchet: A dream problem for Galileo?

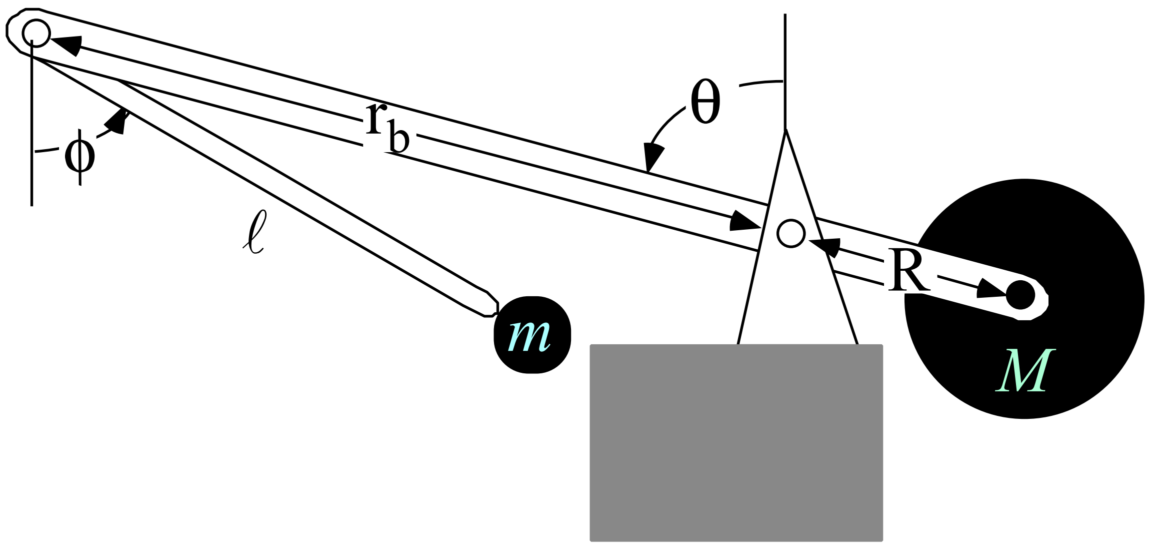
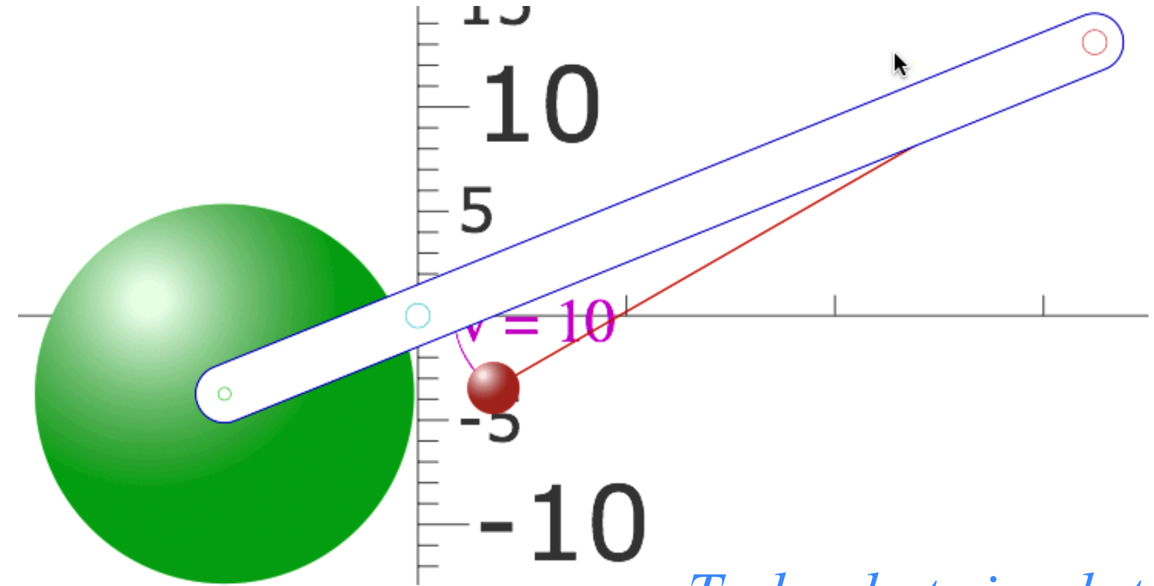
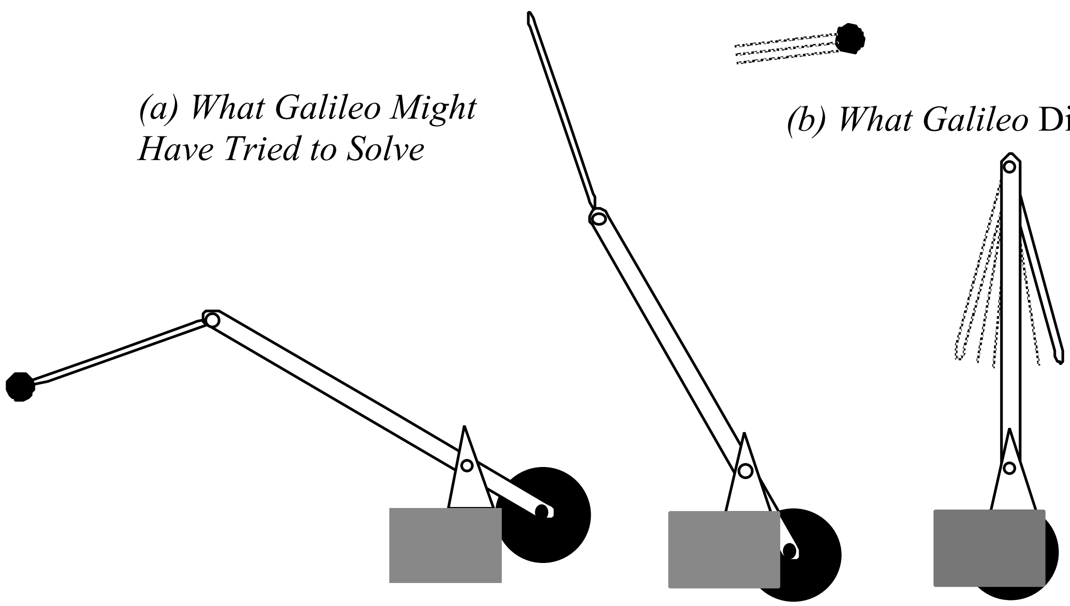


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem

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Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

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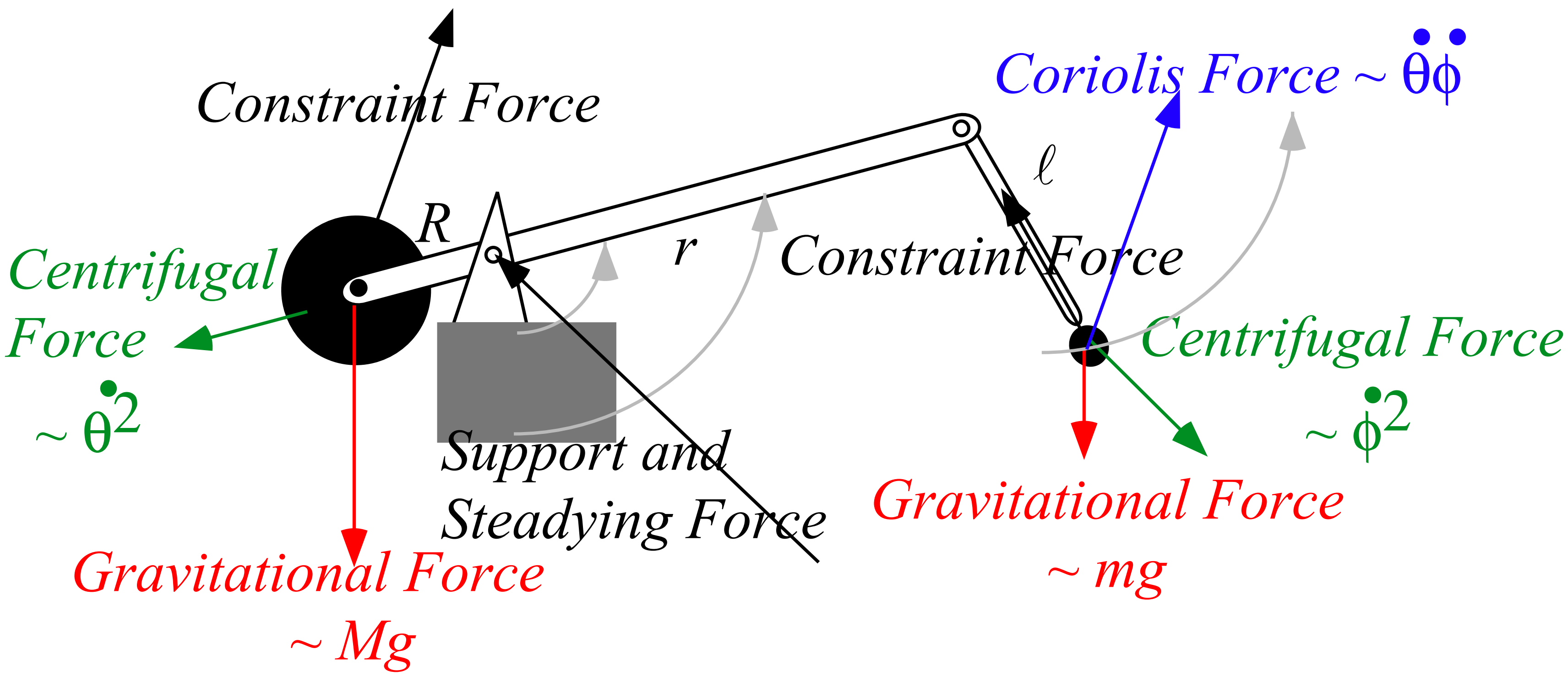
2nd-guessing Riemann equation?

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



Acceleration
and
'Fictitious'
Forces:
Coriolis
Centrifugal

Applied
'Real'
Forces:
Gravity
Stimuli
Friction...

Constraint
'Internal'
Forces:
Stresses
Support...
(Do not contribute.
Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

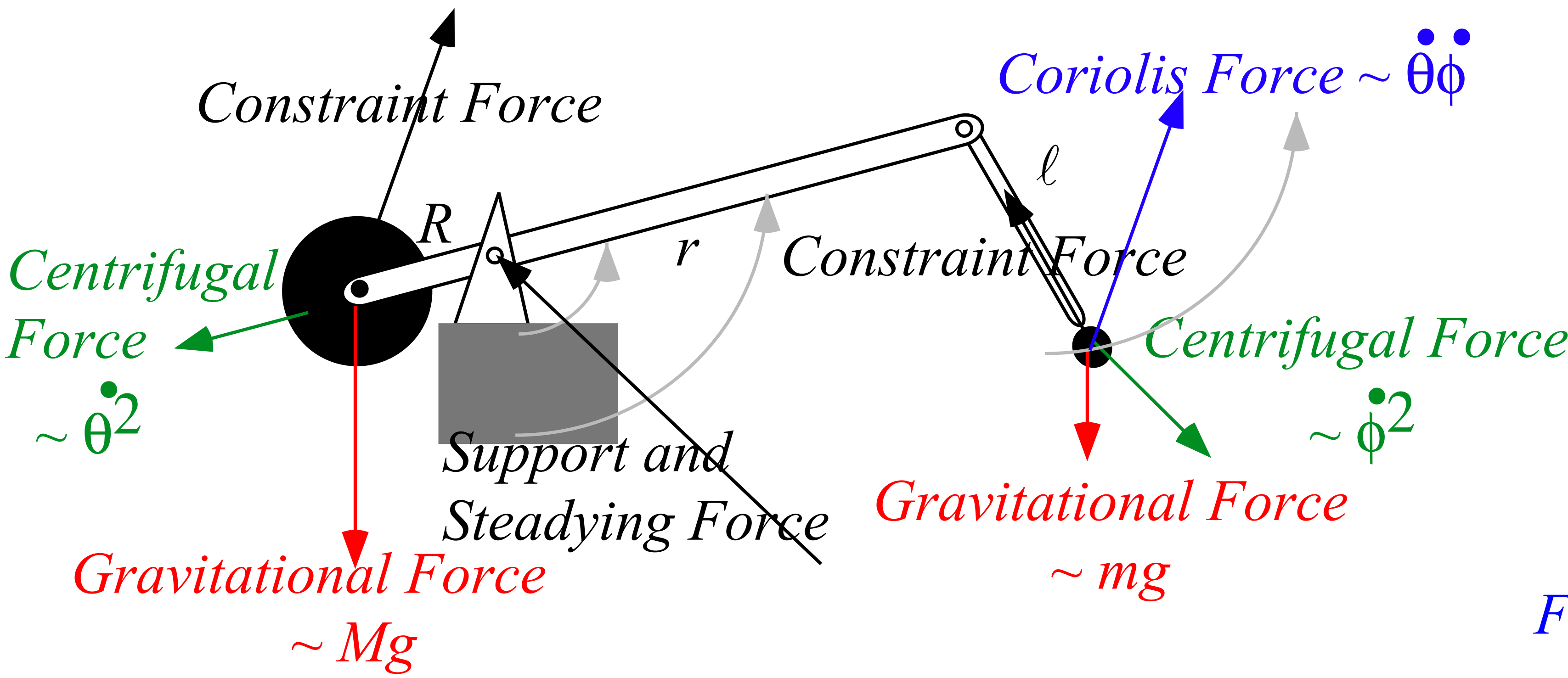
$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

Fig. 2.5.2
(modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot{\phi}} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$L = T - V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
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Trebuchet Cartesian projectile coordinates are double-valued

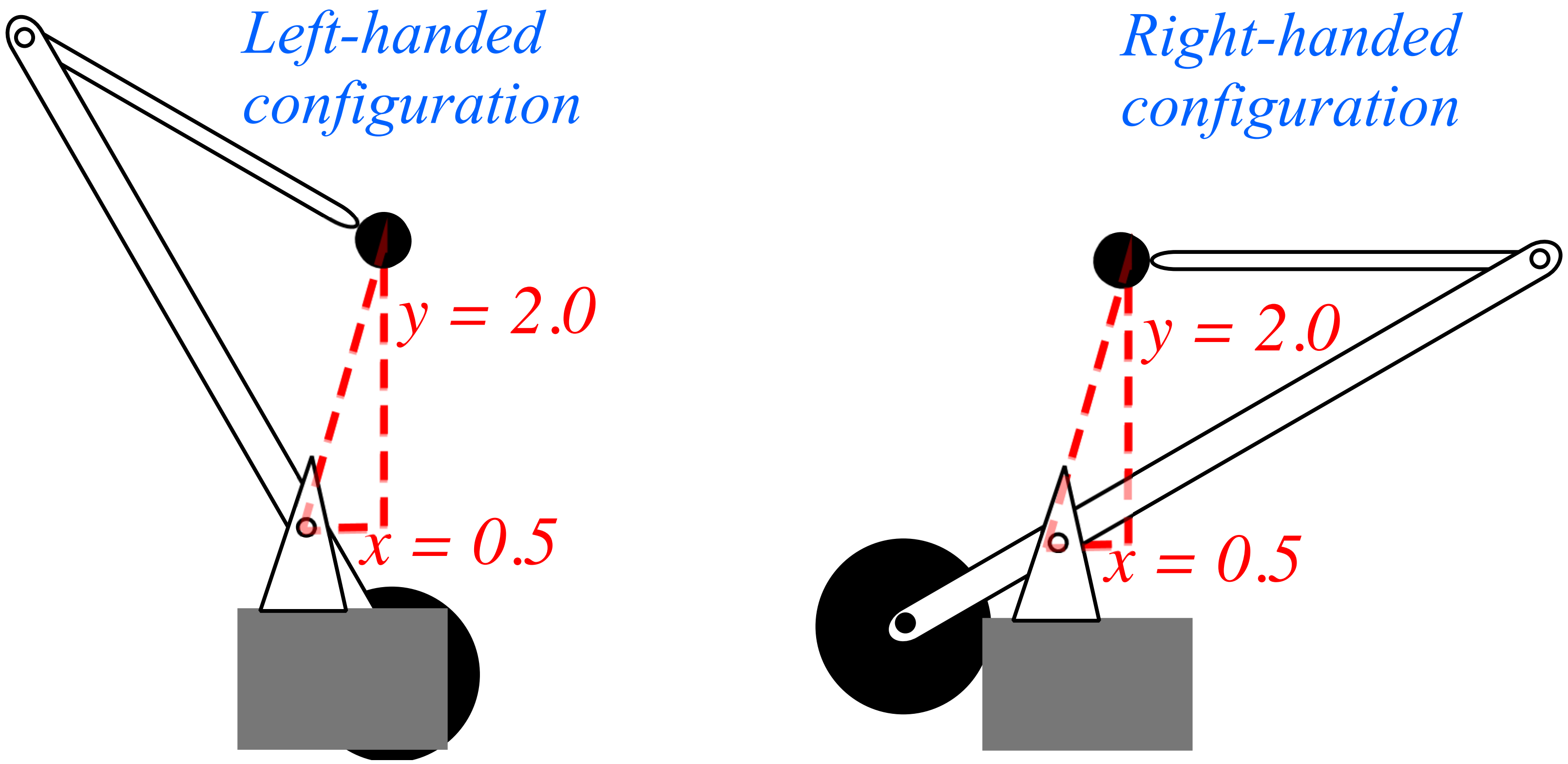

Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

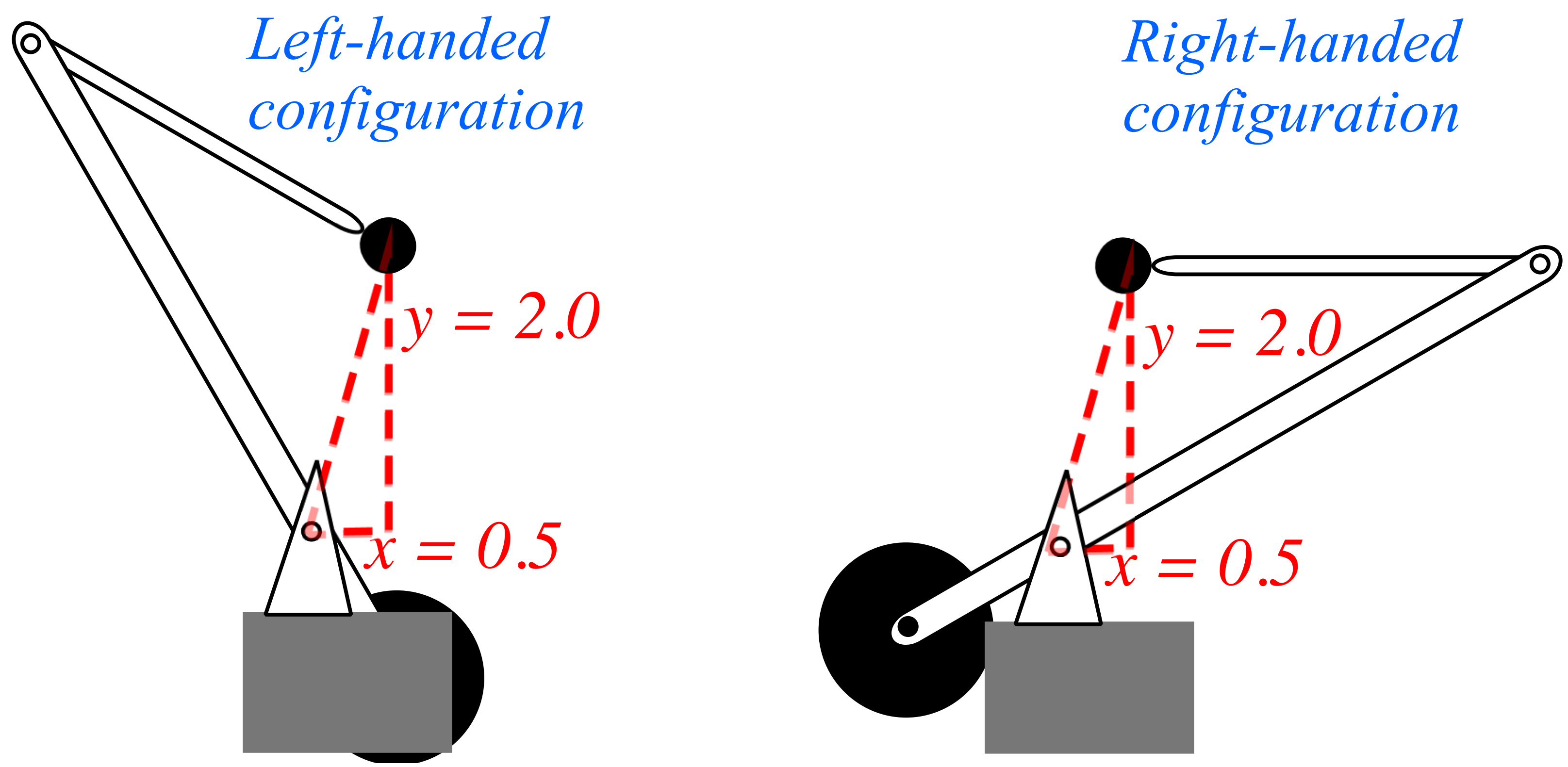

Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

So, for example, are polar coordinates … (for each angle there are two r-values)

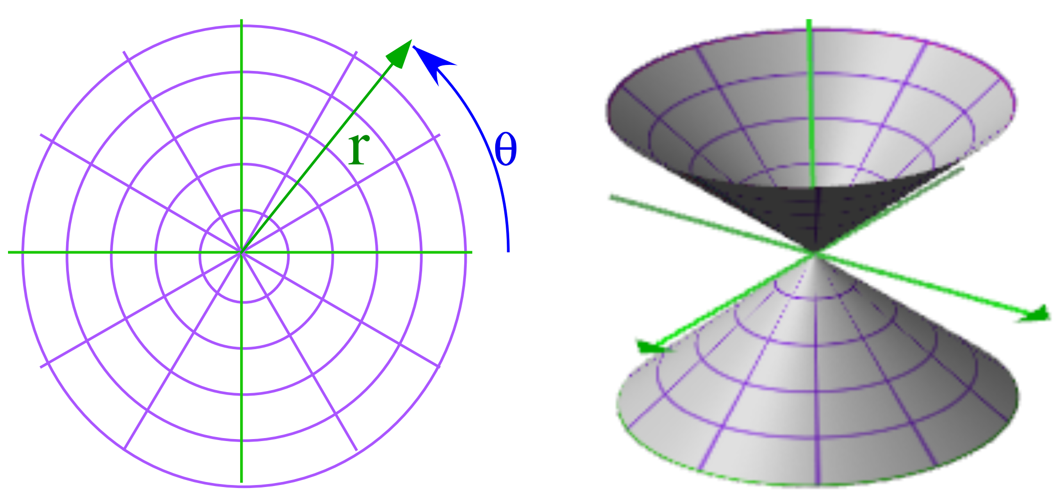

Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.

