

Lecture 24

Tue. 11.13.2014

Introduction to classical oscillation and resonance

(Ch. 1 of Unit 4 11.14.13)

1D forced-damped-harmonic oscillator equations and Green's function solutions

Linear harmonic oscillator equation of motion.

*Linear **damped**-harmonic oscillator equation of motion.*

Frequency retardation and amplitude damping

Figure of oscillator merit (the 5% solution $3/\Gamma$ and other numbers)

*Linear **forced-damped**-harmonic oscillator equation of motion.*

Phase lag and amplitude resonance amplification

Figure of resonance merit: Quality factor $q = \omega_0/2\Gamma$

*Properties of **Green's function** solutions and their mathematical/physical behavior*

Transient solutions vs. Steady State solutions

*Complete **Green's Solution** for the **FDHO** (**Forced-Damped-Harmonic Oscillator**)*

Quality factors: Beat, lifetimes, and uncertainty

*Approximate Lorentz-**Green's Function** for high quality **FDHO** (Quantum propagator)*

Common Lorentzian (a.k.a. Witch of Agnesi)

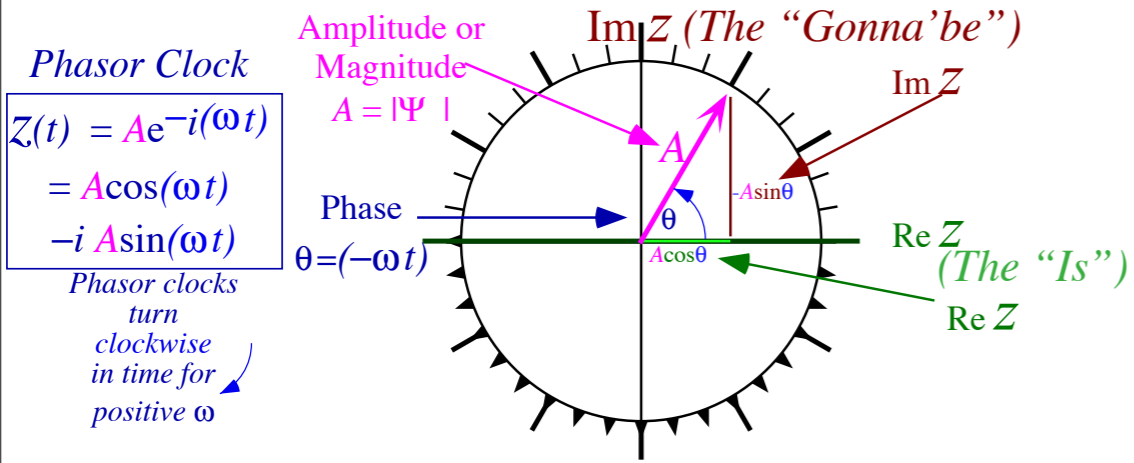
Smith Charts

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external **stimulating force** $\longrightarrow F_{stimulus}(t) = eE(t)$

held back by a **harmonic (linear) restoring force** $\longrightarrow F_{restore} = -kz, (k = \omega_0^2 m),$

retarded by **frictional damping force** $\longrightarrow F_{damping} = -b \frac{dz}{dt}, (b = 2\Gamma m)$

Linear

harmonic oscillator equation of motion.

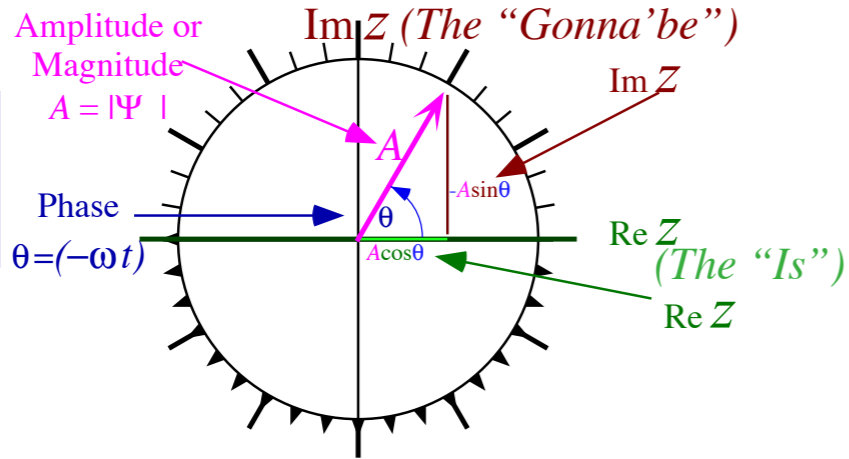
Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

$$= A\cos(\omega t)$$

$$-i A\sin(\omega t)$$

Phasor clocks turn clockwise in time for positive ω



$$F_{total}(t) = m \frac{d^2 z}{dt^2} =$$

$F_{restore}$

$$\frac{d^2 z}{dt^2} =$$

$$\frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + \omega_0^2 z = 0$$

$$+ \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force $\longrightarrow F_{restore} = -kz, (k = \omega_0^2 m),$

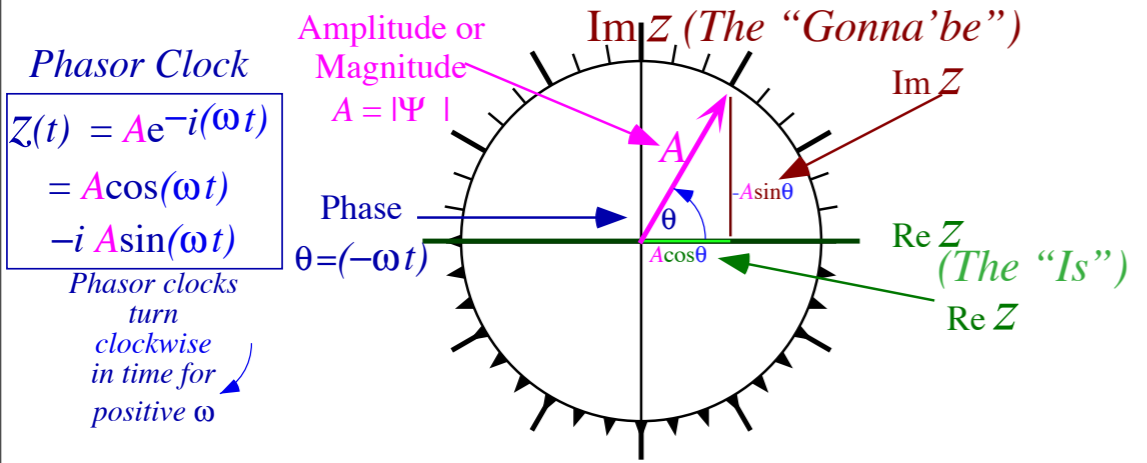
Linear

harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + \omega_0^2 z = 0$$



Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force $\longrightarrow F_{restore} = -kz, (k = \omega_0^2 m),$

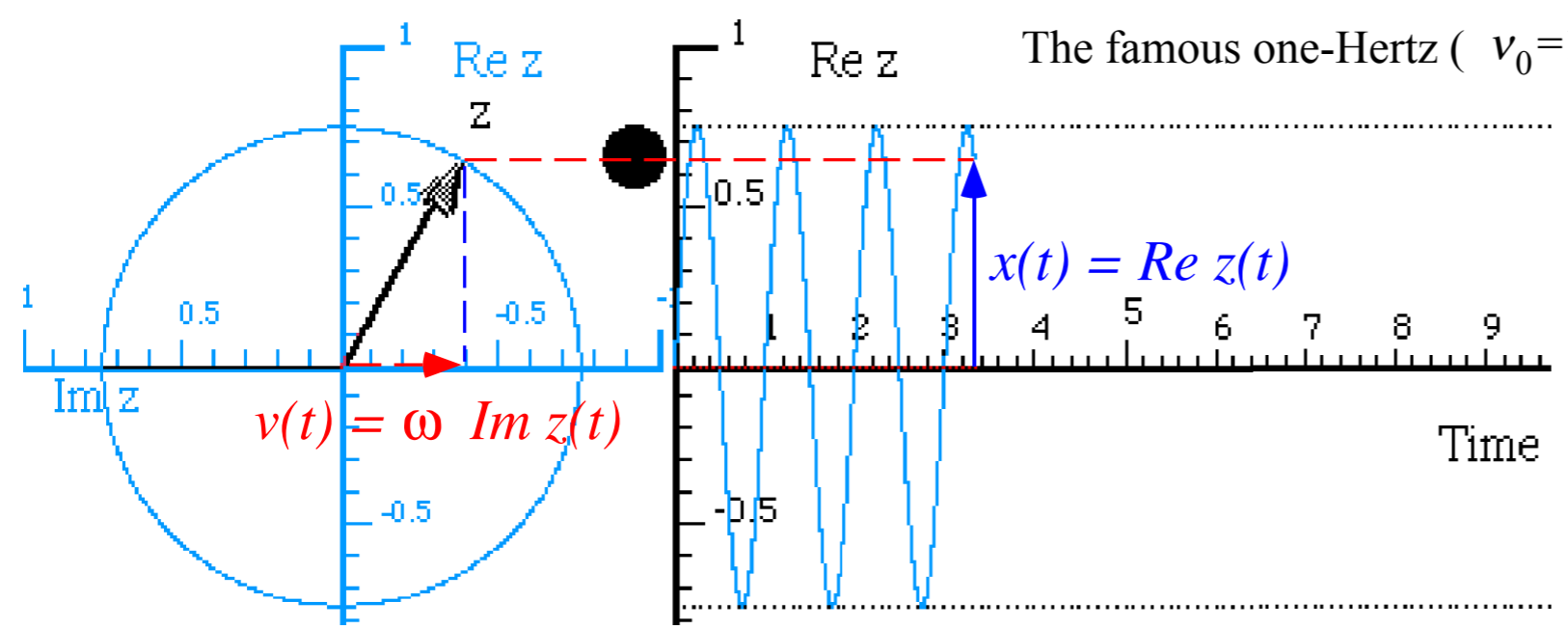


Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0$

Linear *damped-harmonic* oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

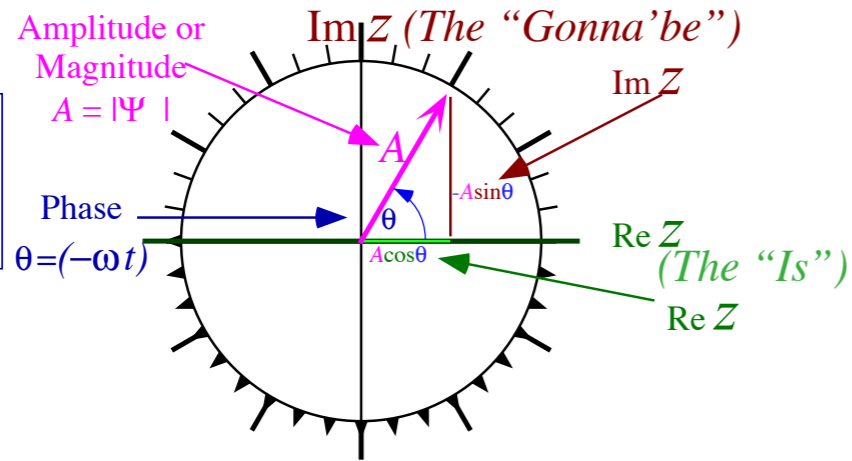
Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

$$= A \cos(\omega t)$$

$$-i A \sin(\omega t)$$

Phasor clocks
turn
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Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force** $\longrightarrow F_{restore} = -kz, \quad (k = \omega_0^2 m),$

retarded by **frictional damping force** $\longrightarrow F_{damping} = -b \frac{dz}{dt}, \quad (b = 2\Gamma m)$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

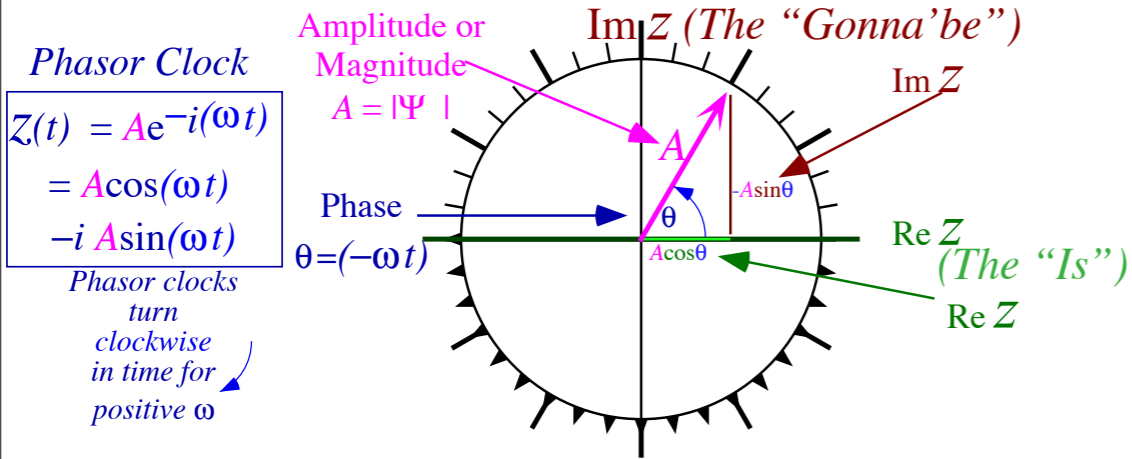
$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z = z(t) = Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2 \right] e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force

$$F_{restore} = -kz$$

retarded by frictional damping force

$$F_{damping} = -b \frac{dz}{dt}$$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

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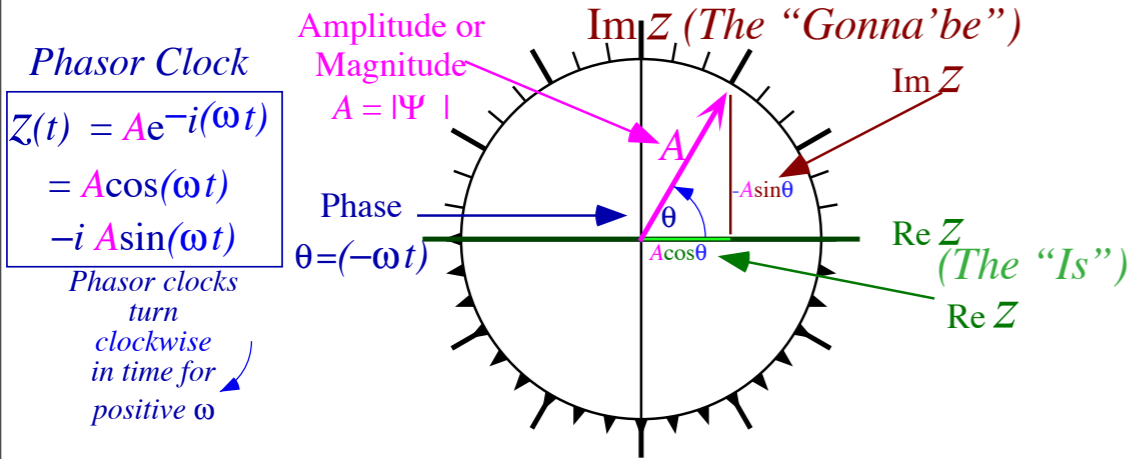
Trick:
Set: $z = z(t) = Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2 \right] e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force $F_{restore} = -kz$

retarded by frictional damping force $F_{damping} = -b \frac{dz}{dt}$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

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$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z = z(t) = A e^{-i\omega t}$

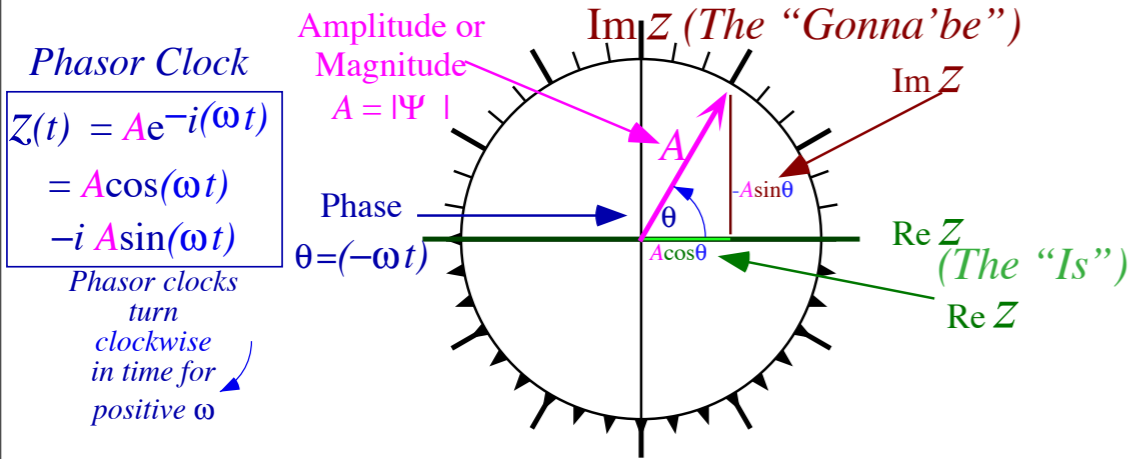
$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2 \right] e^{-i\omega t} = 0$$

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Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force**

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Linear *damped-harmonic oscillator equation of motion.*

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Trick:

Set: $z = z(t) = A e^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2 \right] e^{-i\omega t} = 0$$

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Solve for: $\omega = \omega_{\pm}$

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$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

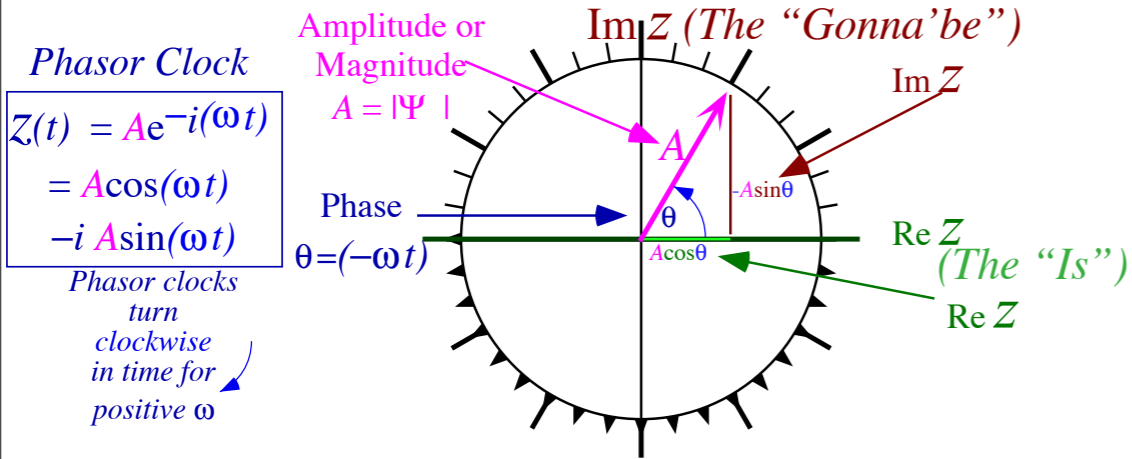
Solution:

$$z(t) = e^{-i \left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2} \right) t}$$

$$= e^{\left(-\Gamma \pm i \sqrt{\omega_0^2 - \Gamma^2} \right) t}$$

$$= e^{-\Gamma t} e^{\pm i \sqrt{\omega_0^2 - \Gamma^2} t}$$

$$= e^{-\Gamma t} e^{\pm i \omega_{\Gamma} t}$$



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force**

$$F_{restore} = -kz$$

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Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

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$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z = z(t) = Ae^{-i\omega t}$

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$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}t}$$

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Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

$$= A\cos(\omega t)$$

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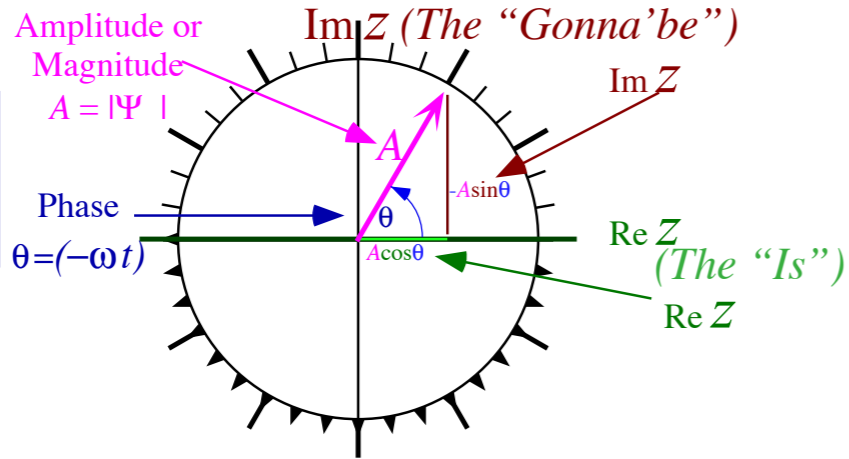
Phasor clocks

turn

clockwise

in time for

positive ω



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force**

$$F_{restore} = -kz$$

retarded by **frictional damping force**

$$F_{damping} = -b \frac{dz}{dt}$$

It oscillates at an angular frequency ω_{Γ} reduced slightly by .05% from ω_0 due to damping $\Gamma = 0.2$.

$$\omega_{\Gamma} = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0 - \frac{1}{2}(\Gamma^2 / \omega_0) + \dots = 6.2831853 - 0.003183 + \dots = 6.280002 + \dots = 6.280001$$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z = z(t) = Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2 \right] e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

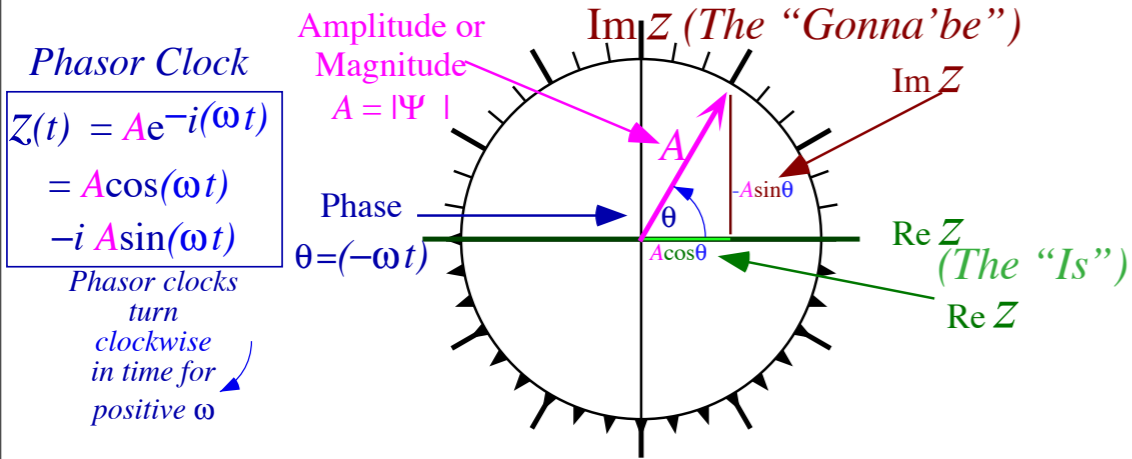
Solution:

$$z(t) = e^{-i \left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2} \right) t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2} \right) t}$$

