Lecture 21

Tue. 11.04 to Thur. 11.06.2014

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- *Hamiltonian formulation*
- *Hamilton's equations*

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
$e=-1.602176\cdot 10^{-19}$*Coulombs*

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

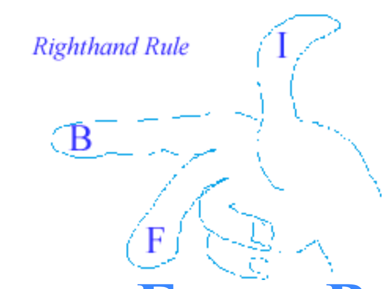
Electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$

scalar potential field $\Phi=\Phi(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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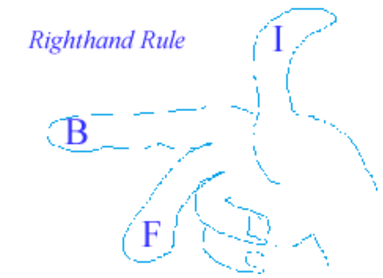

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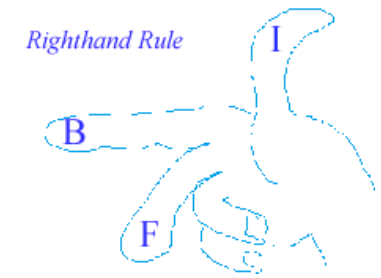

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Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

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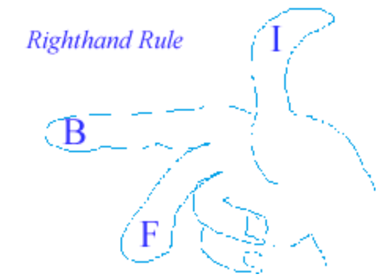

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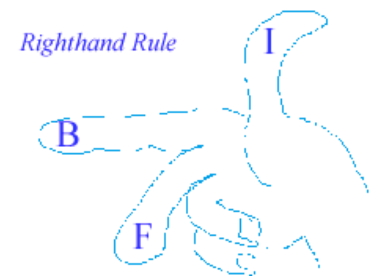

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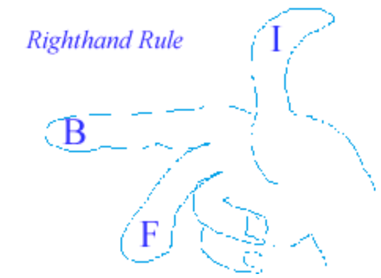

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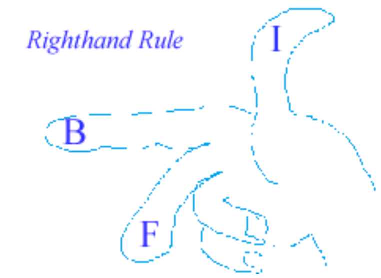

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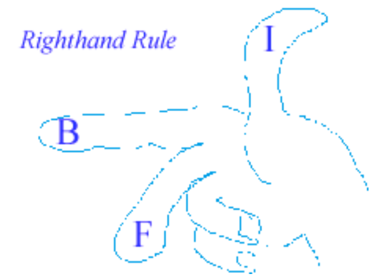

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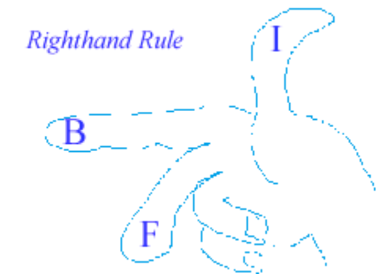

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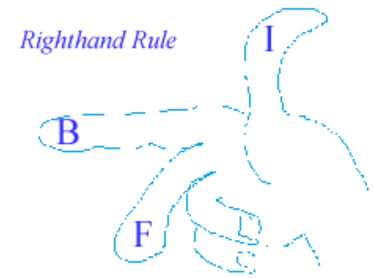

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Converting back to Gibbs's **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

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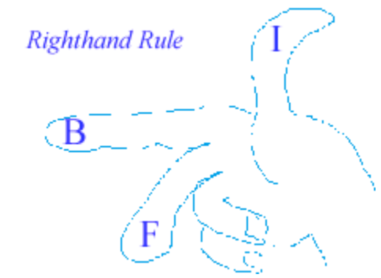

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&= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right) A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}
\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj} = \delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero. $\partial_k v^j \equiv 0$ *iff* : $\nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial\mathbf{r}} = \mathbf{0}$

Vector analysis for particle-in-(A,Φ)-potential

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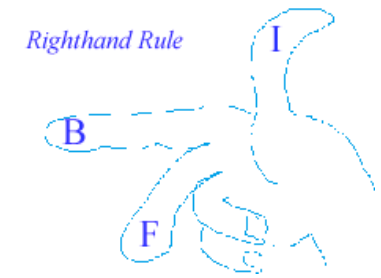

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\varepsilon_{ikj}=\varepsilon_{kij}=1$$
$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i(\nabla\times\mathbf{A})_j$$
$$=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$
$$=\varepsilon_{kij}\varepsilon_{abj}\,v_i\left(\partial_a A_b\right)$$
$$=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$
$$=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)$$
$$=v_b\left(\partial_k A_b\right)-v_a\left(\partial_a A_k\right)$$
$$=\left(\partial_k A_b\right)v_b-v_a\left(\partial_a A_k\right)=(\nabla\mathbf{A})\bullet\mathbf{v}-\mathbf{v}\bullet\nabla\mathbf{A}$$
$$=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-(\nabla\mathbf{v})\bullet\mathbf{A}-\mathbf{v}\bullet\nabla\mathbf{A}$$

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$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-\quad 0\quad-\mathbf{v}\bullet\nabla\mathbf{A}\qquad \textit{for particle mechanics}$$

Summary of Vector analysis for particle-in-$(\mathbf{A},\Phi)$-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k} v_j = (\partial_k A_j) v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j \frac{\partial A_k}{\partial x_j} = (v_j \partial_j A_k)$$

$$\left[\nabla(\mathbf{A}\bullet\mathbf{v})\right]_k = \left[(\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}\right]_k$$

$$\partial_k\left(A_b v_b\right) = \left(\partial_k v_b\right) A_b - \left(\partial_k v_a\right) A_a$$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential*

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- *Hamiltonian formulation*
- *Hamilton's equations*

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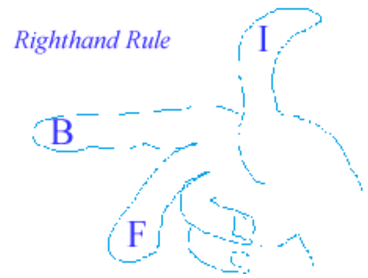

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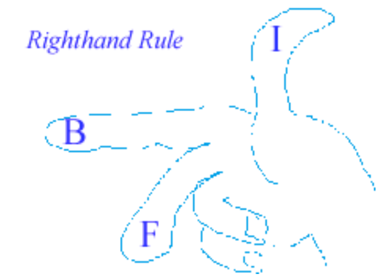

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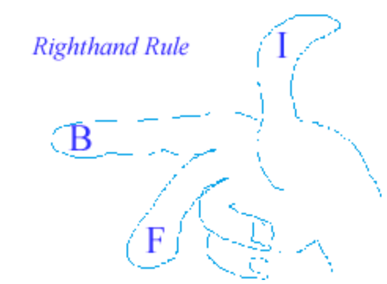

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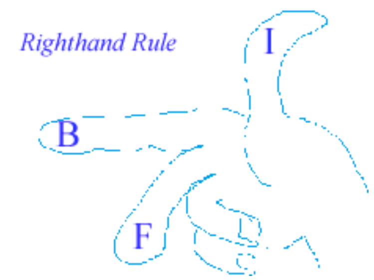

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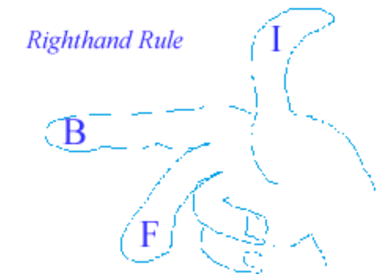

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

Inserting Φ-term that $\partial_{\mathbf{v}}$ zeros :

This step requires that: $\frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0$

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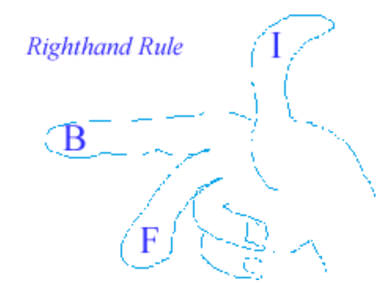

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This step requires that :

$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)\equiv\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

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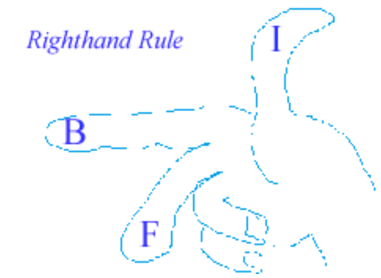

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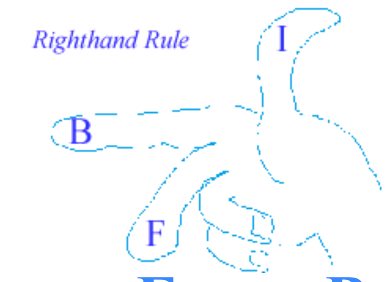

F=*q*v×B=I×B

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) = -e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)}_{} = \frac{\partial}{\partial\mathbf{r}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)}_{}$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}} = \frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv²/2* and *PE=e*Φ.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*

*Lagrangian for particle-in-(**A**,Φ)-potential*

*Hamiltonian for particle-in-(**A**,Φ)-potential*

*Canonical momentum in (**A**,Φ) potential*

Hamiltonian formulation

Hamilton's equations

Hamiltonian for particle-in-(A,Φ)-potential

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Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

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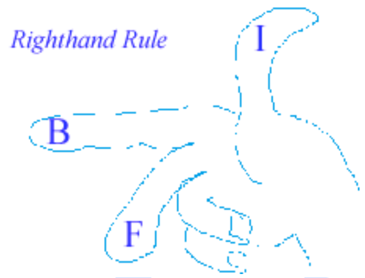

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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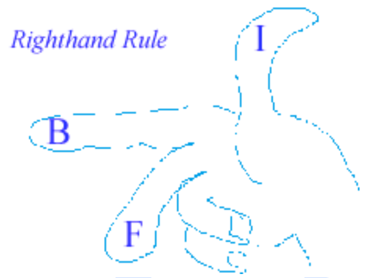

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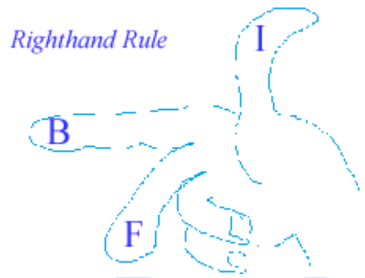

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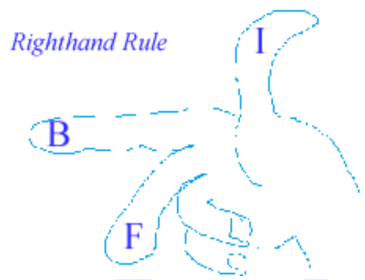

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Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\quad$ $(For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term $-\mathbf{v}\cdot e\mathbf{A}$ leads to an extraordinary *canonical momentum:* $\mathbf{p} = m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.
Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* $\mathbf{p}$ as follows: $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

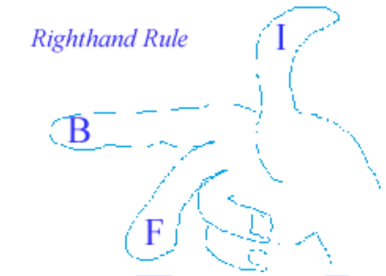
Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + e\mathbf{A}(\mathbf{r},t)\right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

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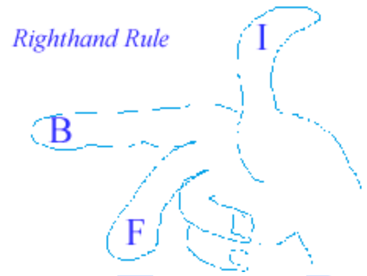