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"Flat" (q^1=θ, q^2=ϕ)-graph of trebuchet loci compared to "rolled-up" toroidal manifold

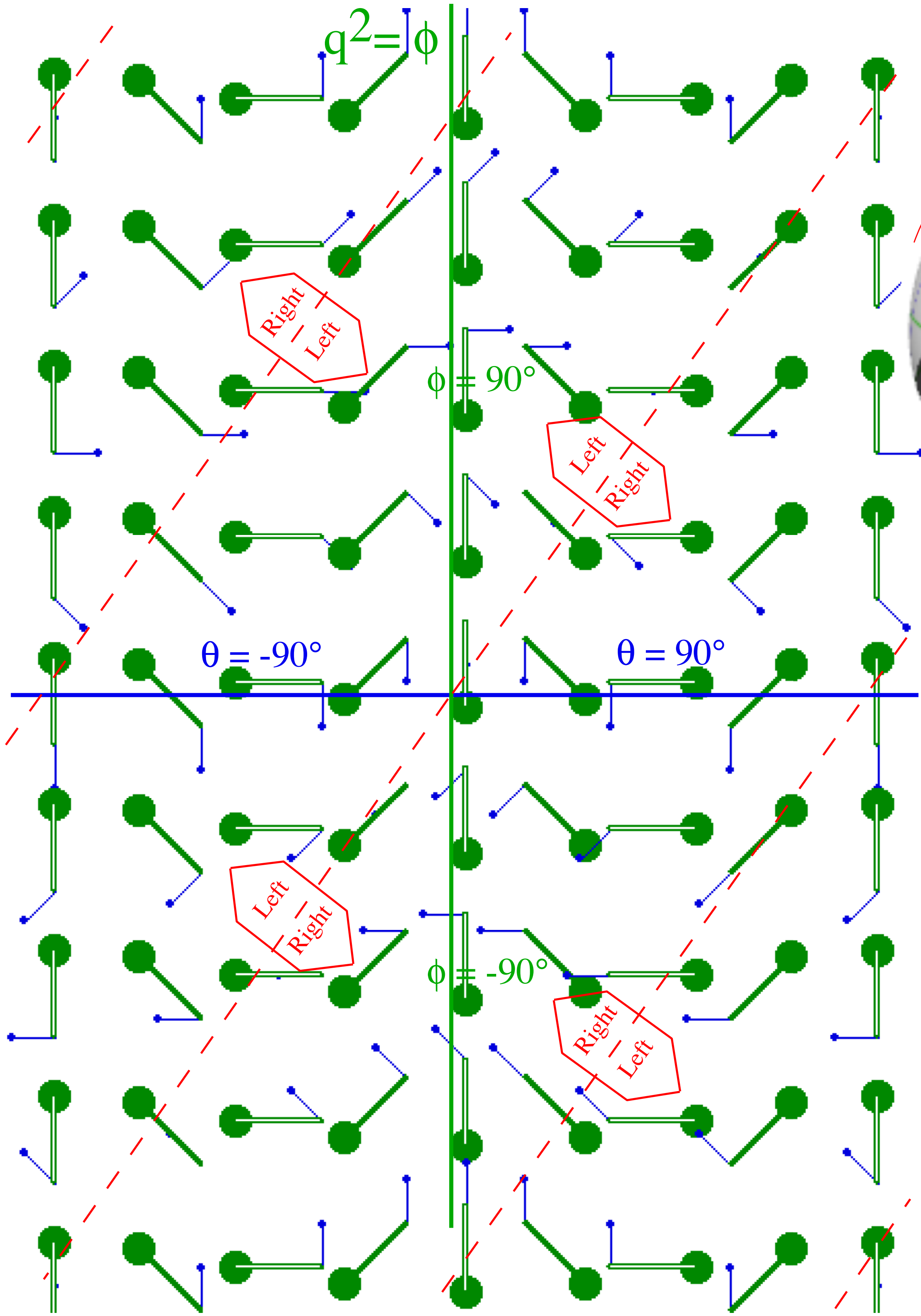


Fig. 3.1.3 "Flattened" (q^1=θ, q^2=ϕ) coordinate manifold for trebuchet

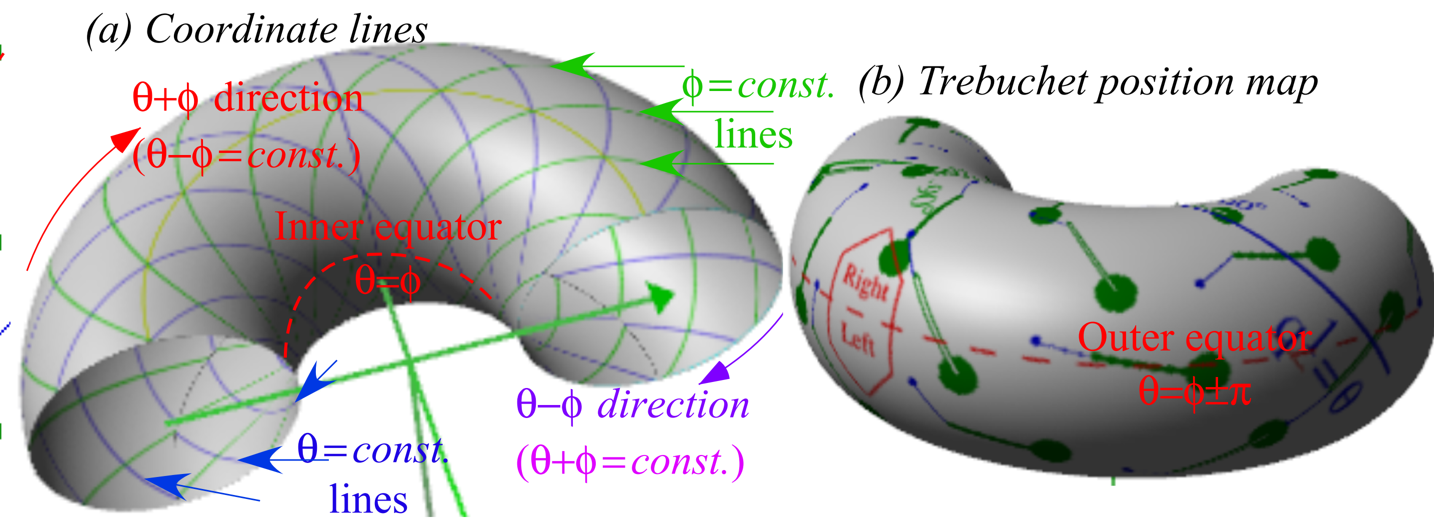


Fig. 3.1.2 Trebuchet torus.
(a) (q^1=θ, q^2=ϕ) coordinate lines.(b)Trebuchet position map and equators.

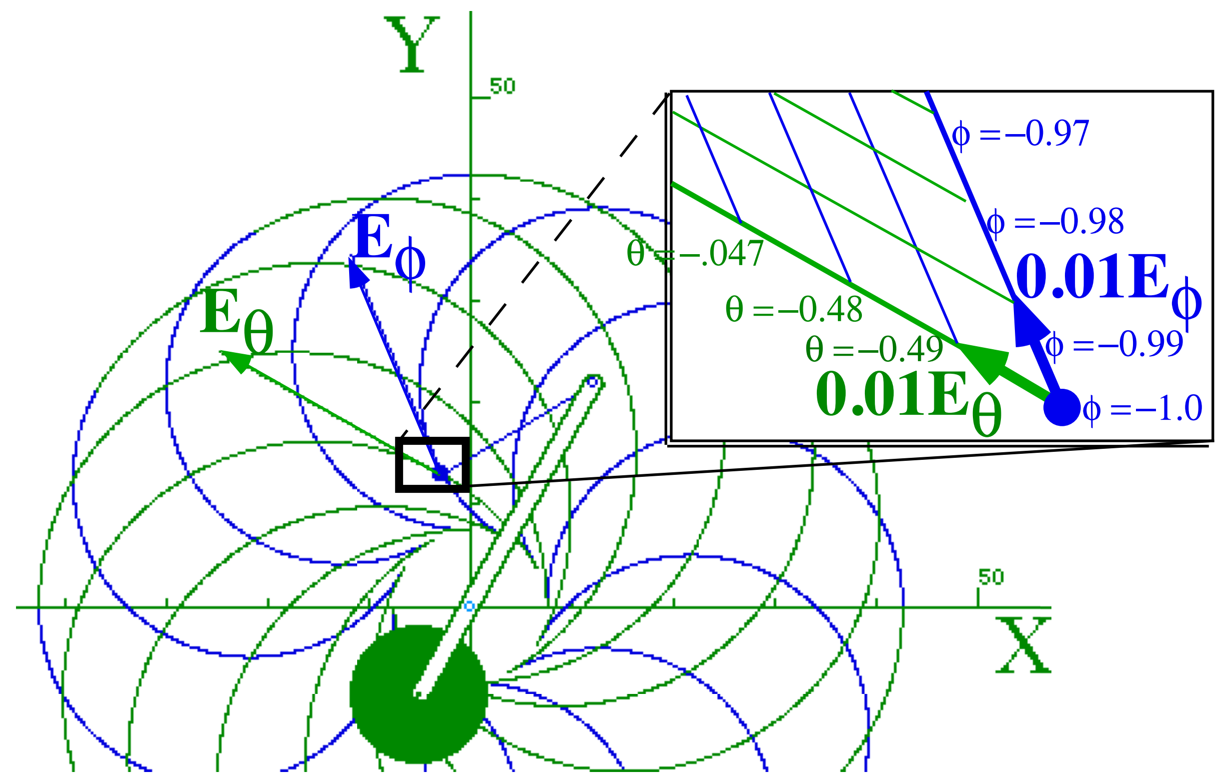
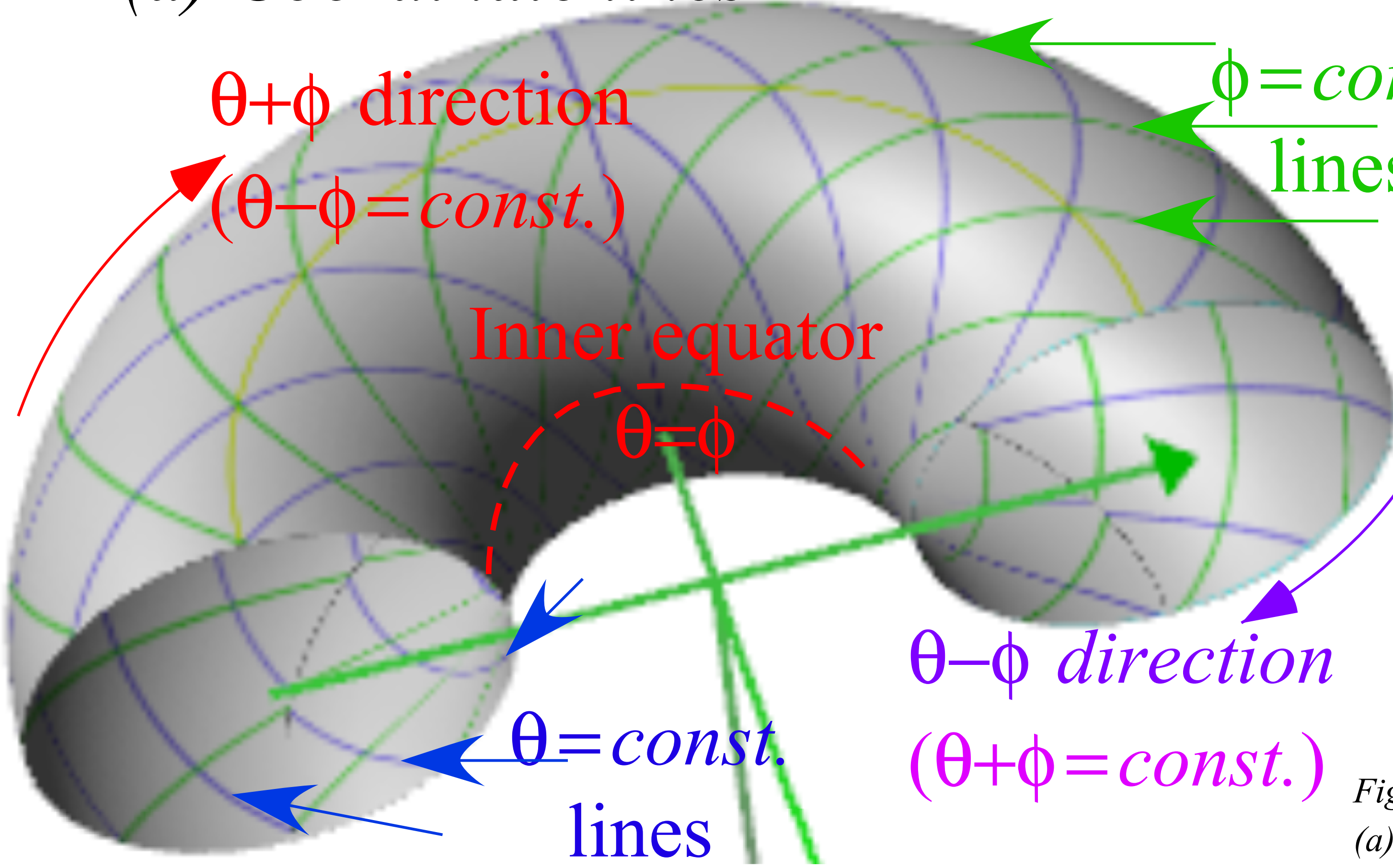


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold of trebuchet positions

(a) Coordinate lines



(b) Trebuchet position map

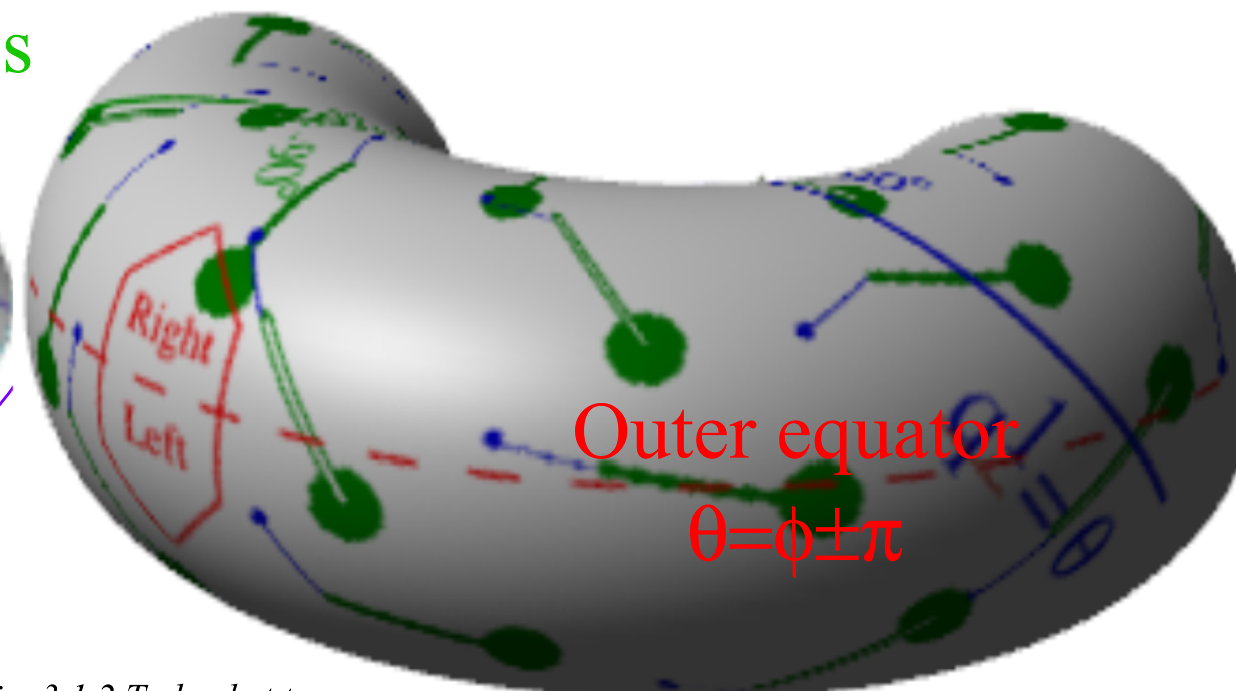
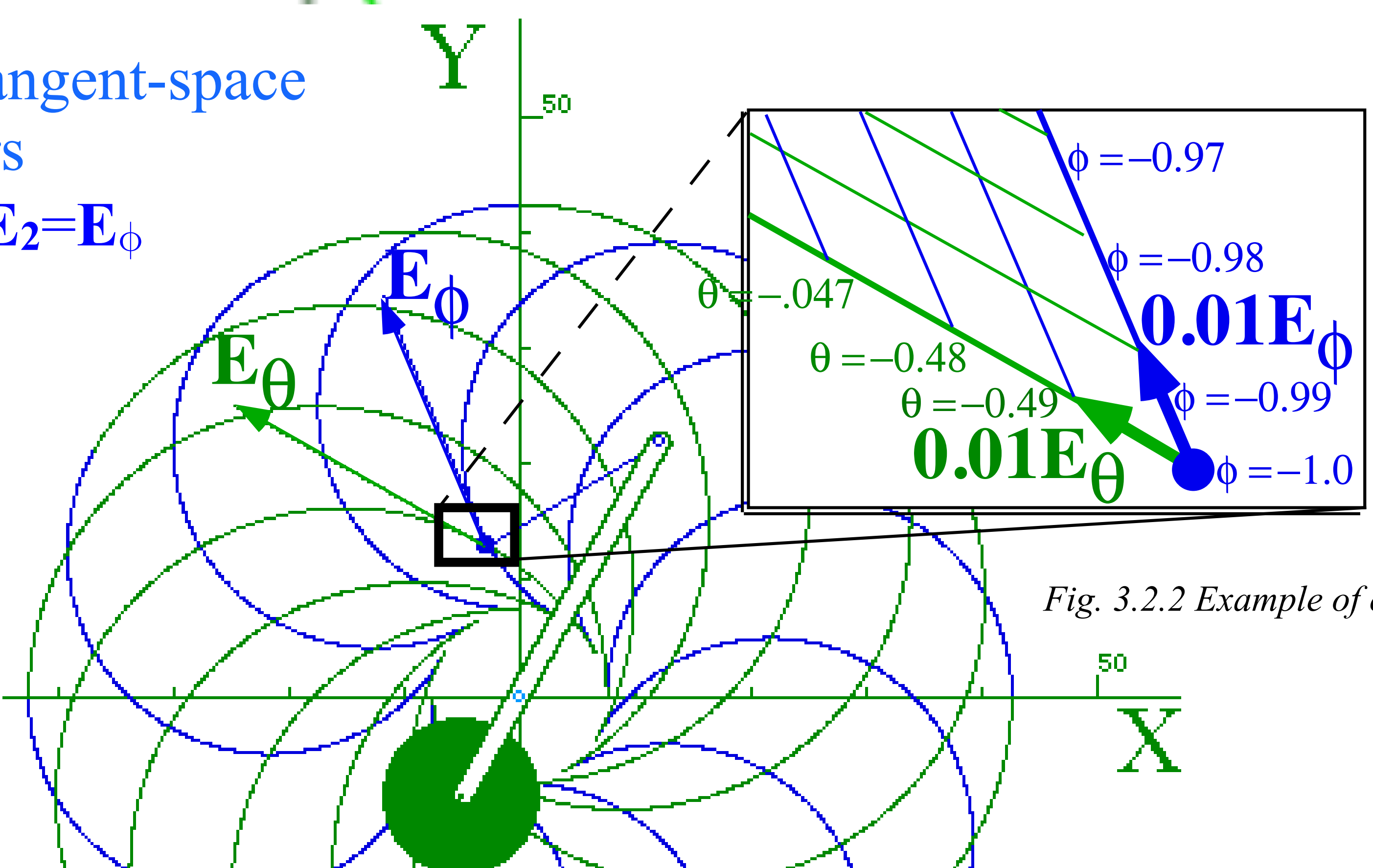


Fig. 3.1.2 Trebuchet torus.
(a) ($q^1=\theta$, $q^2=\phi$) coordinate lines.(b)Trebuchet position map and equators.

Covariant tangent-space
GCC vectors
$\mathbf{E}_1=\mathbf{E}_\theta$ and $\mathbf{E}_2=\mathbf{E}_\phi$



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Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold of trebuchet positions and "Flat" (q^1=θ, q^2=ϕ)-graph

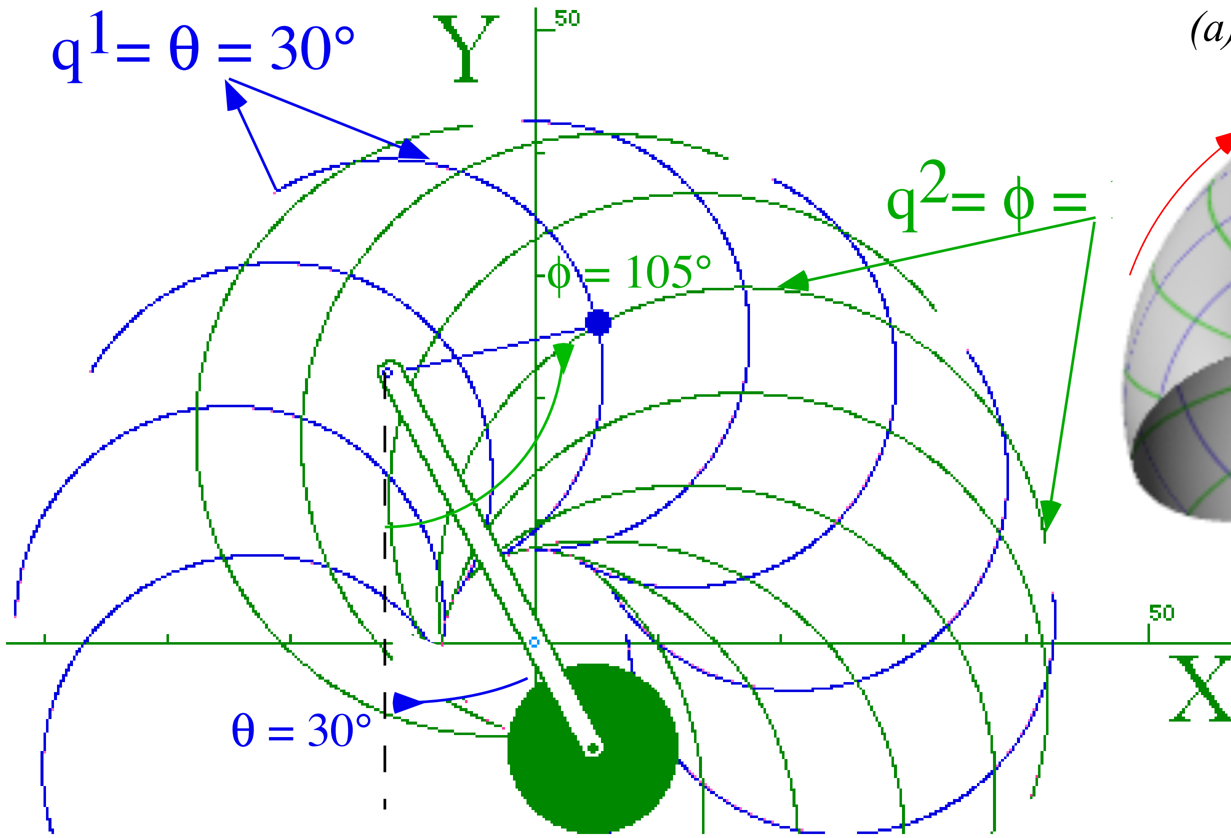
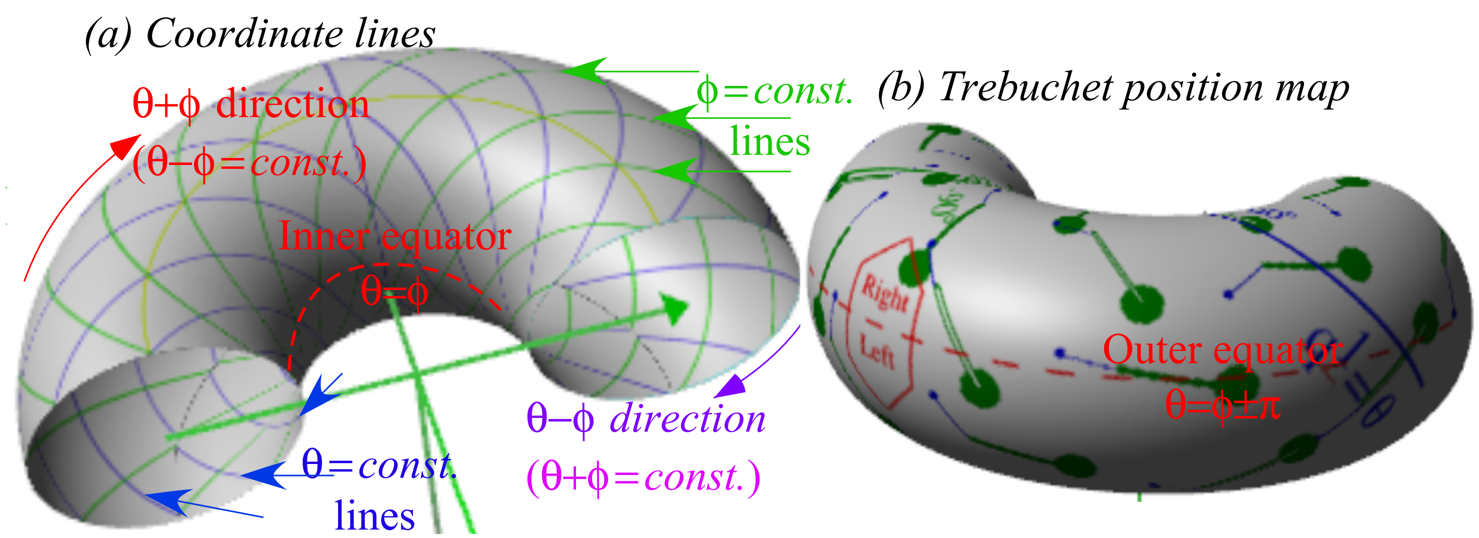


Fig. 3.1.1a (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Left handed sheet.)