$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2} t}$$

$$= e^{-\Gamma t} e^{\pm i\omega_{\Gamma} t}$$



Coordinate $z = z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force

$$F_{restore} = -kz$$

retarded by frictional damping force

$$F_{damping} = -b \frac{dz}{dt}$$

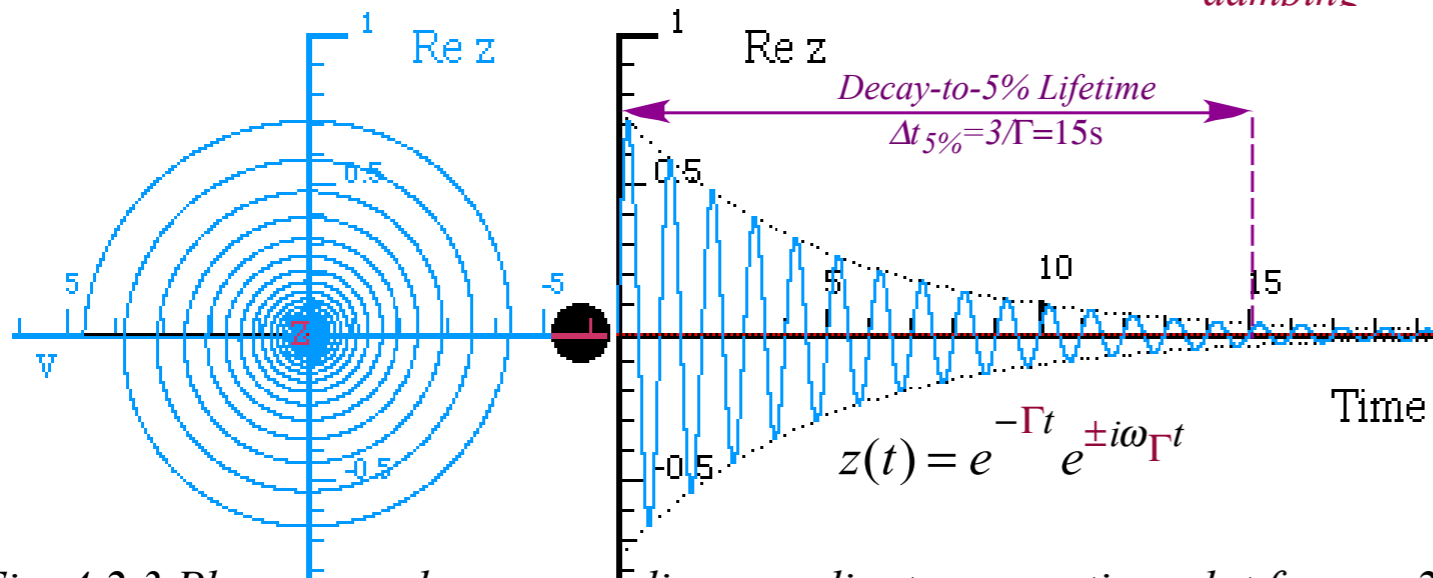


Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0 = 2\pi$ and $\Gamma = 0.2$

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

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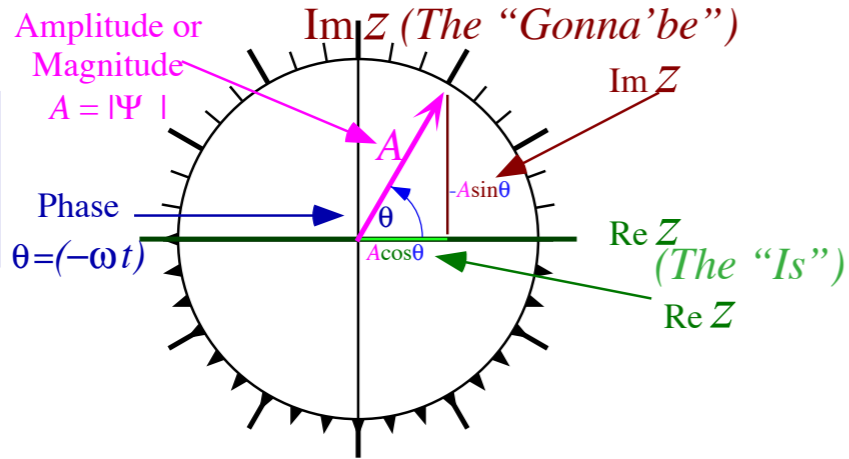
Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

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Phasor clocks turn clockwise in time for positive ω



Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force

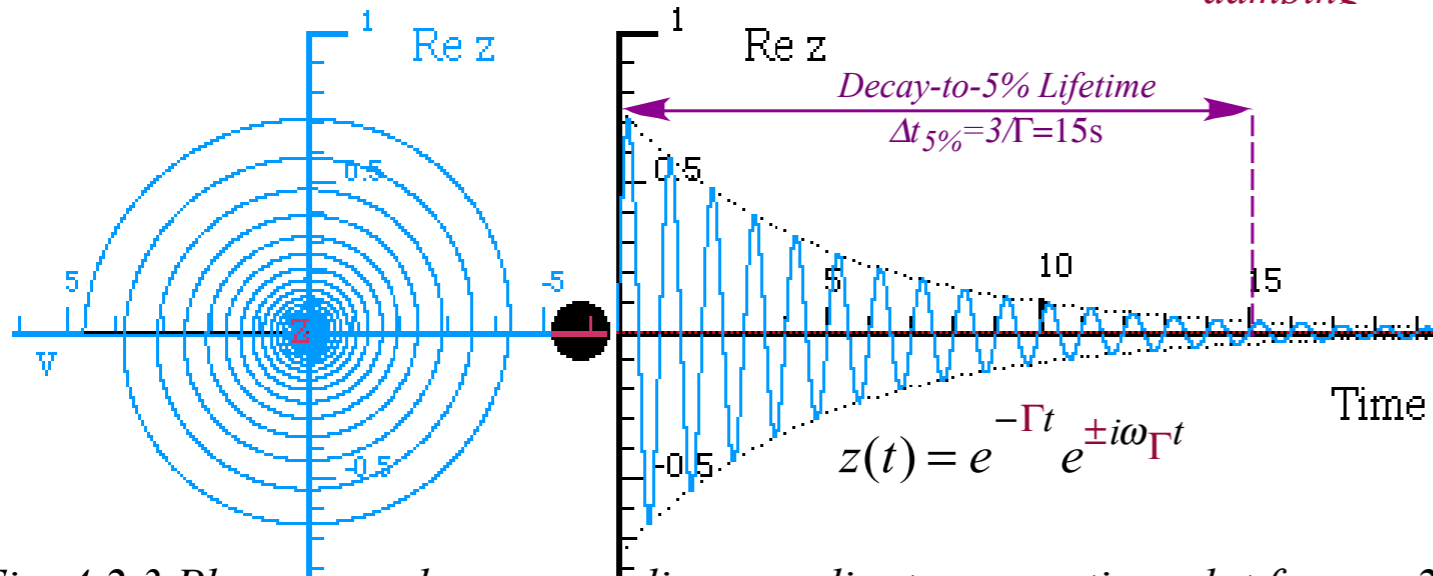
$$F_{restore} = -kz$$

retarded by frictional damping force

$$F_{damping} = -b \frac{dz}{dt}$$

Oscillator Figures of Merit:

Time required to reduce amplitude to 5%



Easy-to-recall 5% approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

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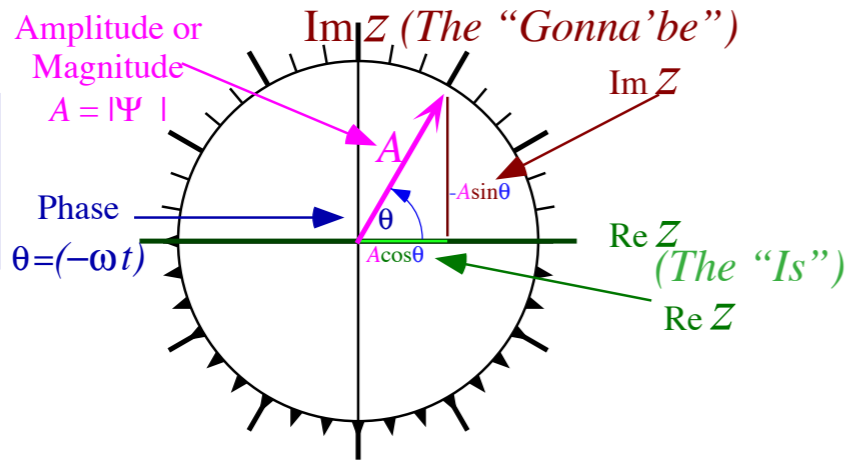
Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

$$= A \cos(\omega t)$$

$$-i A \sin(\omega t)$$

Phasor clocks
turn
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Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force**

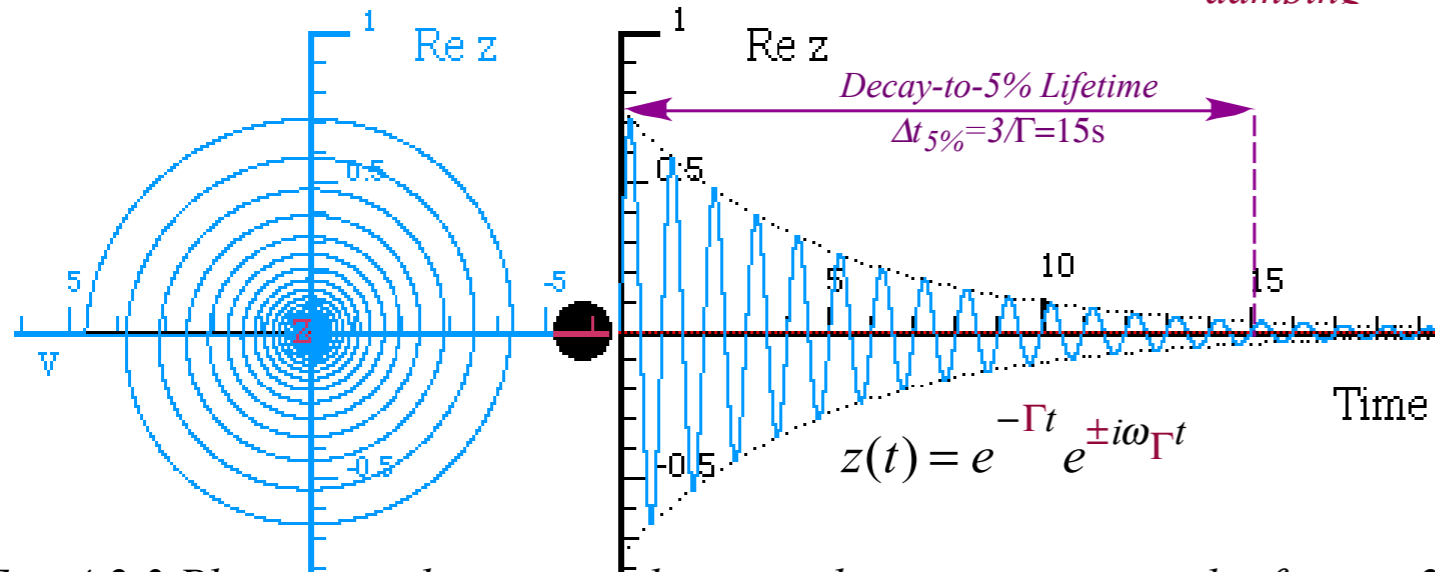
$$F_{restore} = -kz$$

retarded by **frictional damping force**

$$F_{damping} = -b \frac{dz}{dt}$$

Oscillator Figures of Merit:

Time required to
to reduce amplitude
to 5% (or 4.321%)



Easy-to-recall 5% approximation: More precise one:

$$e^{-3} \cong 0.05$$

$$e^{-\pi} \cong 0.04321$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15 \quad t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear *damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = 0$$

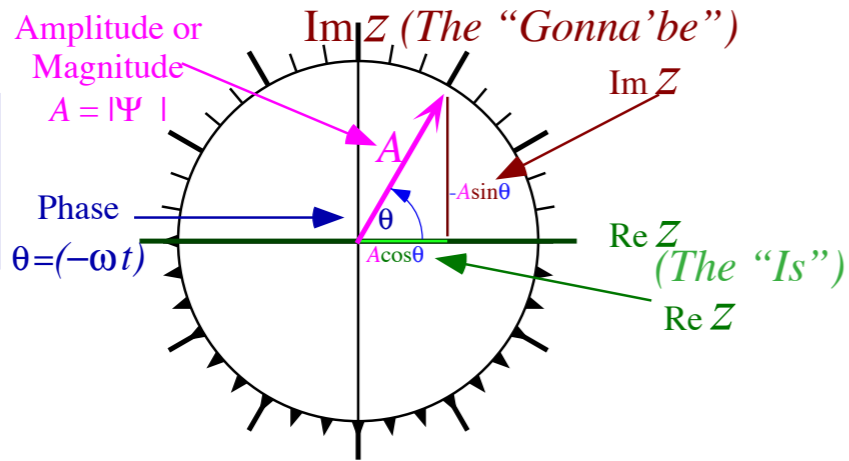
Phasor Clock

$$Z(t) = Ae^{-i(\omega t)}$$

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Phasor clocks turn clockwise in time for positive ω



Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a **harmonic (linear) restoring force**

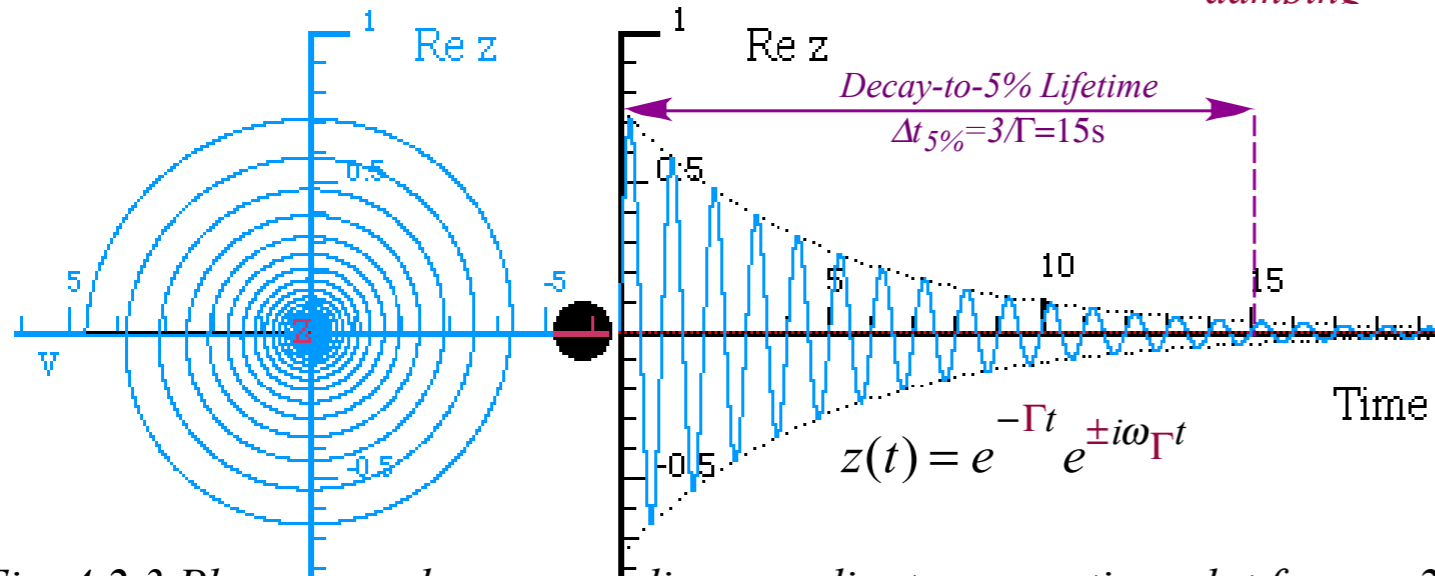
$$F_{restore} = -kz$$

retarded by **frictional damping force**

$$F_{damping} = -b \frac{dz}{dt}$$

Oscillator Figures of Merit:

Number N of oscillations to reduce amplitude to 5% (or 4.321%)



Easy-to-recall 5% approximation: $e^{-3} \cong 0.05$ More precise one: $e^{-\pi} \cong 0.04321$

$$N_{5\%} = \frac{\omega_{\Gamma} \cdot t_{5\%}}{2\pi} = \frac{3\omega_{\Gamma}}{2\pi\Gamma} \sim \frac{\omega_{\Gamma}}{2\Gamma}$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

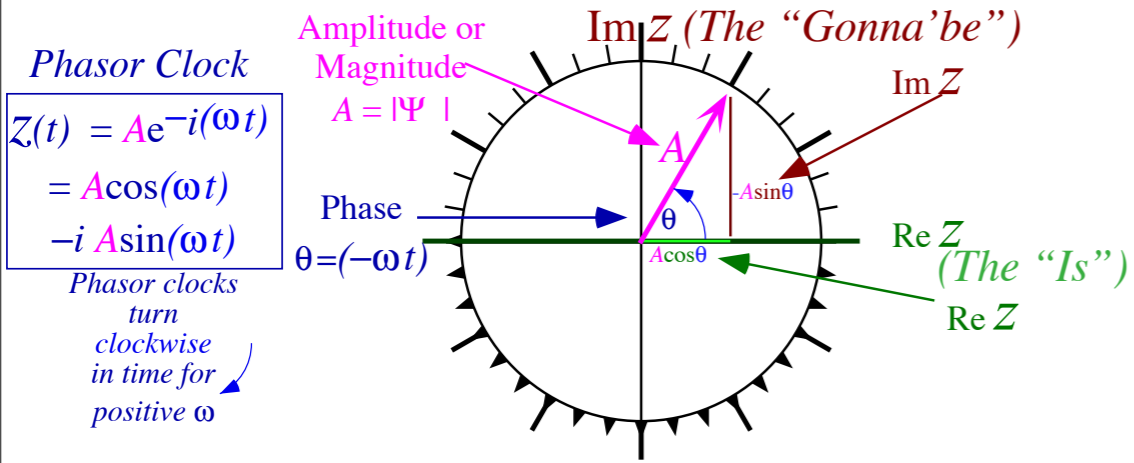
Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external **stimulating force** $\longrightarrow F_{stimulus}(t) = eE(t)$

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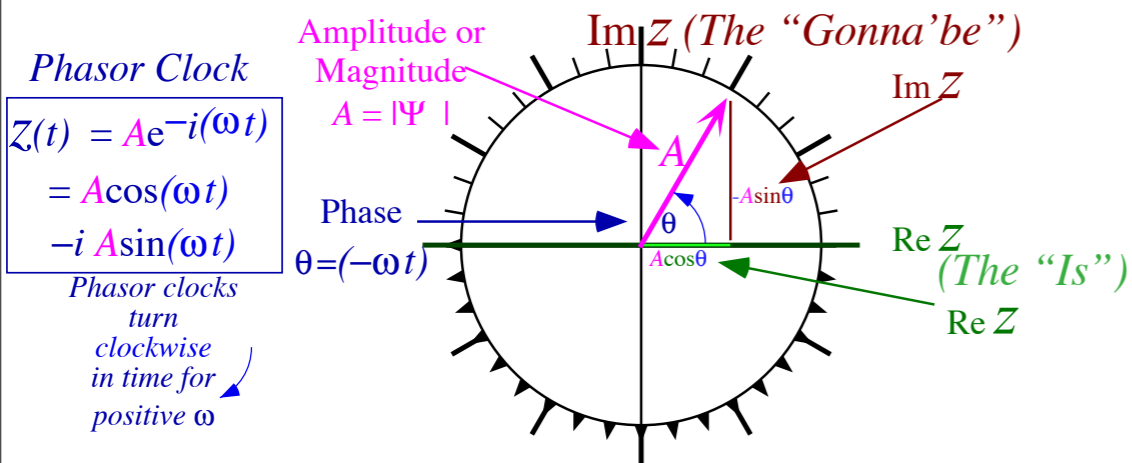
retarded by **frictional damping force** $\longrightarrow F_{damping} = -b \frac{dz}{dt}, (b = 2\Gamma m)$

Linear forced-damped-harmonic oscillator equation of motion.

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Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

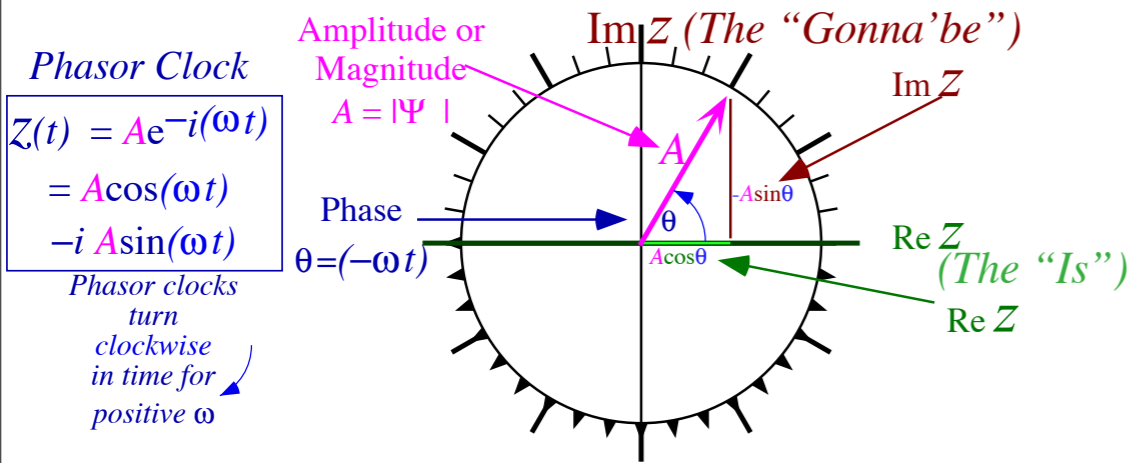
Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

Linear forced-damped-harmonic oscillator equation of motion.

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Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 \right) z = a_{stimulus}$$

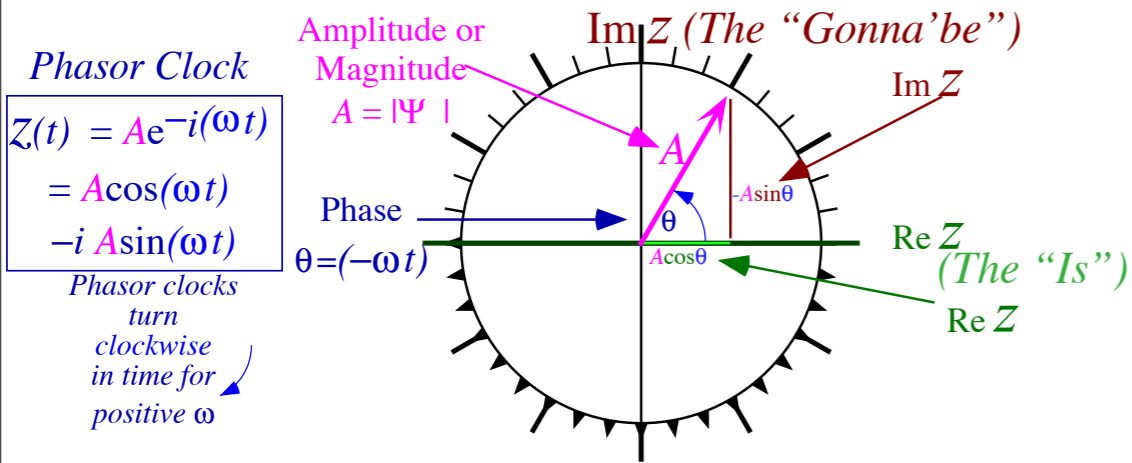
$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2} a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

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$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 \right) z = a_{stimulus}$$

Pretty crazy?

