Lecture 20
Mon. 10.29 thru Wed. 10.31 2018

Introduction to classical oscillation and resonance

(Ch. 1 of Unit 4)

1D forced-damped-harmonic oscillator equations and Green's function solutions

- *Linear harmonic oscillator equation of motion.*
- *Linear damped-harmonic oscillator equation of motion.*
  - *Frequency retardation and amplitude damping*
  - *Figure of oscillator merit (the 5% solution $3/\Gamma$ and other numbers)*
- *Linear forced-damped-harmonic oscillator equation of motion.*
  - *Phase lag and amplitude resonance amplification*
  - *Figure of resonance merit: (angular) Quality factor $q=\omega_0/2\Gamma$*

Properties of Green's function solutions and their mathematical/physical behavior

- *Transient solutions vs. Steady State solutions*

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

- *Quality factors: Beat, lifetimes, and uncertainty*

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

- *Common Lorentzian (a.k.a. Witch of Agnesi)*
- *Smith Charts (Graph paper)*

A running collection of links to course-relevant sites and articles

Physics Web Resources

Comprehensive Harter-Soft Resource Listing

UAF Physics YouTube channel

LearnIt Physics Web Applications

Neat external material to start the class:

- AIP publications
- AJP article on superball dynamics
- AAPT summer reading

"Texts"

Classical Mechanics with a Bang!

Quantum Theory for the Computer Age

Principles of Symmetry, Dynamics, and Spectroscopy

Modern Physics and its Classical Foundations

Classes

2014 AMOP

2017 Group Theory for QM

2018 AMOP

2018 Adv Mechanics

AnalyIt Web Application, posted 10/22/2018 in our *testing* area:
https://modphys.hosted.uark.edu/testing/markup/AnalyItBJS.html

These *are* hot off the presses:

- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell's demon - Kumar-Nature-Letters-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Berredo-Nature-Letters-2018

Slightly Older ones:

- Wave–particle duality of C60 molecules
- Optical vortex knots – One Photon at a Time

"*Relawavity*" and quantum basis of *Lagrangian* & *Hamiltonian* mechanics:

- 2-CW laser wave - BohrIt Web App
- Lagrangian vs Hamiltonian - RelaWavity Web App

- AMOP Ch 0 Space-Time Symmetry - 2019
- Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Older Links from Lectures 14-17

http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html
http://www.sussexcountyonline.com/news/photos/punkinchunkin.html
http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=MontezumasRevenge
https://modphys.hosted.uark.edu/markup/TrebuchetWeb.html?scenario=SeigeOfKenilworth
https://modphys.hosted.uark.edu/pdfs/Journal_Pdfs/Trebuchet-SciAm_273_66_July_1995_chevedden1.pdf
'*Simple*' *Pendulum* Sim: https://modphys.hosted.uark.edu/markup/PendulumWeb.html
'*Cycloid*' *Pendulum*: https://modphys.hosted.uark.edu/markup/CycloidulumWeb.html
Google search on: "Satelite view of Patricia" (Images)
Physics Girl Channel - Fun with Vortex Rings in the Pool: https://www.youtube.com/watch?v=72LWr7BU8Ao
iBall demo - Quasi-periodicity: https://youtu.be/_jntDtULxDc

Previous Links to supplement Lecture 18-19

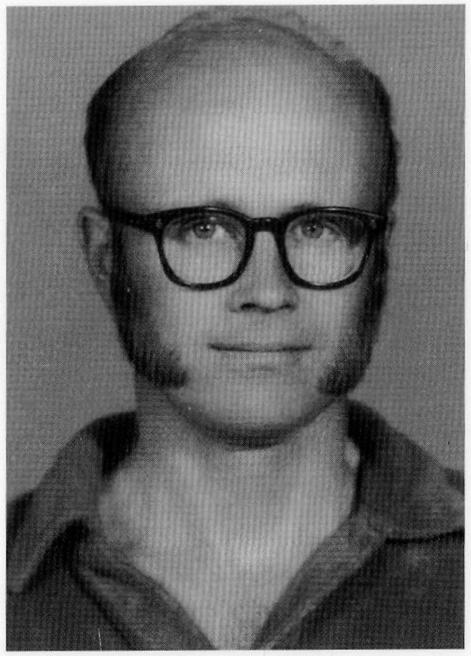
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion
https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2
Mechanical Analog to EM Motion (YouTube video) - https://youtu.be/hTd5FTJ-vRk
CoulIt Web Simulation: Bound-state motion in parabolic coordinates
CoulIt Web Simulation: Bound-state motion in hyperbolic coordinates

Links to supplement Lecture 20

http://nobelprize.org/
https://modphys.hosted.uark.edu/markup/OscillItWeb.html
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=18
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=27
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=31
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=35
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=38
https://modphys.hosted.uark.edu/markup/OscillItWeb.html?scenario=39
Smith Chart

Without resonance…
…we are all blind, deaf, and dumb.

Anonymous



PAULINIA, BRASIL 1976

THE SPEED OF LIGHT IS
299,792,458 METERS PER SECOND!

-- The Purest Light and a Resonance Hero – Ken Evenson (1932-2002) --

Ken Evenson

When travelers punch up their GPS coordinates they owe a debt of gratitude to an under sung hero who, alongside his colleagues and students, often toiled 18 hour days deep inside a laser laboratory lit only by the purest light in the universe.

Ken was an "Indiana Jones" of modern physics. While he may never have been called "Montana Ken," such a name would describe a real life hero from Bozeman, Montana, whose extraordinary accomplishments in many ways surpass the fictional characters in cinematic thrillers like *Raiders of the Lost Arc*.

Indeed, there were some exciting real life moments shared by his wife Vera, one together with Ken in a canoe literally inches from the hundred-foot drop-off of Brazil's largest waterfall. But, such outdoor exploits, of which Ken had many, pale in the light of an in-the-lab brilliance and courage that profoundly enriched the world.

Ken is one of few researchers and perhaps the only physicist to be twice listed in the *Guinness Book of Records*. The listings are not for jungle exploits but for his lab's highest frequency measurement and for a speed of light determination that made *c* many times more precise due to his lab's pioneering work with John Hall in laser resonance and metrology[†].

The meter-kilogram-second (mks) system of units underwent a redefinition largely because of these efforts. Thereafter, the speed of light *c* was set to $299{,}792{,}458 ms^{-1}$. The meter was defined in terms of *c*, instead of the other way around since his time precision had so far trumped that for distance. Without such resonance precision, the Global Positioning System (GPS), the first large-scale wave space-time coordinate system, would not be possible.

Ken's courage and persistence at the Time and Frequency Division of the Boulder Laboratories in the National Bureau of Standards (now the National Institute of Standards and Technology or NIST) are legendary as are his railings against boneheaded administrators who seemed bent on thwarting his best efforts. Undaunted, Ken's lab painstakingly exploited the resonance properties of metal-insulator diodes, and succeeded in literally counting the waves of near-infrared radiation and eventually visible light itself.

Those who knew Ken miss him terribly. But, his indelible legacy resonates today as ultra-precise atomic and molecular wave and pulse quantum optics continue to advance and provide heretofore unimaginable capability. Our quality of life depends on their metrology through the Quality and Finesse of the resonant oscillators that are the heartbeats of our technology.

Before being taken by Lou Gehrig's disease, Ken began ultra-precise laser spectroscopy of unusual molecules such as HO_2, the radical cousin of the more common H_2O. Like Ken, such radical molecules affect us as much or more than better known ones. But also like Ken, they toil in obscurity, illuminated only by the purest light in the universe.

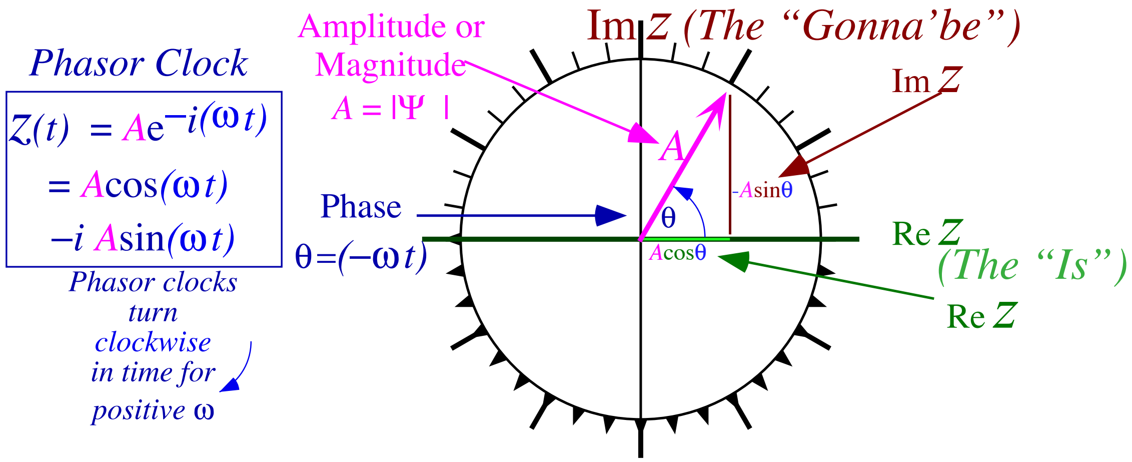
In 2005 the Nobel Prize in physics was awarded to Glauber, Hall, and Hensch[††] for laser optics and metrology.

† K. M. Evenson, J.S. Wells, F.R. Peterson, B.L. Danielson, G.W. Day, R.L. Barger and J.L. Hall,
Phys. Rev. Letters 29, 1346(1972).

†† *The Nobel Prize in Physics, 2005*. http://nobelprize.org/

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

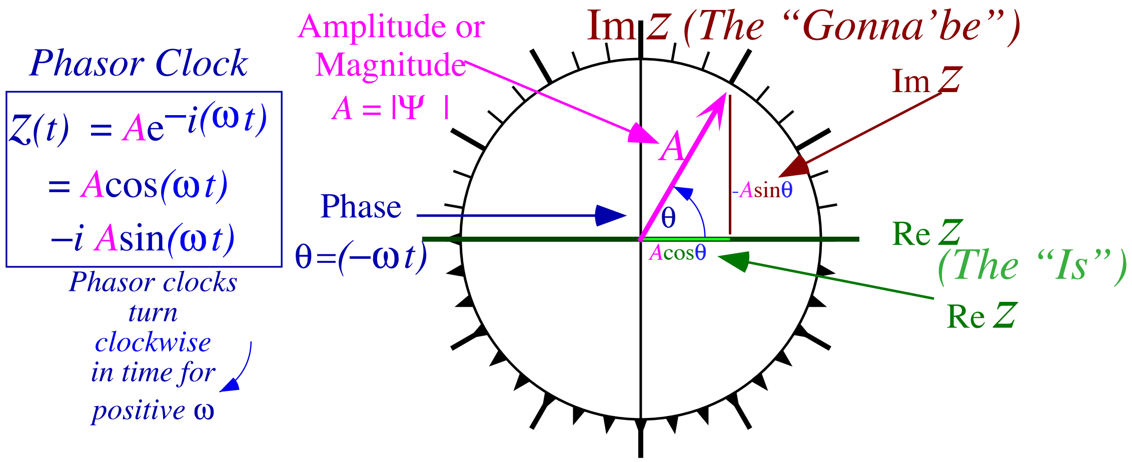
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad (k = \omega_0^2 m),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad (b = 2\Gamma m)$

Linear *harmonic* *oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

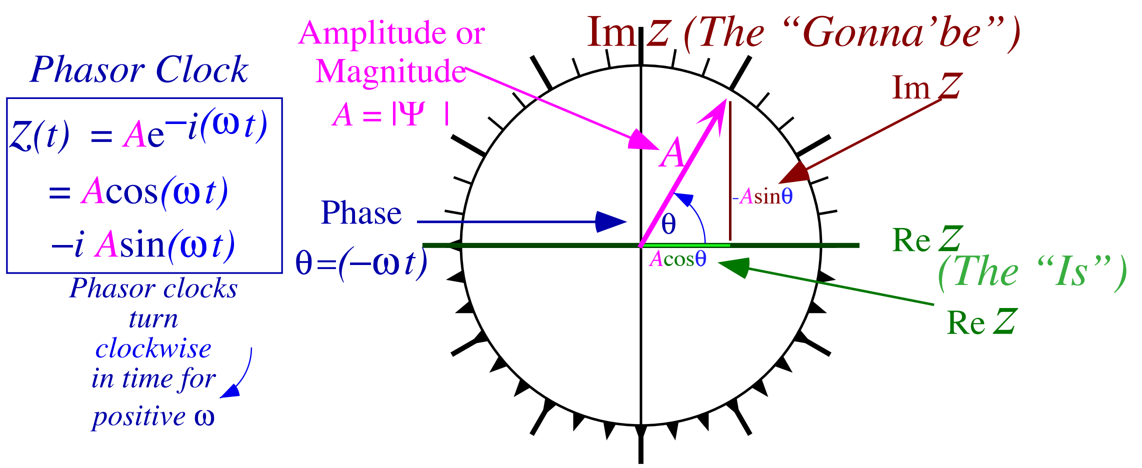
held back by a harmonic (linear) restoring force $\longrightarrow$ $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

Linear harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$



Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

The famous one-Hertz (ν_0=1/s. or: $\omega_0 = 2\pi = 6.2832$rad/s.) oscillator.

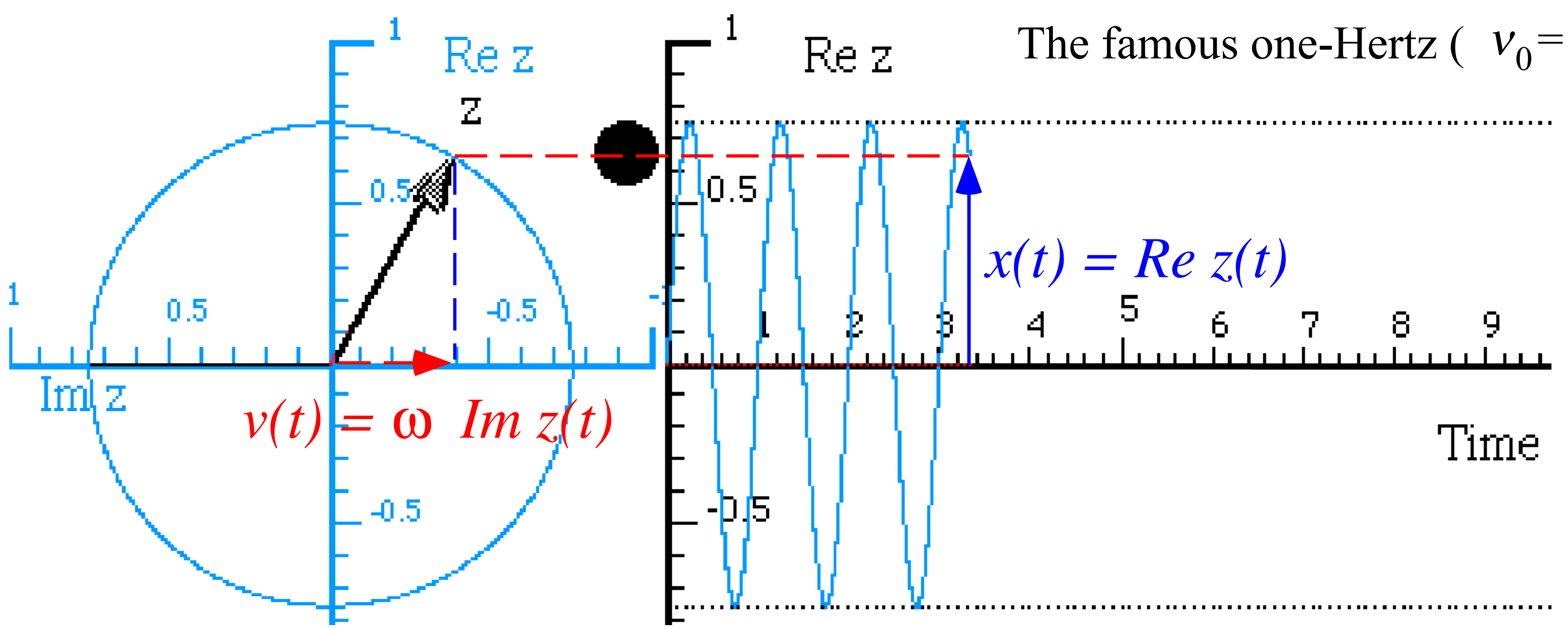


Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0$

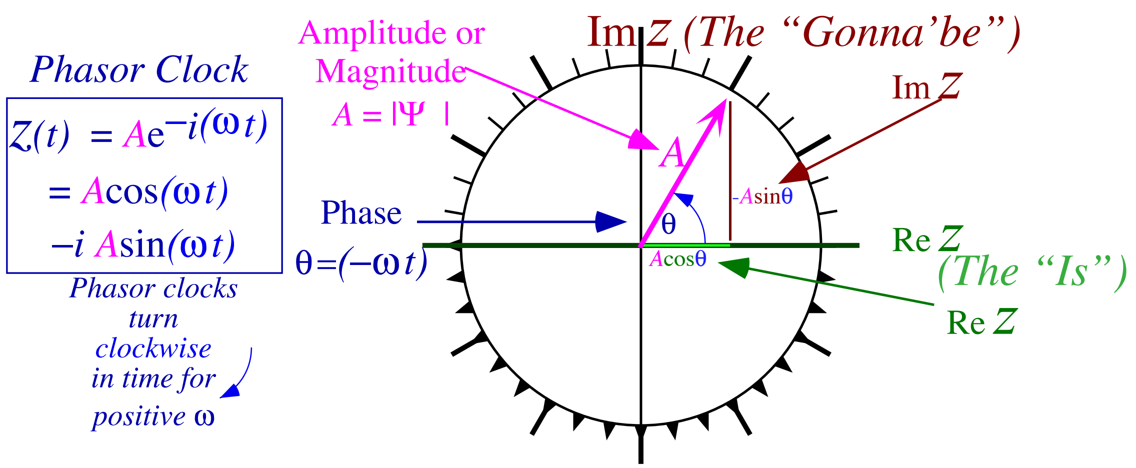
[OscillIt Web Simulation](https://modphys.hosted.uark.edu/markup/OscillItWeb.html)

OscillIt Web Simulation (Generic):
https://modphys.hosted.uark.edu/markup/OscillItWeb.html

Linear *-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

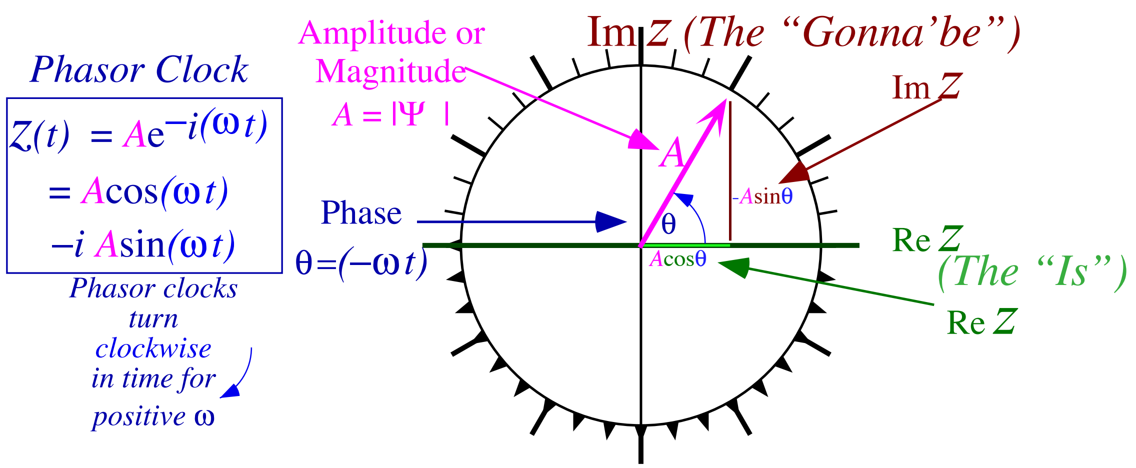
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

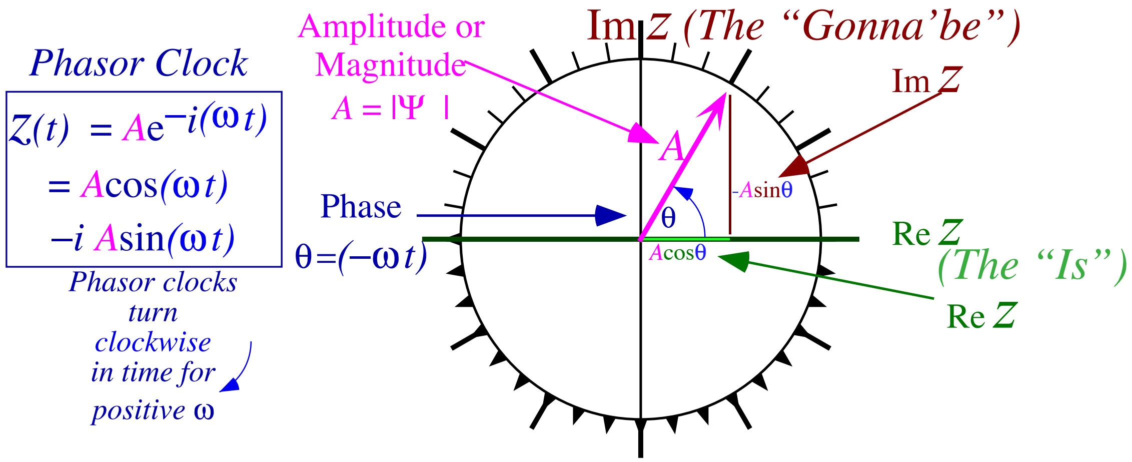
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\dfrac{dz}{dt}$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

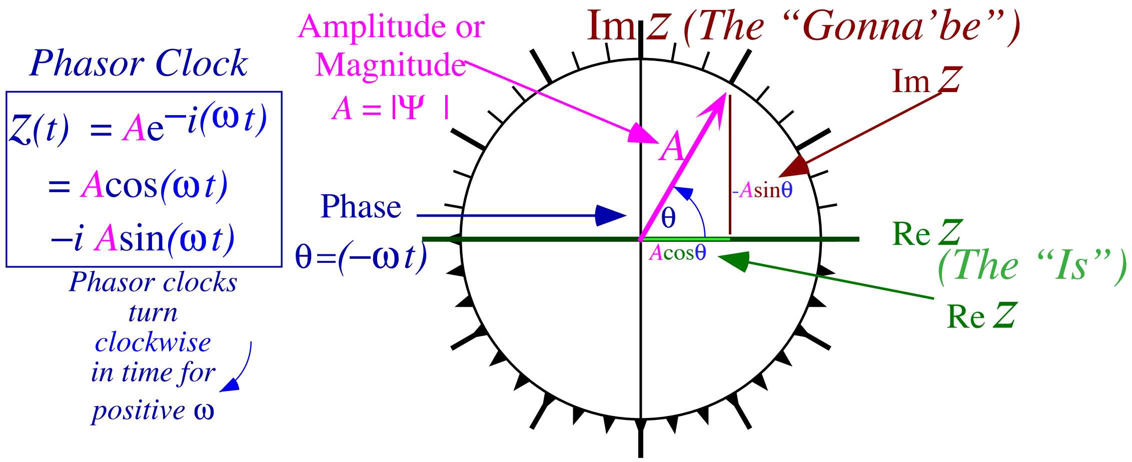
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