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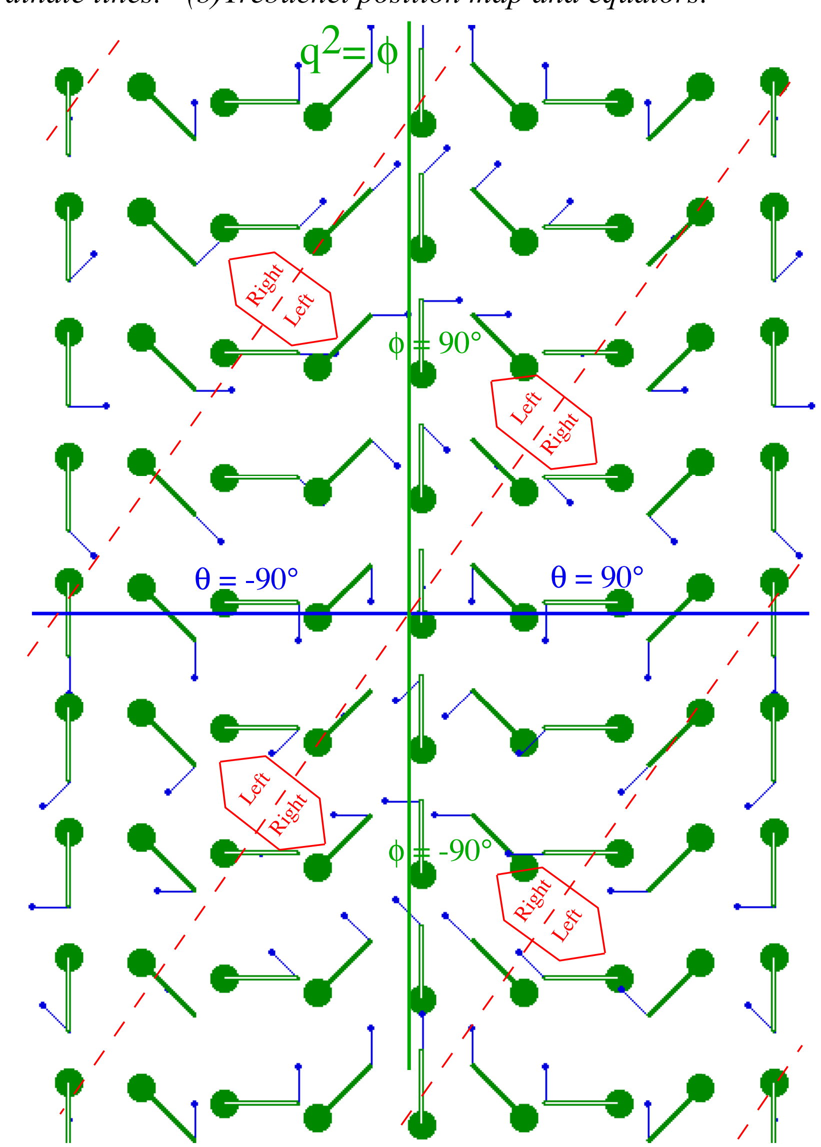

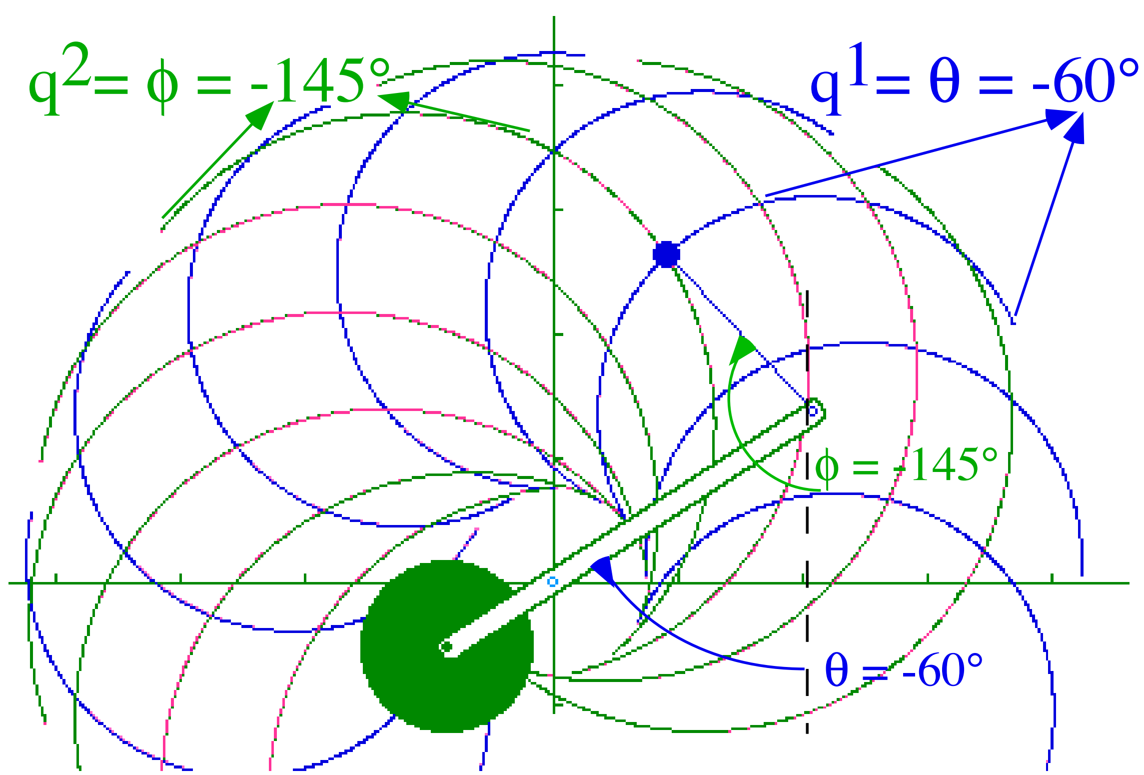

Fig. 3.1.1b (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Right handed sheet.)

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A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

from p. 43 of Lect. 9

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r \;\; \mathbf{E}_2=\mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r \;\; \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J\rangle = \begin{array}{cc} \left(\begin{array}{cc} \dfrac{\partial x^1}{\partial q^1} & \dfrac{\partial x^1}{\partial q^2} \\[2ex] \dfrac{\partial x^2}{\partial q^1} & \dfrac{\partial x^2}{\partial q^2} \end{array}\right) \\ \uparrow \mathbf{E}_1 \;\; \uparrow \mathbf{E}_2 \end{array} = \begin{array}{c} \left(\begin{array}{cc} \dfrac{\partial x}{\partial r}=\cos\phi & \dfrac{\partial x}{\partial \phi}=-r\sin\phi \\[2ex] \dfrac{\partial y}{\partial r}=\sin\phi & \dfrac{\partial y}{\partial \phi}=r\cos\phi \end{array}\right) \\ \uparrow \mathbf{E}_r \qquad\qquad \uparrow \mathbf{E}_\phi \end{array}$$

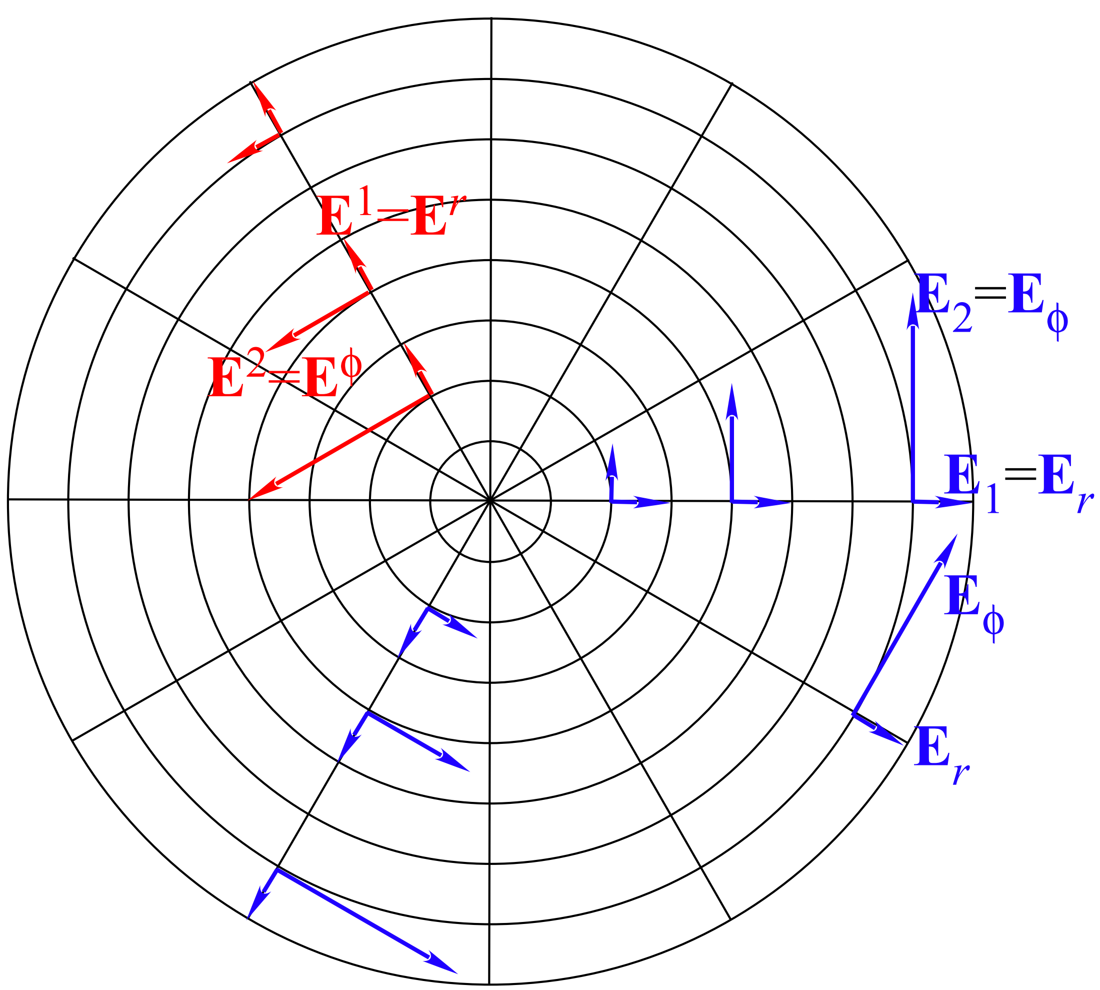
$$\langle K\rangle = \langle J^{-1}\rangle = \begin{array}{cc} \left(\begin{array}{cc} \dfrac{\partial r}{\partial x}=\cos\phi & \dfrac{\partial r}{\partial y}=\sin\phi \\[2ex] \dfrac{\partial \phi}{\partial x}=\dfrac{-\sin\phi}{r} & \dfrac{\partial \phi}{\partial y}=\dfrac{\cos\phi}{r} \end{array}\right) & \begin{array}{l} \leftarrow \mathbf{E}^r=\mathbf{E}^1 \\[4ex] \leftarrow \mathbf{E}^\phi=\mathbf{E}^2 \end{array} \end{array}$$

Derived from polar definition: $x=r\cos\phi$ *and* $y=r\sin\phi$

Inverse polar definition:

$r^2=x^2+y^2$ *and* $\phi=\mathrm{atan2}(y,x)$

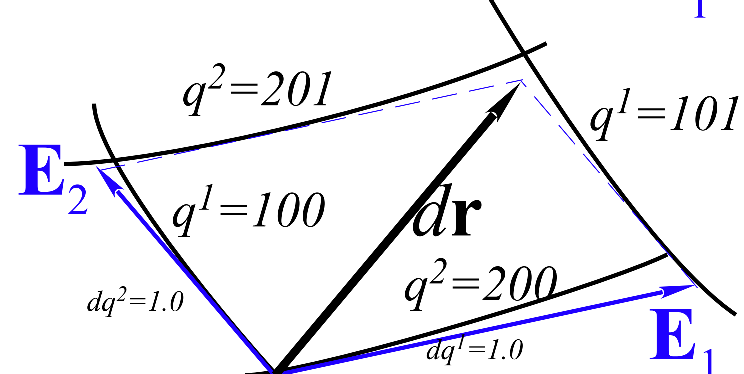
(a) Polar coordinate bases



(b) *Co*variant bases $\{\mathbf{E}_1 \; \mathbf{E}_2\}$

(Tangent)

$d\mathbf{r}=\mathbf{E}_1 dq^1+\mathbf{E}_2 dq^2$

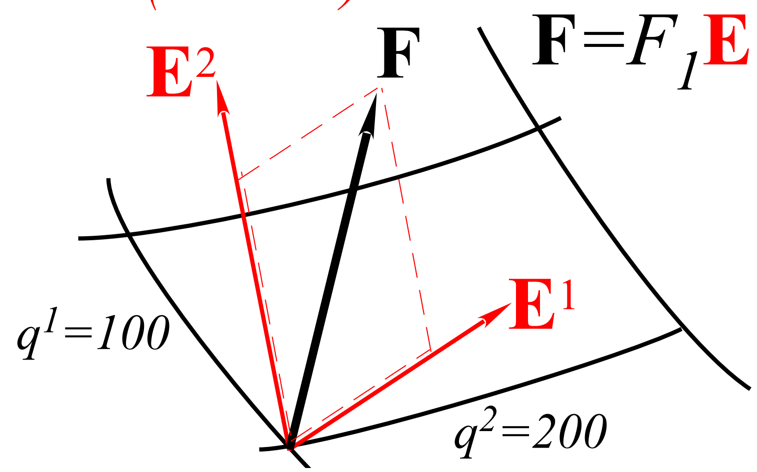


NOTE:These are 2D drawings! <u>No</u> 3D <u>perspective</u>

(c) *Contra*variant bases $\{\mathbf{E}^1 \; \mathbf{E}^2\}$

(Normal)

$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$



Unit 1
Fig. 12.10

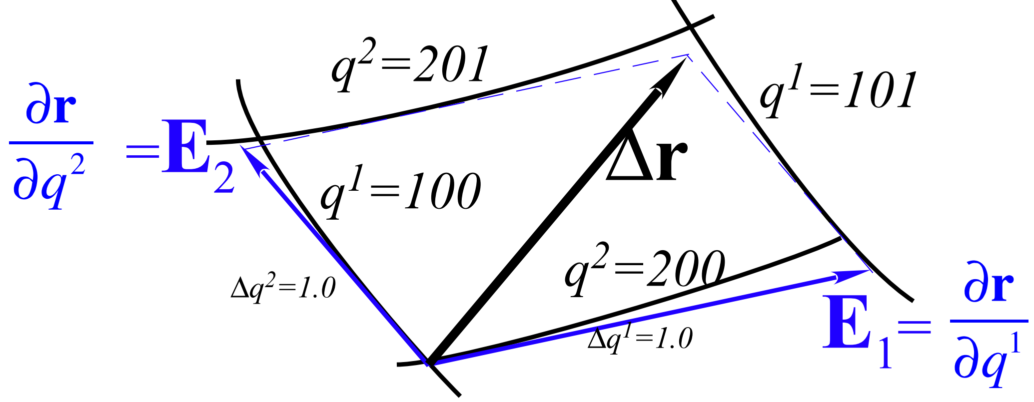
Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs.* *Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

from p. 49 of Lect. 9

*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls
(Tangent) *geometric unit*

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



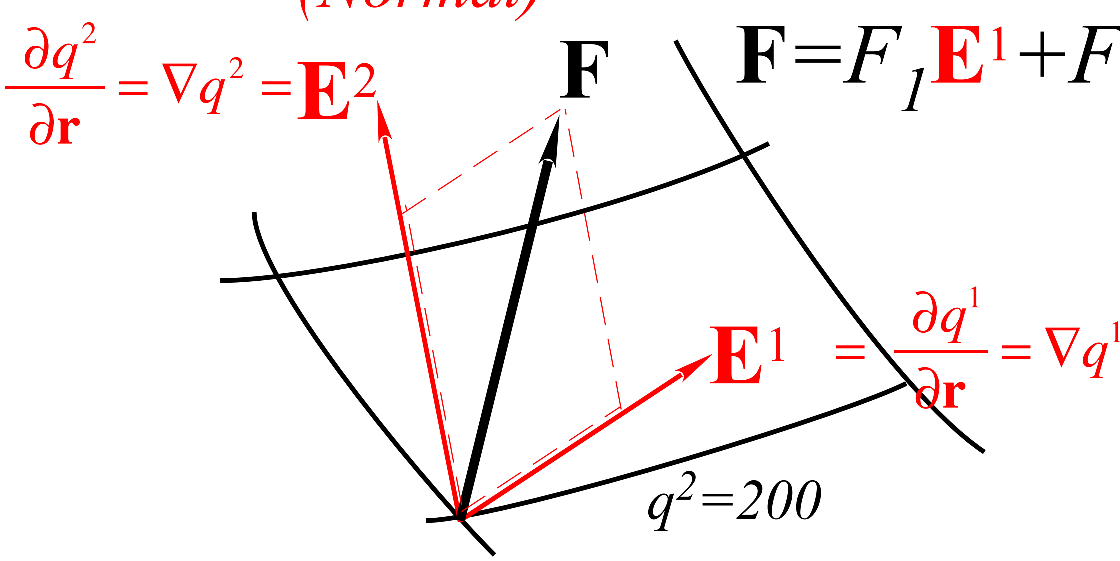
$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3*,... remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells
(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$



NOTE: These are 2D drawings! *No* 3D *perspective*

$\mathbf{E}^1$ *is normal to* q^1*=const. since*
gradient *of* q^1 *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
of all its partial derivatives

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

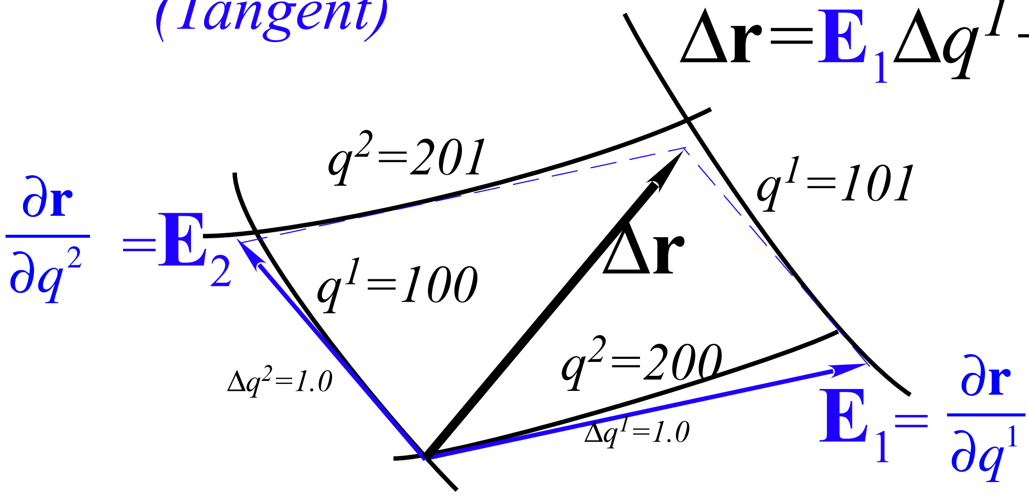
from p. 49 of Lect. 9

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\hat{\text{cell}}$ walls (*geometric unit*)

(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... *remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

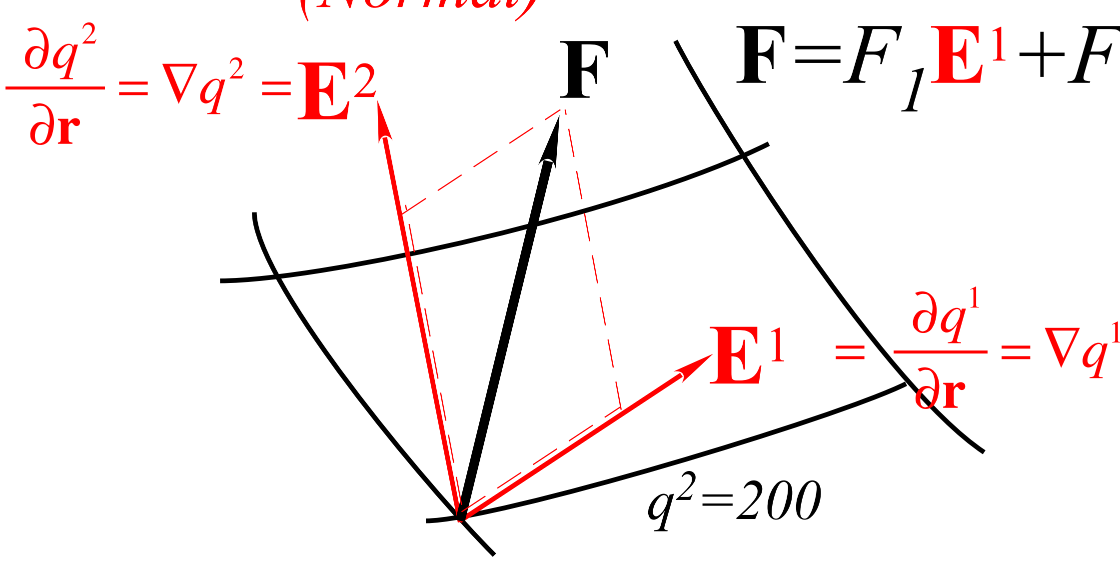
Co-Contra dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ *are orthonormal:*

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$



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$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph

Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

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Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

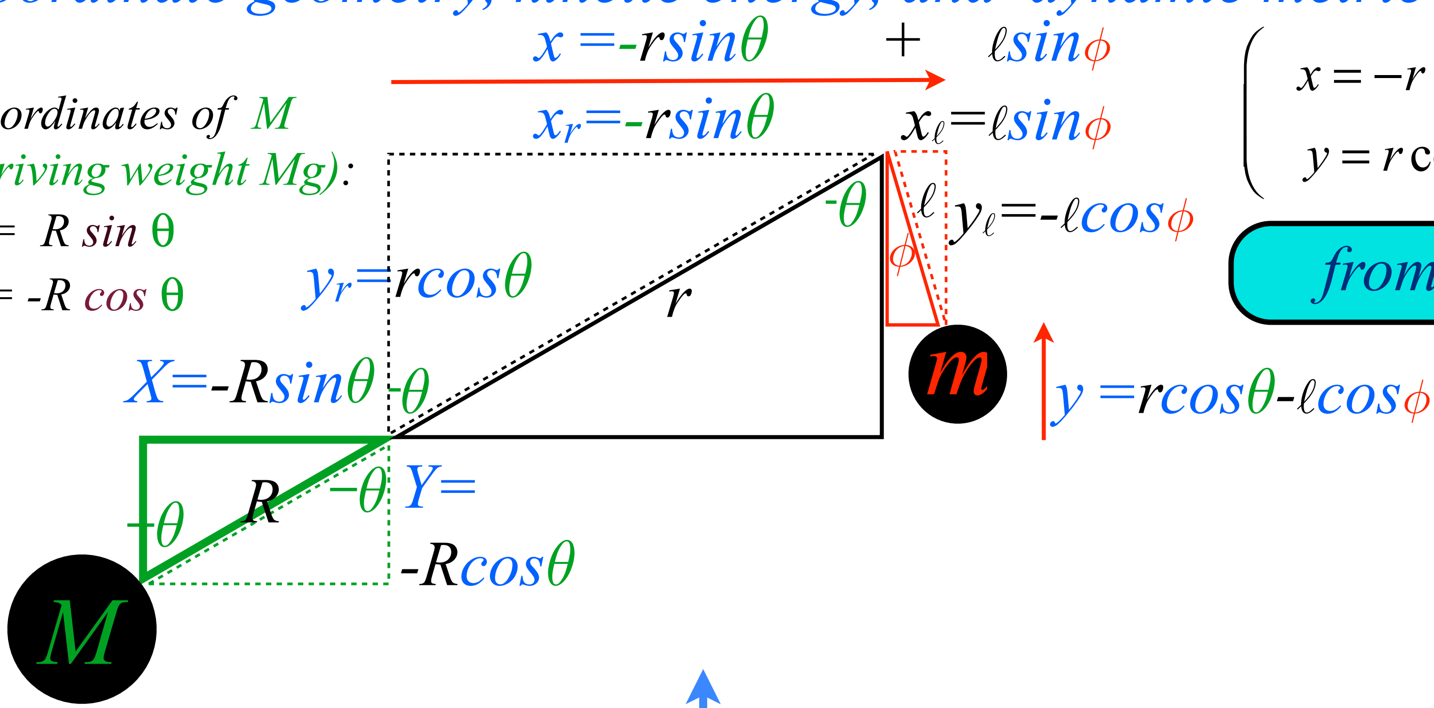
2nd-guessing Riemann equation?

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Coordinates of M (Driving weight Mg):

$X = R \sin\theta$

$Y = -R \cos\theta$



$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

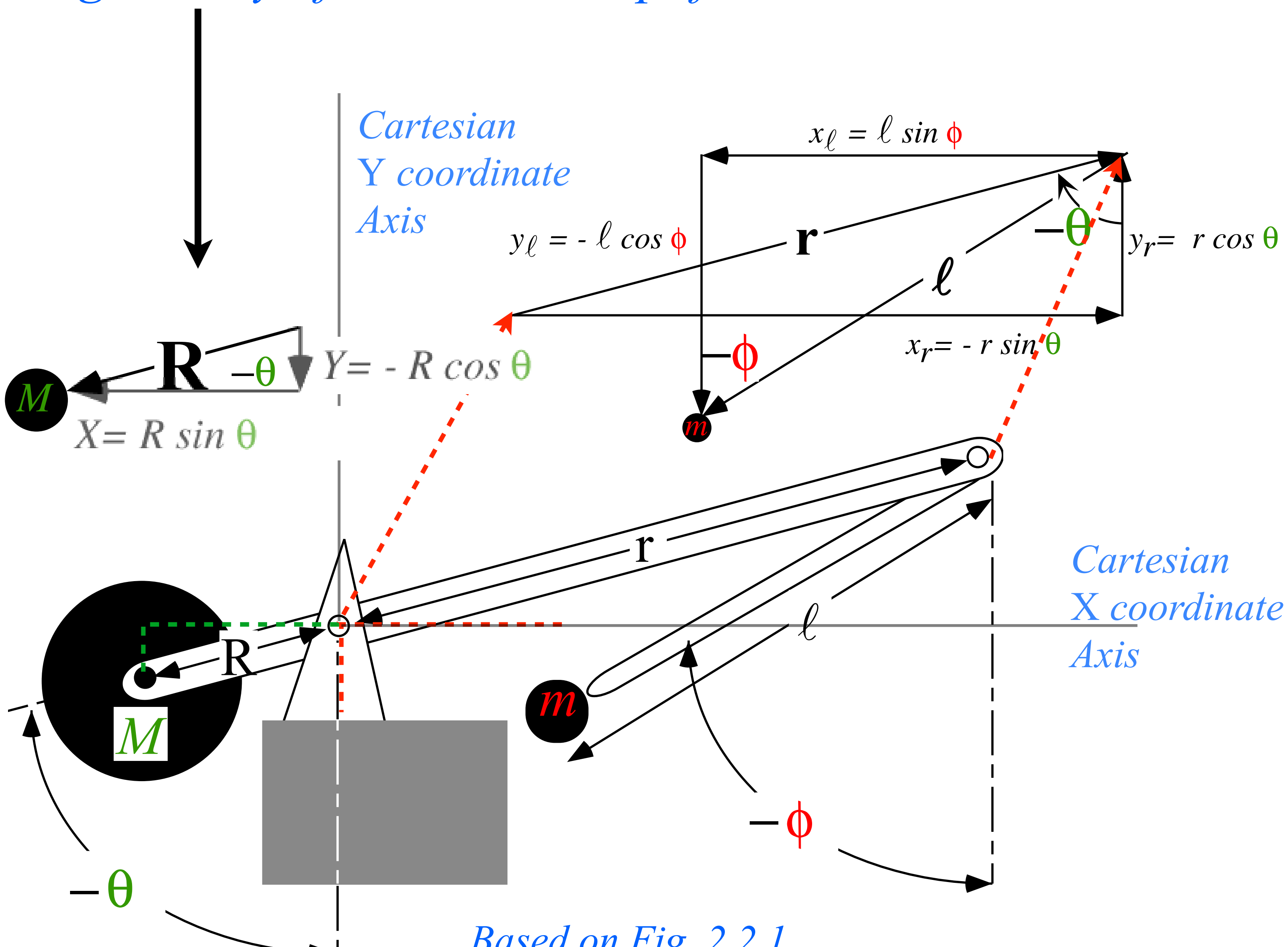
from p. 18 of Lect. 14

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

geometry of trebuchet simplified somewhat...