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$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+\cancel{e\mathbf{A}(\mathbf{r},t)}\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet\cancel{e\mathbf{A}(\mathbf{r},t)}\right)\right)$$

$$H=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}+e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

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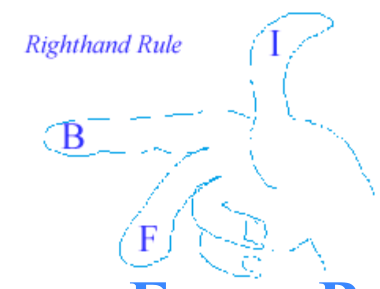

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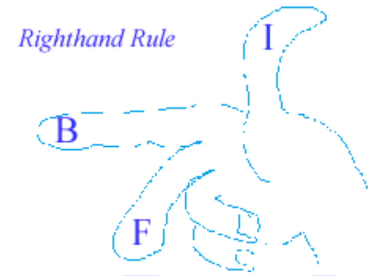
Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

Hamiltonian for charged particle in fields



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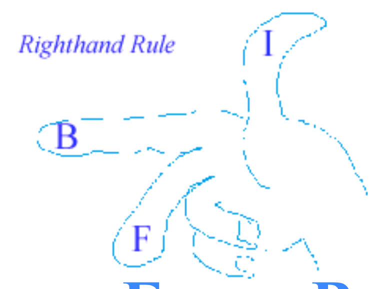

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Hamiltonian for charged particle in fields

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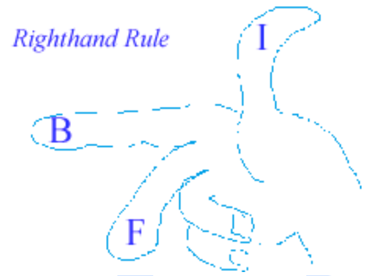

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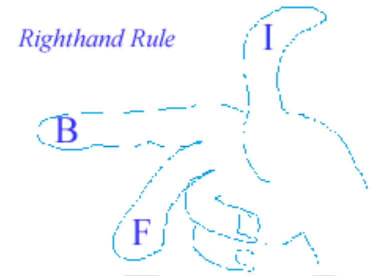

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→ *Hamilton's equations*

Hamilton's equations for charged particle in fields

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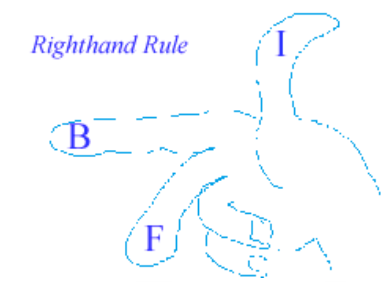
(Correct formally and numerically)

$$H = \frac{\mathbf{p} \bullet \mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p} \bullet \mathbf{A} + \mathbf{A} \bullet \mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A} \bullet \mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)



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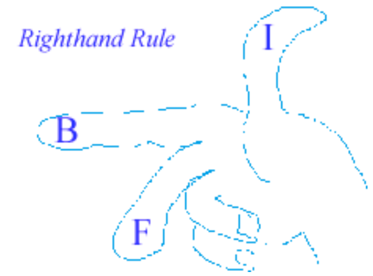
(Expanded)



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Hamilton's $\mathbf{v}$ equation: $$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$

Hamilton's equations for charged particle in fields

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(Expanded)

Hamilton's **v** equation: $\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

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Hamilton's *d***p***/dt* equation: $\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$

(In index notation.)

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(Correct formally and numerically)

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(Expanded)

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(Just copies particle velocity relation.)

Hamilton's $d\mathbf{p}/dt$ equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\frac{\left(p_\mu-eA_\mu\right)}{m}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

Hamilton's equations for charged particle in fields

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Hamilton's $d\mathbf{p}/dt$ equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}_{v_\mu}e\frac{\partial A_\mu}{\partial x_a}-e\frac{\partial\Phi}{\partial x_a}$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\frac{\partial A_a}{\partial t}+E_a\right)$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

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Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum* p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$ (Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$ *(Expanded)*

Hamilton's **v** equation: $$\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$$ *(Just copies particle velocity relation.)*

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(In index notation.)

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$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\boxed{\dot{A}_a-v_\mu\frac{\partial A_a}{\partial x_\mu}}+E_a\right)$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

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(Correct formally and numerically)

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(Expanded)

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$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\underbrace{\frac{\partial A_a}{\partial t}}+E_a\right)$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\dot{p}_a=m\dot{v}_a+\cancel{e\dot{A}_a}=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\boxed{\cancel{\dot{A}_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}+E_a\right)$$

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$$m\dot{v}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}+E_a\right)$$

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$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

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$$E_a=-\frac{\partial \Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$$-\frac{\partial \Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

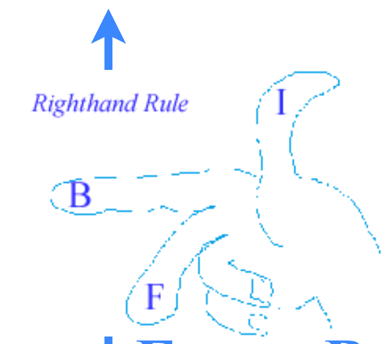
$$\frac{\partial \mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\frac{\partial A_a}{\partial t}=\boxed{\dot{A}_a-\sum_\mu v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of ***momentum*** $\mathbf{p}$. Must replace velocity $\mathbf{v}$ using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\underbrace{\frac{\partial A_a}{\partial t}}+E_a\right)$$

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$$\frac{\partial A_a}{\partial t}=\boxed{\dot{A}_a-\sum_\mu v_\mu\frac{\partial A_a}{\partial x_\mu}}$$

...and now

$$m\dot{v}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}+E_a\right)$$

we come back

$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$ $$\mathbf{B}=\nabla\times\mathbf{A}$$

full circle...

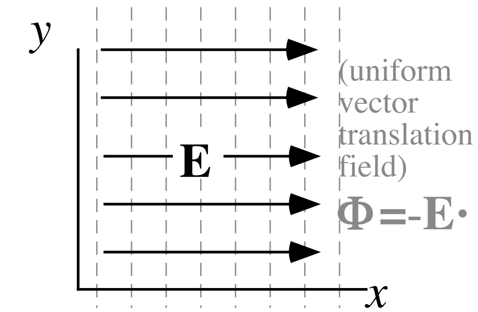
$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\mathbf{v}\bullet\left(\nabla\mathbf{A}\right)-\left(\mathbf{v}\bullet\nabla\right)\mathbf{A}$$ *for particle mechanics*

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = \nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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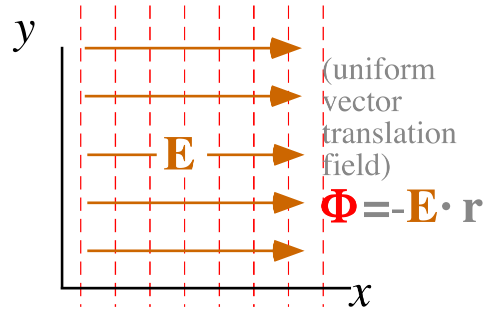

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A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

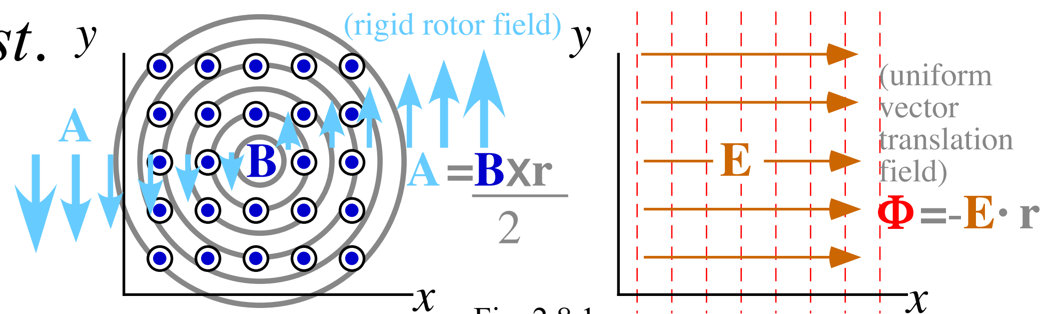

Fig. 2.8.1.

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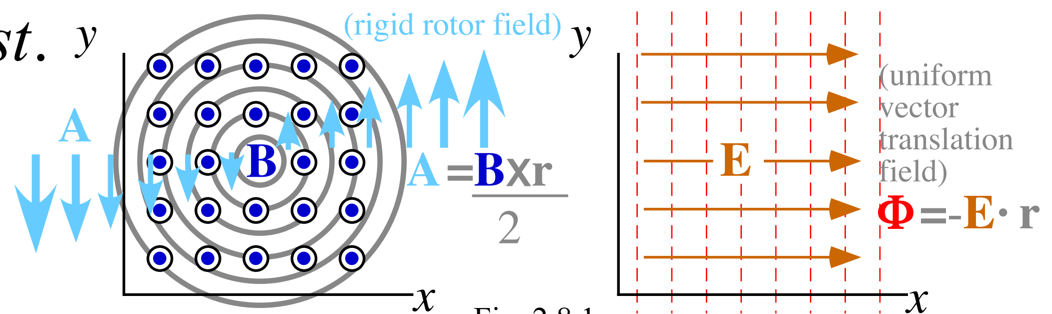
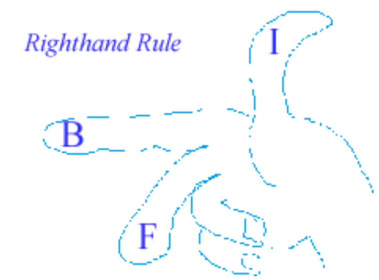


Fig. 2.8.1.

Newtonian electromagnetic equations of motion:

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B}.$$



$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = \mathbf{I}\times\mathbf{B}$$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

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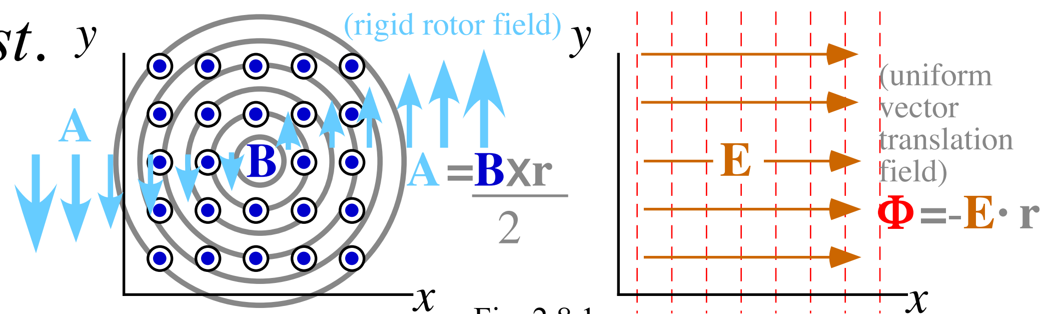

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\boxed{\begin{gathered}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \tfrac{e}{m}E_x \qquad \varepsilon_y = \tfrac{e}{m}E_y \qquad B = \tfrac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{gathered}}$$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

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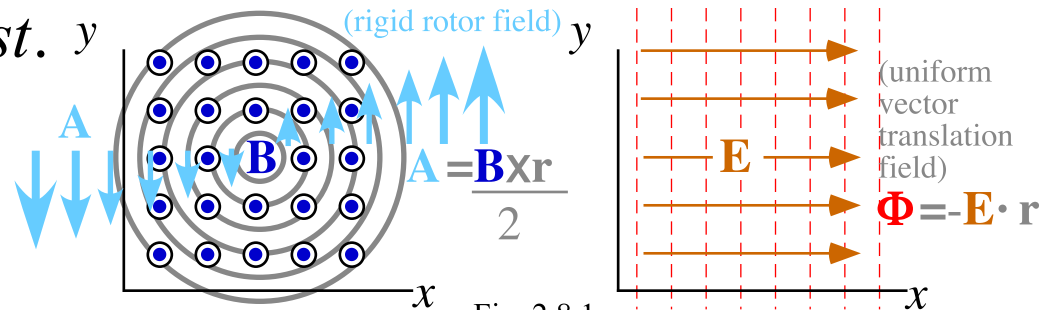