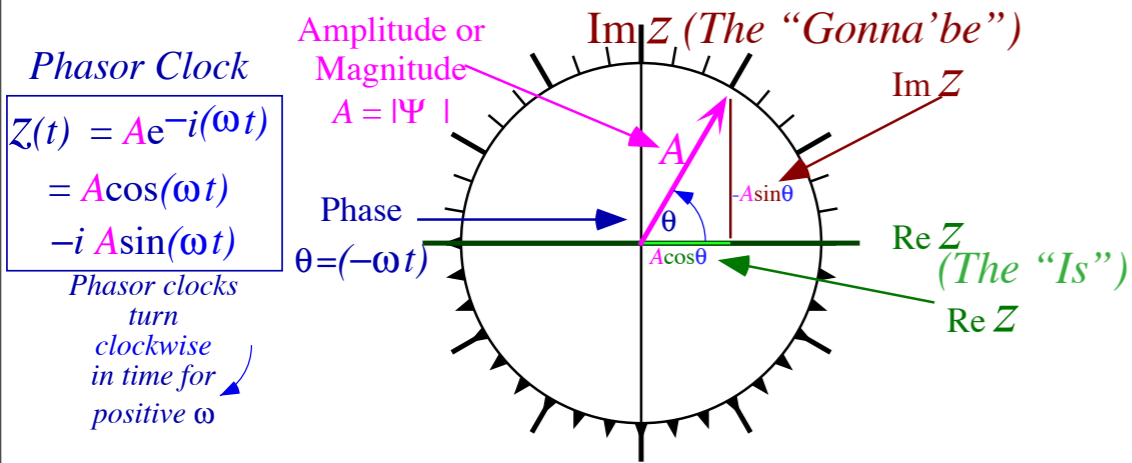
$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2} a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 \right) z = a_{stimulus}$$

Pretty crazy? But not so crazy if

$$a_{stimulus}(t) = |a_{stimulus}| e^{-i\omega_{stimulus} t} = |a_s| e^{-i\omega_s t}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2} a_{stimulus}$$

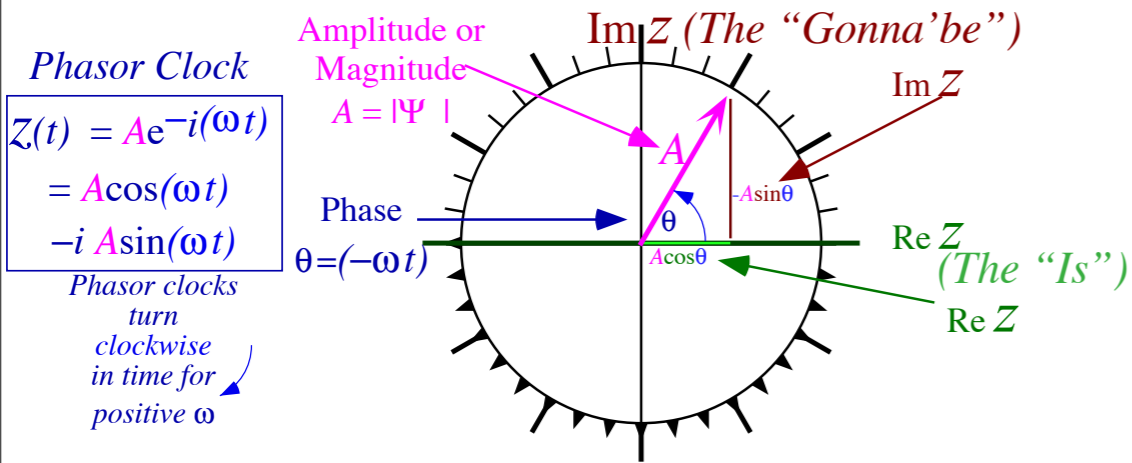
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$$\frac{d^2 z}{dt^2} + 2\Gamma \frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m} E(t)$$



Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 \right) z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

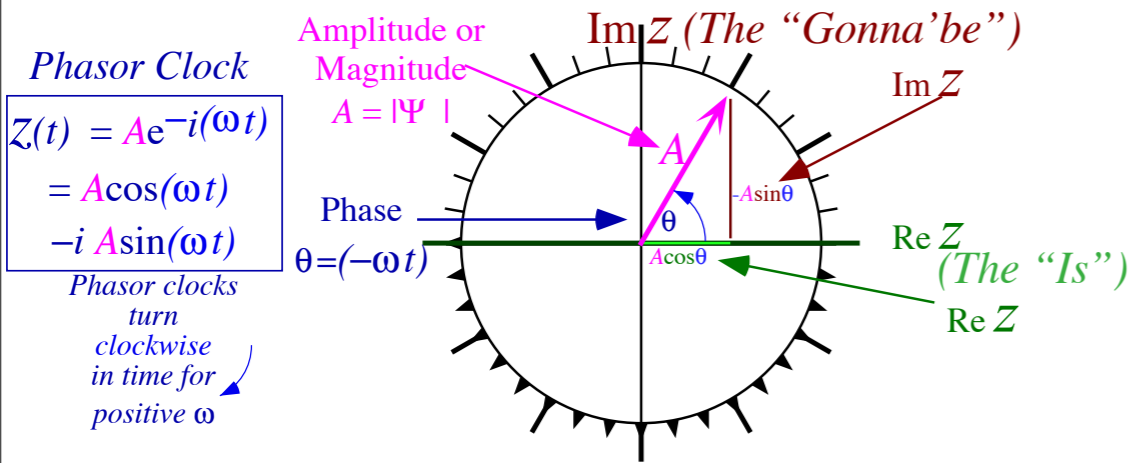
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$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

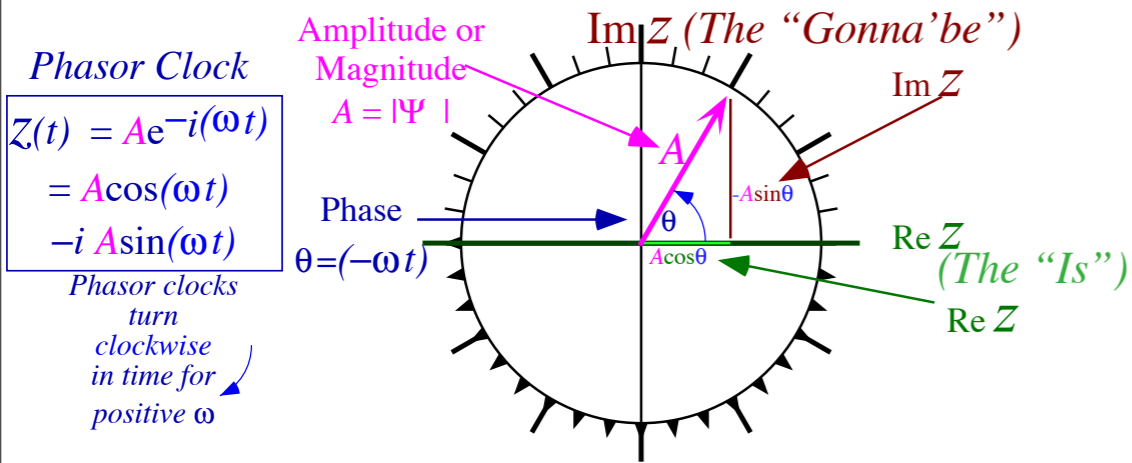
Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m \frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

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Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma \frac{d}{dt} + \omega_0^2 \right) z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}| e^{-i\omega_{stimulus} t} = |a_s| e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

Green's Function for the F-D-H Oscillator (FDHO)

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

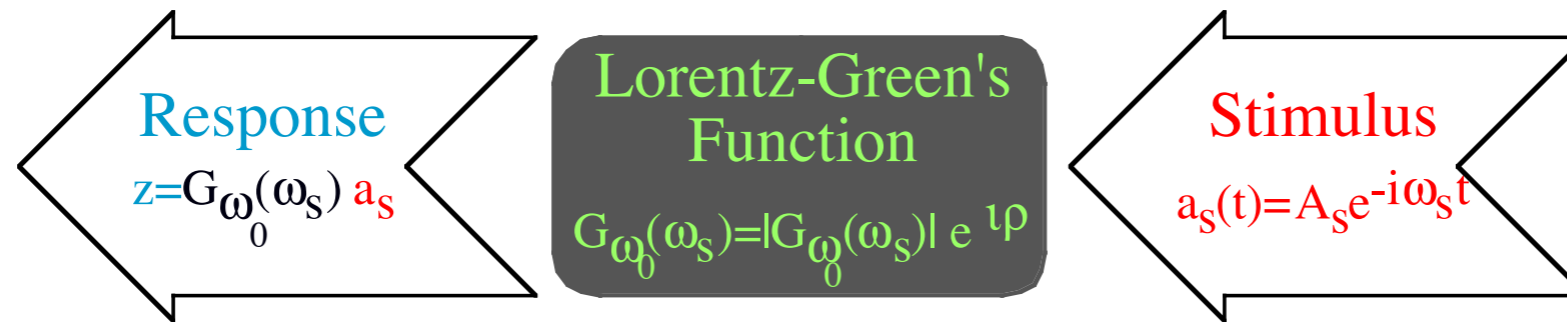


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G :

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

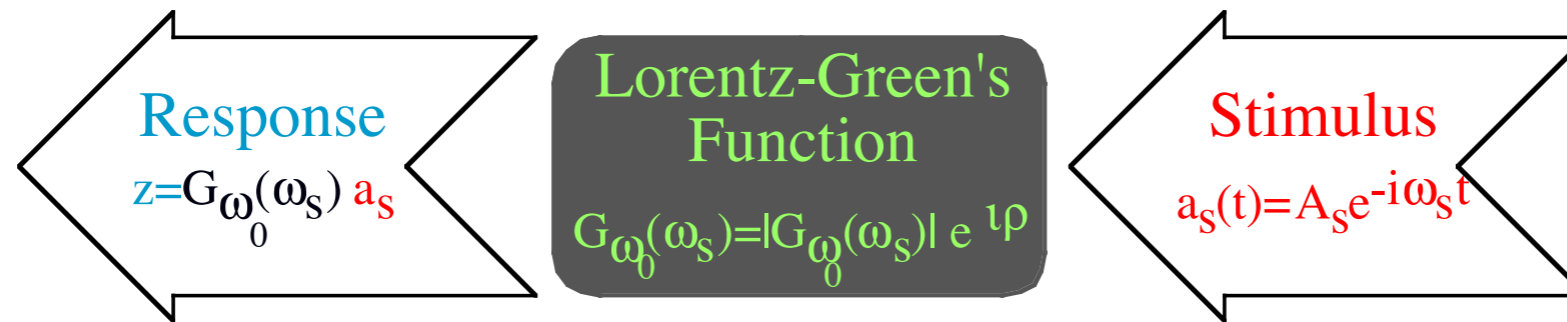


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G : $\frac{1}{x-iy} = \frac{1}{x-iy} \frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2}$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

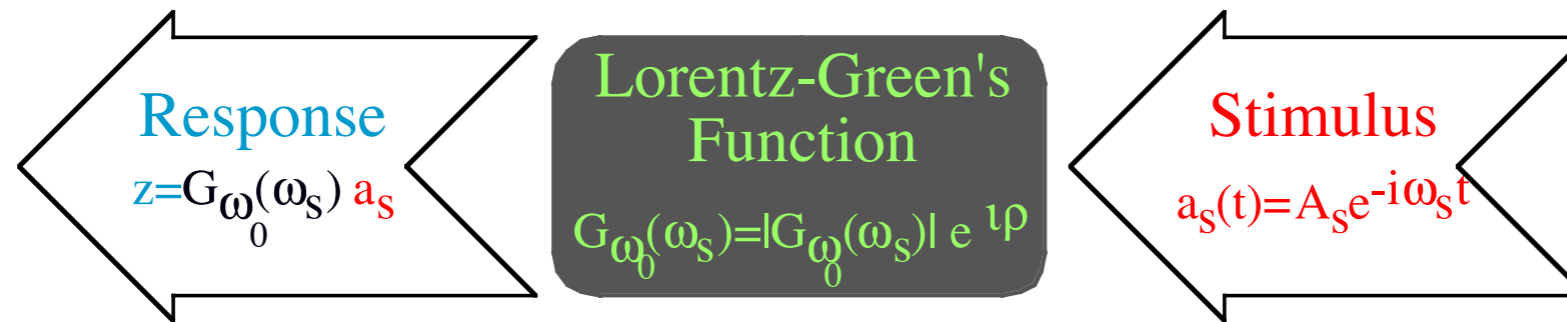


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G : $\frac{1}{x - iy} = \frac{1}{x - iy} \frac{x + iy}{x + iy} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{y}{x^2 + y^2}$

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

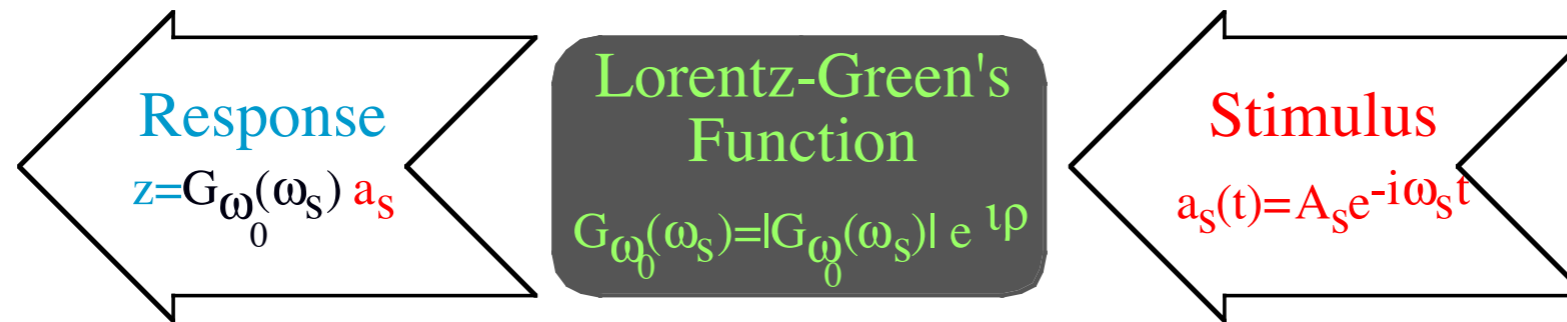


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s) = |G_{\omega_0}(\omega_s)| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G :

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Magnitude $|G_{\omega_0}(\omega_s)|$ and polar angle ρ of the *polar form* of G :

$$|G_{\omega_0}(\omega_s)| = \frac{1}{\sqrt{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}}$$

$$\rho = \tan^{-1} \left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2} \right)$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

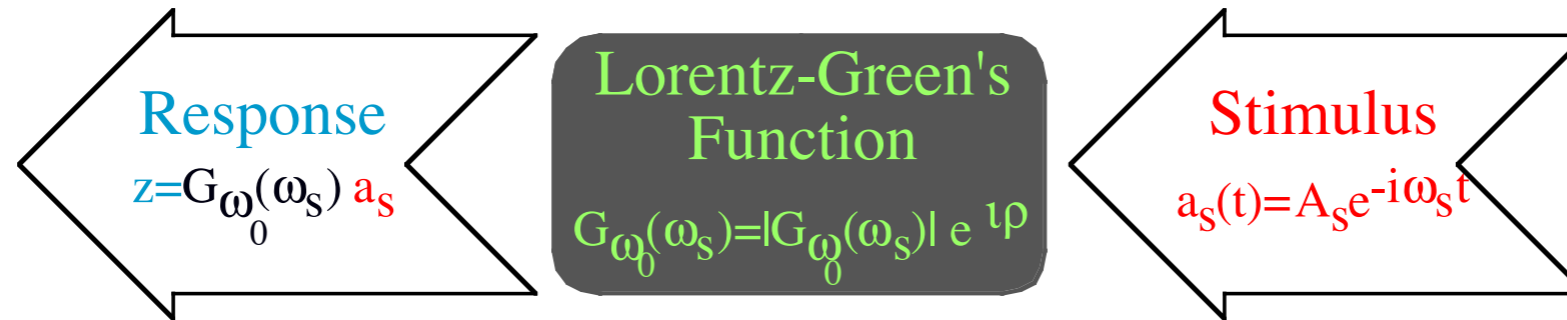


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s) = |G_{\omega_0}(\omega_s)| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G :

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

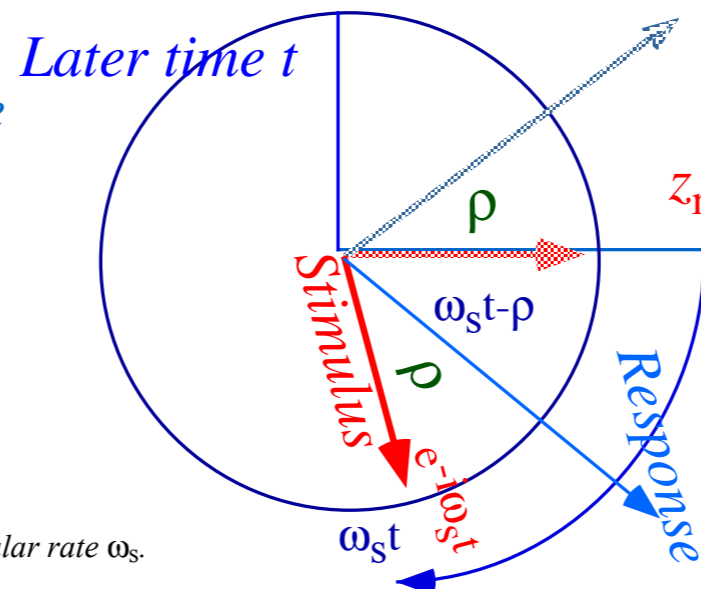
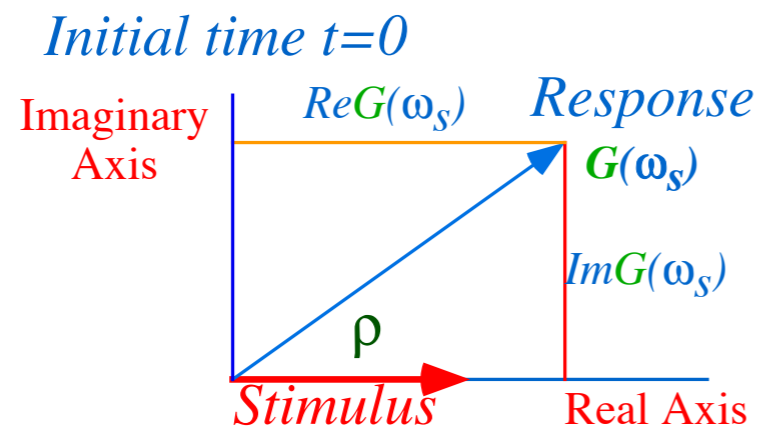
$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Magnitude $|G_{\omega_0}(\omega_s)|$ and *polar angle* ρ of the *polar form* of G :

$$|G_{\omega_0}(\omega_s)| = \frac{1}{\sqrt{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

polar angle ρ is the *phase lag angle* ρ



$$z_{\text{response}}(t) = |G_{\omega_0}(\omega_s)| a(0) e^{-i(\omega_s t - \rho)}$$

Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s .

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

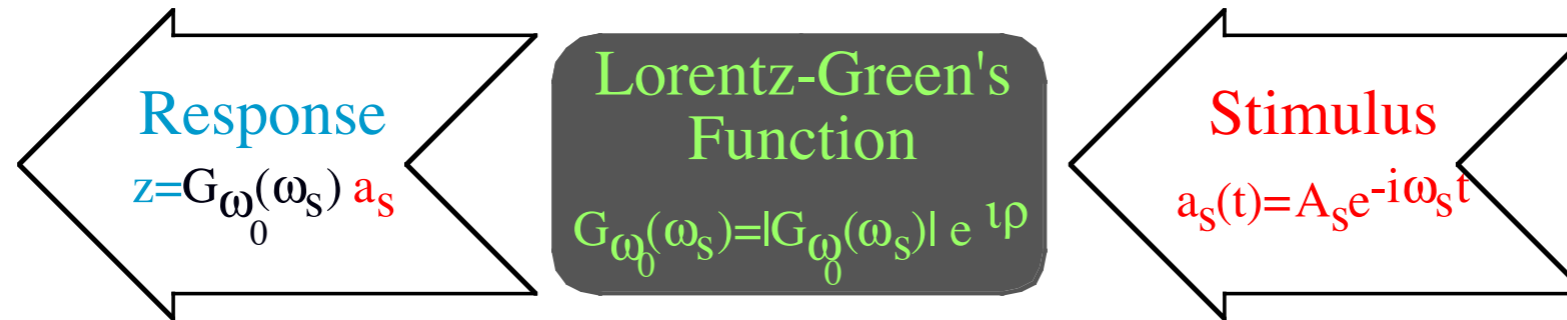


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re } G_{\omega_0}(\omega_s) + i \text{Im } G_{\omega_0}(\omega_s) = |G_{\omega_0}(\omega_s)| e^{i\rho}$$

Real and imaginary parts of the rectangular form of G :

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

Magnitude $|G_{\omega_0}(\omega_s)|$ and *polar angle* ρ of the *polar form* of G :

$$|G_{\omega_0}(\omega_s)| = \frac{1}{\sqrt{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

polar angle ρ is the *phase lag angle* ρ

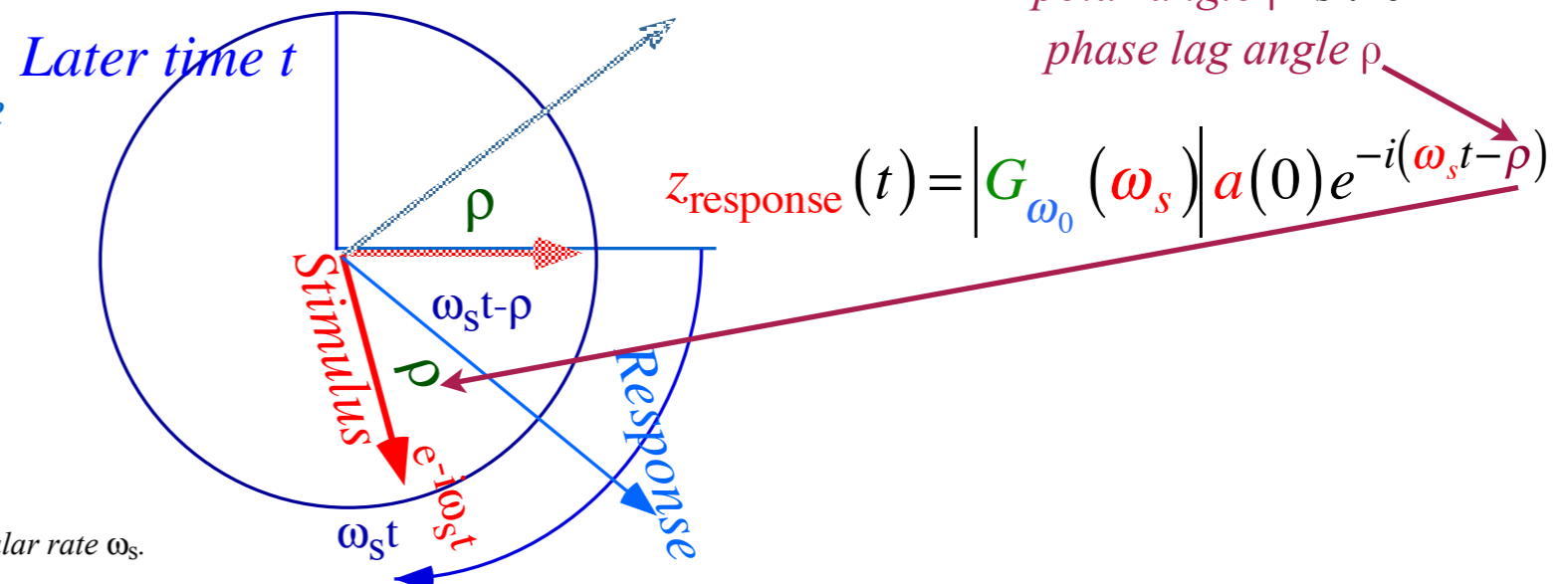
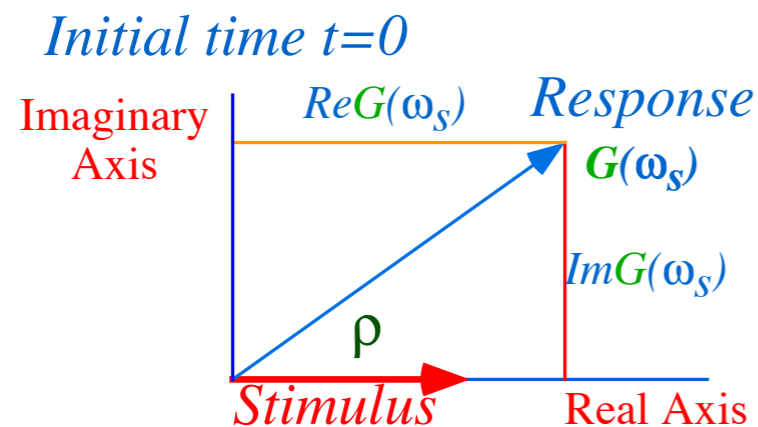


Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s .