Based on Fig. 2.2.1

Jacobian transformation matrix

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}$$

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Jacobian transformation matrix

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

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Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Jacobian transformation matrix

$$x = -r\sin\theta + \ell\sin\phi$$
$$y = r\cos\theta - \ell\cos\phi$$

from p. 18 of Lect. 14

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

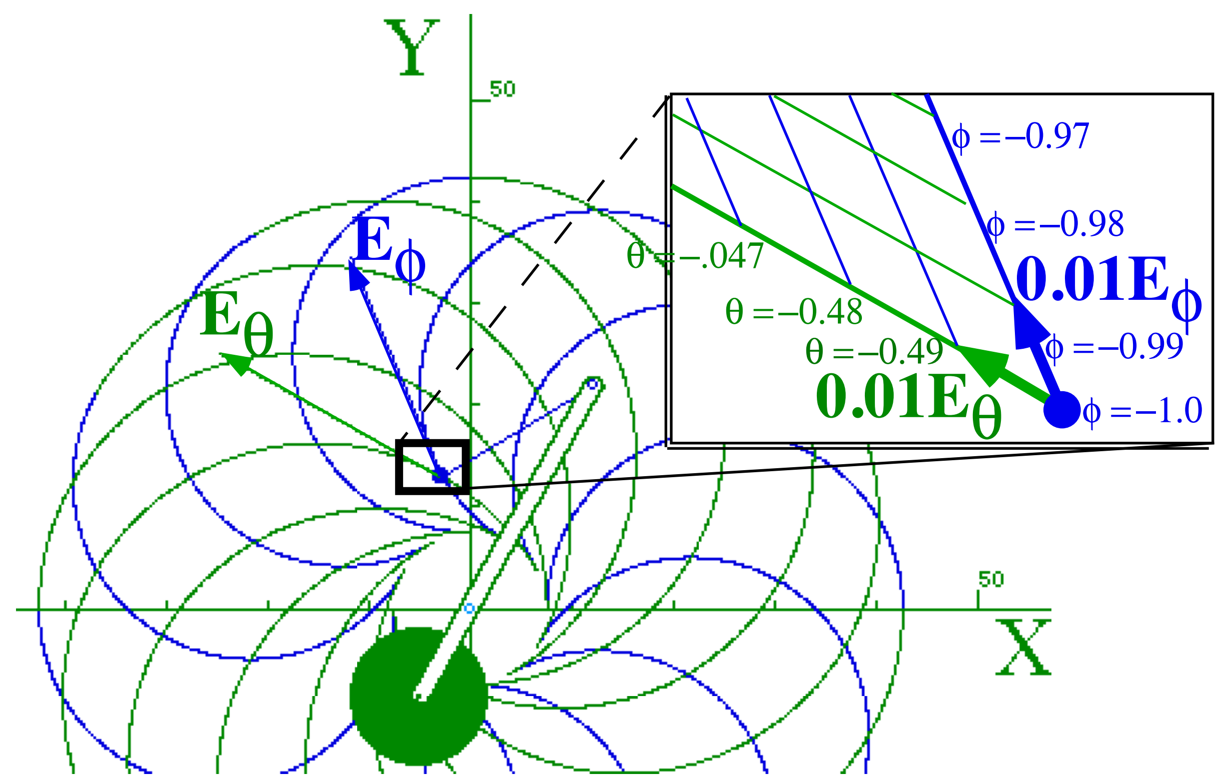

Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
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Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{\ell r \sin\theta \cos\phi - \ell r \sin\phi \cos\theta}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

Covariant vectors $\mathbf{E}_n$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

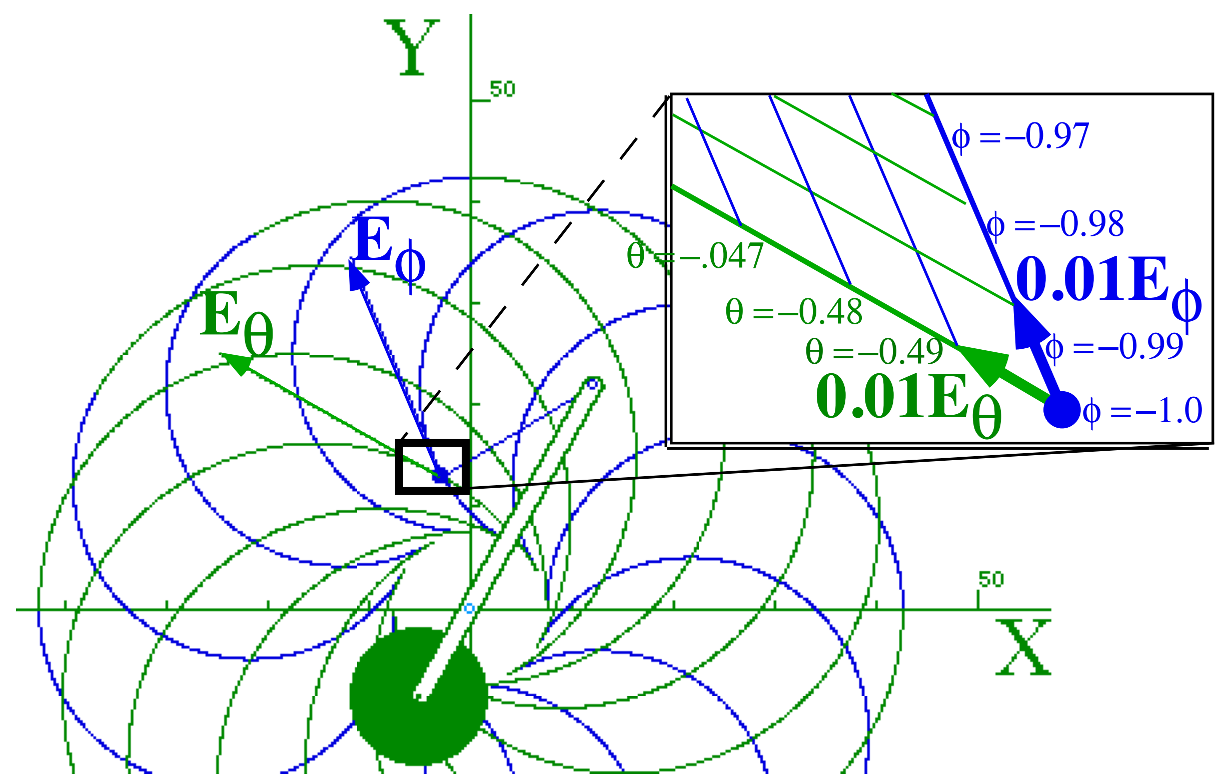


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \quad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

$$\left| \begin{array}{ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} \right| = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{array}{cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array} \right|}{r\ell\sin(\theta-\phi)}$$

$$\left| \begin{array}{ccc} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} \right| = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \left| \begin{array}{cc} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} \right|$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

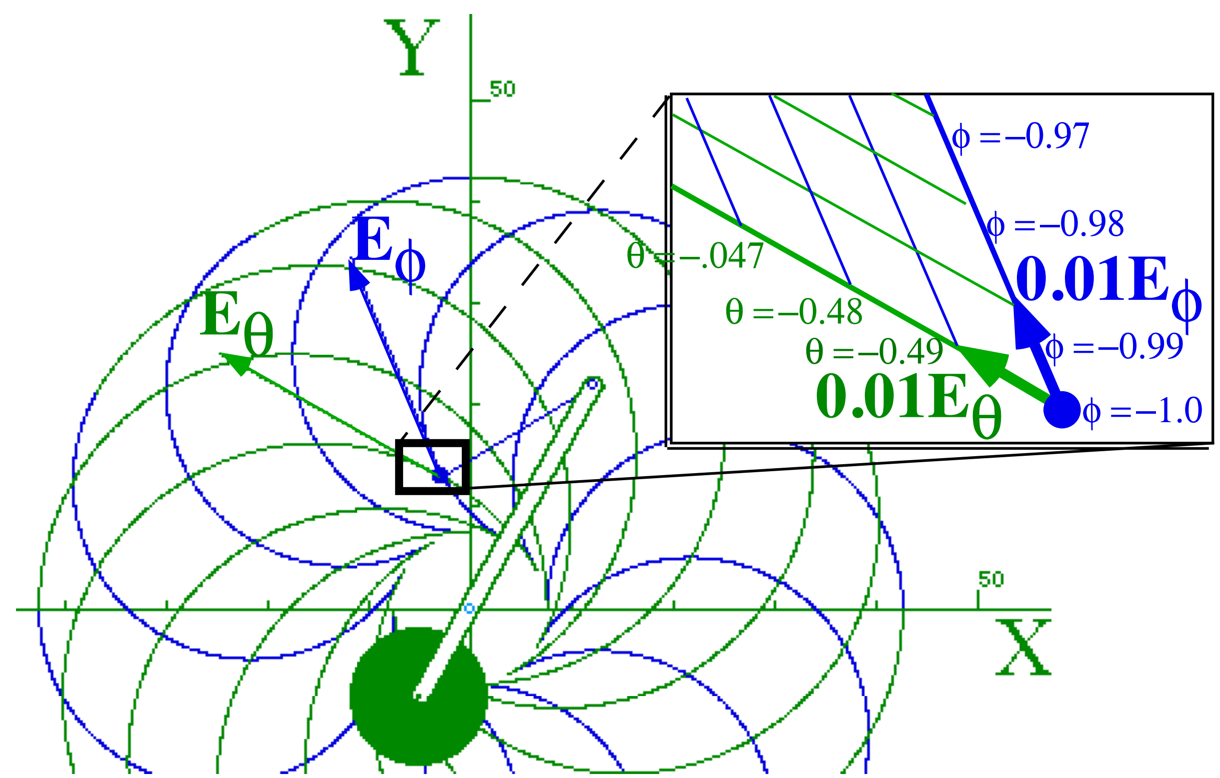


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

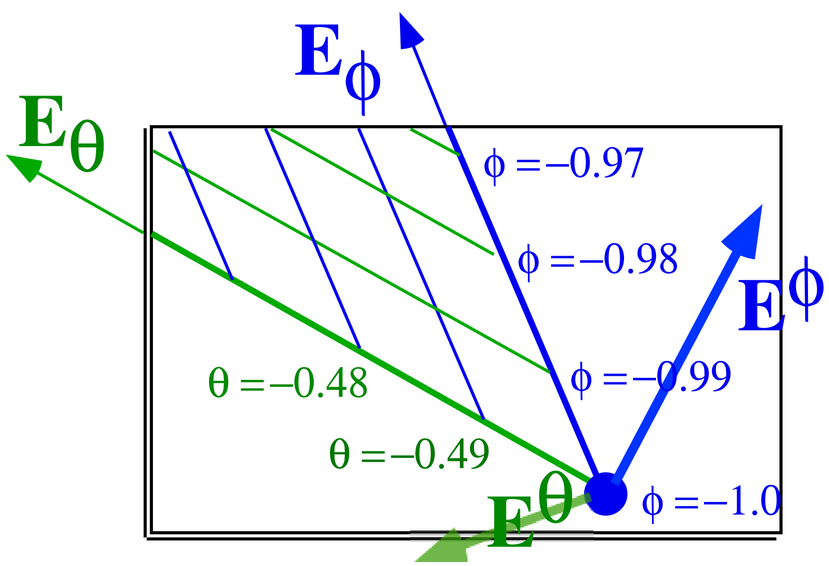
Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors $\mathbf{E}_1=\mathbf{E}_\theta$ and $\mathbf{E}_2=\mathbf{E}_\phi$



Contravariant normal-space GCC vectors $\mathbf{E}^1=\mathbf{E}^\theta$ and $\mathbf{E}^2=\mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

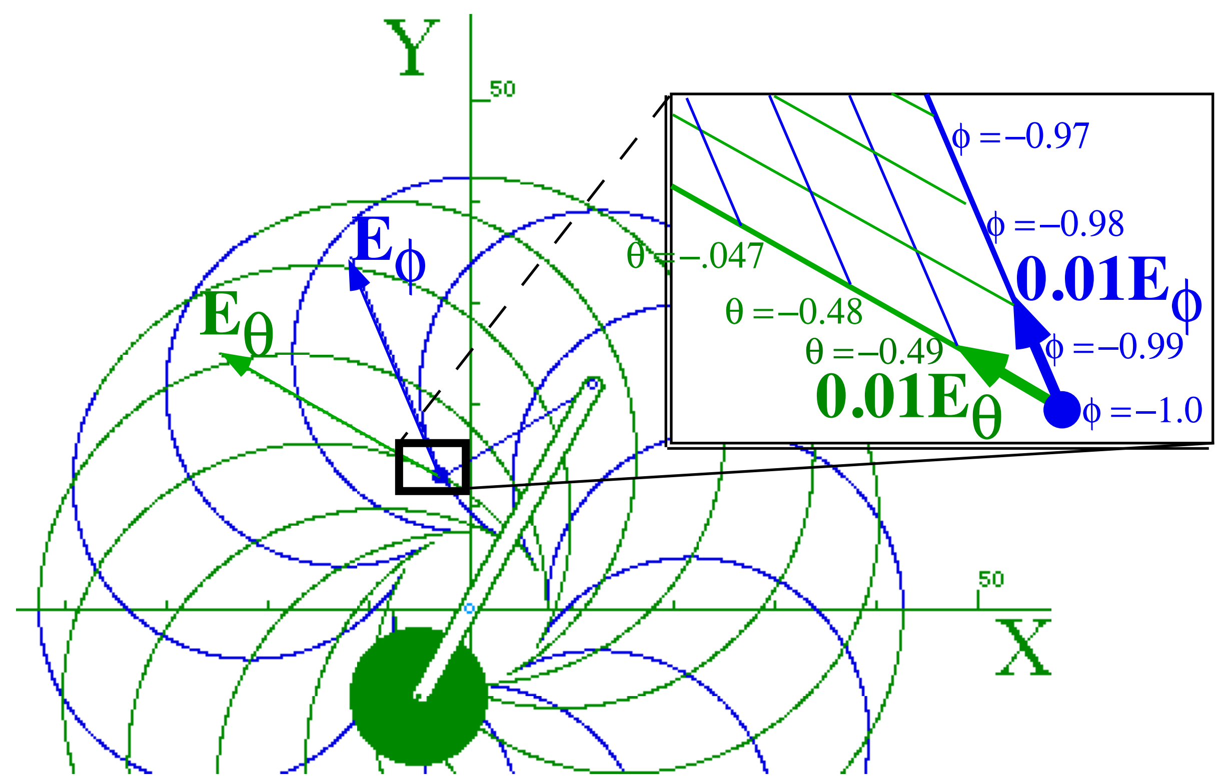


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

$$x = -r\sin\theta + \ell\sin\phi$$
$$y = r\cos\theta - \ell\cos\phi$$

from p. 18 of Lect. 14

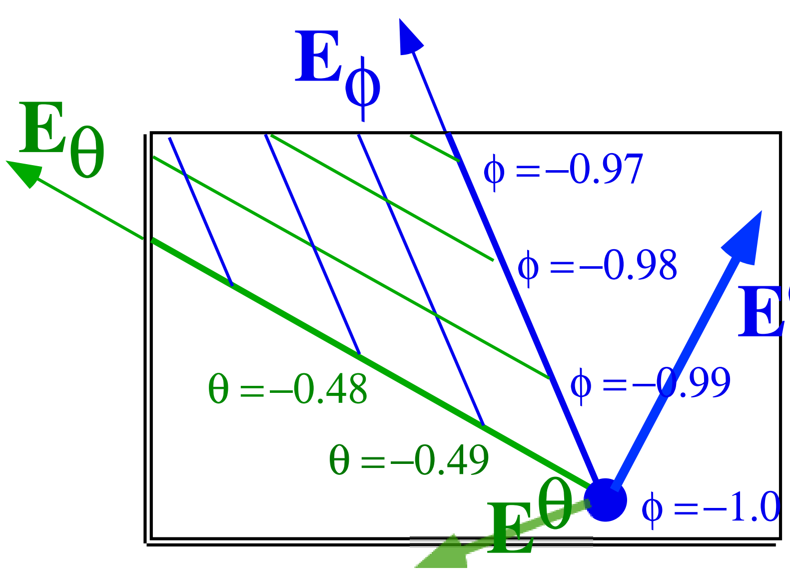
Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$
$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\theta \bullet \mathbf{E}_\phi = 0 = \mathbf{E}_\theta \bullet \mathbf{E}^\phi$$
$$\mathbf{E}^\theta \bullet \mathbf{E}_\theta = 1 = \mathbf{E}_\phi \bullet \mathbf{E}^\phi$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

Covariant tangent-space GCC vectors
$\mathbf{E}_1 = \mathbf{E}_\theta$ and $\mathbf{E}_2 = \mathbf{E}_\phi$



Contravariant normal-space GCC vectors
$\mathbf{E}^1 = \mathbf{E}^\theta$ and $\mathbf{E}^2 = \mathbf{E}^\phi$

Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

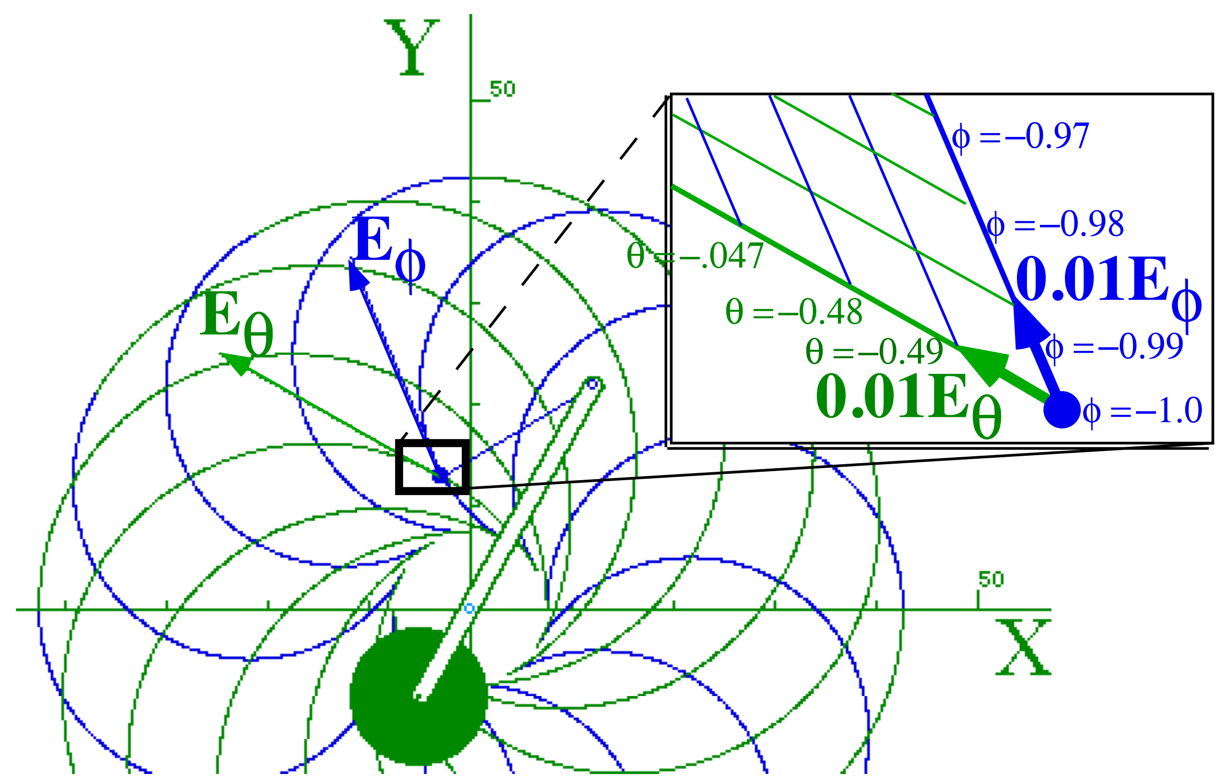


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
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Riemann equation force analysis
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Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant metric tensor g_{mn}

Invariant Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant metric tensor g^{mn}

from p. 53 of Lect. 9

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad \uparrow \mathbf{E}_r \quad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{matrix}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

Using 2x2 inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \dfrac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$

$$\begin{array}{|cccc|c}
\frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & & \mathbf{E}^1 \\
\frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & & \mathbf{E}^2 \\
\vdots & \vdots & \ddots & & \vdots
\end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \qquad \begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

from p. 18 of Lect. 14

$$\begin{array}{|ccc|}
\mathbf{E}_1 & \mathbf{E}_2 & \cdots \\
\hline
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\
\vdots & \vdots & \ddots
\end{array} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

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Contravariant metric $g^{mn} = \mathbf{E}^m \bullet \mathbf{E}^n = g^{nm}$ *versus* *Covariant metric* $g_{mn} = \mathbf{E}_m \bullet \mathbf{E}_n = g_{nm}$

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\theta \bullet \mathbf{E}_\theta & \mathbf{E}_\theta \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_\theta & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix}$$

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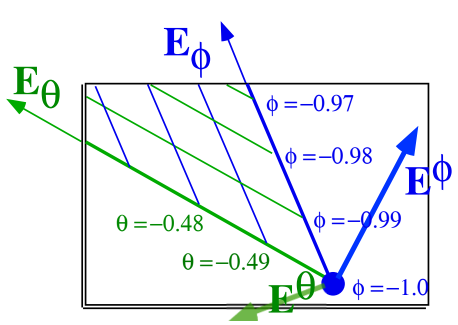

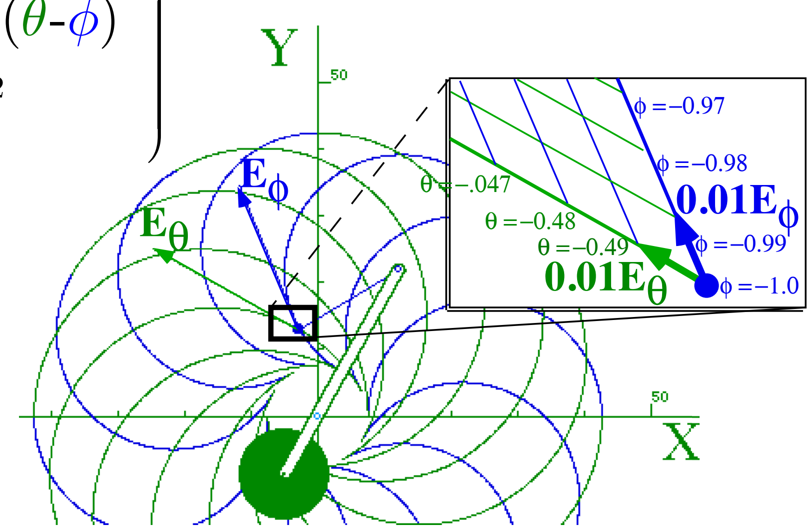

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
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Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs contravariant metric g^{mn} (Lect. 10 p.43-49)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
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$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

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$$\left| \begin{matrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \right| \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\left| \begin{matrix} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{matrix} \right| \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell\sin(\theta-\phi)}$$

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from p. 18 of Lect. 14

$$\begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

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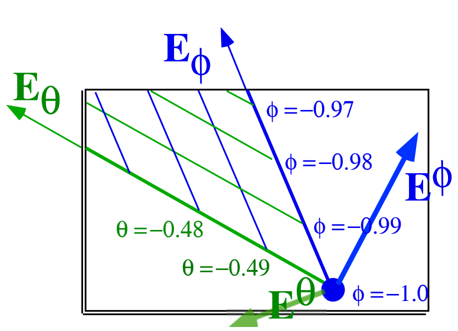

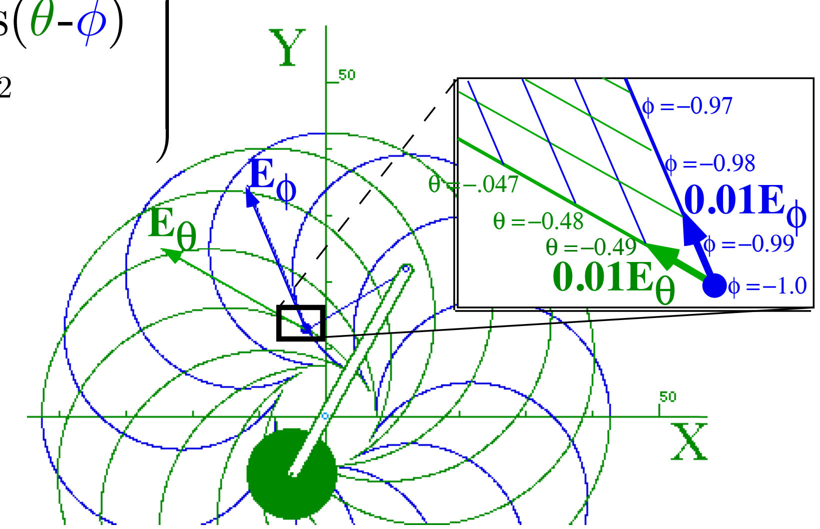

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Using 2x2 inverse

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{vmatrix} D & -B \\ -C & A \end{vmatrix}}{AD-BC}$$

from p. 18 of Lect. 14

$$\begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell\sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