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

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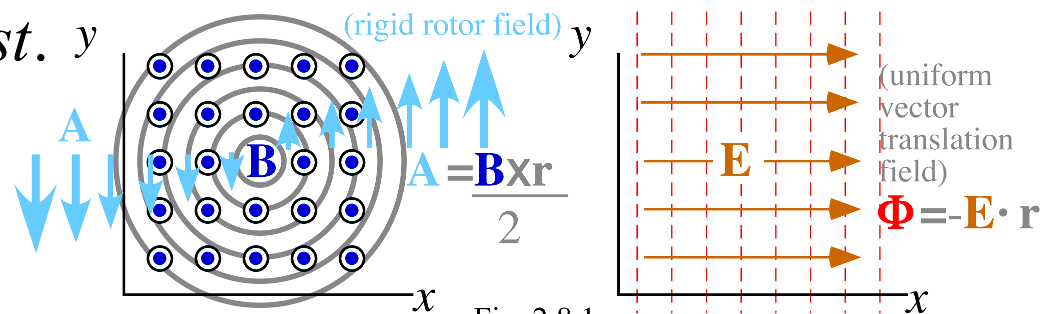

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Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

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$$\dot{\mathbf{v}} = \boldsymbol{\varepsilon} + \mathbf{v}\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

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Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements}: \hat{\mathbf{e}}_\mathbf{x}\rightarrow 1 \quad \text{and}: \hat{\mathbf{e}}_\mathbf{y}\rightarrow i = \sqrt{-1}$$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

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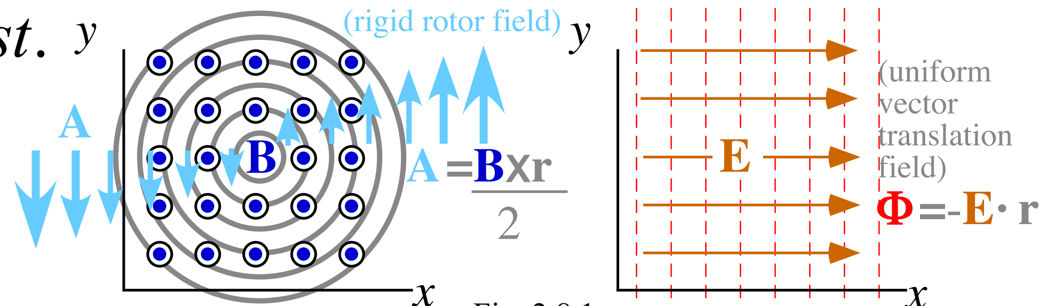

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$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv$$

with replacements : $\hat{\mathbf{e}}_\mathbf{x}\to 1$ and : $\hat{\mathbf{e}}_\mathbf{y}\to i = \sqrt{-1}$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t)$$

where : $\boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

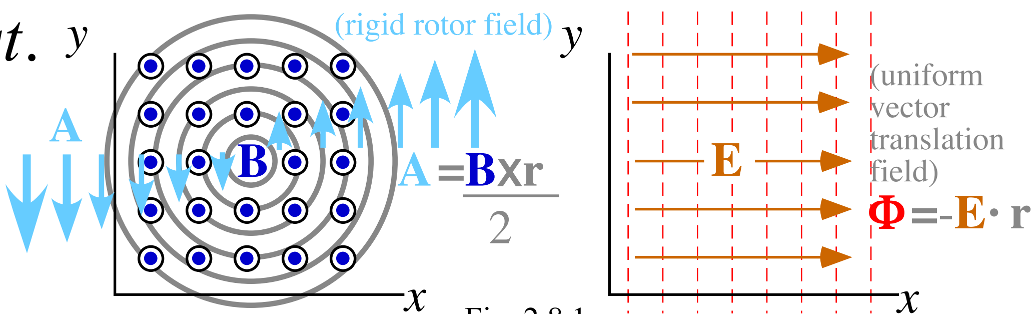

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Gibb's notation:

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

$$\dot{\mathbf{v}} = \boldsymbol{\varepsilon} + \mathbf{v} \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

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$$\dot{v} = \varepsilon - iBv$$ with replacements : $\hat{\mathbf{e}}_\mathbf{x} \to 1$ and : $\hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$

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where : $\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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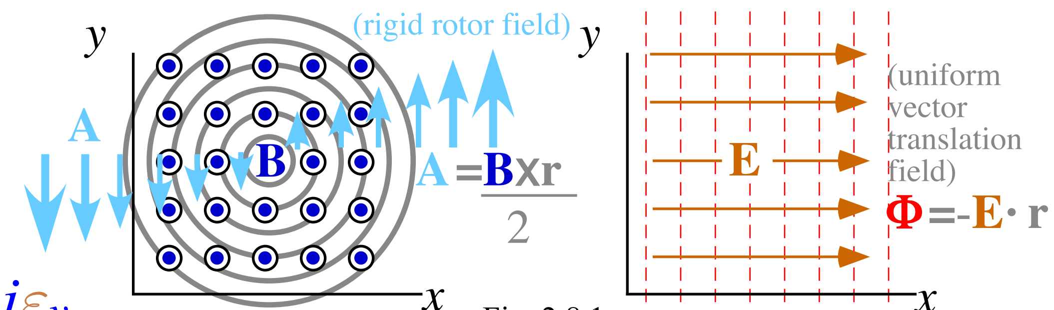
Shorthand Labeling



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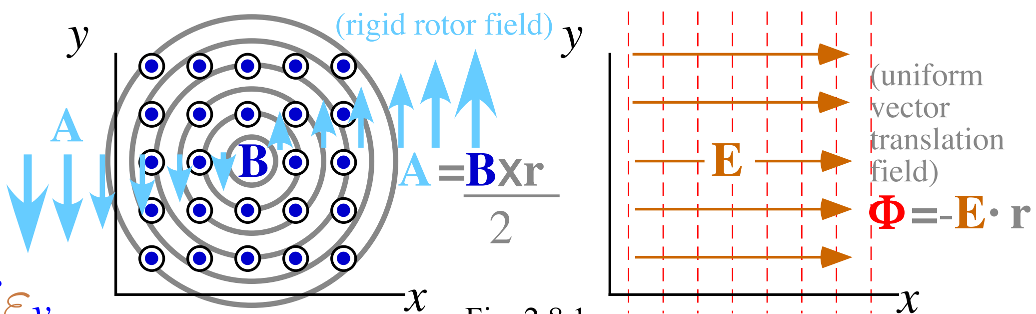

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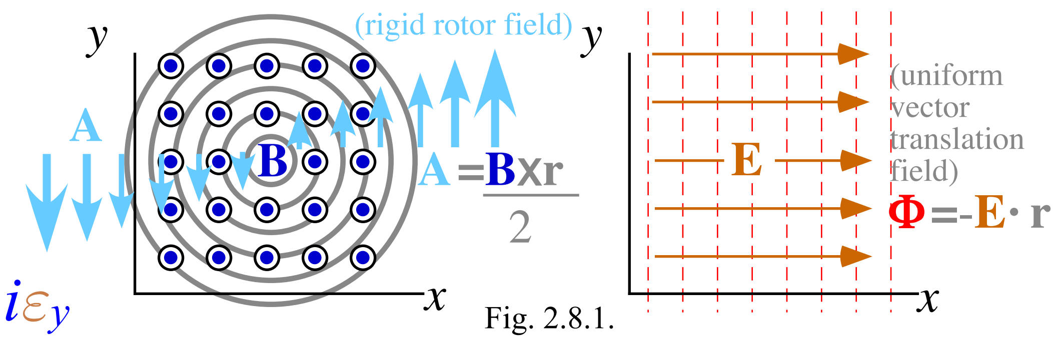

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$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta)$$

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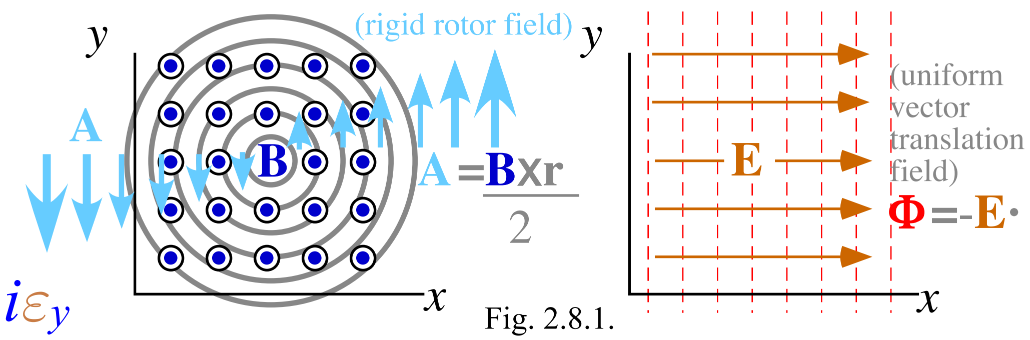

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$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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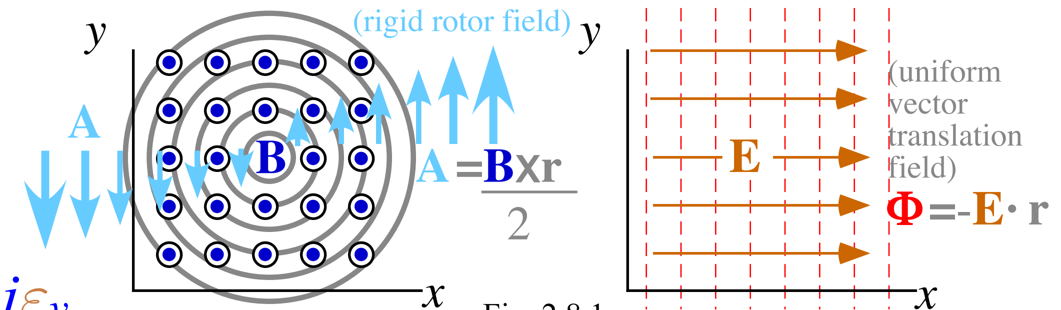
Shorthand Labeling



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$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

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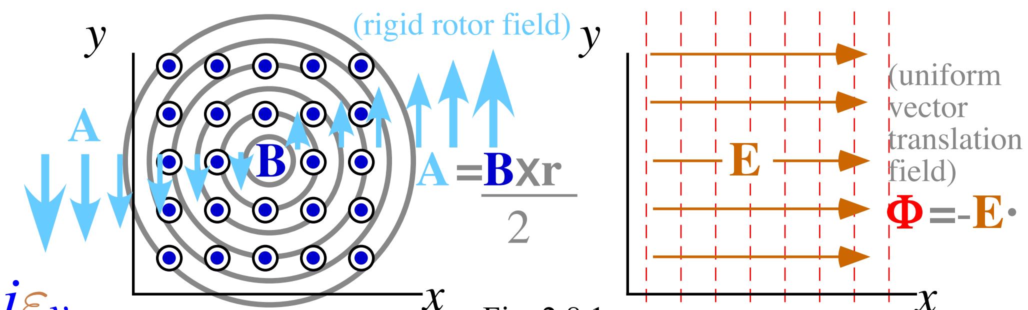

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Crossed E and B field mechanics (Solution by complex variables)

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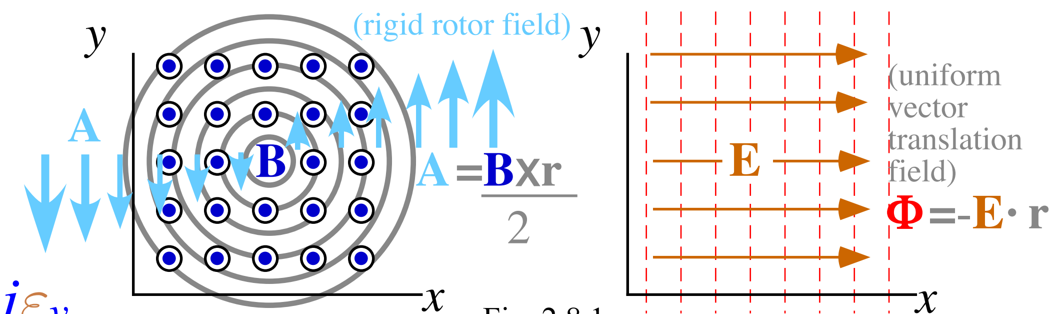