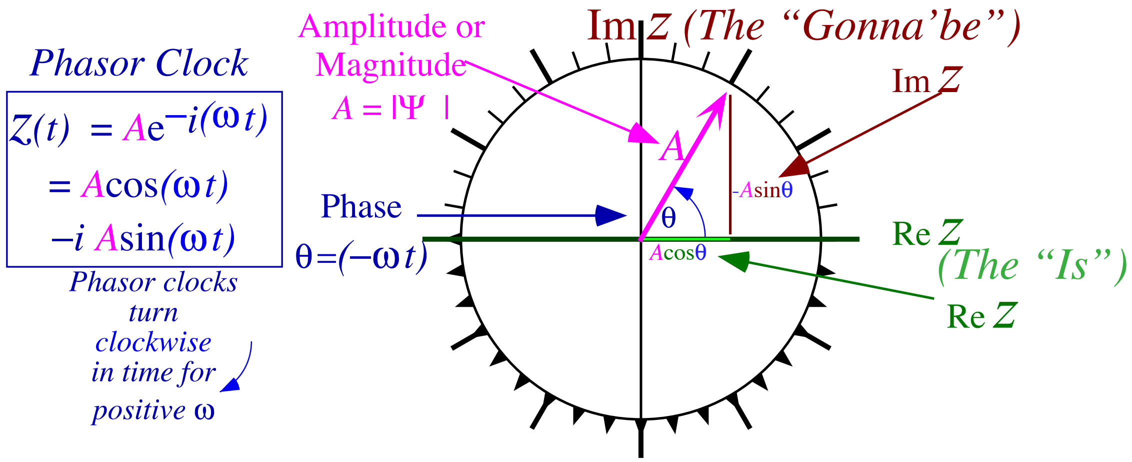
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

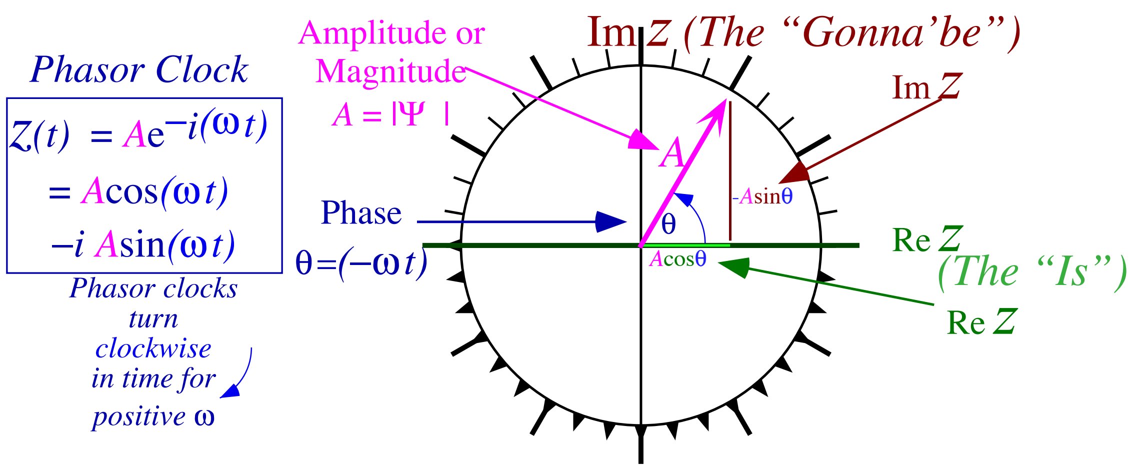
$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}t}$$

$$= e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\, t}$$

$$= e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$$

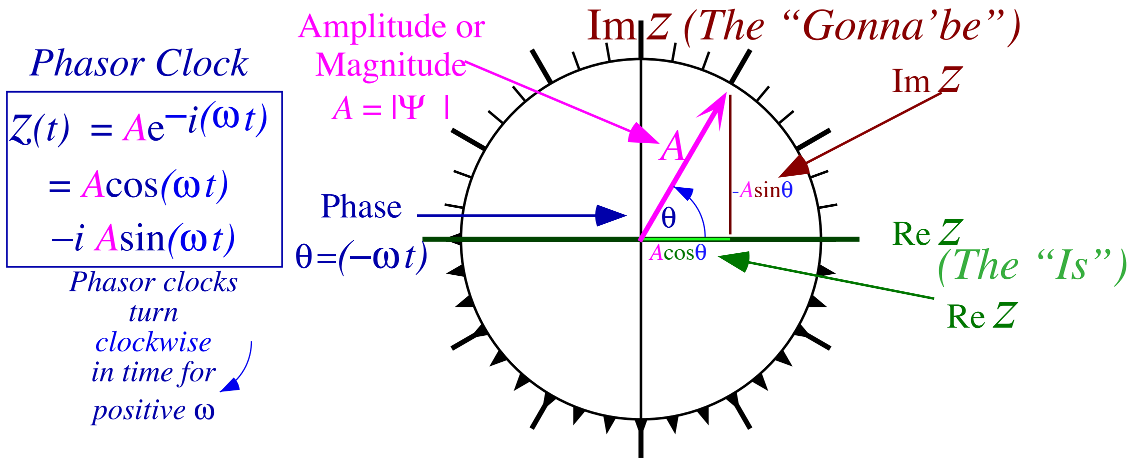
It oscillates at an angular frequency ω_Γ reduced slightly by .05% from ω_0 due to damping Γ =0.2.

$$\omega_\Gamma = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0 - \tfrac{1}{2}(\Gamma^2/\omega_0) + \ldots = 6.2831853 - 0.003183 + .. = 6.280002 + \ldots = 6.280001$$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

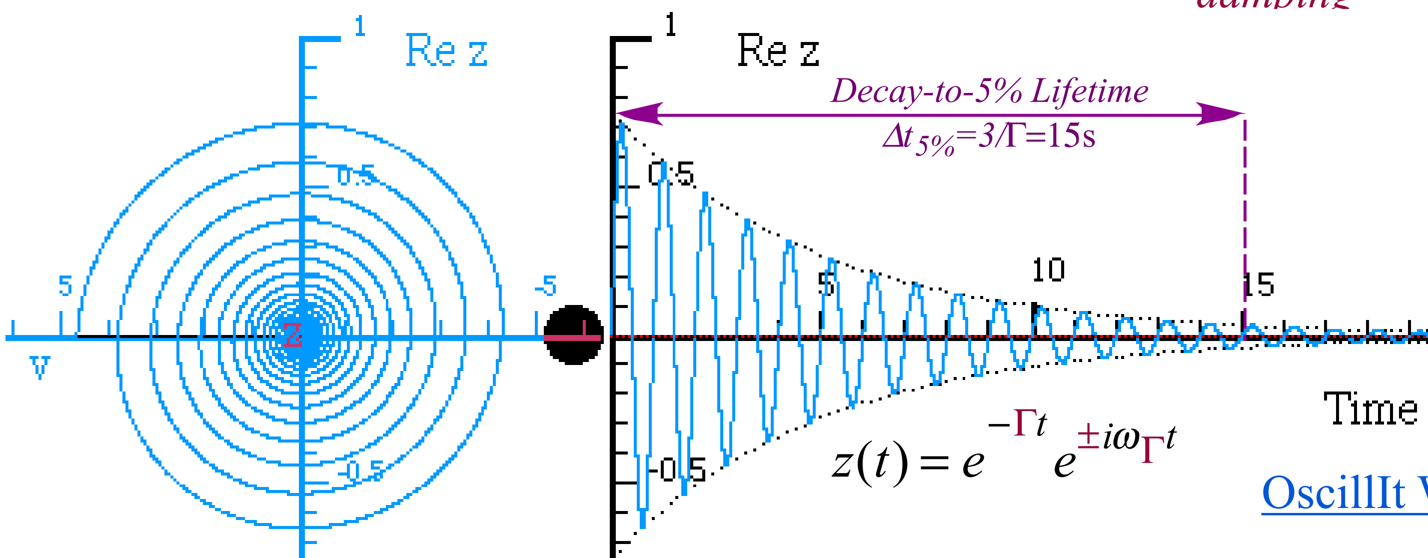
Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\, t}$$

$$= e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$$



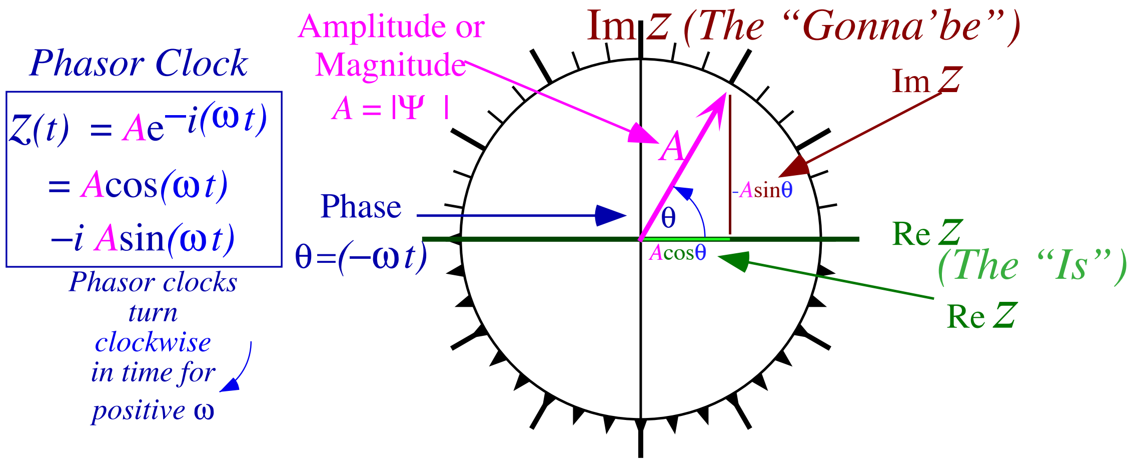
OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

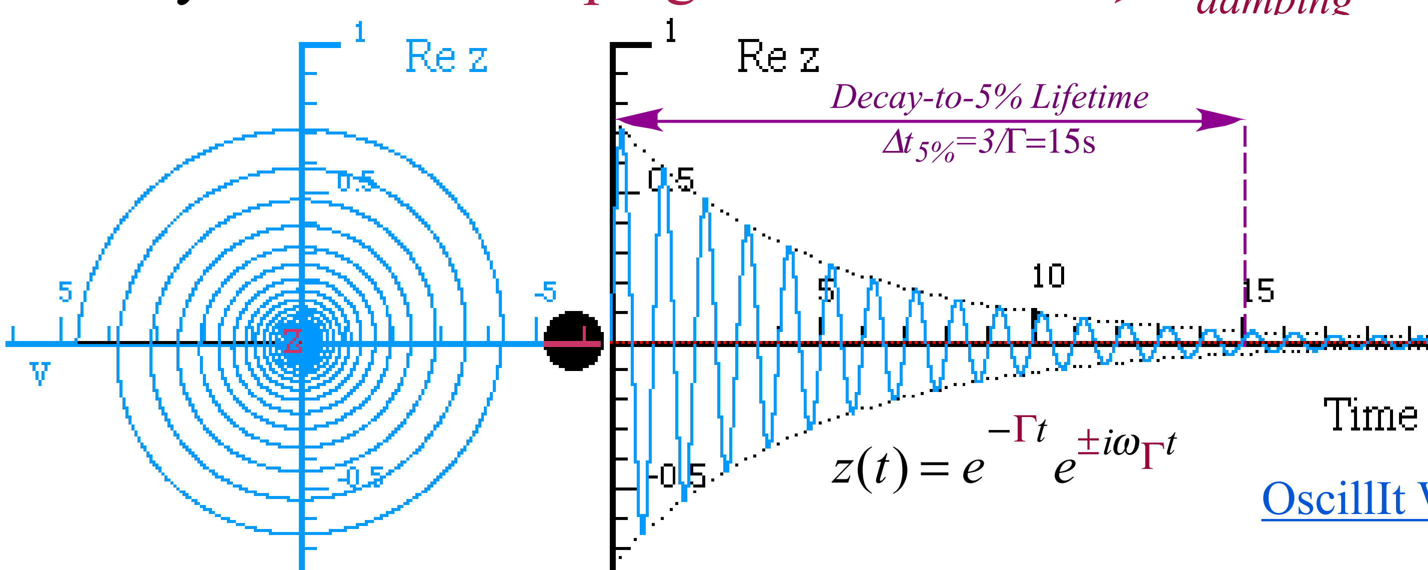
Oscillator Figures of Merit:

Time required to to reduce amplitude to 5%

Easy-to-recall *5%* approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$



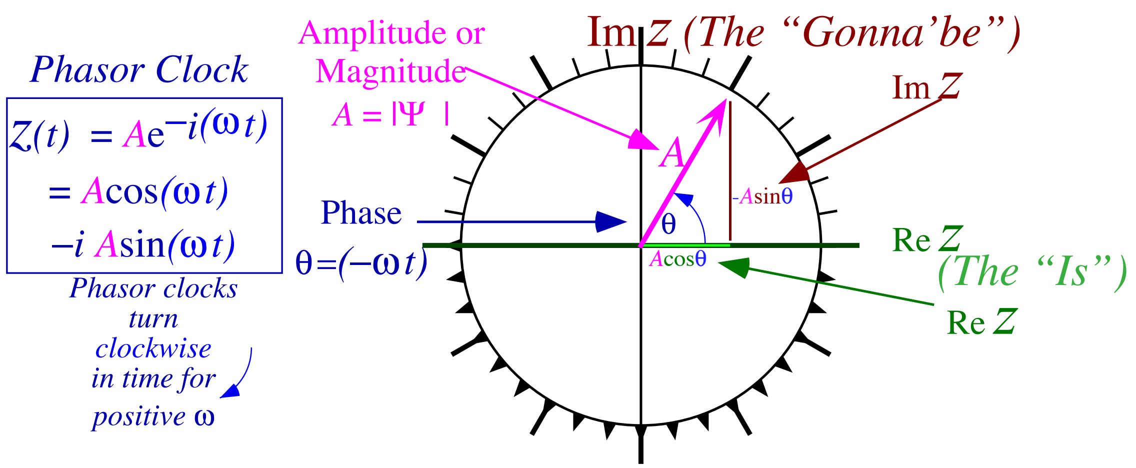
OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear *damped-harmonic* *oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

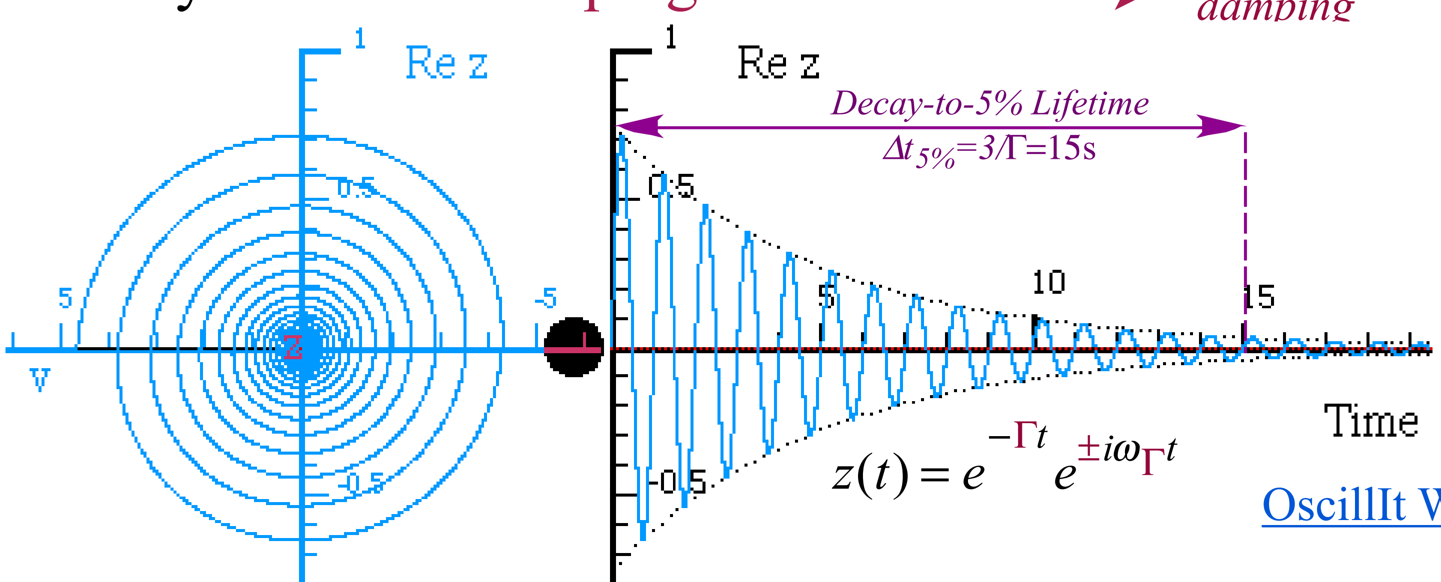
Time required to to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3} \cong 0.05$

More precise one: $e^{-\pi} \cong 0.04321$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$



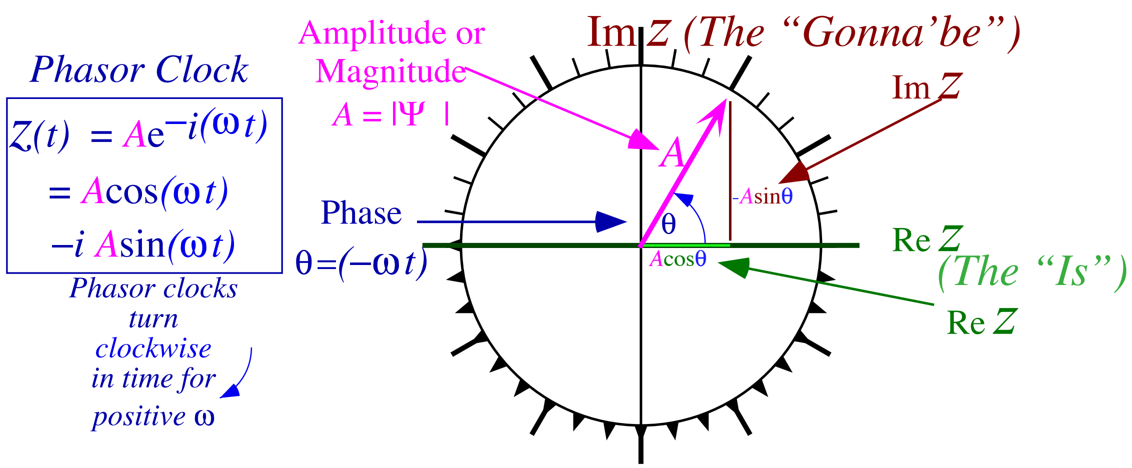
OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

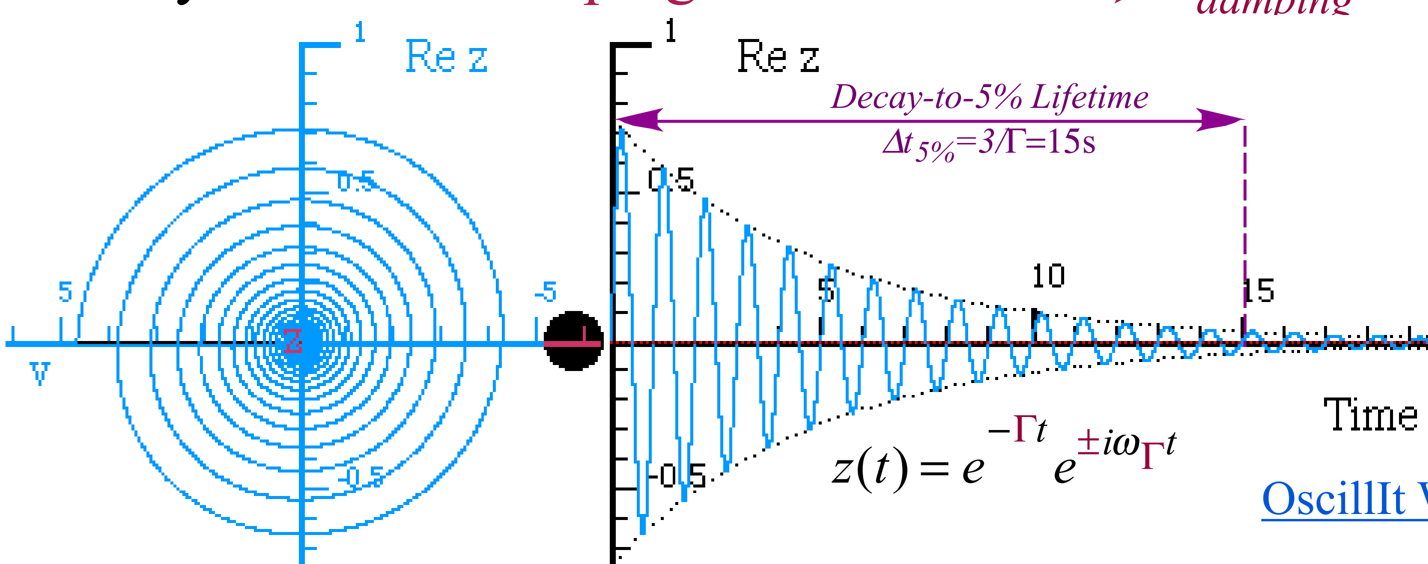
Number *N* of oscillations to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3} \cong 0.05$

More precise one: $e^{-\pi} \cong 0.04321$

$$N_{5\%} = \frac{\omega_\Gamma \cdot t_{5\%}}{2\pi} = \frac{3\omega_\Gamma}{2\pi\Gamma} \sim \frac{\omega_\Gamma}{2\Gamma}$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$



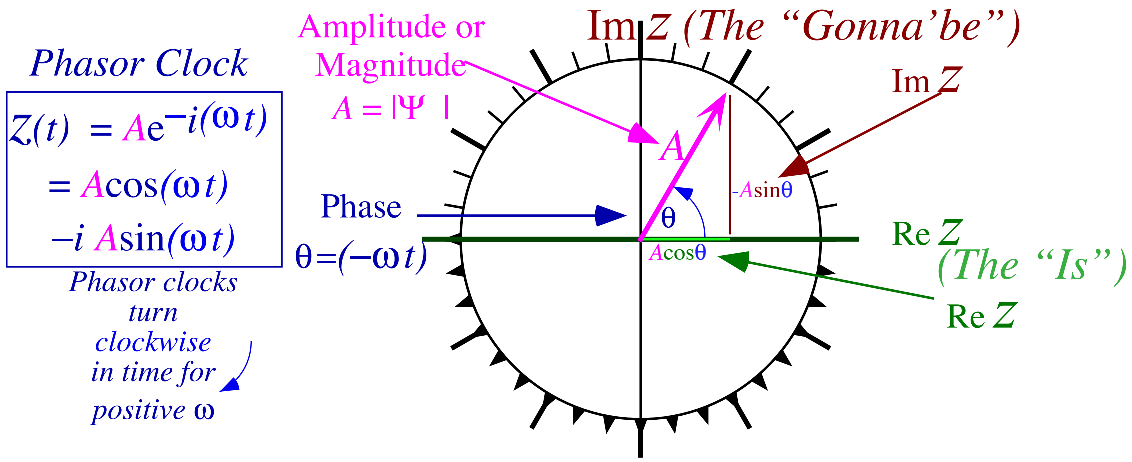
OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0 = 2\pi$ and $\Gamma = 0.2$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$



Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

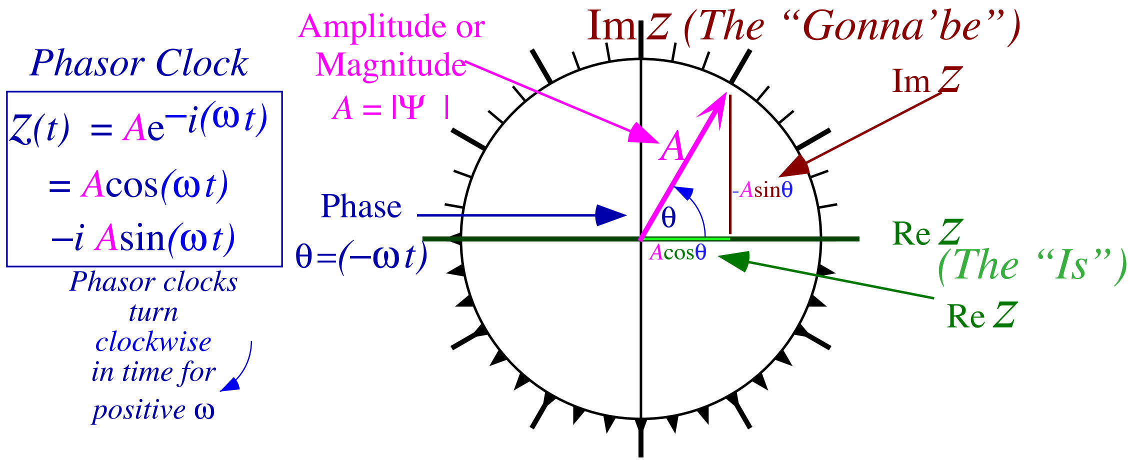
driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad (k = \omega_0^2 m),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad (b = 2\Gamma m)$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