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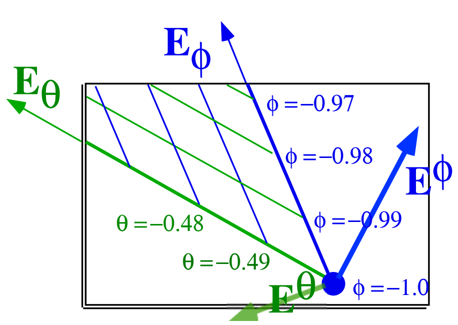
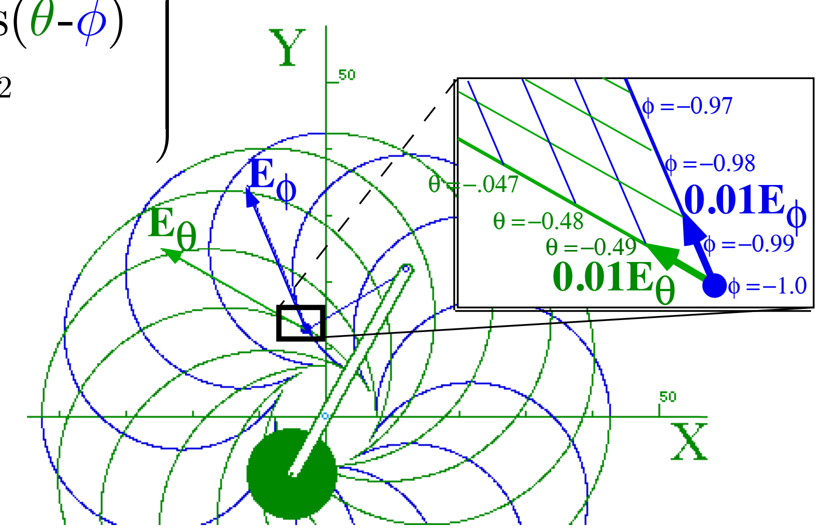
Jacobian $J^T J$*-product gives* g_{mn}



$$J^T J = \begin{array}{c|cc} \mathbf{E}_\theta & -r\cos\theta & -r\sin\theta \\ \mathbf{E}_\phi & \ell\cos\phi & \ell\sin\phi \end{array} \; \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$



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Jacobian J^TJ*-product gives* g_{mn}

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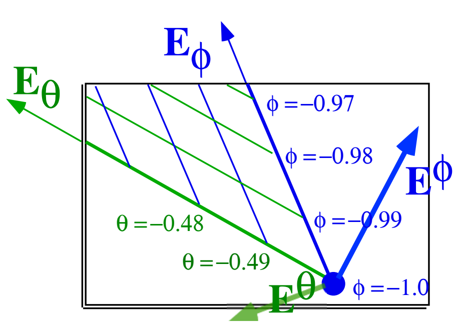

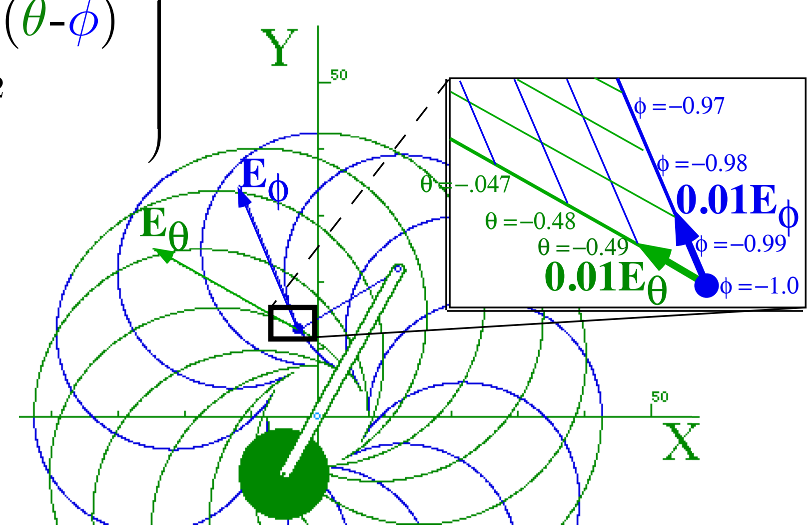

Kajobian KK^T*-product would give* g^{mn}

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
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Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

$$\mathbf{V} = V_\theta\mathbf{E}^\theta + V_\phi\mathbf{E}^\phi$$

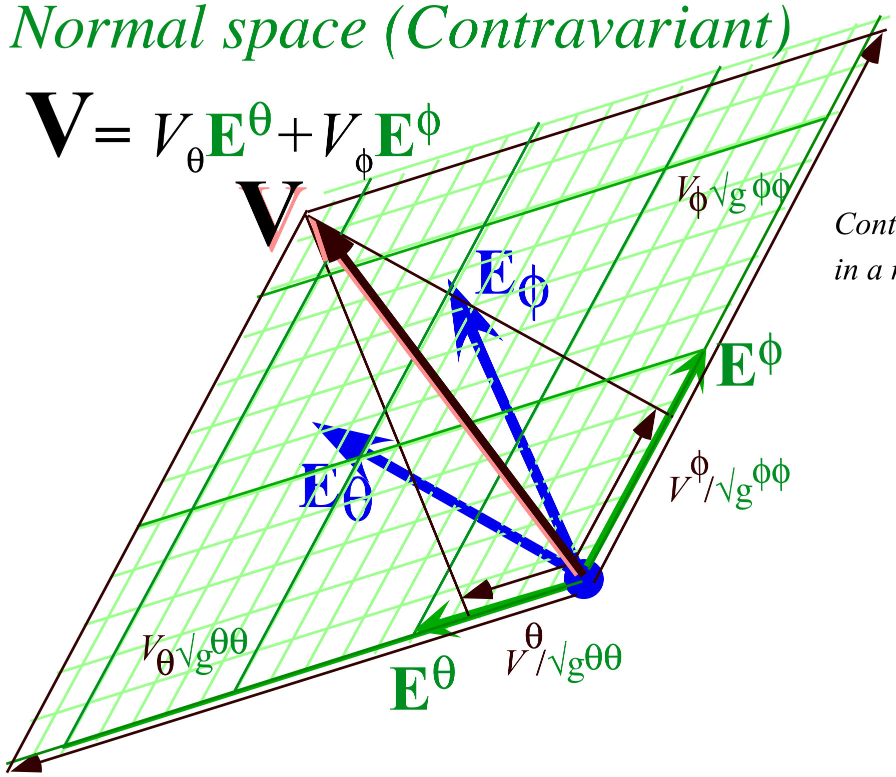

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

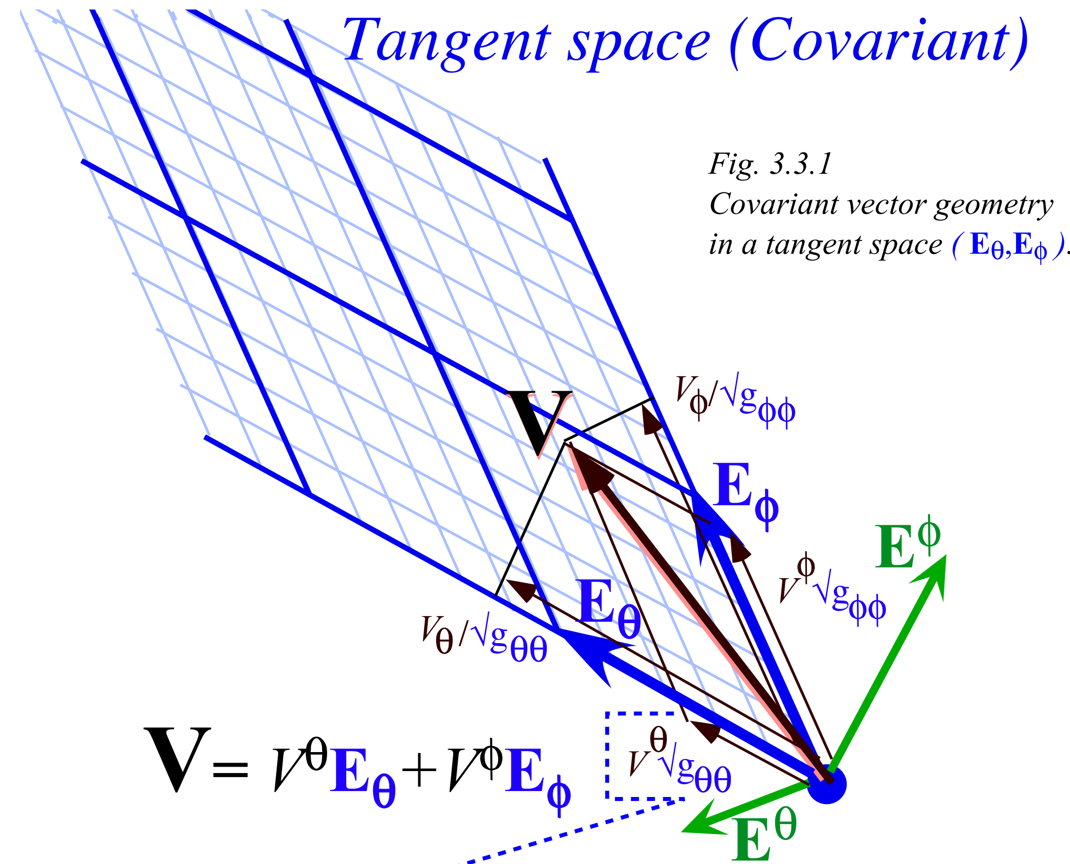

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

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Any vector **U**,**V**,**...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *U^m, V^m,...are contravariant components* and the *U_n , V_n ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

Normal space (Contravariant)

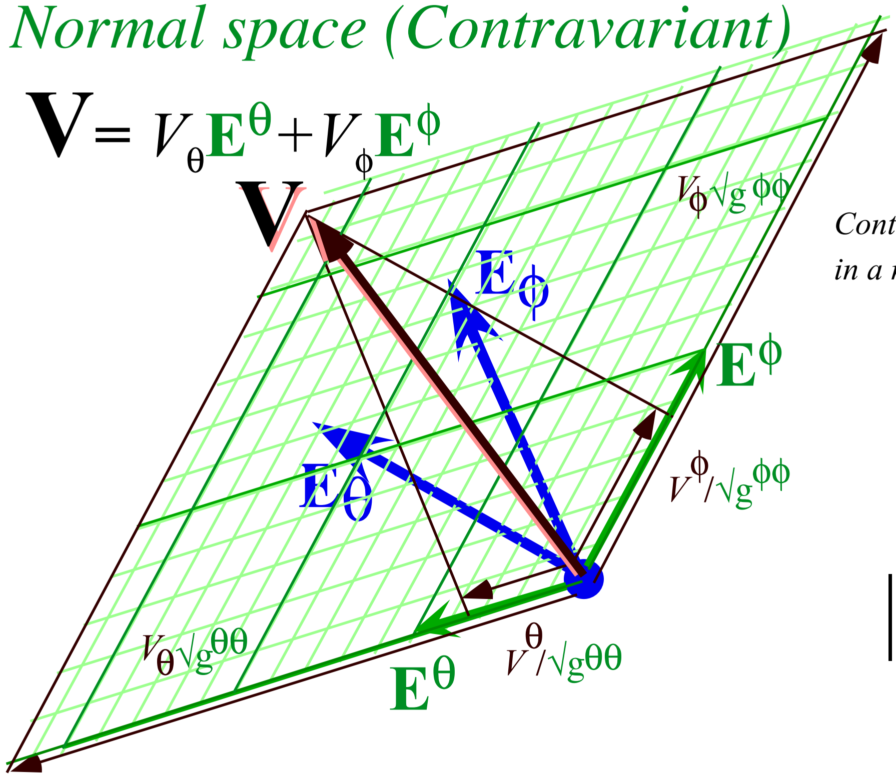

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

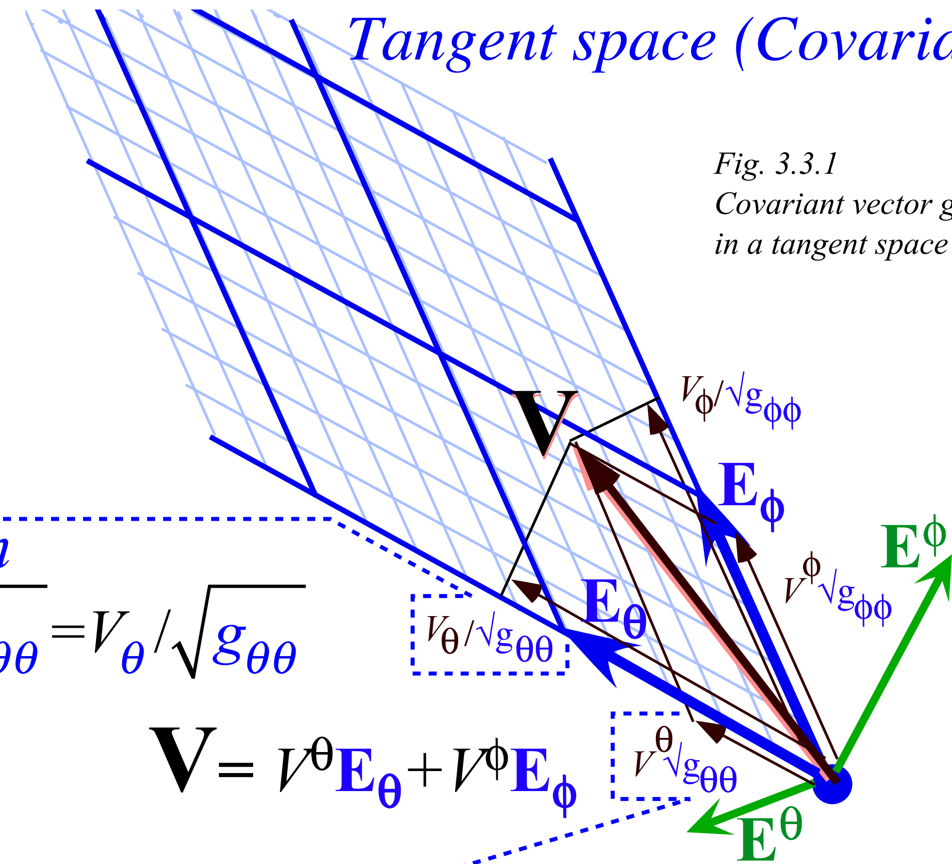

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

covariant parallelogram side

$$\left|V^\theta\,\mathbf{E}_\theta\right| = V^\theta\sqrt{\mathbf{E}_\theta\cdot\mathbf{E}_\theta} = V^\theta\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E}^m$ is written in terms of covariant vectors $\mathbf{E}_n$ as would any vector $\mathbf{V} = V^n\,\mathbf{E}_n$ using dot product $V^n = \mathbf{V}\bullet\mathbf{E}^n$ and metric g_{mn} or g^{mn}...

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

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Normal space (Contravariant)

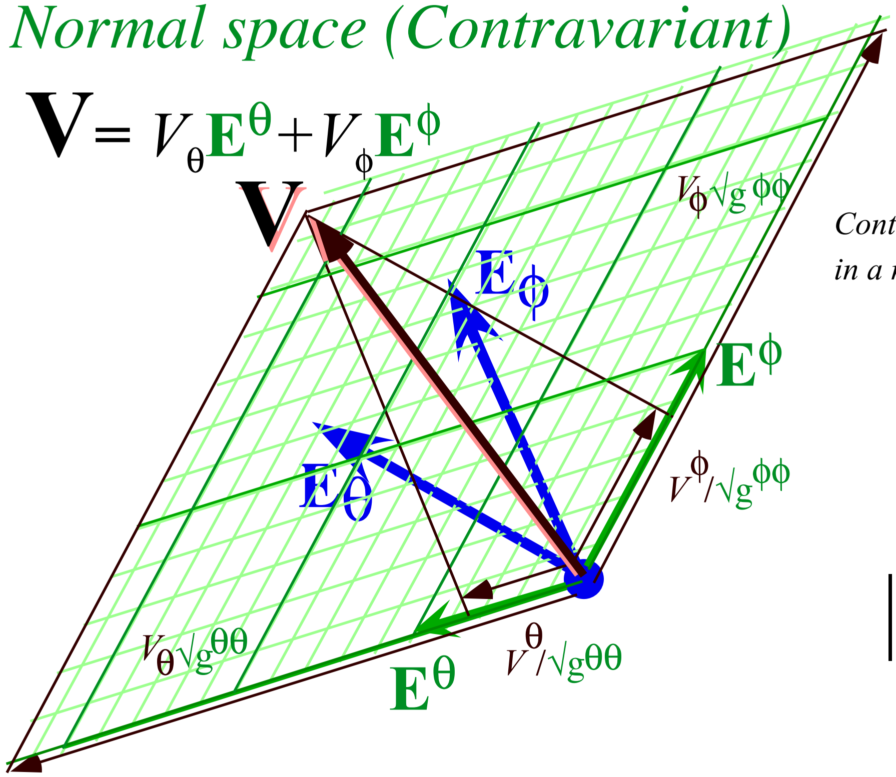

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

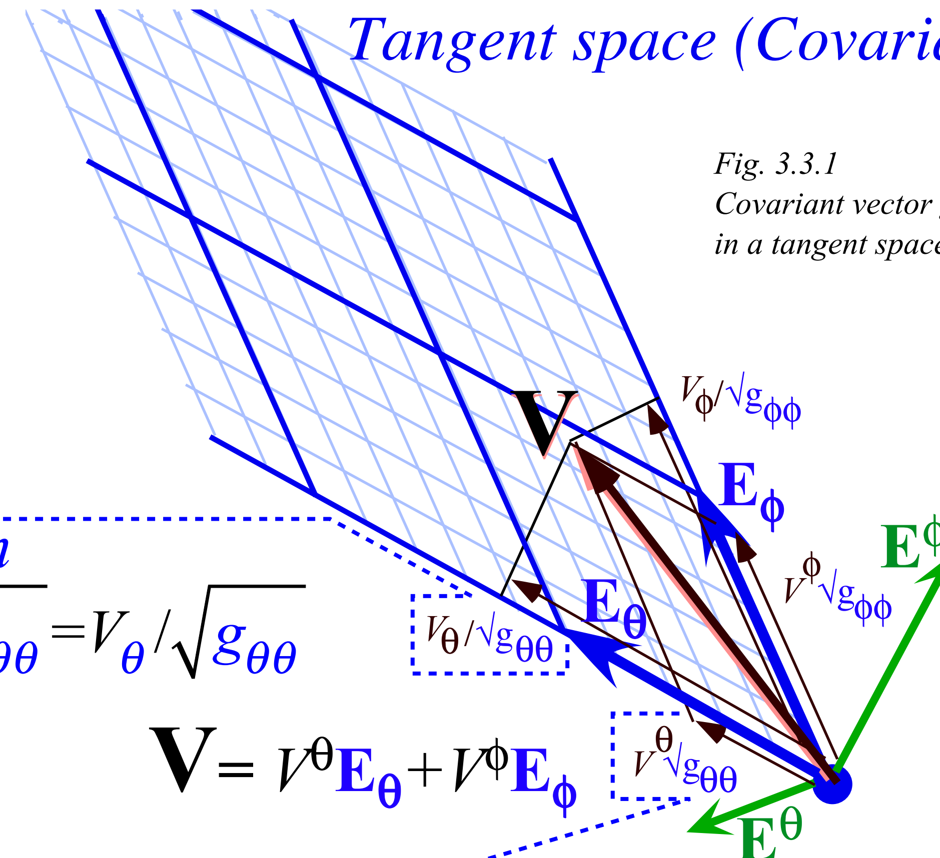

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\hat{\mathbf{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

covariant parallelogram side

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$$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n \text{ implies: } (\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$$

$$\text{so: } \mathbf{E}^m = g^{mn}\,\mathbf{E}_n$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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Normal space (Contravariant)

$$\mathbf{V} = V_\theta \mathbf{E}^\theta + V_\phi \mathbf{E}^\phi$$

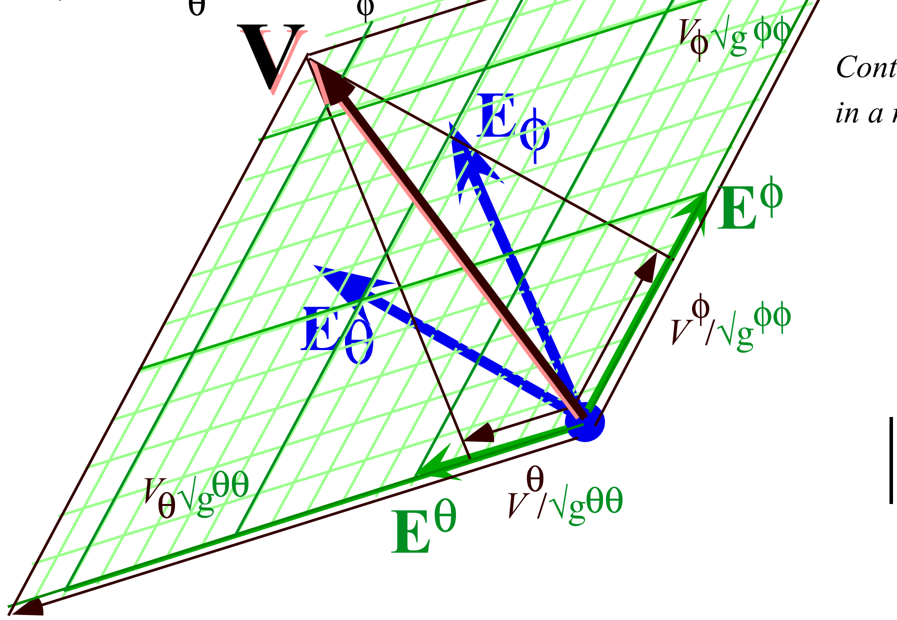

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

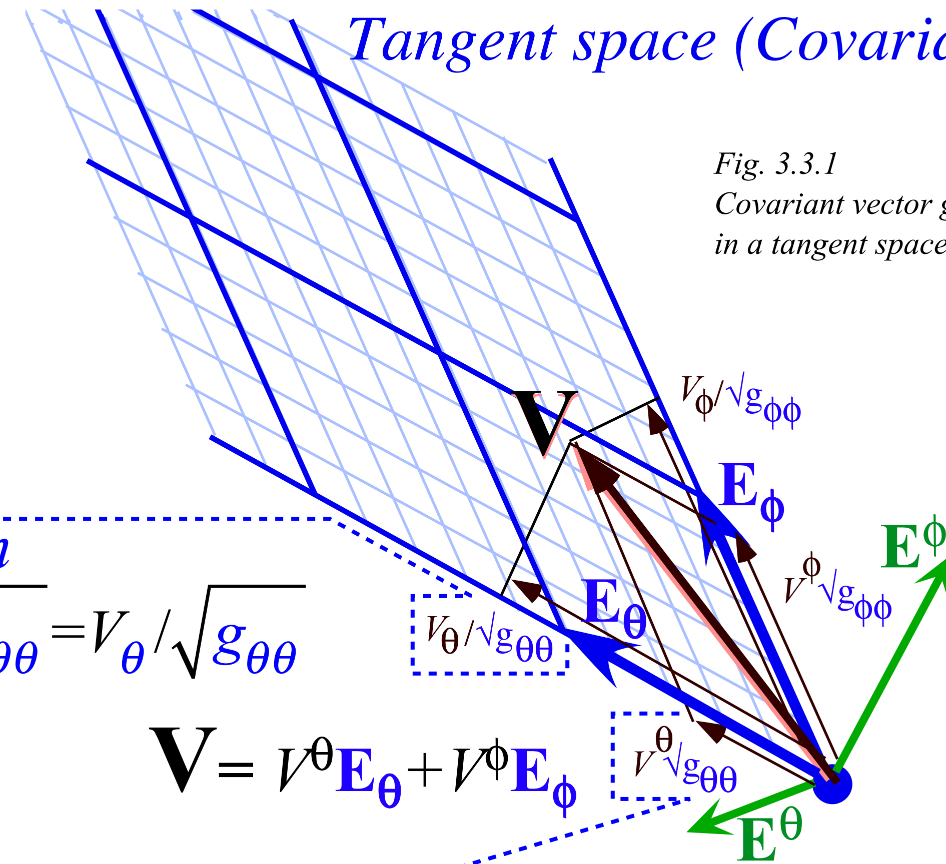

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

covariant projection

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covariant parallelogram side

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$\mathbf{E}^m = (\mathbf{E}^m)^n\mathbf{E}_n$ implies: $(\mathbf{E}^m)^n = \mathbf{E}^m\bullet\mathbf{E}^n = g^{mn}$

so: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$

and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$ ...the same for covariant vectors

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ *and contravariant* $\mathbf{E}^m$ *vectors: Jacobian J vs. Kajobian K*
Covariant metric g_{mn} *vs. contravariant metric* g^{mn} *(Lect. 9 p.53)*
Tangent $\{\mathbf{E}_n\}$ *space vs. Normal* $\{\mathbf{E}^m\}$ *space*
Covariant vs. contravariant coordinate transformations
Metric g_{mn} *tensor geometric relations to length, area, and volume*

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} *(Lect. 14 p. 77)*
Review and application of trebuchet covariant forces F_θ *and* F_ϕ *(Lect. 14 p. 69)*
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},, \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}},\ \textit{etc.}$$

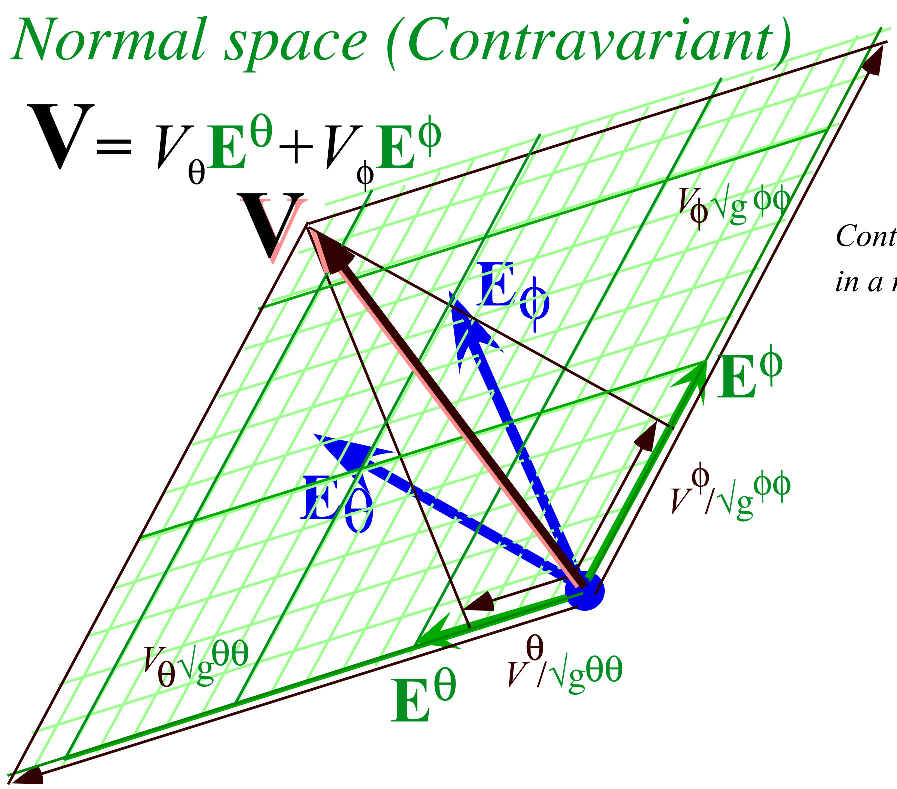

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

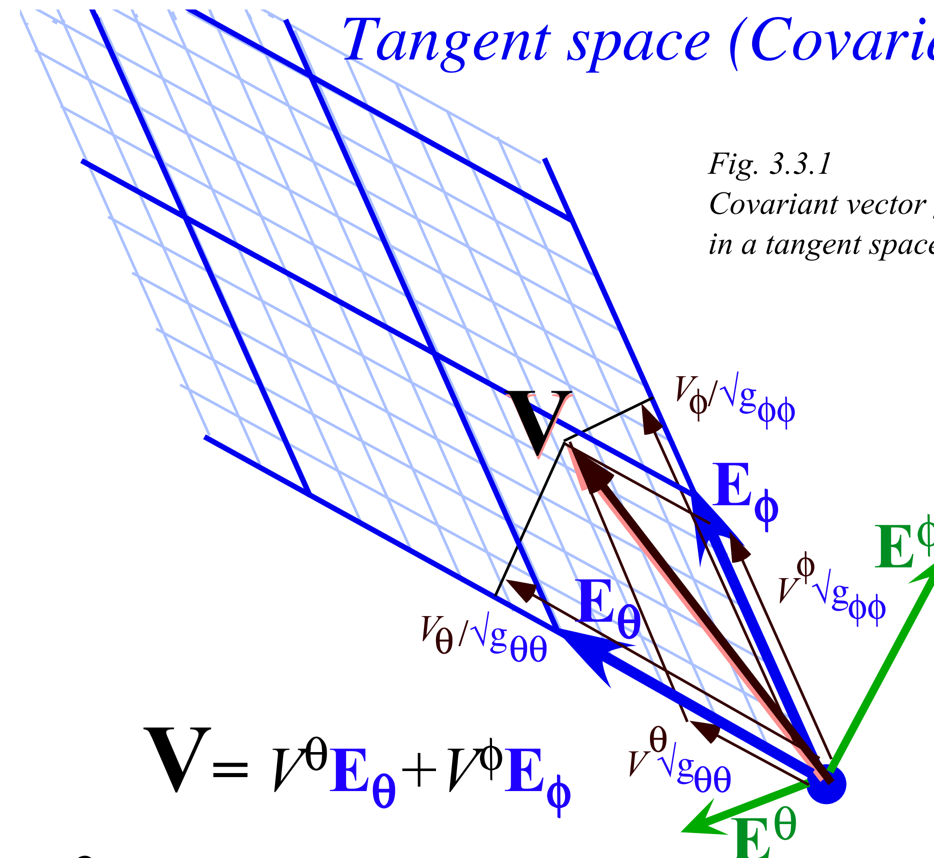

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1,q^2,\cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1,\bar{q}^2,\cdots\}$ using a *"chain-saw-sum rule"*....