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complex form

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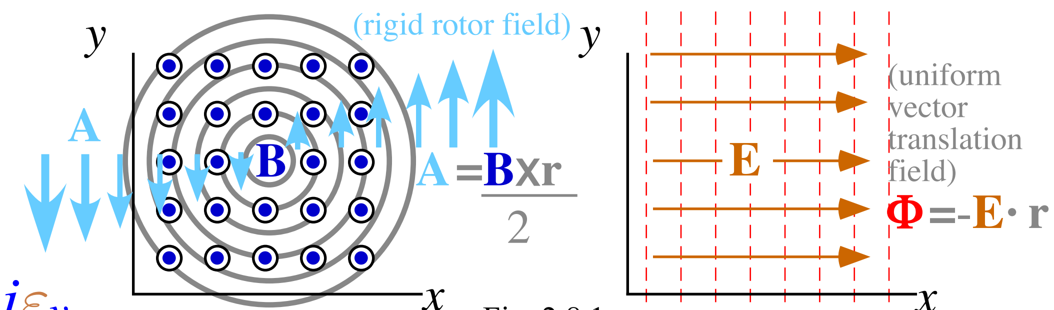

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$$x(t) + iy(t) = e^{-iB\cdot t}\left(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}\right) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

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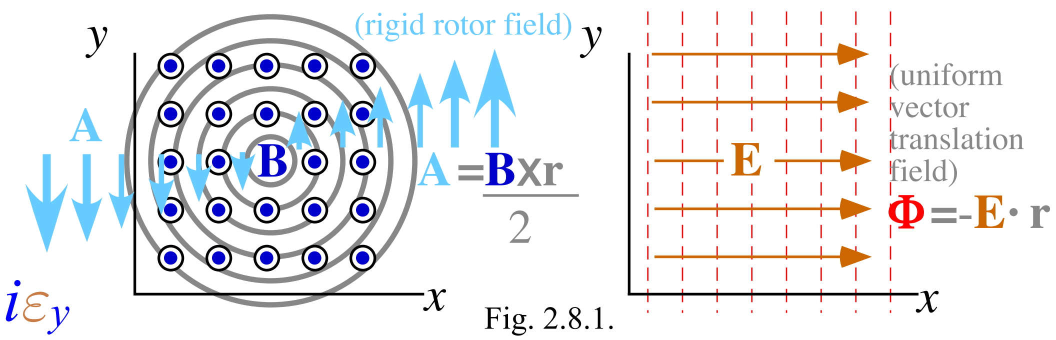

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$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}\left(v(0) + i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}\left(v(0) + i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - \left(\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2}\right) \quad \textit{complex form}$$

$$x(t) + iy(t) = e^{-iB\cdot t}\left(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}\right) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where}: \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

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$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2}) \quad \textit{complex form}$$

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2} \qquad \textit{complex form}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where :} \; \beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}$$

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vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$ *complex form*

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$ *complex form*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$ *vector form*

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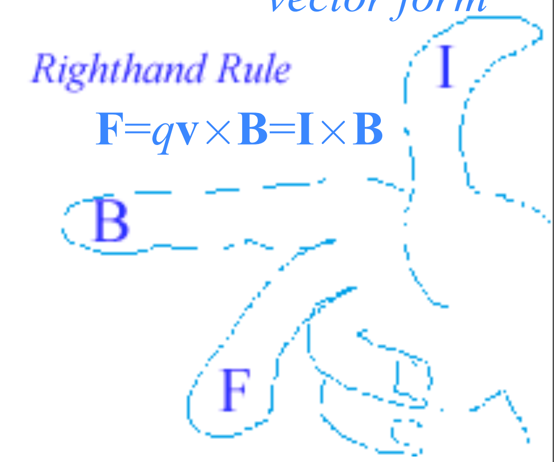
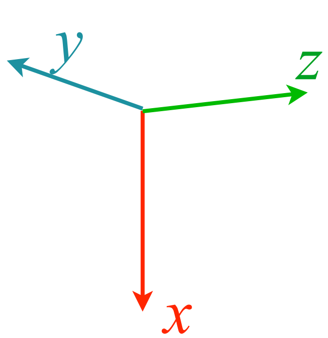
complex form

$$x(t)+iy(t)= \quad e^{-iB\cdot t} \quad (i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}) \quad -i\frac{\varepsilon}{B}\cdot t \quad +x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form





Crossed E and B field mechanics (Solution by complex variables)

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complex form

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$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0)-\left(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2}\right)$$

complex form

$$x(t)+iy(t) = e^{-iB\cdot t}\left(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}\right) - i\frac{\varepsilon}{B}\cdot t + x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

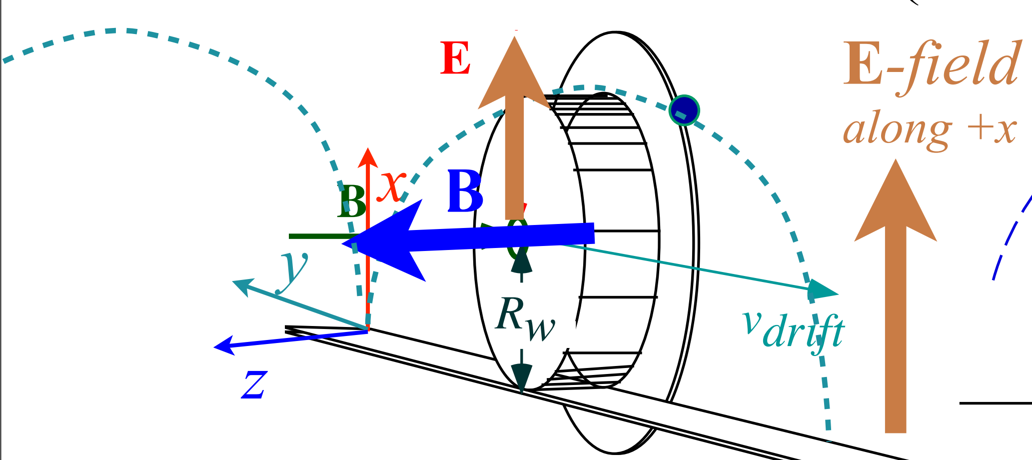
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix} + \begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

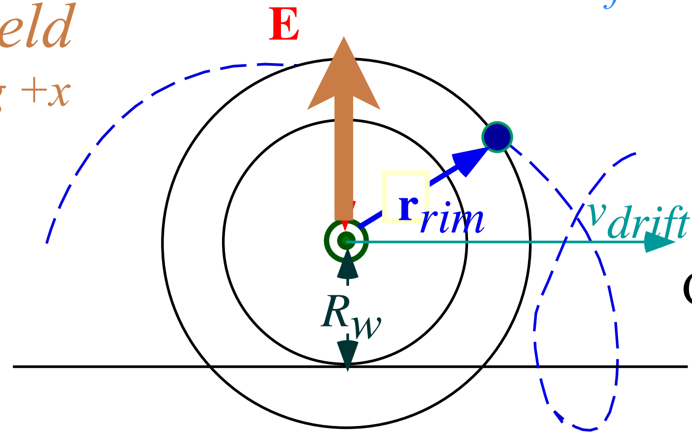
vector form

Cycloid example:
initial $(x(0), y(0)) = (0,0)$
and $(v_x(0), v_y(0)) = (0,0)$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \text{is on rim of a wheel of radius } R_W = E/B^2$$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix} + \begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$





Constant v_{drift} in -y-direction
$v_{drift} = -E/B$

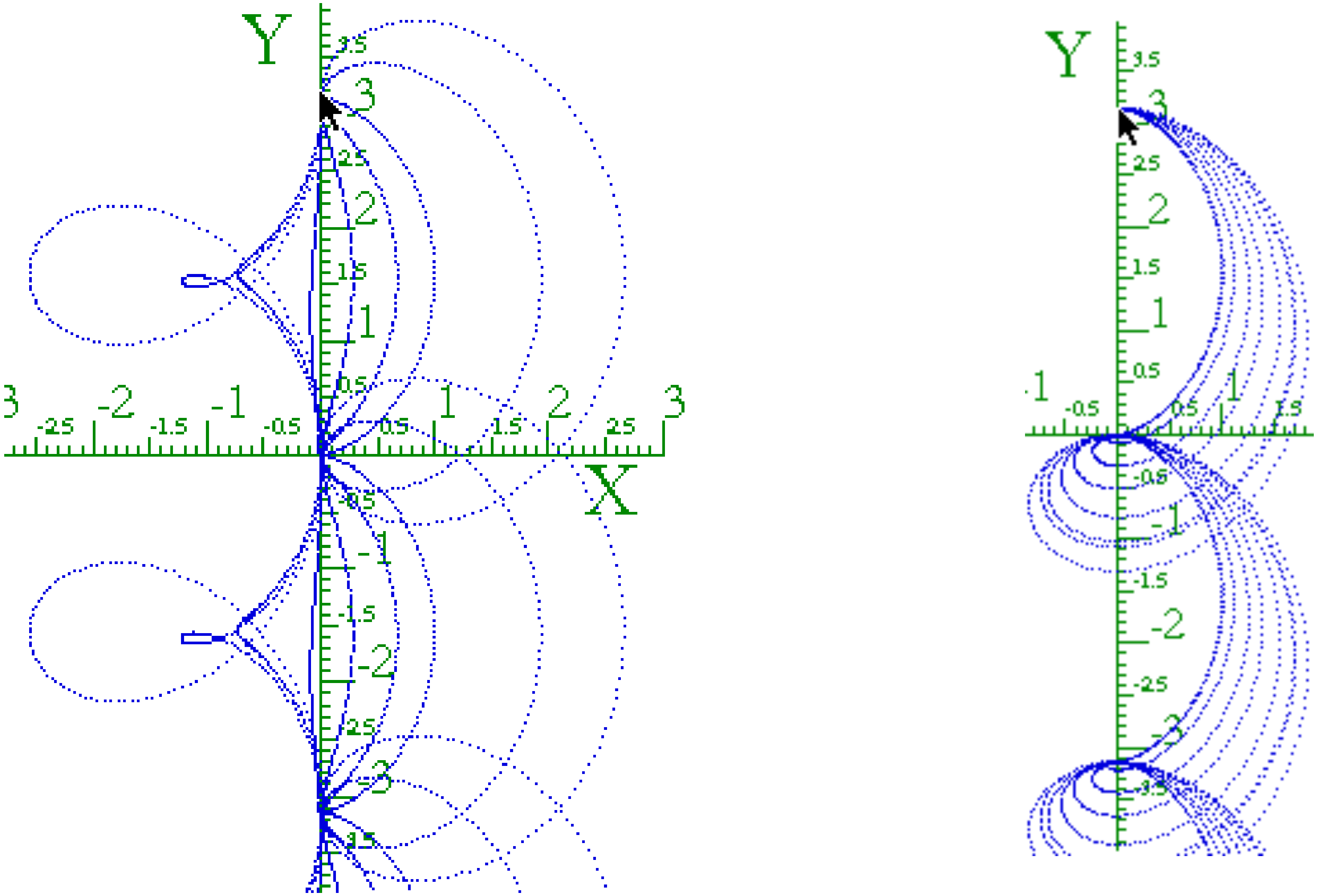
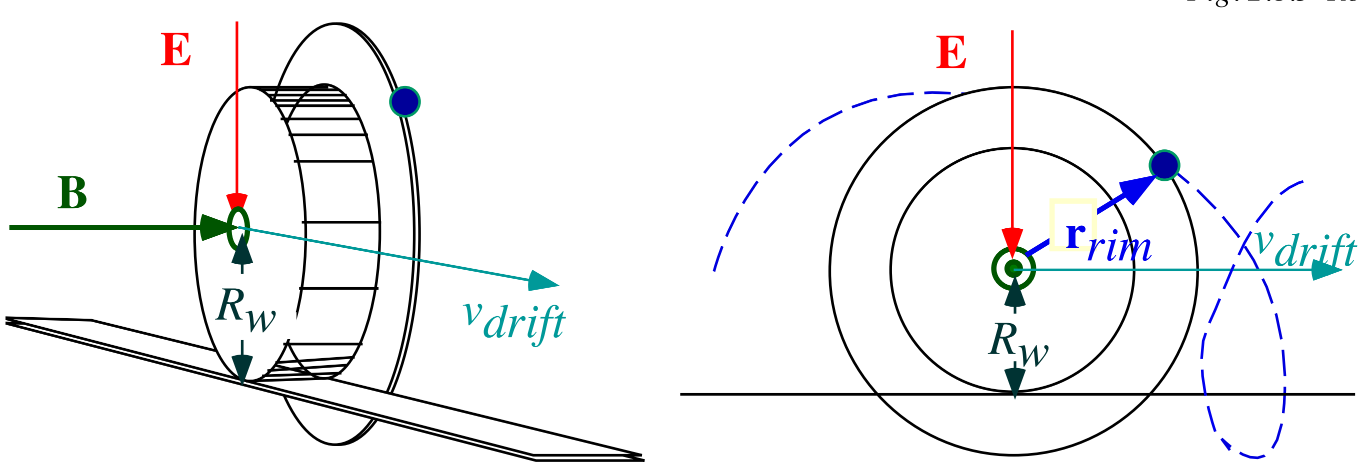