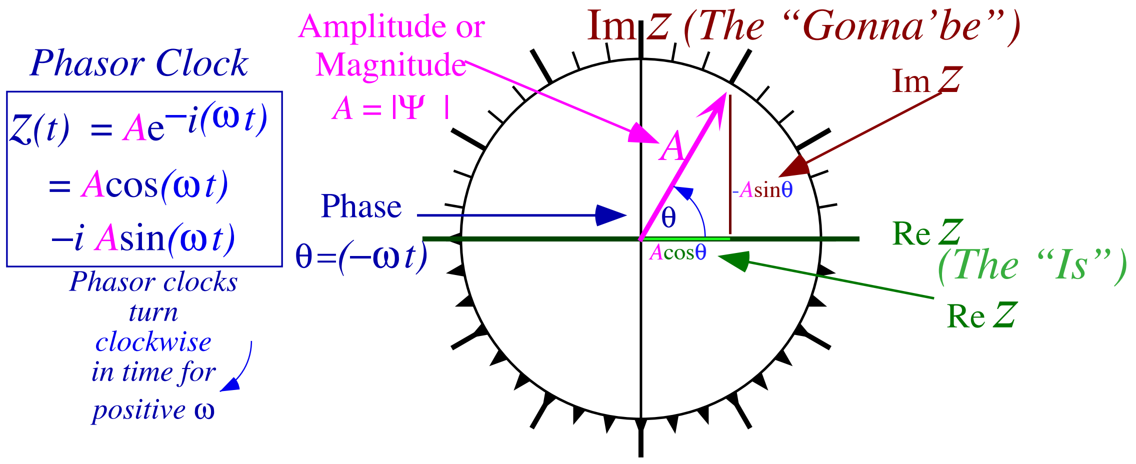
Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

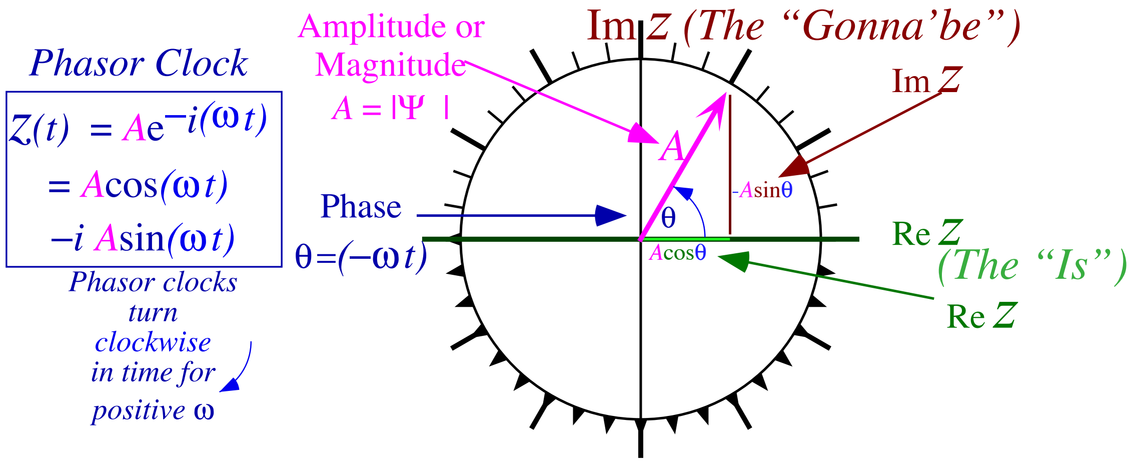
Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right) z = a_{stimulus}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2} a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

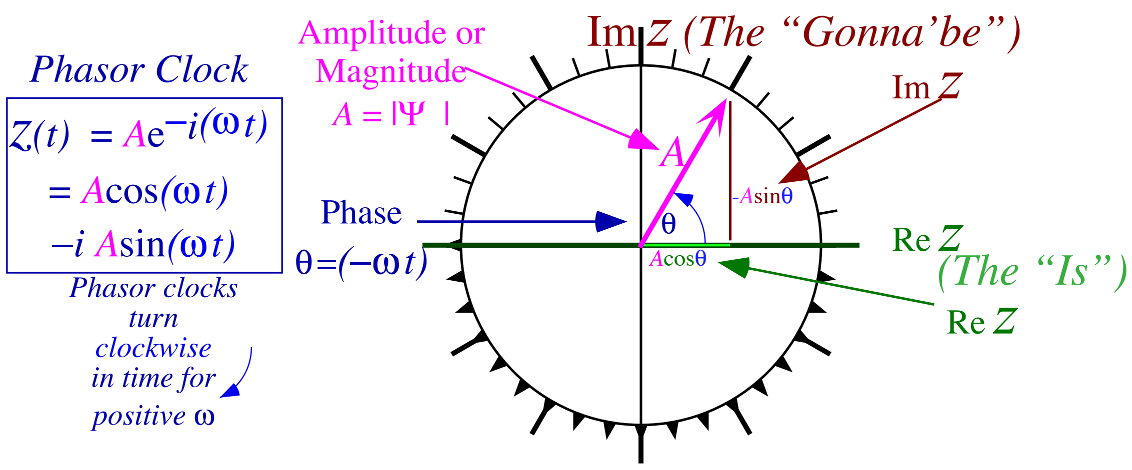
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy?

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}\,a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

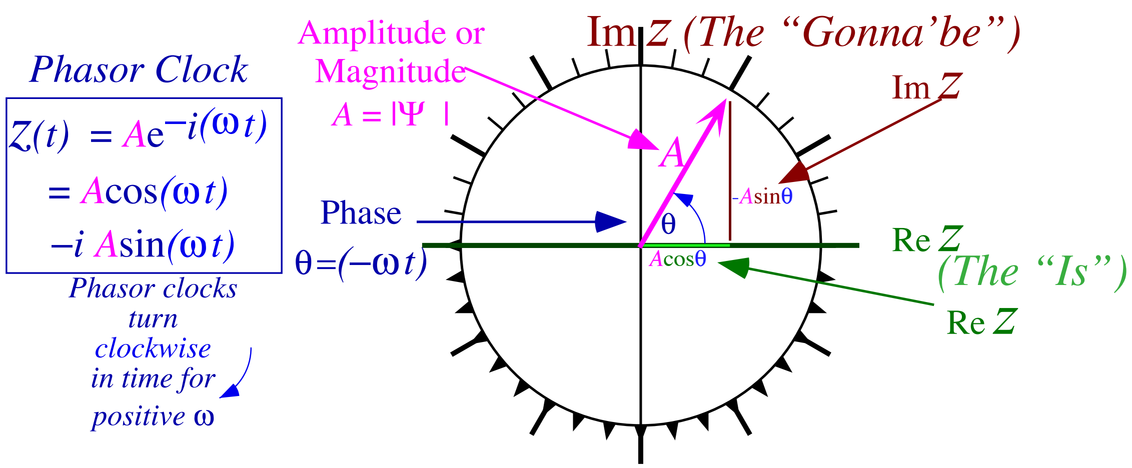
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}\,a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

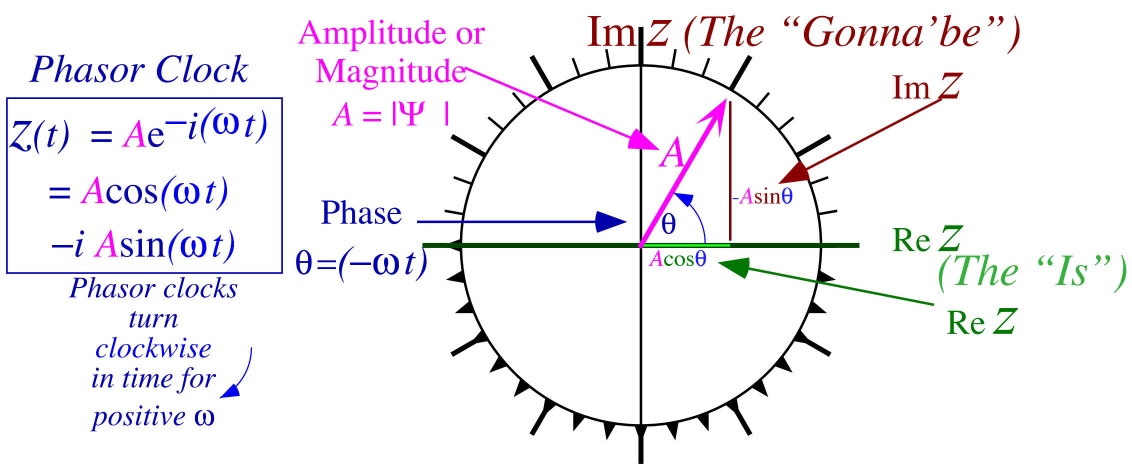
Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

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Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

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Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^{\ 2} - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^{\ 2} - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

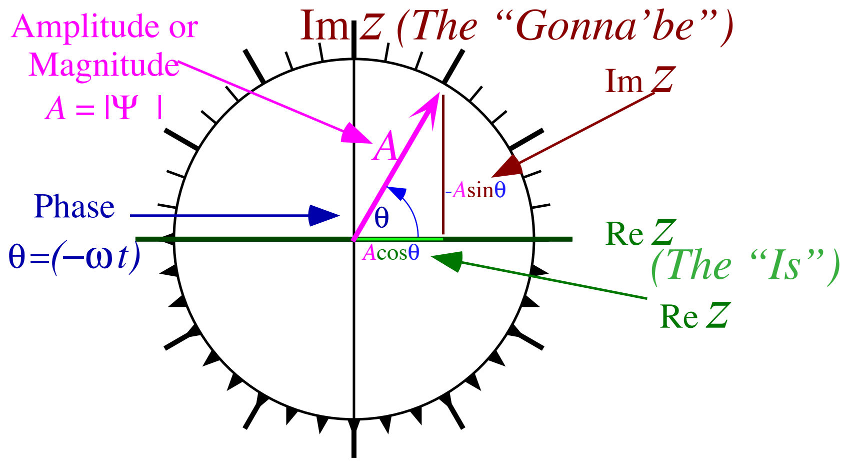
$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

Phasor Clock

$Z(t) = Ae^{-i(\omega t)}$
$= A\cos(\omega t)$
$-i\,A\sin(\omega t)$

Phasor clocks turn clockwise in time for positive ω



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^{\ 2} - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^{\ 2} - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s)\cdot a_s$$

Green's Function for the F-D-H Oscillator (FDHO)

George Green (14 July 1793 – 31 May 1841)

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

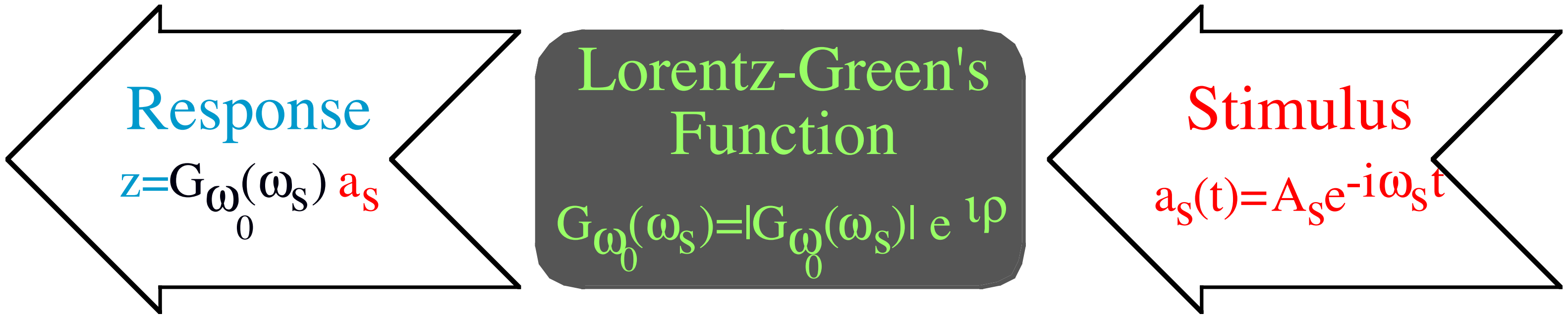


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\,G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of *G*:

Hendrik A. Lorentz

July 18, 1853. - February 4, 1928

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

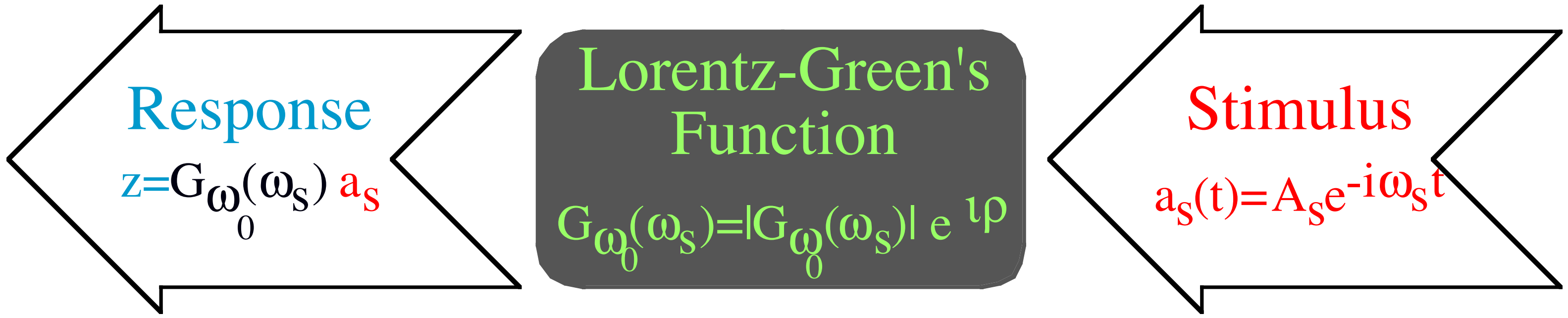


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of *G*: $\frac{1}{x-iy}=\frac{1}{x-iy}\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

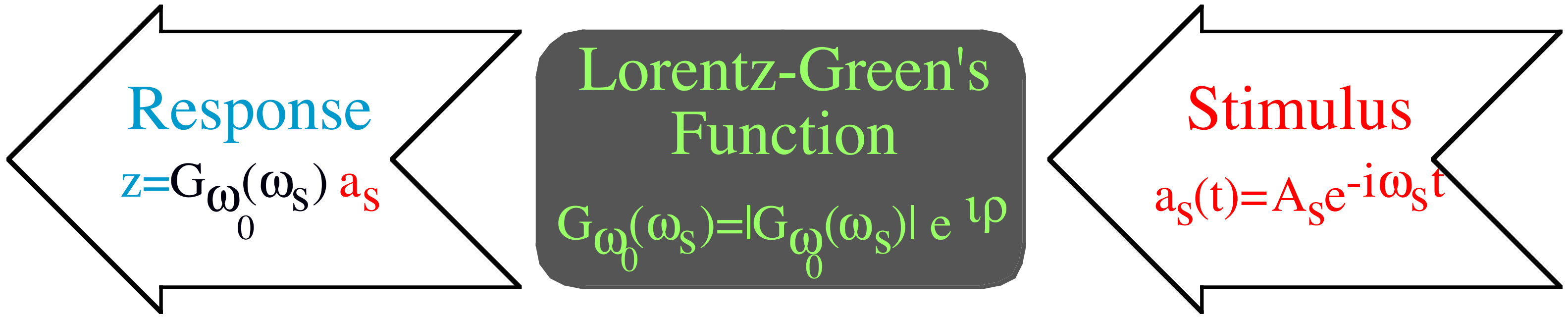


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \operatorname{Re} G_{\omega_0}(\omega_s) + i \operatorname{Im} G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of *G*:

$$\frac{1}{x-iy} = \frac{1}{x-iy}\frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

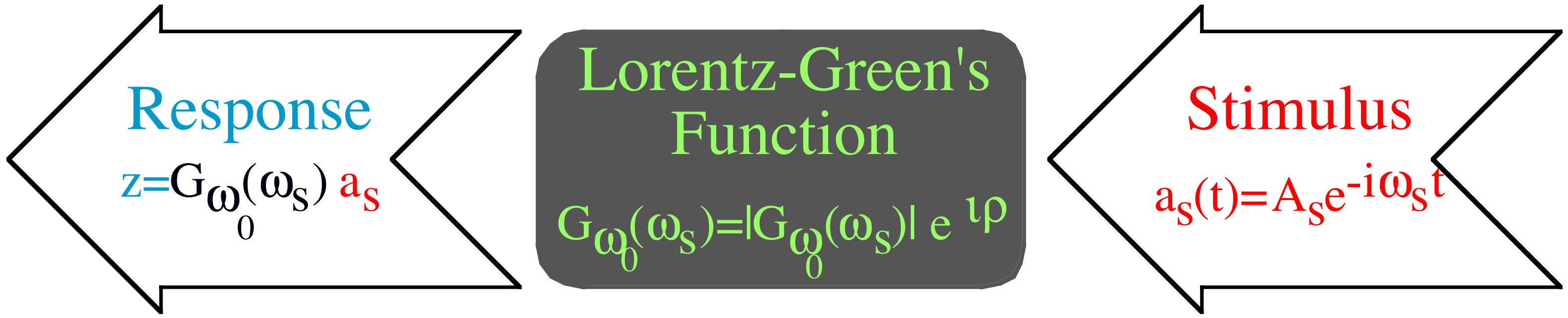


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)=\left|G_{\omega_0}(\omega_s)\right|e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and polar angle ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}$$

$$\rho=\tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

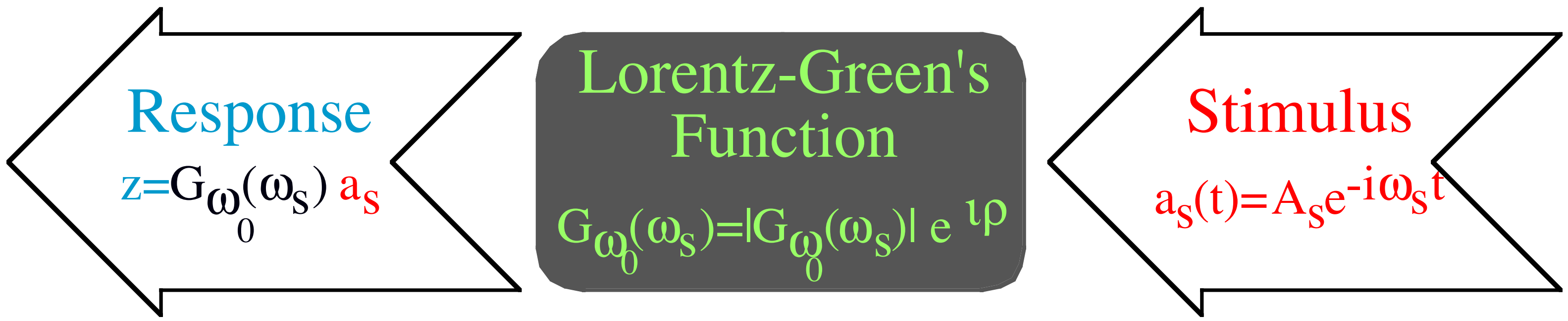


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

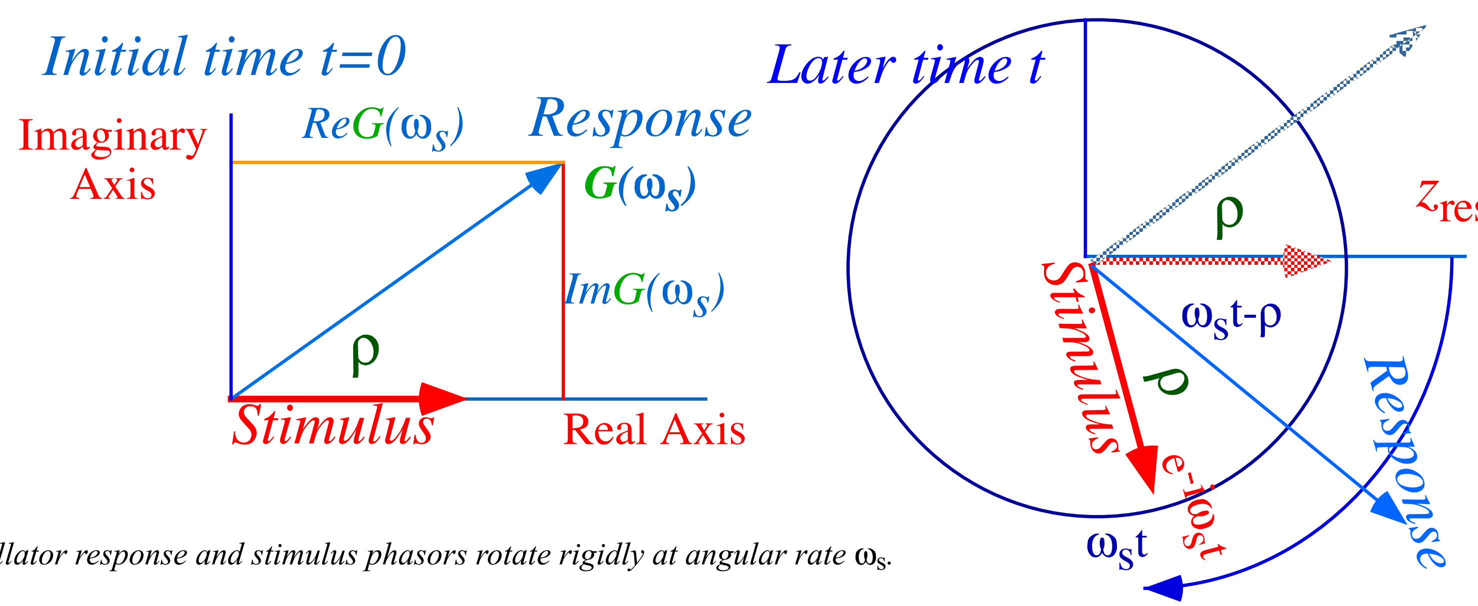
$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$



polar angle ρ is the *phase lag angle* ρ

$$z_{\text{response}}(t) = \left|G_{\omega_0}(\omega_s)\right| a(0)\, e^{-i(\omega_s t - \rho)}$$

Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

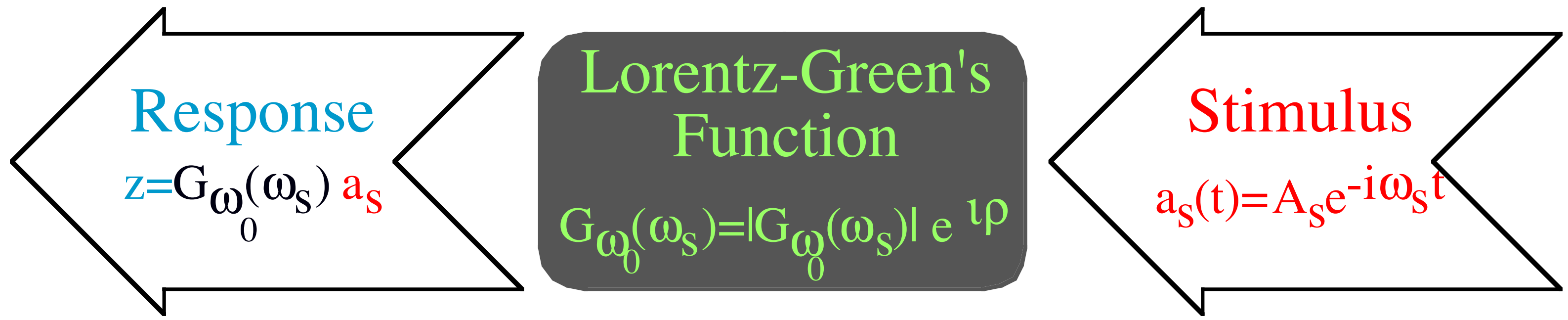


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\, \mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of *G*:

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of *G*:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

