Lecture 25
Fri. 11.18.2016

Introduction to Orbital Dynamics

(Ch. 2-4 of Unit 5 12.01.15)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Review: "3steps from Hell" (Lect. 7 Ch. 9 Unit 1)

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

(A mystery similarity appears)

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Geometry and Symmetry of Coulomb orbits

Detailed elliptic geometry

Detailed hyperbolic geometry

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- ➔ *Effective potentials for IHO and Coulomb orbits*
    - *Stable equilibrium radii and radial/angular frequency ratios*
    - *Classical turning radii and apogee/perigee parameters*
    - *Polar coordinate differential equations*
    - *Quadrature integration techniques*
        - *Detailed orbital functions*
    - *Relating orbital energy-momentum to conic-sectional orbital geometry*
        - *Kepler equation of time and phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

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where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

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Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

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Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

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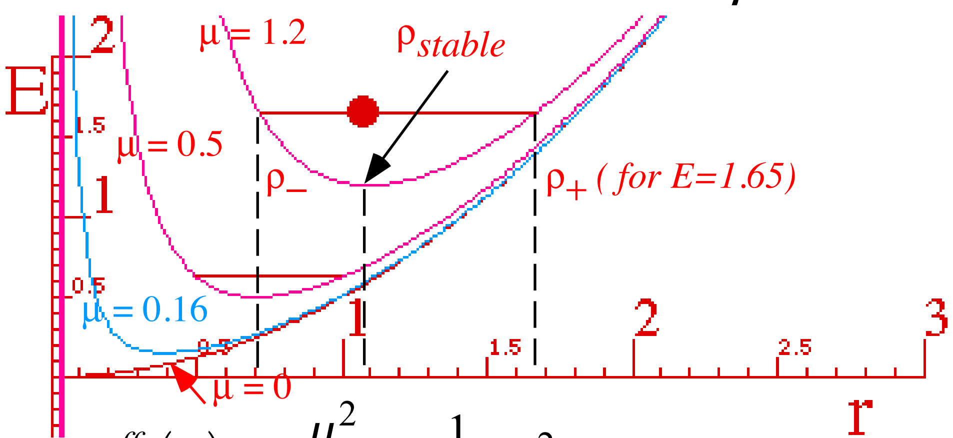

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Web Simulation
OscillatorPE - IHO

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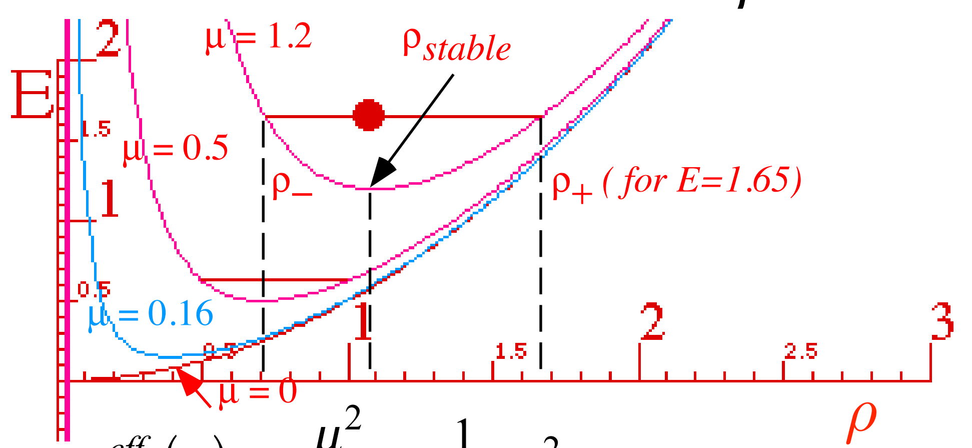

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Web Simulation OscillatorPE - IHO

Effective potential for Coulomb $V(\rho)=-k/\rho$

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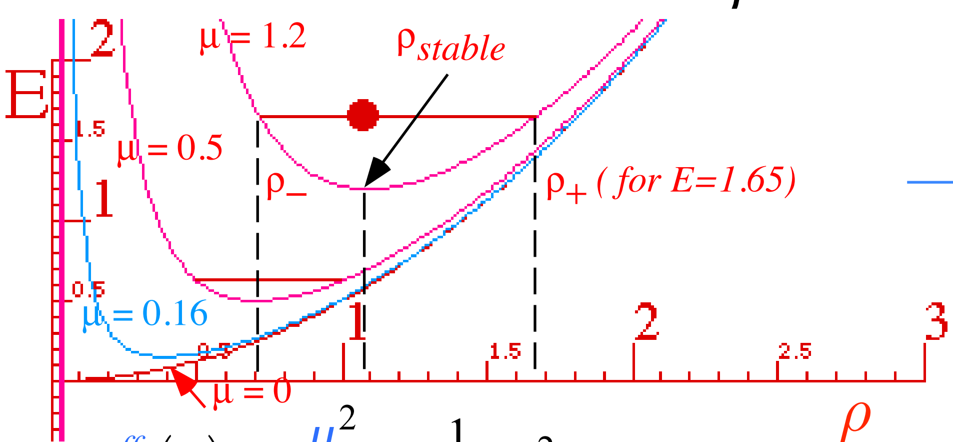

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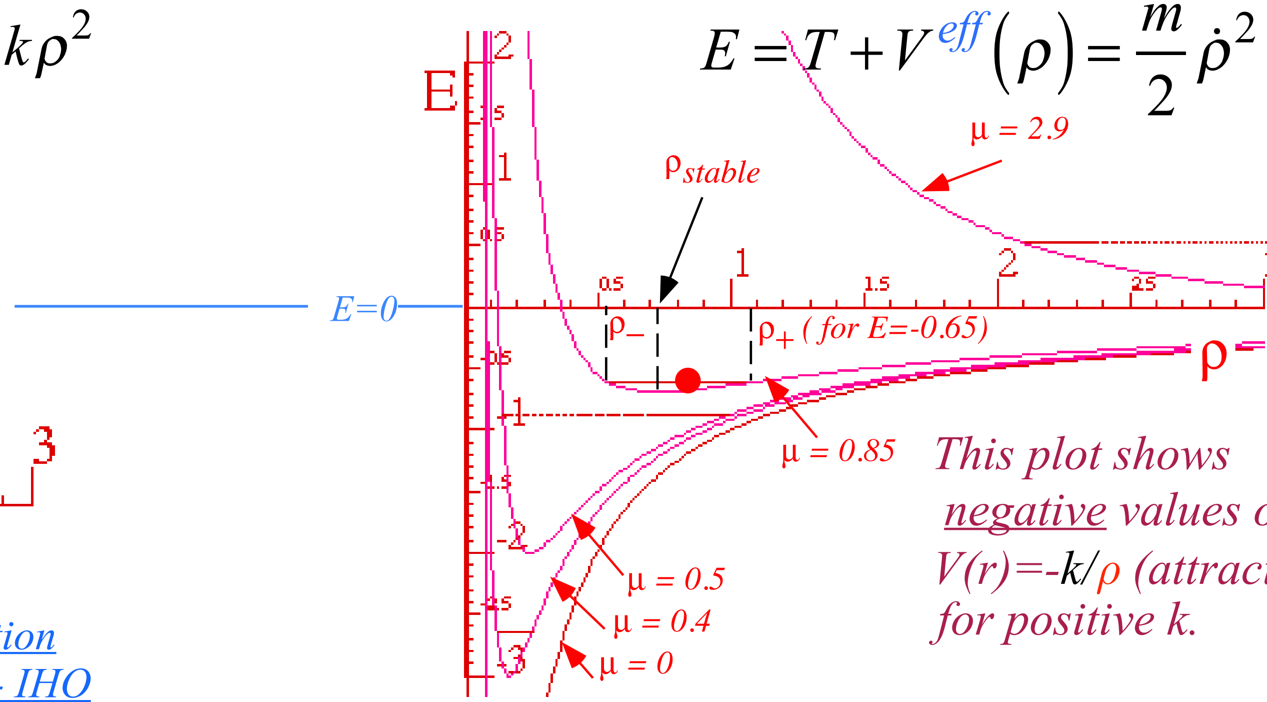

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

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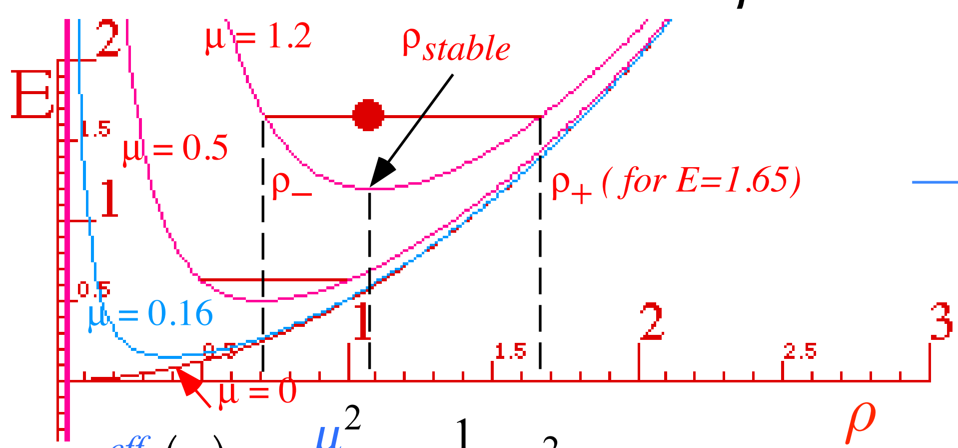

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[Web Simulation OscillatorPE - IHO]

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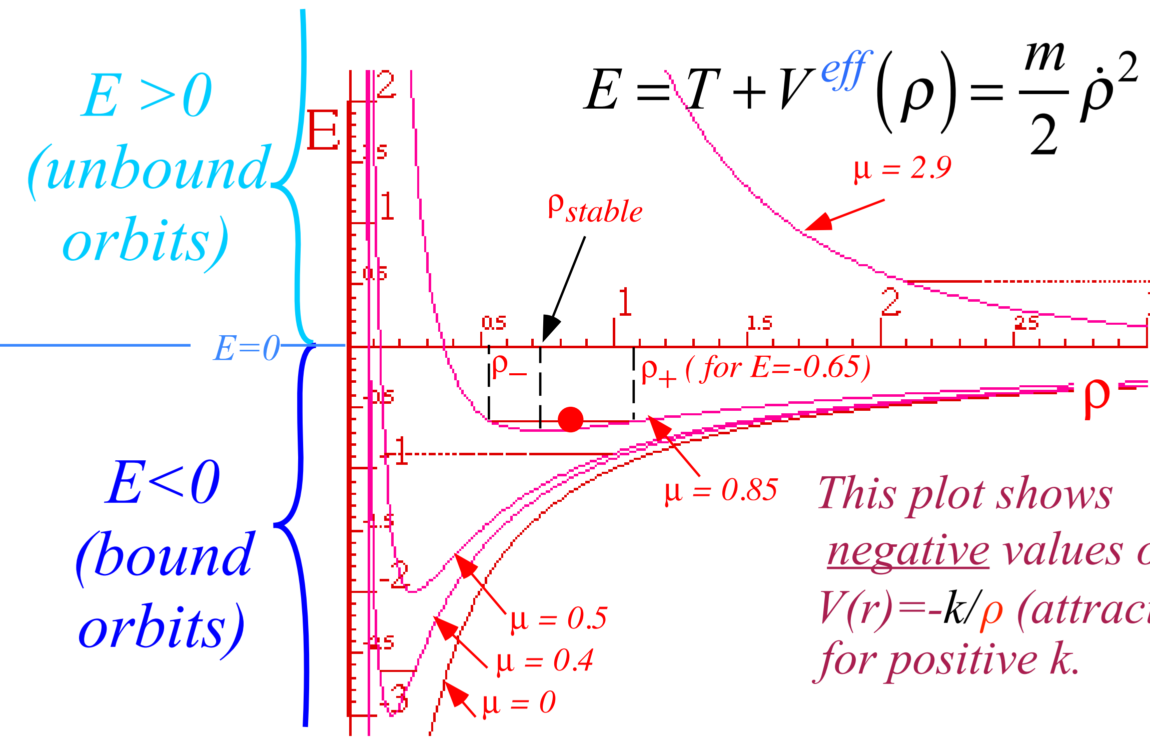

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[Web Simulation OscillatorPE - Coulomb]

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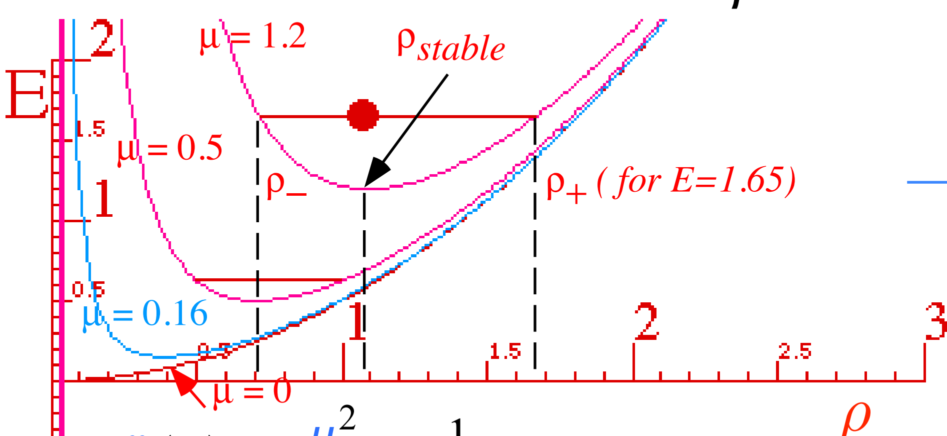

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Web Simulation OscillatorPE - IHO

E >0 (unbound orbits)

E=0

E<0 (bound orbits)

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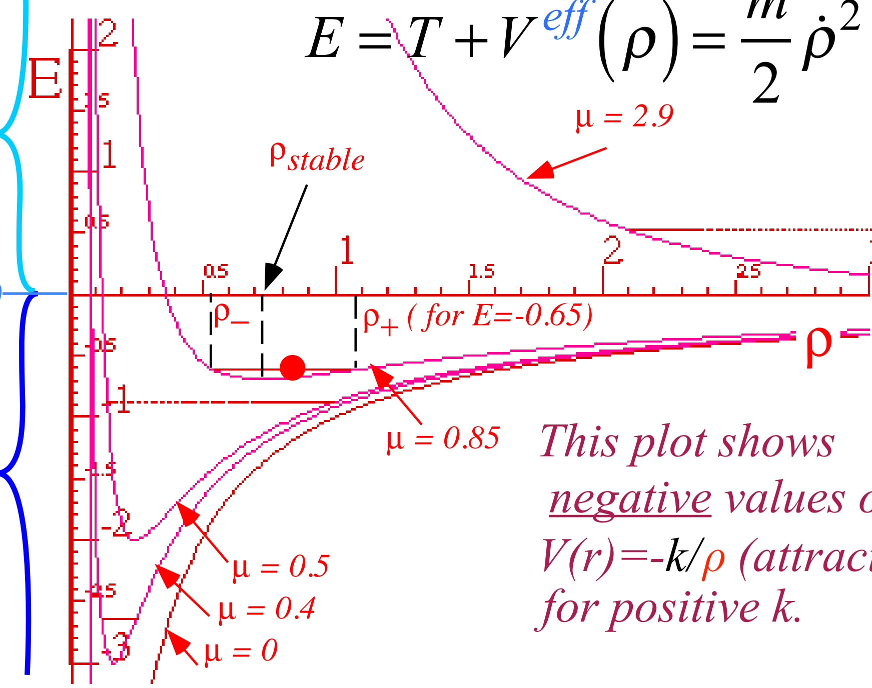

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This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Web Simulation OscillatorPE - Coulomb

In either case: *IHO or Coulomb orbit blows up if* k *is negative.*

Orbits in Isotropic Oscillator and Coulomb Potentials

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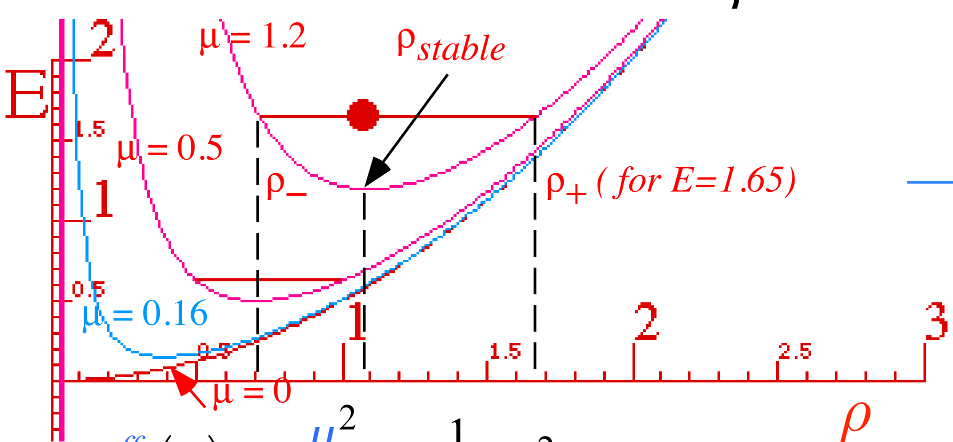

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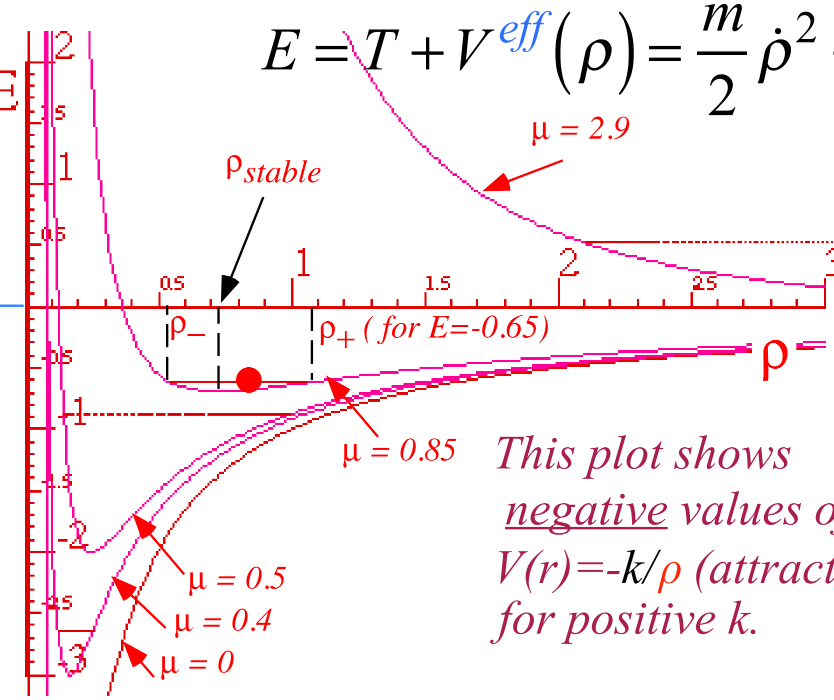

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Web Simulation
OscillatorPE - Coulomb

In either case: *IHO or Coulomb orbit blows up if k is negative.*

NOTE: Our Coulomb field is attractive if k is positive
That is, if $-k/\rho$ is negative.

Coulomb $V(\rho)=-k/\rho$
(Explicit minus (-) convention)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits ➔ *Review: "3steps from Hell" (Lect. 7 Ch. 9 Unit 1)*

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

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Kepler equation of time and phase geometry

Review: "Three (equal) steps from Hell" (Lect. 7 Ch. 9 Unit 1)

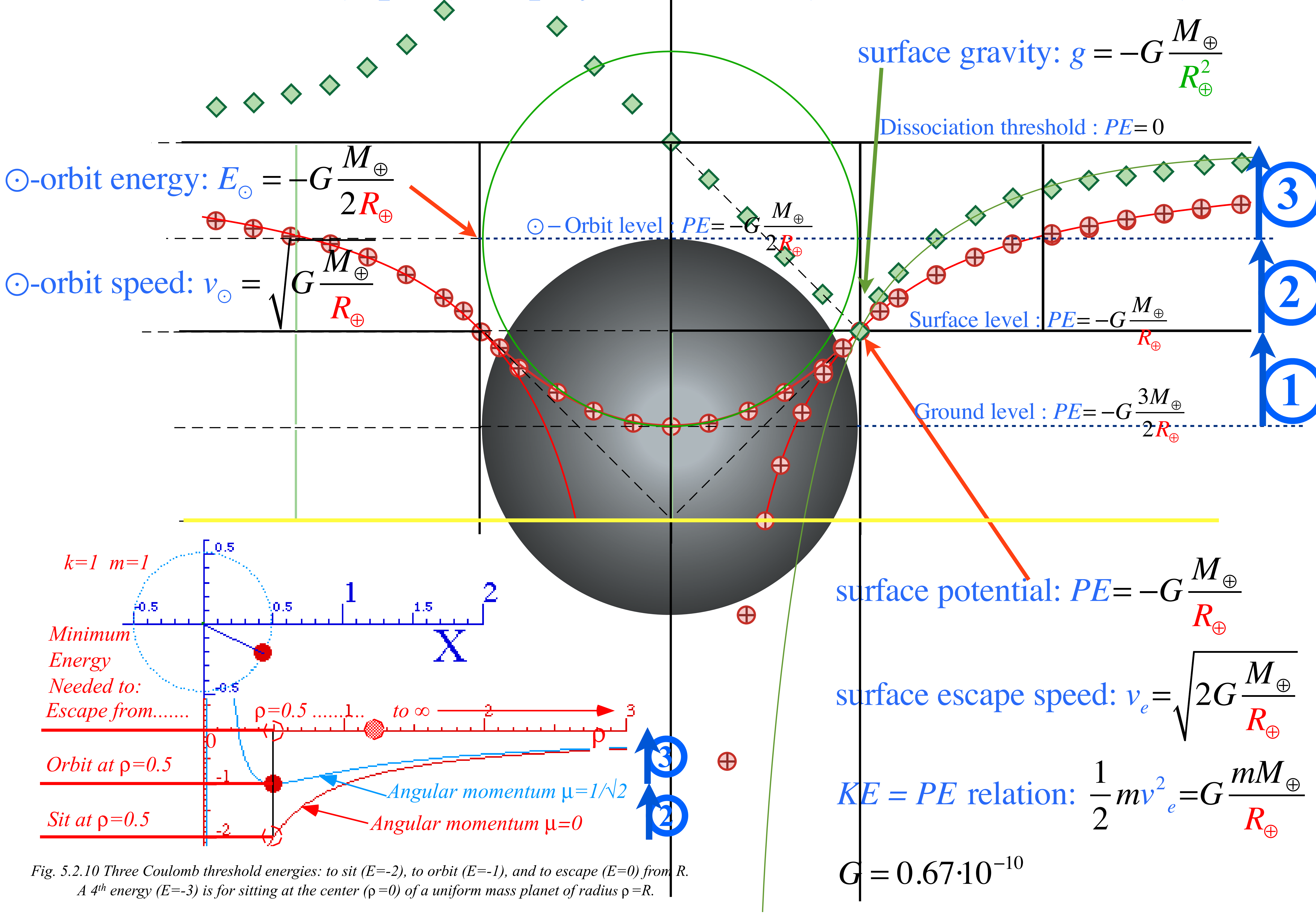

Fig. 5.2.10 Three Coulomb threshold energies: to sit (E=-2), to orbit (E=-1), and to escape (E=0) from R. A 4th energy (E=-3) is for sitting at the center (ρ =0) of a uniform mass planet of radius ρ =R.