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (E=1/2, B=1)

Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits



Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

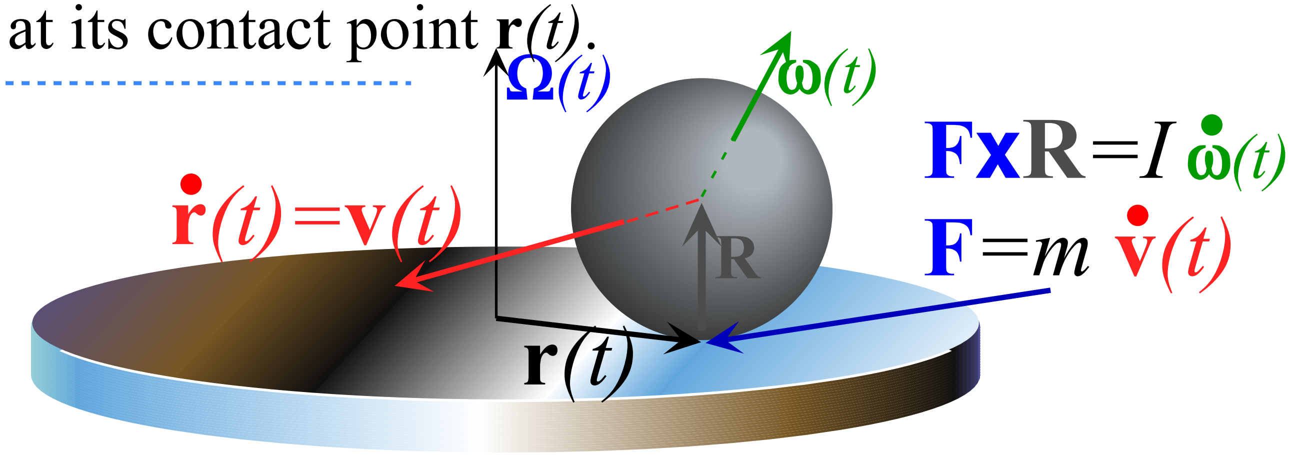
→ *Mechanical analog of cyclotron and FBI rule*

Cycloid geometry and flying sticks

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals

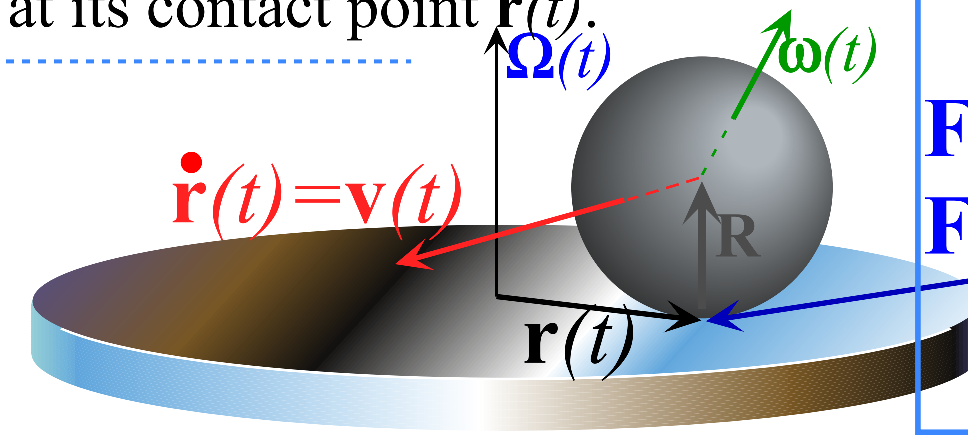
table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathrm{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

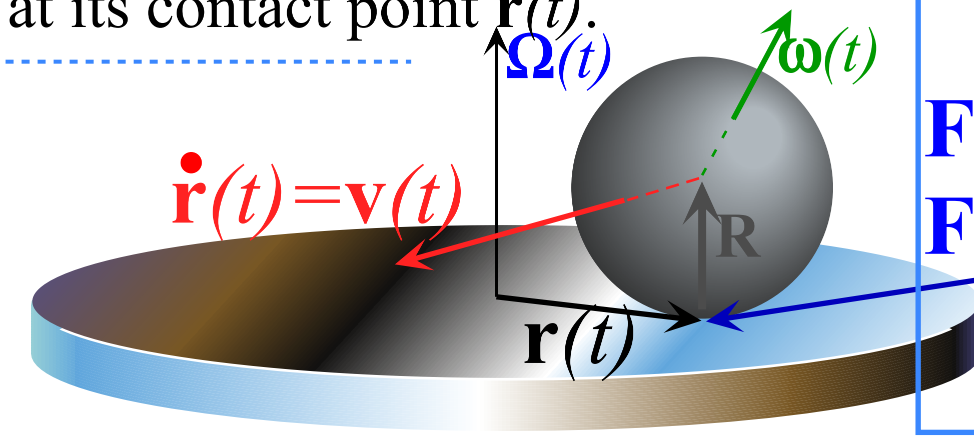
Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\omega}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\omega}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

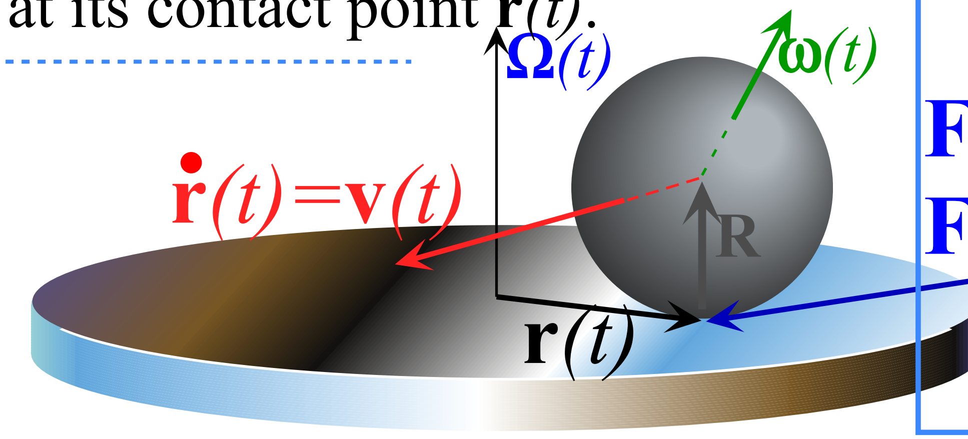
No-slipping: $\mathbf{v}(t)$-$\omega(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

*Torque-and-*F=ma
equations of motion:

$$
\begin{aligned}
I\dot{\omega}(t) &= \mathbf{F}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R
\end{aligned}
$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

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Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t) = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\mathbf{R} \quad = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
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Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Equations of Motion:

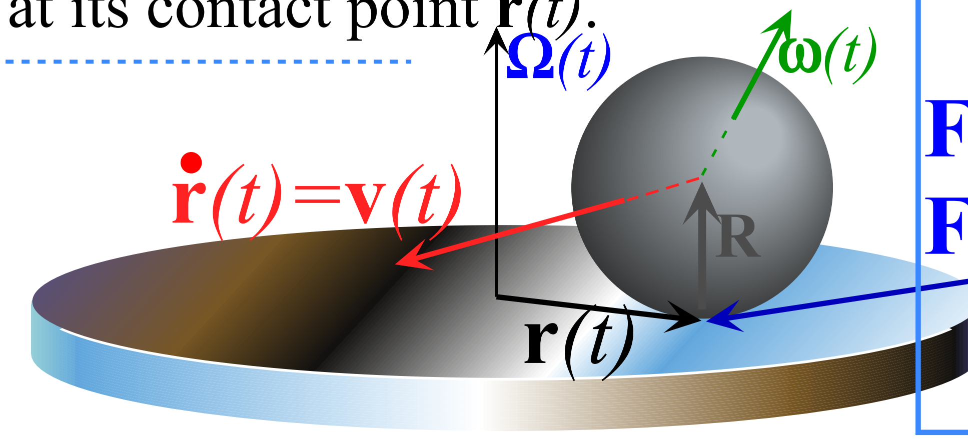
rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint



Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

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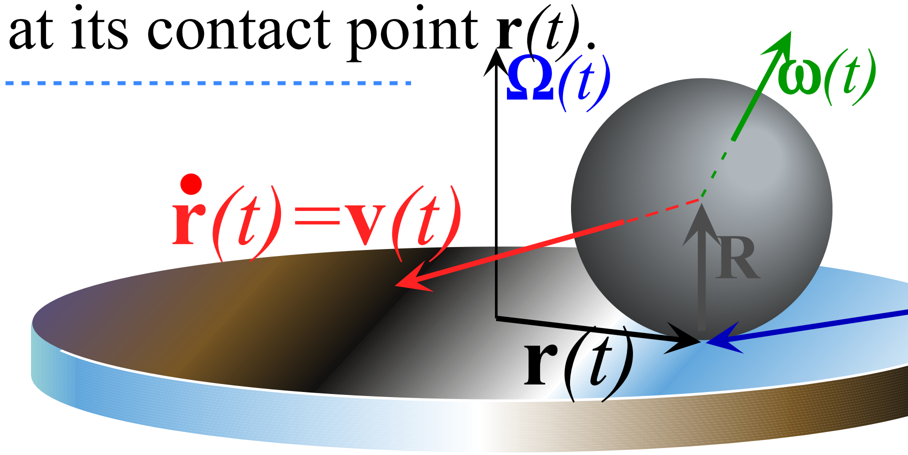
Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

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$$\mathbf{v}(t) = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\mathbf{R} \quad = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R = \boldsymbol{\Omega}\times\mathbf{v}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) \;+\; \dot{\boldsymbol{\omega}}(t) \quad \times\hat{\mathbf{z}}R$$

use: $$\dot{\boldsymbol{\omega}}(t) = \frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$$

Mechanical analog of cyclotron and FBI rule

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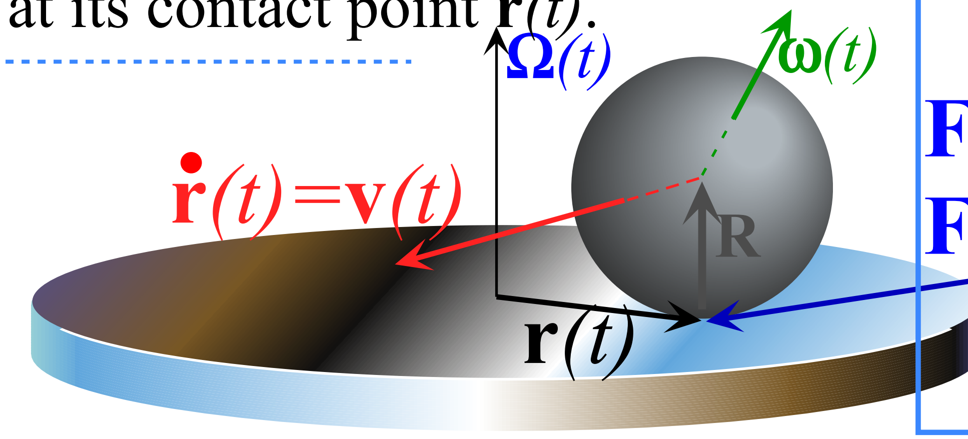

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$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t) \times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