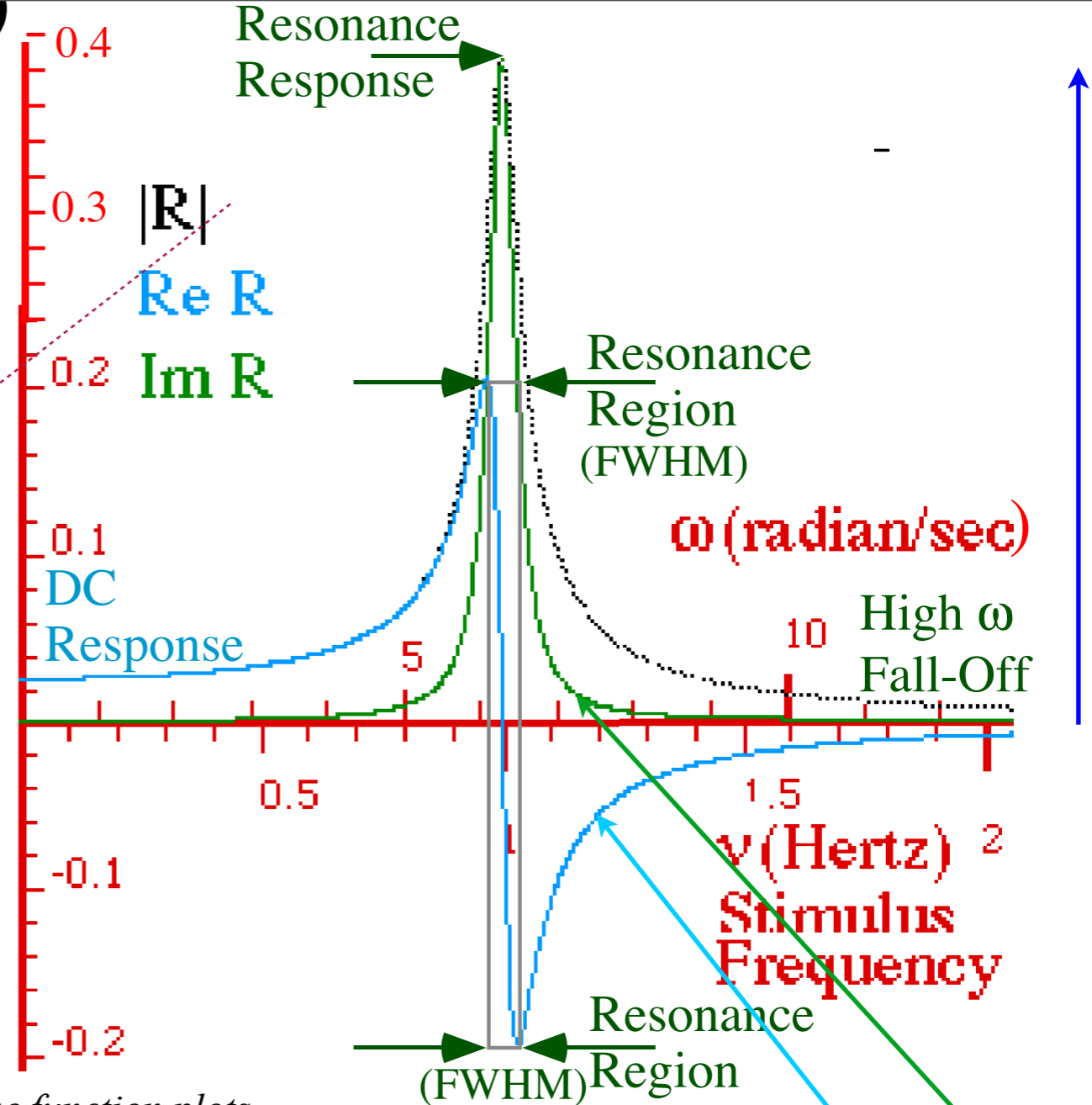
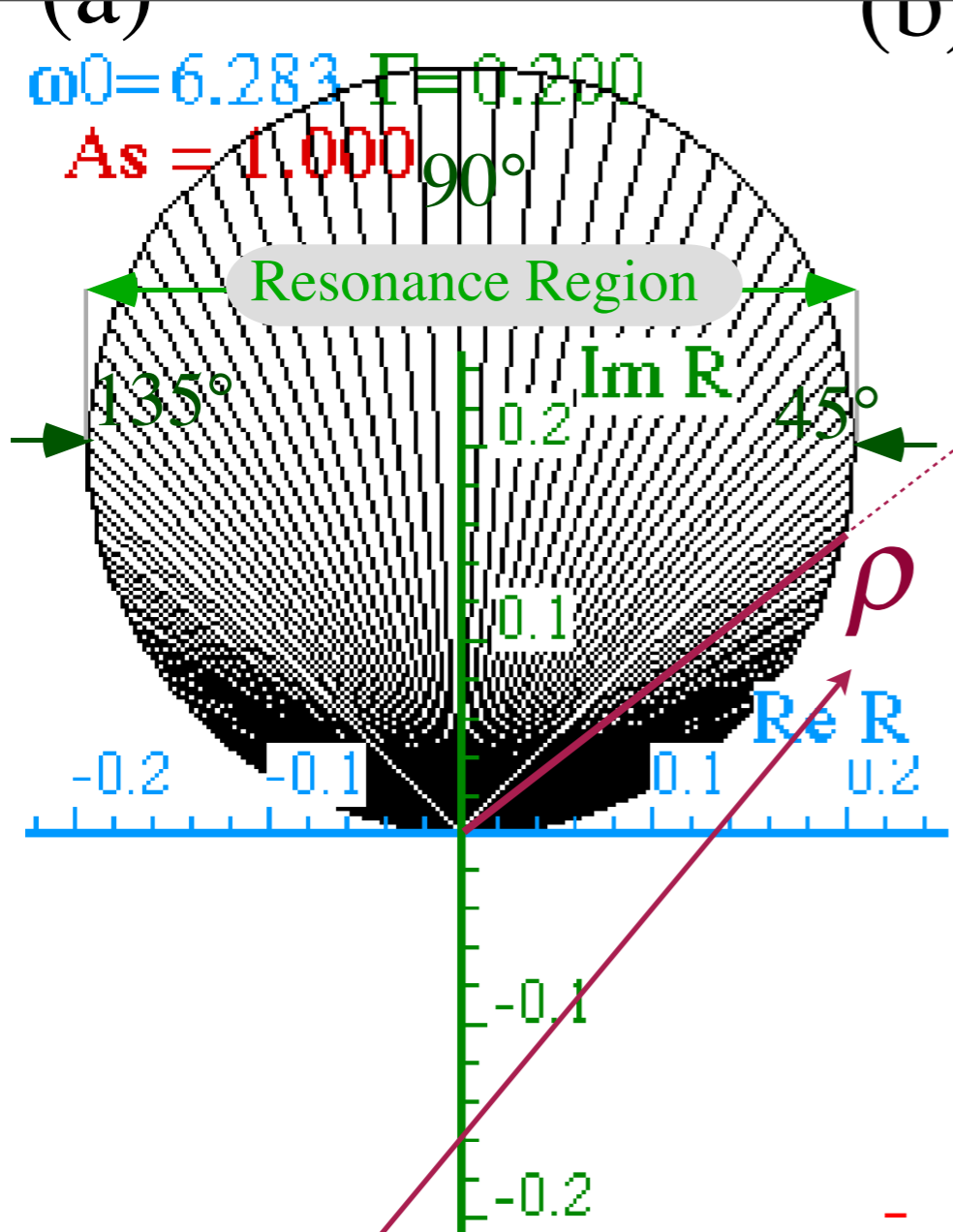


Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1} \left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2} \right)$$

$$Re G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

$$Im G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2}$$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\text{angular quality factor})$$

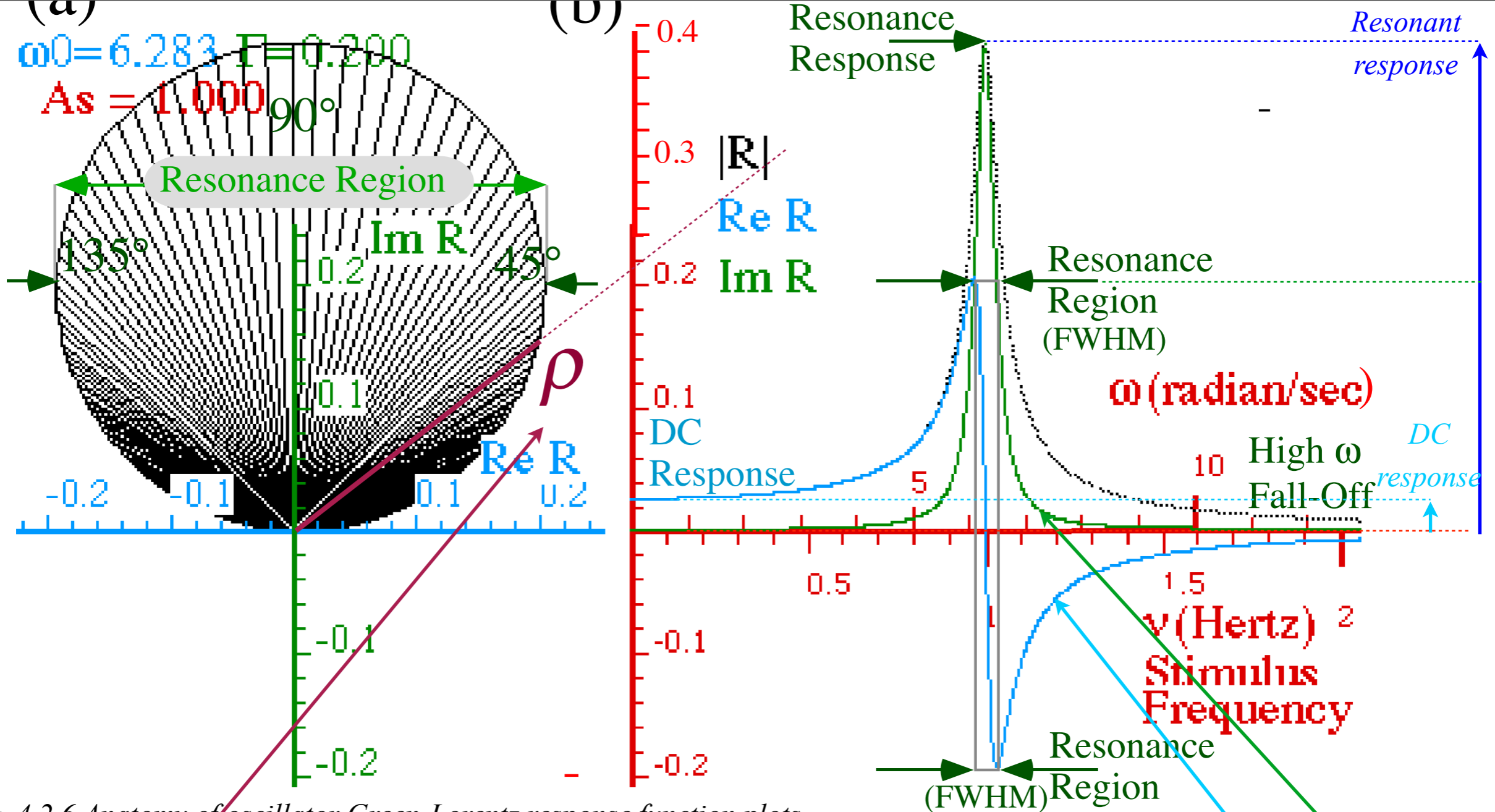


Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1} \left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2} \right)$$

$$\text{Re } G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2} \quad \text{Real part}$$

$$\text{Im } G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{(\omega_0^2 - \omega_s^2)^2 + (2\Gamma\omega_s)^2} \quad \text{Imaginary part}$$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\text{angular quality factor})$$

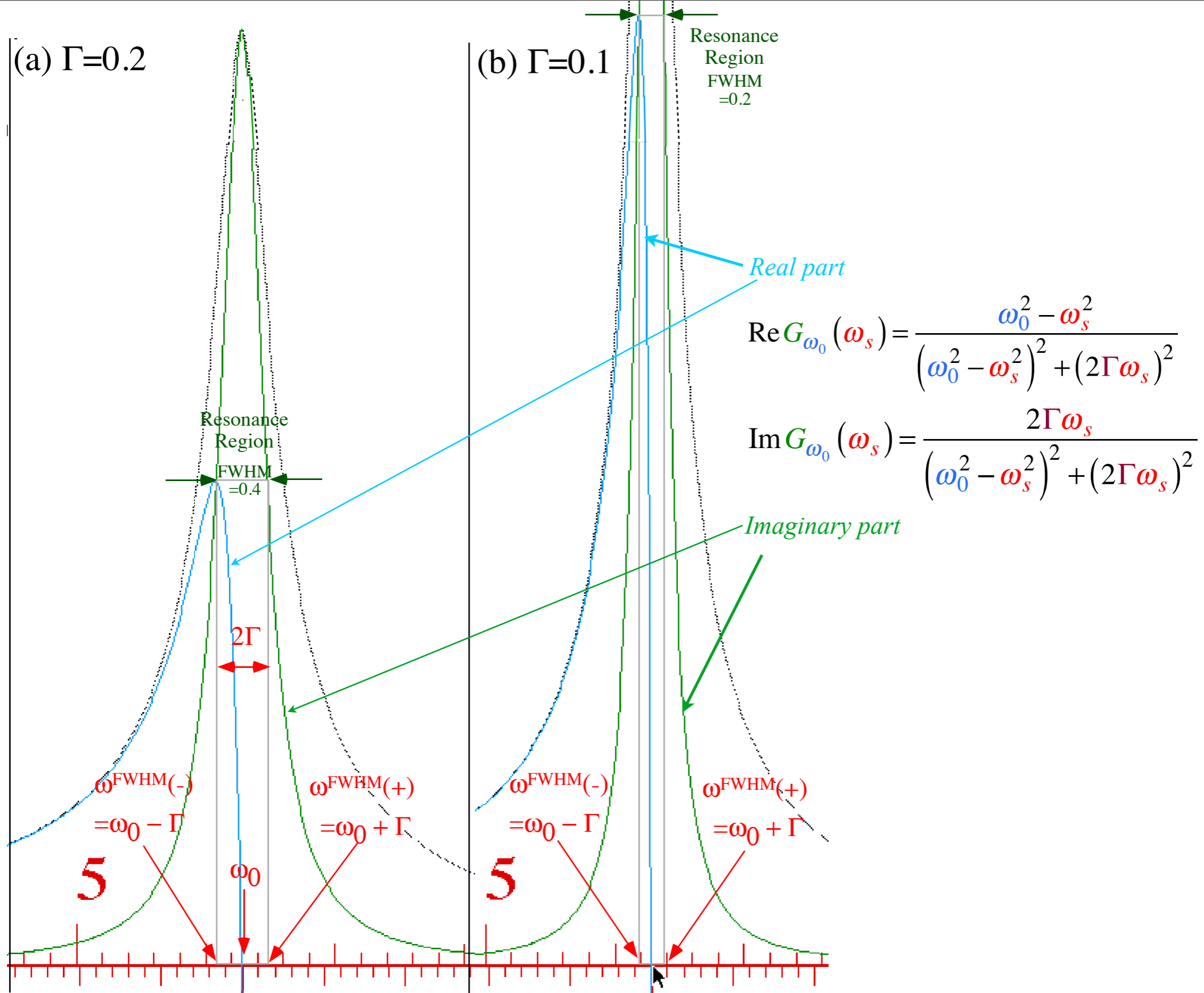


Fig. 4.2.7 Comparing Lorentz-Green resonance region for (a) $\Gamma=0.2$ and (b) $\Gamma=0.1$.

Maximum and minimum points of $\text{Re}G(\omega)$ and inflection points of $\text{Im}G(\omega)$ are near region boundaries $\omega^{\text{FWHM}(\pm)} = \omega_0 \pm \Gamma$.

Complete *Green's Solution* for the *FDHO* (*Forced-Damped-Harmonic Oscillator*)

$$\begin{aligned} z(t) &= z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t) \\ &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + G_{\omega_0}(\omega_s) a(0) e^{-i\omega_s t} \\ &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + \left| G_{\omega_0}(\omega_s) \right| a(0) e^{-i(\omega_s t - \rho)} \end{aligned}$$

Known as “homogeneous” solution (no force)
Let's you set initial values or boundary conditions

Known as “inhomogeneous” solution
Not function of initial values. Marches to stimulus only.

Complete *Green's Solution* for the *FDHO* (*Forced-Damped-Harmonic Oscillator*)

$$\begin{aligned}z(t) &= z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t) \\ &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + G_{\omega_0}(\omega_s) a(0) e^{-i\omega_s t} \\ &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + \left| G_{\omega_0}(\omega_s) \right| a(0) e^{-i(\omega_s t - \rho)}\end{aligned}$$

Known as “homogeneous” solution (no force)
Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as “inhomogeneous” solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$\begin{aligned}
 z(t) &= z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t) \\
 &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + G_{\omega_0}(\omega_s) a(0) e^{-i\omega_s t} \\
 &= Ae^{-\Gamma t} e^{-i\omega_{\Gamma} t} + \left| G_{\omega_0}(\omega_s) \right| a(0) e^{-i(\omega_s t - \rho)}
 \end{aligned}$$

Known as "homogeneous" solution (no force)
 Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "inhomogeneous" solution
 Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

Stimulus: $A_s = 0.5000$ $\omega = 6.2832$
 Response: $R = 0.1989$ $\rho = 1.5708$