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

➔ *Stable equilibrium radii and radial/angular frequency ratios*

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For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

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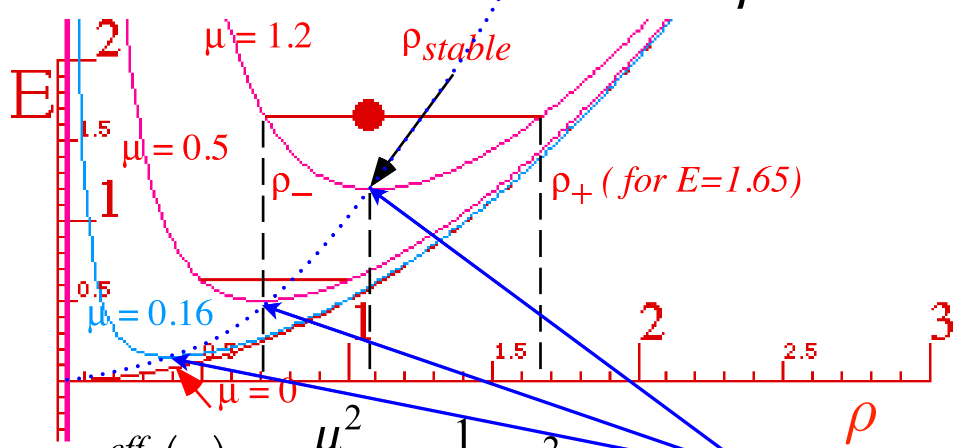

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Web Simulation
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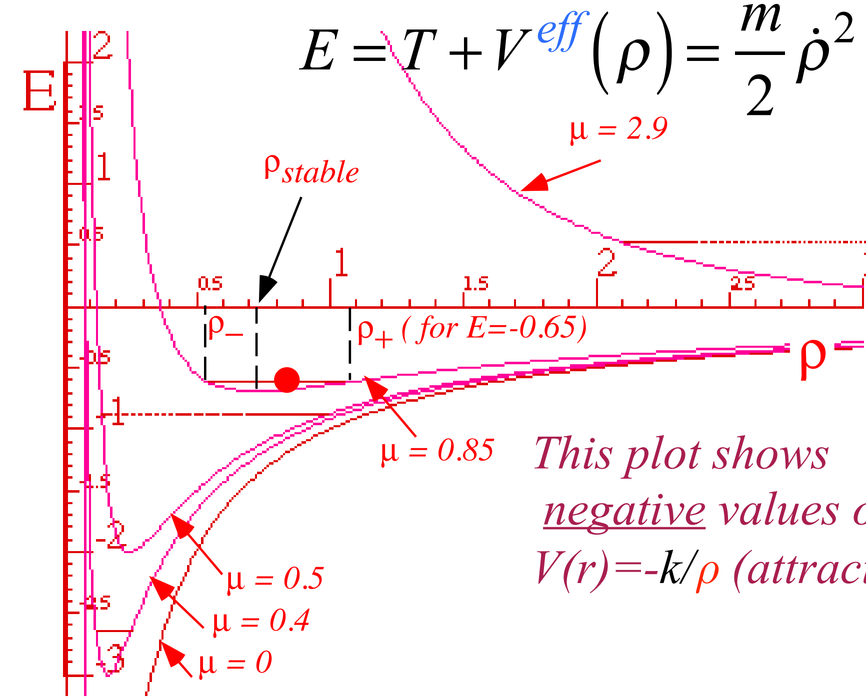

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Web Simulation
OscillatorPE - Coulomb

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \qquad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

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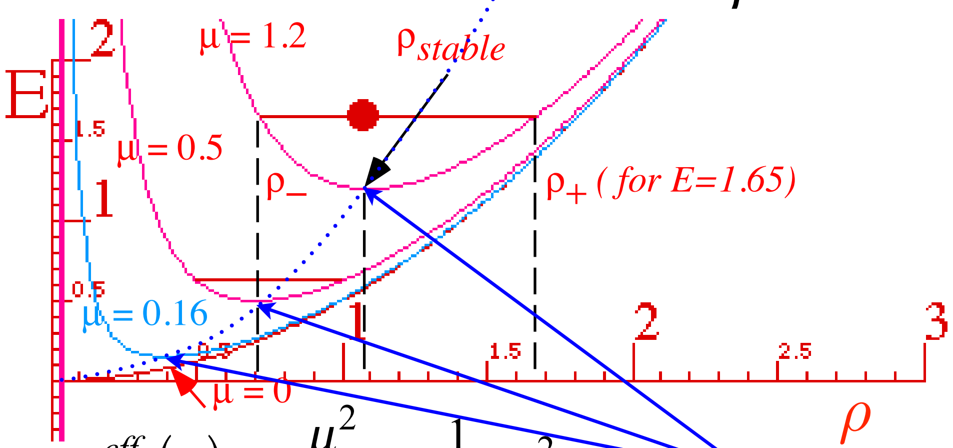

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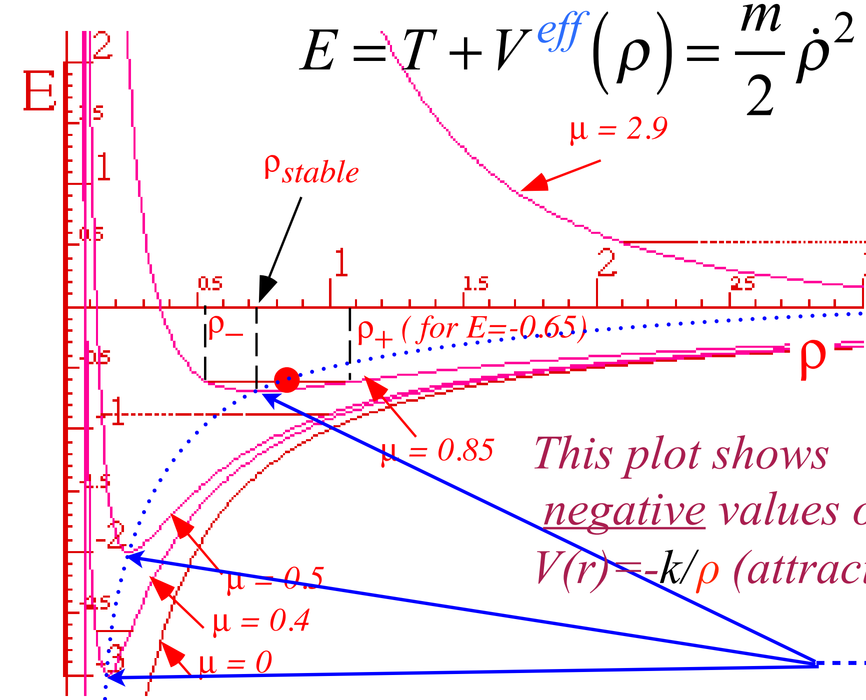

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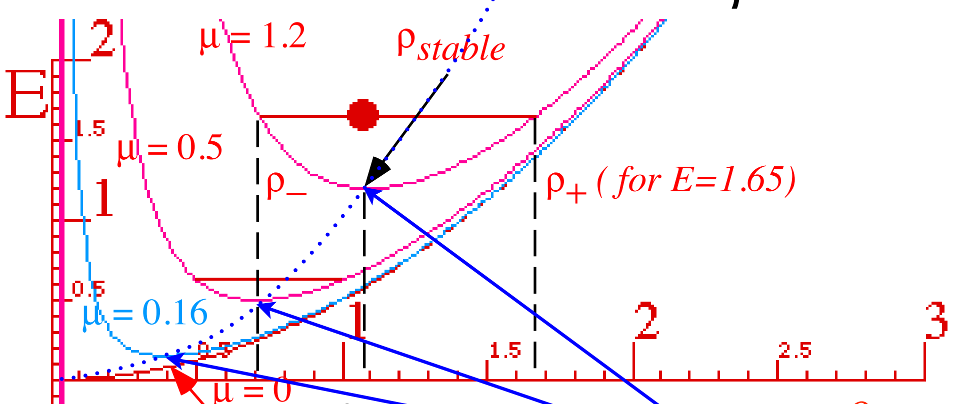

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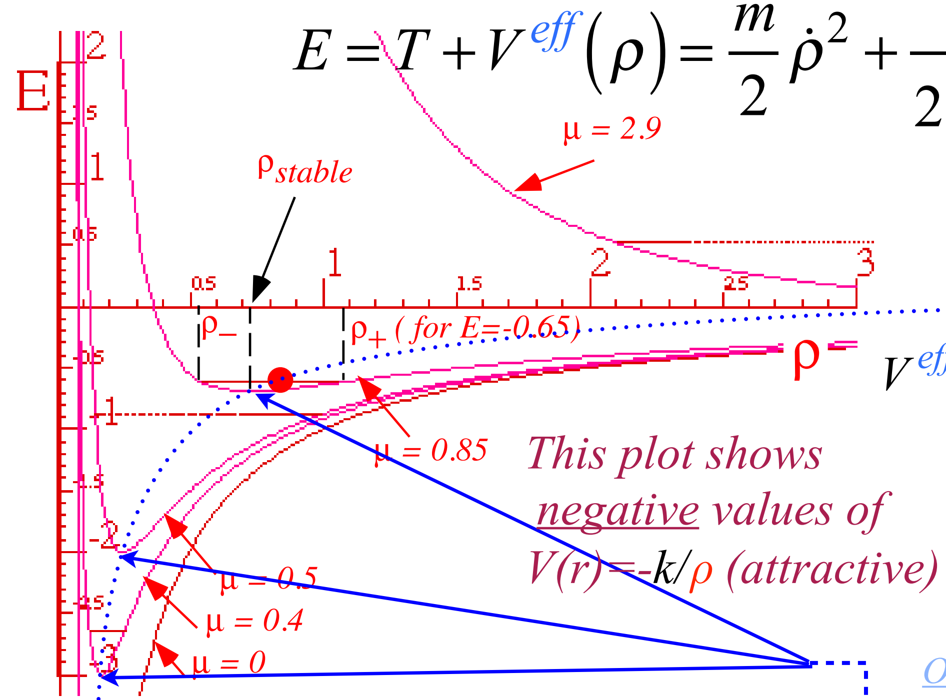

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Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

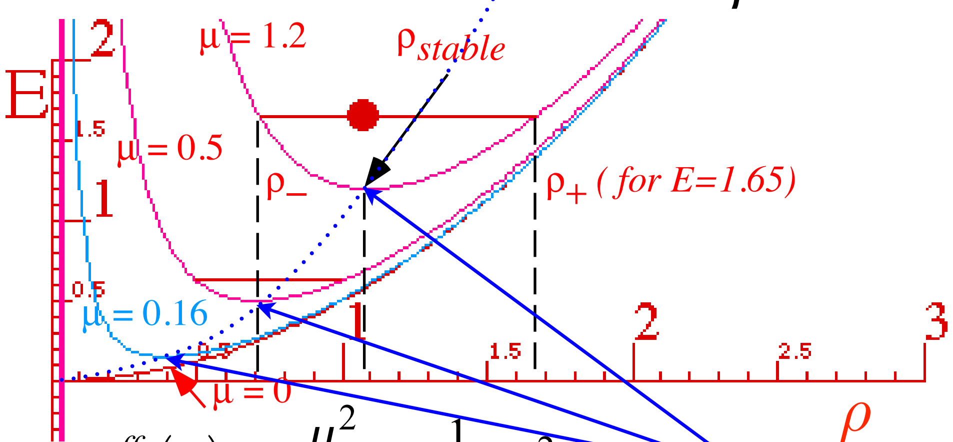
$$T=\frac{m}{2}g_{\rho\rho}\dot\rho^2+\frac{m}{2}g_{\phi\phi}\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{m}{2}\rho^2\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=const=\mu \qquad \boxed{\dot\phi=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

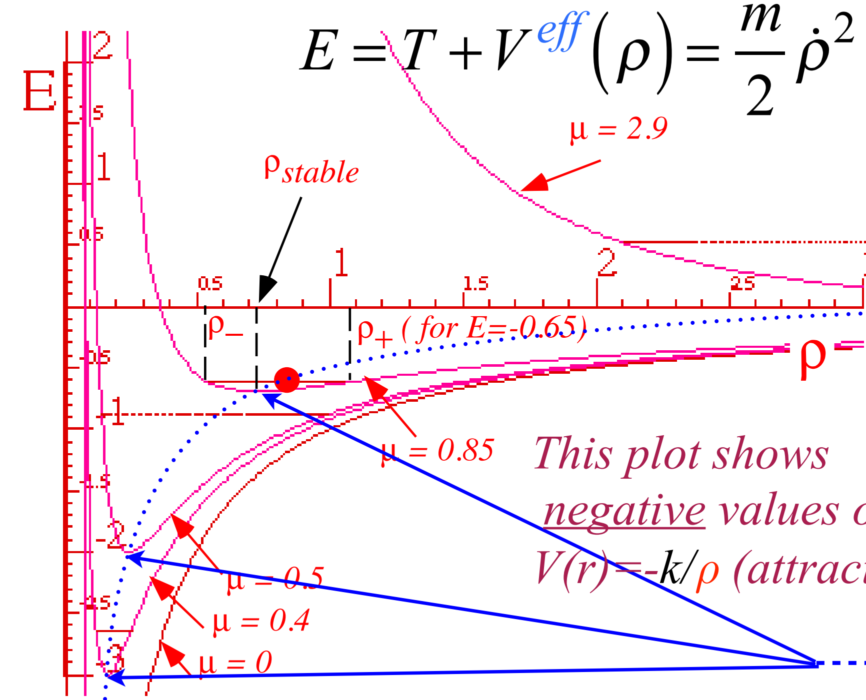


$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

[Web Simulation OscillatorPE - IHO]

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

[Web Simulation OscillatorPE - Coulomb]

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

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$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}-\frac{k}{\rho_{stable}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

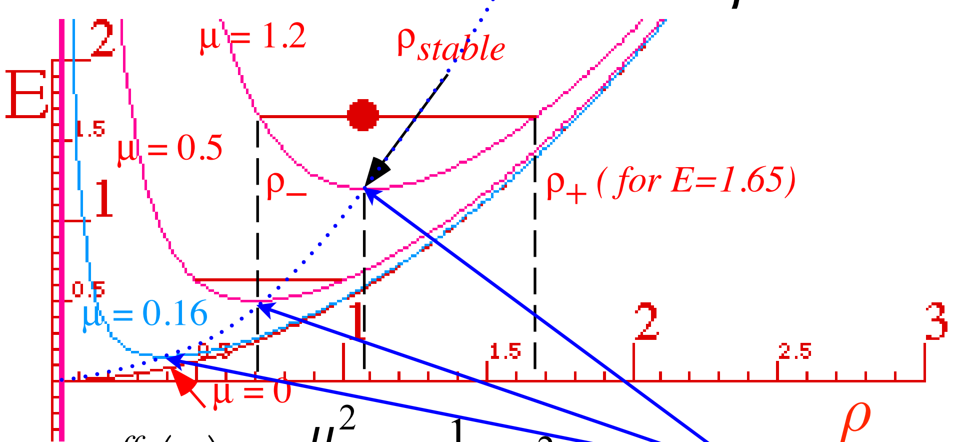
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

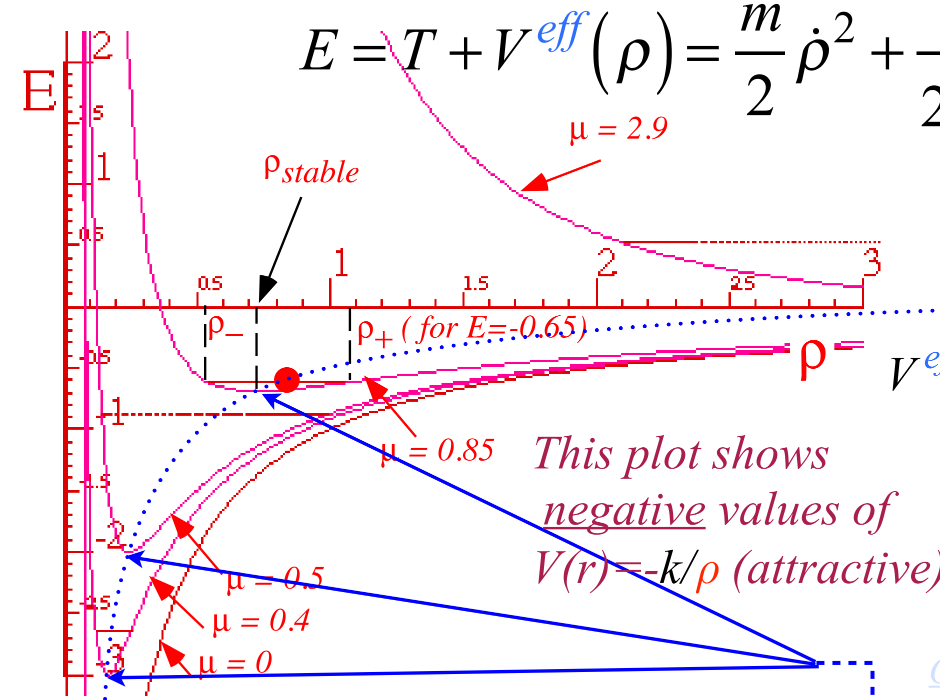


$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Web Simulation OscillatorPE - IHO

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Web Simulation OscillatorPE - Coulomb

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

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$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

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Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\sqrt{\dfrac{k}{m}}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

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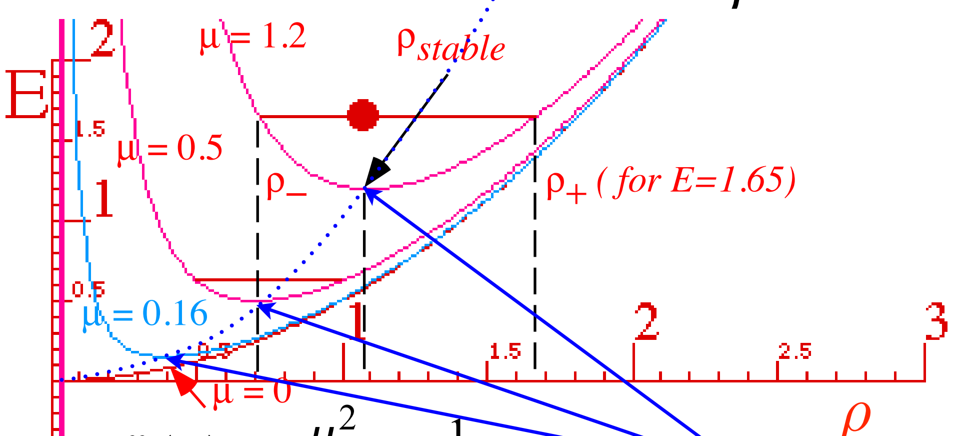

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Web Simulation
OscillatorPE - IHO

Effective potential for Coulomb $V(\rho)=-k/\rho$

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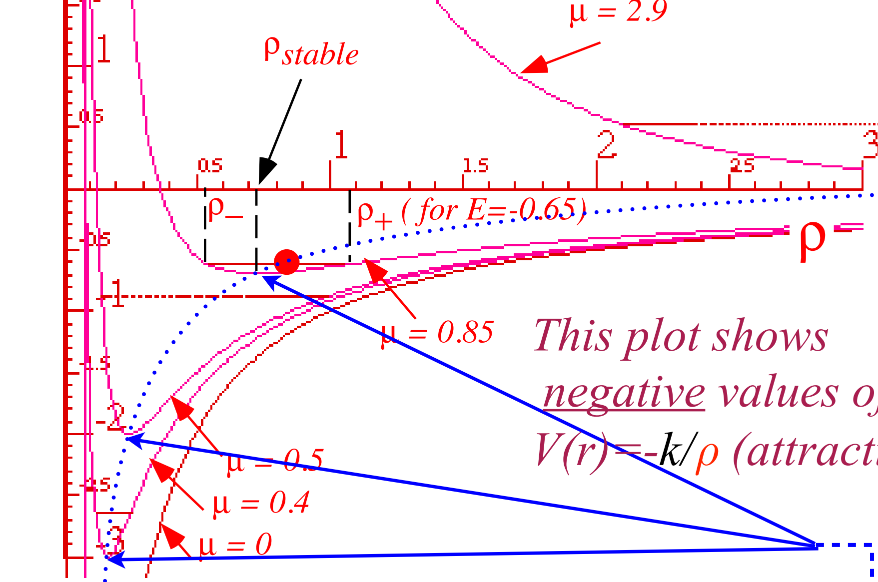

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Web Simulation
OscillatorPE - Coulomb

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

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Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}{}^2}=\sqrt{\frac{k}{m}}$

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Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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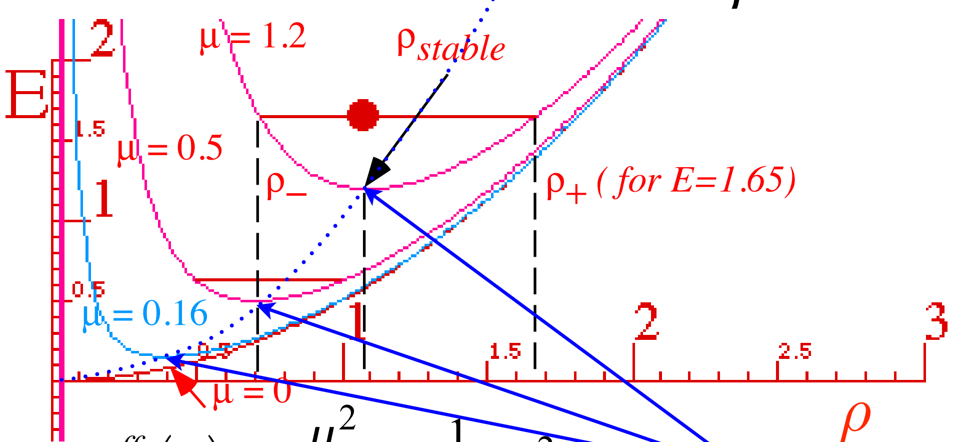

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

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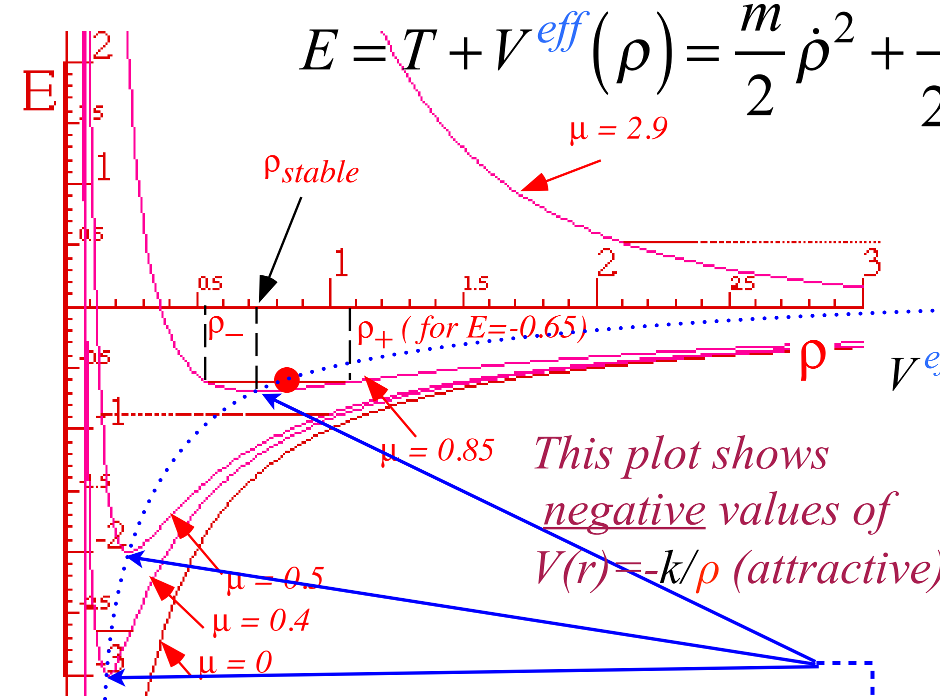

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

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Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}{}^2}=\sqrt{\frac{k}{m}}$

$\omega_{\rho_{stable}}:\omega_\phi=2:1$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}{}^2}=\frac{\mu}{m}\frac{m^2k^2}{\mu^4}=\frac{mk^2}{\mu^3}$

$\omega_{\rho_{stable}}:\omega_\phi=1:1$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