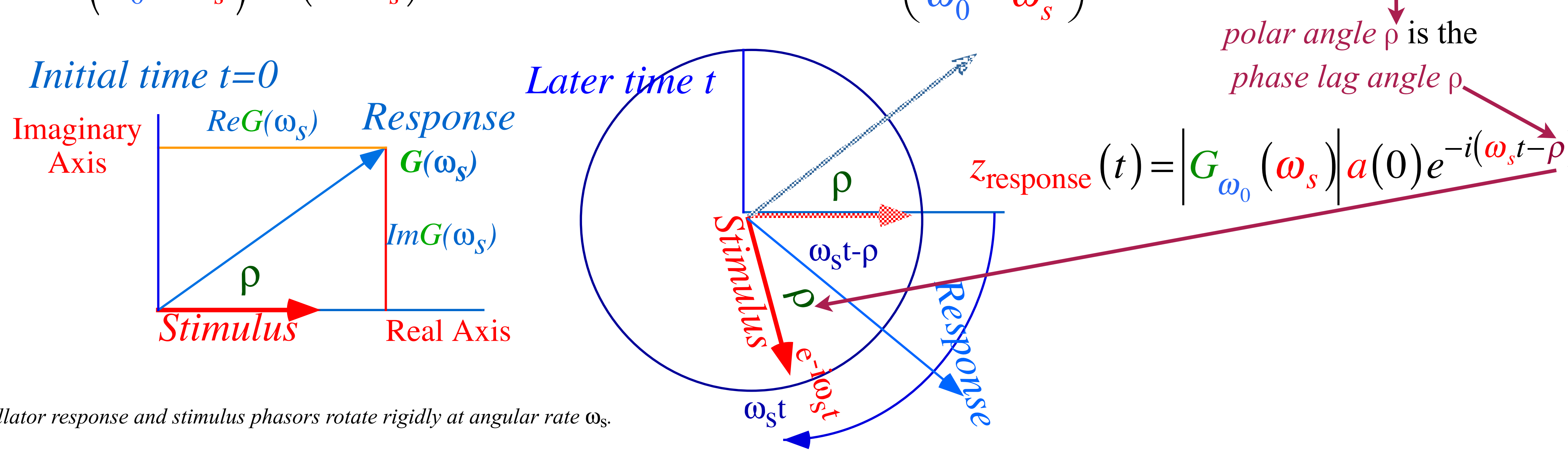
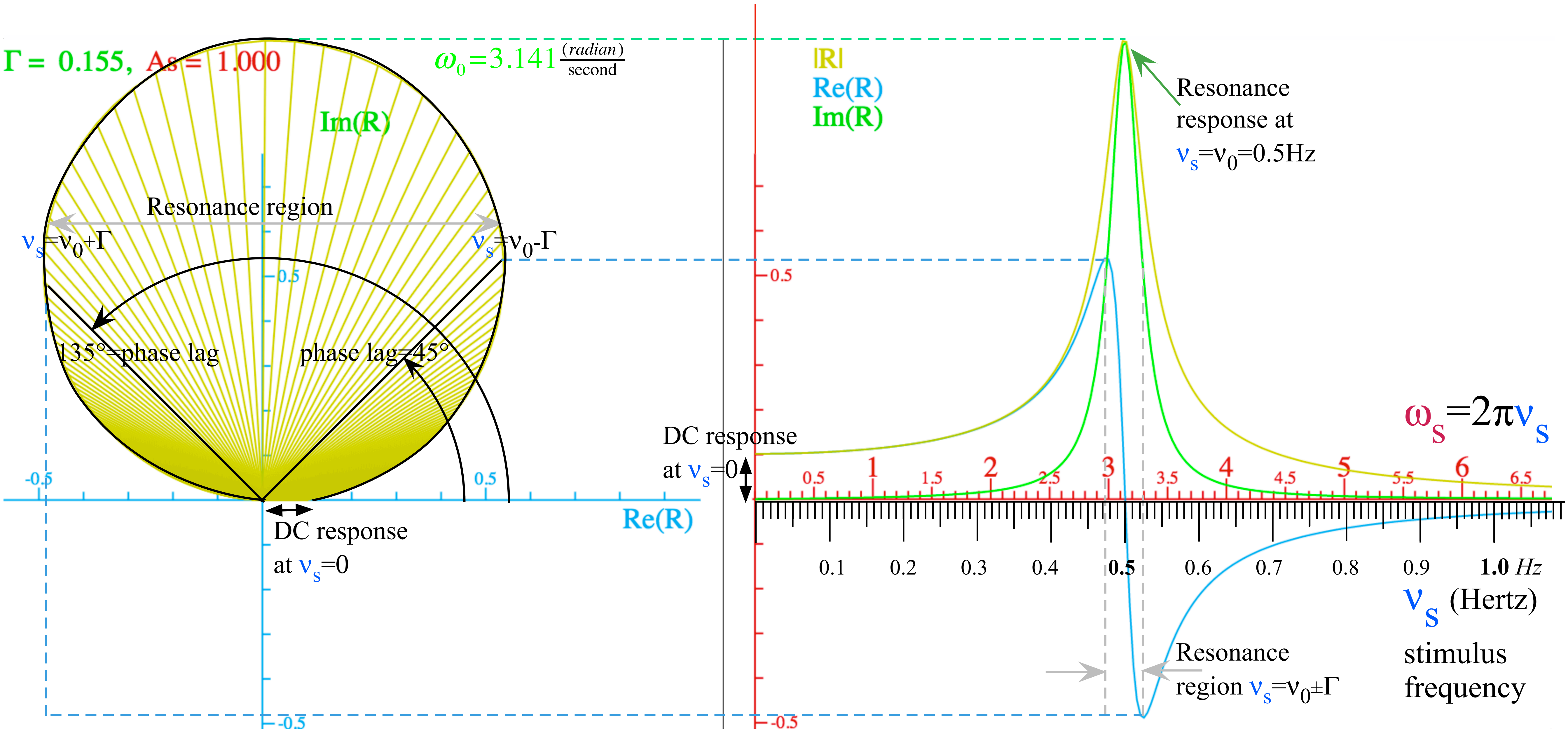


Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.

$$Lorentz\text{-}Green\text{'s function for } \nu_0 = 0.5\,Hz \text{ or}: \omega_0 = \pi \frac{(radian)}{second}$$



OscillIt Web Simulation: Lorentz Response Function

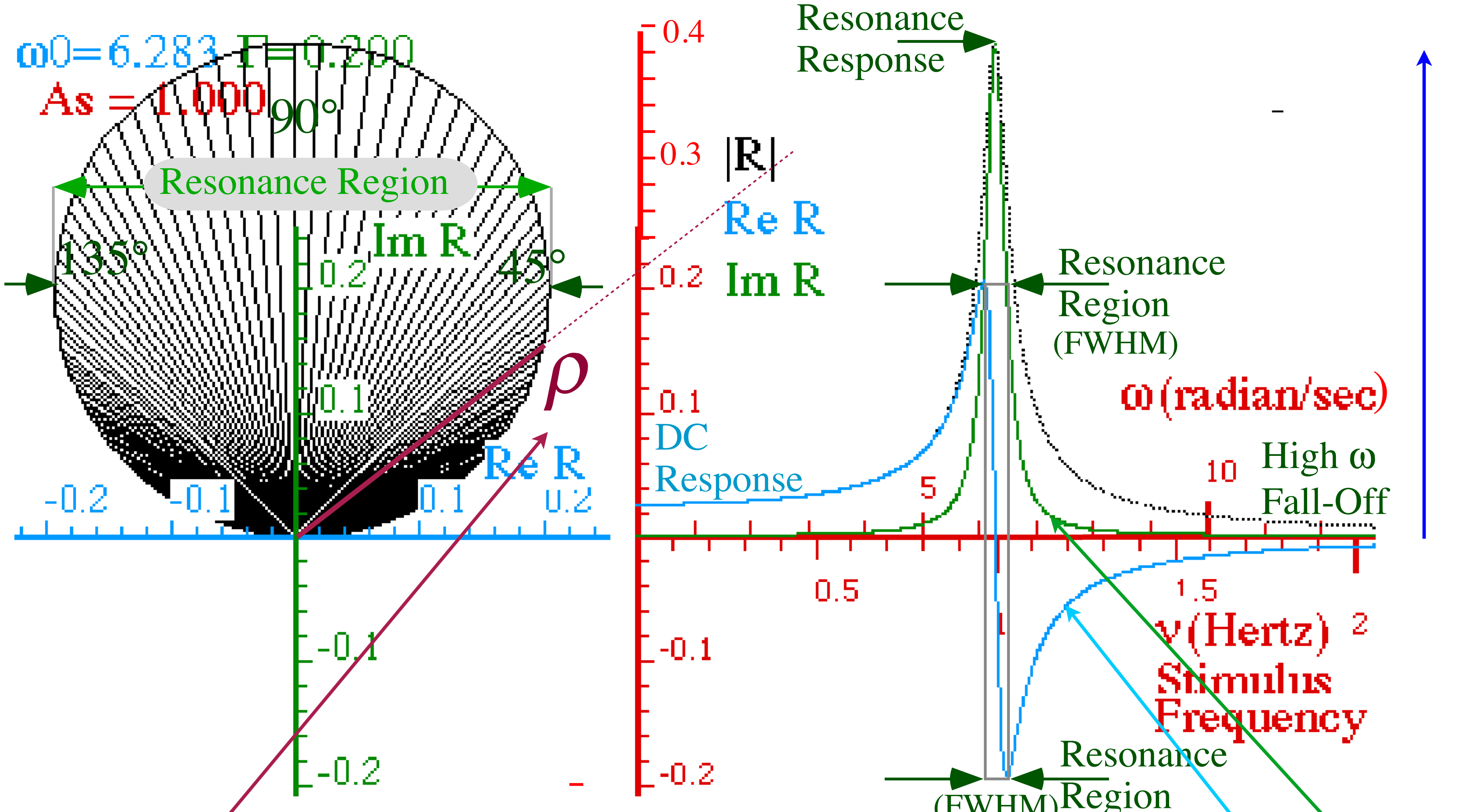


Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Real part

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

OscillIt Web Simulation: Lorentz Response Function

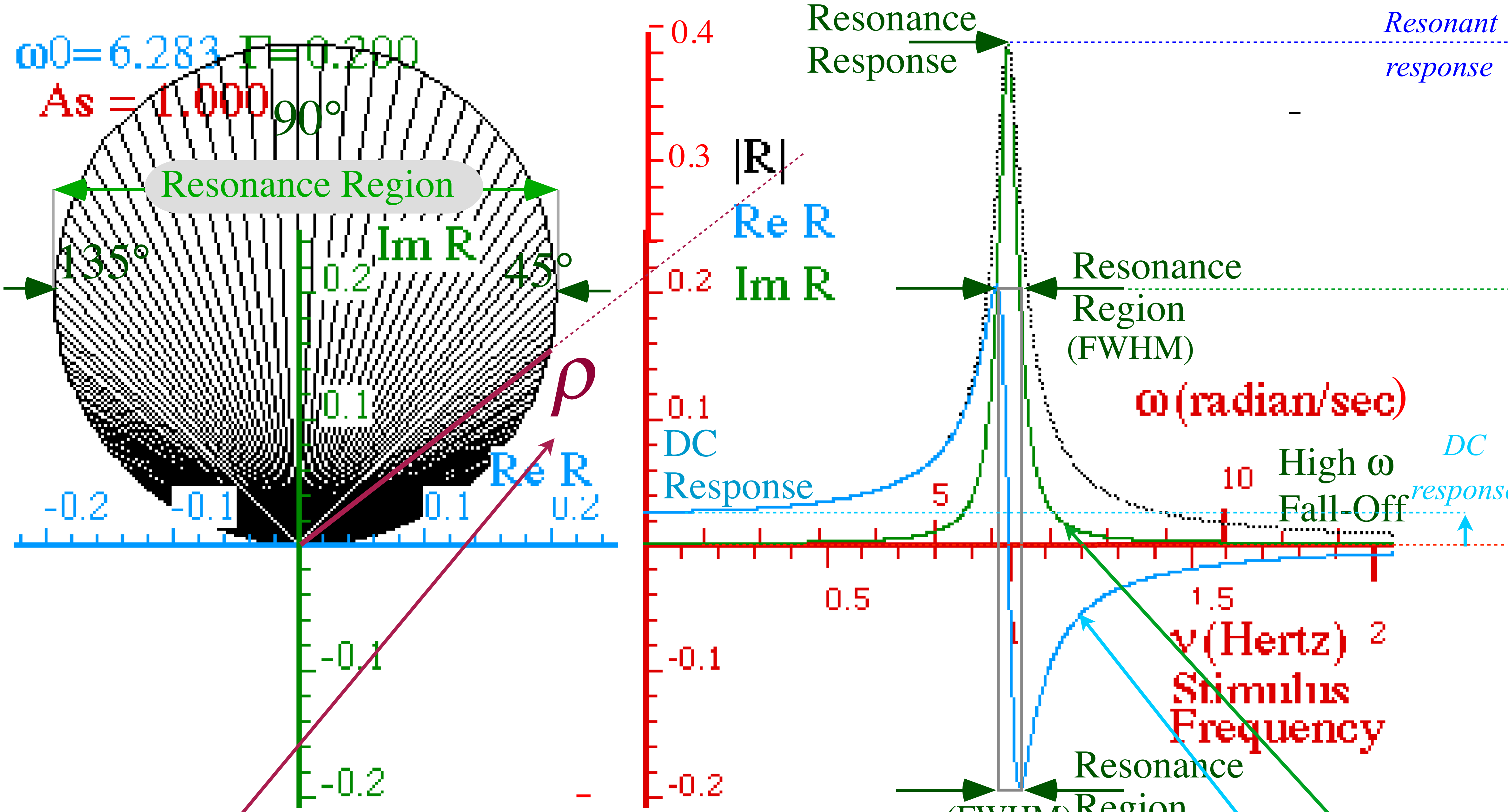


Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Real part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

OscillIt Web Simulation: Lorentz Response Function

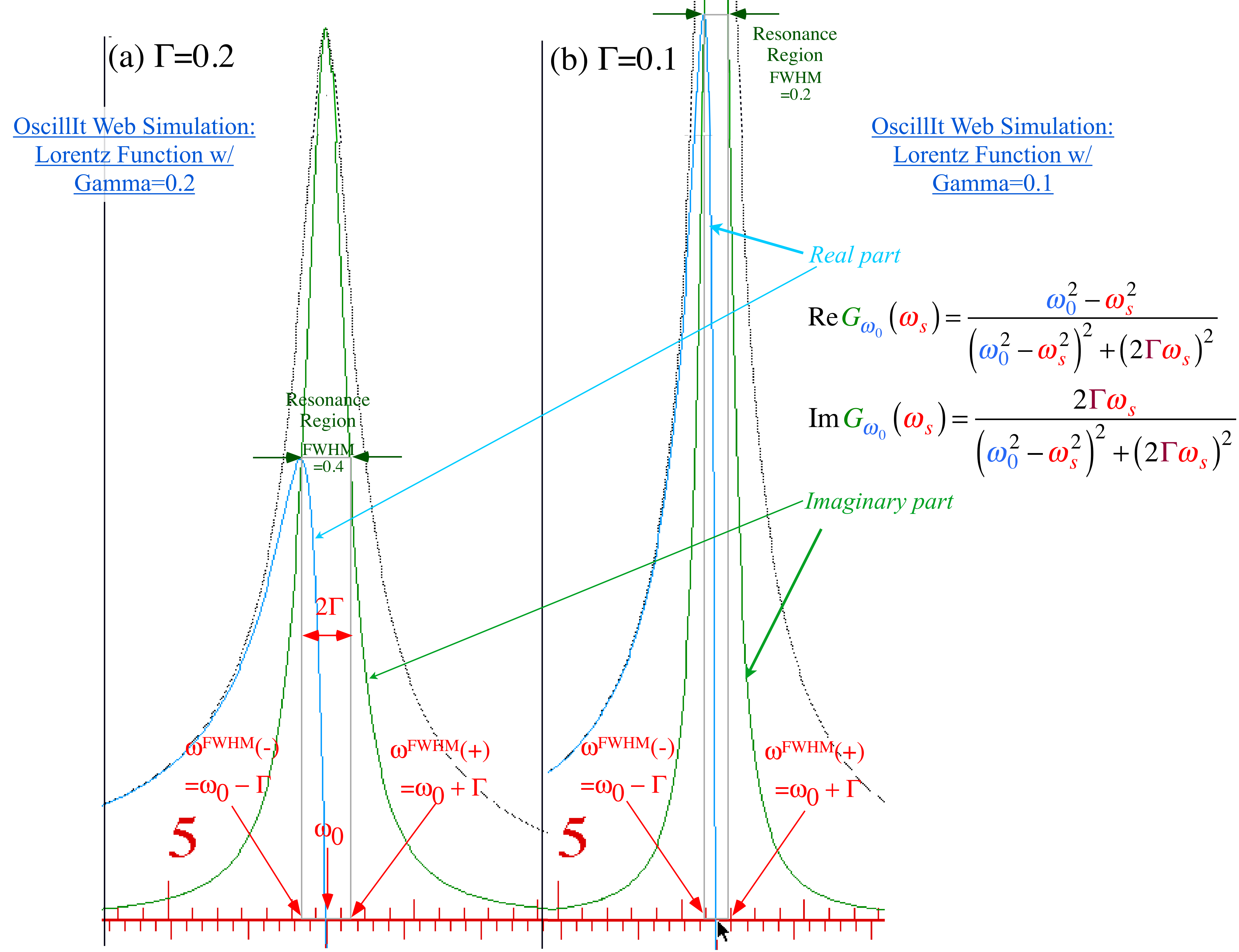


Fig. 4.2.7 Comparing Lorentz-Green resonance region for (a) Γ=0.2 and (b) Γ=0.1.
Maximum and minimum points of ReG(ω) *and inflection points of* ImG(ω) *are near region boundaries* $\omega^{FWHM}(\pm)=\omega_0\pm\Gamma$.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

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Known as *Transient* solution since it dies-off as time advances past initial conditions

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Let's you set initial values or boundary conditions

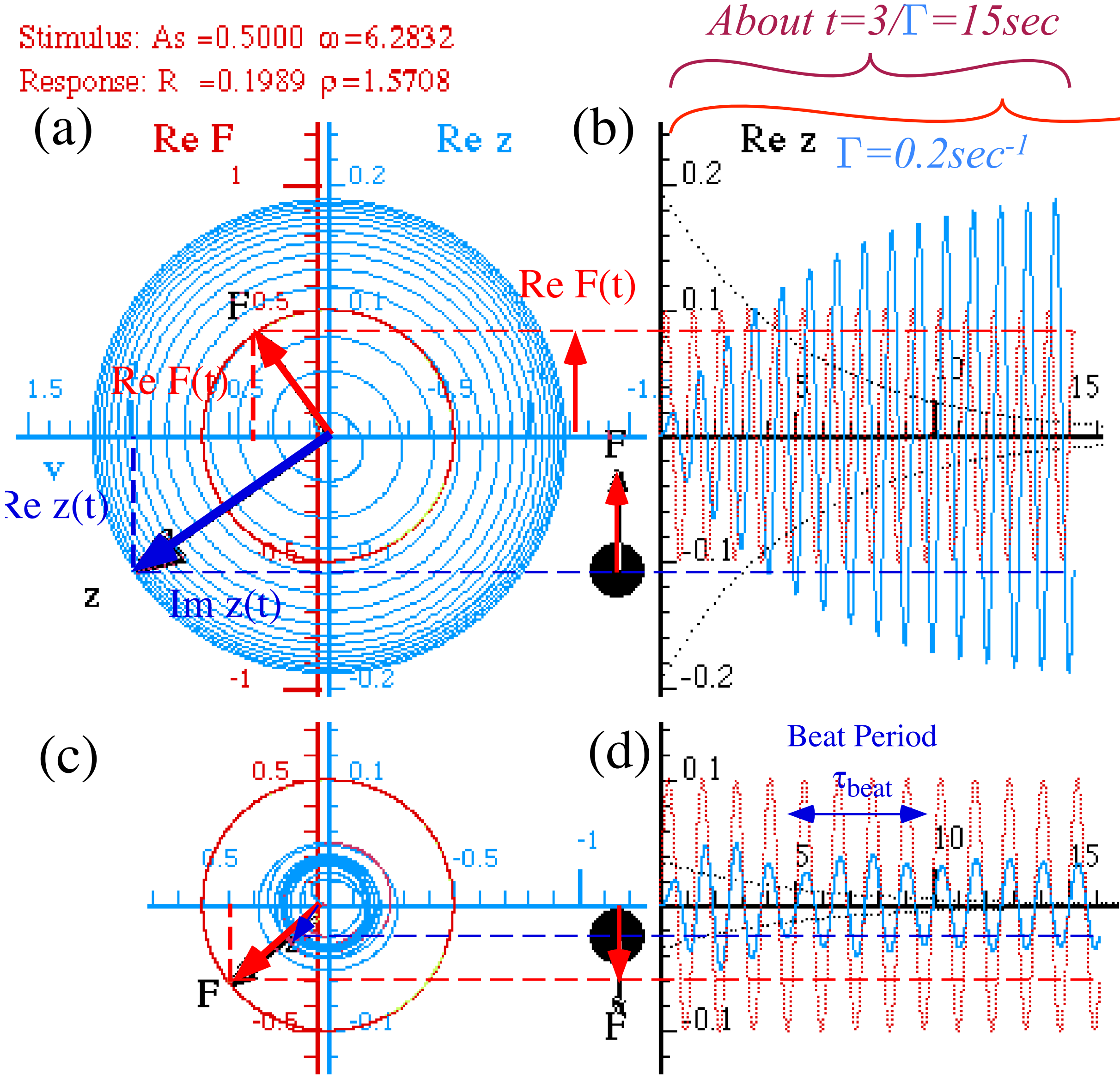
Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

About t=3/Γ=15sec

About t = forever

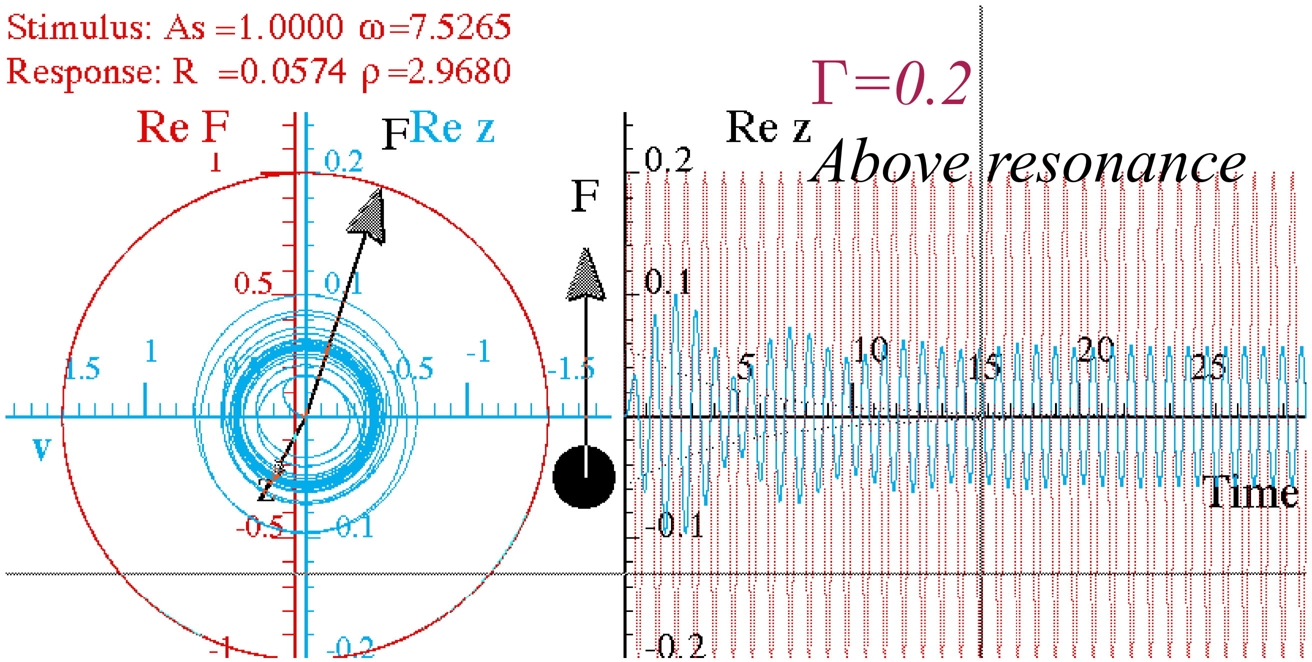


[OscillIt (On Resonance) Simulation]