About $t = 3/\Gamma = 15 \text{ sec}$

About $t = \text{forever}$

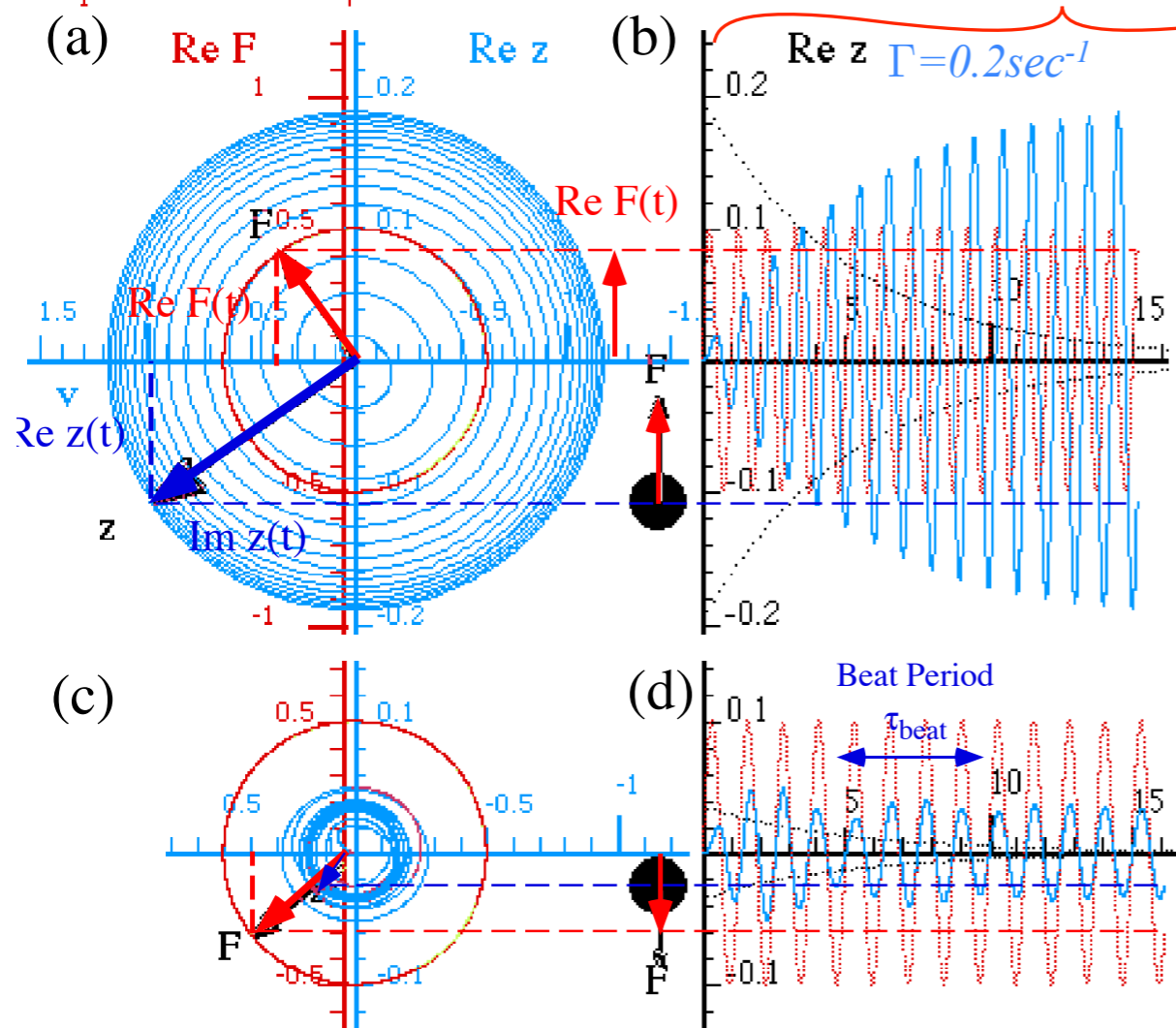


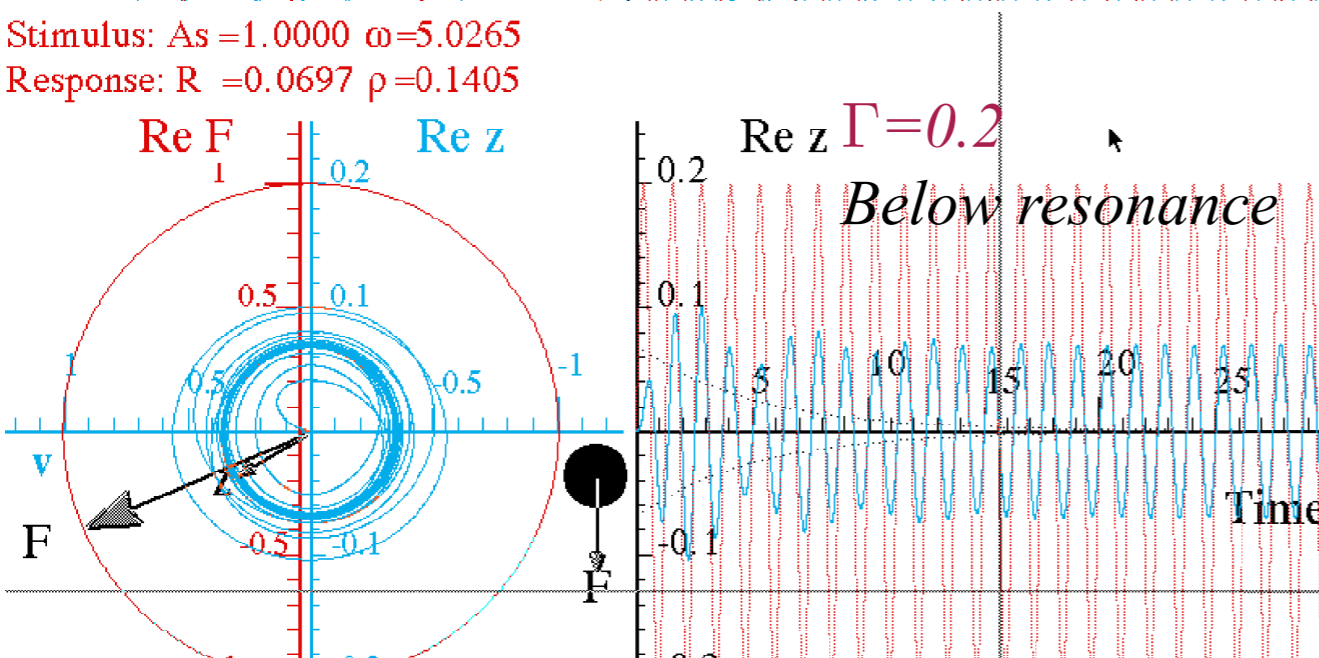
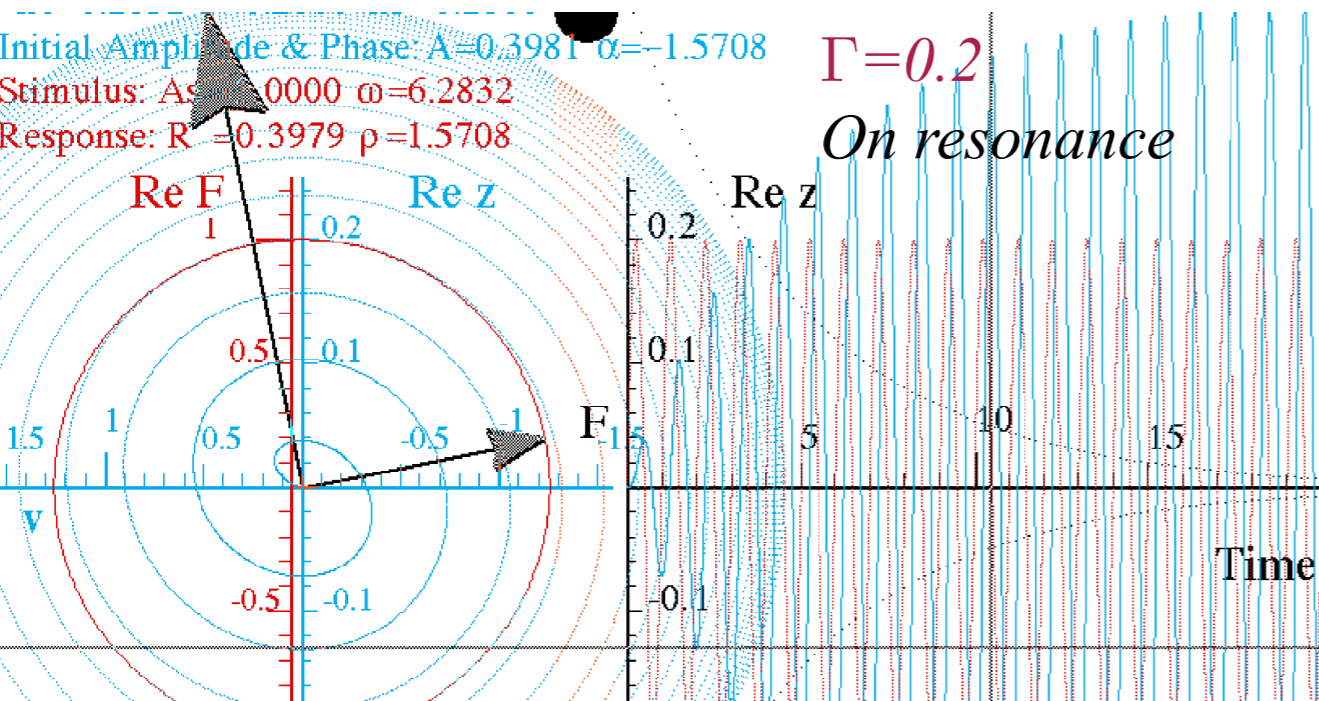
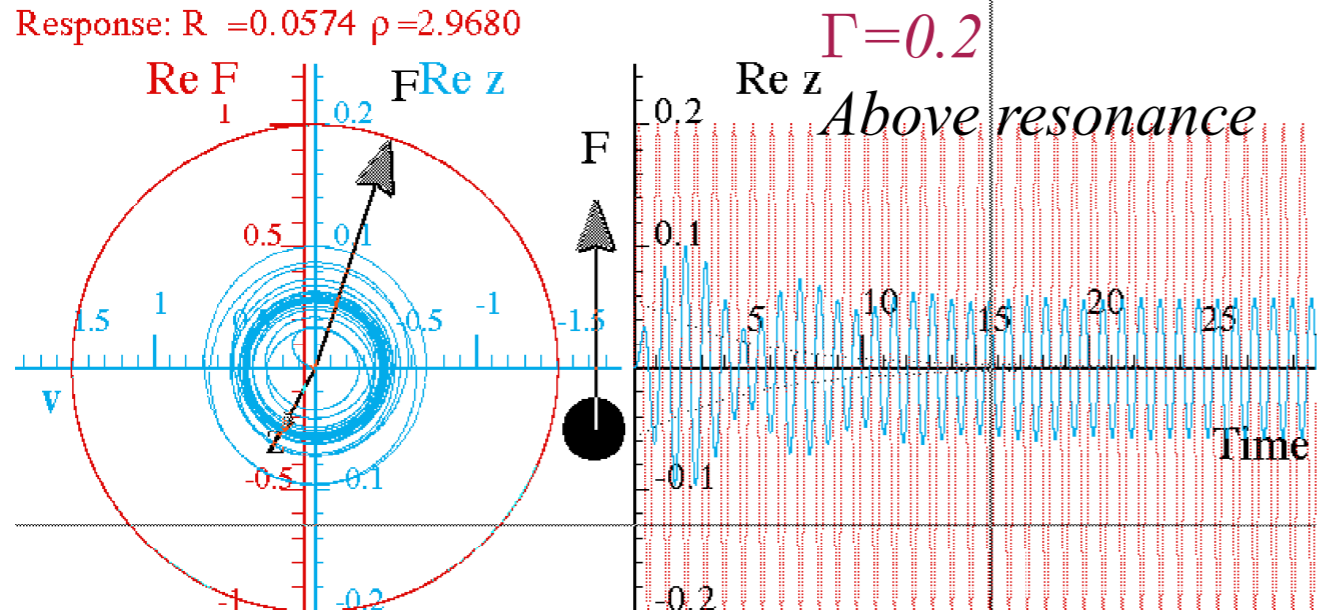
Fig. 4.2.8 On Resonance (a) Response z -phasor lags $\rho = 90^\circ$ behind stimulus F -phasor.

($\omega_s = \omega_0 = 2\pi$, $\omega_0 = 2\pi$, and $\Gamma = 0.2$). (b) Time plots of $\text{Re } z(t)$ and $\text{Re } F(t)$

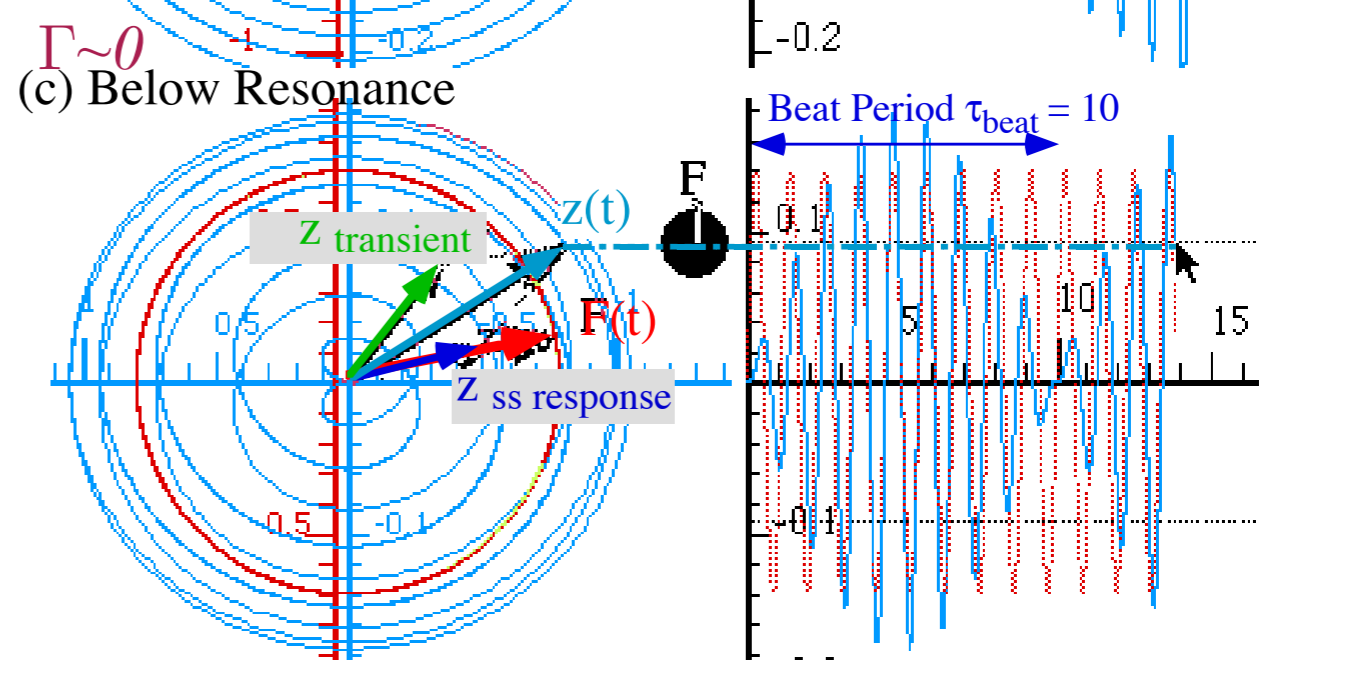
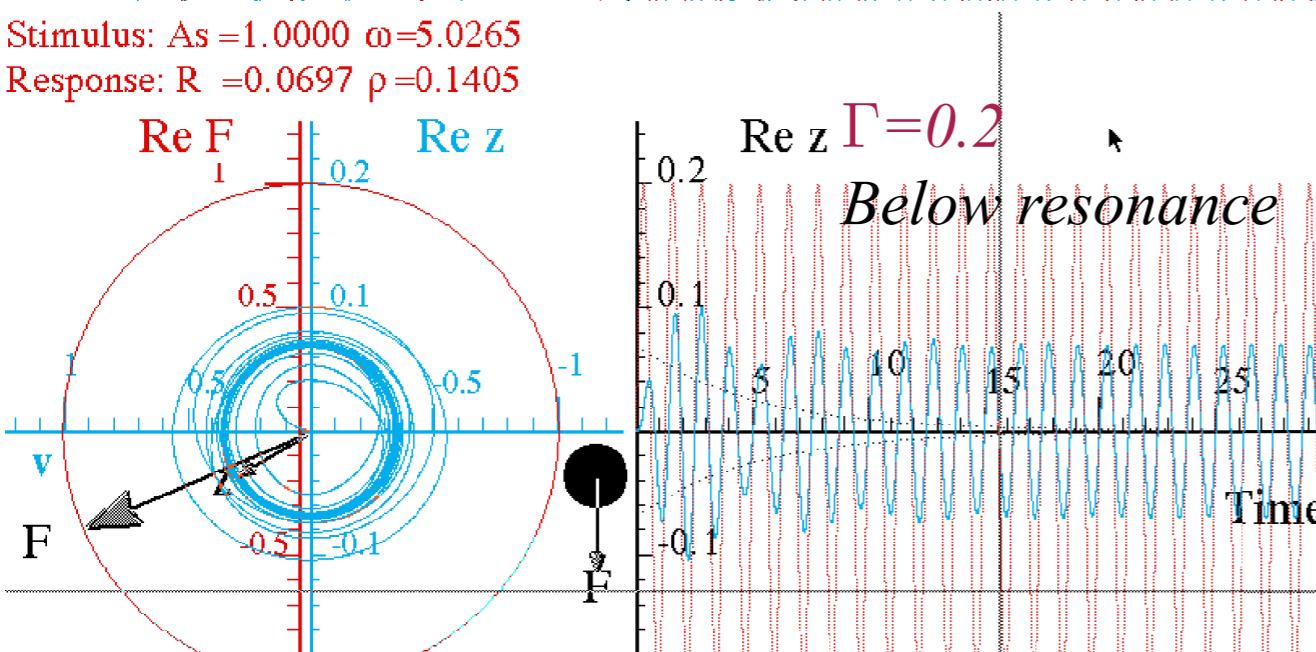
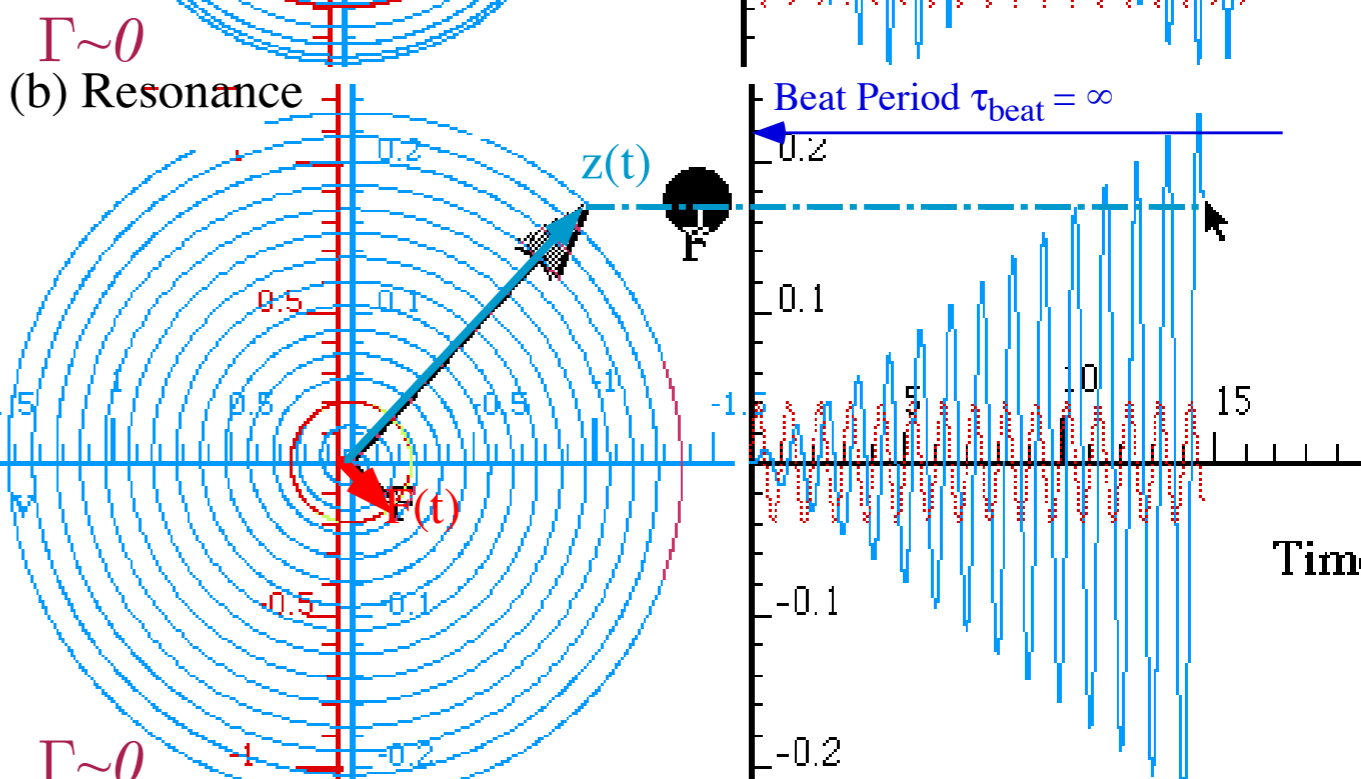
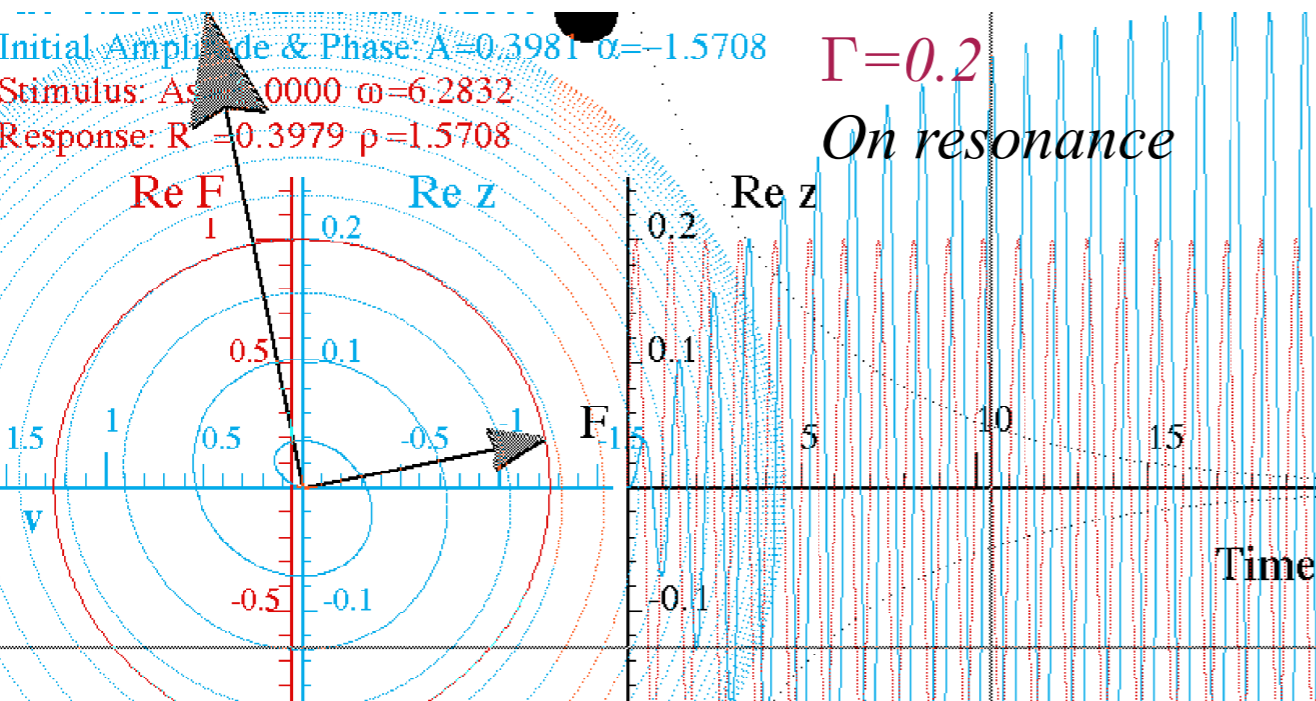
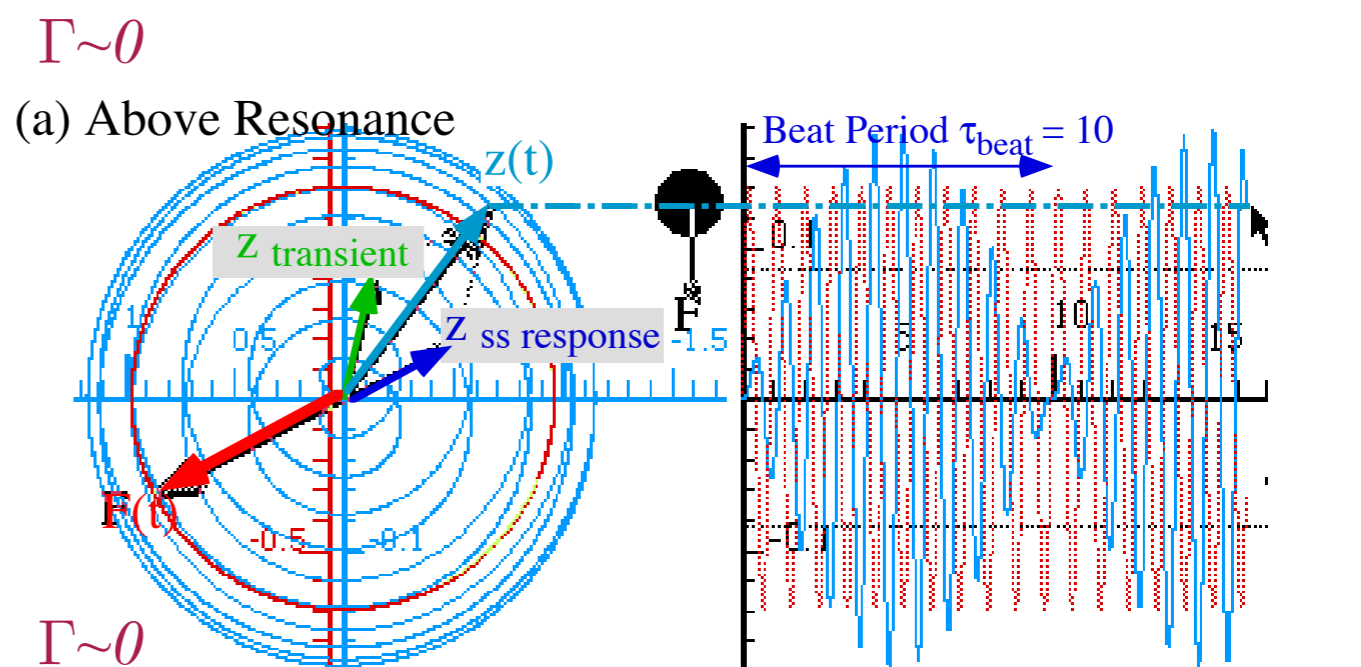
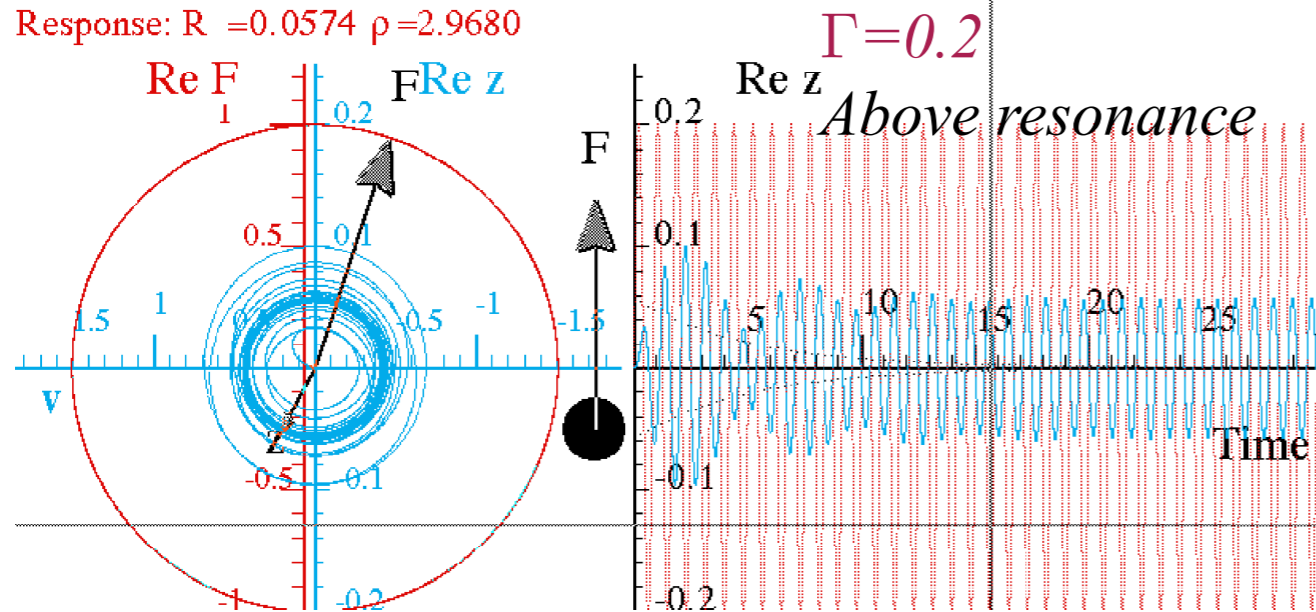
Fig. 4.2.8 Below Resonance (c) Response z -phasor lags $\rho = 8.05^\circ$ behind stimulus F -phasor.

($\omega_s = 5.03$, $\omega_0 = 2\pi$, and $\Gamma = 0.2$). (d) Time plots of $\text{Re } z(t)$ and $\text{Re } F(t)$. Beats are barely visible.