➜ *Classical turning radii and apogee/perigee parameters*

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

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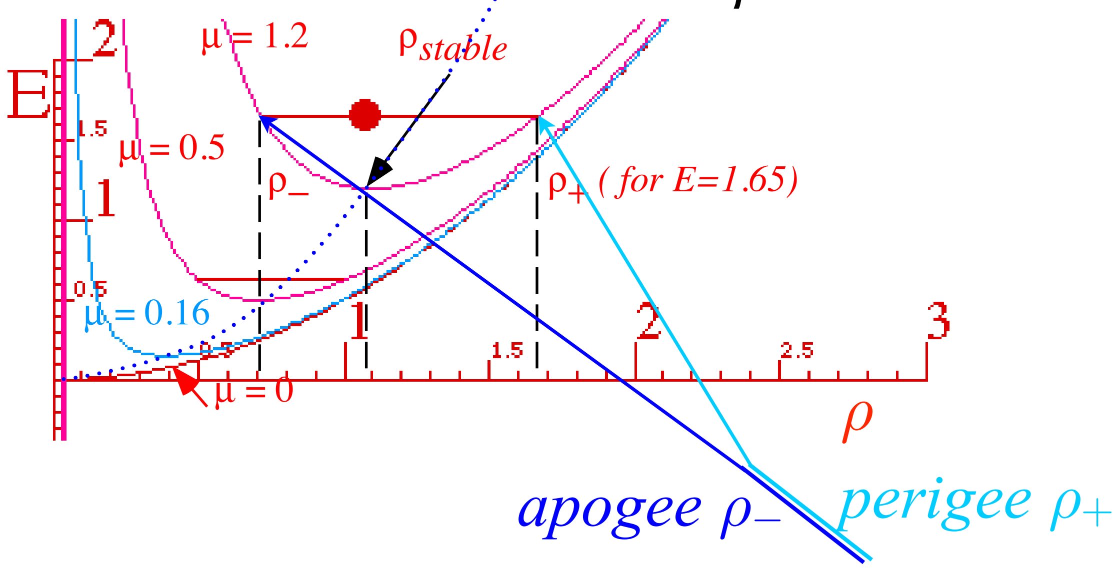
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



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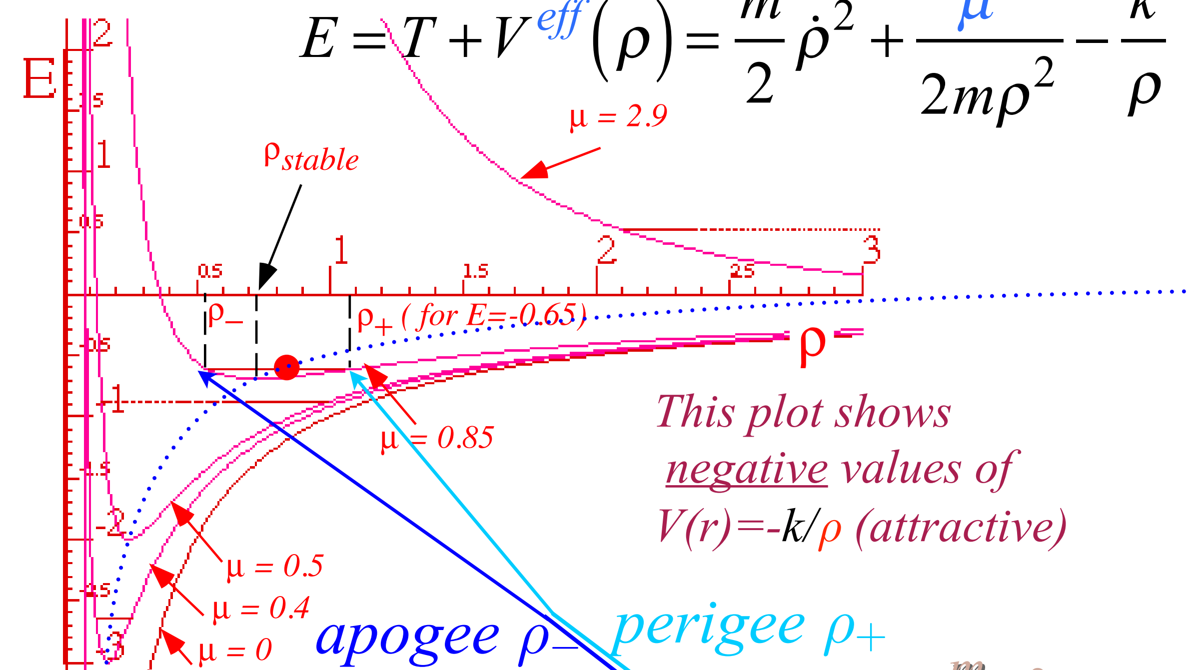

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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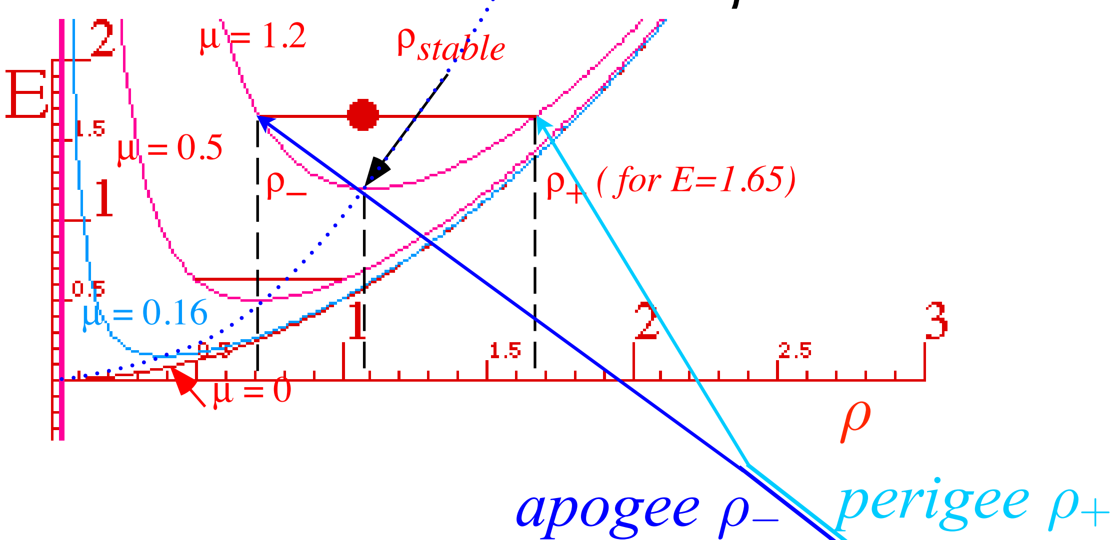

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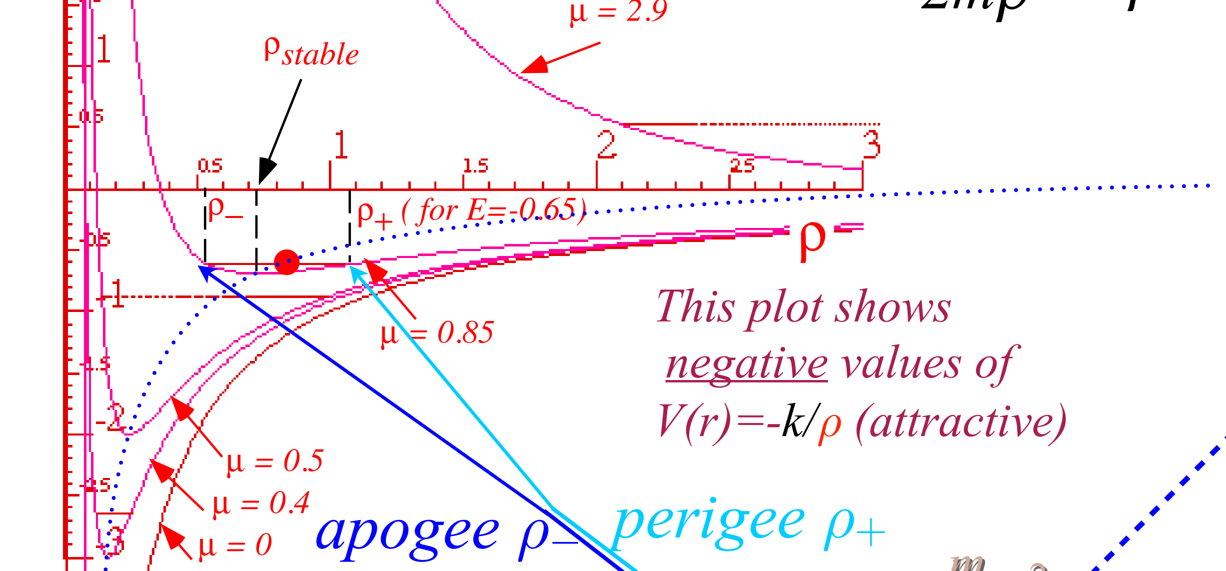

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Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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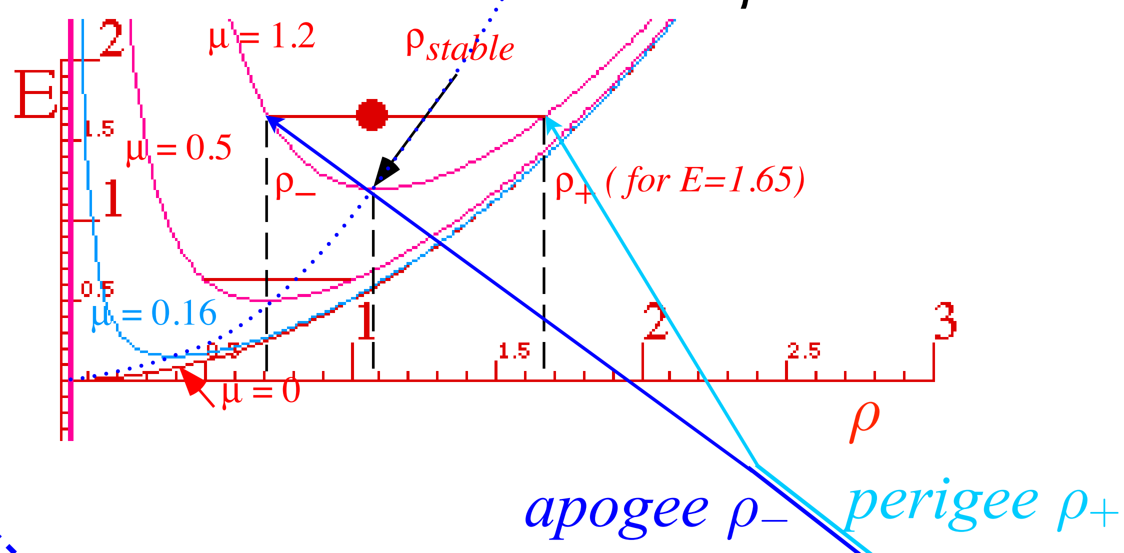
For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ | Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



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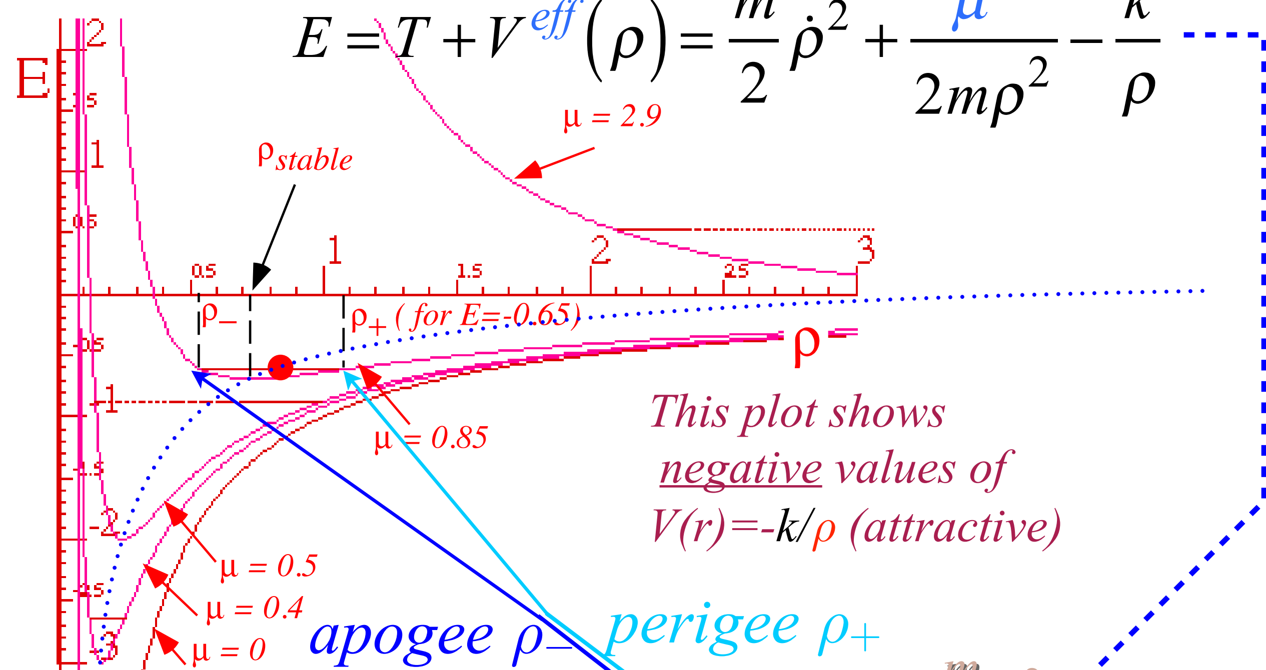

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$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

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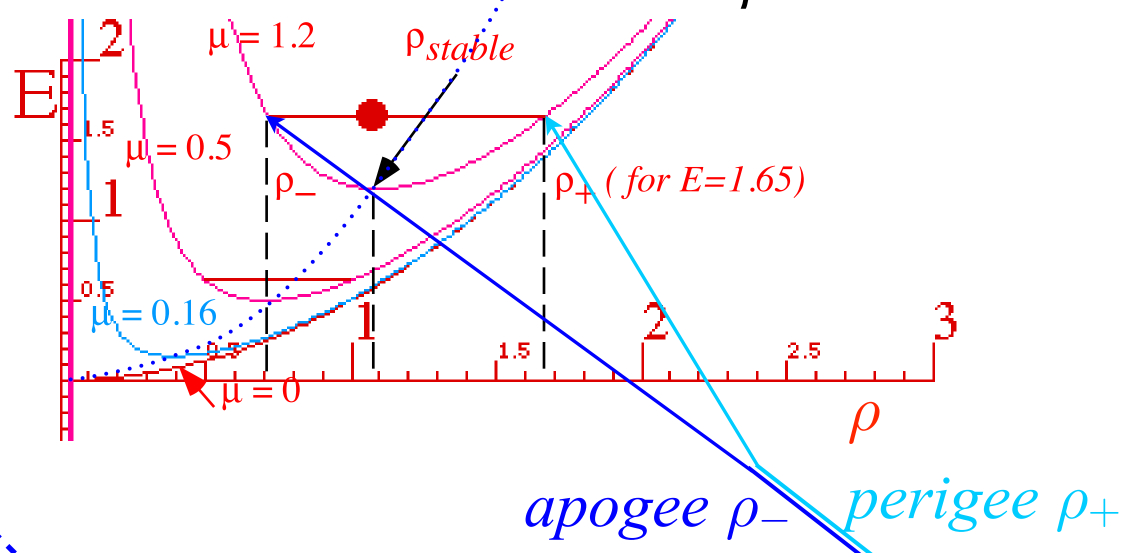
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



Effective potential for Coulomb $V(\rho)=-k/\rho$

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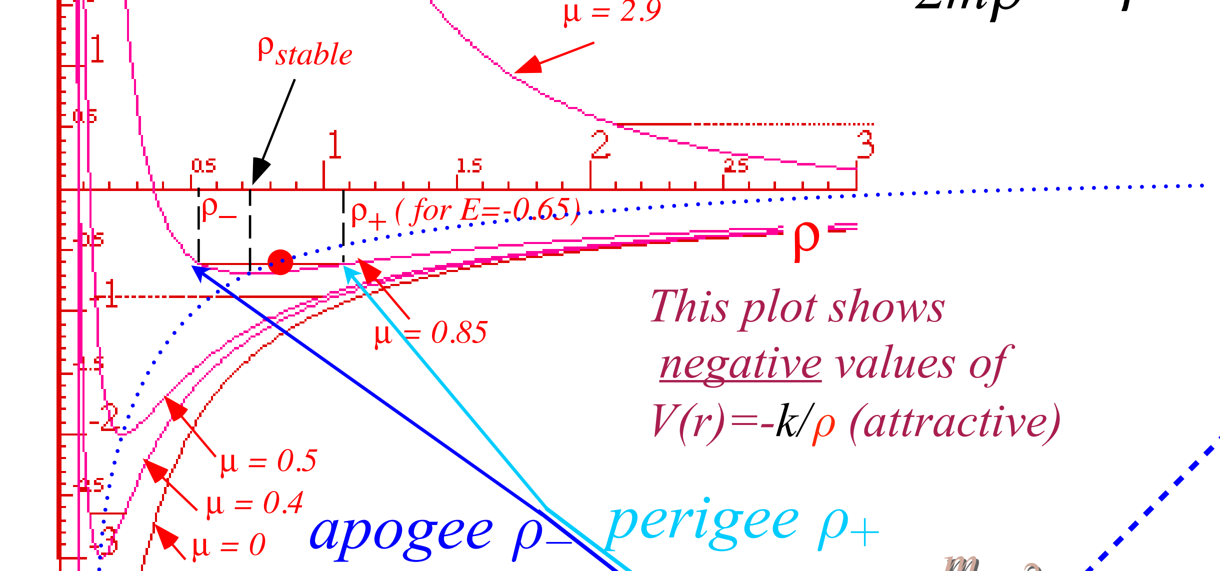

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$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4\quad\text{or else: }0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0\quad\text{or else: }0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

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Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

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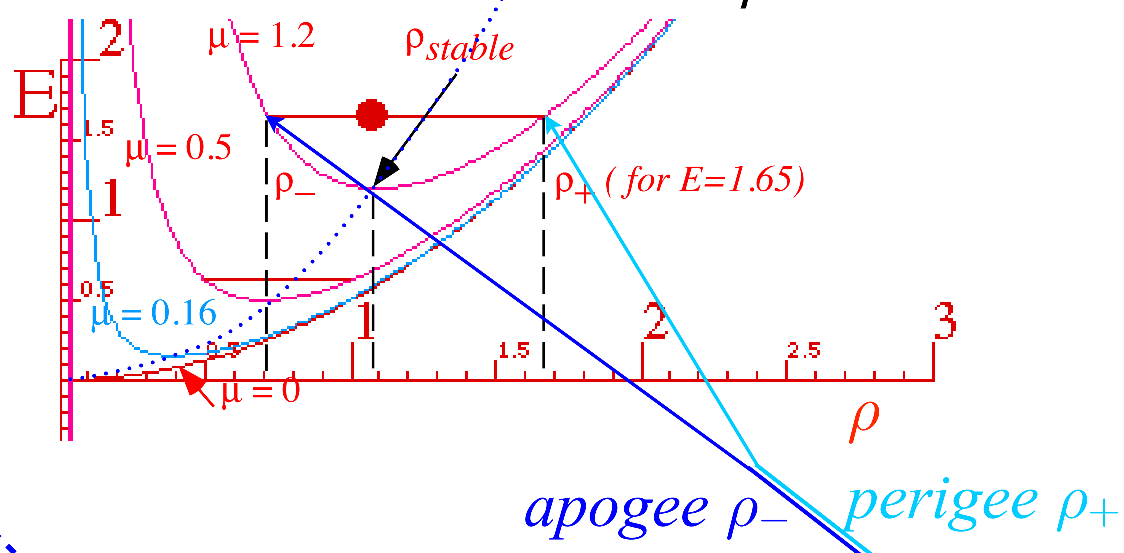

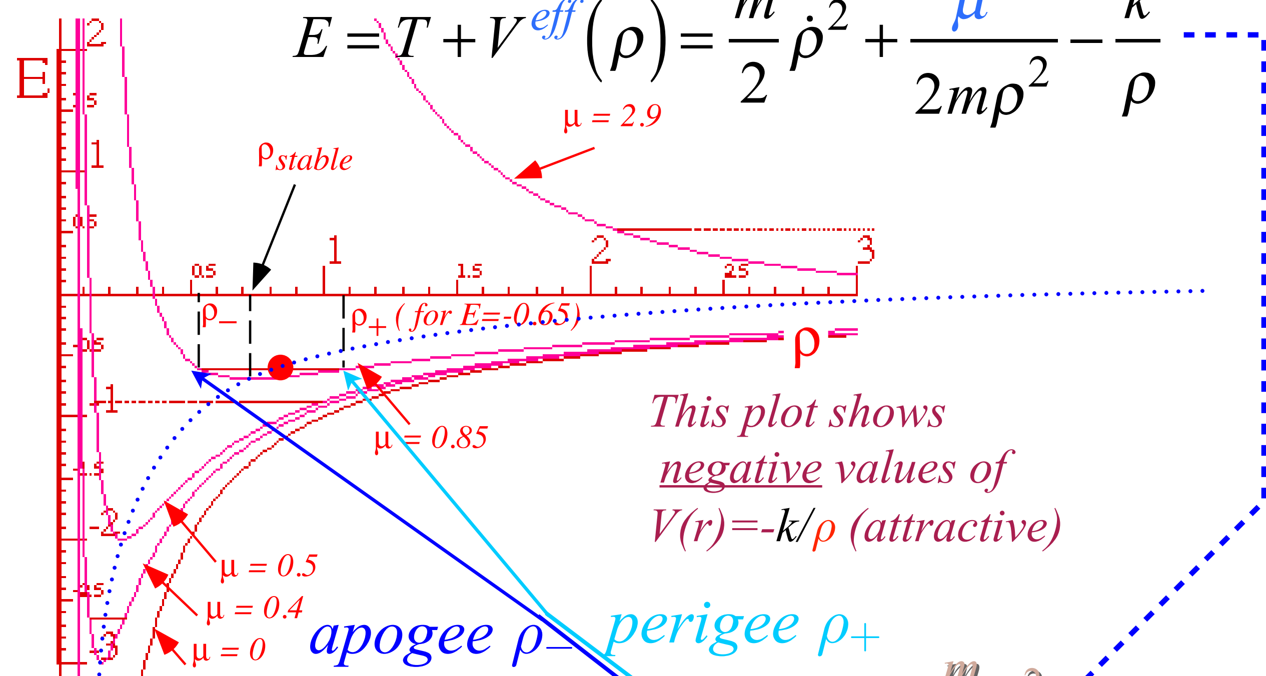

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot\rho^2$ is zero.

$$0=-E+\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad\text{or else:}\quad 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2} \qquad 0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0 \quad\text{or else:}\quad 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad\text{or else:}\quad \frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

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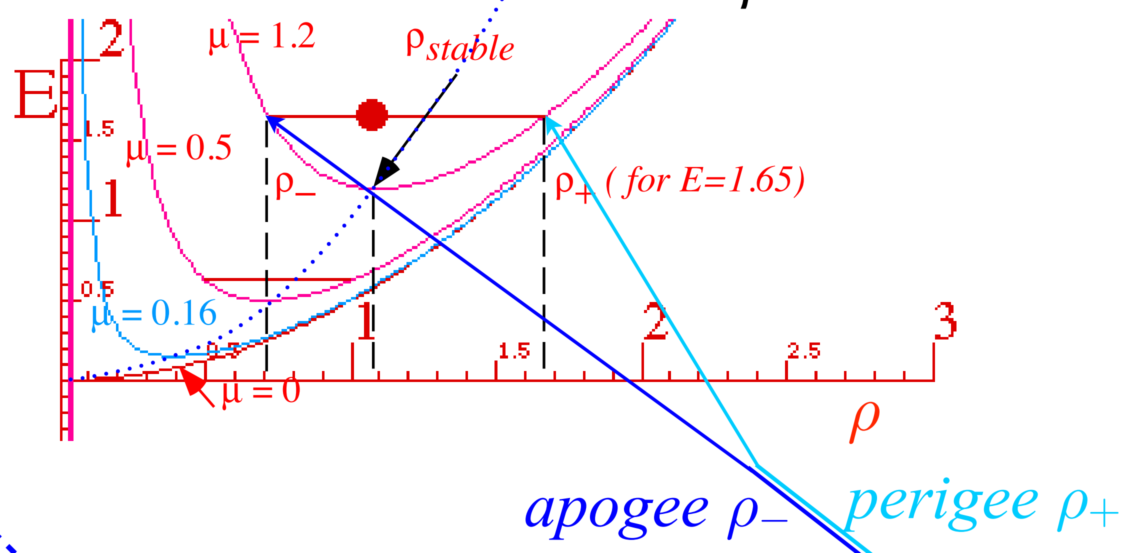