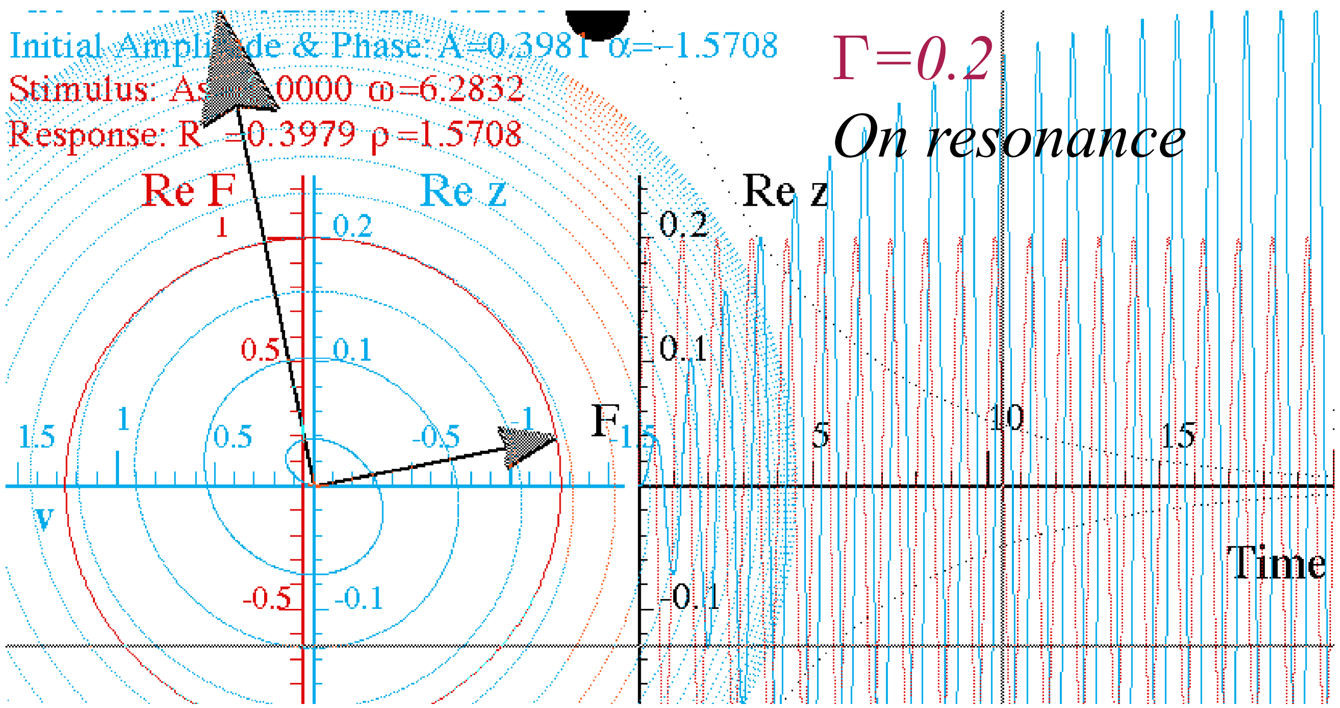
Fig. 4.2.8 On Resonance (a)Response z-phasor lags ρ=90° behind stimulus F-phasor. ($\omega_s=\omega_0=2\pi$, $\omega_0=2\pi$, and $\Gamma=0.2$). (b) Time plots of Re *z(t) and* Re *F(t)*

Fig. 4.2.8 Below Resonance (c)Response z-phasor lags ρ=8.05° behind stimulus F-phasor. ($\omega_s=5.03$, $\omega_0=2\pi$, and $\Gamma=0.2$). (d) Time plots of Re *z(t) and* Re *F(t). Beats are barely visible.*

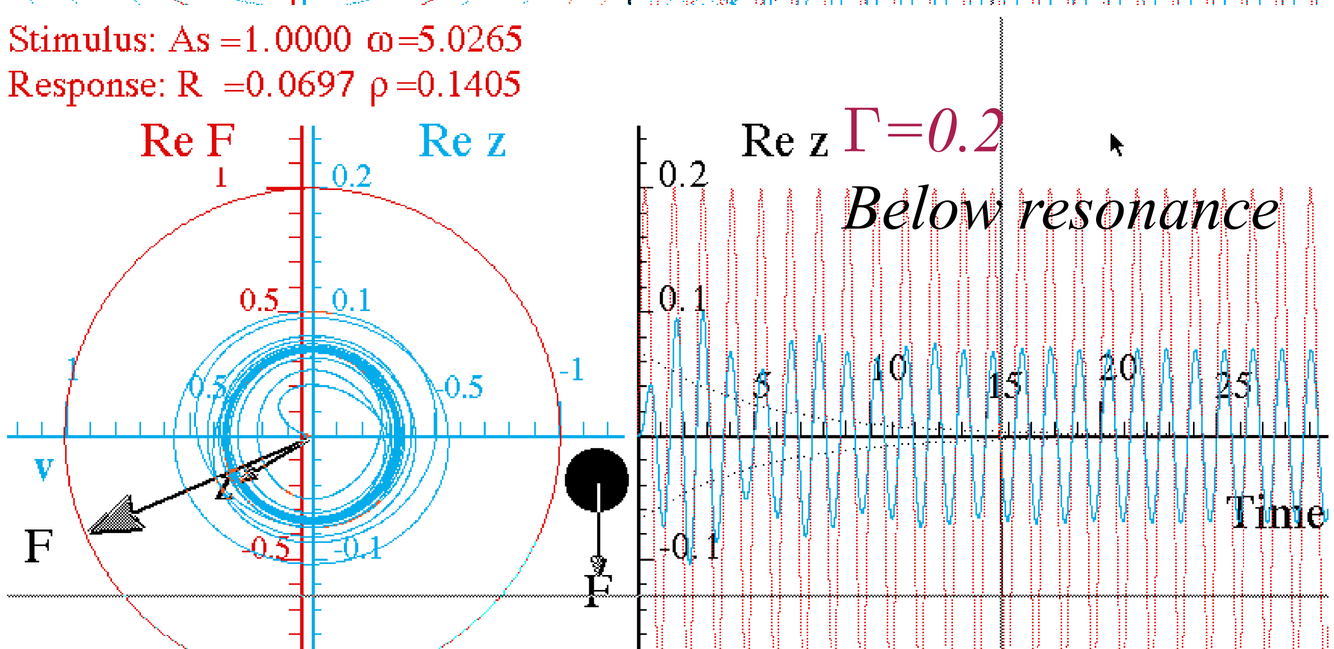
[OscillIt (Way Below Resonance) Simulation]



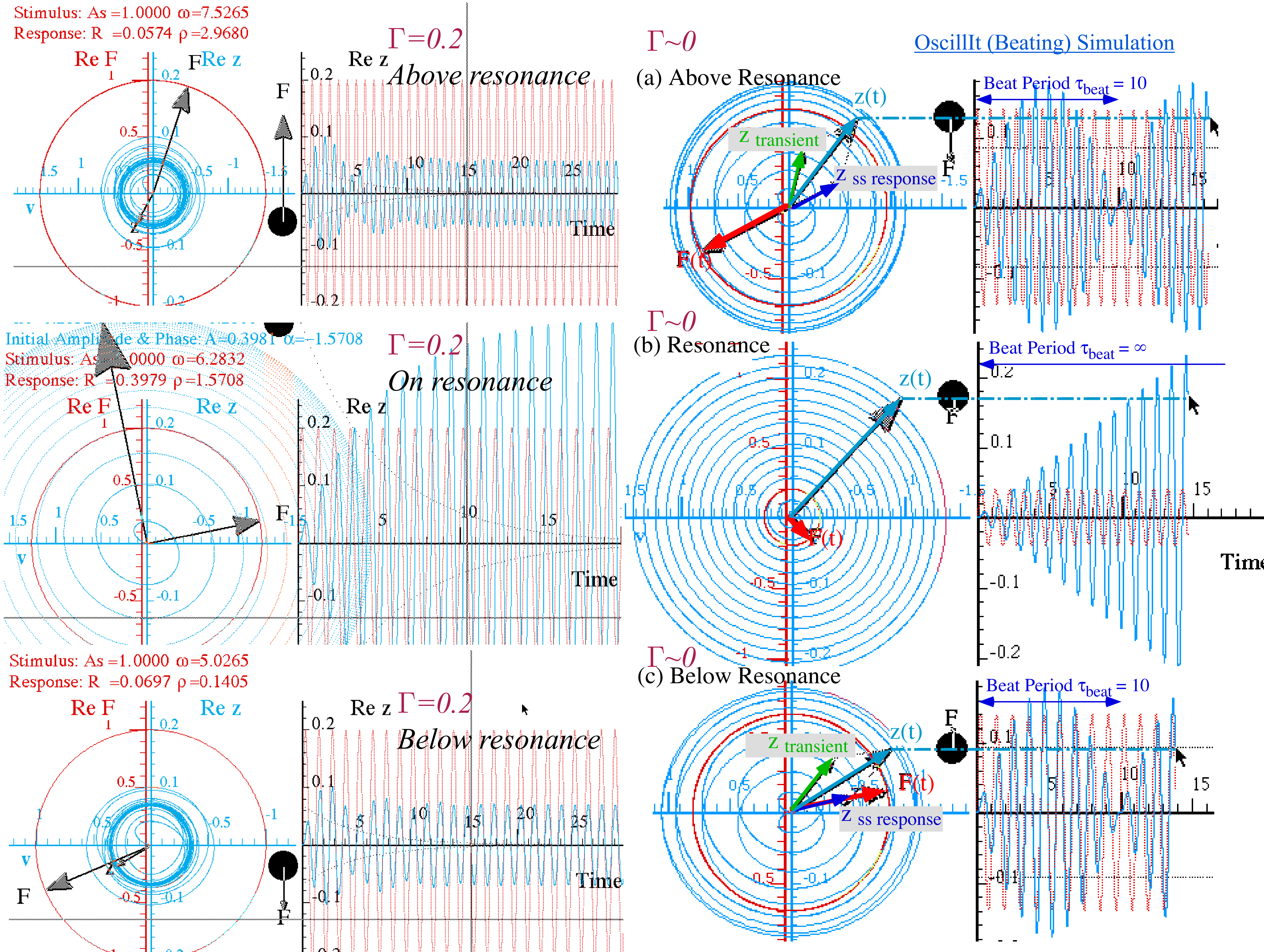
[OscillIt (Way Above Resonance) Simulation]



[OscillIt (On Resonance) Simulation]



[OscillIt (Way Below Resonance) Simulation]



Stimulus: As =1.0000 ω=7.5265
Response: R =0.0574 ρ=2.9680
Γ=0.2
Above resonance
Re F
Re z
F
Time
Initial Amplitude & Phase: A=0.3981 α=-1.5708
Stimulus: As =1.0000 ω=6.2832
Response: R =0.3979 ρ=1.5708
Γ=0.2
On resonance
Stimulus: As =1.0000 ω=5.0265
Response: R =0.0697 ρ=0.1405
Γ=0.2
Below resonance
Γ~0
OscillIt (Beating) Simulation
(a) Above Resonance
z(t)
z transient
z ss response
F(t)
Beat Period τ_beat = 10
(b) Resonance
Beat Period τ_beat = ∞
(c) Below Resonance

Lorentz-Green's Function for high quality FDHO

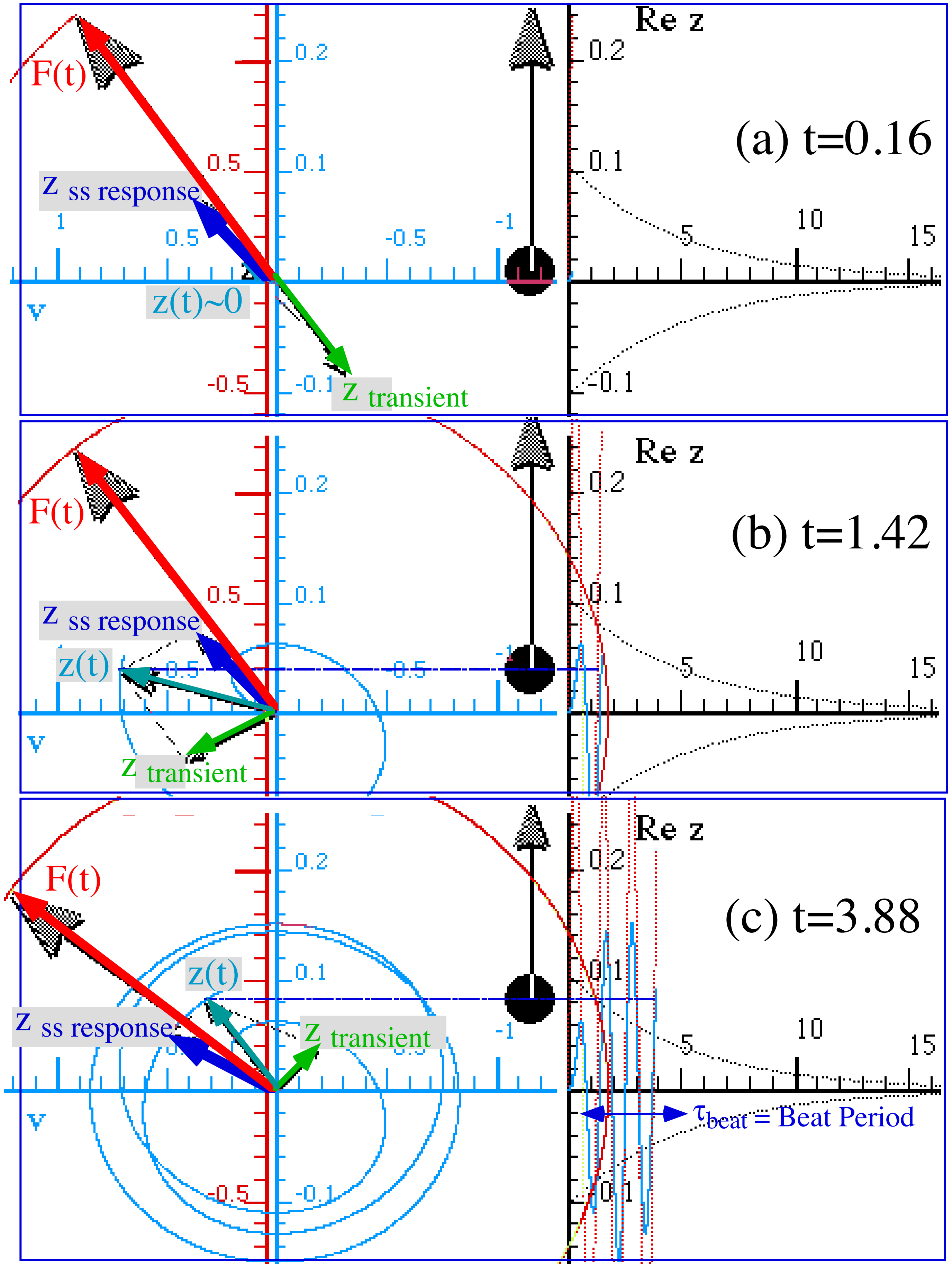


Fig. 4.2.9 Beat formation.

Transient phasor $z_{transient}$ catches up with F-phasor and passes it.

OscillIt (Beating) Simulation

Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

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$$\overset{t_{5\%} = 3/\Gamma = \textit{Lifetime}}{\left(\begin{array}{c}\text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude}\end{array}\right)} \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{array}{c}\text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \textit{ Lifetime}\end{array}$$

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$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma}\cdot\frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The "Heartbeat Count" measure of lifetime

Oscillator figures of merit: quality factors Q and $q=2\pi Q$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

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$$\begin{pmatrix} t_{5\%} = 3/\Gamma = \textit{Lifetime} \\ \text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude} \end{pmatrix} \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix} \text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \textit{ Lifetime} \end{matrix}$$

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The "Heartbeat Count" measure of lifetime

Energy decay (proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$ $\qquad dE = -2\Gamma E$

Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

$t_{5\%} = 3/\Gamma = Lifetime$

$$\left(\begin{matrix}\text{for decaying oscillator}\\ \text{to lose 95\% of}\\ \text{amplitude}\end{matrix}\right) \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix}\text{number } n_{5\%}\\ \text{of oscillations}\\ \text{in a } t_{5\%} \textit{ Lifetime}\end{matrix}$$

$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma} \cdot \frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The "Heartbeat Count" measure of lifetime

Energy decay (proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$ $\qquad dE = -2\Gamma E$

Relative amount of energy lost each cycle period $\left(\tau_0 = \frac{1}{\upsilon_0}\right)$

$$= \tau_0\left(\frac{-dE}{E}\right) = \frac{2\Gamma}{\upsilon_0} \equiv \frac{1}{Q} = \frac{2\pi}{q}$$

$$Q = (\textit{Standard angular quality factor}) = \frac{q}{2\pi}$$

Oscillator figures of merit: Uncertainty 1/q

To see a beat we need $\tau_{\text{half-beat}}$ to be less than $\tau_{5\%}$ or 3/Γ. (Here we approximate π~3.0, again.)

$$\pi / |\omega_s - \omega_0| < 3 / \Gamma \qquad\qquad |\omega_s - \omega_0| > \Gamma$$

This means ω-detuning error is greater than or equal to the decay rate Γ.

Any detuning less than Γ is virtually undetectable.
Total ω uncertainty is ±Γ or twice Γ (that is: FWHM $\Delta\omega = 2\Gamma$).

The *relative frequency uncertainty* $\frac{2\Gamma}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \frac{1}{q} = \frac{\Delta\upsilon}{\upsilon_0}$

is the *inverse* of the *angular quality factor q*.

Linear frequency uncertainty is:

$$\Delta\upsilon = \Delta\omega / 2\pi = \Gamma / \pi$$

If we think of the 5% or 4.321% lifetime of a musical note as its time uncertainty Δt, then:

$$\Delta t \Delta\upsilon = 3 / \pi \approx 1$$

$$\Delta t = t_{5\%} = 3 / \Gamma \qquad\qquad \Delta t = t_{4.321\%} = \pi / \Gamma$$

Very precise measures of imprecision

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

Complex detuning-decay $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0}L(\Delta - i\Gamma)$$

Complex detuning-decay $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \quad \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0}L(\Delta - i\Gamma)$$

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$$= |L|e^{i\rho} = |L|\cos\rho + i|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

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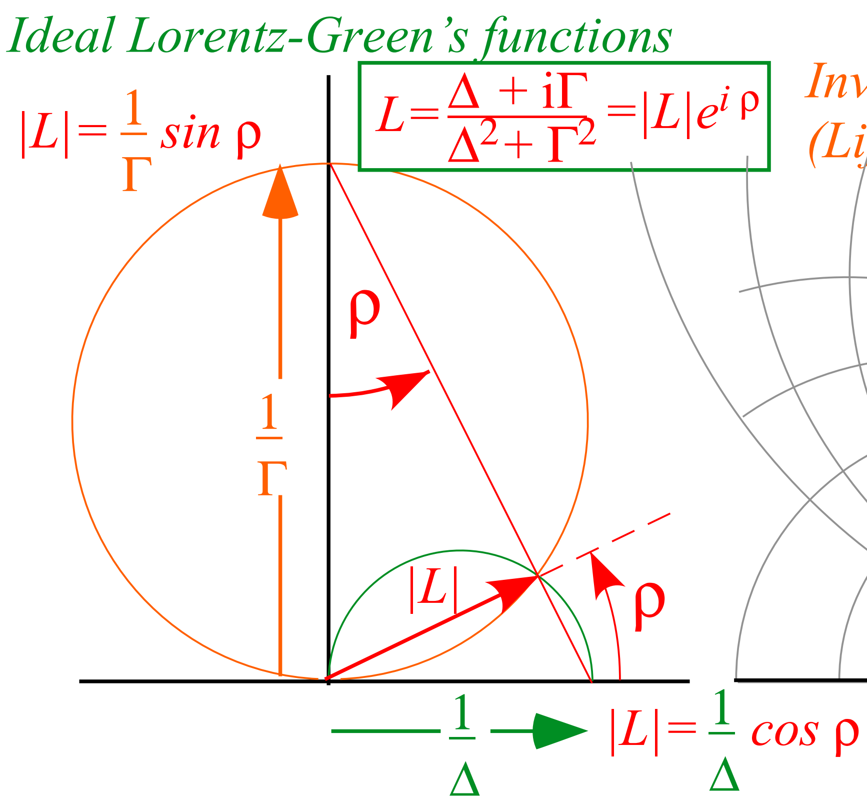
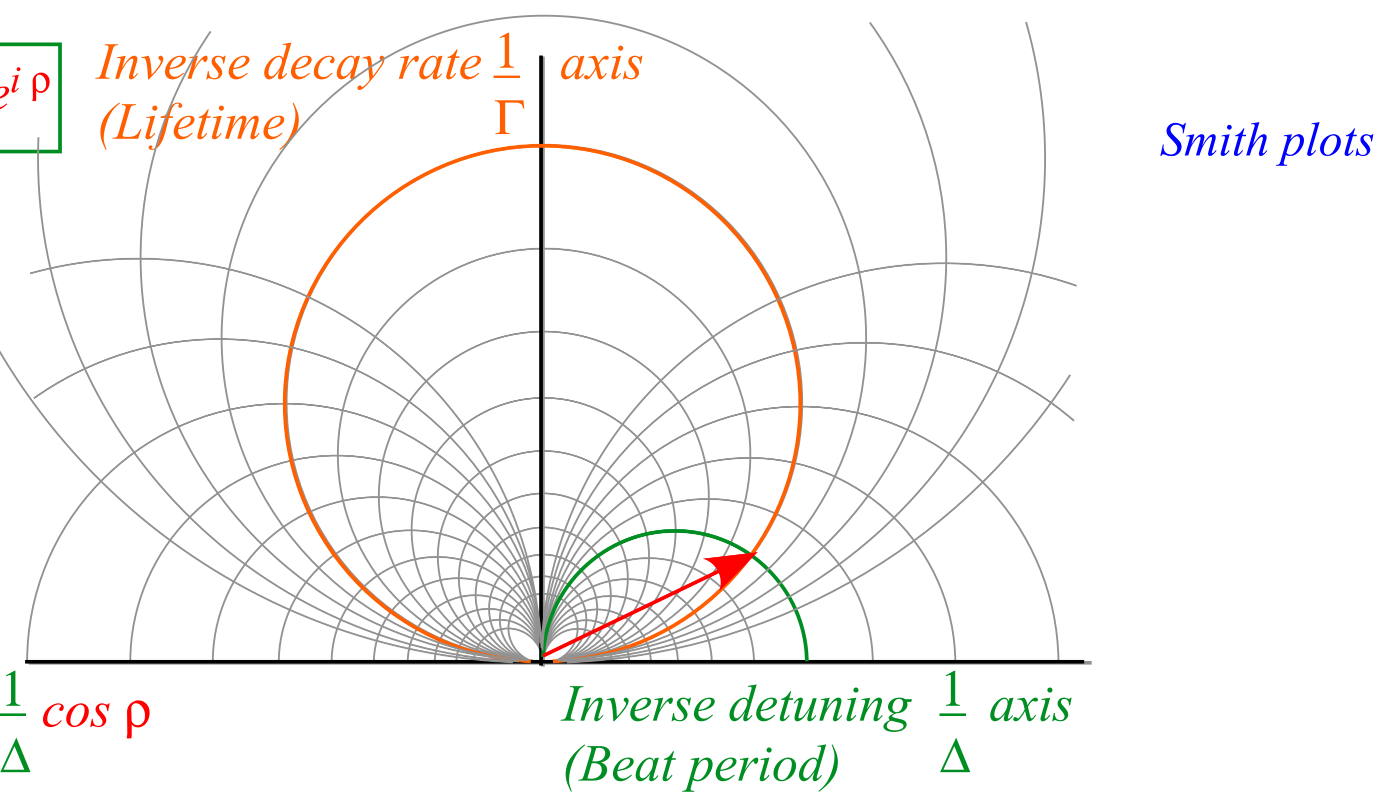
$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$





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$$= |L|e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$