Stimulus: $A_s = 1.0000$ $\omega = 7.5265$
 Response: $R = 0.0574$ $\rho = 2.9680$



Stimulus: $A_s = 1.0000$ $\omega = 7.5265$
 Response: $R = 0.0574$ $\rho = 2.9680$



Lorentz-Green's Function for high quality *FDHO*

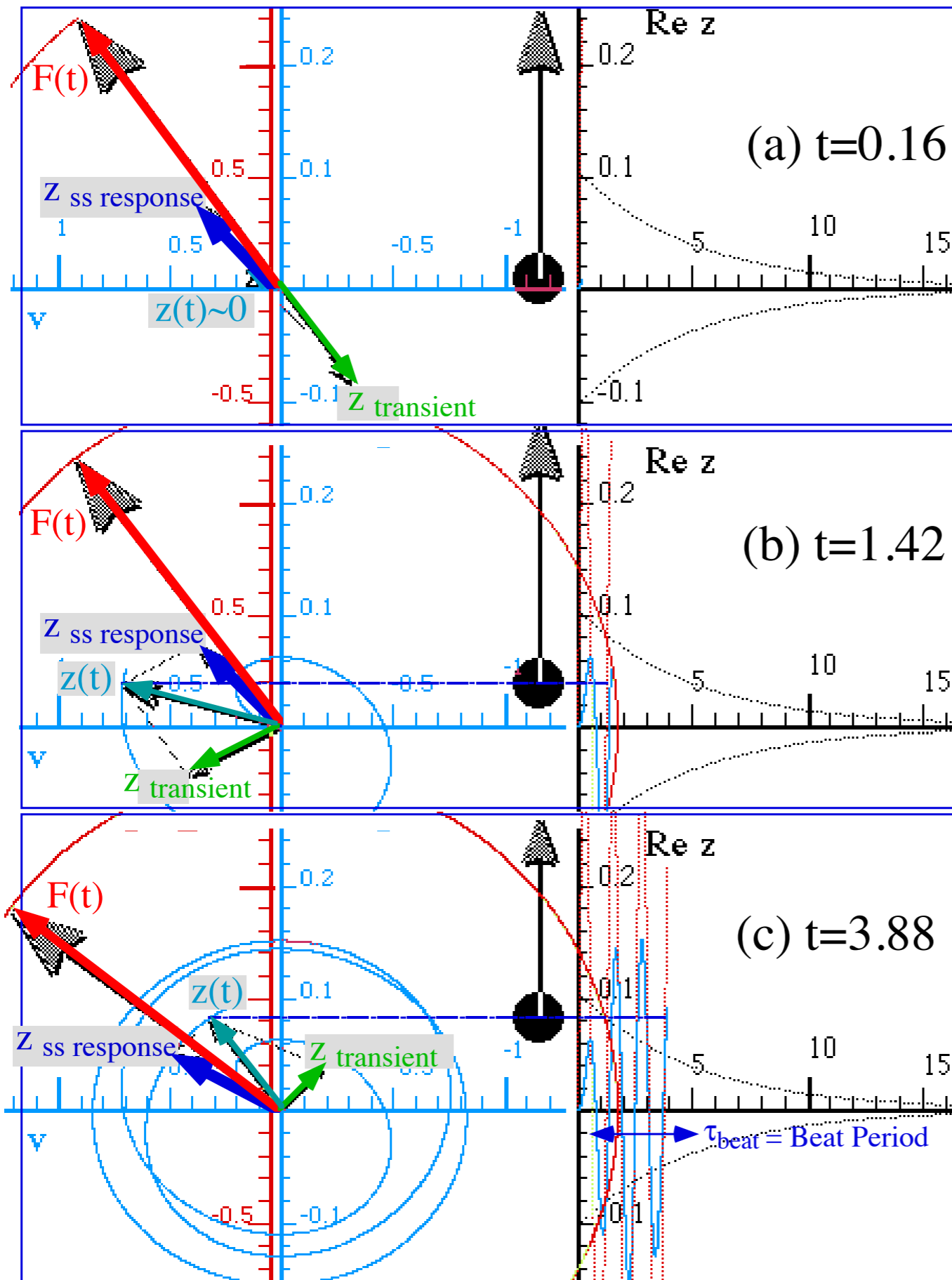


Fig. 4.2.9 Beat formation.

Transient phasor $z_{\text{transient}}$ catches up with F -phasor and passes it.

Oscillator figures of merit: quality factors Q and $q=2\pi Q$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\text{angular quality factor})$$

$$\text{Amplification factor } q = \omega_0/2\Gamma$$

Natural oscillation frequency is approximately $\nu_0 = \omega_0/2\pi$ (for $\omega_0 \gg \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

Oscillator figures of merit: quality factors Q and $q=2\pi Q$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0}(\omega_s = \omega_0)|}{|G_{\omega_0}(0)|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\text{angular quality factor})$$

$$\text{Amplification factor } q = \omega_0/2\Gamma$$

Natural oscillation frequency is approximately $\nu_0 = \omega_0/2\pi$ (for $\omega_0 \gg \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

$$\left(\begin{array}{l} t_{5\%} = 3/\Gamma = \text{Lifetime} \\ \text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude} \end{array} \right) \text{times} \left(\nu_0 = \frac{\omega_0}{2\pi} \right) = \begin{array}{l} \text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \text{ Lifetime} \end{array}$$

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The “Heartbeat Count”
measure of lifetime

Oscillator figures of merit: quality factors Q and $q=2\pi Q$

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The “Heartbeat Count”
measure of lifetime

Energy decay
(proportional to the square of oscillator amplitude): $(e^{\Gamma t})^2 = e^{-2\Gamma t} \quad dE = -2\Gamma E$

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The “Heartbeat Count”
measure of lifetime

Energy decay

(proportional to the square of oscillator amplitude): $(e^{\Gamma t})^2 = e^{-2\Gamma t} \quad dE = -2\Gamma E$

Relative amount

of energy lost
each cycle period $= \tau_0 \left(\frac{-dE}{E} \right) = \frac{2\Gamma}{\nu_0} \equiv \frac{1}{Q} = \frac{2\pi}{q}$
 $\left(\tau_0 = \frac{1}{\nu_0} \right)$

$$Q = (\text{Standard angular quality factor}) = \frac{q}{2\pi}$$

Oscillator figures of merit: Uncertainty 1/q

To see a beat we need $\tau_{\text{half-beat}}$ to be less than $\tau_{5\%}$ or $3/\Gamma$. (Here we approximate $\pi \sim 3.0$, again.)

$$\pi / |\omega_s - \omega_0| < 3 / \Gamma$$

$$|\omega_s - \omega_0| > \Gamma$$

This means ω -detuning error is greater than or equal to the decay rate Γ .

Any detuning less than Γ is virtually undetectable.

Total ω uncertainty is $\pm\Gamma$ or twice Γ (that is: FWHM $\Delta\omega = 2\Gamma$). Linear frequency uncertainty is:

The *relative frequency uncertainty* $\frac{2\Gamma}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \frac{1}{q} = \frac{\Delta\nu}{\nu_0}$ $\Delta\nu = \Delta\omega / 2\pi = \Gamma / \pi$

is the *inverse* of the *angular quality factor* q .

If we think of the 5% or 4.321% lifetime of a musical note as its time uncertainty Δt , then:

$$\Delta t \Delta \nu = 3 / \pi \approx 1$$

$$\Delta t = t_{5\%} = 3 / \Gamma$$

$$\Delta t = t_{4.321\%} = \pi / \Gamma$$

Very precise measures of imprecision

Approximate Lorentz-Green's Function for high quality *FDHO* (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow{\omega_s \rightarrow \omega_0} \frac{1}{2\omega_s} \frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0} \frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

Complex detuning-decay $\delta = \Delta - i\Gamma$ variable δ is defined with the real detuning $\Delta = \omega_0 - \omega_s$

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$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \text{Re } L + i \text{Im } L = \frac{\Delta}{\Delta^2 + \Gamma^2} + i \frac{\Gamma}{\Delta^2 + \Gamma^2} = |L|^2 \Delta + i |L|^2 \Gamma$$

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Complex detuning-decay $\delta = \Delta - i\Gamma$ variable δ is defined with the real detuning $\Delta = \omega_0 - \omega_s$

$$\begin{aligned} L(\Delta - i\Gamma) &= \frac{1}{\Delta - i\Gamma} = \text{Re } L + i \text{Im } L = \frac{\Delta}{\Delta^2 + \Gamma^2} + i \frac{\Gamma}{\Delta^2 + \Gamma^2} = |L|^2 \Delta + i |L|^2 \Gamma \\ &= |L| e^{i\rho} = |L| \cos \rho + i |L| \sin \rho = \frac{\cos \rho}{\sqrt{\Delta^2 + \Gamma^2}} + i \frac{\sin \rho}{\sqrt{\Delta^2 + \Gamma^2}} \text{ where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}} \end{aligned}$$

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Ideal Lorentz-Green's functions

$$|L| = \frac{1}{\Gamma} \sin \rho$$

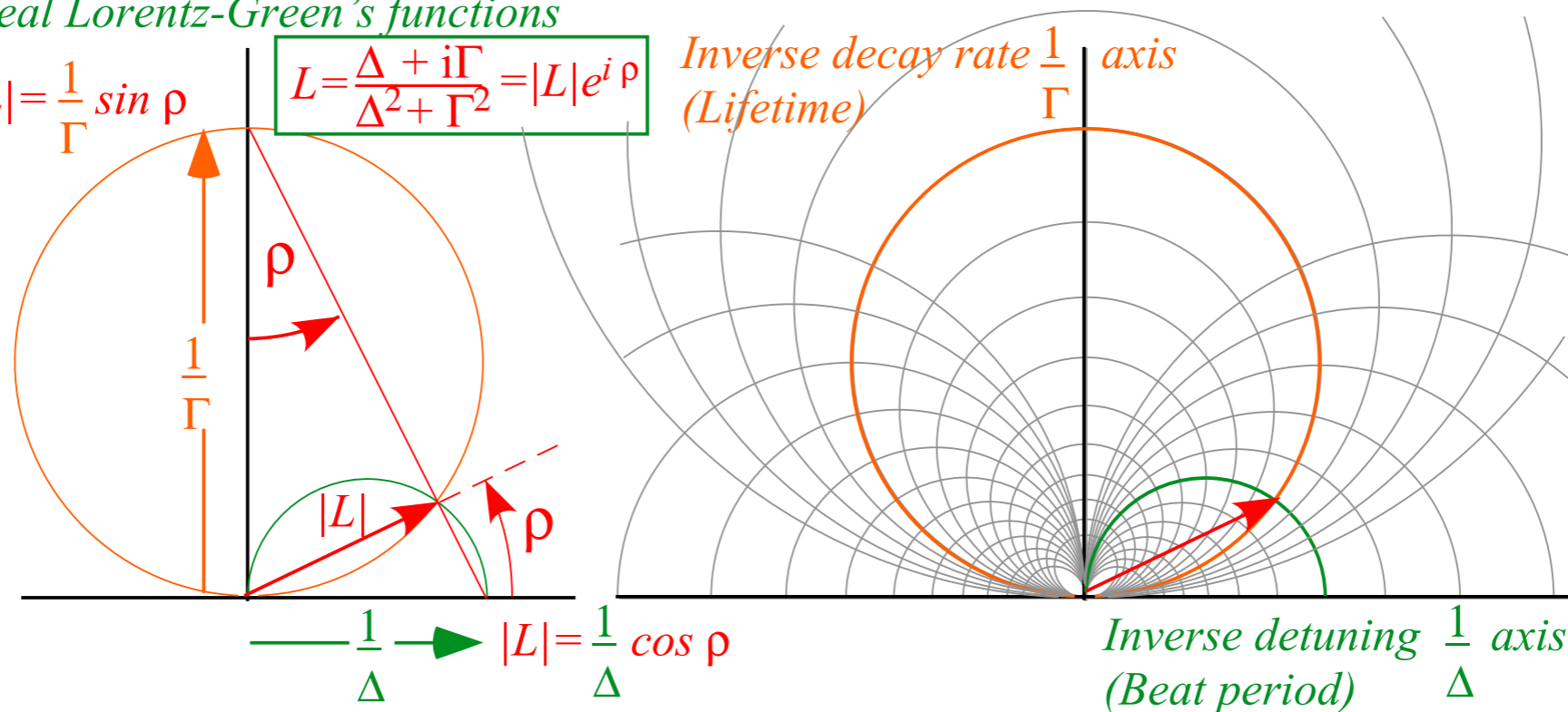
$$L = \frac{\Delta + i\Gamma}{\Delta^2 + \Gamma^2} = |L| e^{i\rho}$$

Inverse decay rate $\frac{1}{\Gamma}$ axis
(Lifetime)

Smith plots

$$|L| = \frac{1}{\Gamma} \sin \rho$$

$$|L| = \frac{1}{\Delta} \cos \rho$$



Approximate Lorentz-Green's Function for high quality *FDHO* (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow{\omega_s \rightarrow \omega_0} \frac{1}{2\omega_s} \frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0} \frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

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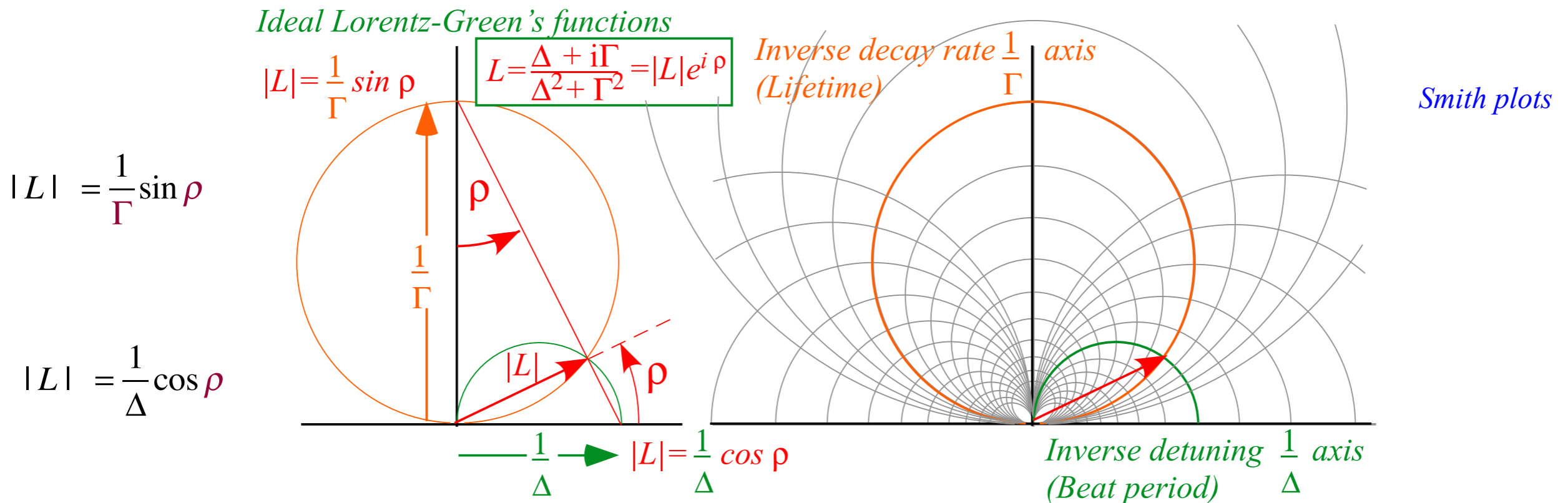


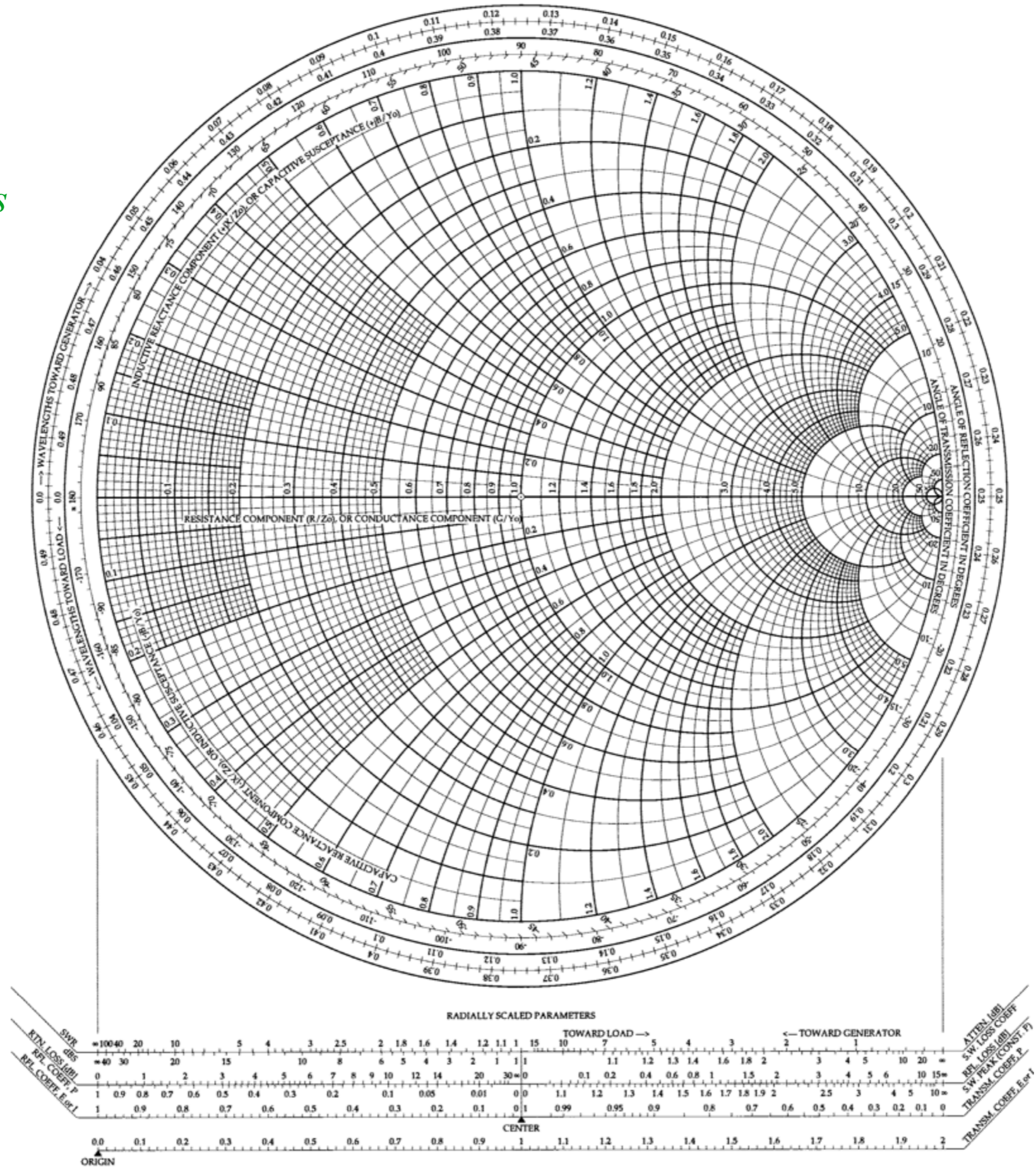
Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ vs. beat-period $1/\Delta$ coordinates)

Constant Δ and Γ curves in Fig. 4.2.13 are orthogonal circles of $1/z$ -dipolar coordinates. Recall Fig. 1.10.11.

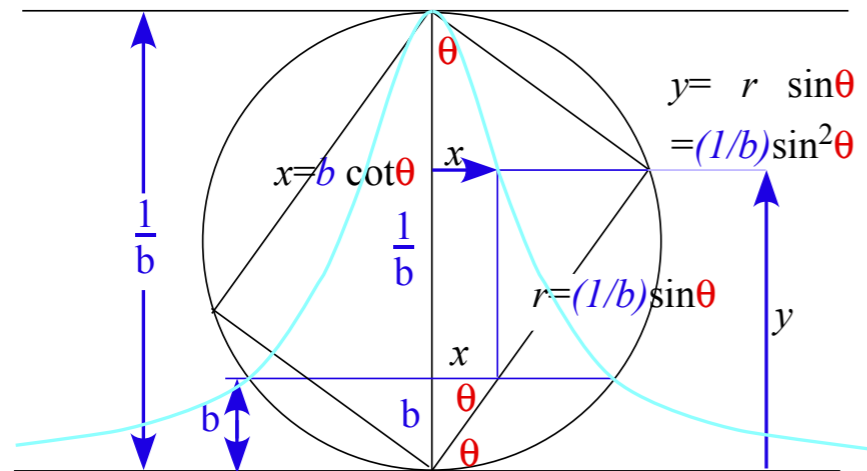
An FDHO Green's
Function
Slide rule

A plot of
 $f(z) = 1/z$

For wavy
"Ohm's Laws"
 $V = I \cdot Z$
 $I = V/Z$



The Common Lorentzian (a.k.a. The Witch of Agnesi)



$$x^2 = b^2 \cot^2 \theta = b^2 \frac{\cos^2 \theta}{\sin^2 \theta} = b^2 \frac{1 - \sin^2 \theta}{\sin^2 \theta} = \frac{b^2}{\sin^2 \theta} - b^2$$

$$x^2 + b^2 = \frac{b^2}{\sin^2 \theta} = \frac{b}{y} \quad y = \frac{b}{x^2 + b^2}$$

*Common Lorentzian function I.
(imaginary "absorbive" part)*

Maria Gaetana Agnesi



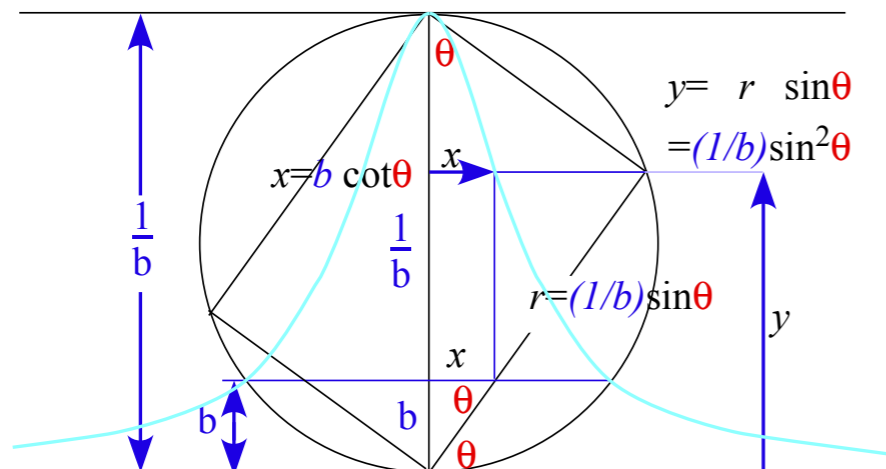
Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics

The Common Lorentzian (a.k.a. The Witch of Agnesi)