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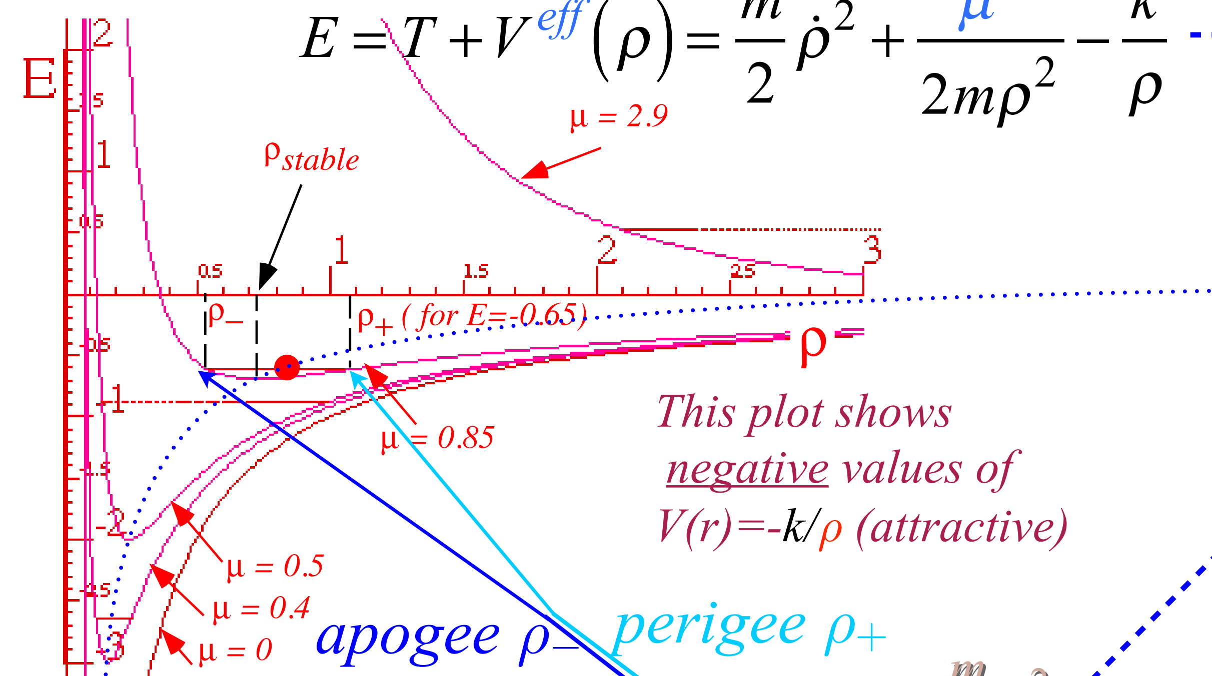

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Notice mysterious similarity: E→k and k→2E

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

➡ *Polar coordinate differential equations*

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

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Let: $\frac{1}{\rho}=u$ so: $\begin{cases} -\frac{d\rho}{\rho^2}=du \\ d\rho=-\frac{du}{u^2}\end{cases}$

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Common:

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Common steps

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$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Coulomb:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{2k\sqrt{x}}{m}}}$$

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  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
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(A mystery similarity appears)

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$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

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Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

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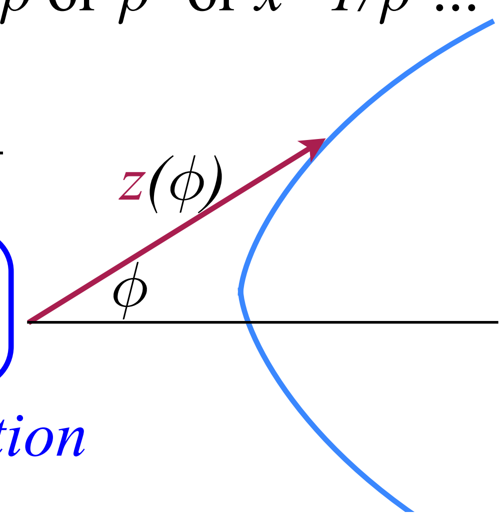
radial-polar-coordinate orbit function



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$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}\qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

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$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

IHOscillator: $A=\frac{\mu^2}{m^2}$ $\quad B=-\frac{2E}{m}$ $\quad C=\frac{k}{m}$ $\quad D=-\frac{\mu}{2m}$

Coulomb: $A=\frac{\mu^2}{m^2}$ $\quad B=\frac{2k}{m}$ $\quad C=-\frac{2E}{m}$ $\quad D=-\frac{\mu}{m}$

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For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

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Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\dfrac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

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$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

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$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

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$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

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Algebra details on following pages

Orbits in Isotropic Oscillator and Coulomb Potentials

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Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}\qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\quad \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Oscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Algebra details on following pages

Algebra details and checks

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}$$

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{\frac{2E}{m}}{2\frac{\mu^2}{m^2}}=\frac{E}{\mu^2/m}=\frac{z_++z_-}{2}$$

$$\beta=\frac{\sqrt{\left(\frac{2E}{m}\right)^2-4\frac{\mu^2}{m^2}\frac{k}{m}}}{2\frac{\mu^2}{m^2}}$$

$$\beta=\frac{\sqrt{\left(\frac{2Em}{m\cdot m}\right)^2-4\frac{\mu^2}{m^2}\frac{km}{m^2}}}{2\frac{\mu^2}{m^2}}=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}=\frac{z_+-z_-}{2}$$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-\frac{2k}{m}}{2\frac{\mu^2}{m^2}}=\frac{-k}{\mu^2/m}=\frac{z_++z_-}{2}$$

$$\beta=\frac{\sqrt{\left(\frac{2k}{m}\right)^2+4\frac{\mu^2}{m^2}\frac{2E}{m}}}{2\frac{\mu^2}{m^2}}=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}=\frac{z_+-z_-}{2}$$

Checking that roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$) from p.27-29.

$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad \text{or else:}\quad \frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$\rho_\pm=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}\quad \text{or else:}\quad \frac{1}{\rho_\pm}=\frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

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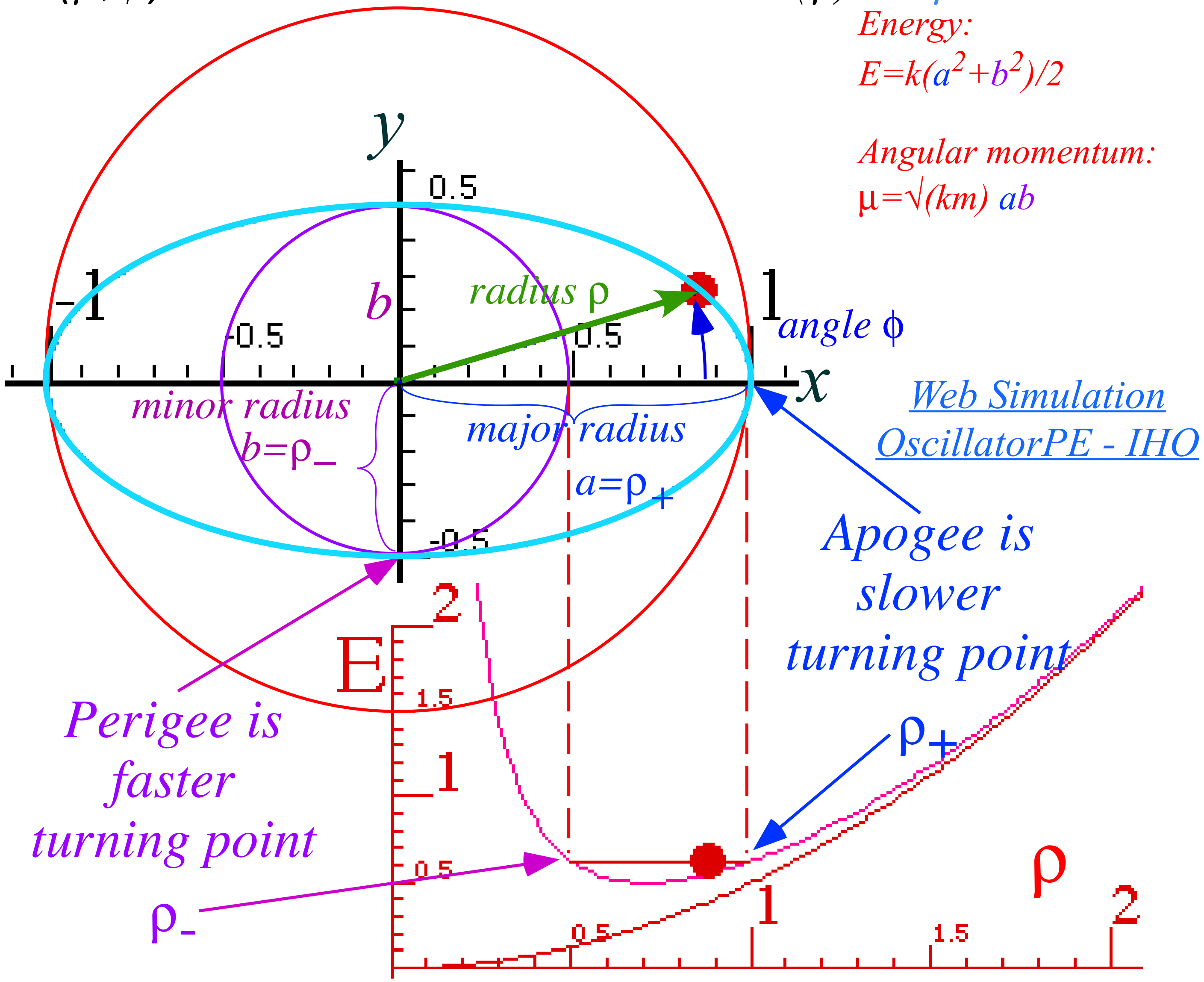
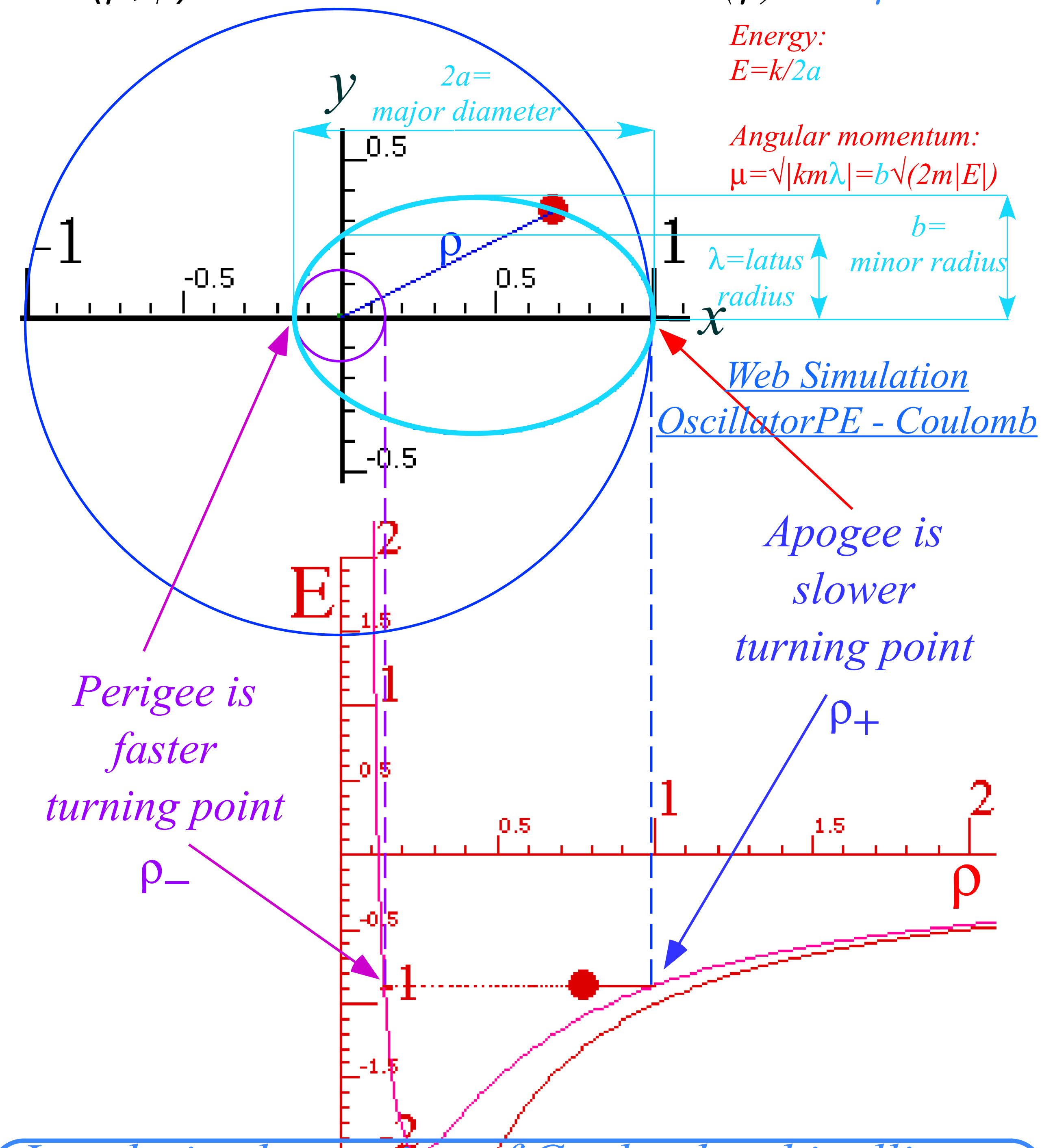
Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



Just derived equation of IHO orbit ellipse

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

$$\frac{1}{\rho^2}=+\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos(2\phi)$$

One of many equations of center-centered ellipse

Just derived equation of Coulomb orbit ellipse

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

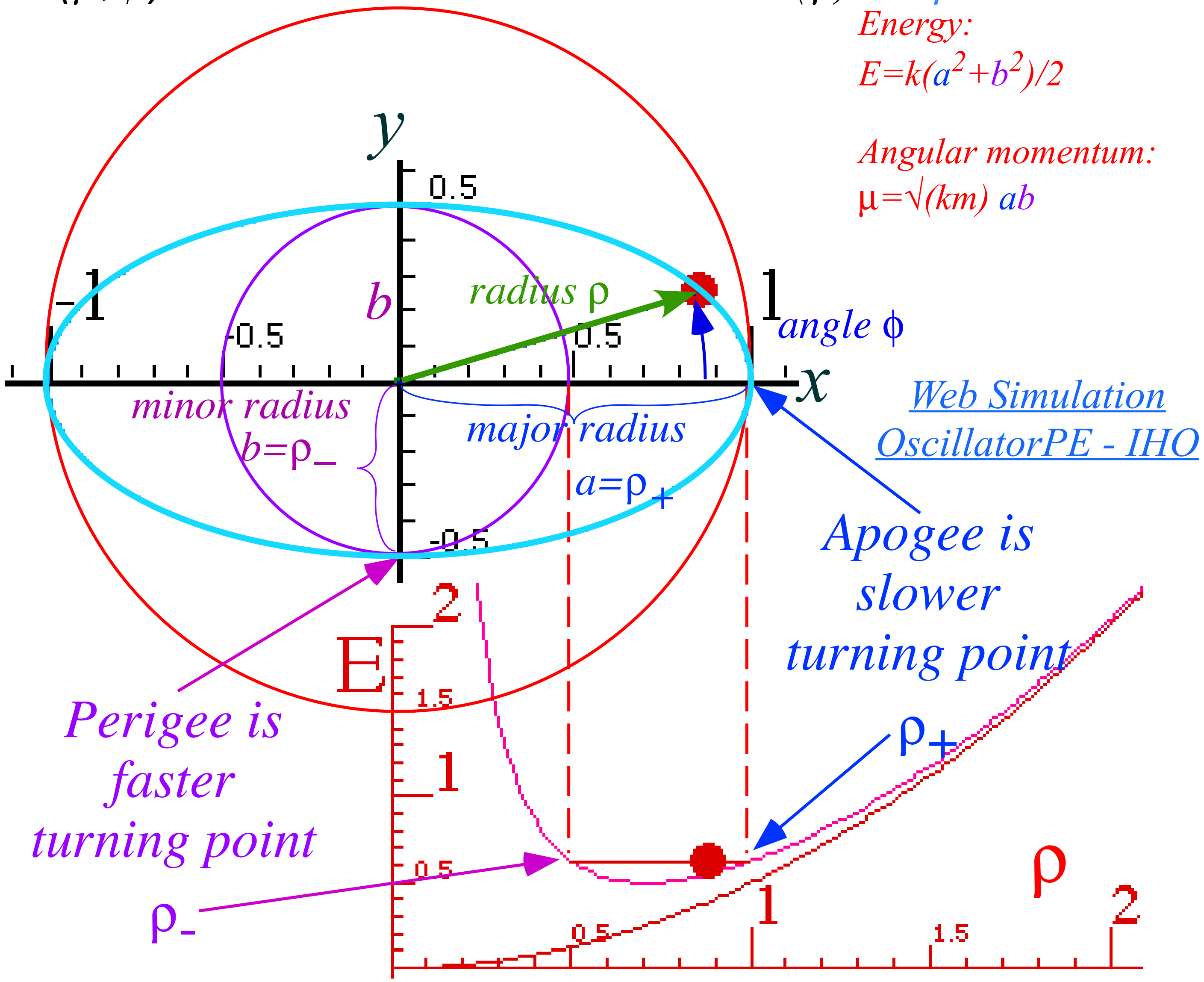
(to be discussed shortly)

$$\frac{1}{\rho}=\frac{a}{b^2}-\frac{\sqrt{a^2-b^2}}{b^2}\cos\phi$$

One of many equations of focus-centered ellipse

Orbits in Isotropic Oscillator and Coulomb Potentials

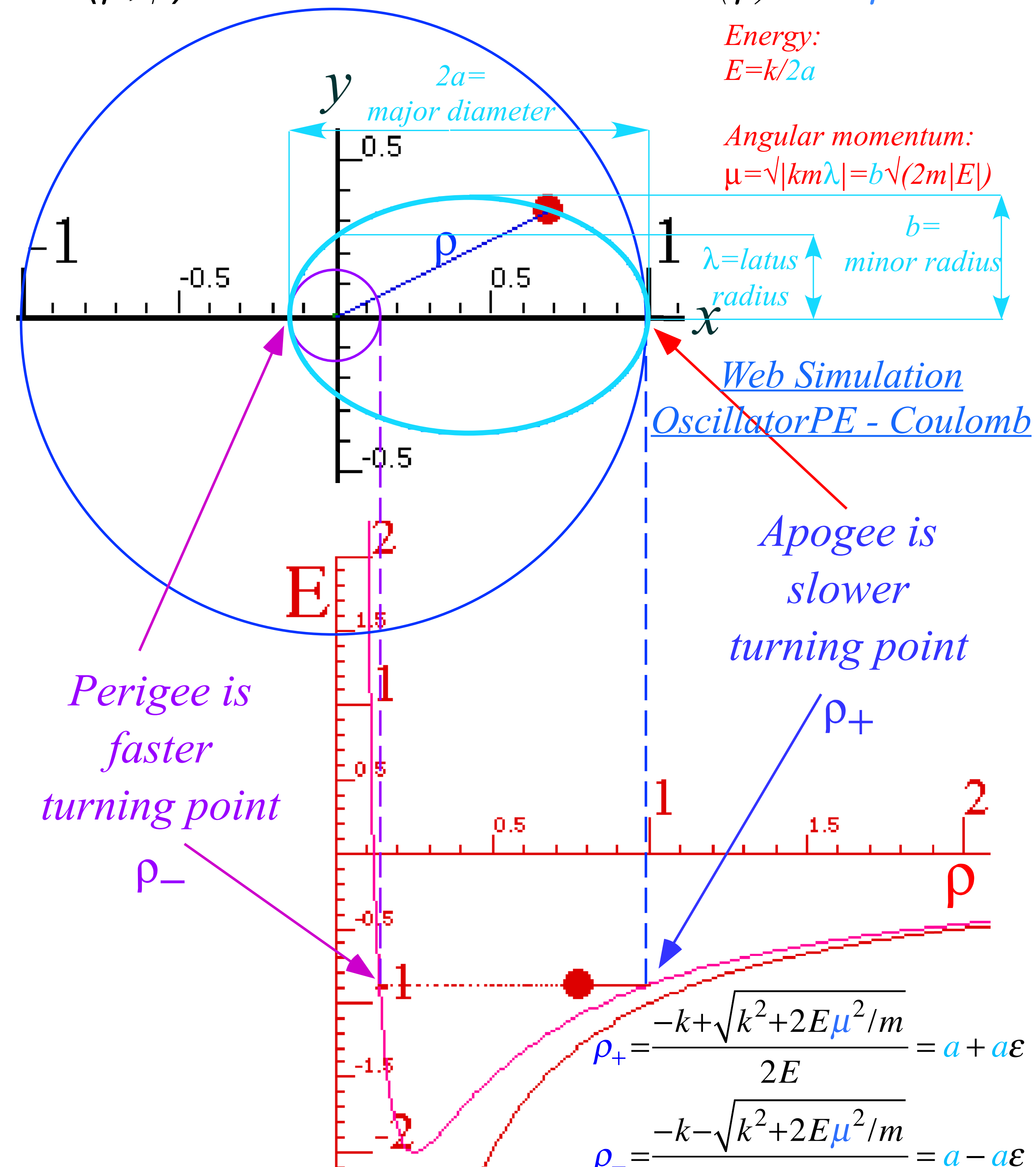
(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$



$$\rho_+^2 = \frac{E + \sqrt{E^2 - k\mu^2/m}}{k} = a^2$$

$$\rho_-^2 = \frac{E - \sqrt{E^2 - k\mu^2/m}}{k} = b^2$$

(ρ,ϕ) orbits for Coulomb $V(\rho) = -k/\rho$



$$\rho_+ = \frac{-k + \sqrt{k^2 + 2E\mu^2/m}}{2E} = a + a\varepsilon$$

$$\rho_- = \frac{-k - \sqrt{k^2 + 2E\mu^2/m}}{2E} = a - a\varepsilon$$

(from p.29 or p.57)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

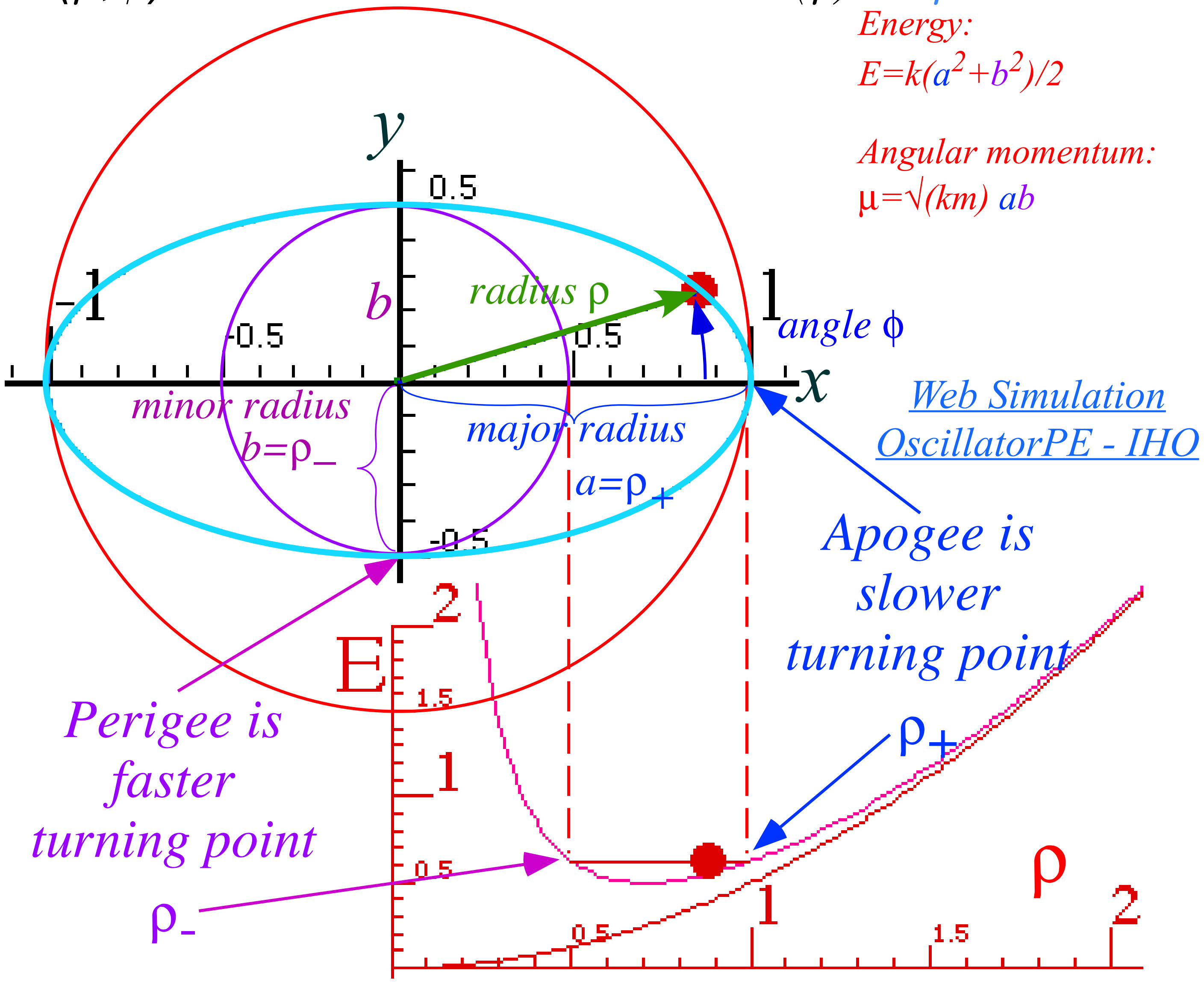
$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$