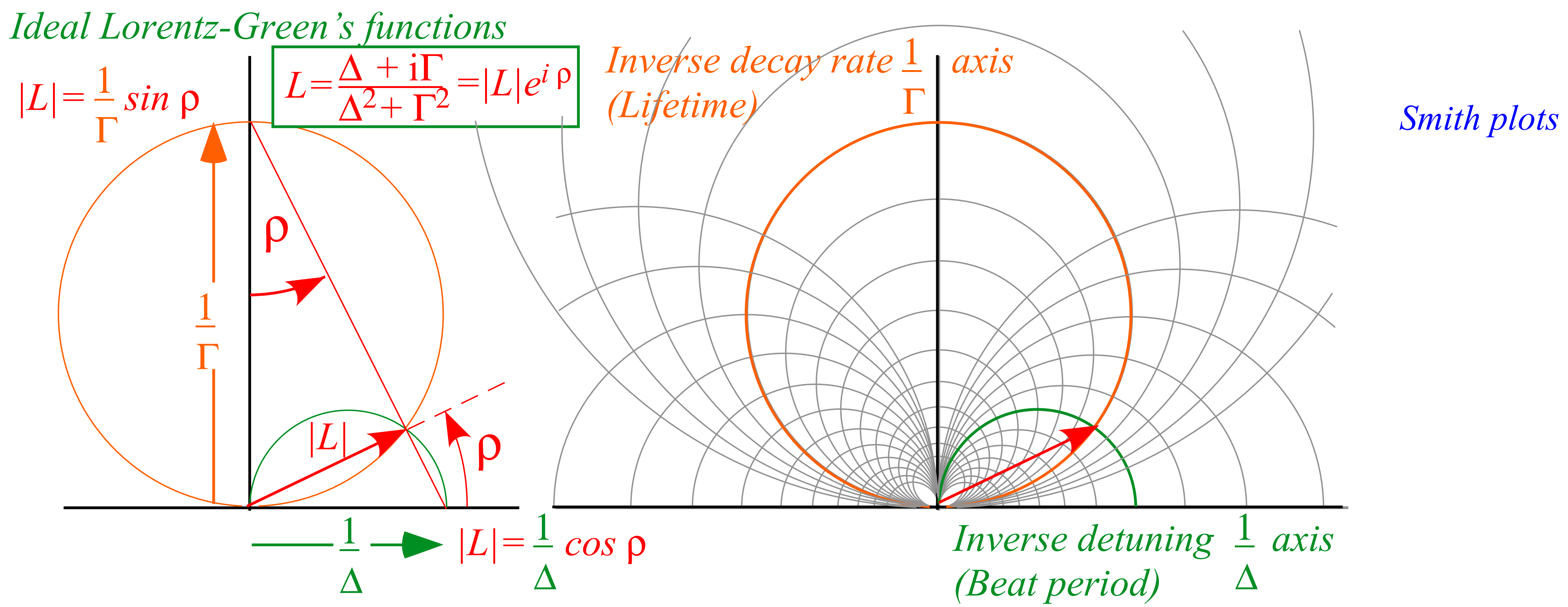


Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

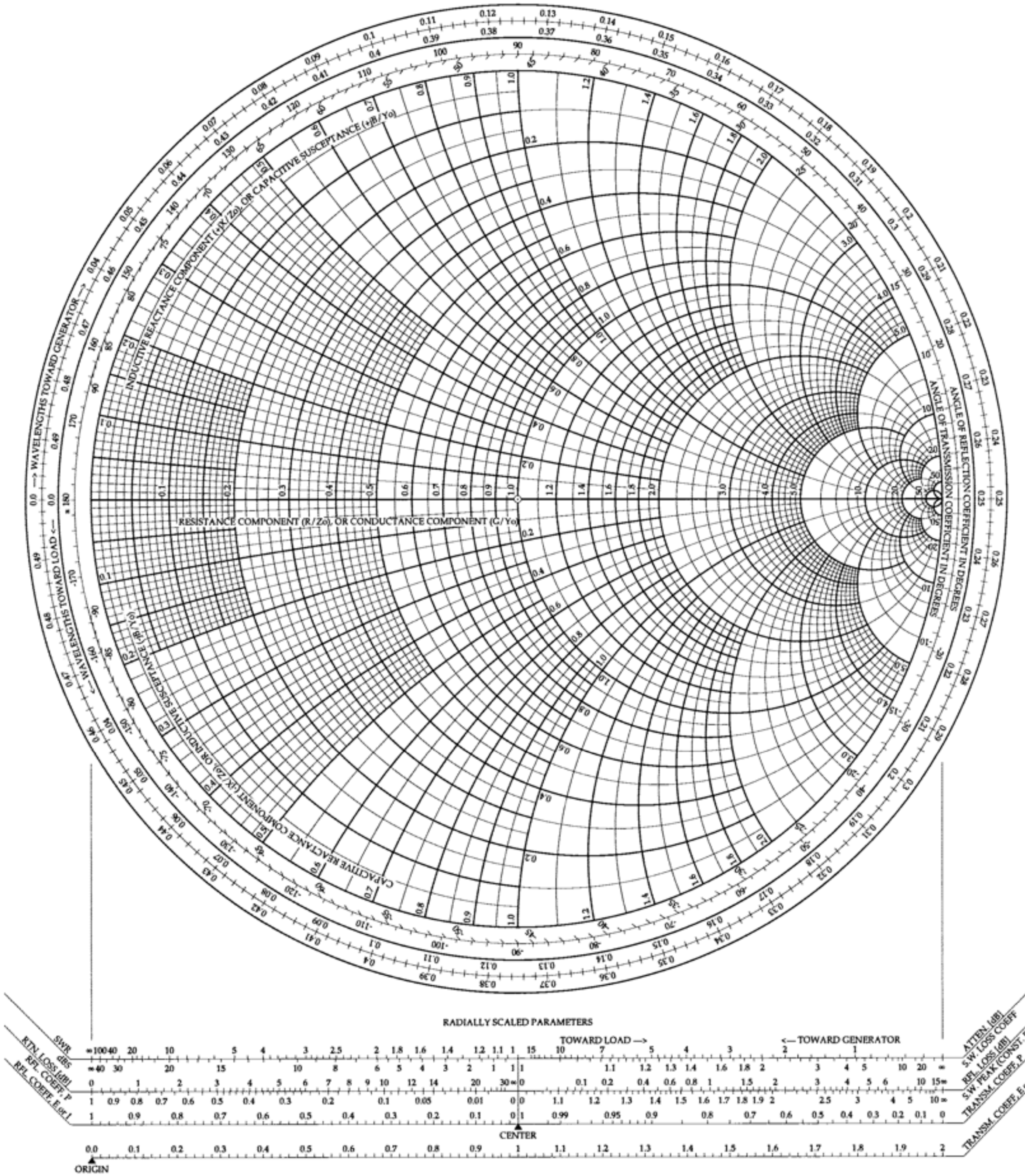
Constant Δ and Γ curves in Fig. 4.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

SMITH CHART (Invented by Phillip H. Smith 1905-1987)

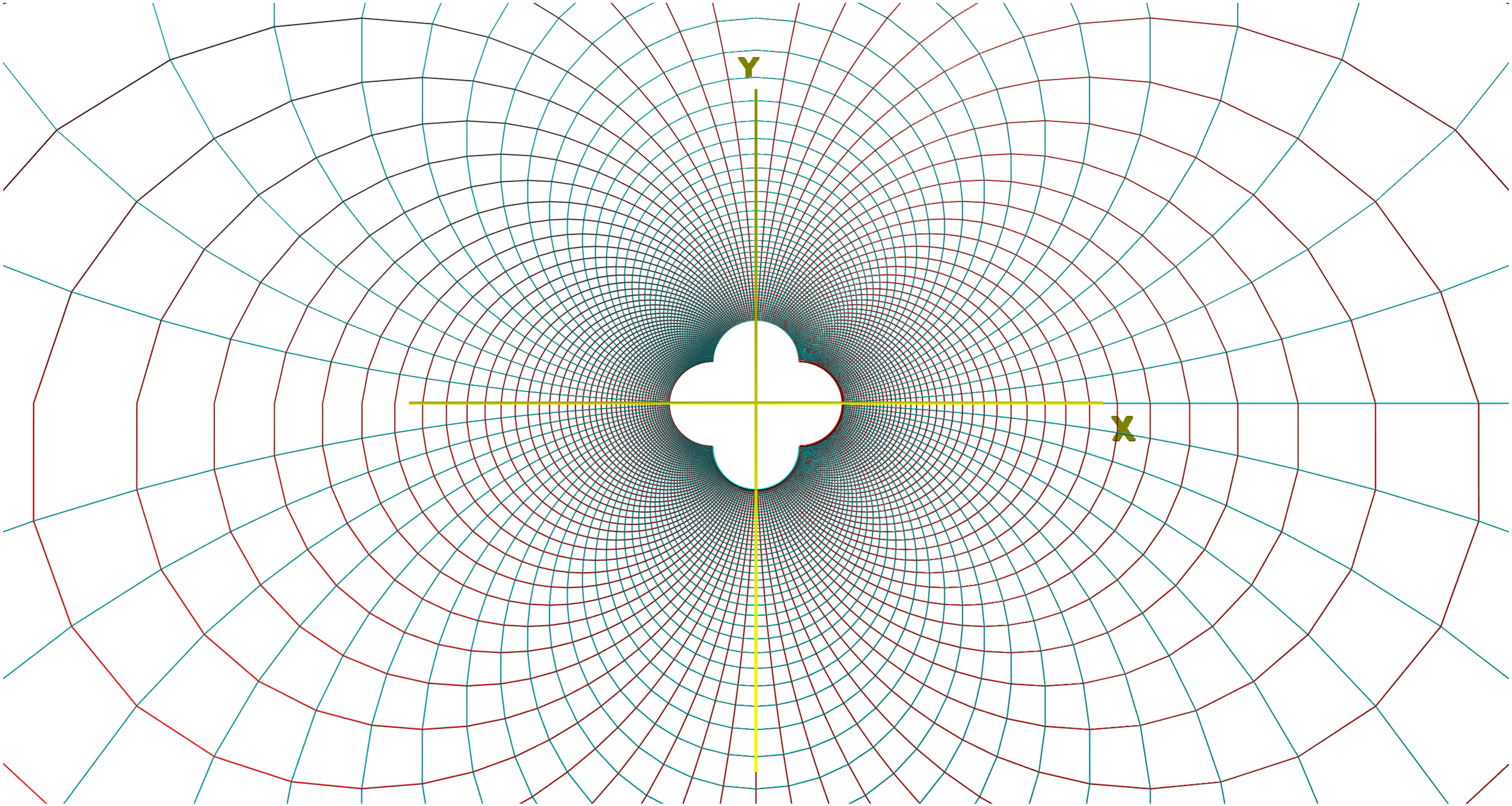
An FDHO Green's Function Slide rule

A plot of $f(z) = 1/z$

For wavy "Ohm's Laws"
$V = I \cdot Z$
$I = V/Z$



Smith plot: Graph paper



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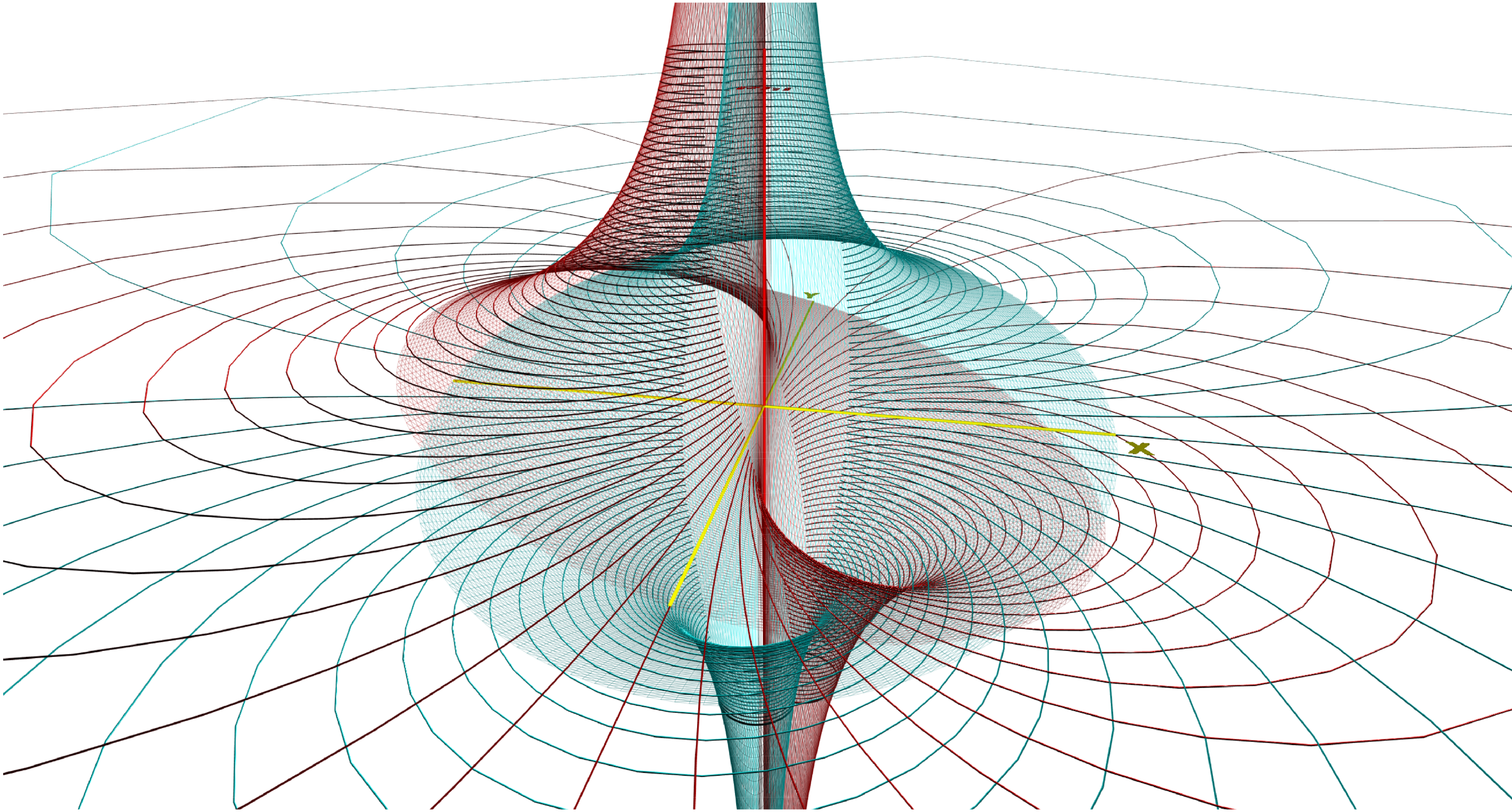
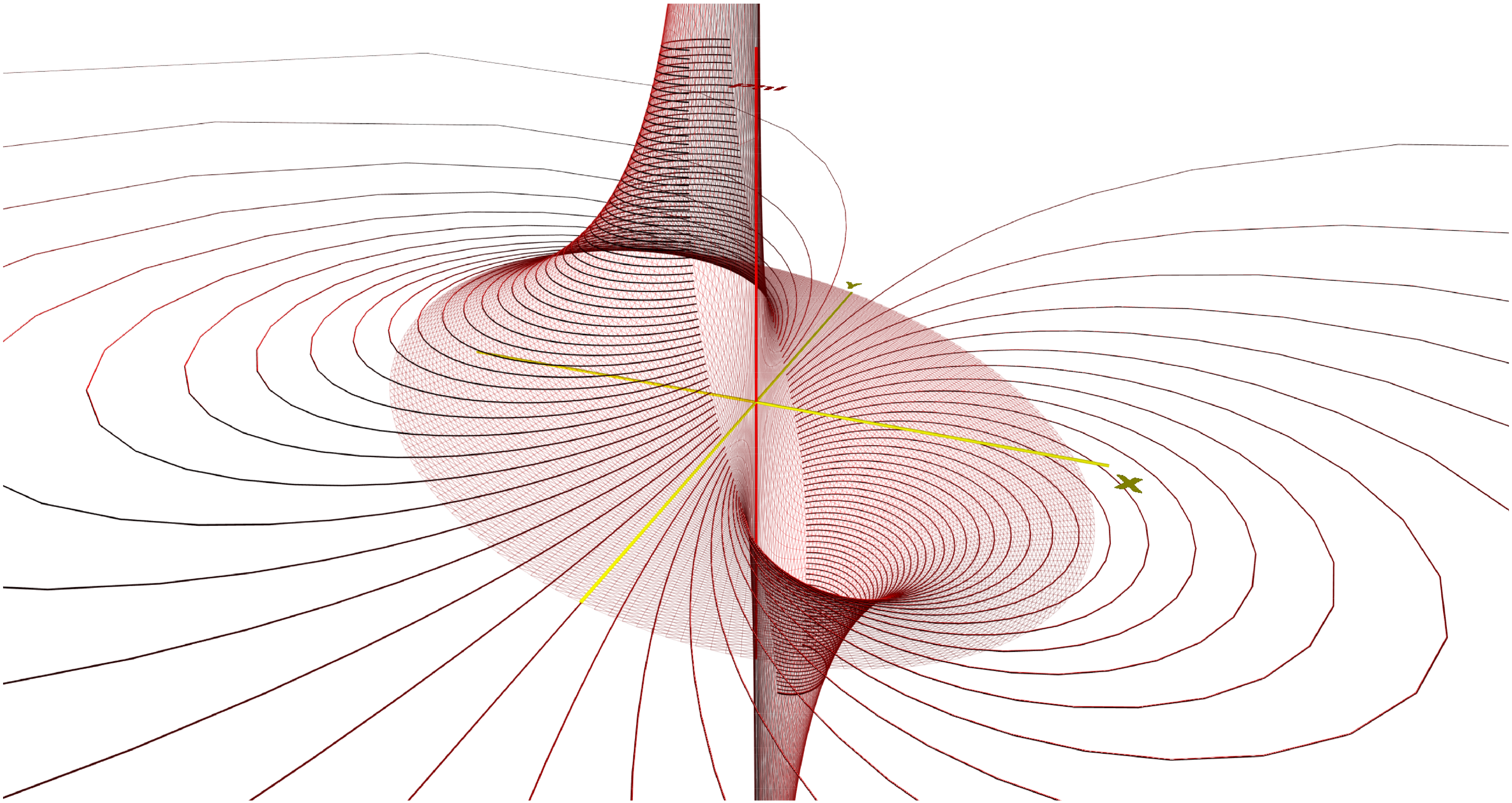
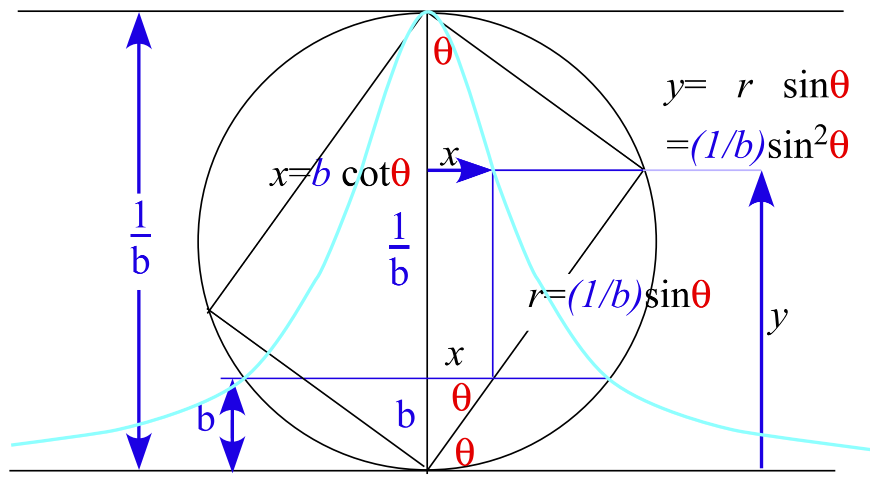

X



y
x

The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

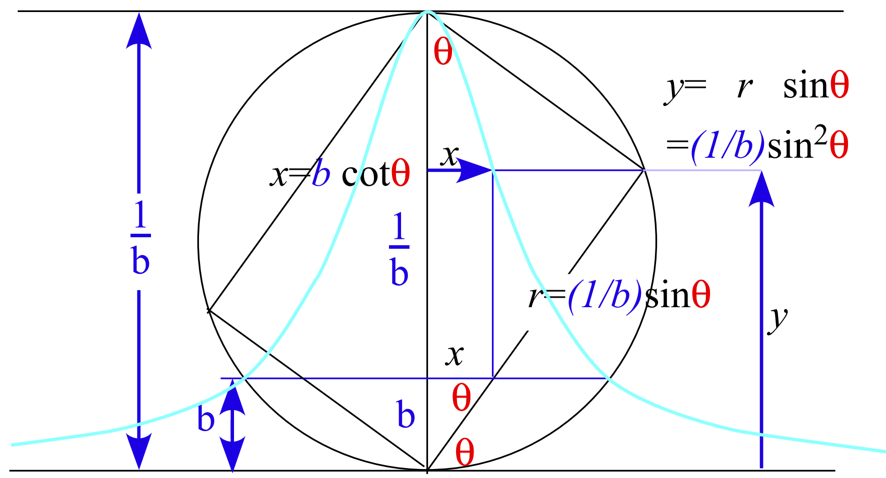
$$y=\frac{b}{x^2+b^2}$$

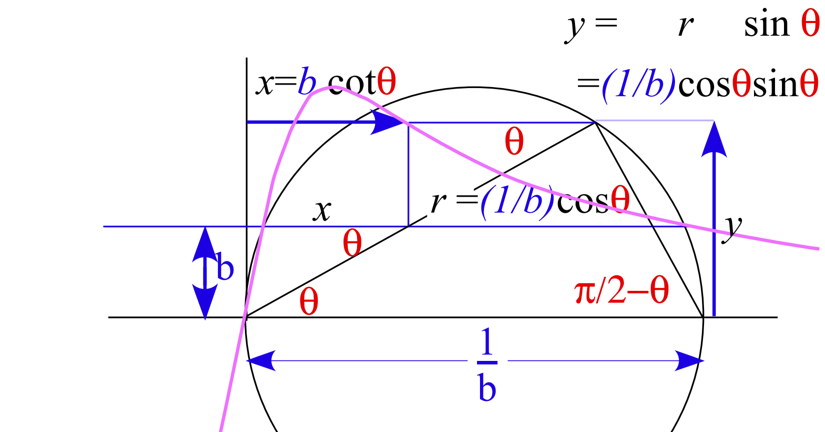
Common Lorentzian function I.
(imaginary "absorbtive" part)

Maria Gaetana Agnesi

Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics

The Common Lorentzian (a.k.a. *The Witch of Agnesi*)





$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

Common Lorentzian function I.
(imaginary "absorbtive" part)

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

Common Lorentzian function II.
(real "refractory" part)

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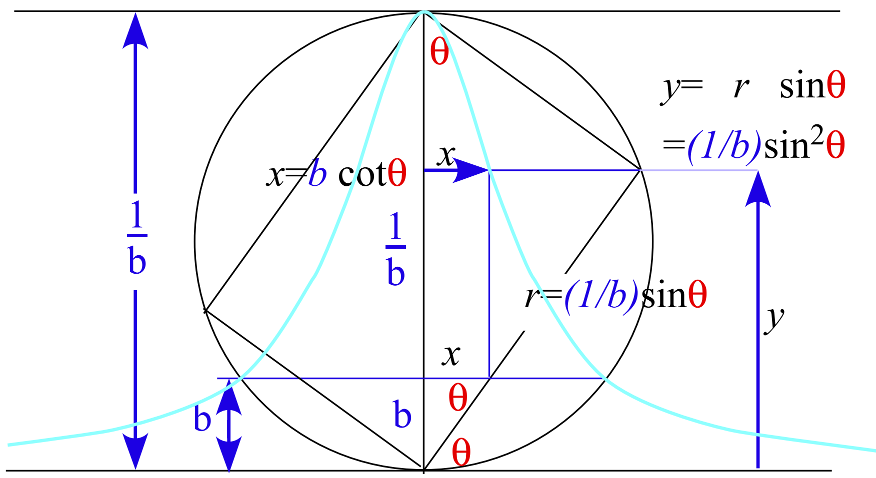
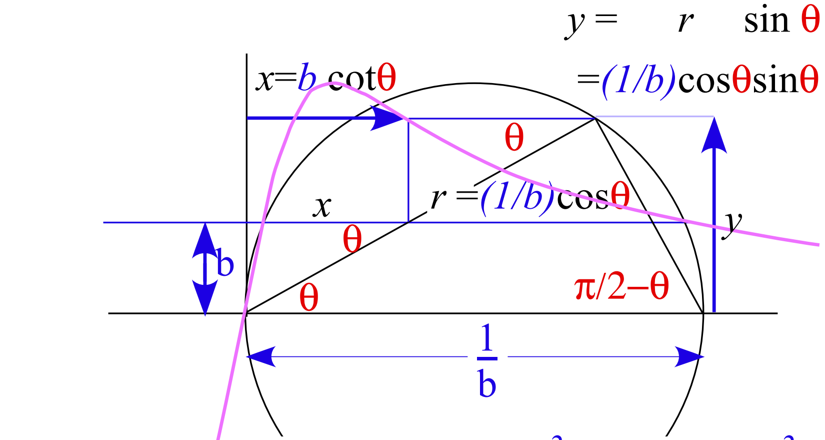
Fields	Mathematics