Maria Gaetana Agnesi



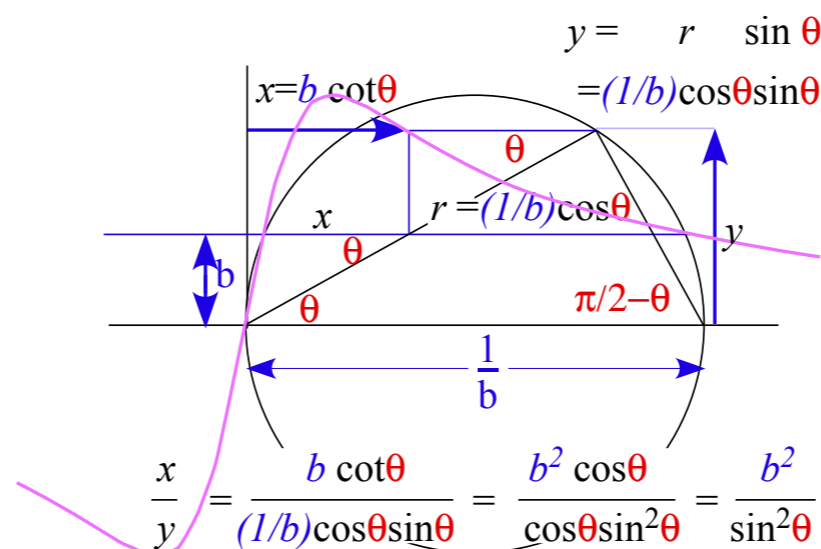
Born May 16, 1718
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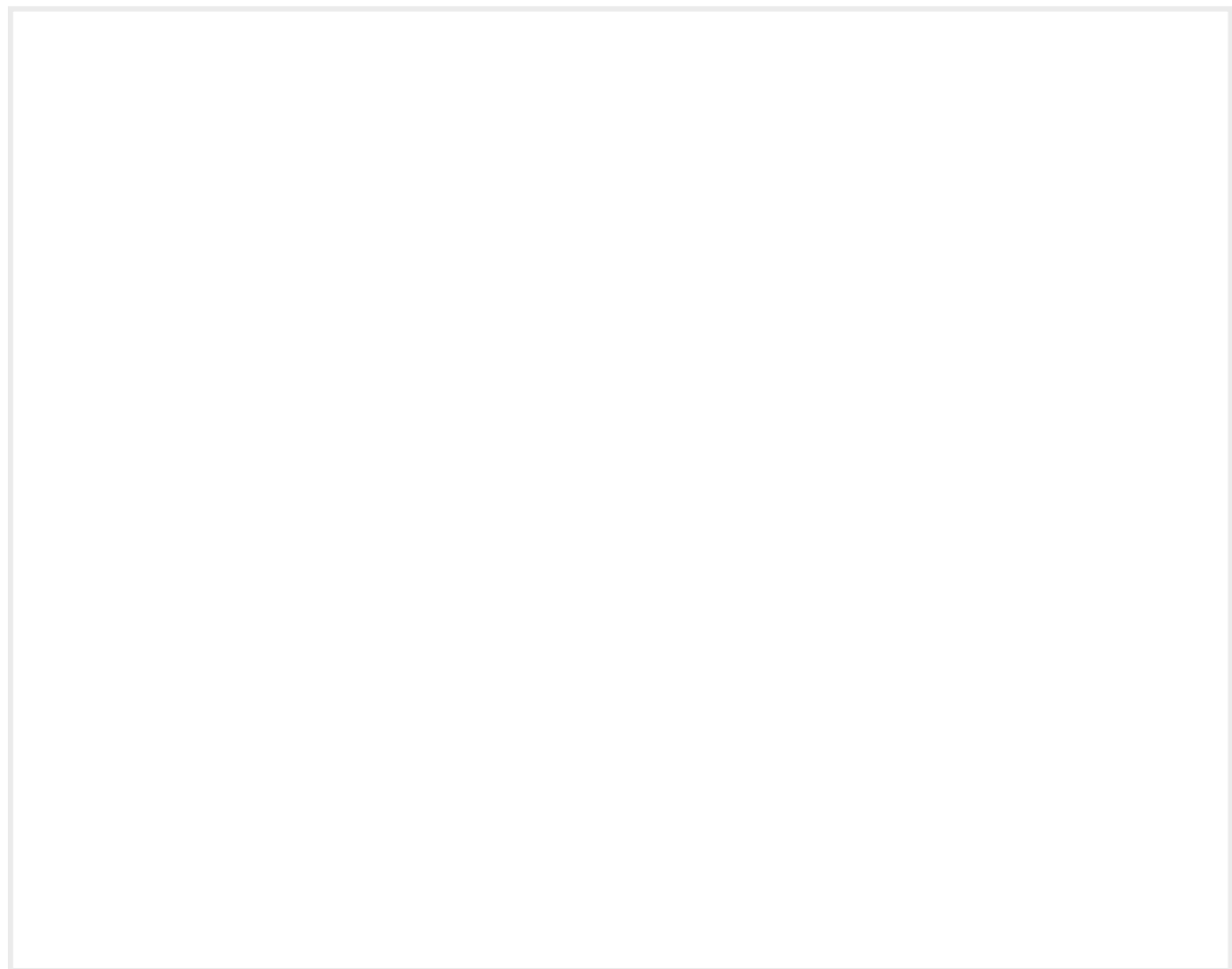
Common Lorentzian function I.
(imaginary "absorbive" part)



$$\frac{x}{y} = \frac{b \cot \theta}{(1/b) \cos \theta \sin \theta} = \frac{b^2 \cos \theta}{\cos \theta \sin^2 \theta} = \frac{b^2}{\sin^2 \theta}$$

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Common Lorentzian function II.
(real "refractory" part)

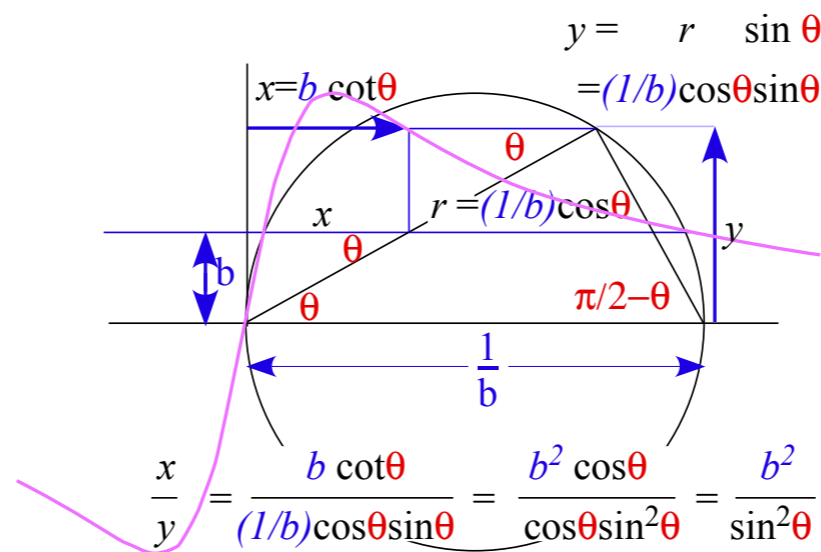
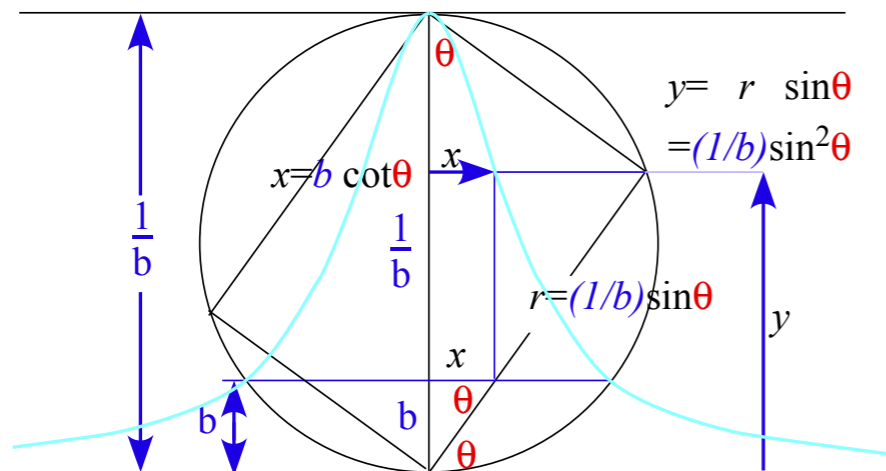


The Common Lorentzian (a.k.a. The Witch of Agnesi)

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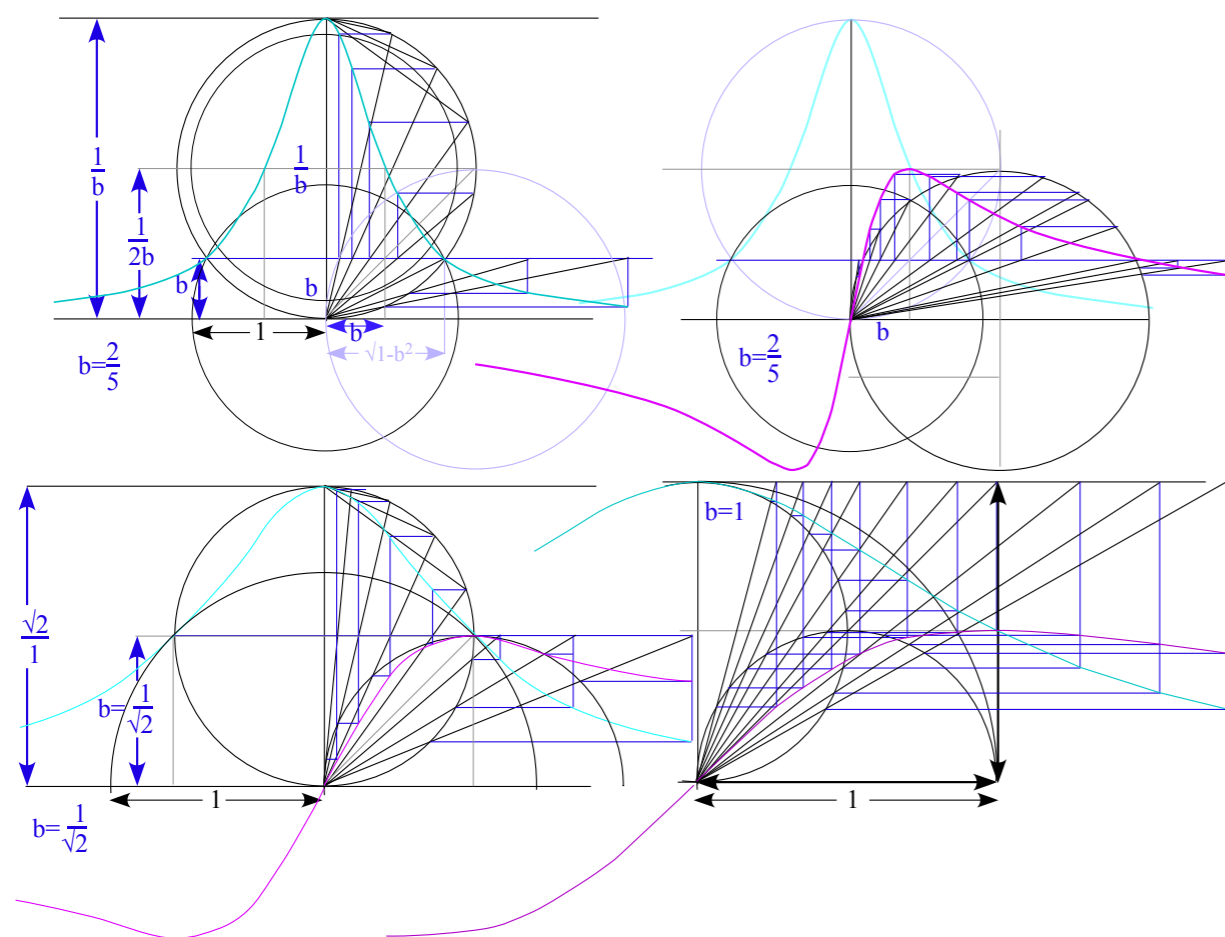
$$\frac{x}{y} = \frac{b \cot \theta}{(1/b) \cos \theta \sin \theta} = \frac{b^2 \cos \theta}{\cos \theta \sin^2 \theta} = \frac{b^2}{\sin^2 \theta}$$

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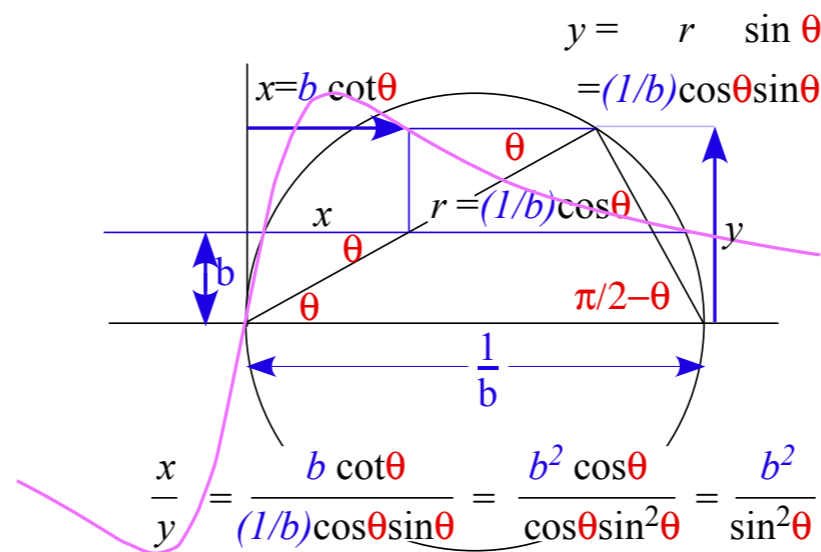
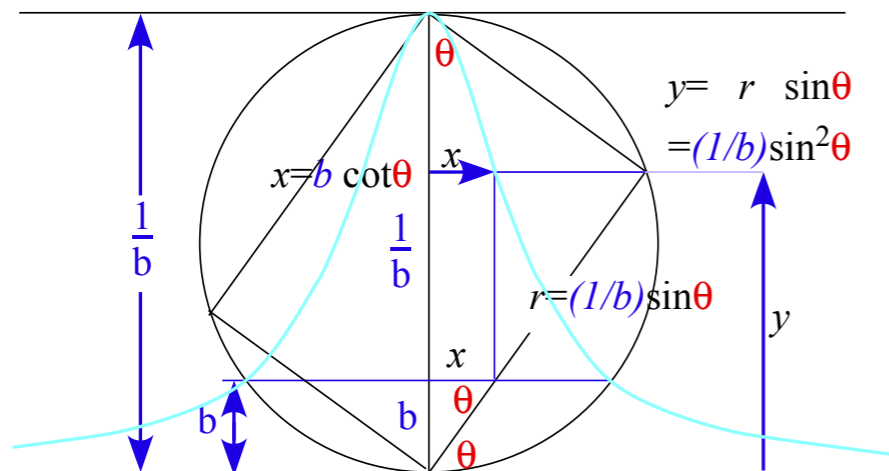


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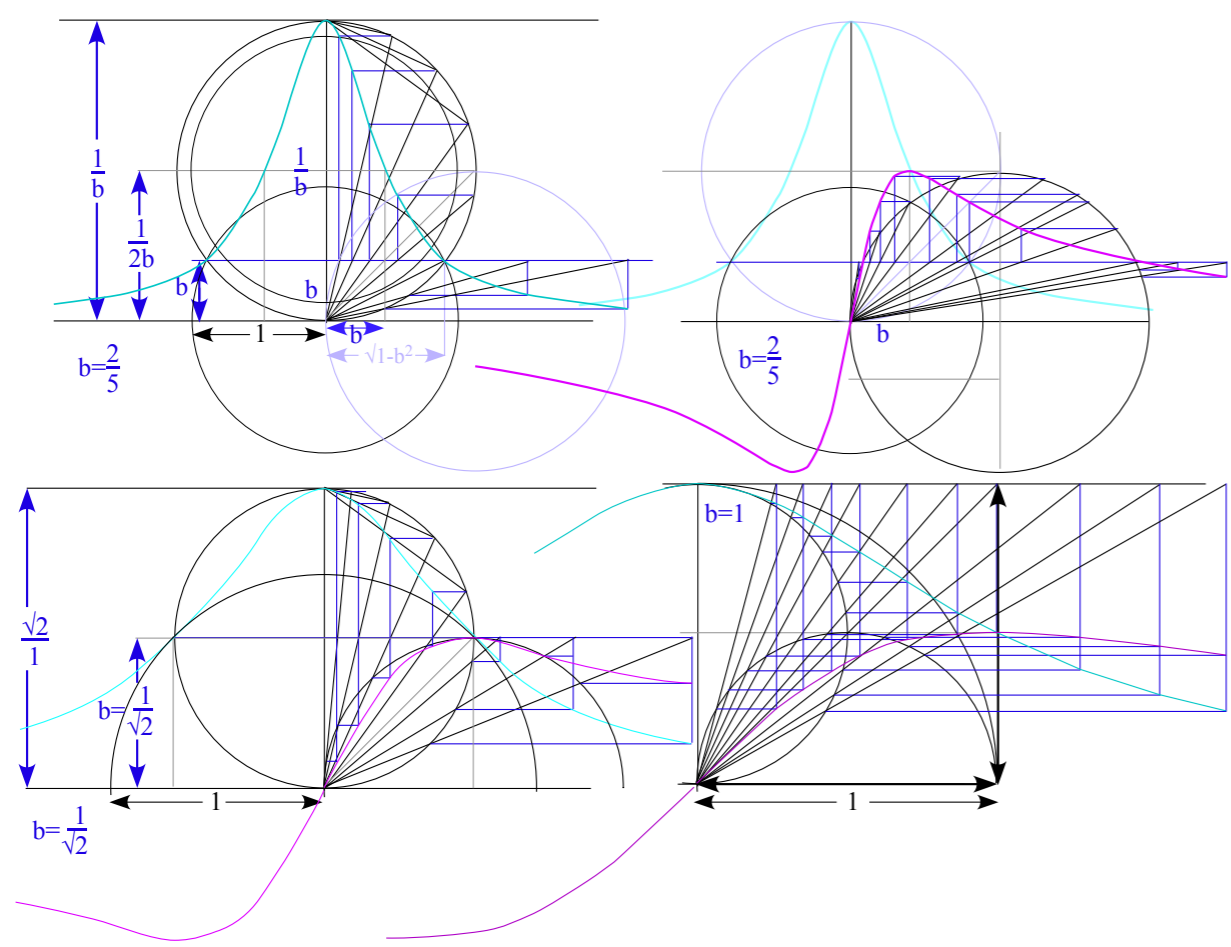
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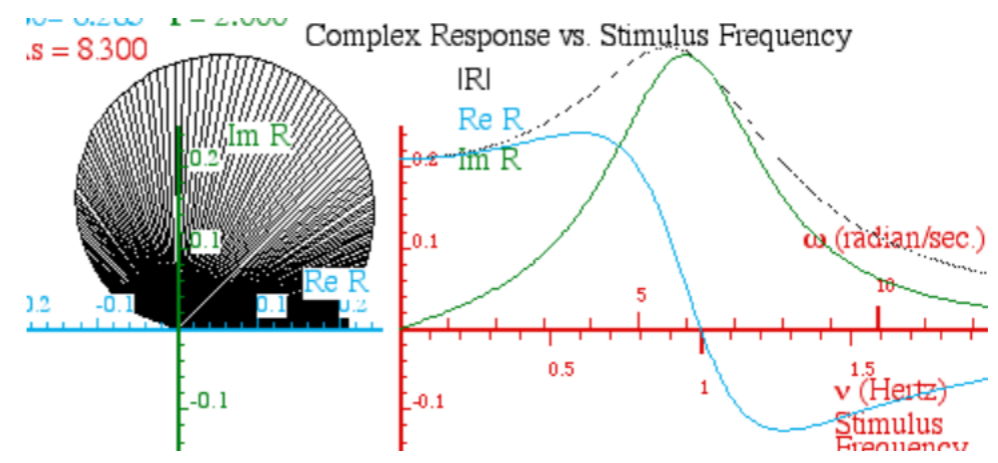
Common Lorentzian function I.
(imaginary "absorbptive" part)

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Common Lorentzian function II.
(real "refractory" part)



Compare ideal Lorentzians ($\Gamma=0.2$) with a very non-ideal one ($\Gamma=2$)

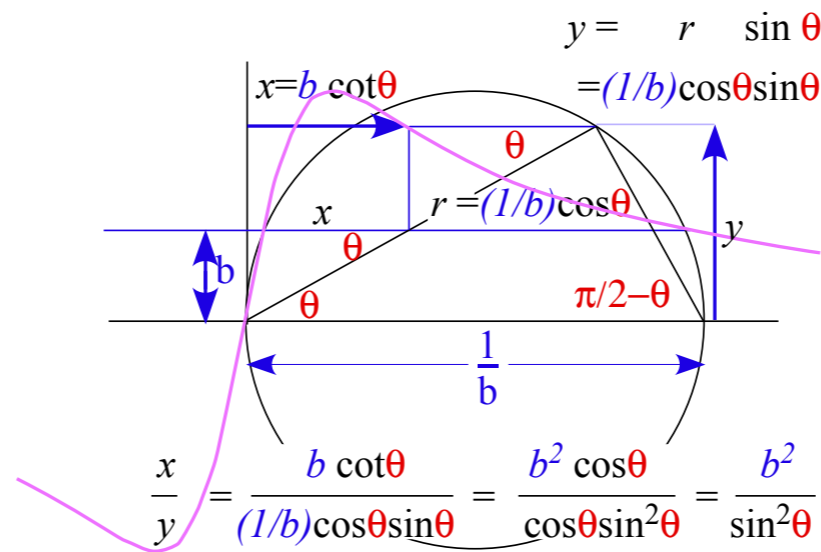
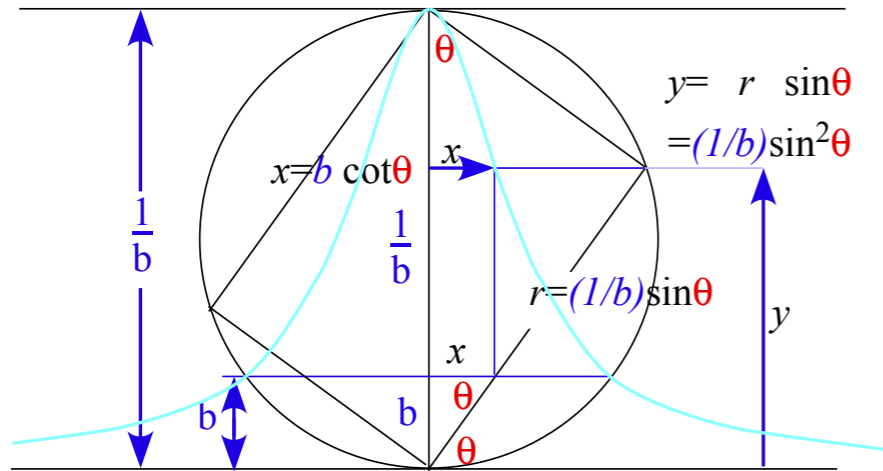


The Common Lorentzian (a.k.a. The Witch of Agnesi)

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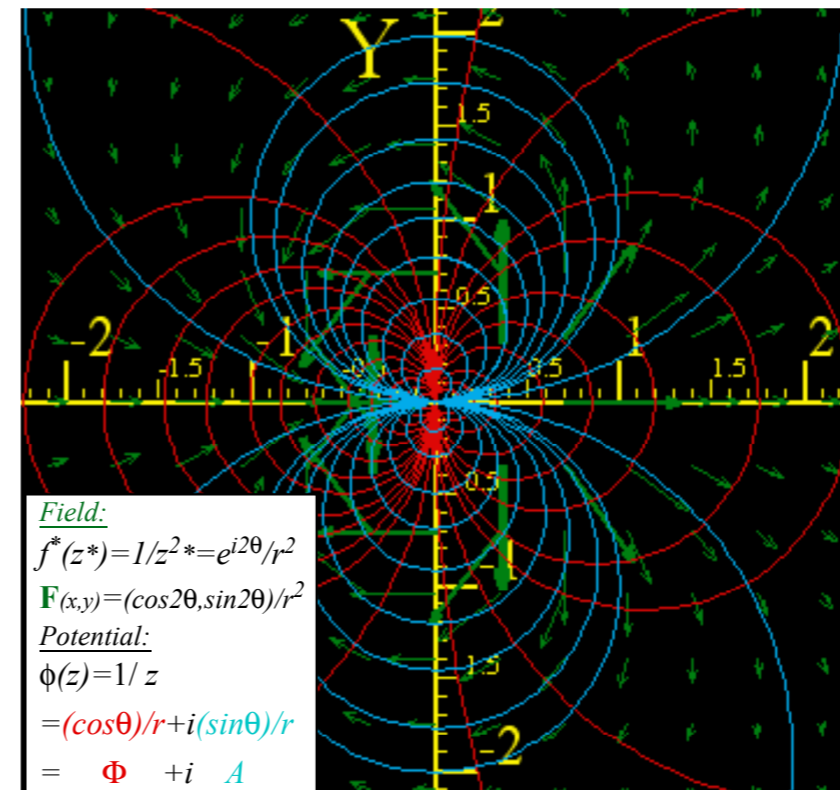
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Common Lorentzian function I.
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$$x^2 + b^2 = \frac{b^2}{\sin^2 \theta} = \frac{x}{y} \quad \boxed{y = \frac{x}{x^2 + b^2}}$$