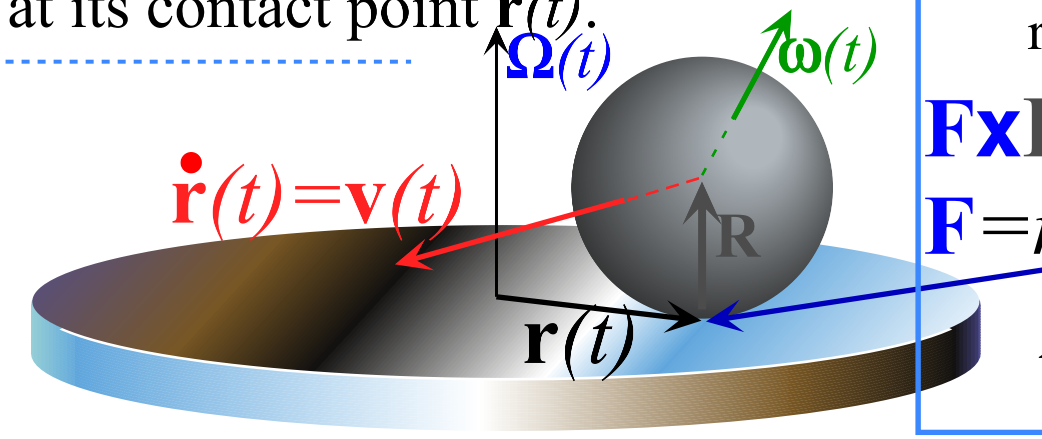
$$=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

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Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

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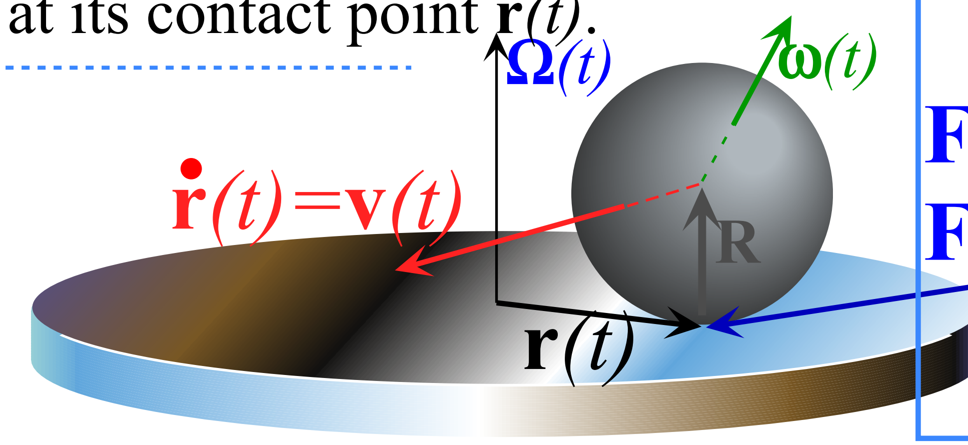

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since $\dot{\mathbf{v}}(t)$ always in table plane

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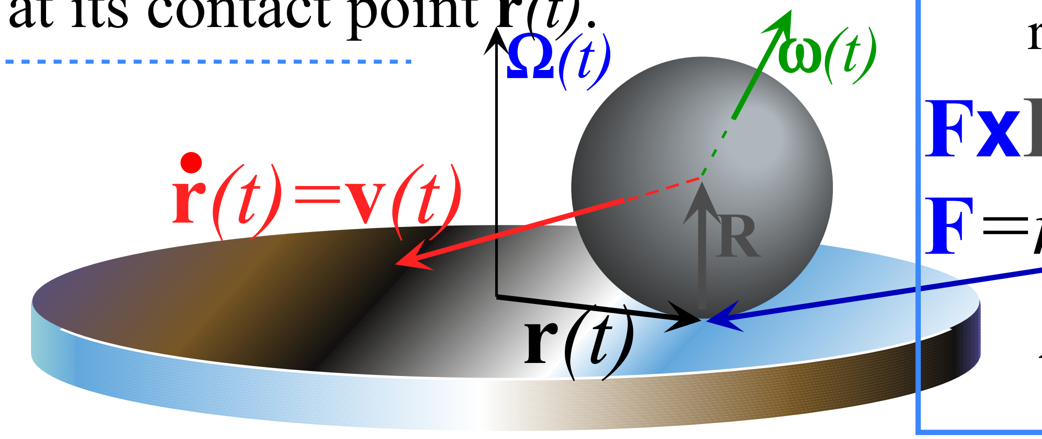

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$$=\boldsymbol{\Omega}\times\mathbf{v}(t)+\quad 0\quad-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or : $\dot{\mathbf{v}}(t)=\dfrac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}\times\mathbf{v}(t)$

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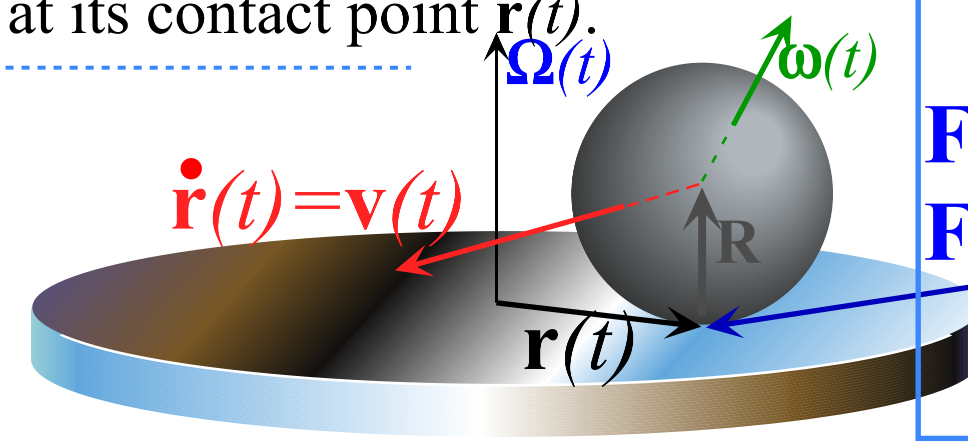

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$$= \boldsymbol{\Omega}\times\mathbf{v}(t) + \quad 0 \quad - \frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

$$\left(1 + \frac{mR^2}{I}\right)\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

$\mathbf{F}=\mathbf{B}\times\mathbf{v}$ *mechanical analog:*

$$\dot{\mathbf{v}}(t) = \frac{\boldsymbol{\Omega}}{1 + \frac{mR^2}{I}}\times\mathbf{v}(t)$$

Crossed E and B field mechanics

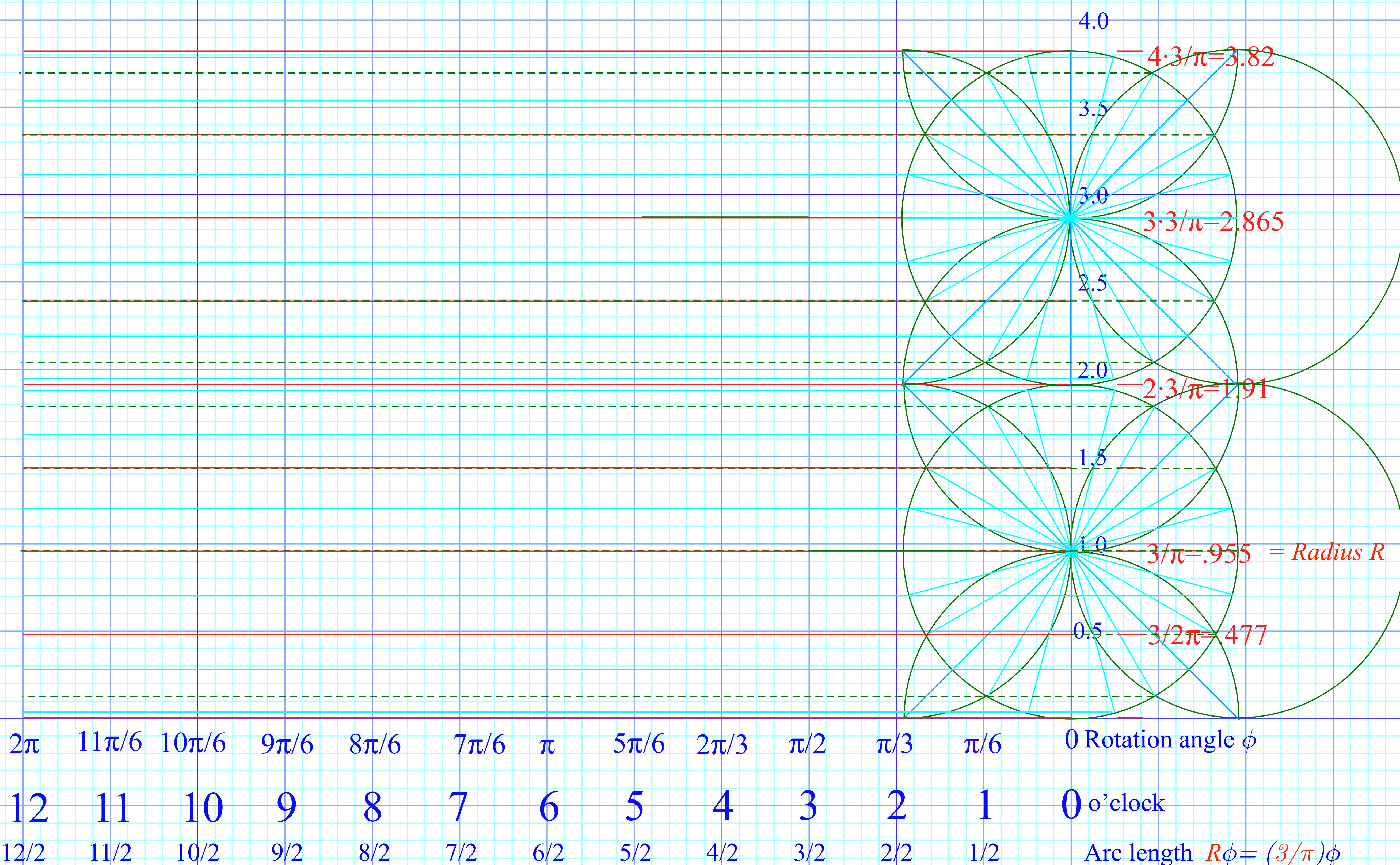
Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

→ *Cycloid geometry and flying sticks*

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled-out circumference $R\phi= (3/\pi)m\pi/n=3m/n$.



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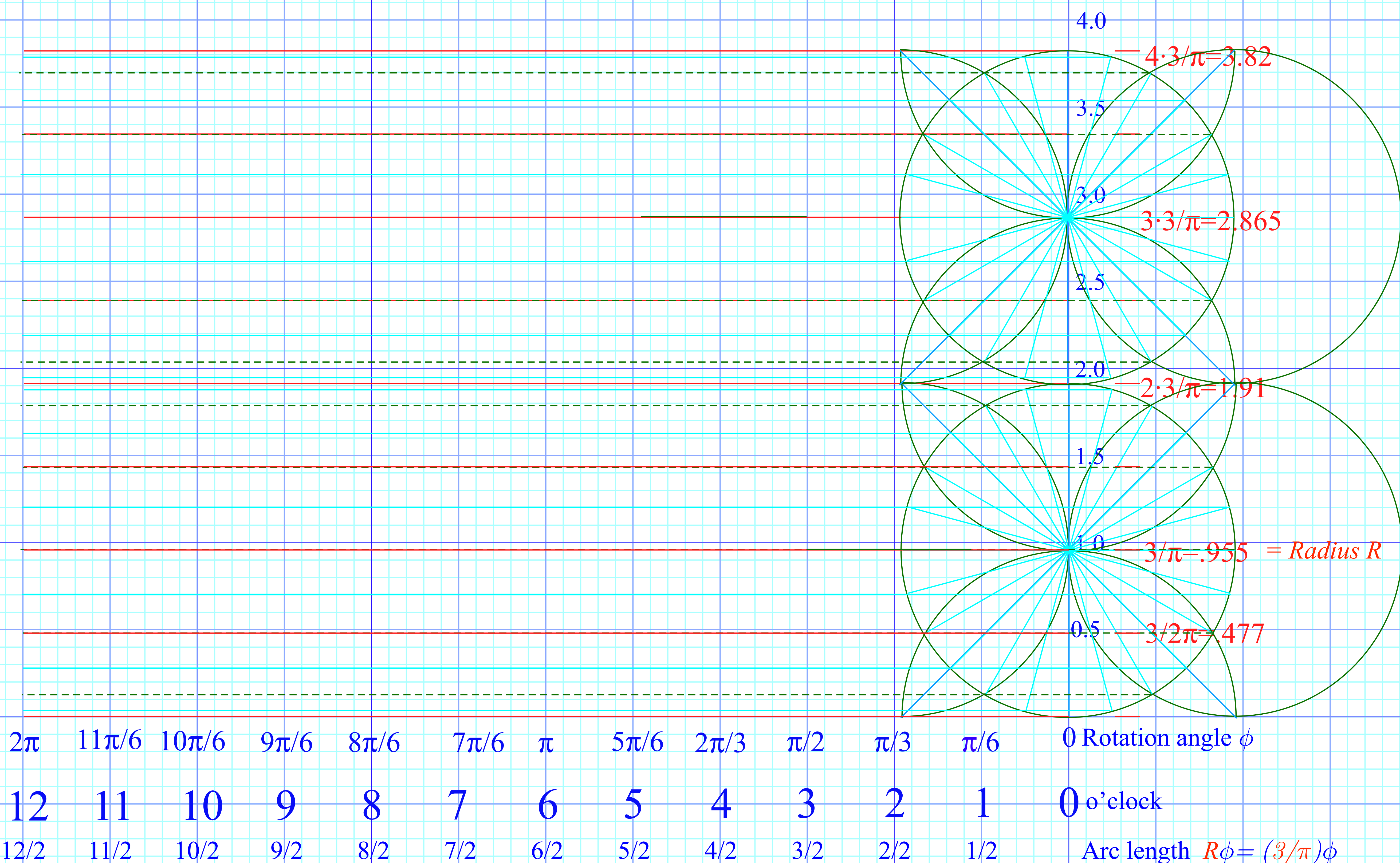