The Common Lorentzian (a.k.a. *The Witch of Agnesi*)





$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}\;b^2$

$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$

$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$

$$y=\frac{b}{x^2+b^2}$$

Common Lorentzian function I.
(imaginary "absorbtive" part)

$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$

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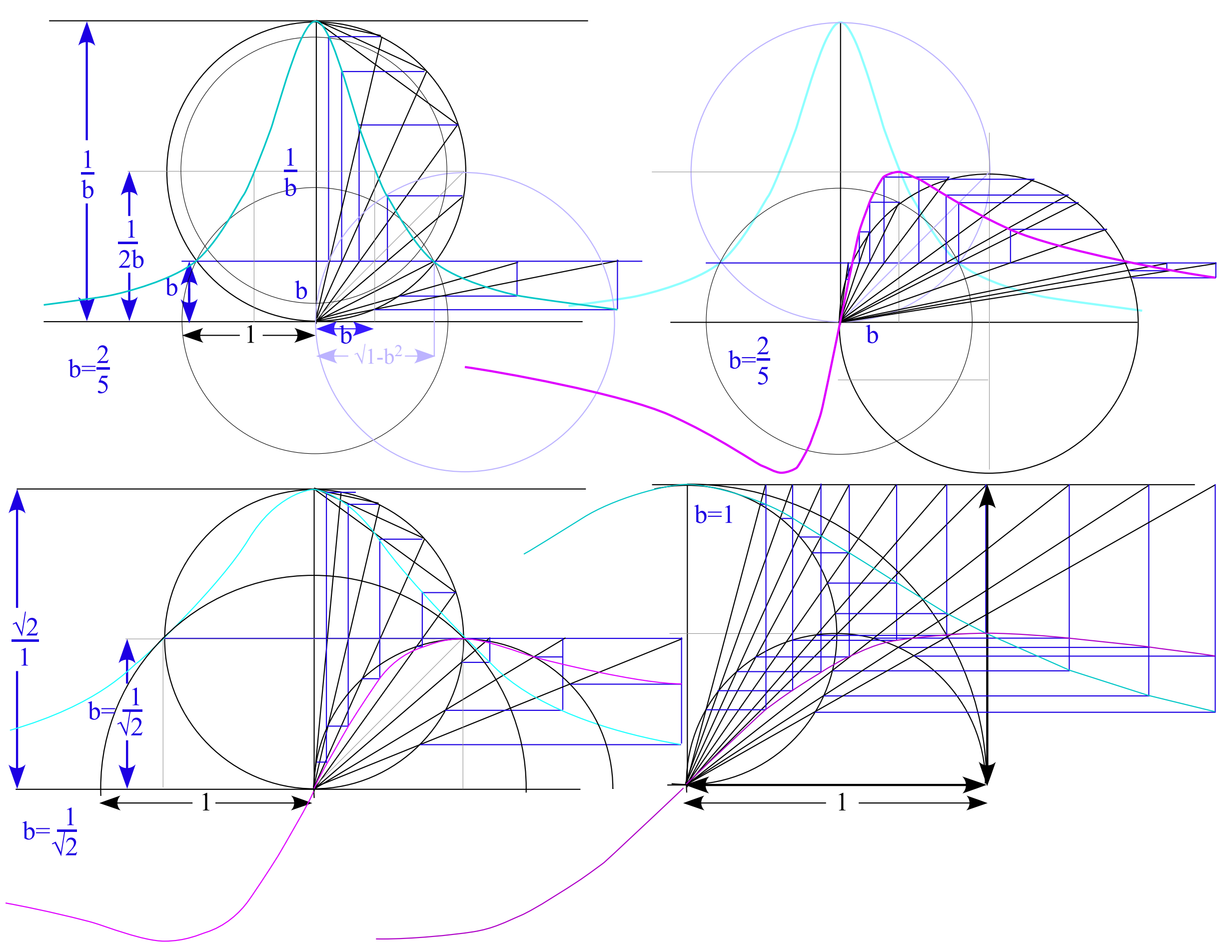
Common Lorentzian function II.
(real "refractory" part)

Maria Gaetana Agnesi

Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics



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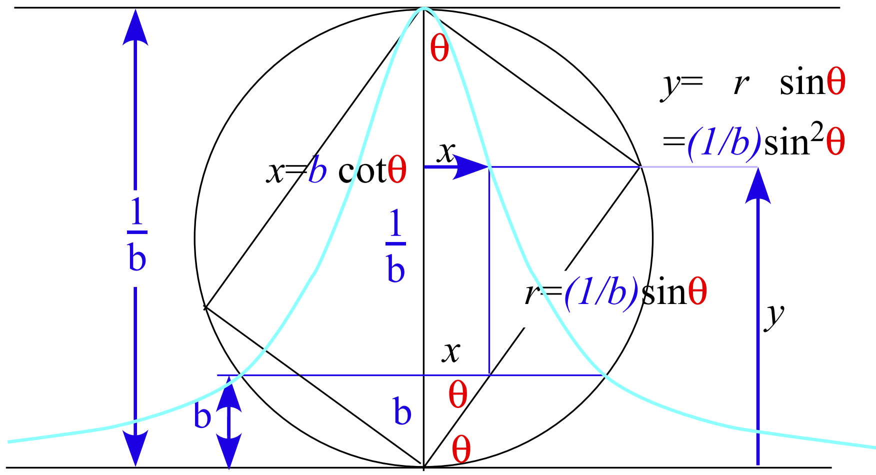

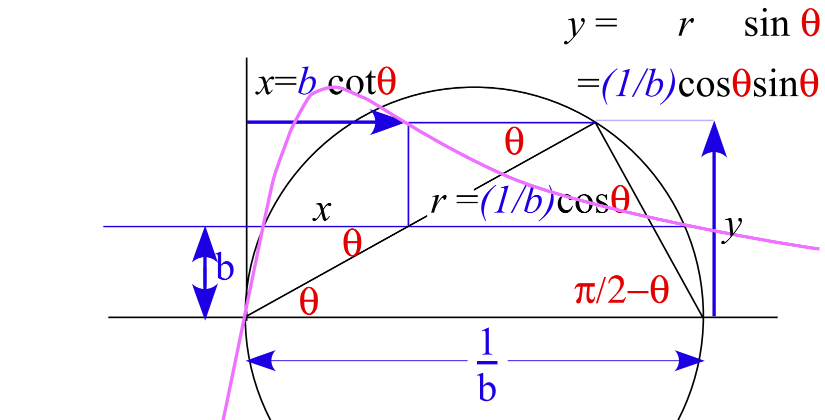

$$\frac{x}{y} = \frac{b\cot\theta}{(1/b)\cos\theta\sin\theta} = \frac{b^2\cos\theta}{\cos\theta\sin^2\theta} = \frac{b^2}{\sin^2\theta}$$

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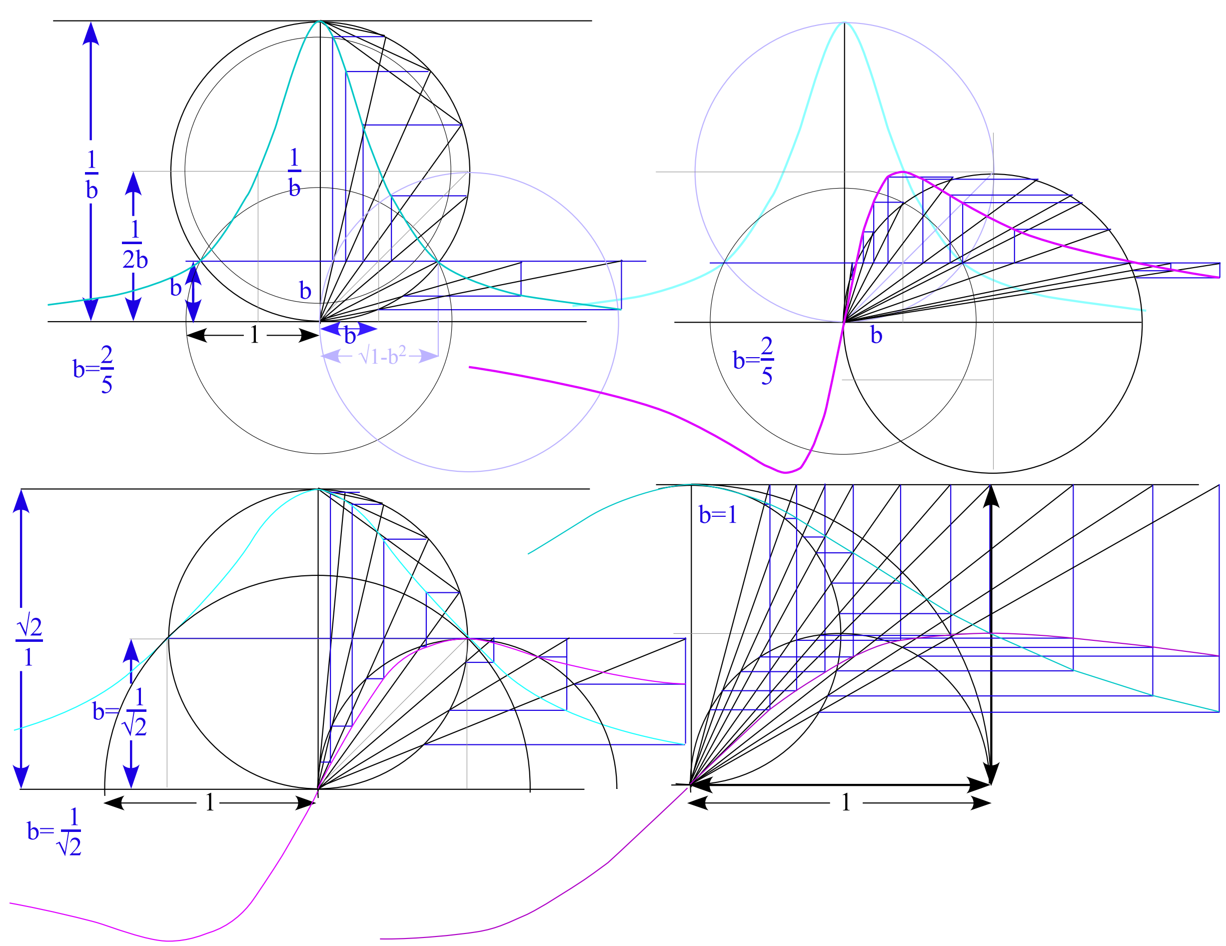
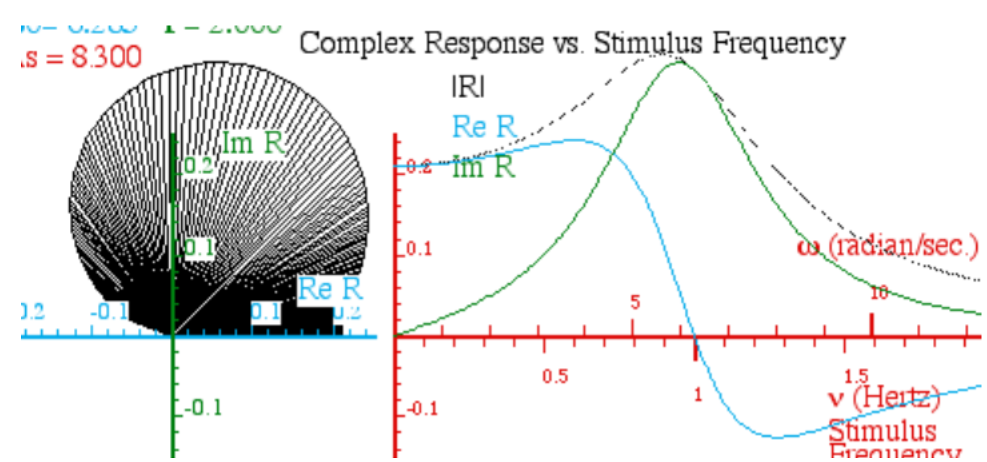
(real "refractory" part)



Underlined below are links to the OscillIt Web Simulations

Compare ideal Lorentzians (Γ=0.2)
with a very non-ideal one (Γ=2)



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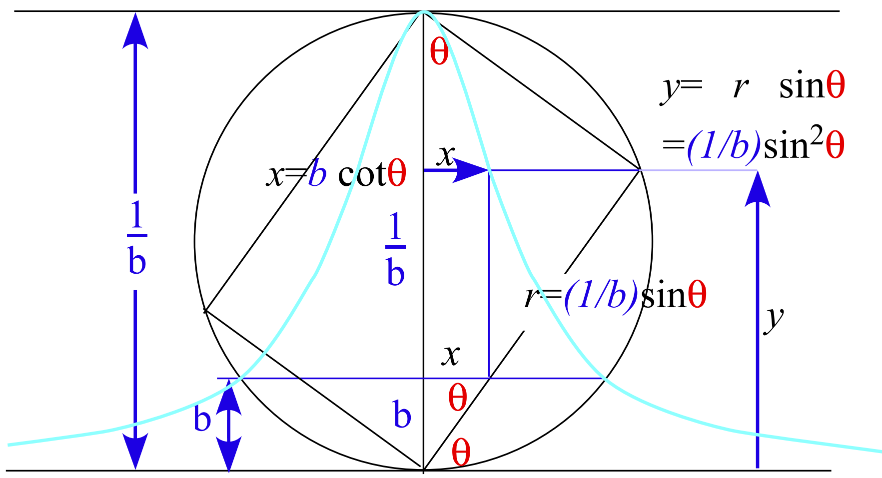

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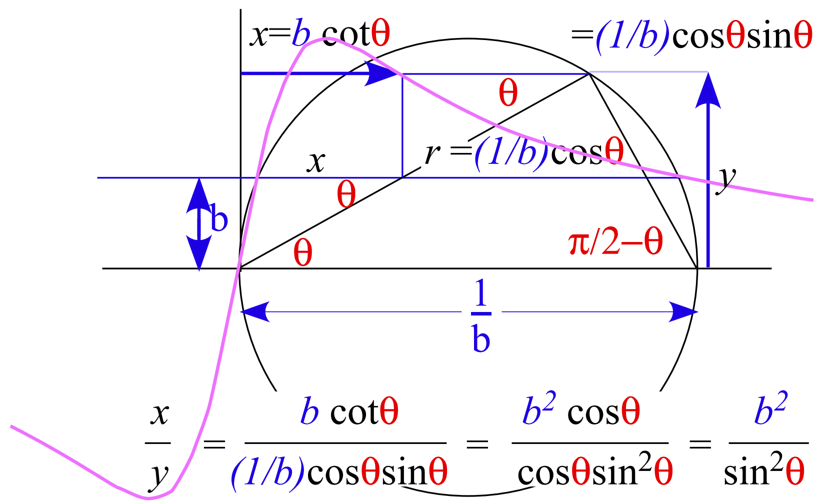

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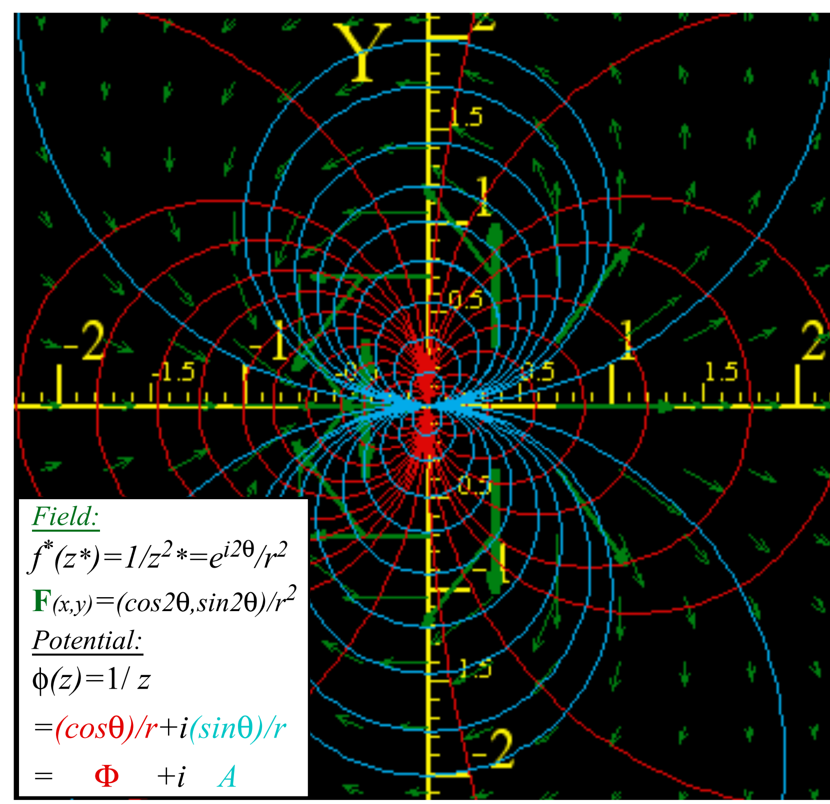


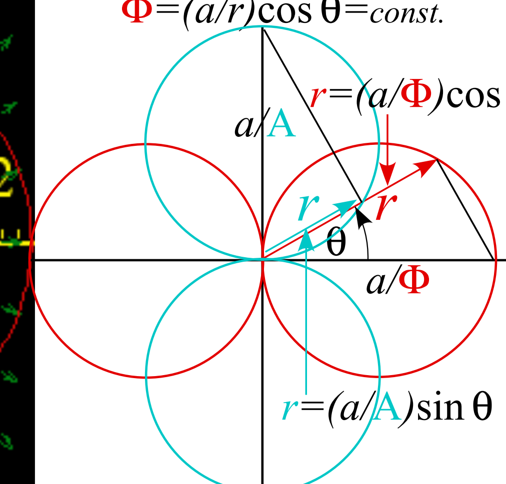
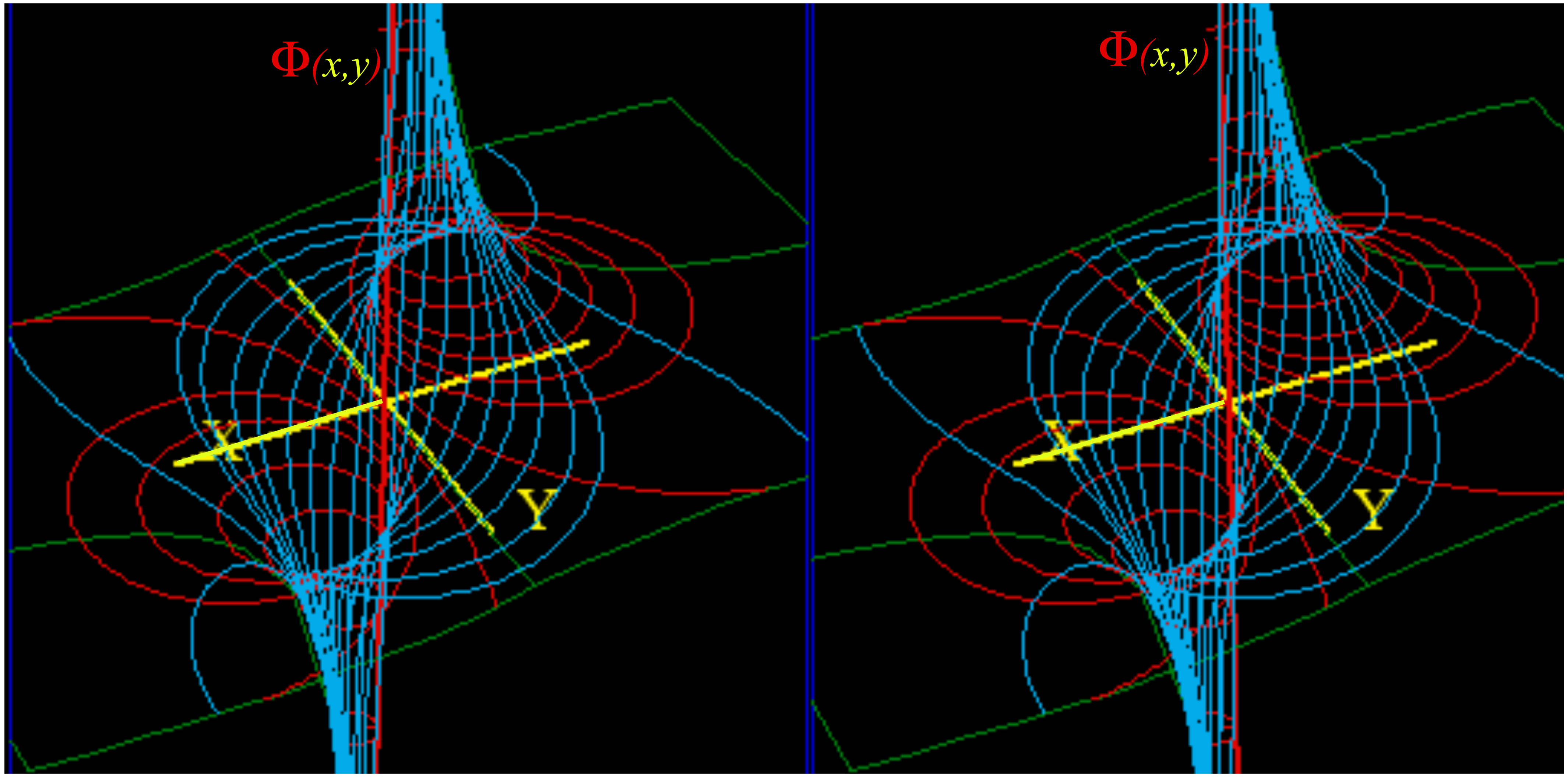
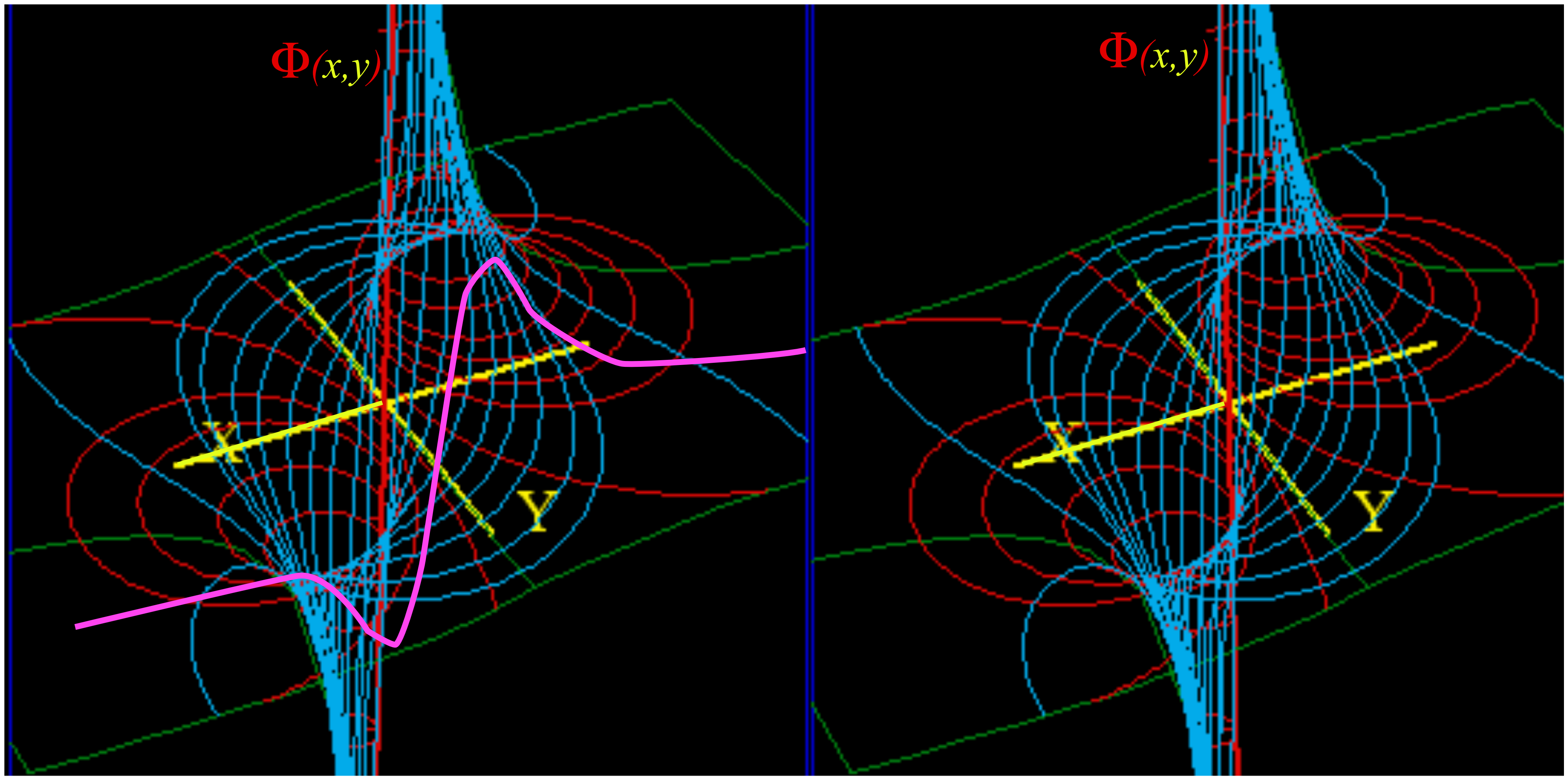