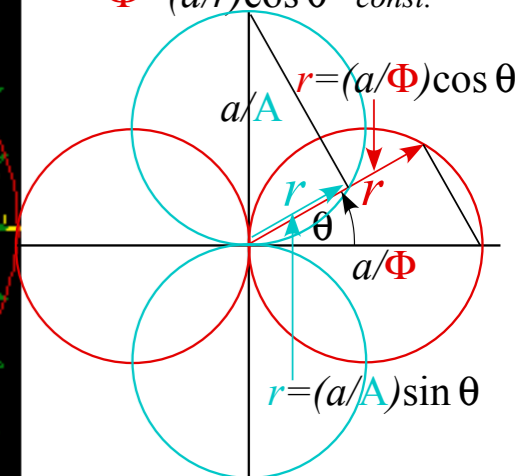
Common Lorentzian function II.
(real "refractory" part)



Field:
 $f^*(z^*) = 1/z^{2*} = e^{i2\theta}/r^2$
 $\mathbf{F}(x,y) = (\cos 2\theta, \sin 2\theta)/r^2$
Potential:
 $\phi(z) = 1/z$
 $= (\cos \theta)/r + i(\sin \theta)/r$
 $= \Phi + i A$

Scalar potentials

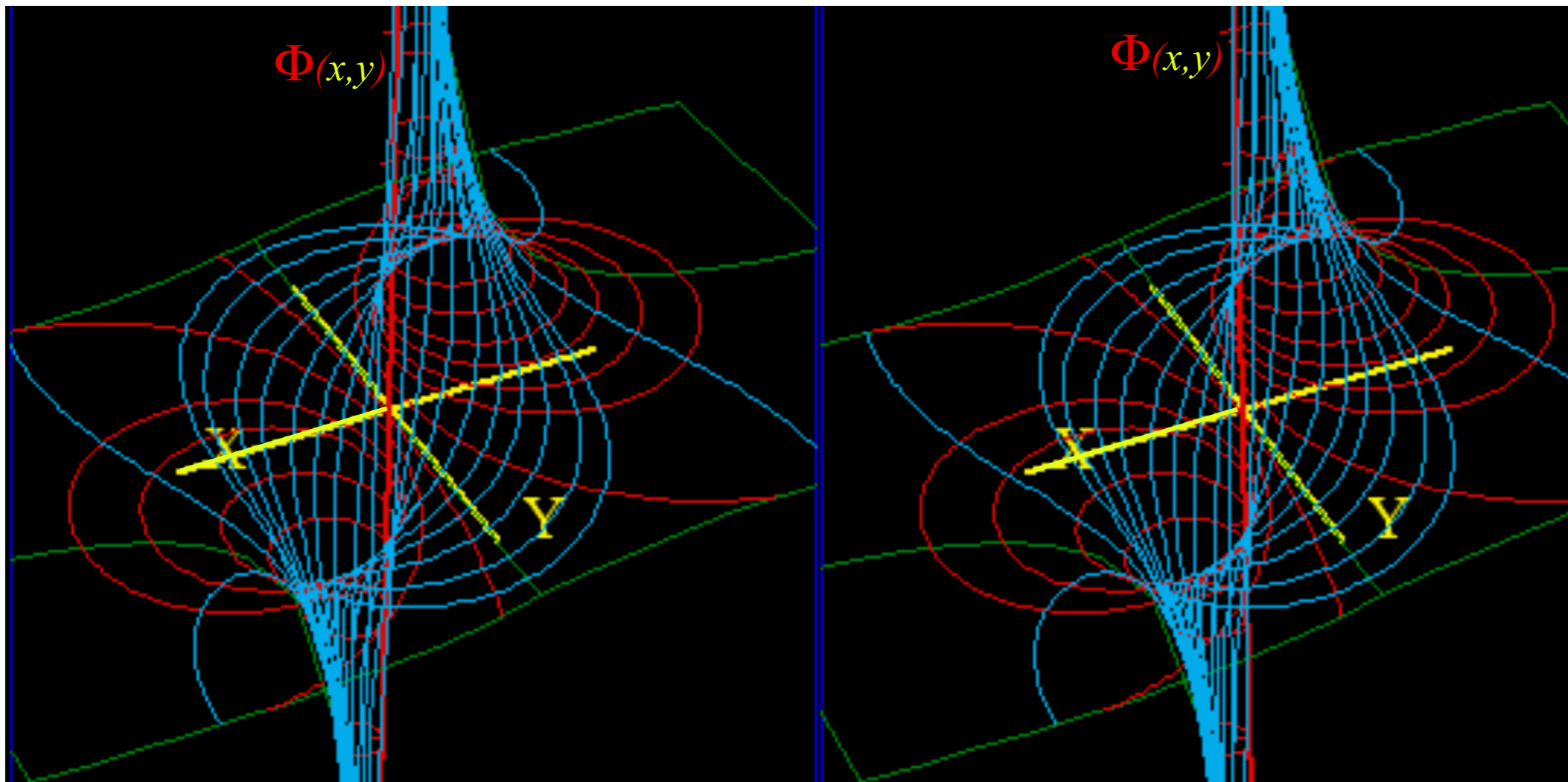
$$\Phi = (a/r) \cos \theta = \text{const.}$$



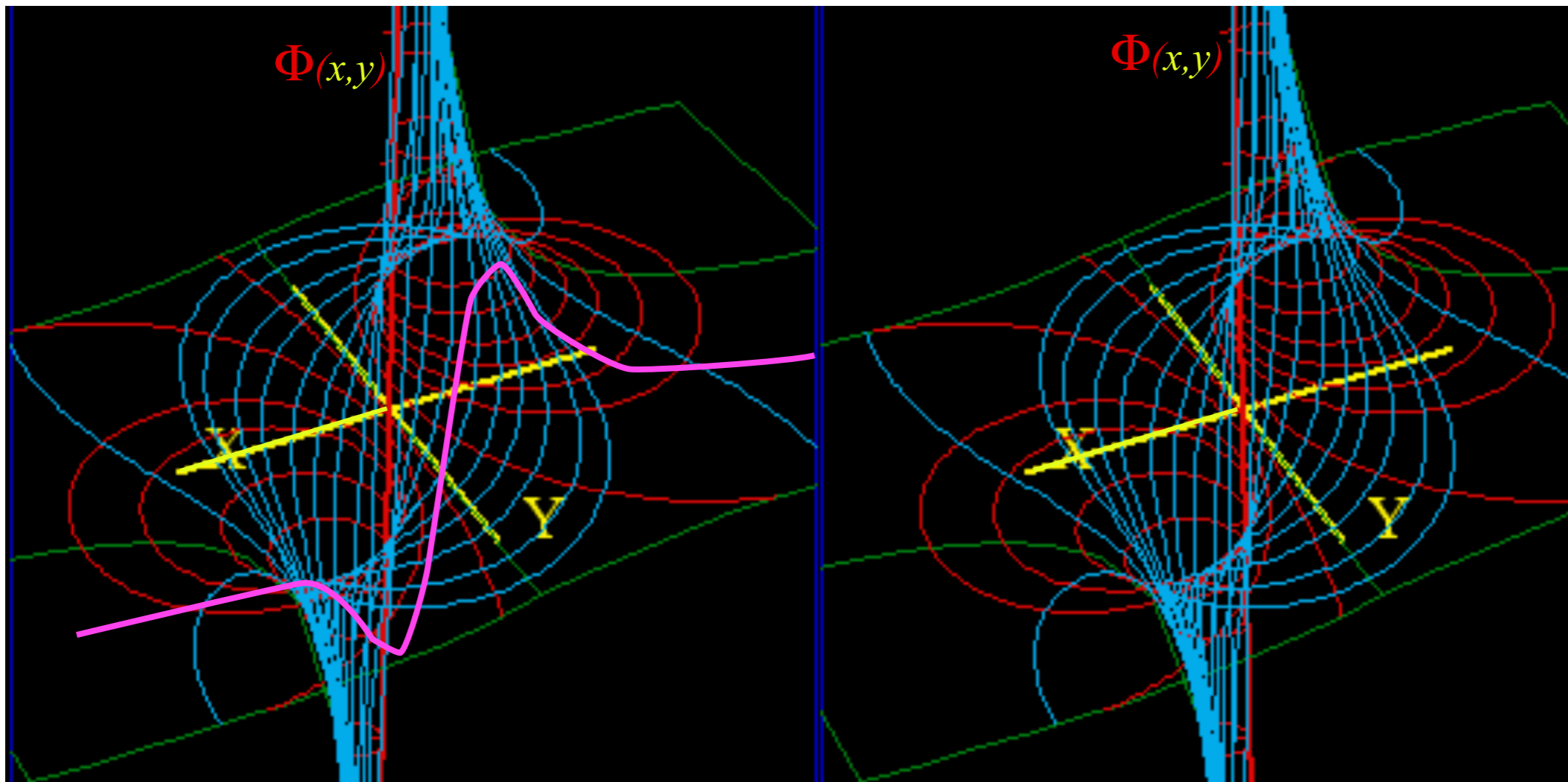
Vector potentials

$$A = (a/r) \sin \theta = \text{const.}$$

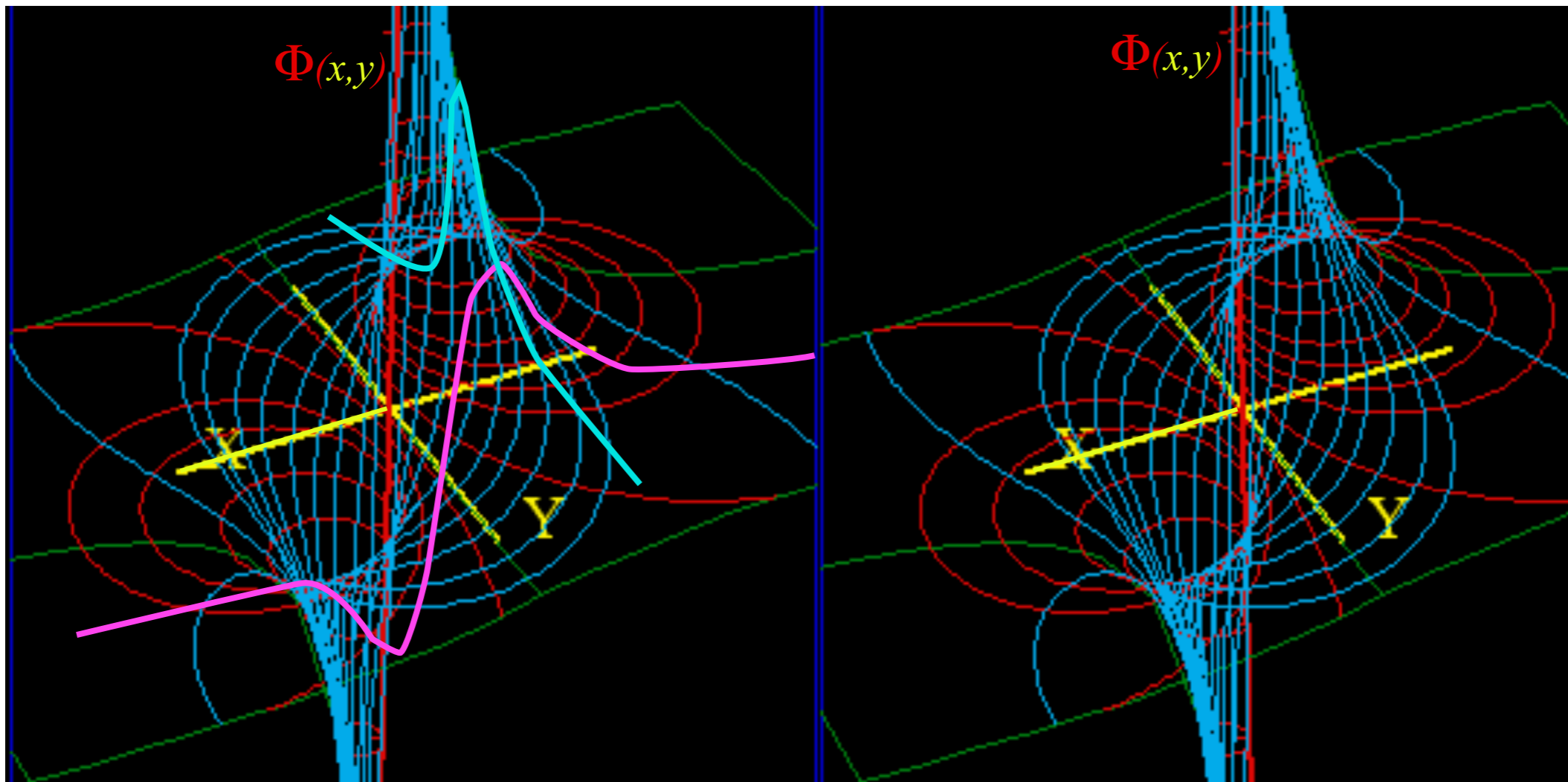
Fig. 10.11 Dipole \mathbf{F} -field $f(z) = 1/z^2$ and scalar potential ($\Phi = \text{const.}$)-circles orthogonal to ($A = \text{const.}$)-circles.



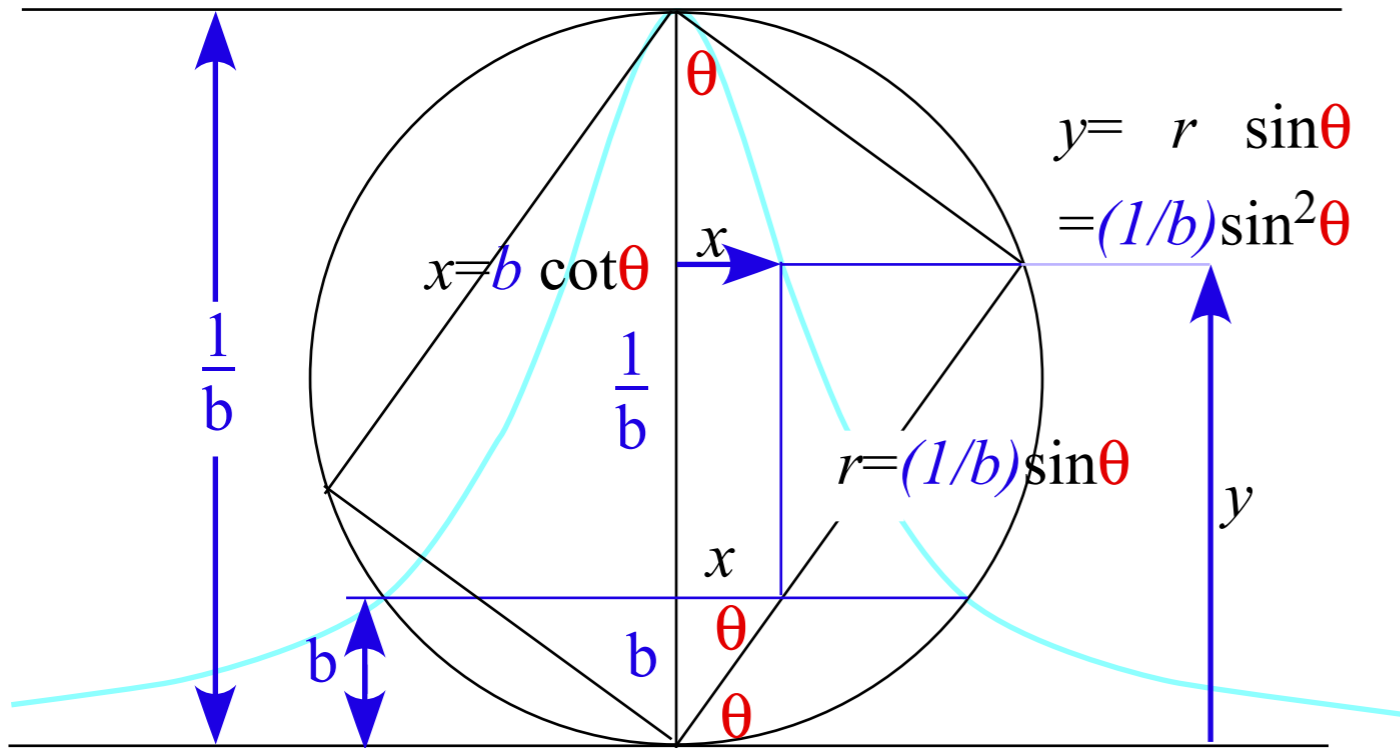
From: Fig. 1.10.12



From: Fig. 1.10.12



From: Fig. 1.10.12

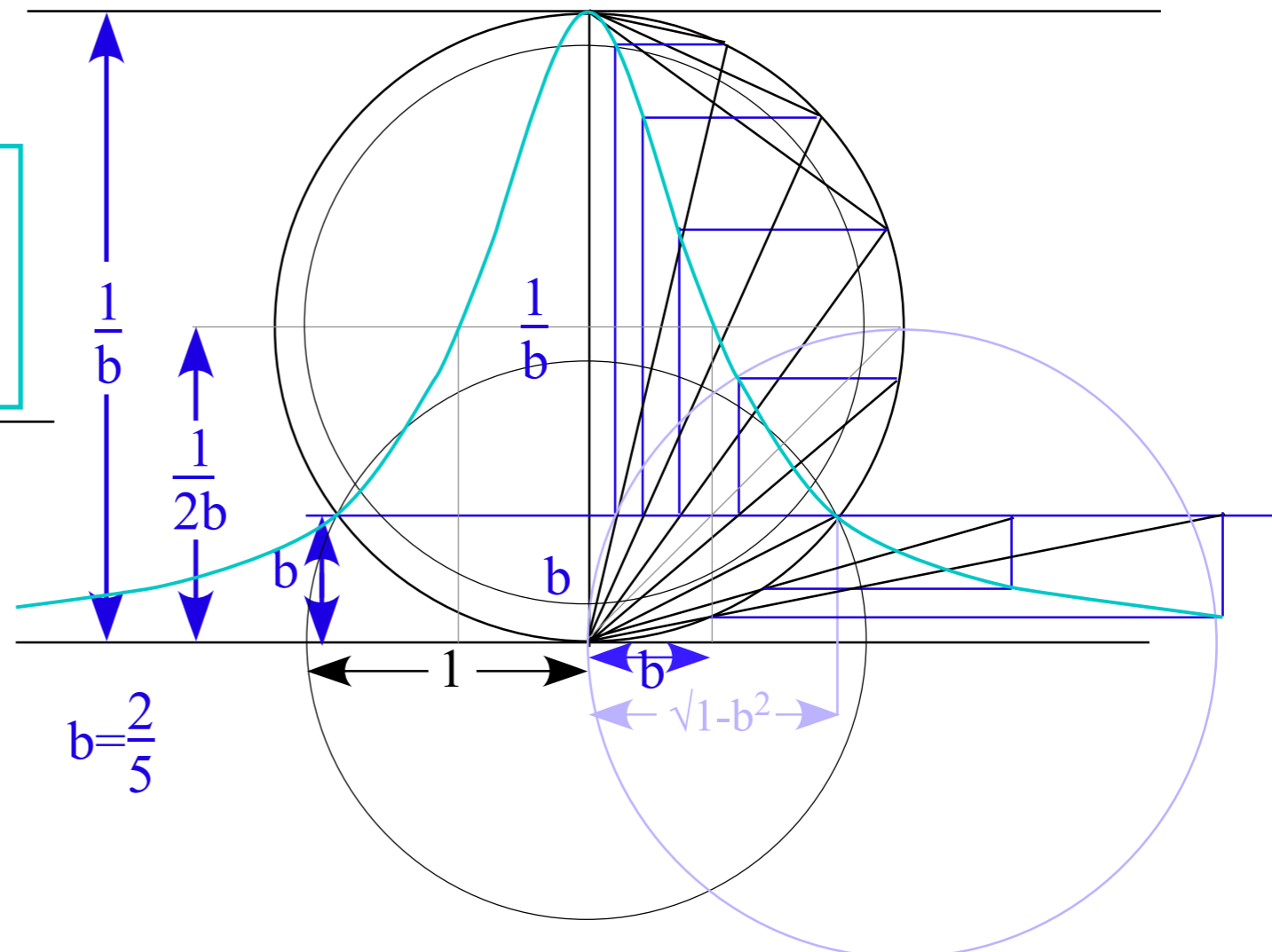


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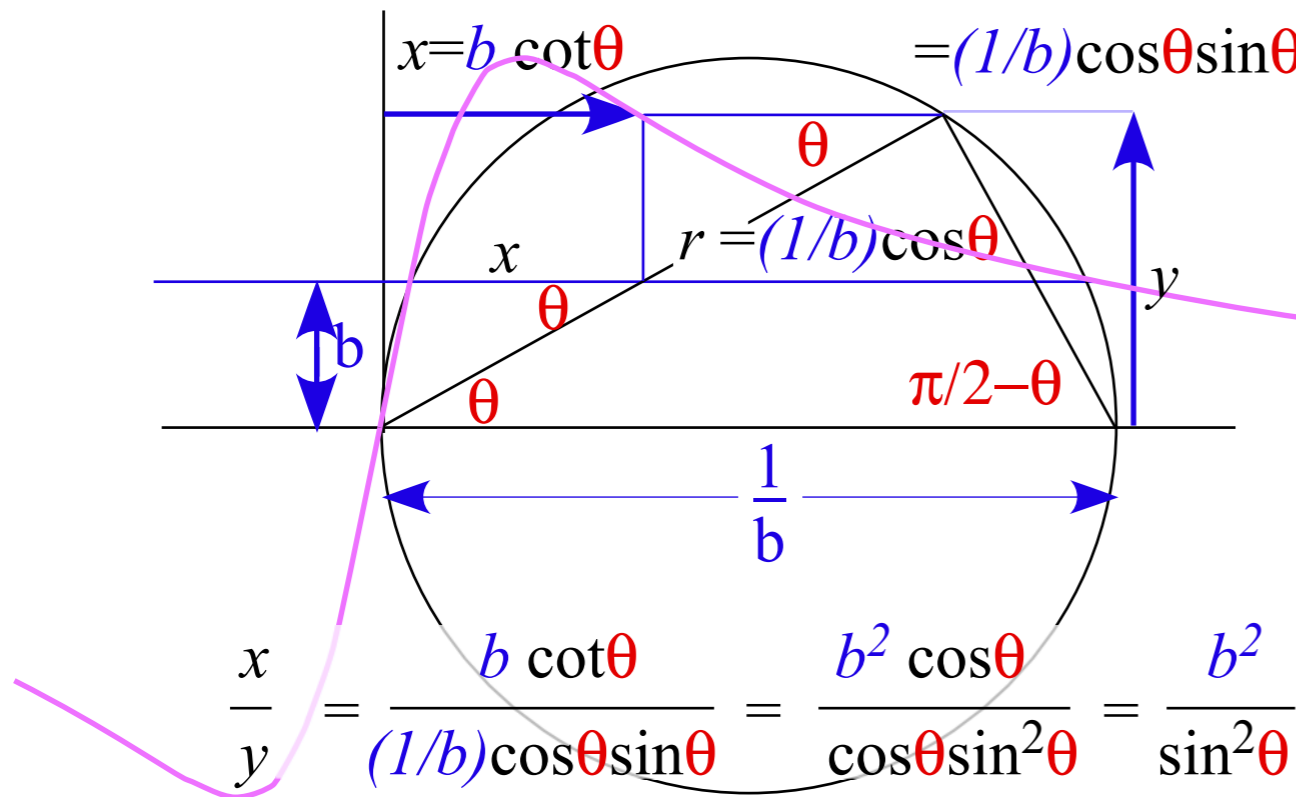
*Common Lorentzian function I.
(imaginary "absorbitive" part)*



$$b = \frac{2}{5}$$

$$y = r \sin \theta$$

$$= (1/b) \cos \theta \sin \theta$$



$$x^2 + b^2 = \frac{b^2}{\sin^2 \theta} = \frac{x}{y}$$

$$y = \frac{x}{x^2 + b^2}$$

*Common Lorentzian function II.
(real "refractory" part)*