Energy: $E=k(a^2+b^2)/2$

Angular momentum: $\mu=\sqrt{(km)}\,ab$

Web Simulation OscillatorPE - IHO

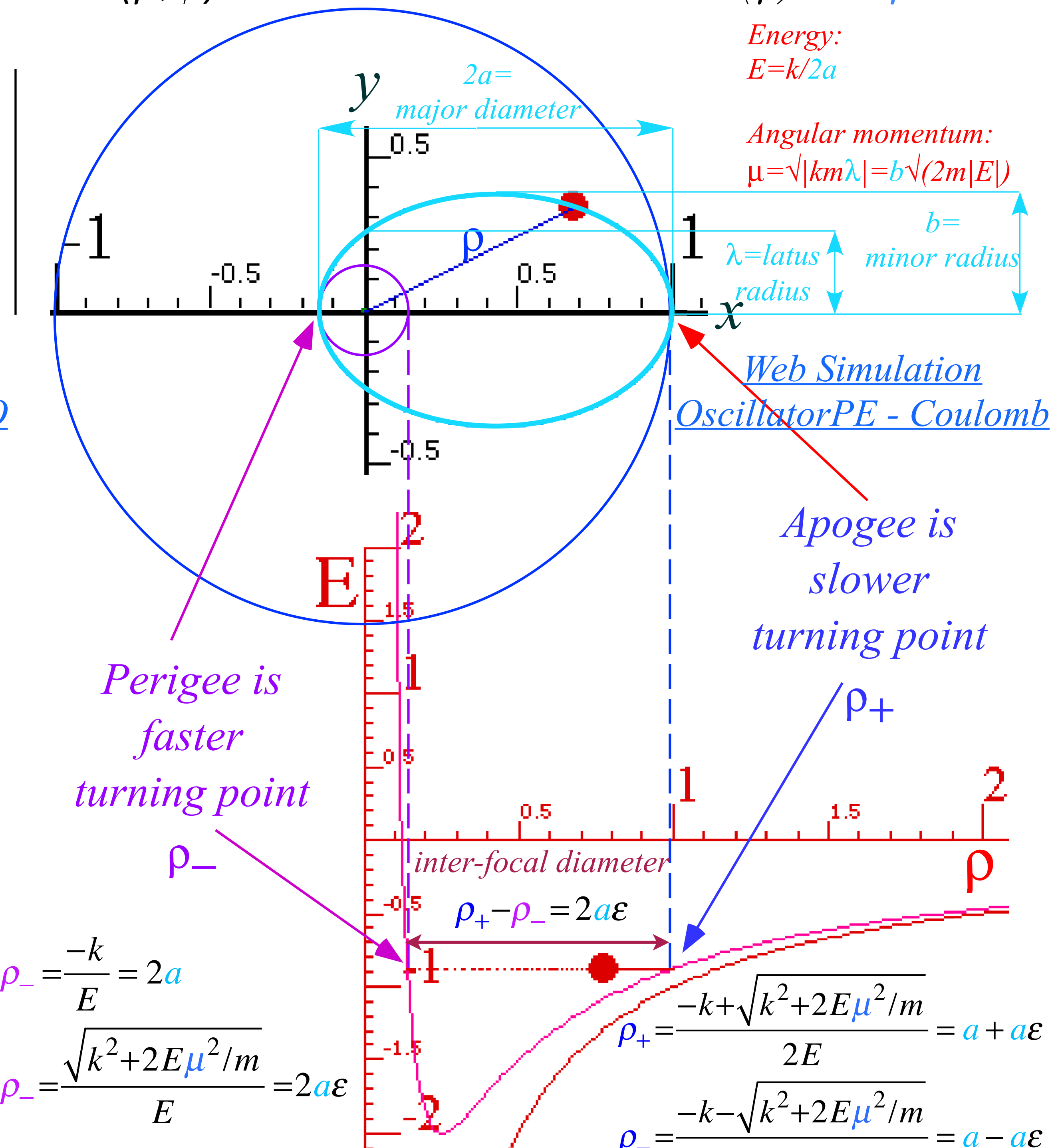


(ρ,ϕ) orbits for Coulomb $V(\rho) = -k/\rho$

Energy: $E=k/2a$

Angular momentum: $\mu=\sqrt{|km\lambda|}=b\sqrt{(2m|E|)}$

Web Simulation OscillatorPE - Coulomb



$$\rho_+^2 = \frac{E+\sqrt{E^2-k\mu^2/m}}{k} = a^2 \qquad \rho_+^2+\rho_-^2 = \frac{2E}{k} = a^2+b^2$$

$$\rho_-^2 = \frac{E-\sqrt{E^2-k\mu^2/m}}{k} = b^2 \qquad \rho_+^2-\rho_-^2 = \frac{2\sqrt{E^2-k\mu^2/m}}{k} = a^2-b^2 = a\varepsilon$$

$$\rho_+^2\rho_-^2 = \frac{E^2 - E^2-k\mu^2/m}{k^2} = a^2b^2 = \frac{-\mu^2}{km}$$

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a$$

$$\rho_+ - \rho_- = \frac{\sqrt{k^2+2E\mu^2/m}}{E} = 2a\varepsilon$$

$$\rho_+ = \frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E} = a + a\varepsilon$$

$$\rho_- = \frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E} = a - a\varepsilon$$

$$\rho_+\rho_- = \frac{k^2-k^2-2E\mu^2/m}{(2E)^2} = a^2 - a^2\varepsilon^2 = \frac{-\mu^2}{2Em} = b^2$$ (from p.29 or p.57)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

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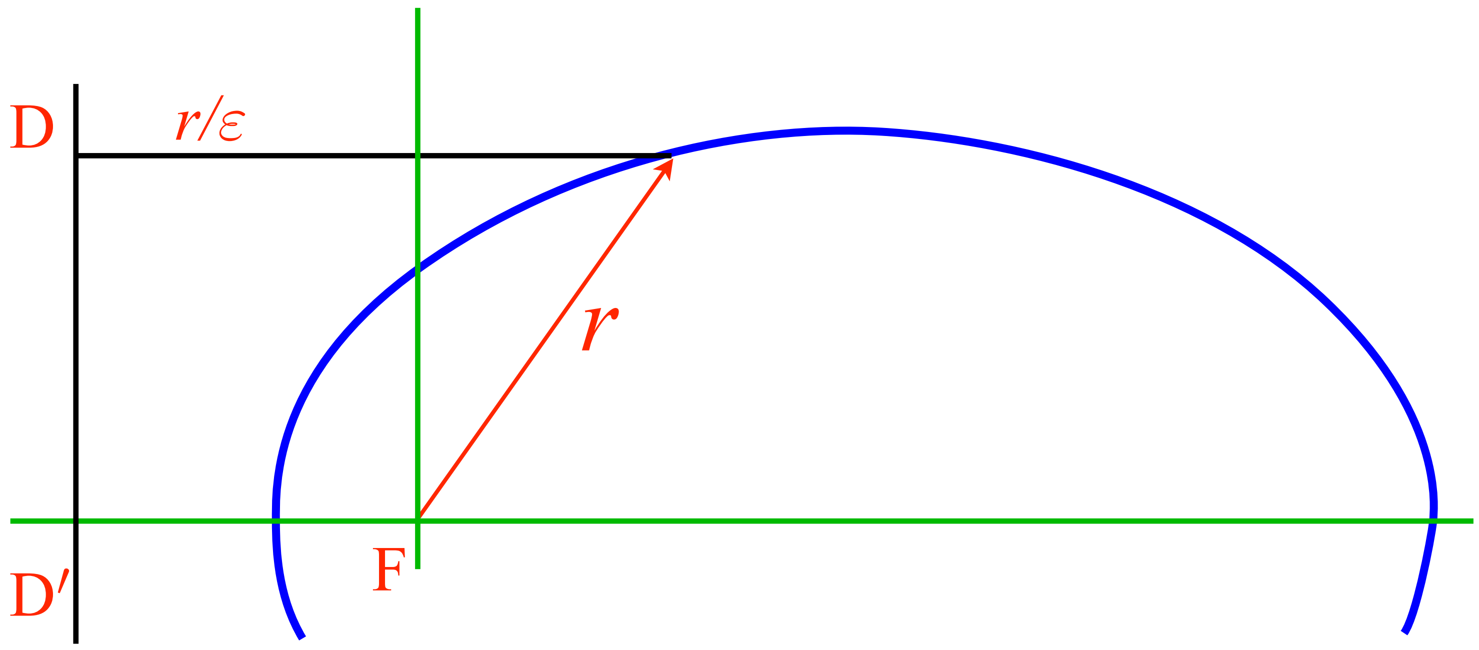
Geometry of ALL Coulomb conic section orbits (Let: r = ρ here)

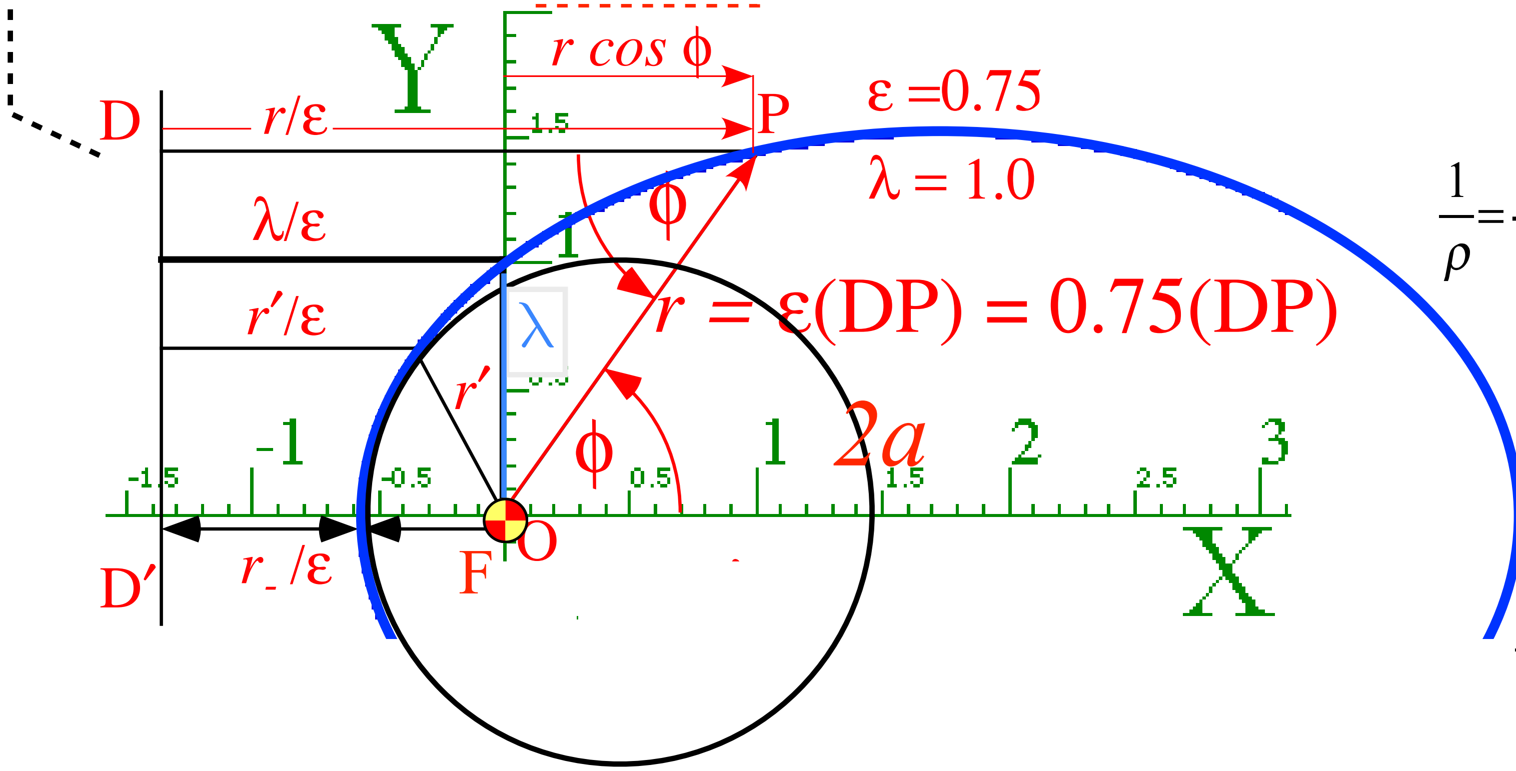


All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = $\boldsymbol{\varepsilon}$·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.59 physics:

$$\frac{1}{r} = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by: ***Eccentricity*** ε

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Geometry of ALL Coulomb conic section orbits (Let:$r \equiv \rho$ here)

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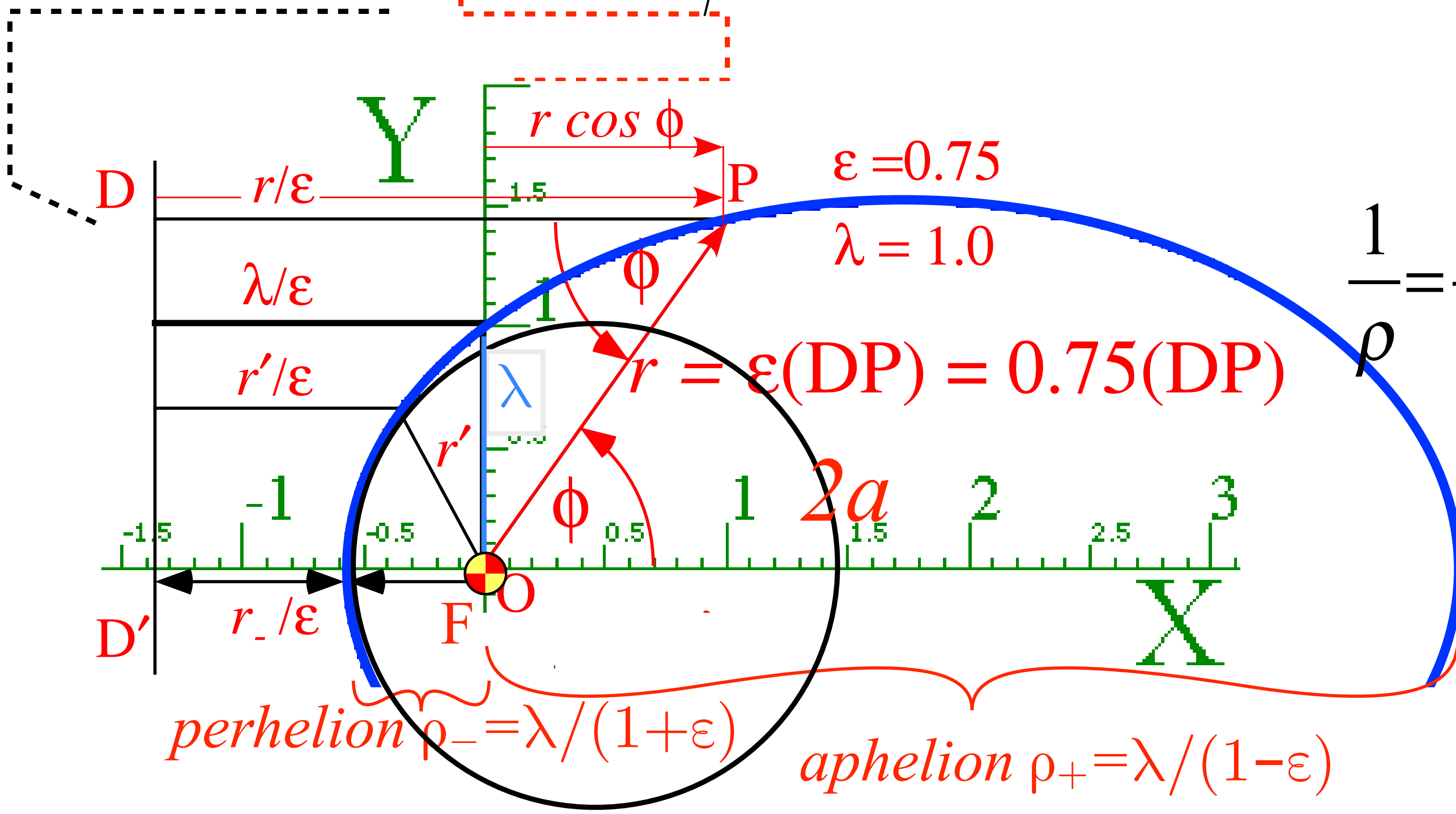

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$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

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Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

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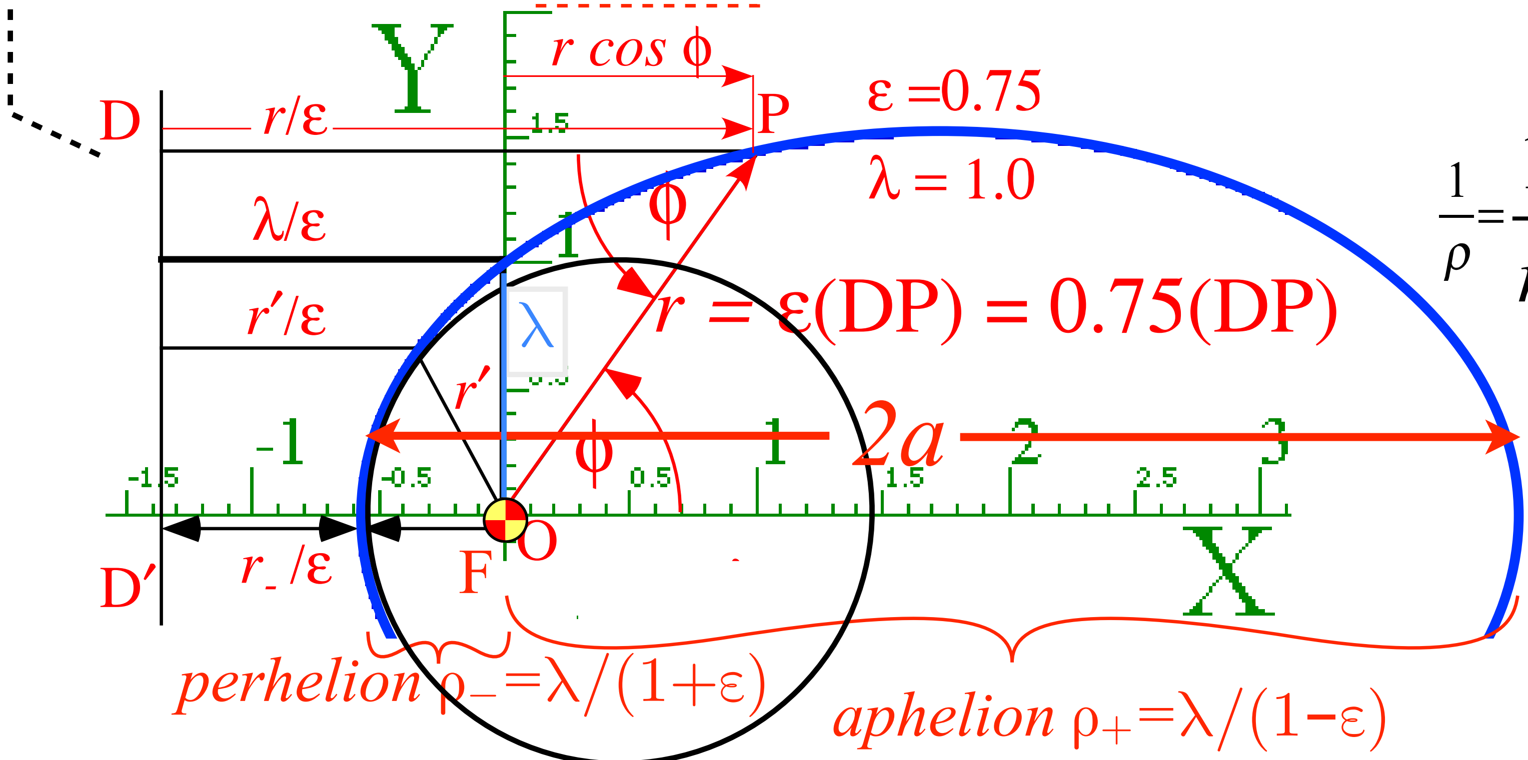

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***Major axis**:* $\rho_+ + \rho_- = 2a$

$$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$$

All conics defined by: ***Eccentricity*** ε

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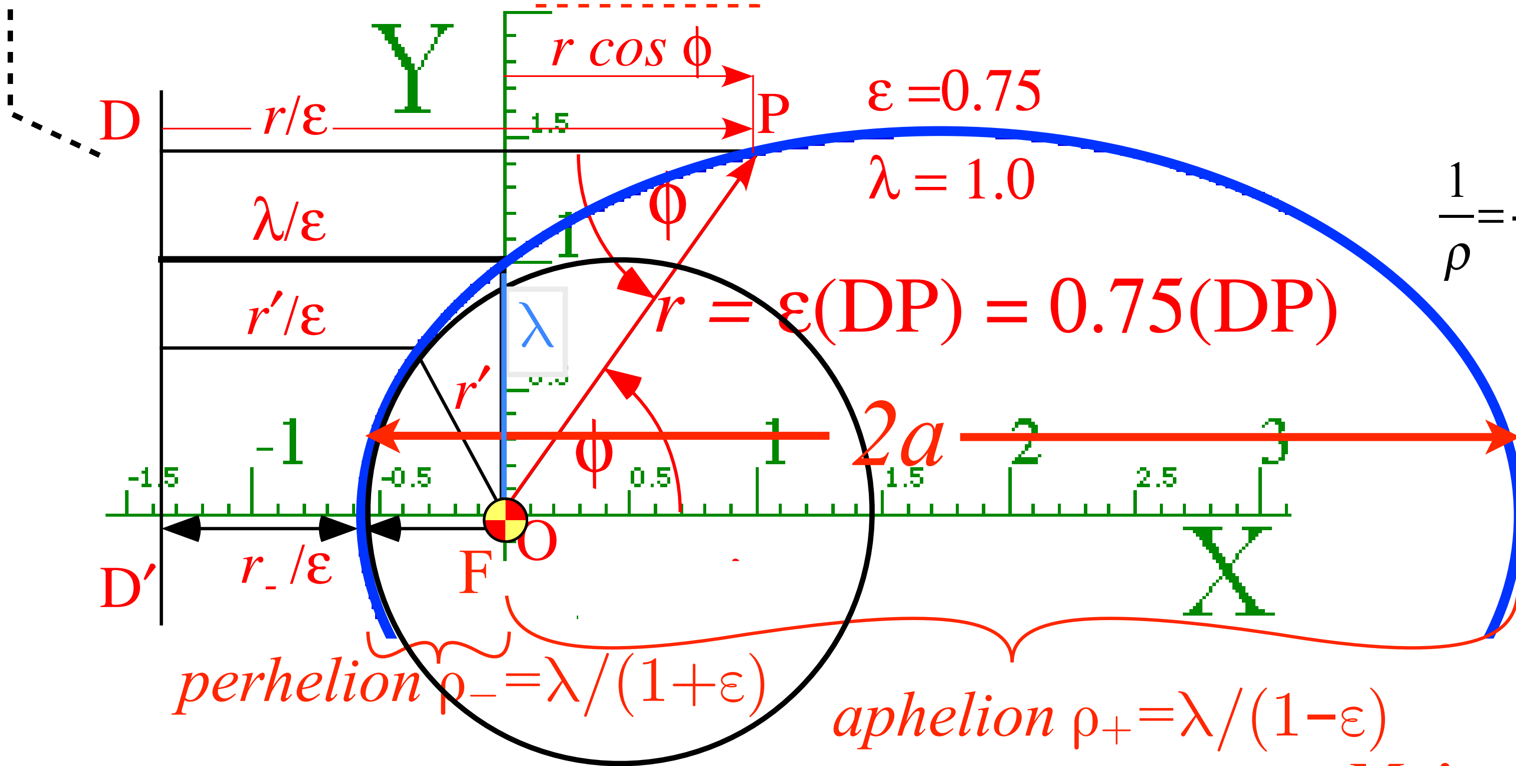
$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