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ , \quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

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$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}},\ , \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}},\ etc.$$

Normal space (Contravariant)

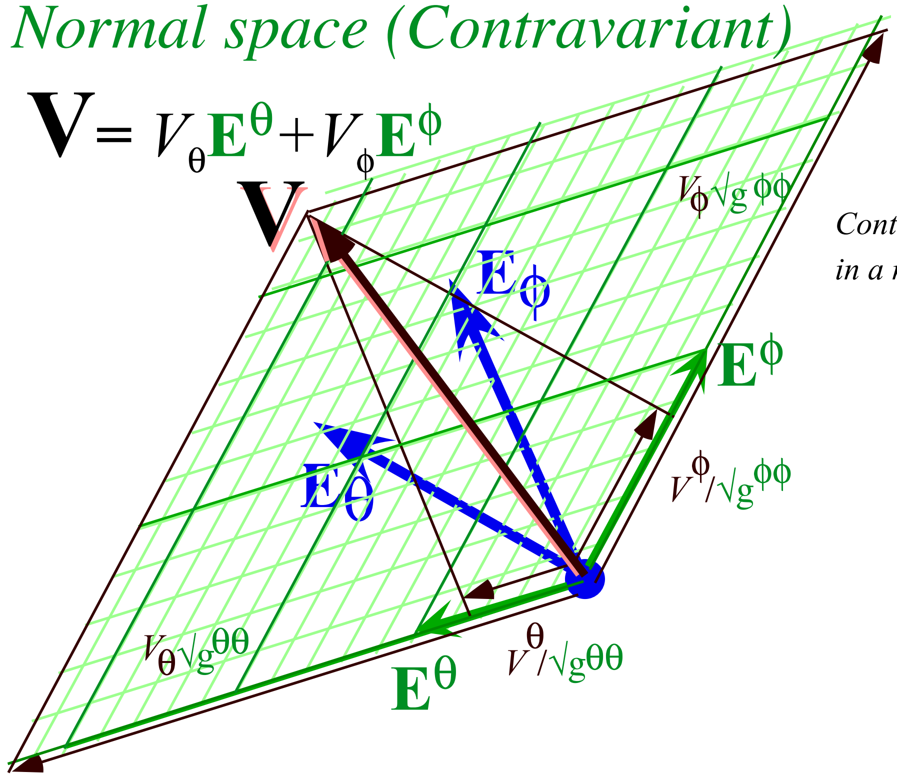

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta$,$\mathbf{E}^\phi$).

Tangent space (Covariant)

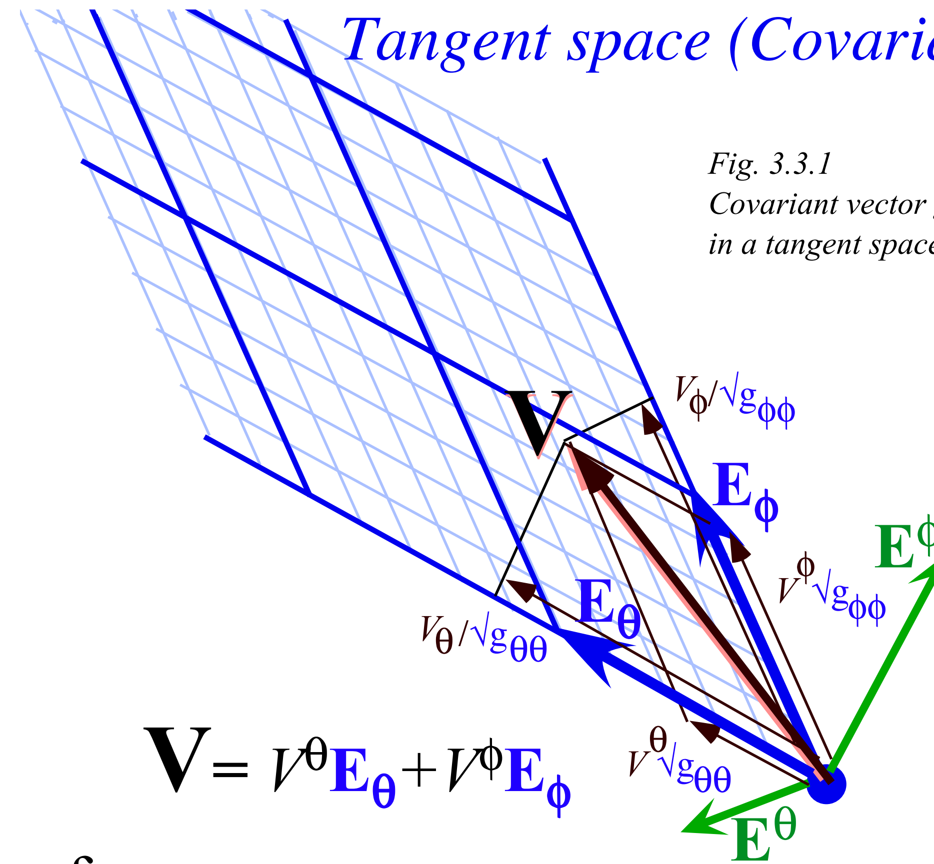

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta$,$\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

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$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\phantom{\partial \mathbf{r}}}{\partial q^m}\frac{\partial \mathbf{r}}{} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}\ ,\quad \text{or:}\ \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

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where the *Um, Vm,...are contravariant components*

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and the *Un , Vn ,...are covariant components*

$U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

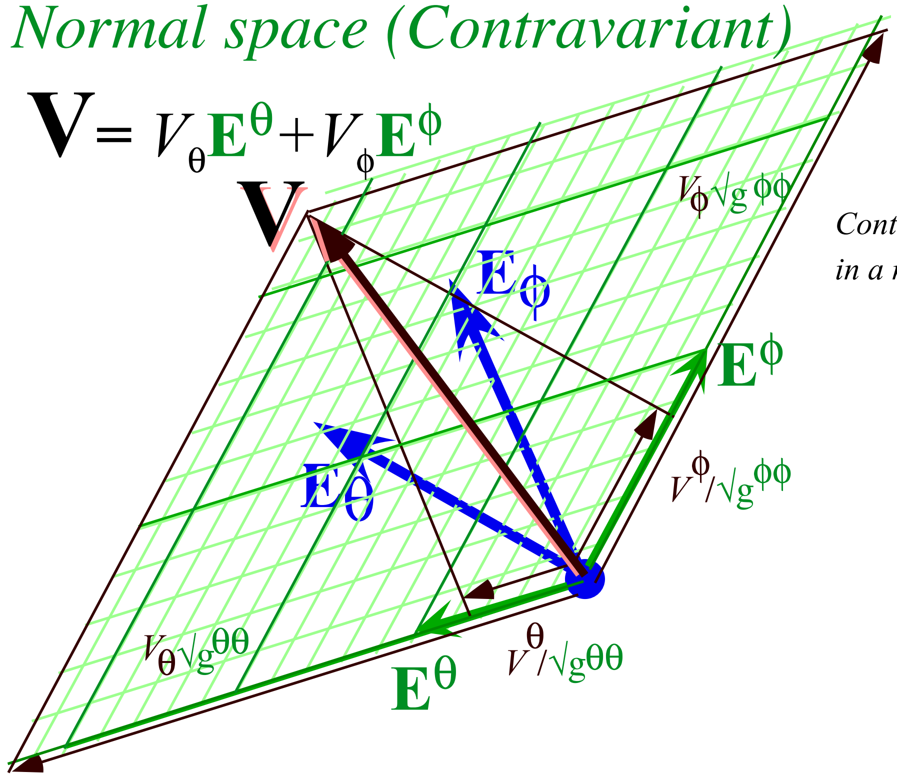
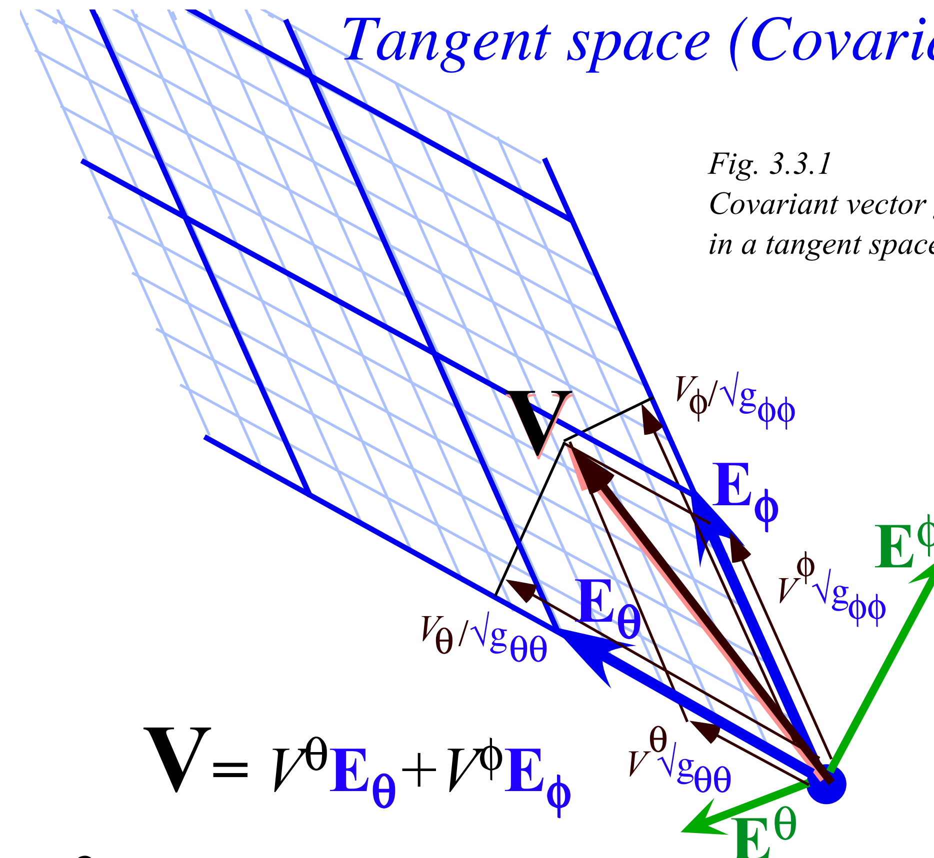


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).



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implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

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implies: $V_m = \dfrac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{V}_{\bar{m}}$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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and the *Un , Vn ,...are covariant components*

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Normal space (Contravariant)

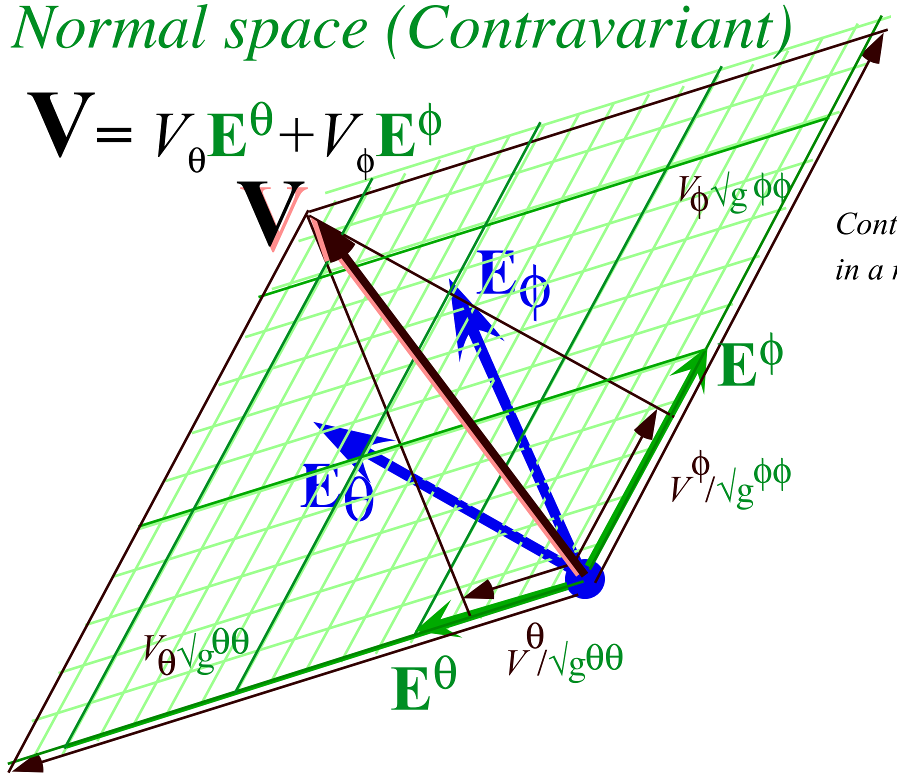

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

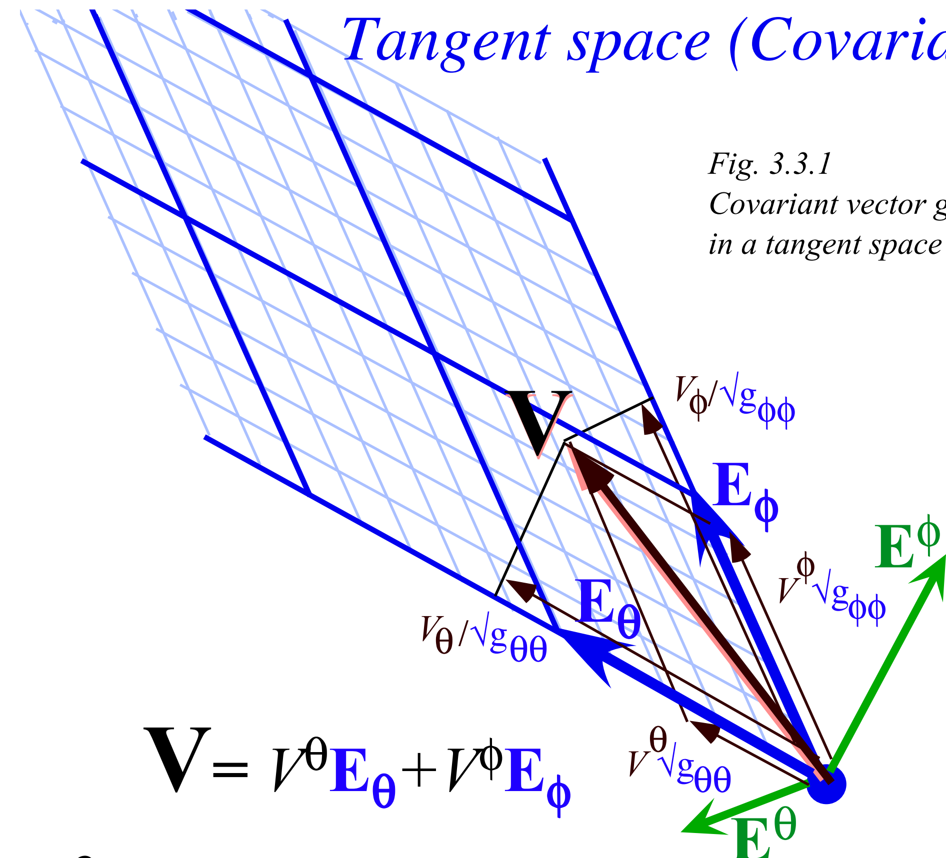

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

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implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \bar{V}^{\bar{m}}$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}} |\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}} \langle m|\bar{m}\rangle\langle\bar{m}|$$

implies: $\langle m|\Psi\rangle = \sum_{\bar{m}} \langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$

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$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m} \frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or:} \quad \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}} |\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}} \langle\bar{m}|m\rangle|\bar{m}\rangle$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

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$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}},\ , \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}},\ etc.$$

Normal space (Contravariant)

$$\mathbf{V} = V_\theta\mathbf{E}^\theta + V_\phi\mathbf{E}^\phi$$

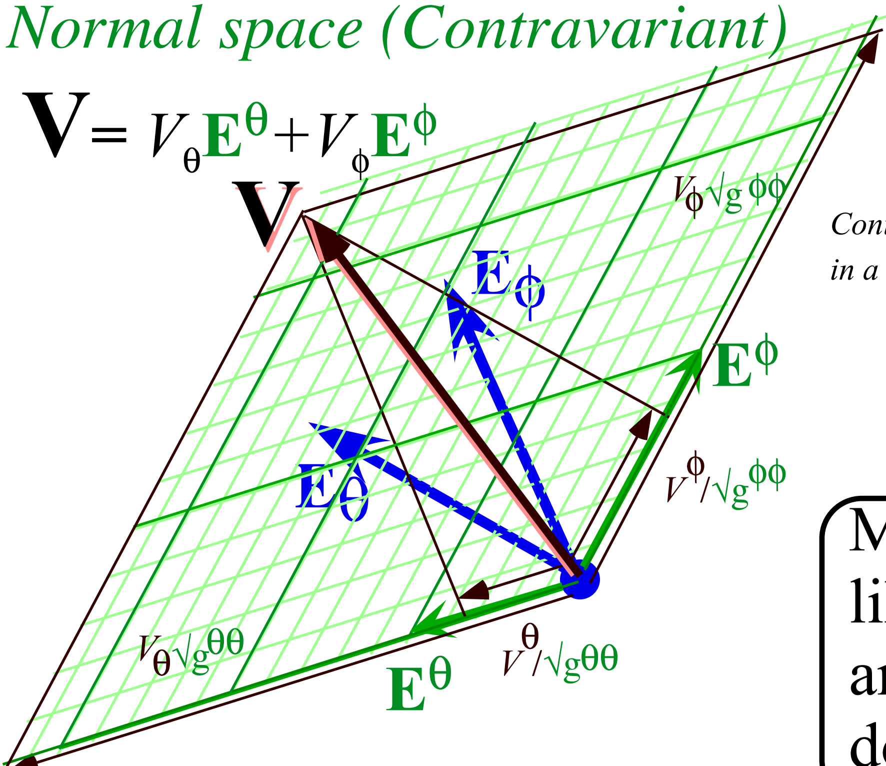

Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

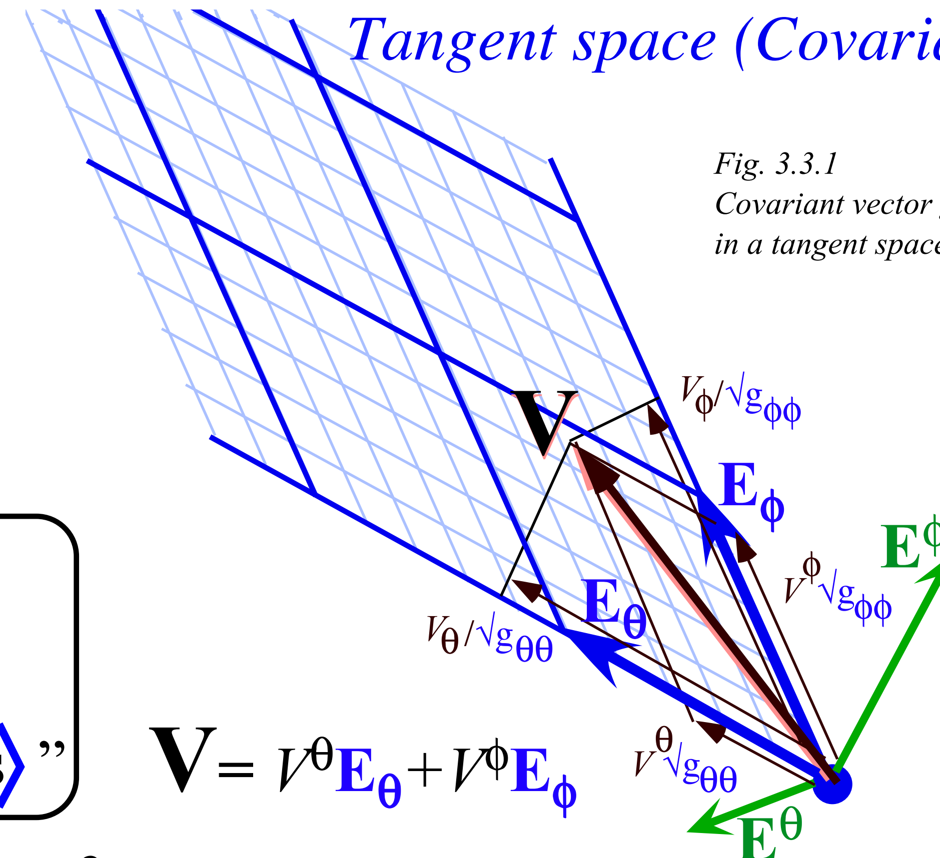

Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Metric relations
like: $\mathbf{E}^m = g^{mn}\,\mathbf{E}_n$
and: $\mathbf{E}_n = g_{mn}\,\mathbf{E}^m$
don't exist for "$\langle$bra-kets$\rangle$"

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ using a *"chain-saw-sum rule"*….