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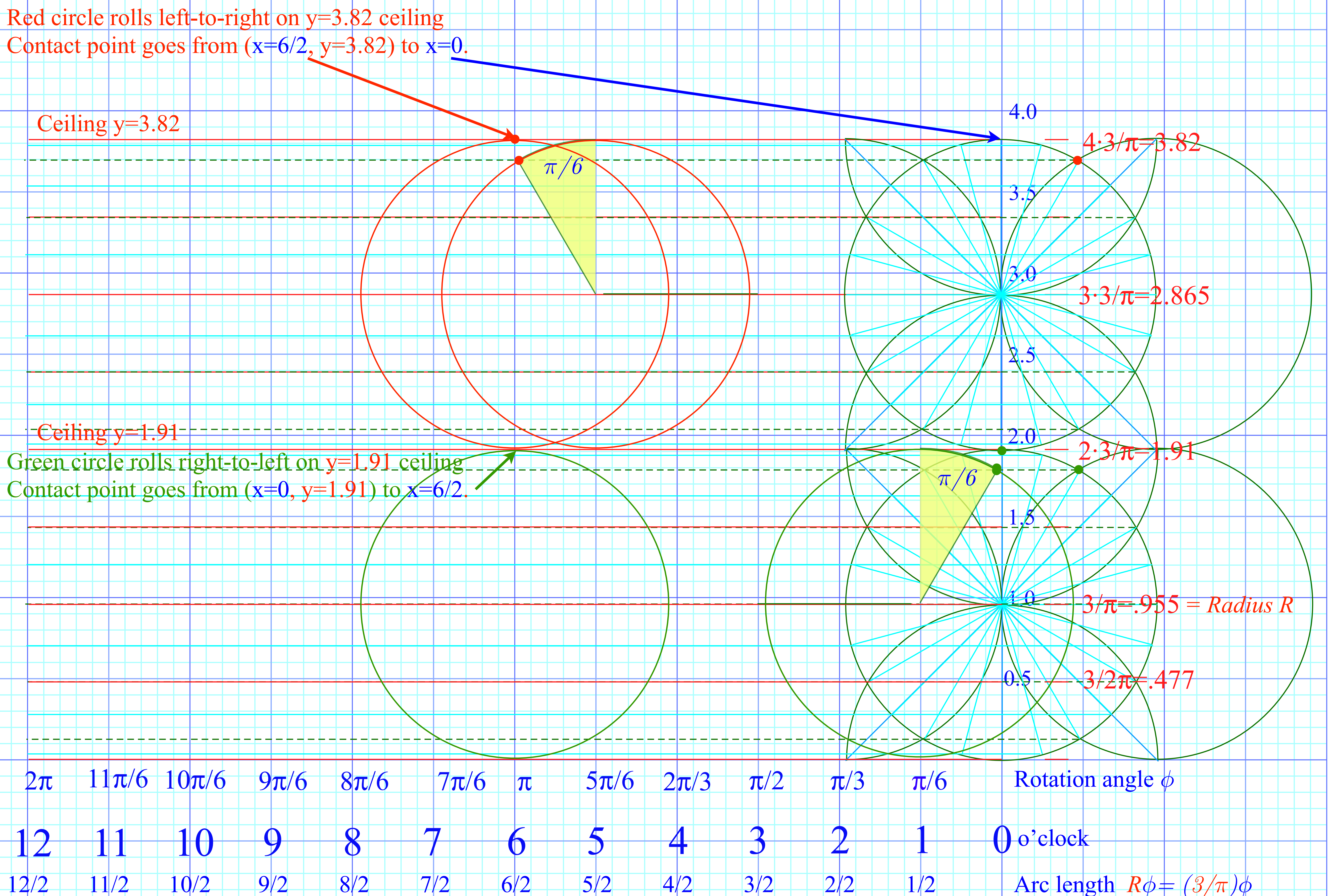

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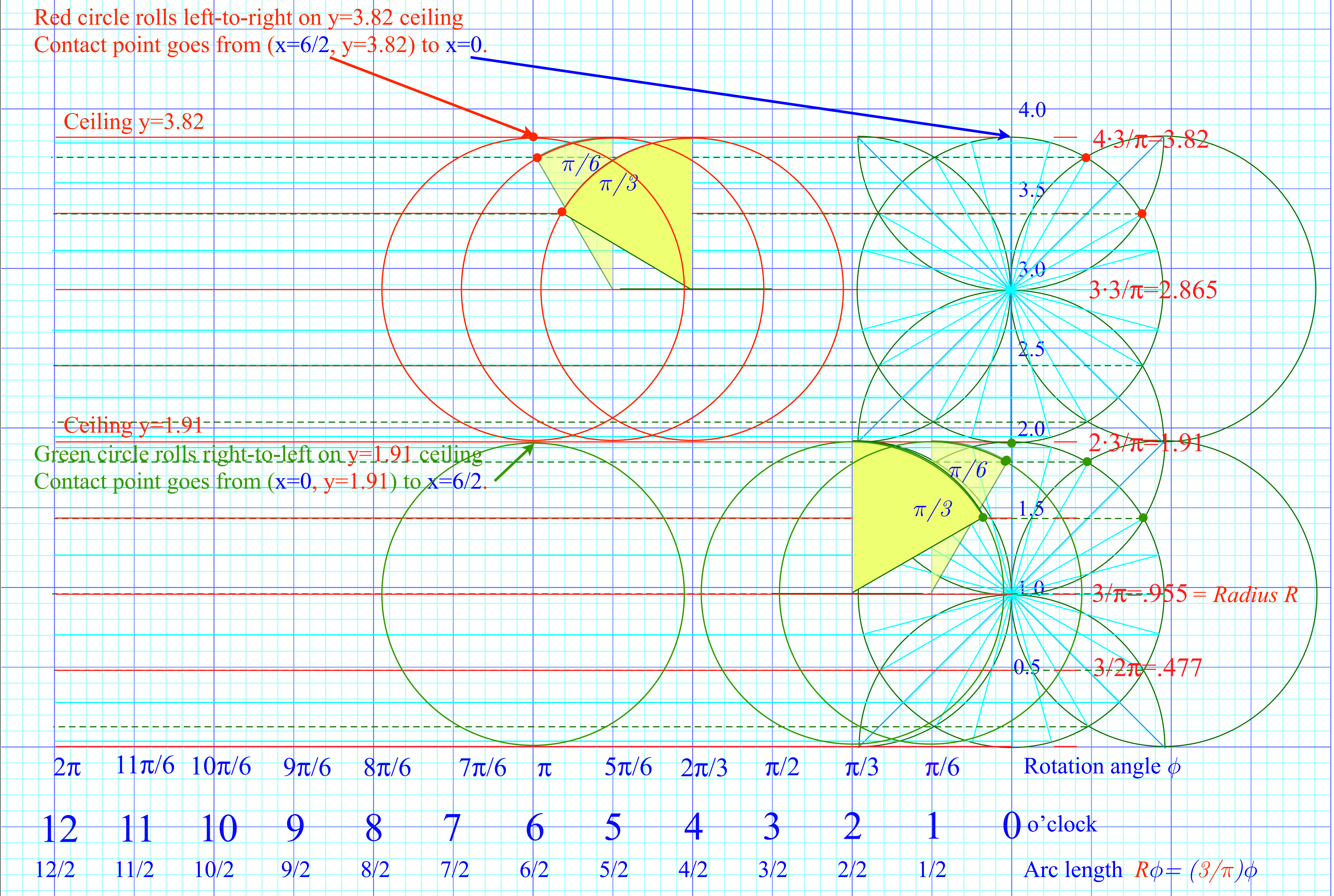

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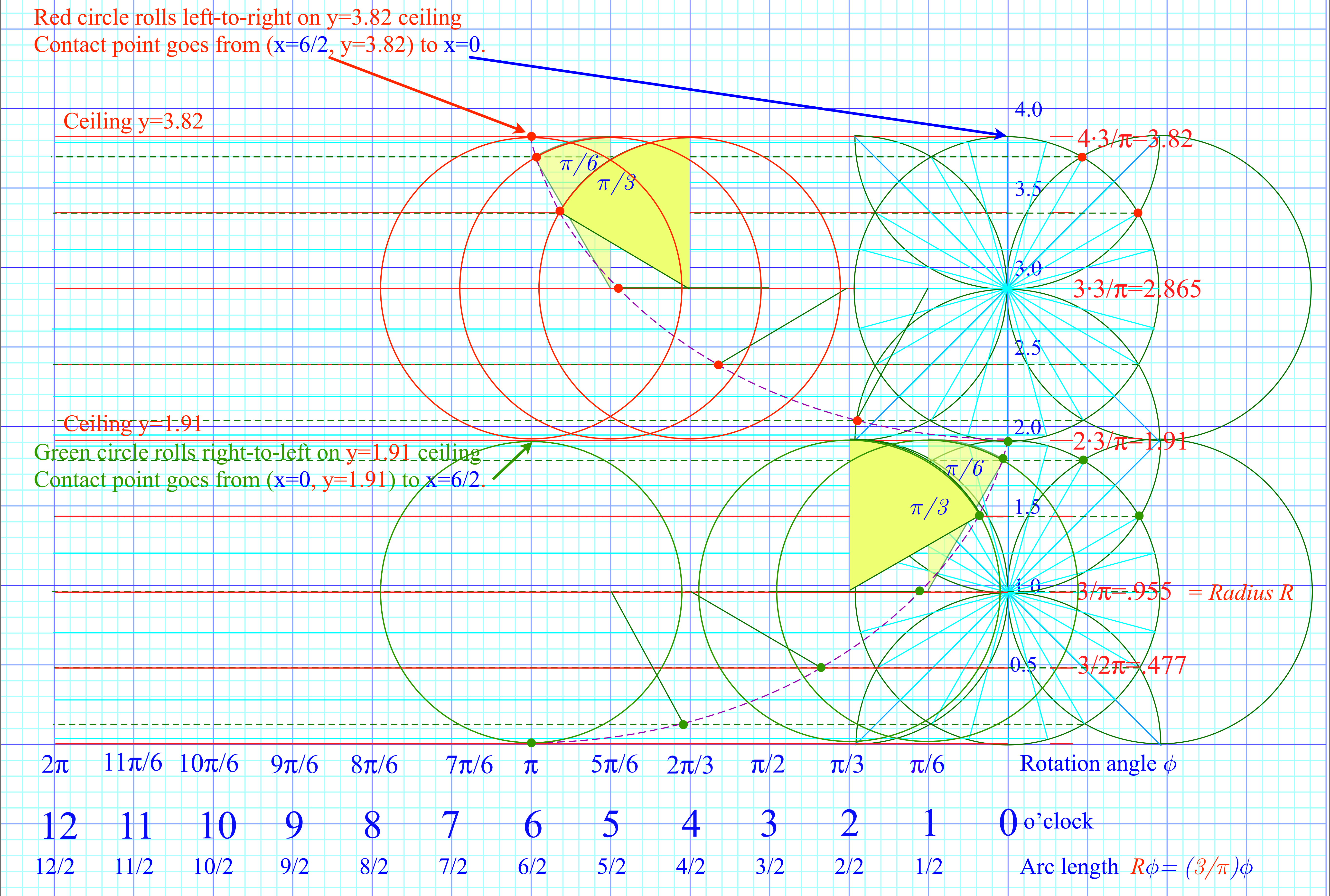