***Major axis**:* $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

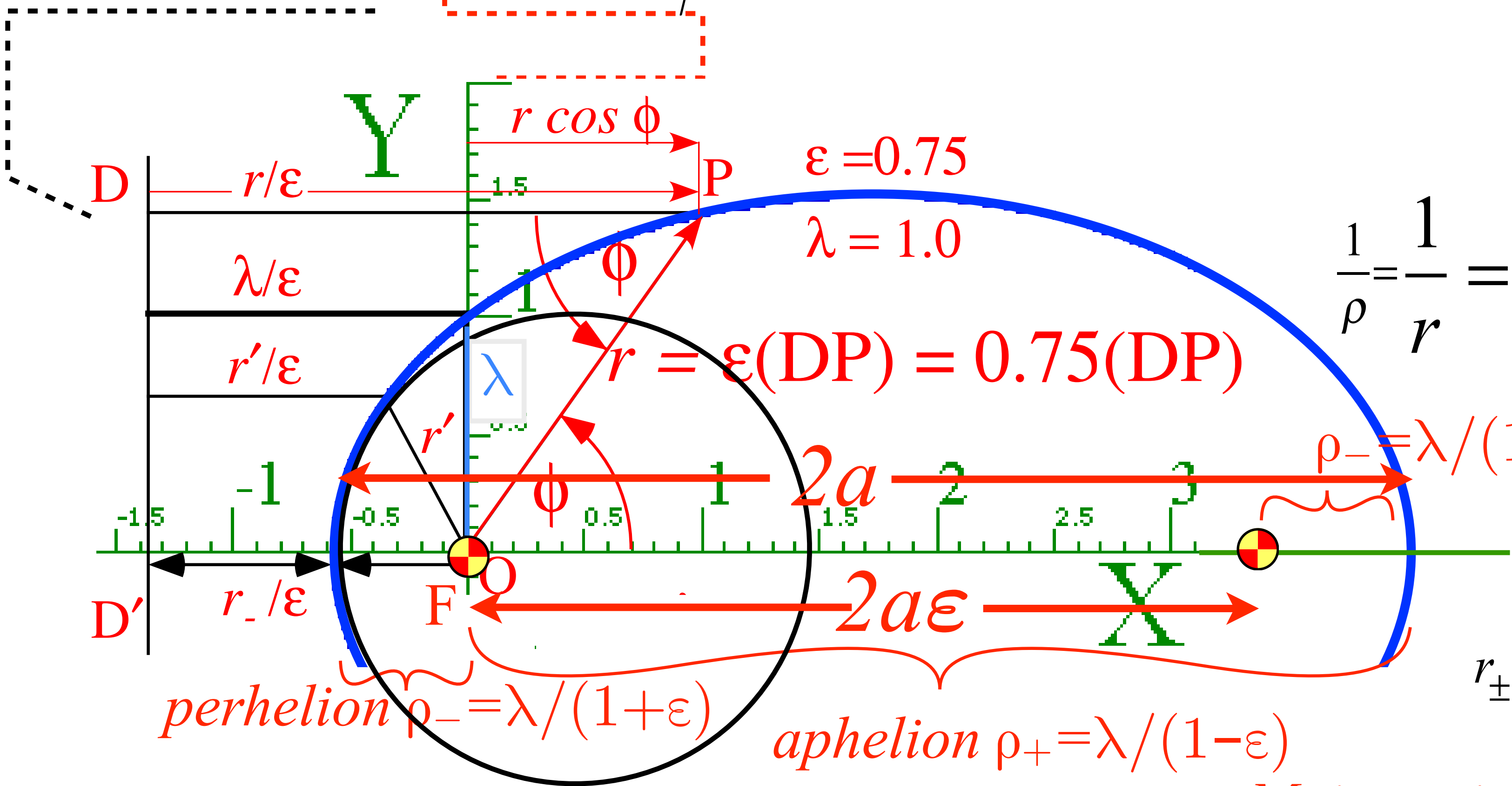
Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad E = \frac{-k}{2a}$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

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$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

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Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad E = \frac{-k}{2a}$$

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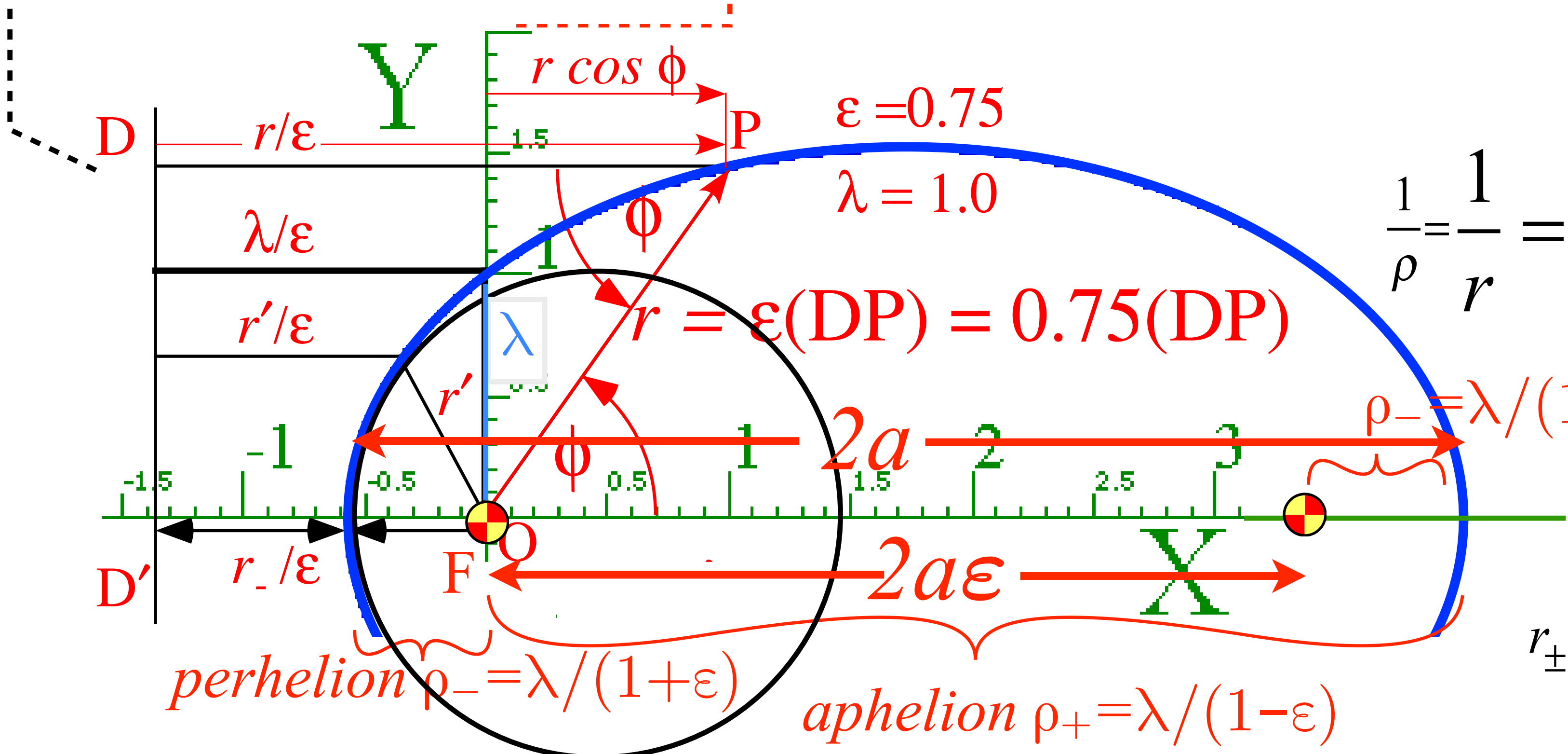

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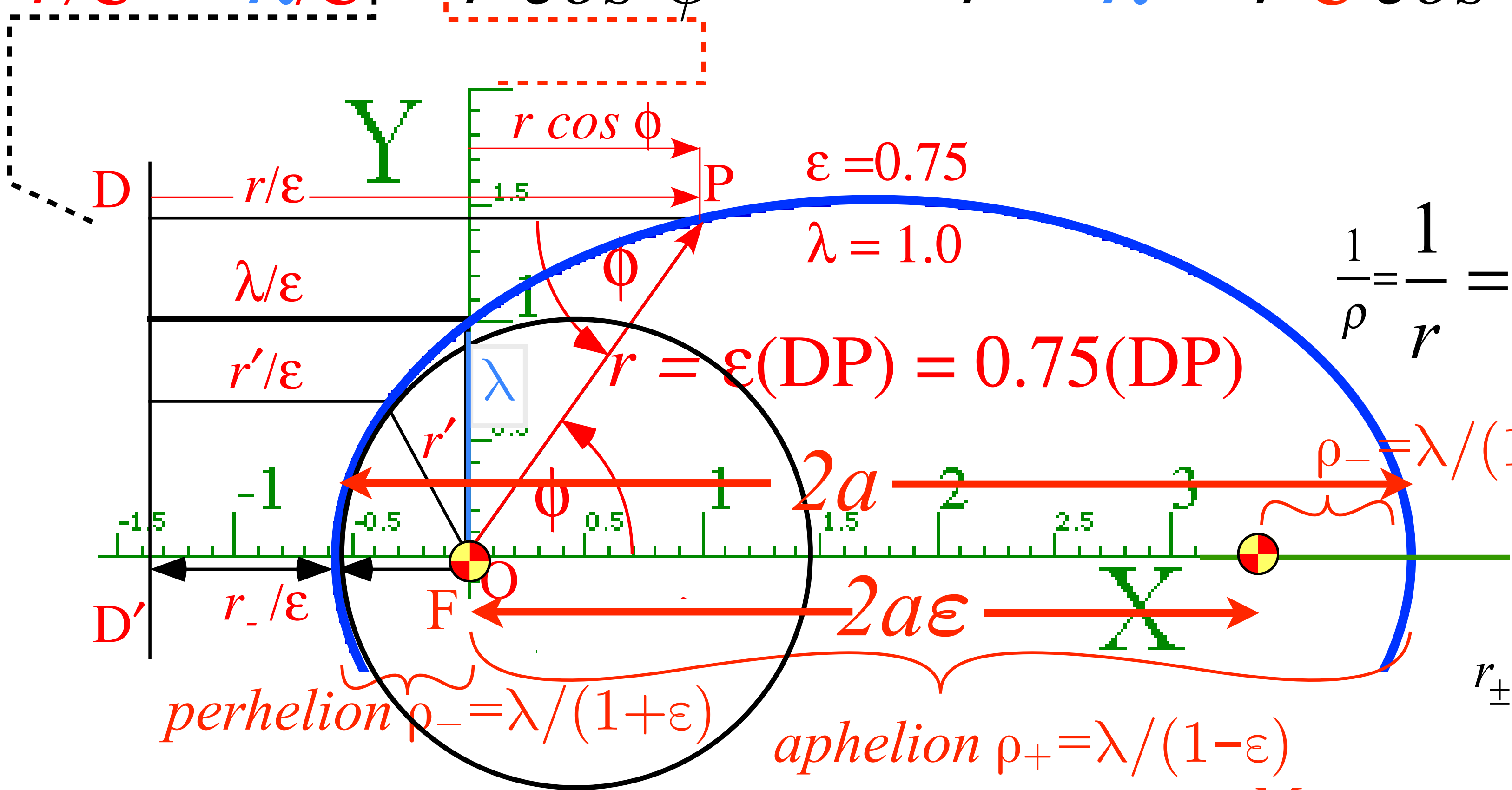

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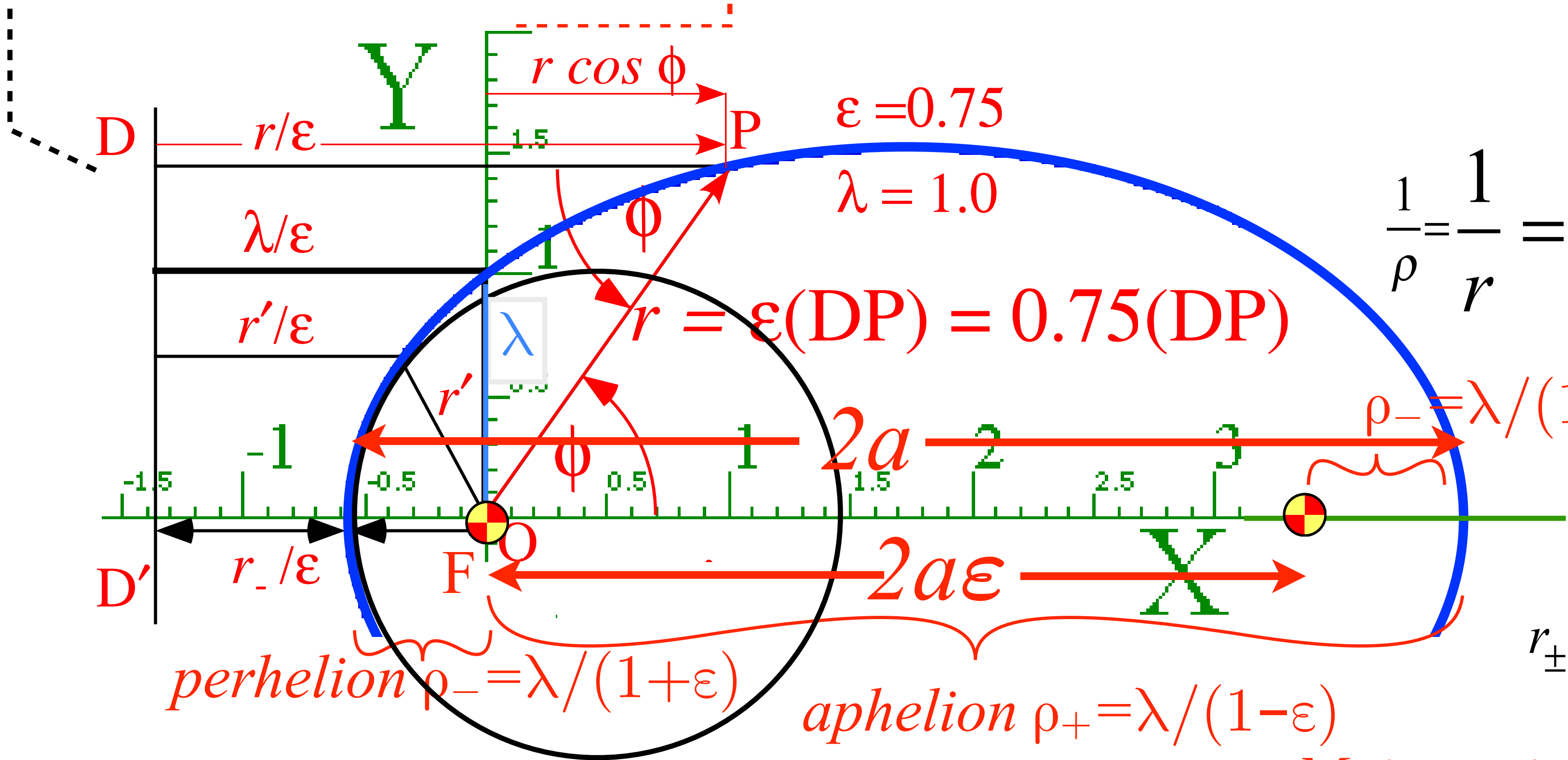

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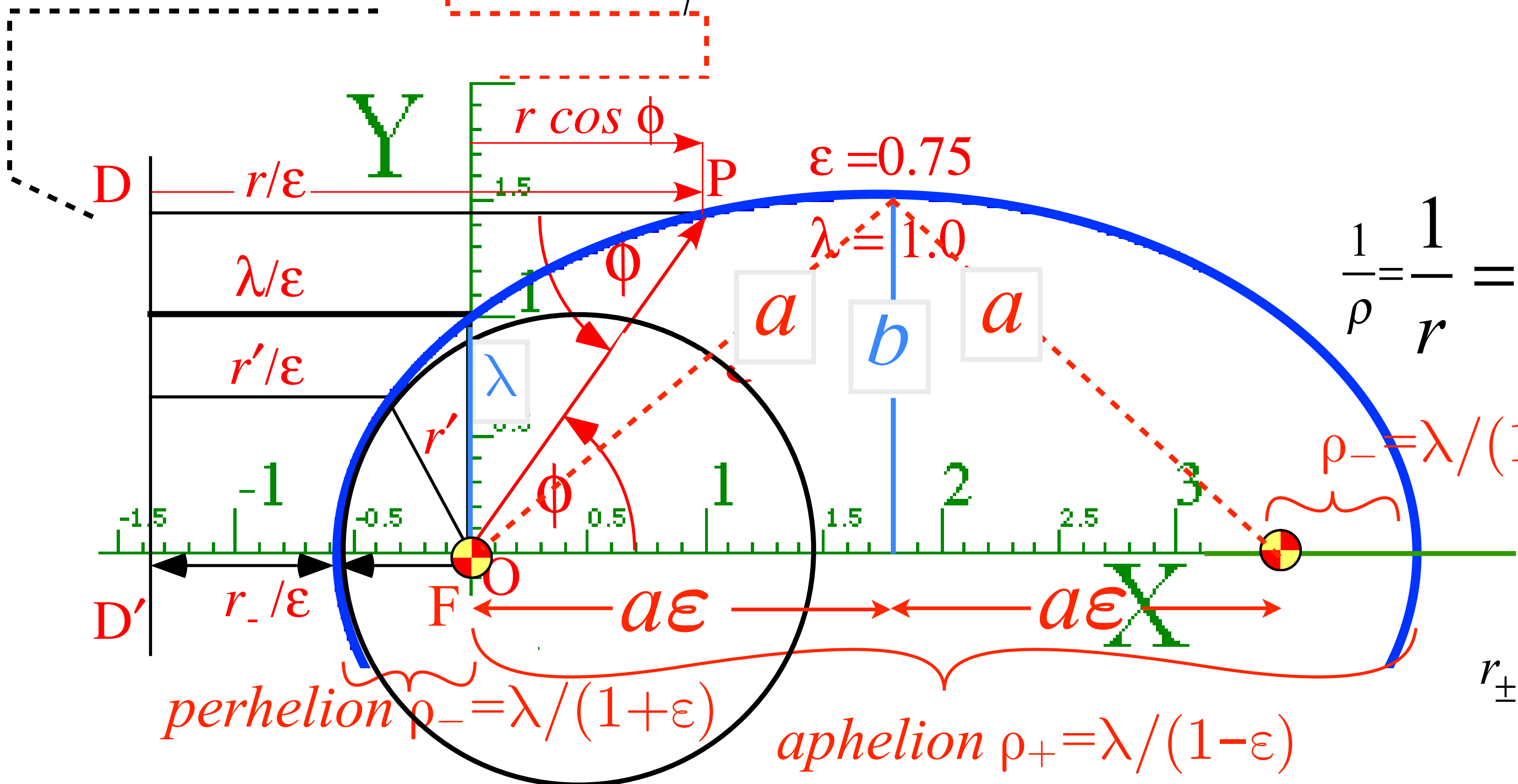

ρ₋=λ/(1+ε) *perhelion*

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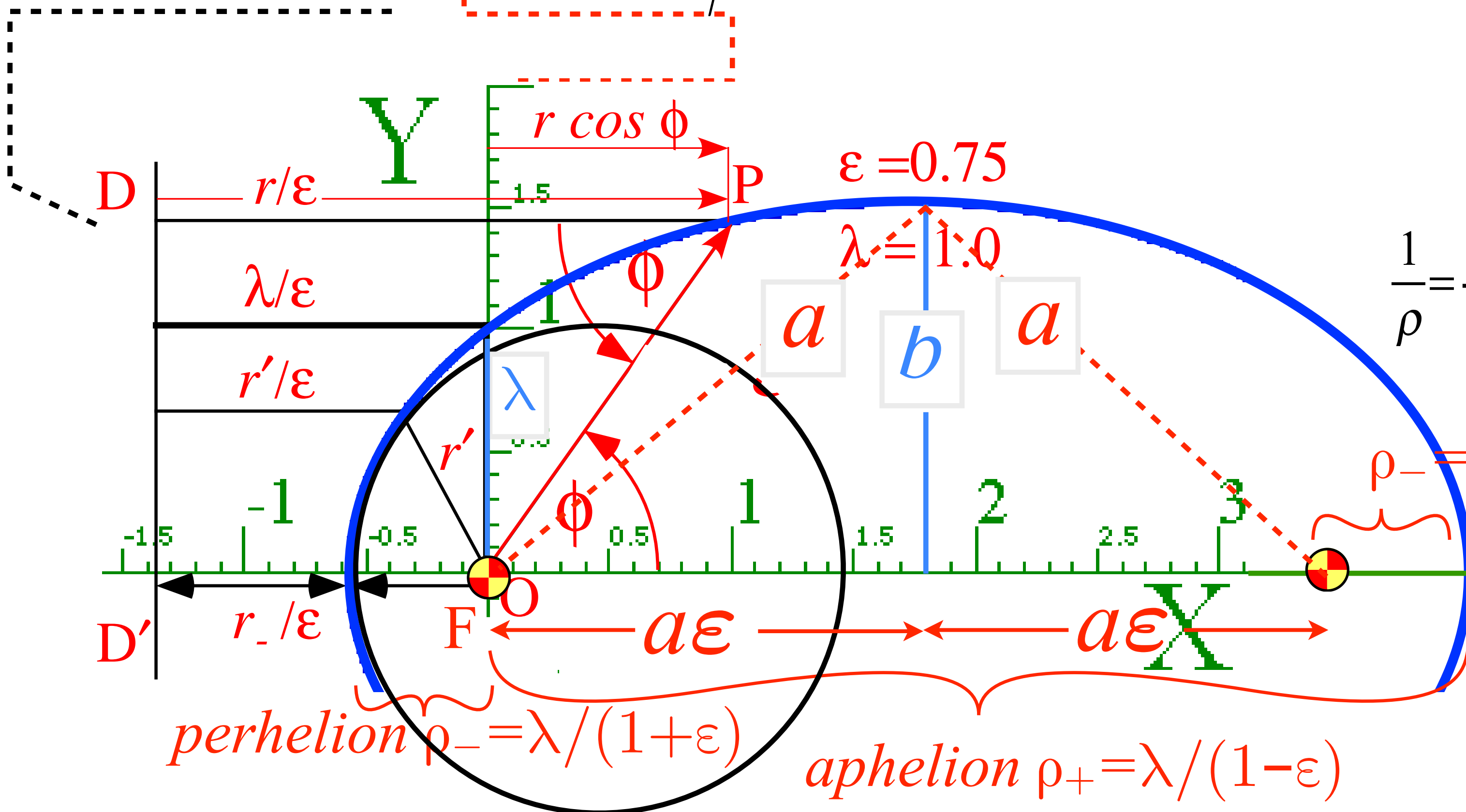