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad\ \partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or:} \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}} \quad \text{implies:}\ V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad\ \partial \mathbf{r}}{\partial q^m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or:} \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies:}\ \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

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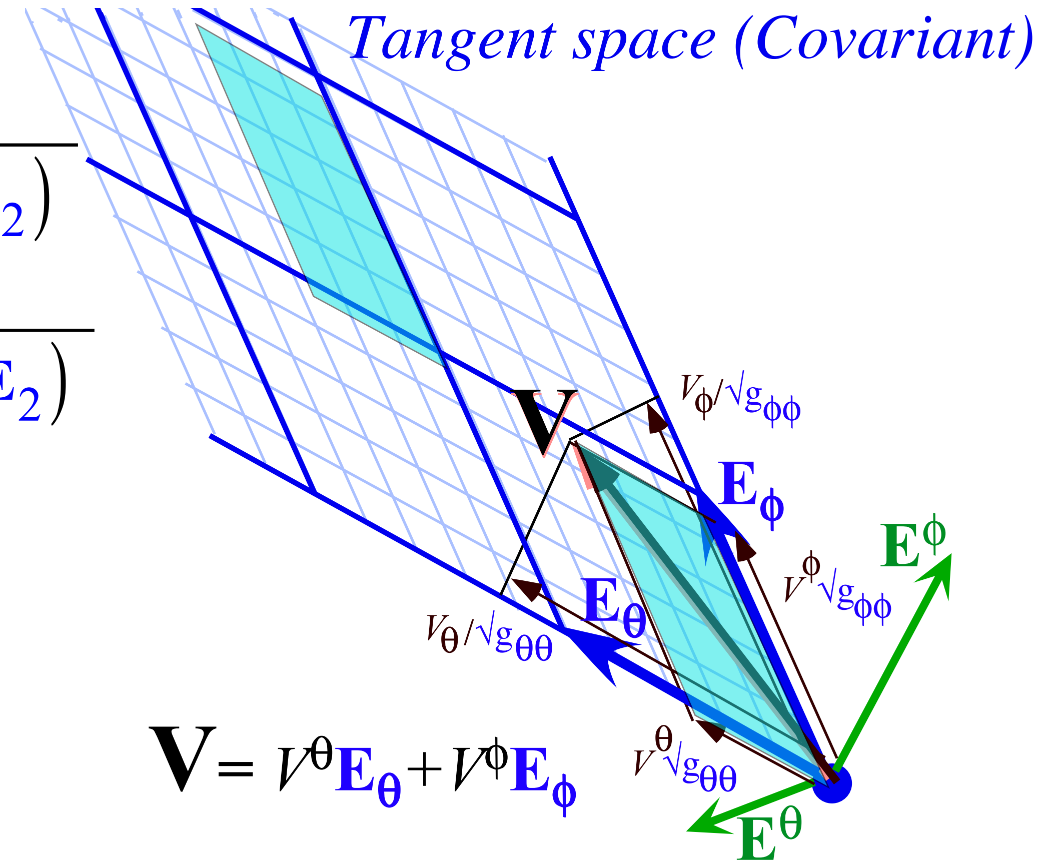
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

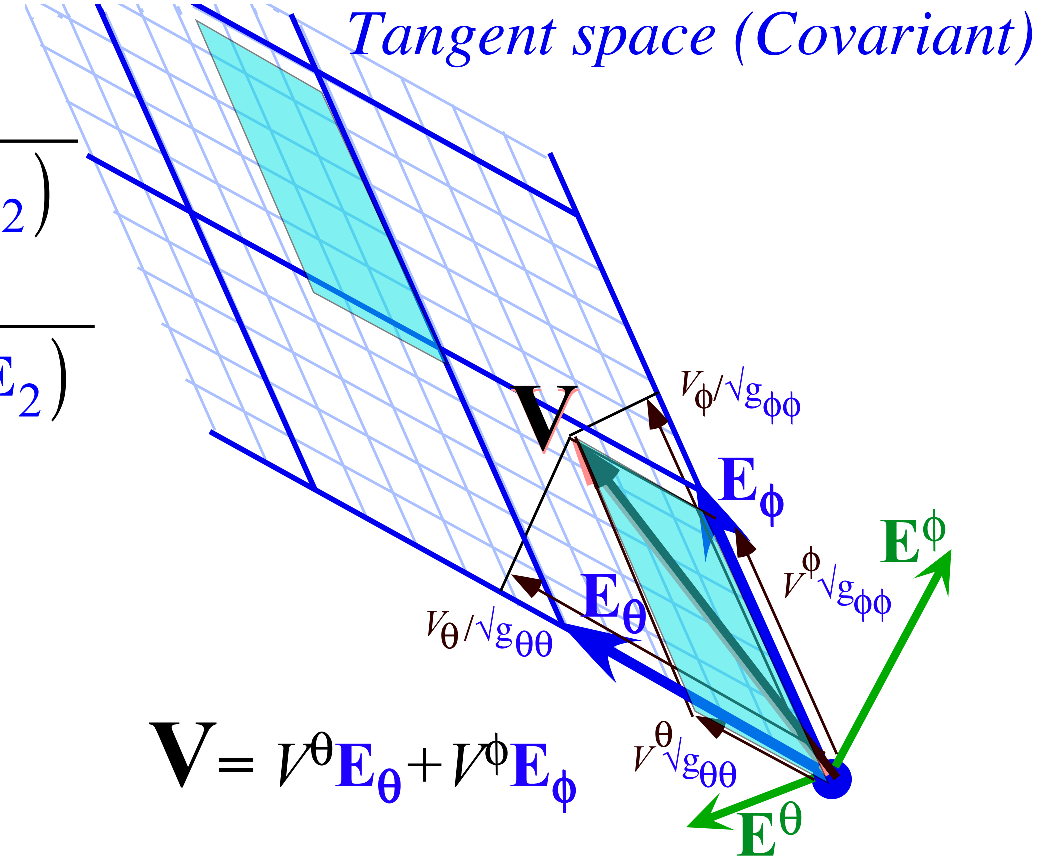
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right|=V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2\right)=V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}}$$

where: $g_{12}=\mathbf{E}_1\bullet\mathbf{E}_2=g_{21}$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

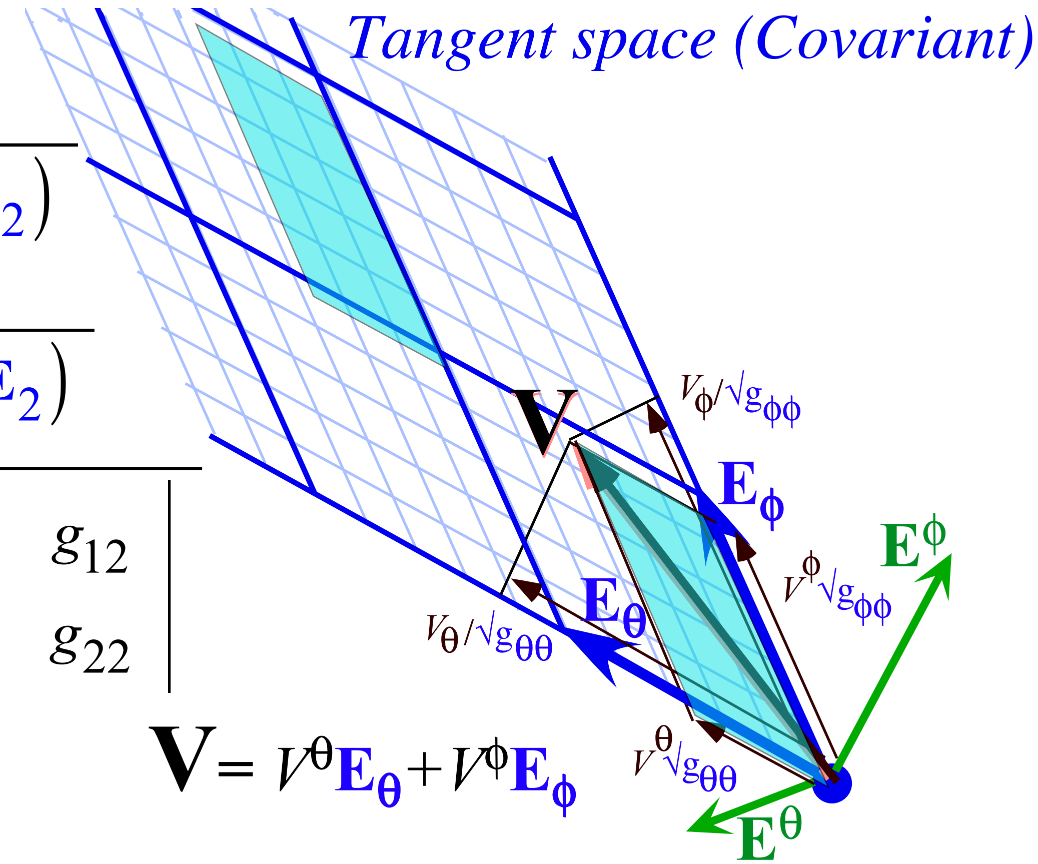
Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

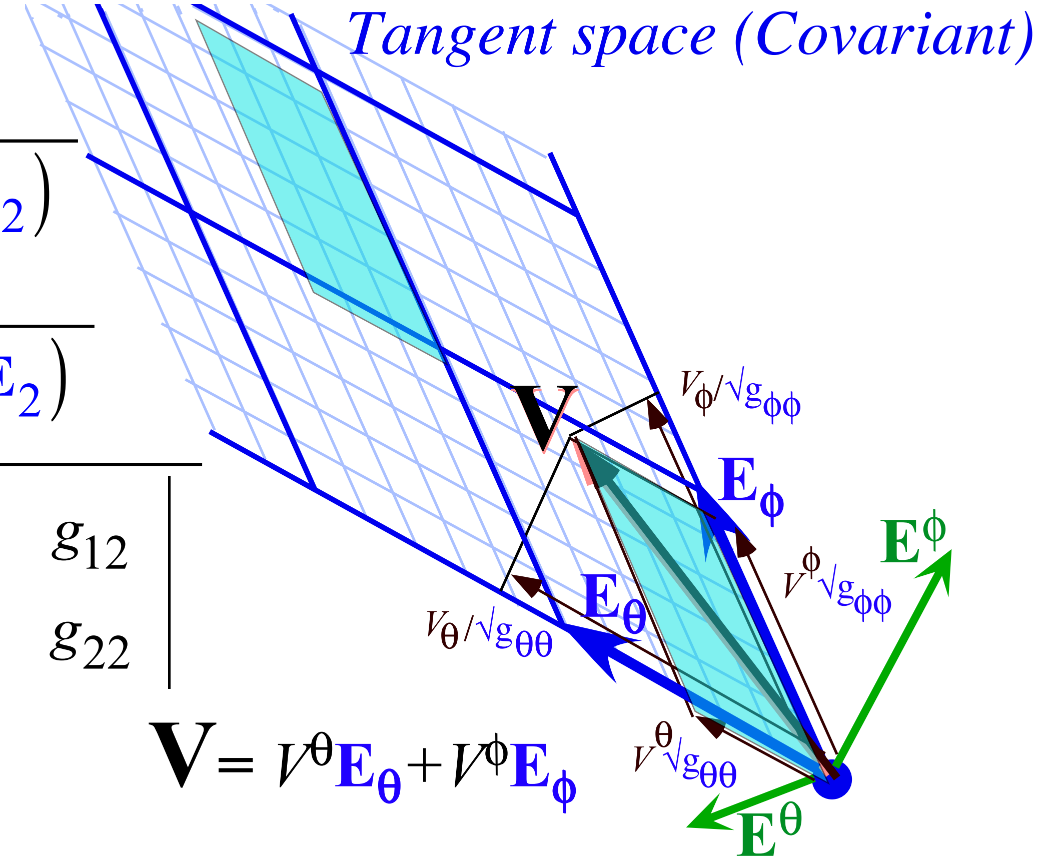
$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$



Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Tangent space (Covariant) area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$



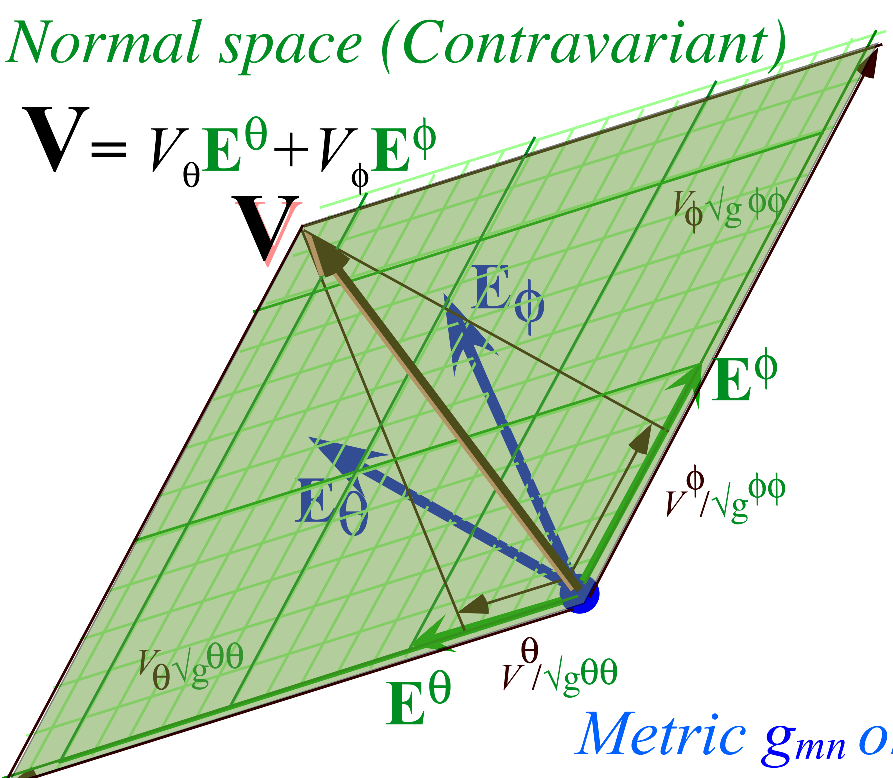
$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right) - \left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22} - g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$

Normal space (Contravariant) area spanned by $V_1\mathbf{E}^1$ and $V_2\mathbf{E}^2$



$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right| = V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right) - \left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$= V_1V_2\sqrt{g^{11}g^{22} - g^{12}g^{21}} = V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1\bullet\mathbf{E}^2 = g^{21}$

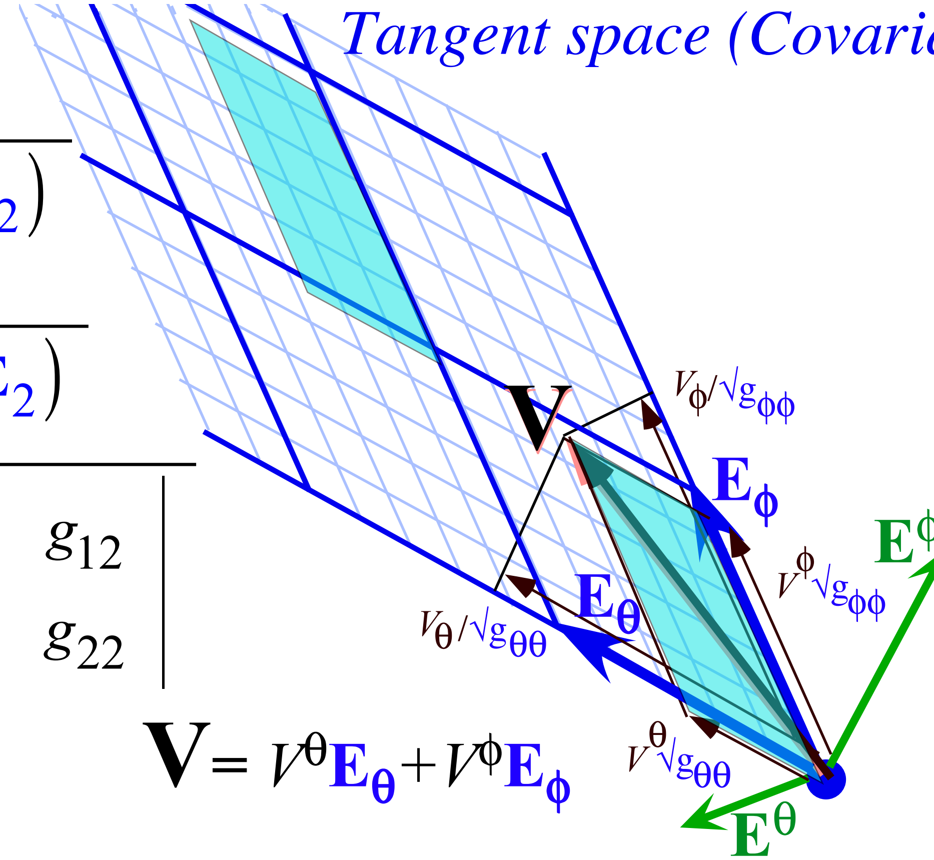
Tangent space (Covariant)

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\left|\mathbf{E}_1\times\mathbf{E}_2\right| = V^1V^2\sqrt{\left(\mathbf{E}_1\times\mathbf{E}_2\right)\bullet\left(\mathbf{E}_1\times\mathbf{E}_2\right)}$$

$$Area\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2\right) = V^1V^2\sqrt{\left(\mathbf{E}_1\bullet\mathbf{E}_1\right)\left(\mathbf{E}_2\bullet\mathbf{E}_2\right)-\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)\left(\mathbf{E}_1\bullet\mathbf{E}_2\right)}$$

$$= V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{21}} = V^1V^2\sqrt{\det\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}}$$

where: $g_{12} = \mathbf{E}_1\bullet\mathbf{E}_2 = g_{21}$



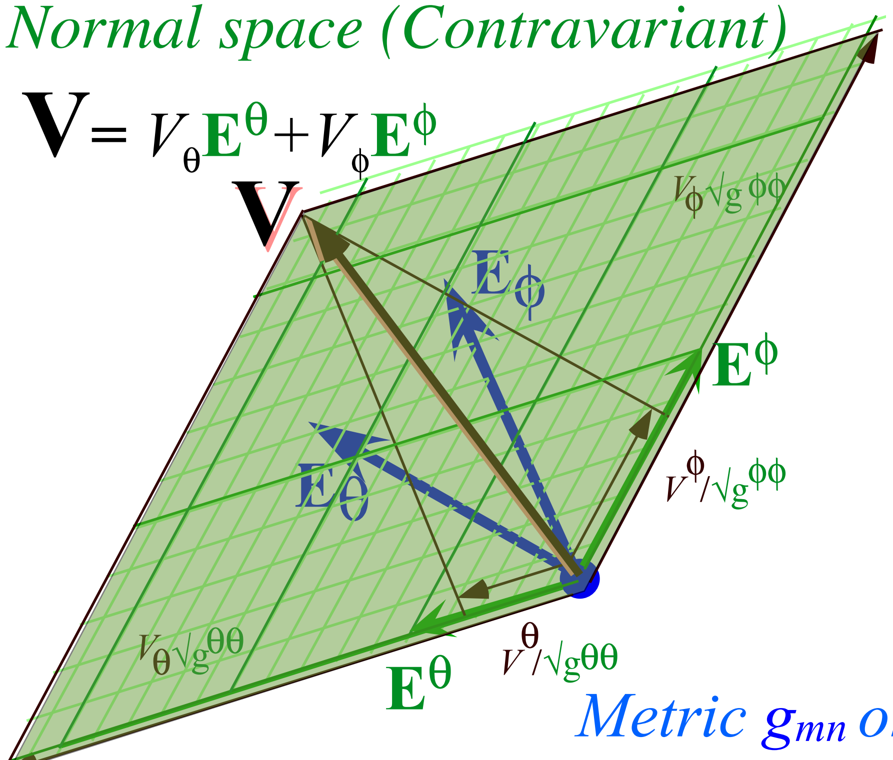
$$\mathbf{V} = V^\theta\mathbf{E}_\theta + V^\phi\mathbf{E}_\phi$$

Determinant product rule: $det|\mathbf{g}_{cov}|\cdot det|\mathbf{g}^{cont}| = 1$ since $(\mathbf{g}_{cov})^{-1} = \mathbf{g}^{cont}$ or : $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}\cdot\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

(Recall: $\mathbf{E}_n\bullet\mathbf{E}^n = 1$) $\quad(\mathbf{g}_{cov}) \cdot (\mathbf{g}^{cont}) = (\mathbf{1})$

Normal space (Contravariant)

$$\mathbf{V} = V_\theta\mathbf{E}^\theta + V_\phi\mathbf{E}^\phi$$



$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\left|\mathbf{E}^1\times\mathbf{E}^2\right| = V_1V_2\sqrt{\left(\mathbf{E}^1\times\mathbf{E}^2\right)\bullet\left(\mathbf{E}^1\times\mathbf{E}^2\right)}$$

$$Area\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2\right) = V_1V_2\sqrt{\left(\mathbf{E}^1\bullet\mathbf{E}^1\right)\left(\mathbf{E}^2\bullet\mathbf{E}^2\right)-\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)\left(\mathbf{E}^1\bullet\mathbf{E}^2\right)}$$

$$= V_1V_2\sqrt{g^{11}g^{22}-g^{12}g^{21}} = V_1V_2\sqrt{\det\begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{vmatrix}}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

where: $g^{12} = \mathbf{E}^1\bullet\mathbf{E}^2 = g^{21}$

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

3D Covariant Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3\left|\mathbf{E}_1 \times \mathbf{E}_2 \bullet \mathbf{E}_3\right| = V^1V^2V^3 \det\begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

Covariant metric matrix is product of J-matrix and its transpose J^T

$$\mathbf{g}_{cov} \equiv \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1V^2V^3 \det\left|J\right| = V^1V^2V^3\sqrt{\det\left|\mathbf{g}_{cov}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and <u>volume</u>

3D Contravariant Kajobian determinant K-rows are $\mathbf{E}^1$, $\mathbf{E}^2$ and $\mathbf{E}^3$.

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3\left|\mathbf{E}^1 \times \mathbf{E}^2 \bullet \mathbf{E}^3\right| = V_1V_2V_3 \det\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{vmatrix}$$

Contravariant metric matrix is product of K-matrix and its transpose K^T

$$\mathbf{g}^{cont} \equiv \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^2}{\partial x^1} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial q^1}{\partial x^2} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^3}{\partial x^2} \\ \frac{\partial q^1}{\partial x^3} & \frac{\partial q^2}{\partial x^3} & \frac{\partial q^3}{\partial x^3} \end{pmatrix} = K \bullet K^T$$

Then determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives:

$$Volume\left(V_1\mathbf{E}^1, V_2\mathbf{E}^2, V_3\mathbf{E}^3\right) = V_1V_2V_3 \det\left|K\right| = V_1V_2V_3\sqrt{\det\left|\mathbf{g}^{cont}\right|}$$

Metric g_{mn} or g^{mn} tensor geometric relations to length, area, and volume

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

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Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
2nd-guessing Riemann equation?

Canonical momentum and γ_{mn} tensor

Review of p_θ , p_ϕ vs γ_{mn} from p. 77 of Lect. 14

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm} \qquad QED$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

<u>Everything that can move</u> contributes. (<u>Very</u> easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt}p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

p-dot part of Lagrange 2^{nd} equations

$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

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From preceding Lagrange 1^{st} equations

Lagrange equation force analysis

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$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

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$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

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$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

From preceding Lagrange 1st equations

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.

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Geometric and topological properties of GCC transformations (Mostly from Unit 3.)

Trebuchet Cartesian projectile coordinates are double-valued

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Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K

Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)

Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space

Covariant vs. contravariant coordinate transformations

Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)

Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)

Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)

Riemann equation derivation for trebuchet model

Riemann equation force analysis

2nd-guessing Riemann equation?

Force, Work, and Acceleration

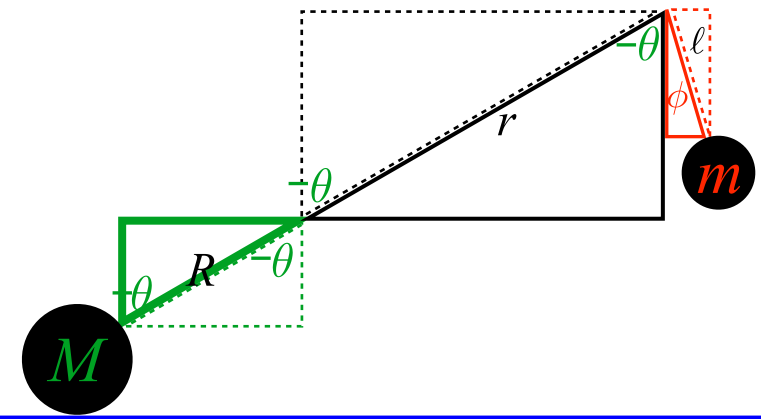
$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Review of F_θ , F_ϕ vs F_x, F_y, F_X, F_Y from p. 69 of Lect. 14

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &+ M\ddot{Y}\,dY &+ F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &+ M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &+ m\ddot{x}\,dx &+ F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &+ m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &+ m\ddot{y}\,dy &+ F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &+ m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$



Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given

($F_X = 0$, $F_Y = -Mg$)

($F_x = 0$, $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

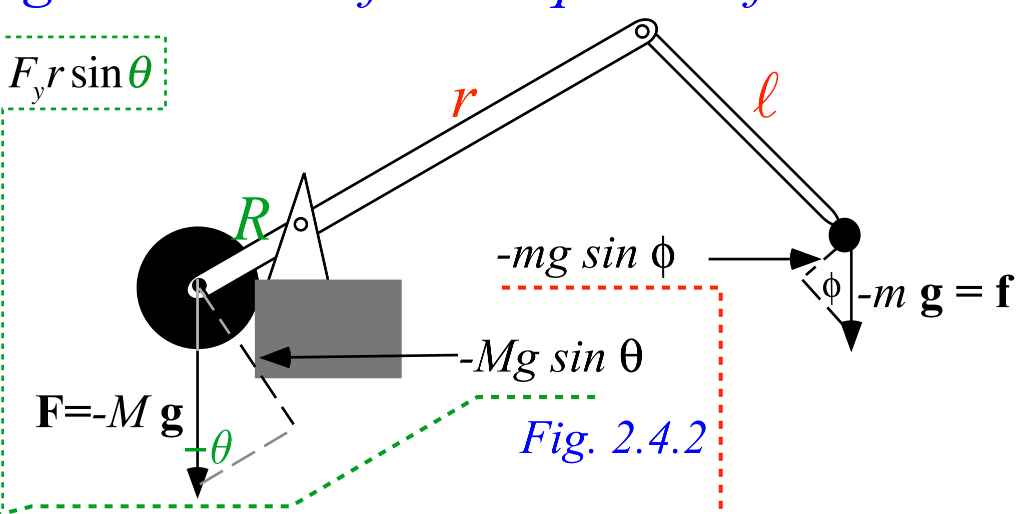


Fig. 2.4.2

Q: Are there ± sign errors here?
A: No. Beam in $-\theta$ position.