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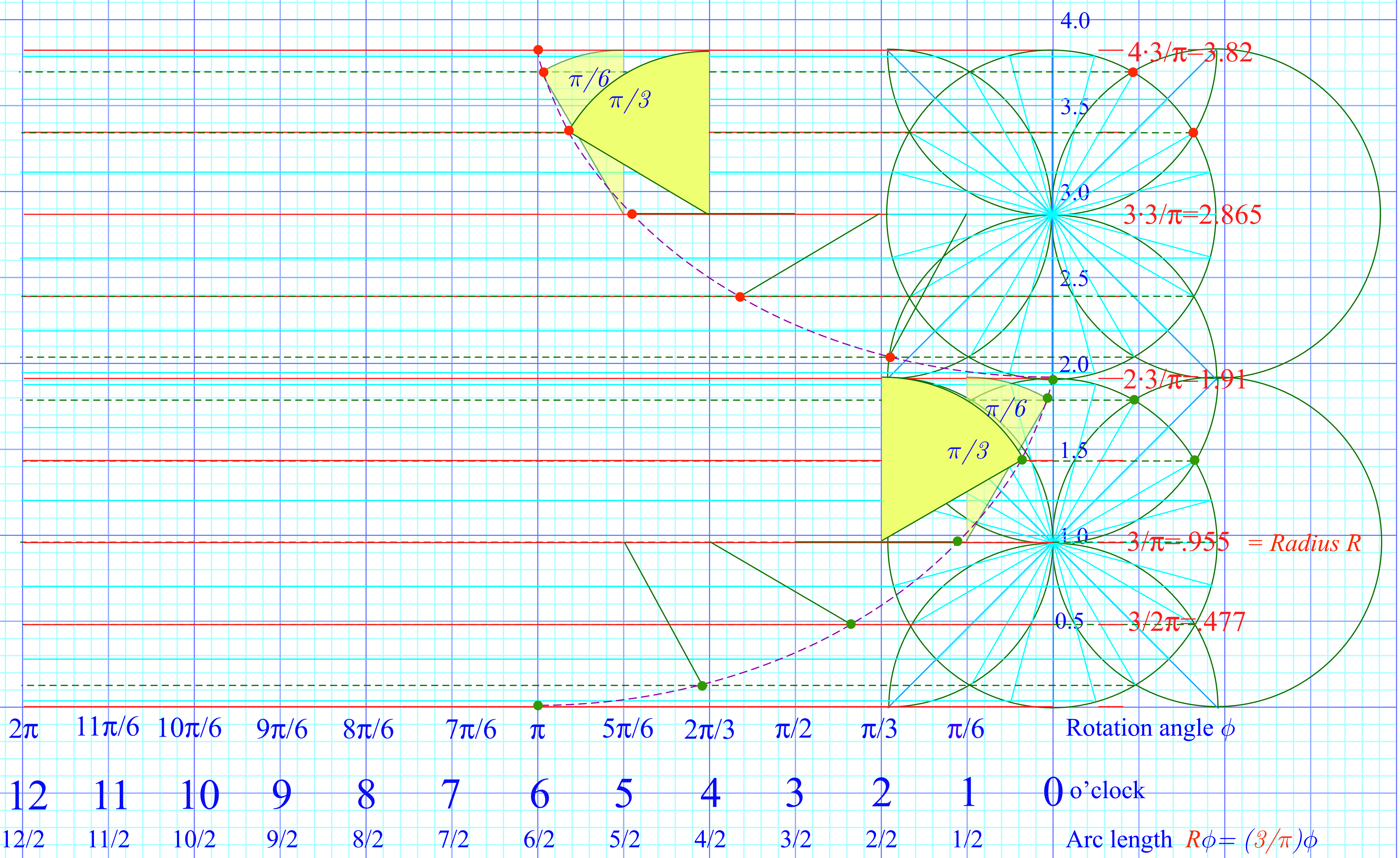

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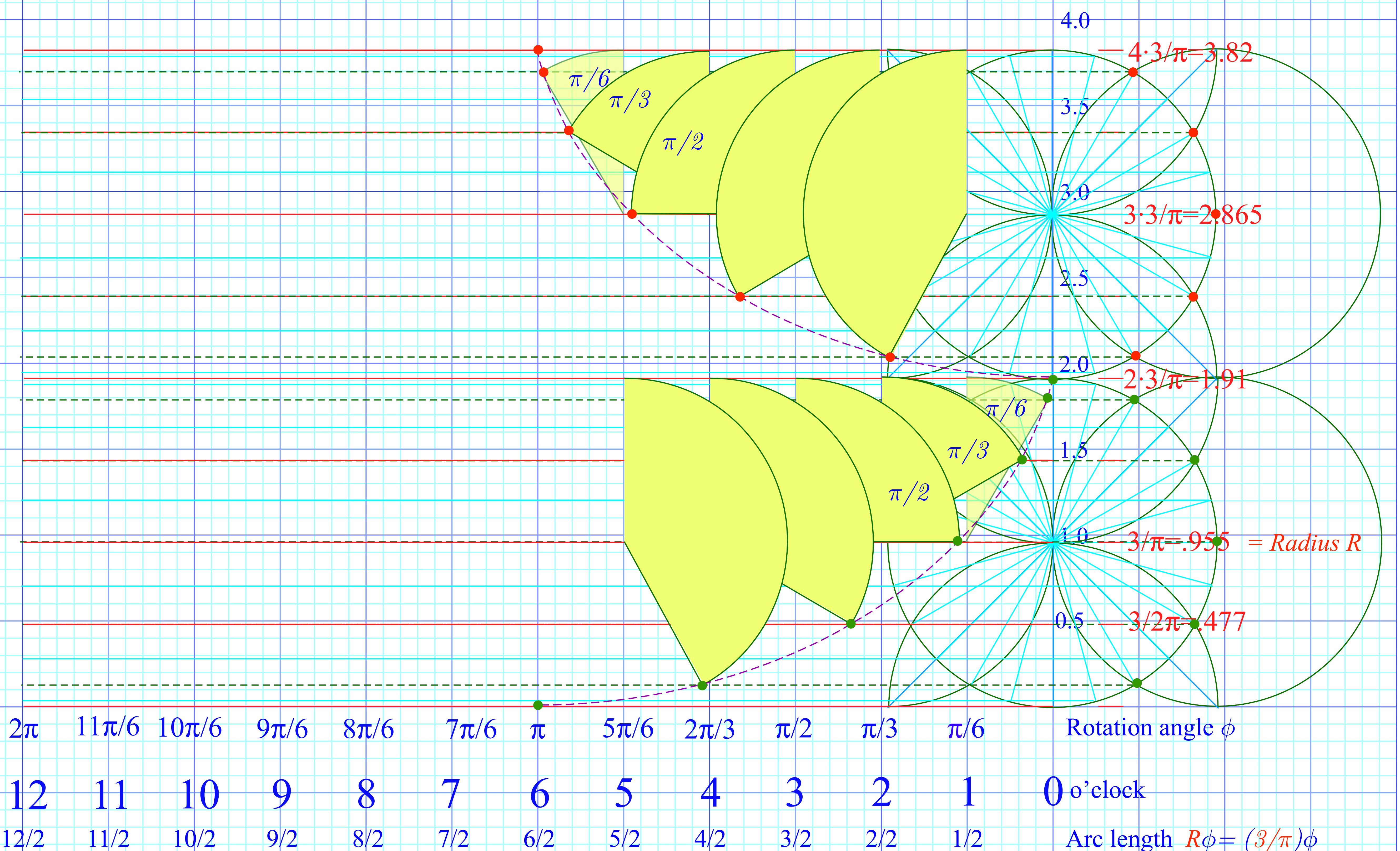

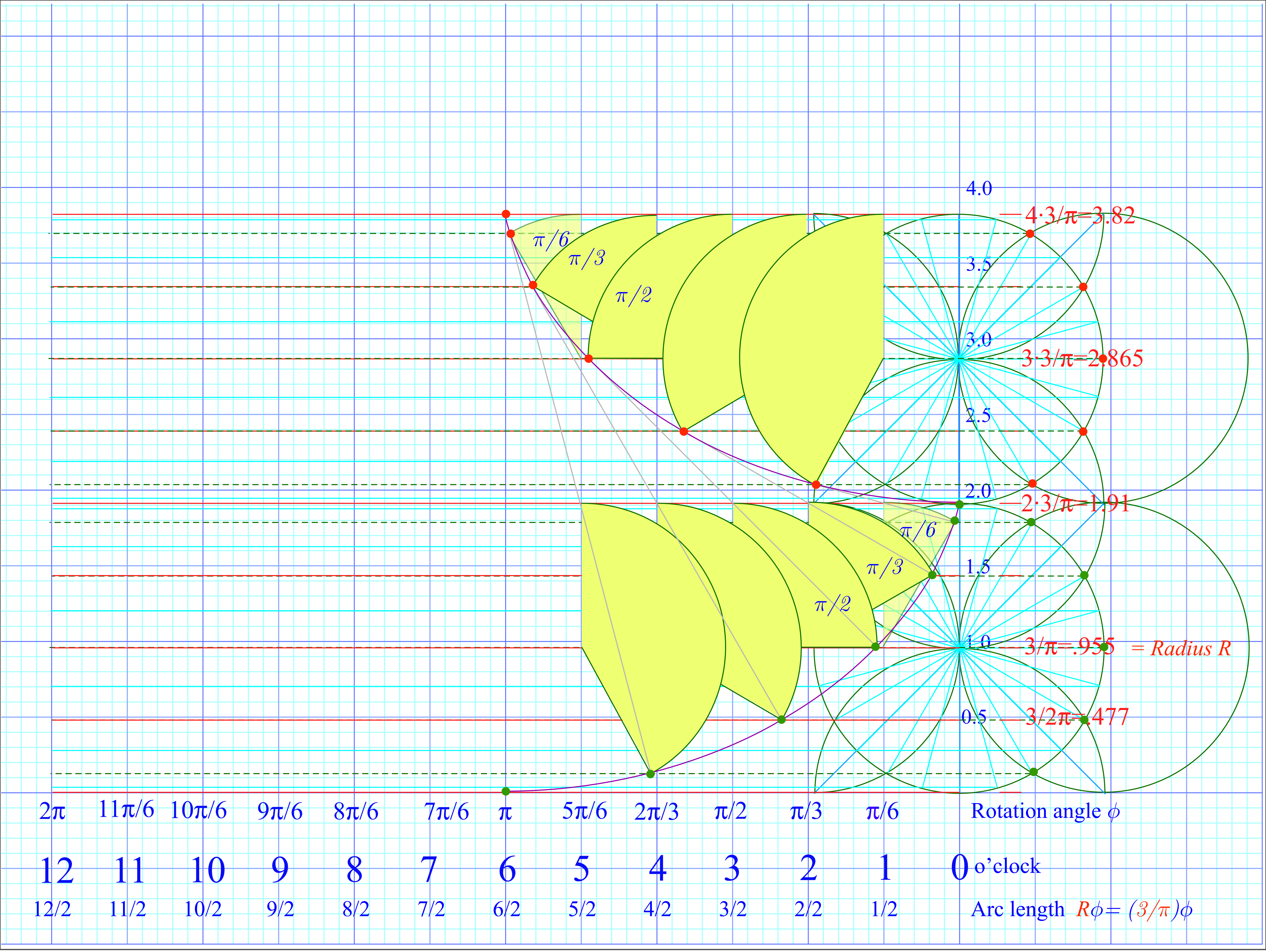
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
π/6
π/3
π/2
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle φ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rφ= (3/π)φ

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