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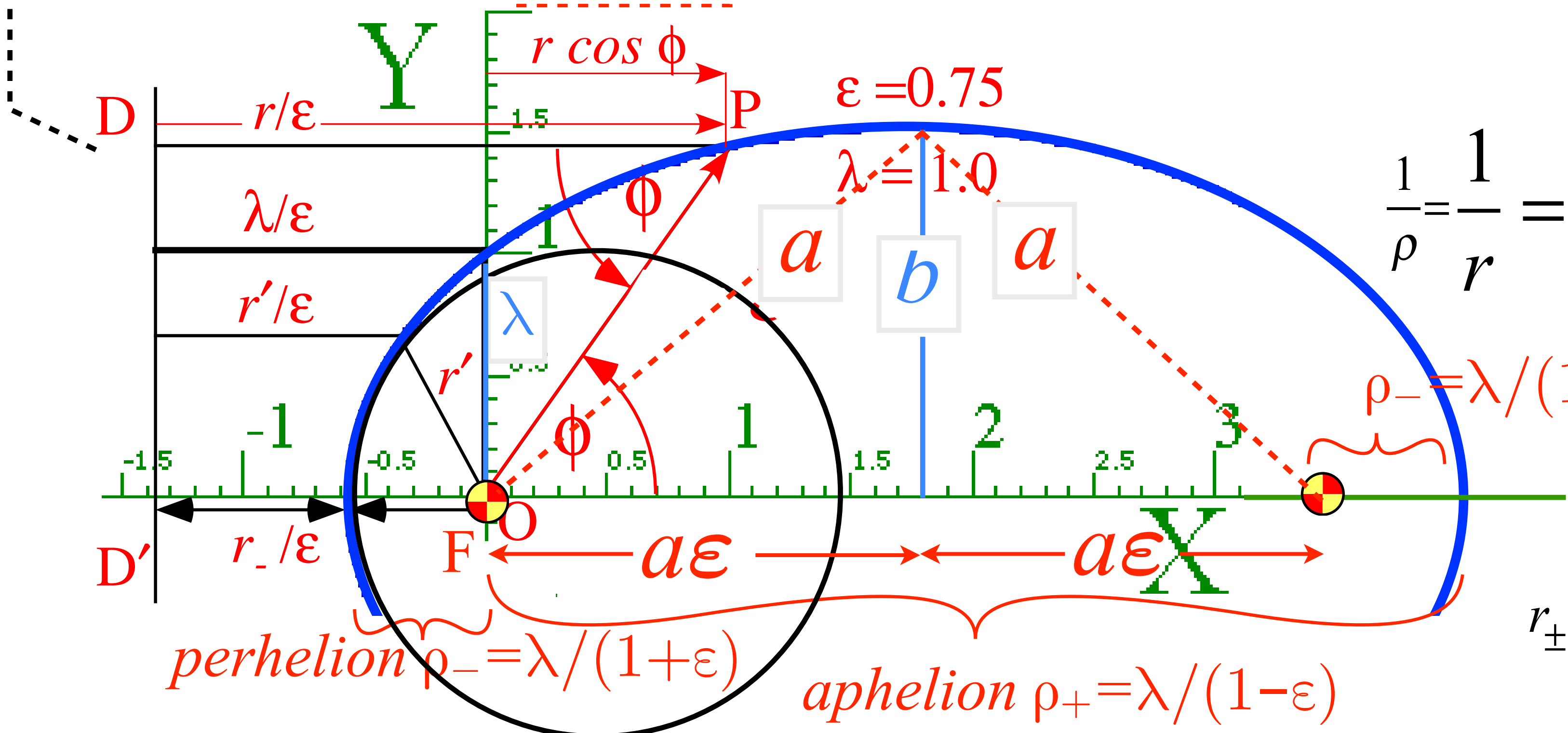

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(x,y) *parameters*	*physical parameters*	(r,ϕ) *parameters*
major radius $a = \dfrac{k}{2E}$	*Energy* $E = \dfrac{k}{2a}$	*eccentricity* $\varepsilon = \sqrt{\dfrac{k^2 m + 2L^2 E}{k^2 m}}$
minor radius $b = \dfrac{L}{\sqrt{2m\lvert E\rvert}}$	*∠-momentum* $L = \sqrt{km\lambda} \equiv \mu$	*latus radius* $\lambda = \dfrac{L^2}{km}$

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Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - ➔ *Kepler equation of time and phase geometry*

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

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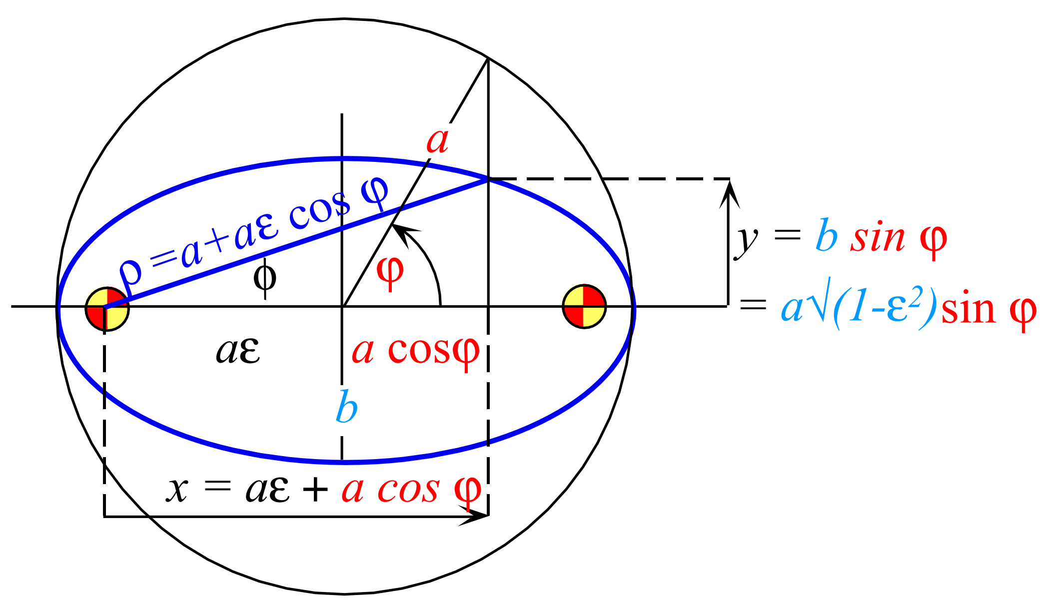
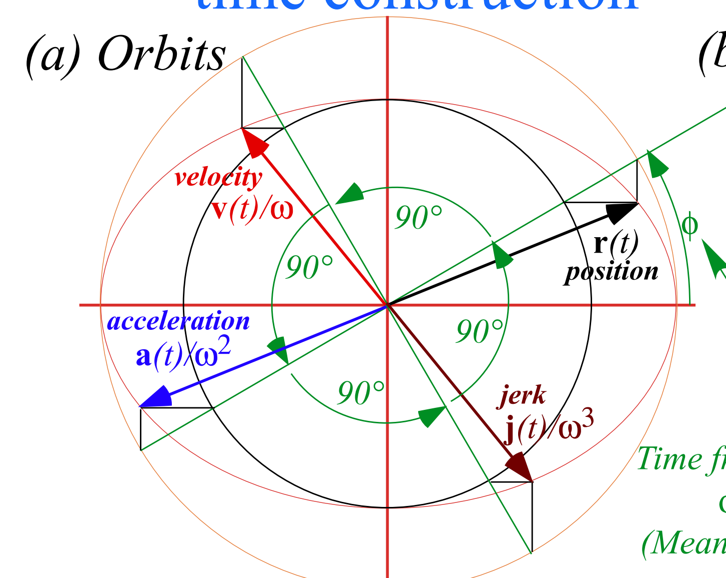


Unit 1 Ch. 9
Recall IHO orbit
time construction



[RelaWavity Web Simulation - Orbits]

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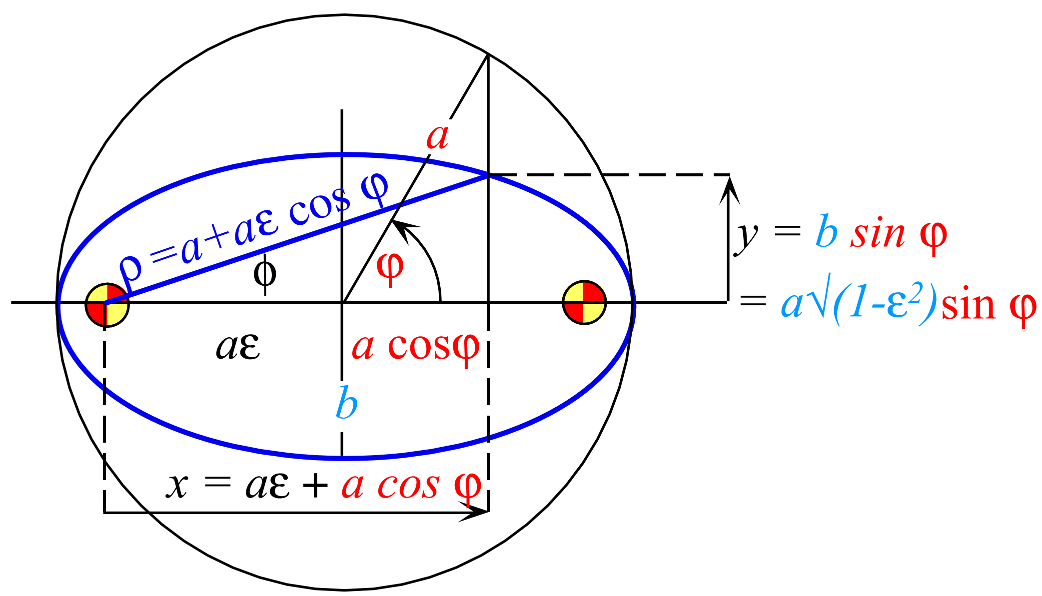

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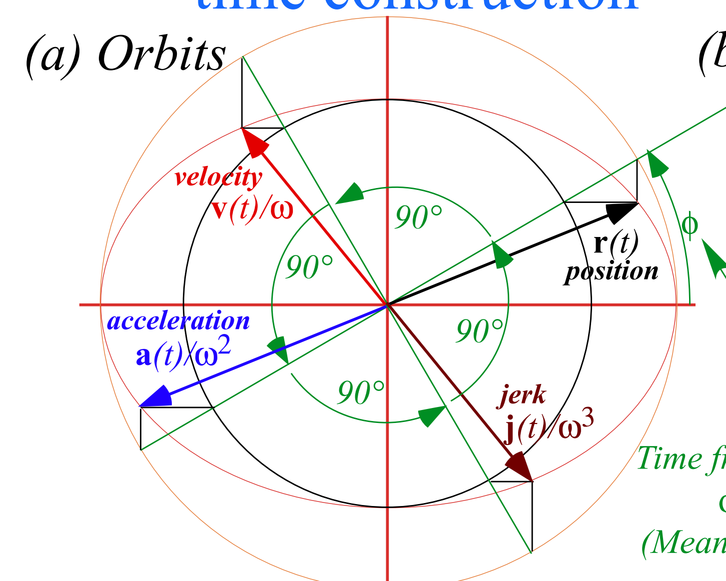

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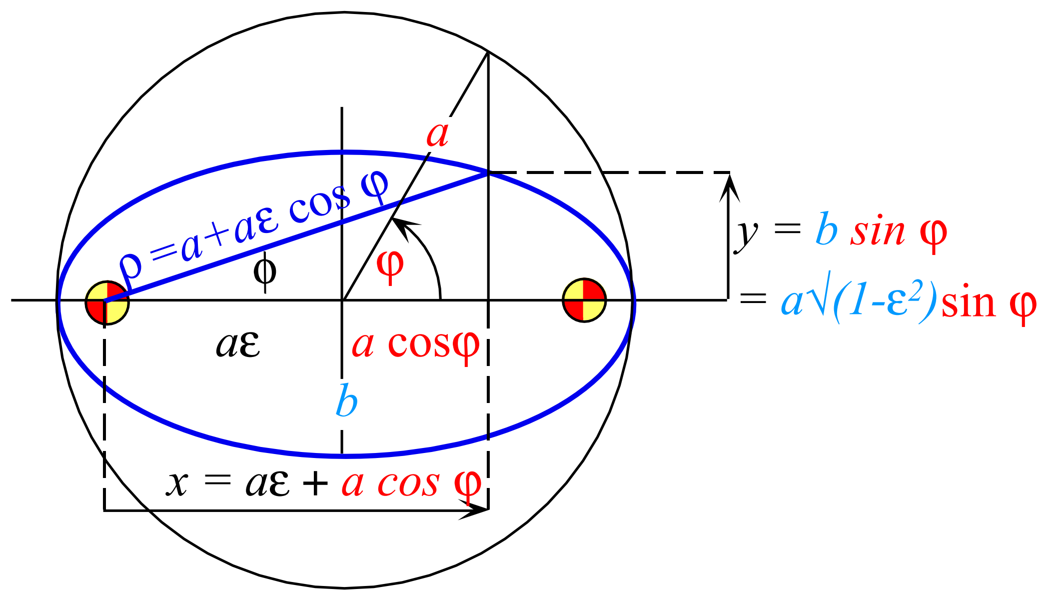

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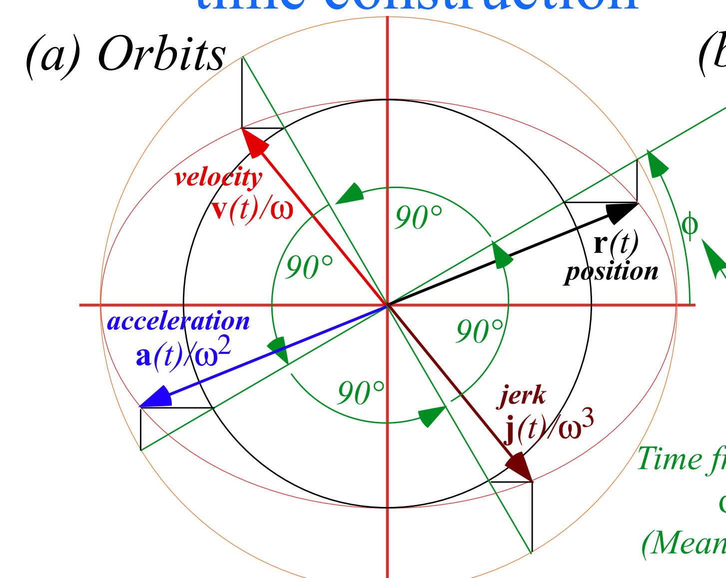

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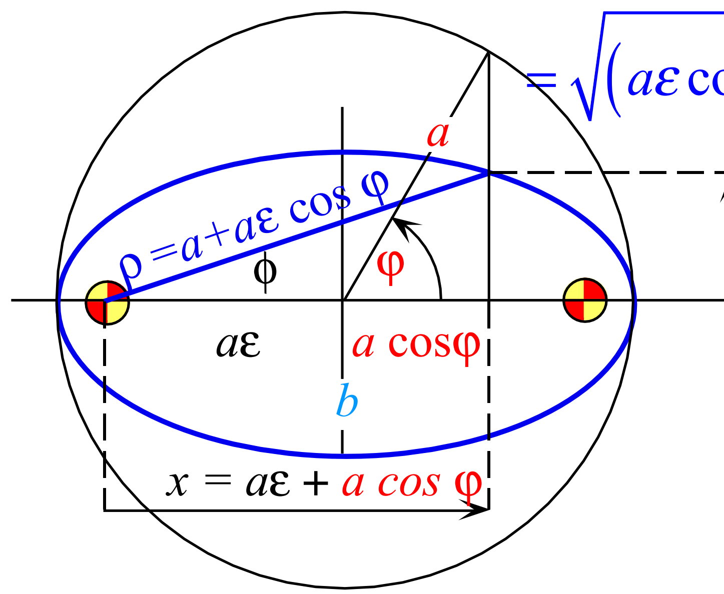
$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$



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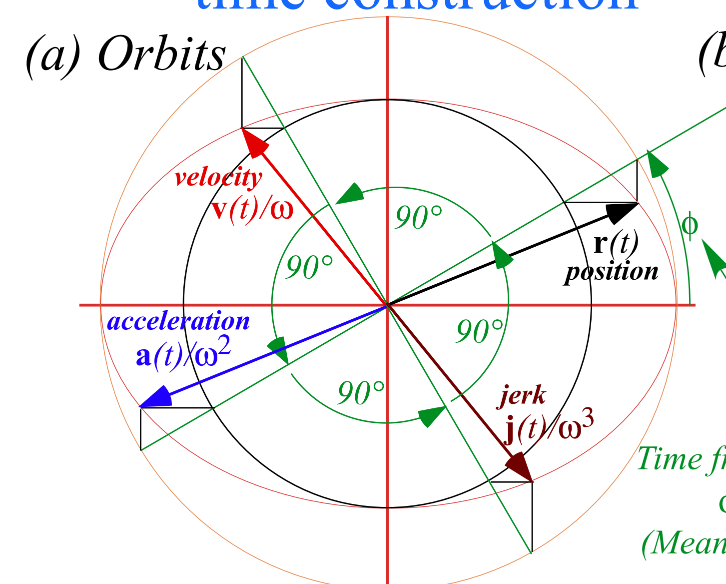

RelaWavity Web Simulation - Orbits

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

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$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$

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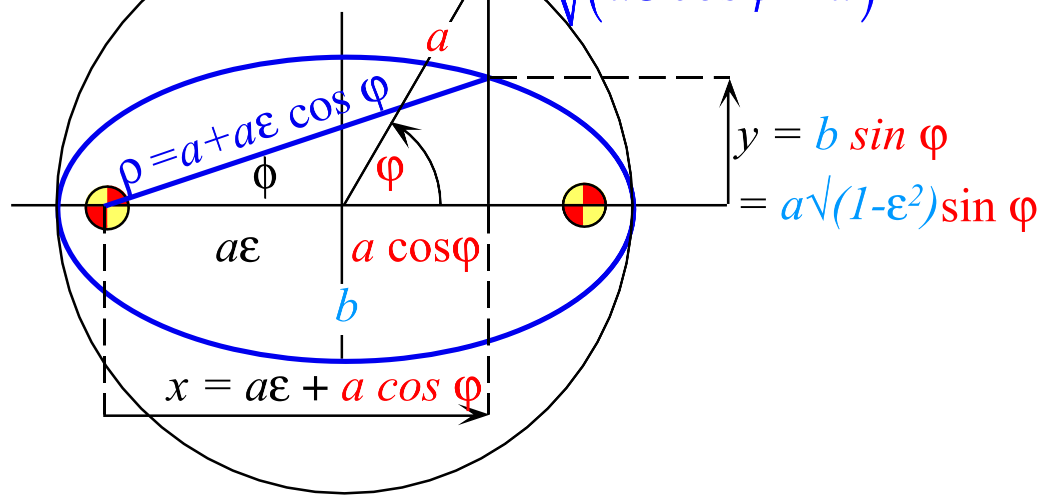
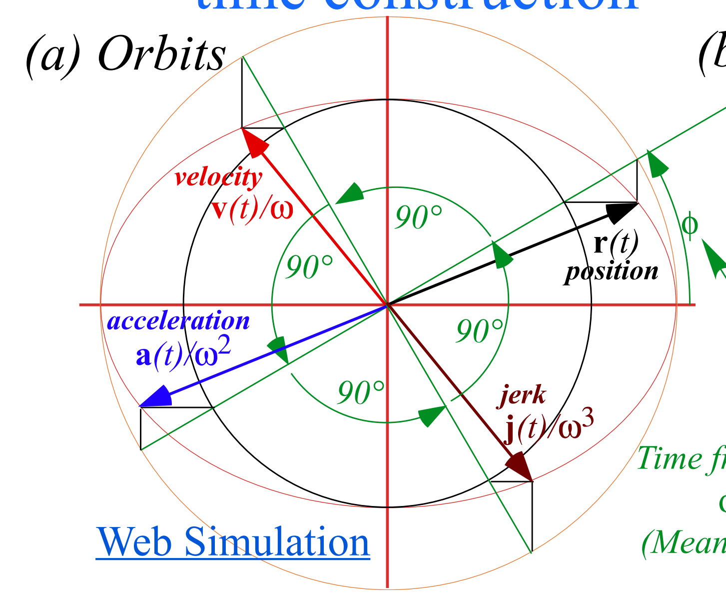


Unit 1 Ch. 9
Recall IHO orbit
time construction



[Web Simulation]

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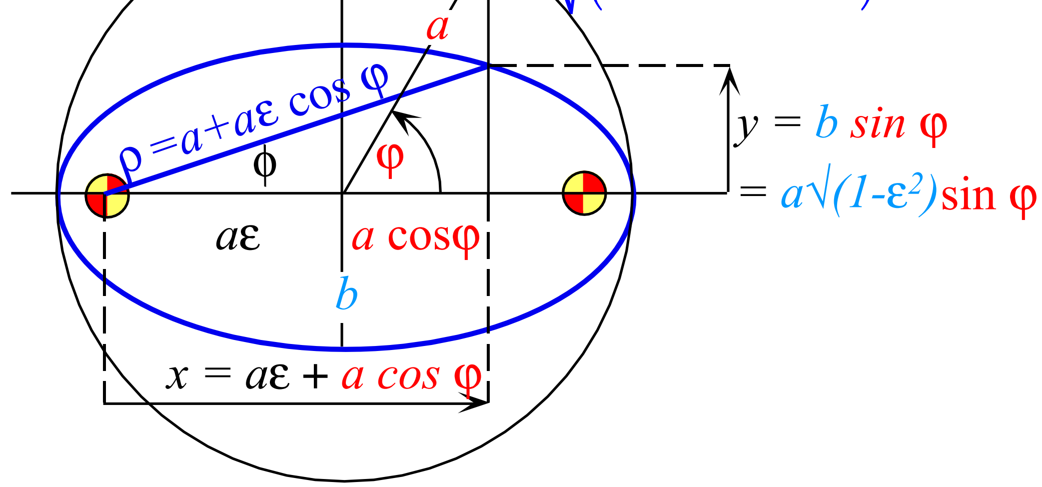
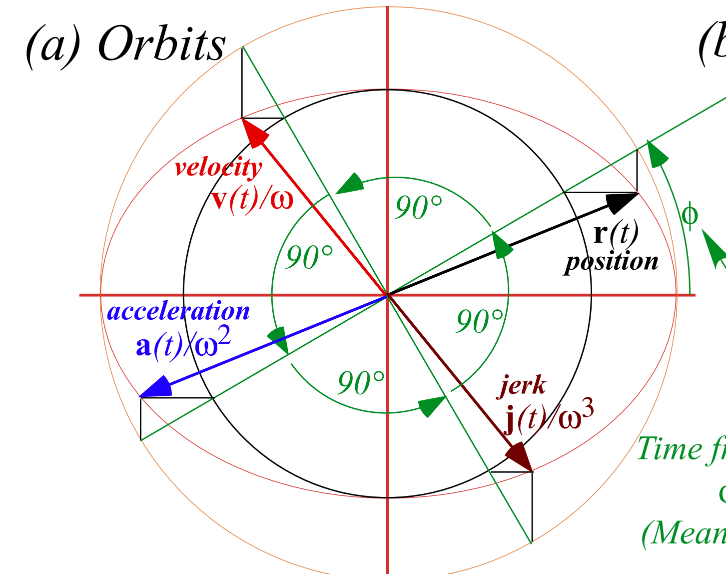


Unit 1 Ch. 9
Recall IHO orbit time construction



RelaWavity Web Simulation - Orbits

$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)\, a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}}$$

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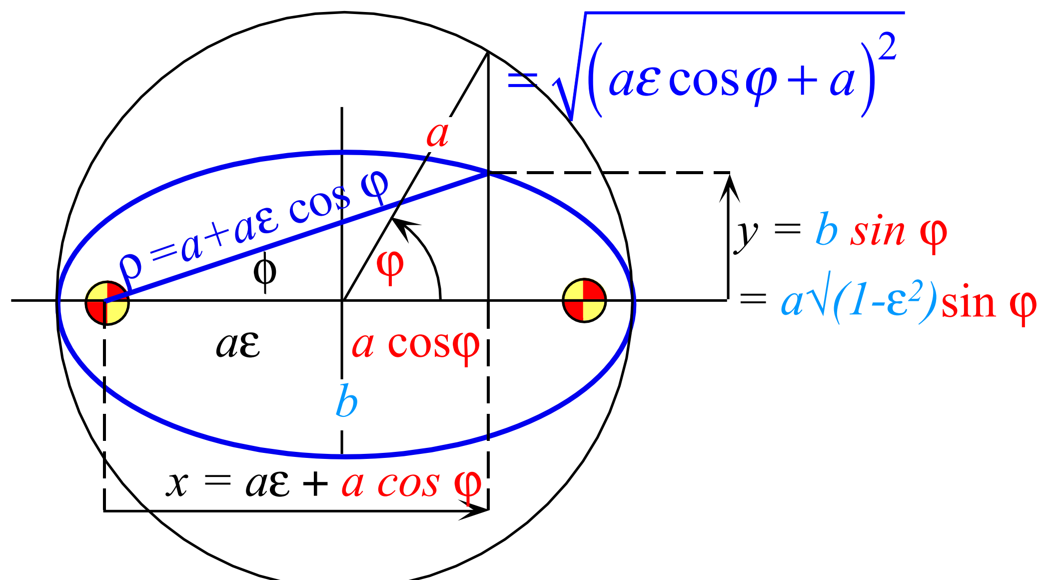