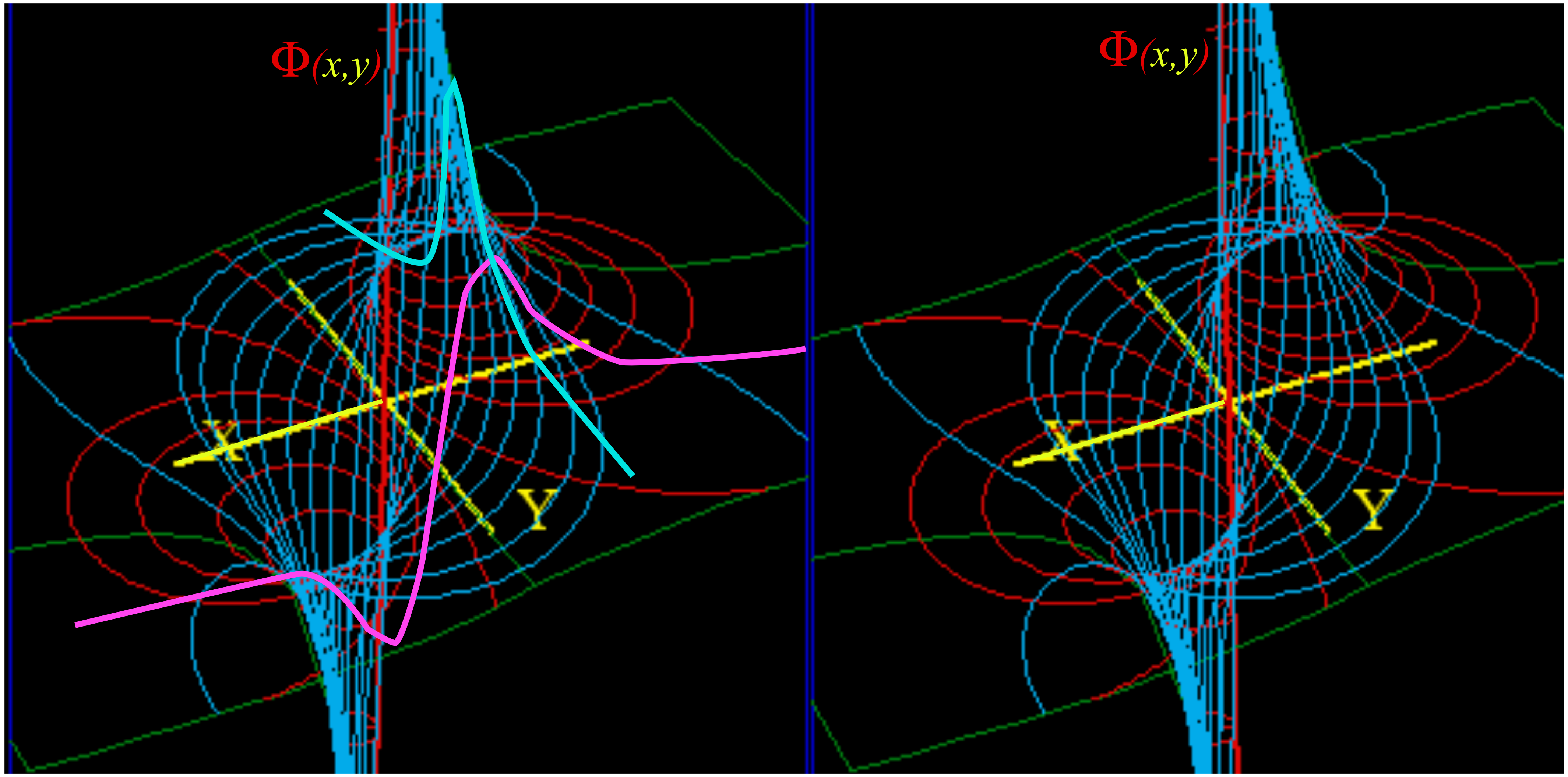
*Fig. 10.11 Dipole **F**-field $f(z)=1/z^2$ and scalar potential (Φ=const.)-circles orthogonal to (A=const.)-circles.*



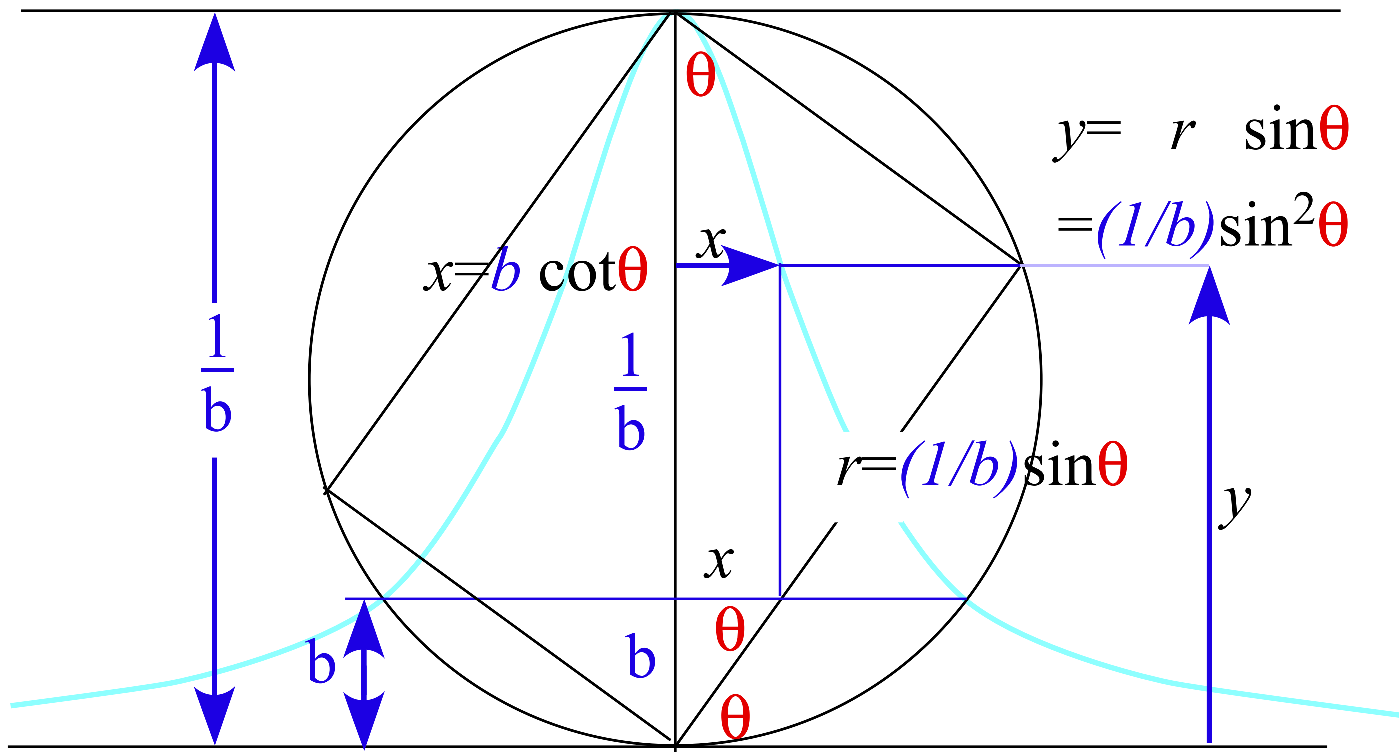
From: Fig. 1.10.12



From: Fig. 1.10.12



From: Fig. 1.10.12



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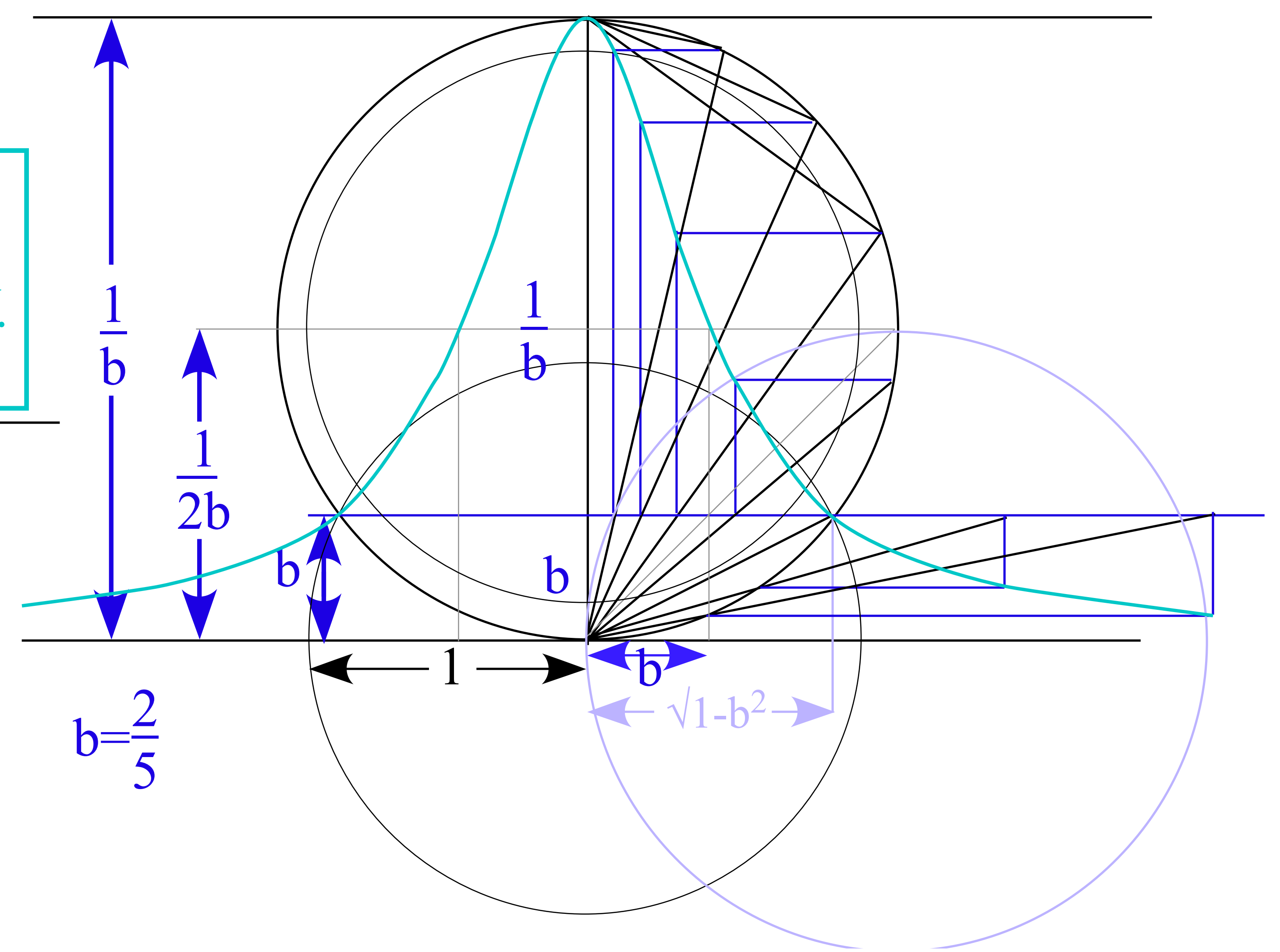
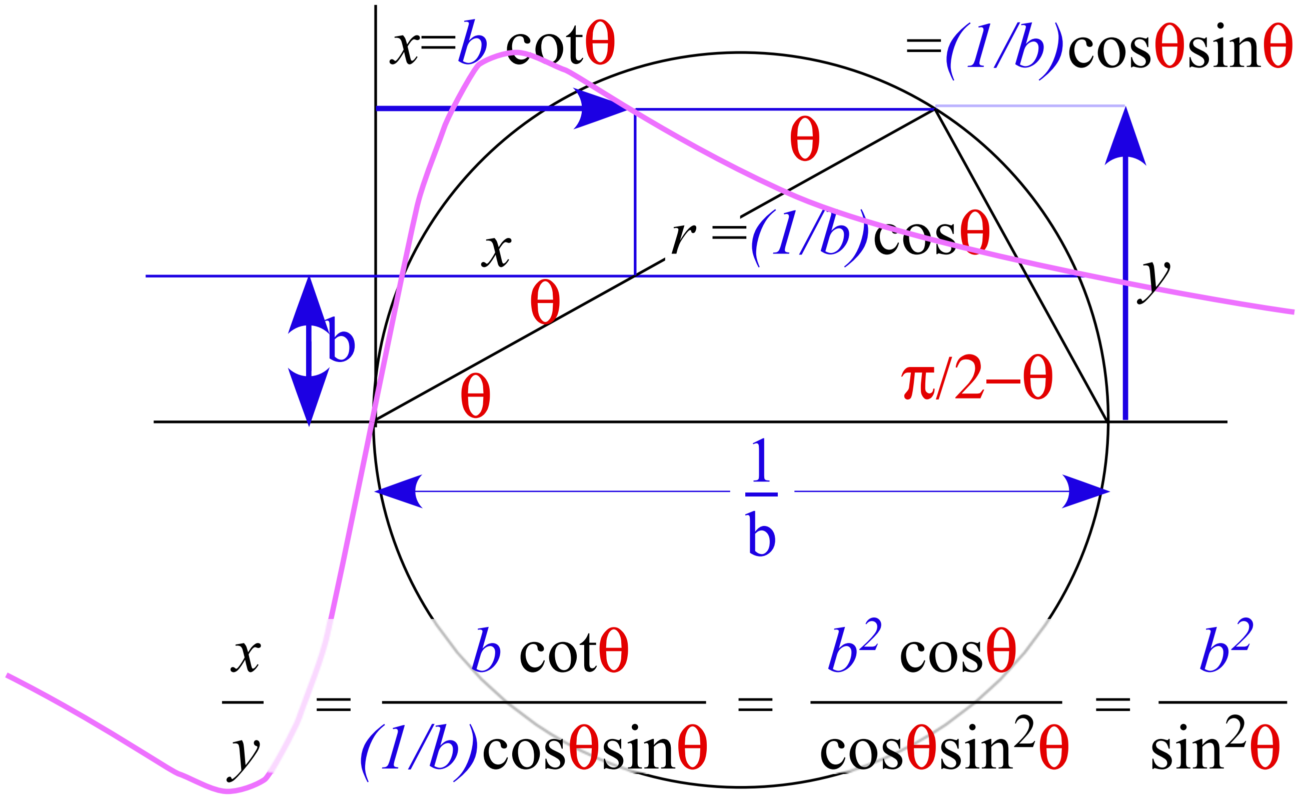


$$y = r \sin\theta$$
$$= (1/b)\cos\theta\sin\theta$$



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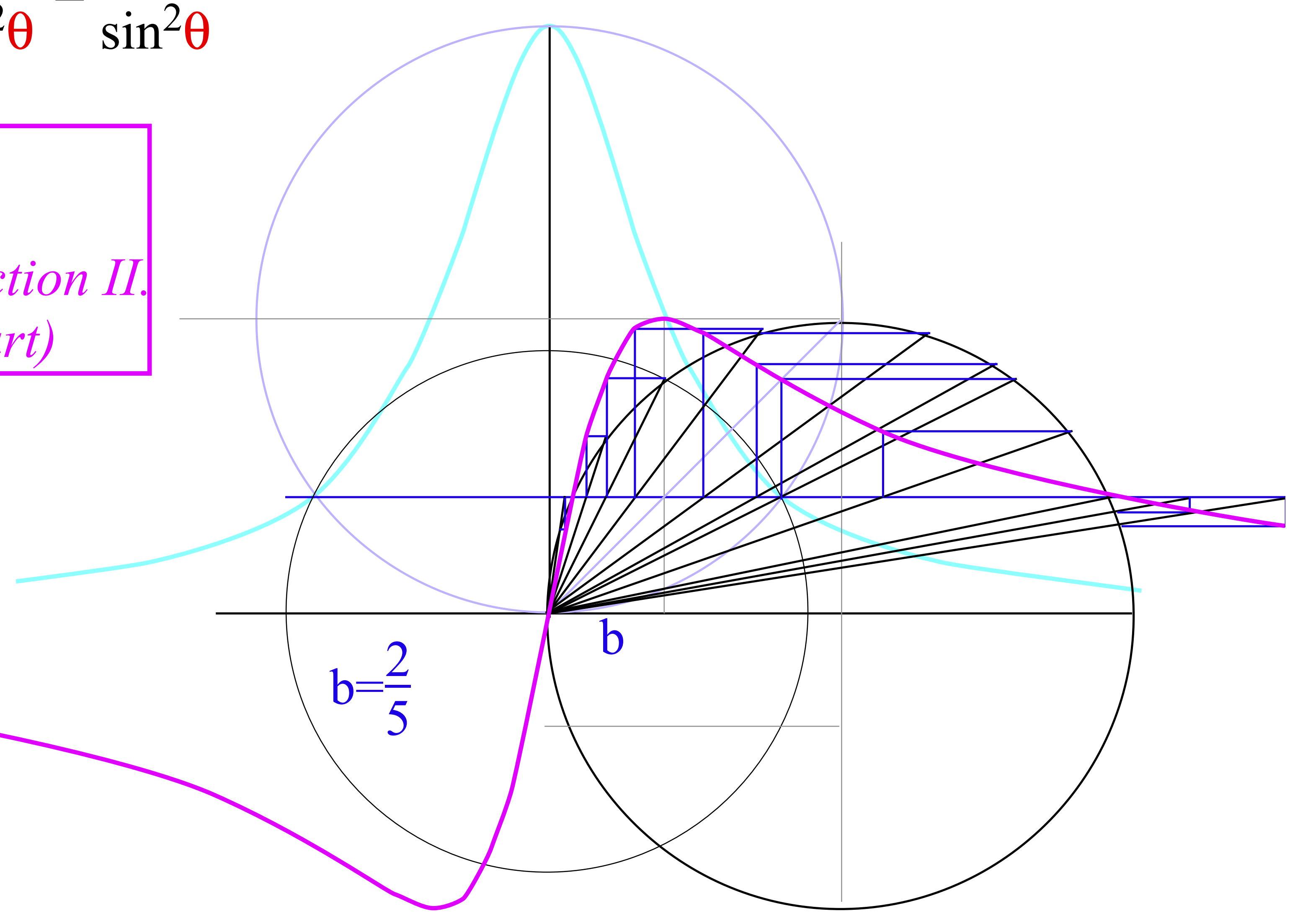

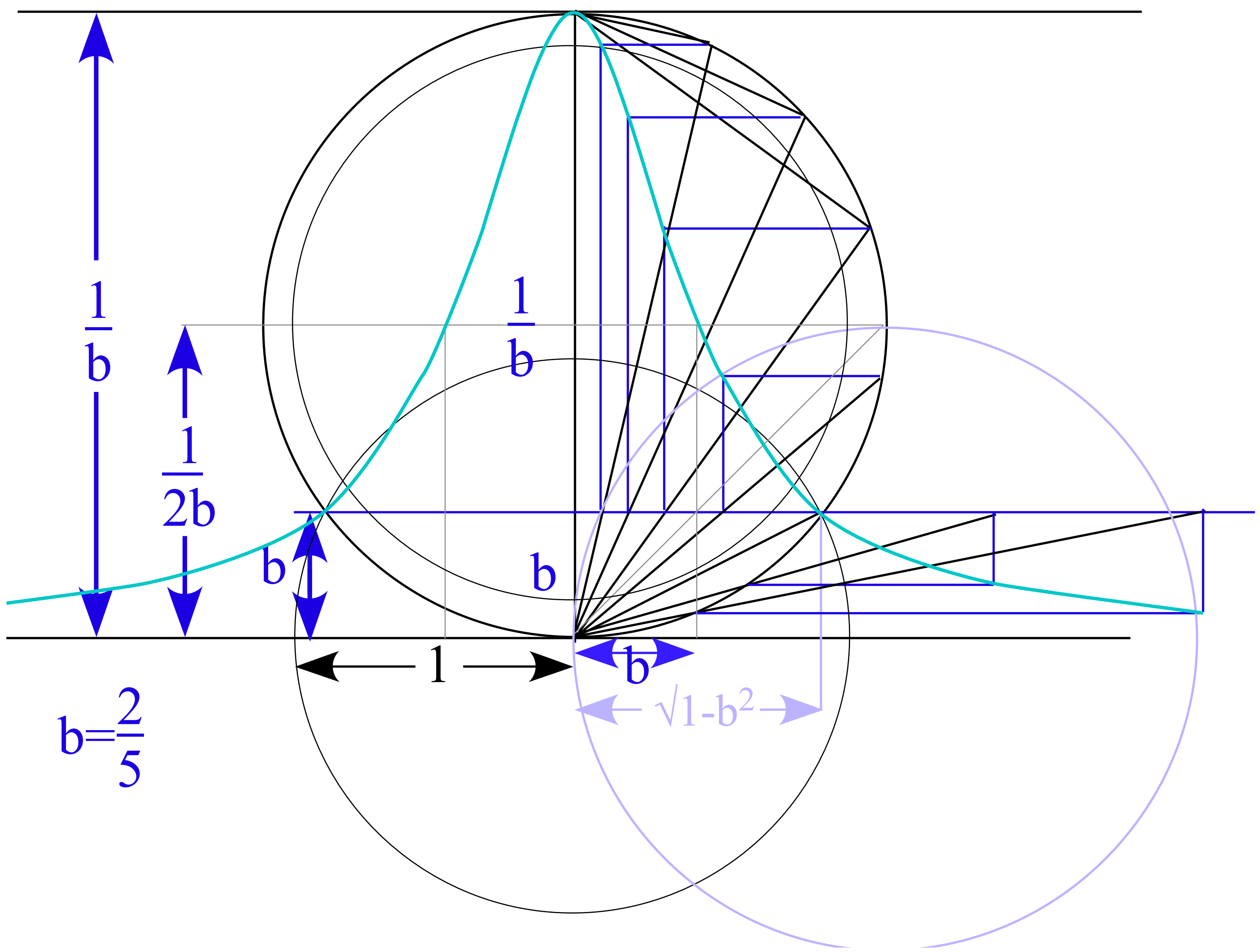

1/b
1/b
1/2b
b
b
1
b
√1-b²
b=2/5