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given

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These are competing torques on main beam R... ... and a torque on throwing lever ℓ

Lagrange equation force analysis

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Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

The rest of Lagrange 2nd equations

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

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$$\boxed{= F_{\phi} - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_{μ} from p.69 of Lect. 15 (see above)

$$F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$F_{\phi} = -mg\ell\sin\phi$$

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$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (see above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$\boxed{= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} - mr\ell\dot{\phi}^2\sin(\theta-\phi)} \quad \boxed{= F_\theta \cancel{+ mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$\boxed{= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} + mr\ell\dot{\theta}^2\sin(\theta-\phi)} \quad \boxed{= F_\phi \cancel{- mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (see above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

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Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= \boxed{F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (see above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

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Lagrange equation force analysis

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[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$\boxed{= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

gravity forces F_μ from p.69 of Lect. 14 (see above)

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}}-\frac{\partial T}{\partial q^{\mu}}=\dot{p}_{\mu}-\frac{\partial T}{\partial q^{\mu}}=F_{\mu}$$

$$\dot{p}_{\theta}=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)}=\boxed{F_{\theta}=-MgR\sin\theta+mgr\sin\theta}$$

$$\dot{p}_{\phi}=\boxed{m\ell^2\ddot{\phi}\quad-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)}=\boxed{F_{\phi}=-mg\ell\sin\phi}$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
→ *Riemann equation derivation for trebuchet model*
Riemann equation force analysis
2nd-guessing Riemann equation?

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_{\theta} = \boxed{-MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_{\phi} = \boxed{-mg\ell\sin\phi}$$

Riemann equation force analysis

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$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2 \sin(\theta - \phi)} = F_{\phi} = \boxed{-mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2 \sin(\theta - \phi) \\ -mr\ell\dot{\theta}^2 \sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi) = F_{\phi} = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta - \phi) \\ -mr\ell\dot{\theta}^2\sin(\theta - \phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

This uses the γ_{mn} *tensor :* $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

Riemann equation force analysis solves for GCC accelerations $\ddot{\theta}$ *and* $\ddot{\phi}$

$$\dot{p}_{\theta} = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_{\phi} = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_{\phi} = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

This uses the γ_{mn} *tensor :*

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}*-matrix... Let's consolidate ...*

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$$

$$\dot{p}_\theta=\boxed{\left(MR^2+mr^2\right)\ddot{\theta}-mr\ell\ddot{\phi}\cos(\theta-\phi)-mr\ell\dot{\phi}^2\sin(\theta-\phi)=F_\theta=-MgR\sin\theta+mgr\sin\theta}$$

$$\dot{p}_\phi=\boxed{m\ell^2\ddot{\phi}\quad-mr\ell\ddot{\theta}\cos(\theta-\phi)+mr\ell\dot{\theta}^2\sin(\theta-\phi)=F_\phi=-mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}-\begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}=\begin{pmatrix}F_\theta\\ F_\phi\end{pmatrix}$$

$$=\begin{pmatrix}-MgR\sin\theta+mgr\sin\theta\\ -mg\ell\sin\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}=\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} \longleftarrow \begin{matrix}\text{“Super-Inertia”} \\ I_S\end{matrix}$$

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} \quad - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

... and apply it...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}}$$

Riemann equation form

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ *becomes* $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta}$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} \quad - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi}$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$ $\quad I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \boxed{\left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)} = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = \boxed{m\ell^2\ddot{\phi} \quad - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)} = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

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$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} \longleftarrow \text{"Super-Inertia" } I_S$$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" (q^1=θ, q^2=ϕ)-manifold and "Flat" (x=θ, y=ϕ)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
→ *Riemann equation force analysis*
2nd-guessing Riemann equation?

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

$$Let: (\theta-\phi) = -\frac{\pi}{2} \qquad \text{so:} \qquad I_S = m\ell^2\left[MR^2+mr^2\right] \quad \text{and let: } \omega \equiv \dot{\theta} = \dot{\phi}$$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} -\omega^2 \\ \omega^2 \end{pmatrix} mr\ell$$

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]}\begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results

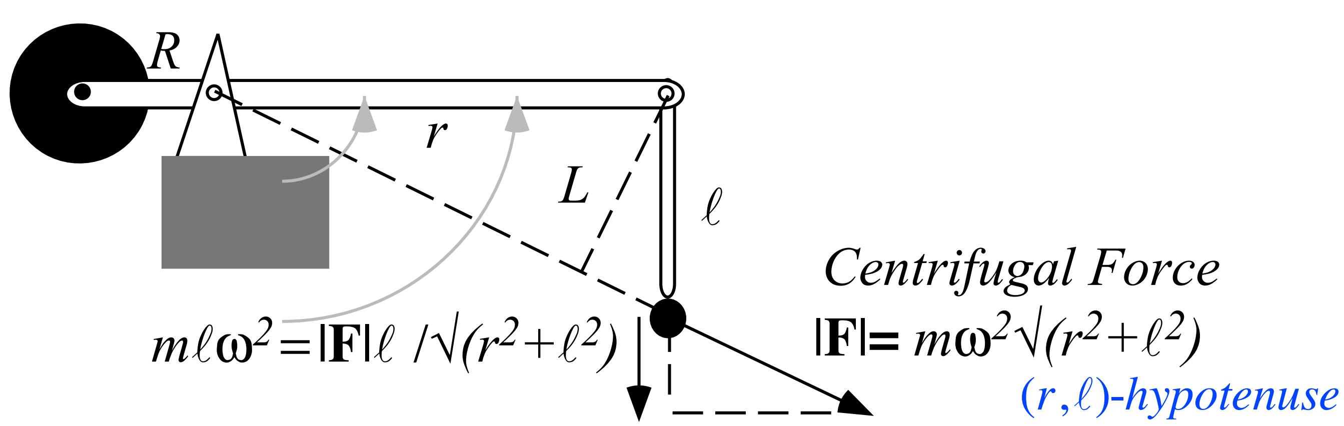


Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu}-\frac{\partial T}{\partial q^\mu}=\dot{p}_\mu-\frac{\partial T}{\partial q^\mu}=F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu=\ddot{q}^\nu \ldots$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix}=\boxed{\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta+mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi-mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}\quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta=0=F_\phi$

$$I_s=m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]$$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)=\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix}mr\ell\sin(\theta-\phi)$$

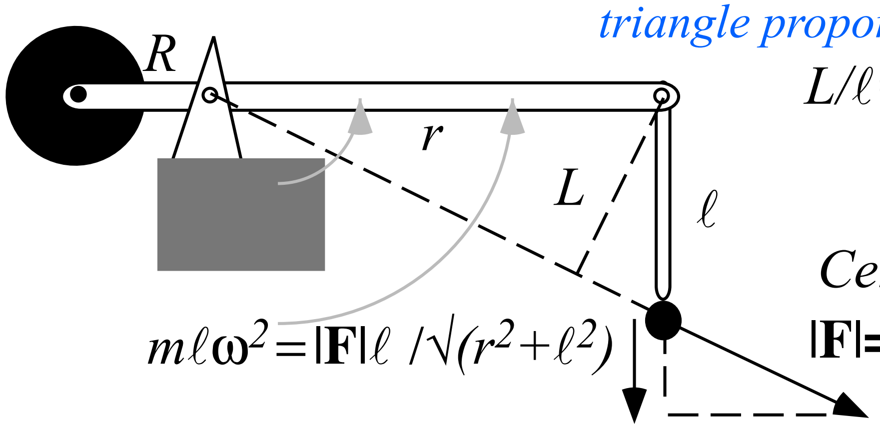
$Let:(\theta-\phi)=-\frac{\pi}{2}$ so: $I_S=m\ell^2\left[MR^2+mr^2\right]$ and let: $\omega\equiv\dot{\theta}=\dot{\phi}$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}-\dot{\phi}^2\\ \dot{\theta}^2\end{pmatrix}mr\ell=\begin{pmatrix}m\ell^2 & 0\\ 0 & MR^2+mr^2\end{pmatrix}\begin{pmatrix}-\omega^2\\ \omega^2\end{pmatrix}mr\ell$$

$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}-\dot{\phi}^2\\ \dot{\theta}^2\end{pmatrix}mr\ell=\frac{\begin{pmatrix}m\ell^2 & 0\\ 0 & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]}\begin{pmatrix}-mr\ell\omega^2\\ mr\ell\omega^2\end{pmatrix}=\begin{pmatrix}\frac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$$

Trying to 2nd-guess Riemann results

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.



triangle proportionality:

$L/\ell=r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}|=m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega\equiv\dot{\theta}=\dot{\phi},\ \theta=\frac{-\pi}{2},\ \phi=0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot q^\mu} - \frac{\partial T}{\partial q^\mu} = \dot p_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot p_\mu = \ddot q^\nu \ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot p_\theta \\ \dot p_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot\theta \\ \ddot\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot\phi^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot\theta^2 \sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$F_\theta = 0 = F_\phi$ $\qquad I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$

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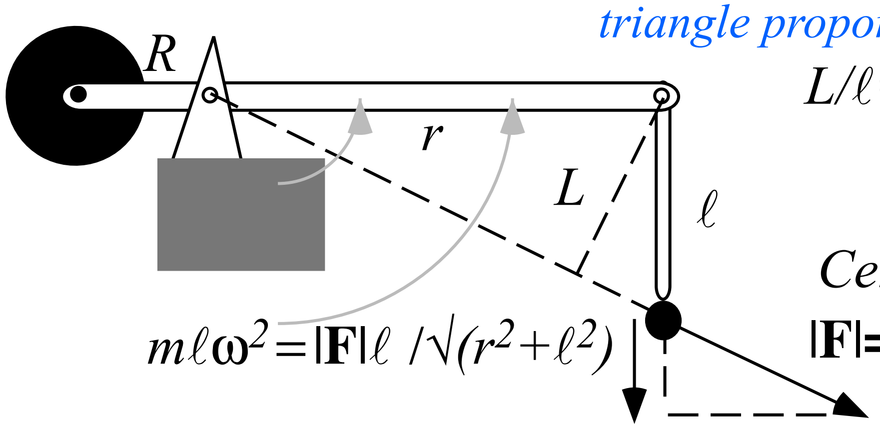
$$\begin{pmatrix} \ddot\theta \\ \ddot\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} -\dot\phi^2 \\ \dot\theta^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]}\begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results (Gravity-free case)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



triangle proportionality:

$L/\ell = r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^2\ddot\phi = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot\phi = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot\theta = \dot\phi,\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Move to top of page...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ *becomes* $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \boxed{\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix} \quad \text{Riemann equation form}}$$

Gravity-free case:

$$I_s = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$$

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$$I_S \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix} \begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

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$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\right]} \begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2+mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results (Gravity-free case)

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The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

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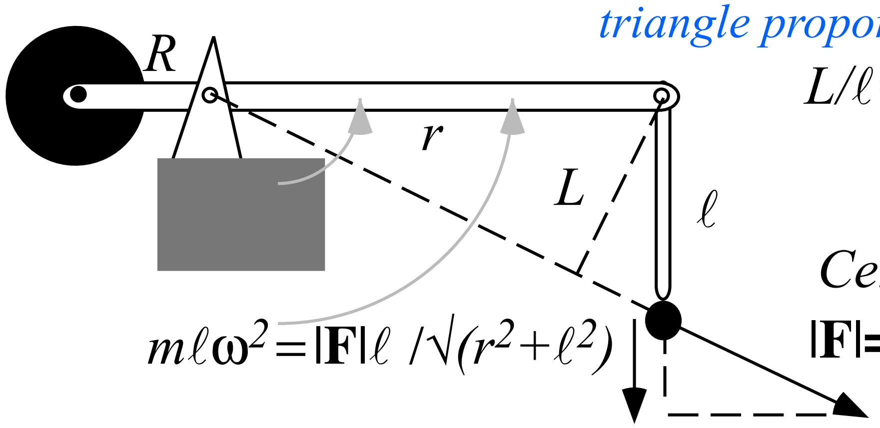

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Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

(r,ℓ)-hypotenuse

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

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Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Trying to 2nd-guess Riemann results (Gravity-free case)

Move to top of page...

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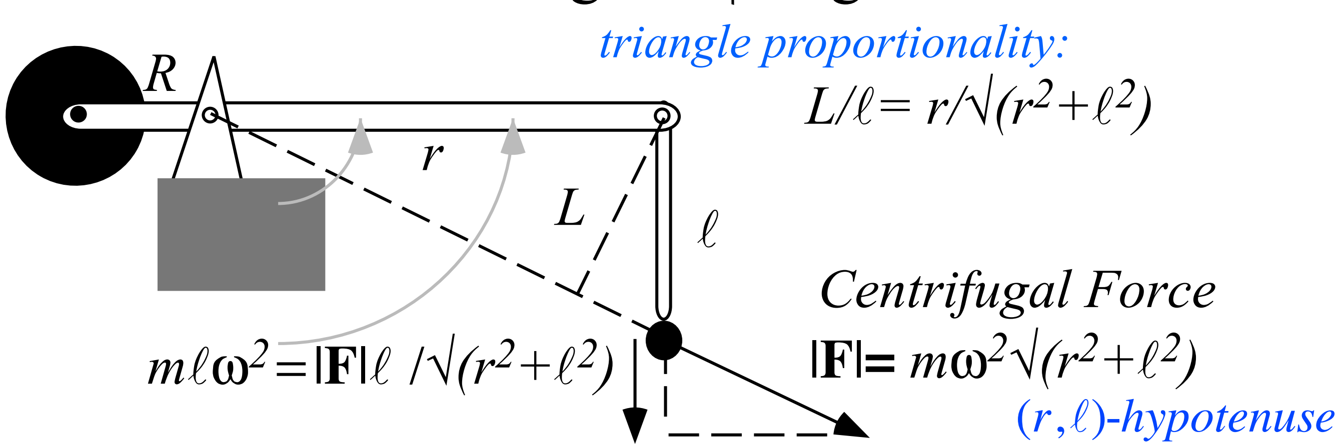

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or: $\ddot{\phi} = FL\,/\,m\ell^2 = \omega^2 r\,/\,\ell$

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Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

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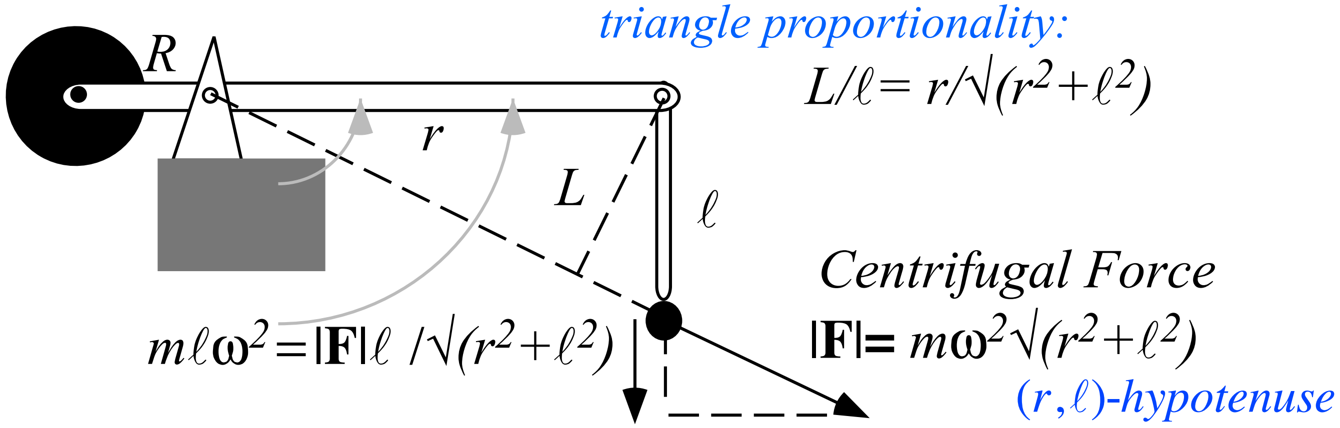

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Fig. 2.5.1 Centrifugal force for state of motion $(\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0)$

Review (Mostly Unit 2.): Was the Trebuchet a dream problem for Galileo? Not likely.
Forces in Lagrange force equation: total, genuine, potential, and/or fictitious

Geometric and topological properties of GCC transformations (Mostly from Unit 3.)
Trebuchet Cartesian projectile coordinates are double-valued
Toroidal "rolled-up" ($q^1=\theta$, $q^2=\phi$)-manifold and "Flat" ($x=\theta$, $y=\phi$)-graph
Review of covariant $\mathbf{E}_n$ and contravariant $\mathbf{E}^m$ vectors: Jacobian J vs. Kajobian K
Covariant metric g_{mn} vs. contravariant metric g^{mn} (Lect. 9 p.53)
Tangent $\{\mathbf{E}_n\}$space vs. Normal $\{\mathbf{E}^m\}$space
Covariant vs. contravariant coordinate transformations
Metric g_{mn} tensor geometric relations to length, area, and volume

Lagrange force equation analysis of trebuchet model (Mostly from Unit 2.)
Review of trebuchet canonical (covariant) momentum and mass metric γ_{mn} (Lect. 14 p. 77)
Review and application of trebuchet covariant forces F_θ and F_ϕ (Lect. 14 p. 69)
Riemann equation derivation for trebuchet model
Riemann equation force analysis
→ *2nd-guessing Riemann equation?*

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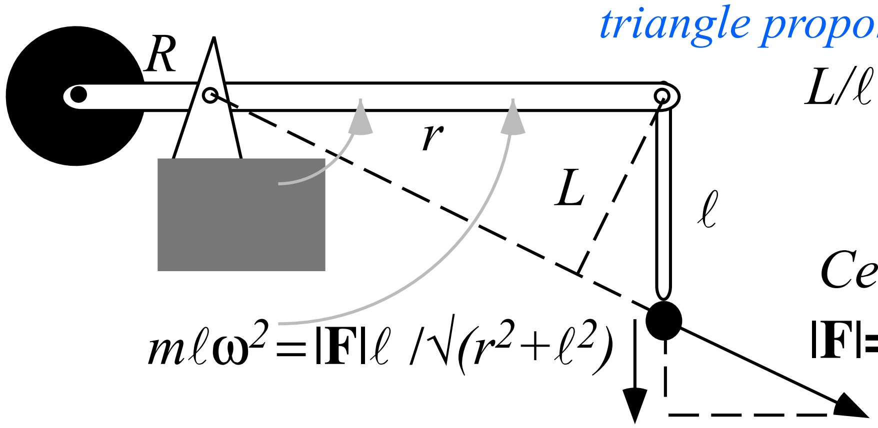

triangle proportionality:

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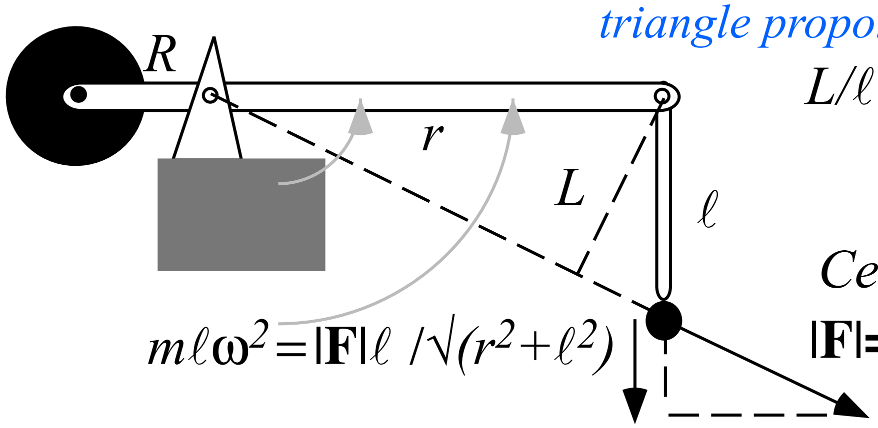

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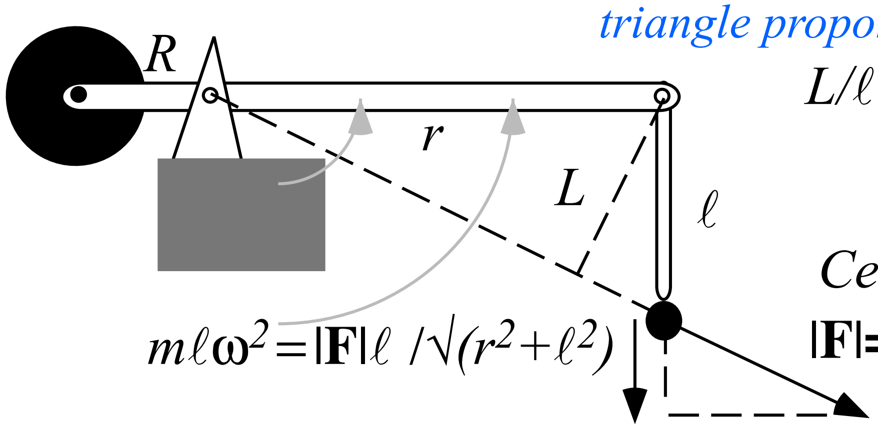

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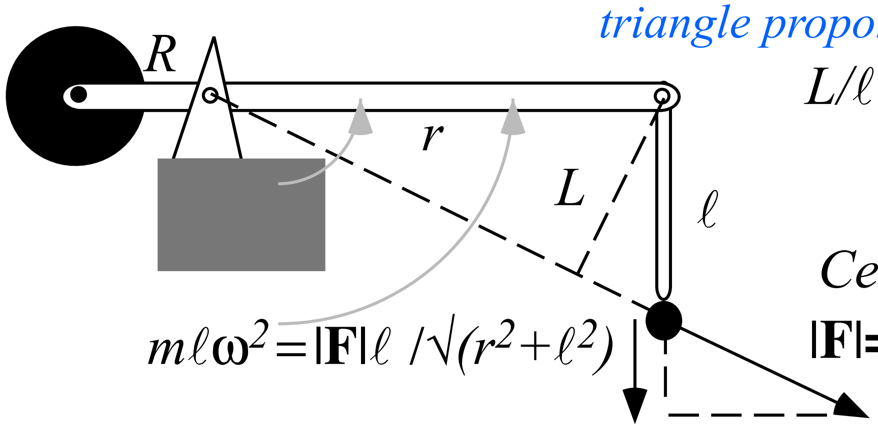

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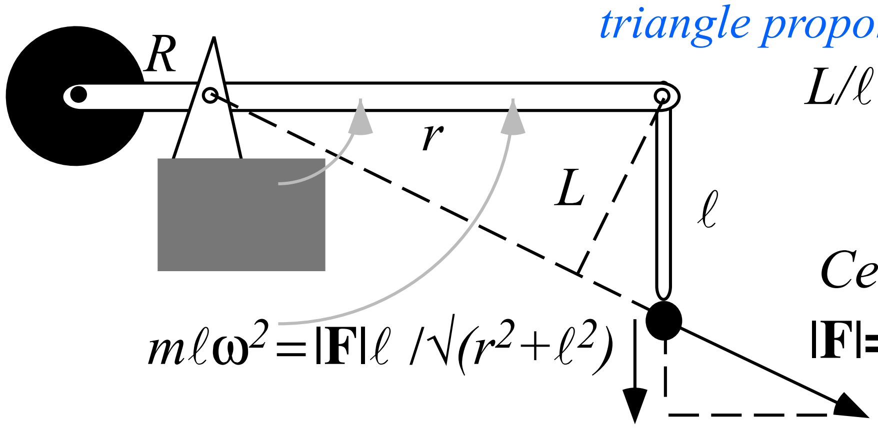

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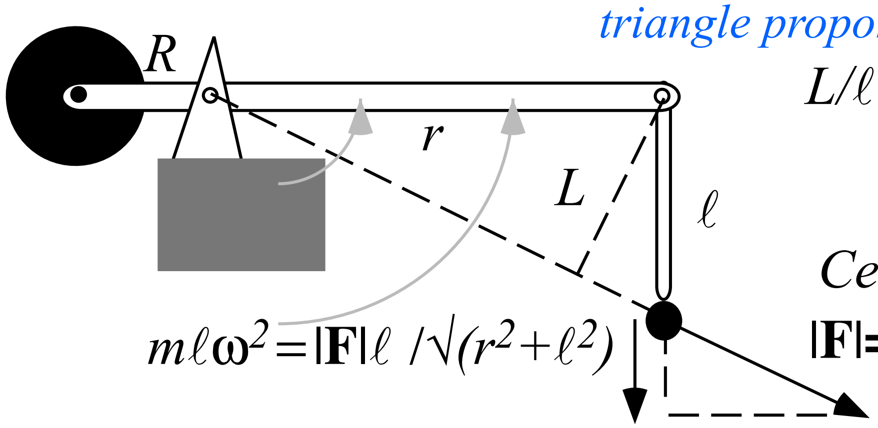

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Checks with $\ddot{\theta}$ Riemann equation

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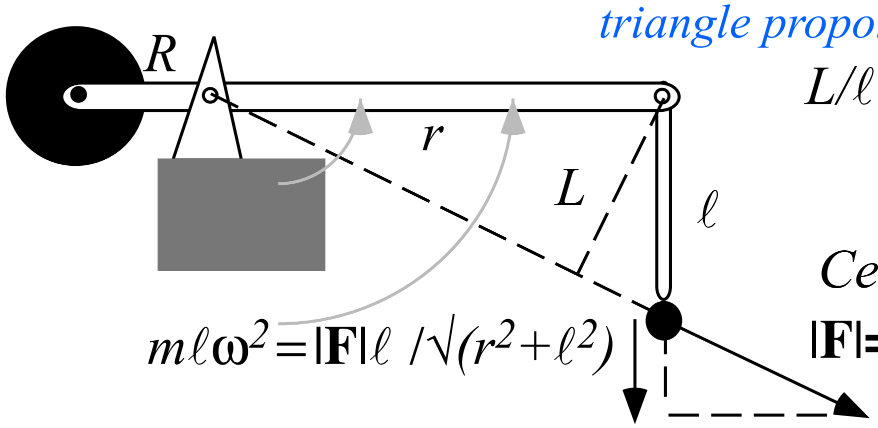

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Note the time derivative of total momentum is zero if outside torques are zero.(twirling skater analogy)

$$\dot{p}_\theta + \dot{p}_\phi = 0, \text{ if } F_\theta = 0 = F_\phi$$