Unit 1 Ch. 9
Recall IHO orbit
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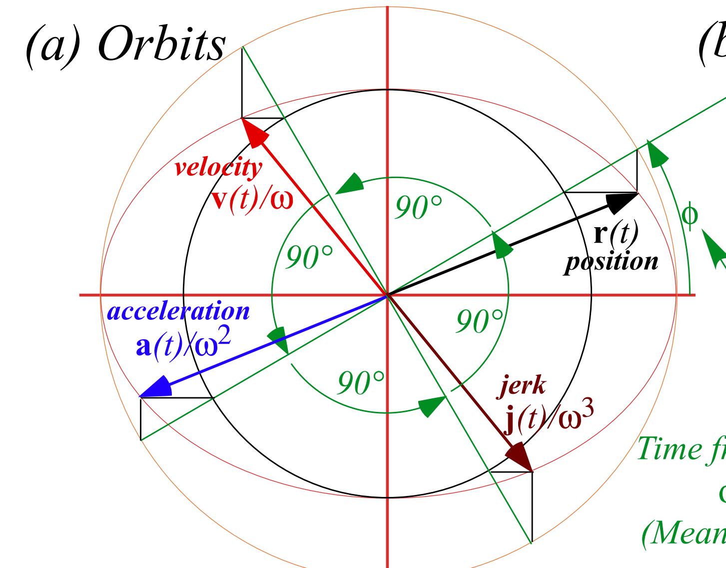

[RelaWavity Web Simulation - Orbits]

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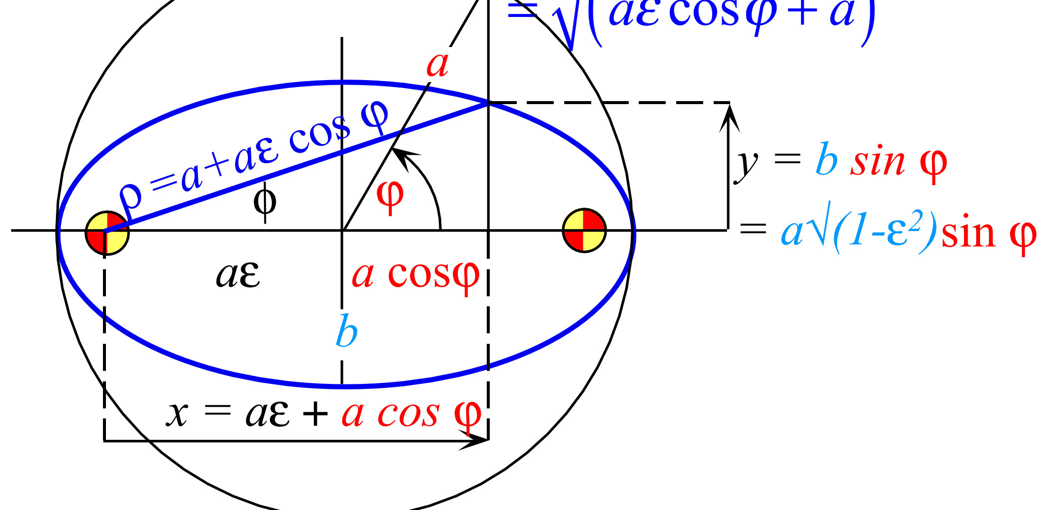
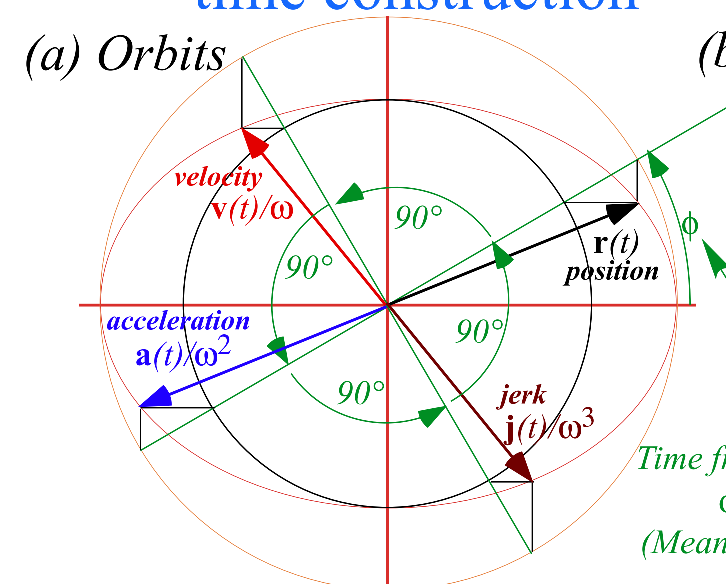


Unit 1 Ch. 9
Recall IHO orbit time construction



RelaWavity Web Simulation - Orbits

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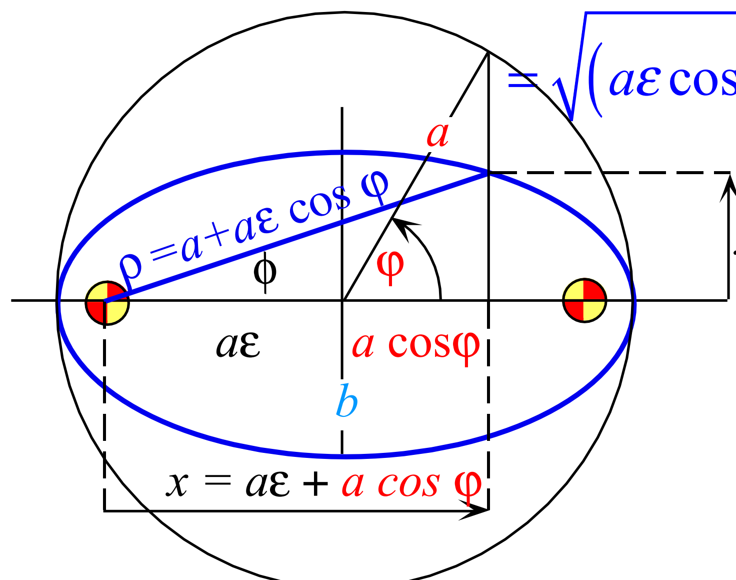

Unit 1 Ch. 9
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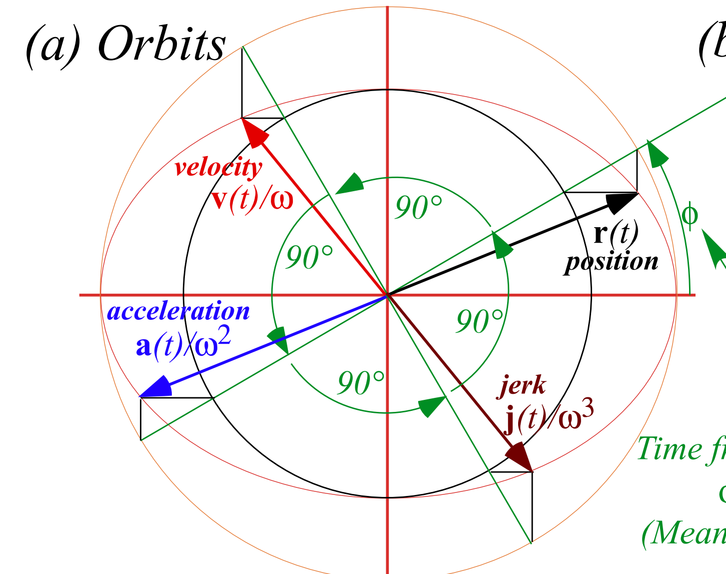

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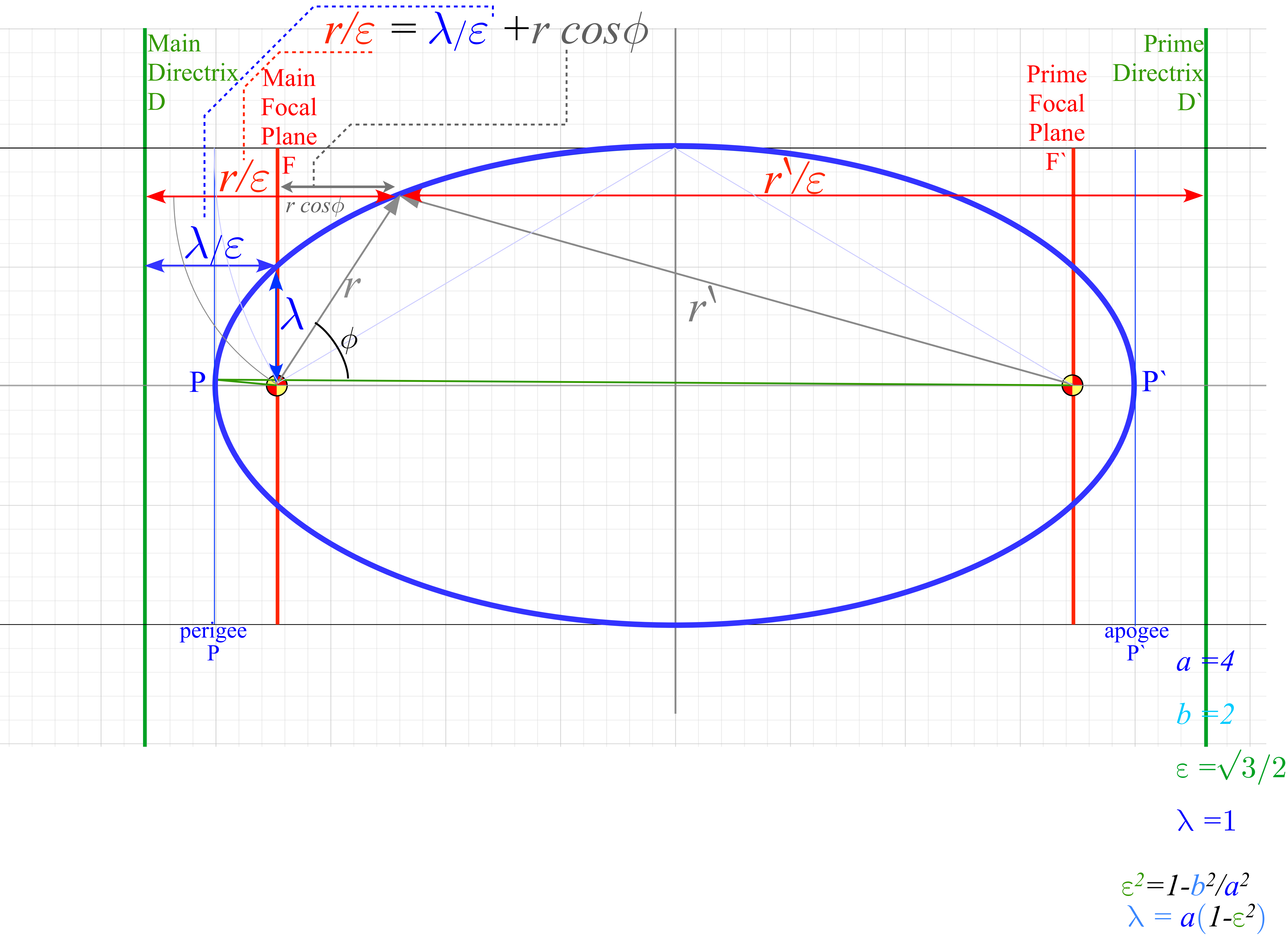
$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)\, d\varphi = \sqrt{\frac{ma^3}{k}} (\varphi + \varepsilon\sin\varphi)$$

Kepler's equations of orbital time

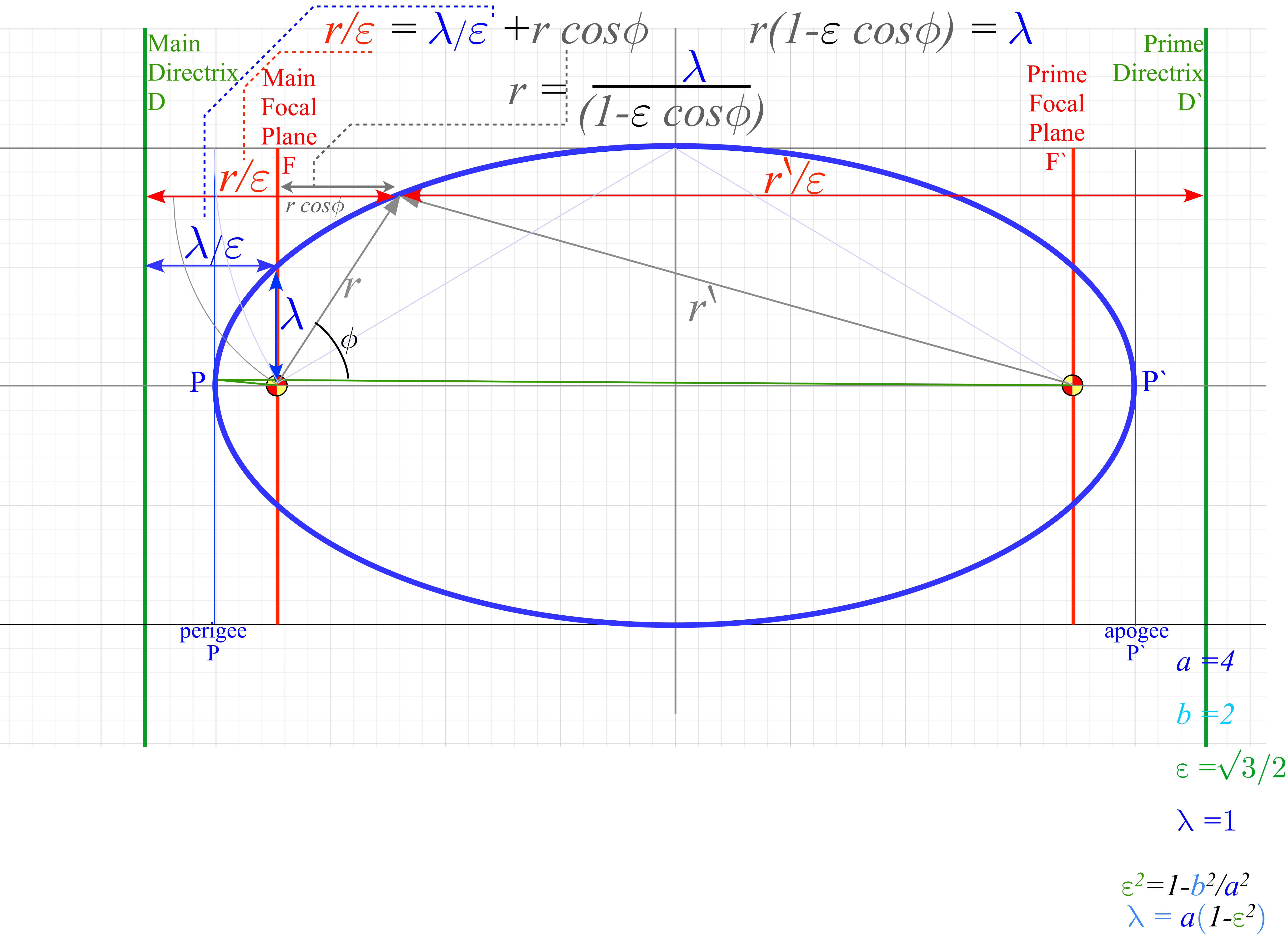
$$\textit{Orbit Period} = T = \frac{2\pi}{\omega_\varphi} = 2\pi\sqrt{\frac{ma^3}{k}}$$

Geometry and Symmetry of Coulomb orbits

- ➔ *Detailed elliptic geometry*
- *Detailed hyperbolic geometry*



r/ε = λ/ε + r cosϕ
Main Directrix D
Main Focal Plane
F
r/ε
r cosϕ
λ/ε
λ
r
ϕ
r`
r`/ε
Prime Focal Plane F`
Prime Directrix D`
P
P`
perigee P
apogee P`
a = 4
b = 2
ε = √3/2
λ = 1
ε² = 1 - b²/a²
λ = a(1 - ε²)


r/ε = λ/ε +r cosϕ
r(1-ε cosϕ) = λ
r = λ / (1-ε cosϕ)
Main
Directrix
D
Main
Focal
Plane
F
Prime
Focal
Plane
F`
Prime
Directrix
D`
r/ε
r`/ε
r cosϕ
λ/ε
λ
r
r`
ϕ
P
P`
perigee
P
apogee
P`
a =4
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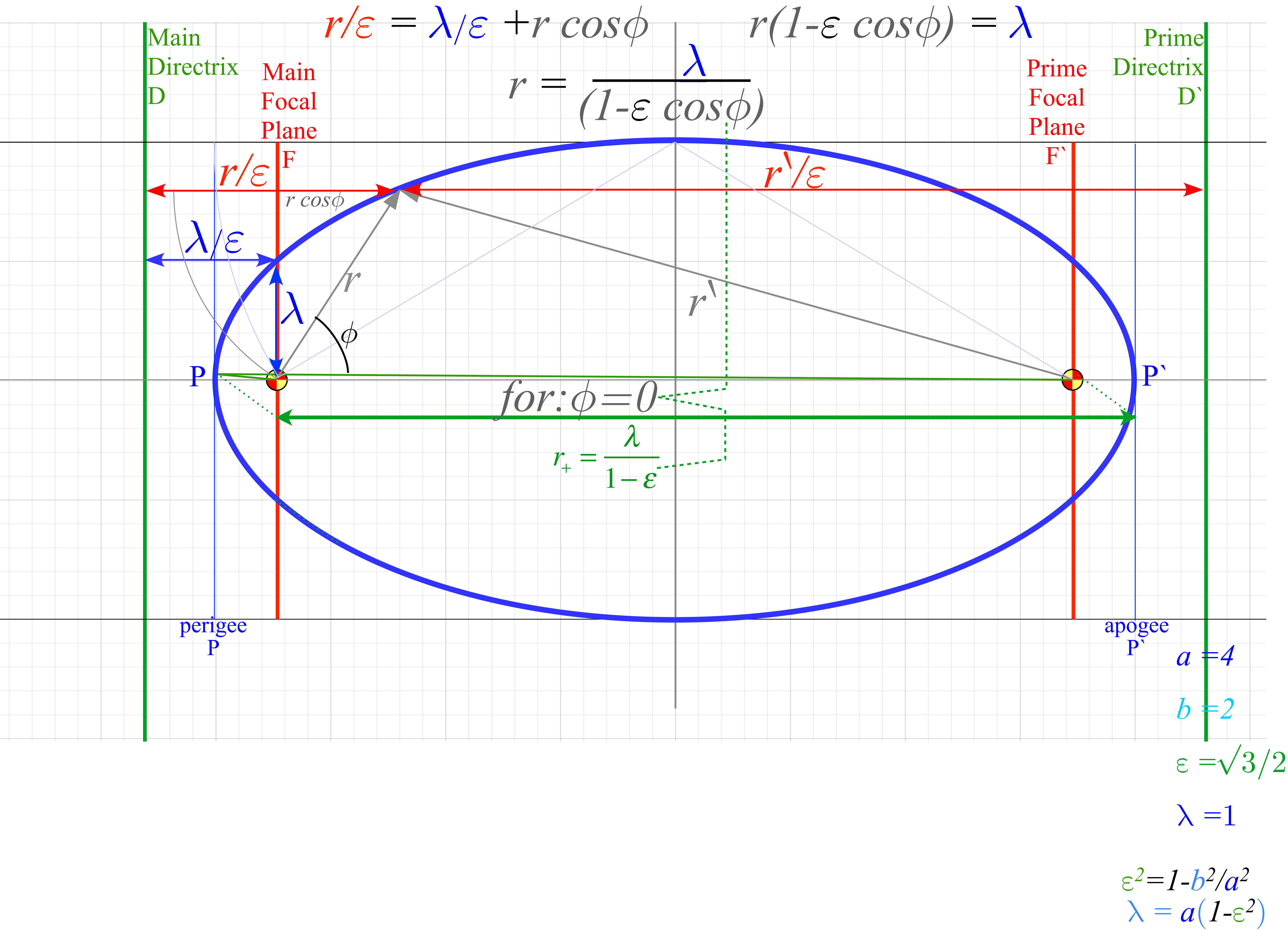
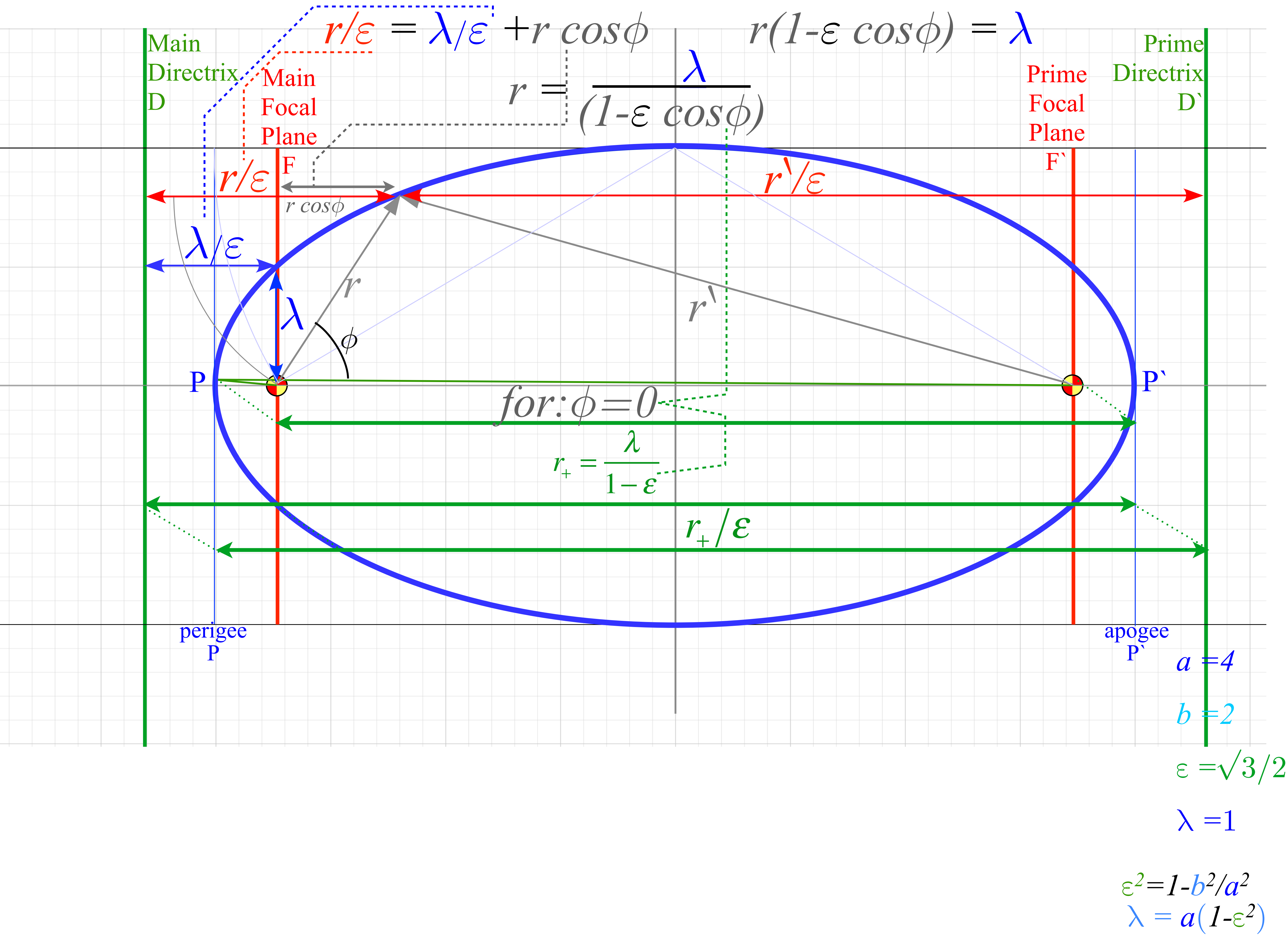


r/ε = λ/ε +r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main
Directrix
D
Main
Focal
Plane
F
Prime
Focal
Plane
F`
Prime
Directrix
D`
r/ε
r cosφ
r`/ε
λ/ε
λ
r
φ
r`
P
P`
for:φ=0
r+ = λ / 1−ε
perigee
P
apogee
P`
a =4
b =2
ε =√3/2
λ =1
ε2=1-b2/a2
λ = a(1-ε2)



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r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r`/ε
r cosφ
λ/ε
λ
r
r`
φ
P
P`
for:φ=0
r+ = λ / (1−ε)
r+/ε
perigee P
apogee P`
a =4
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λ =1
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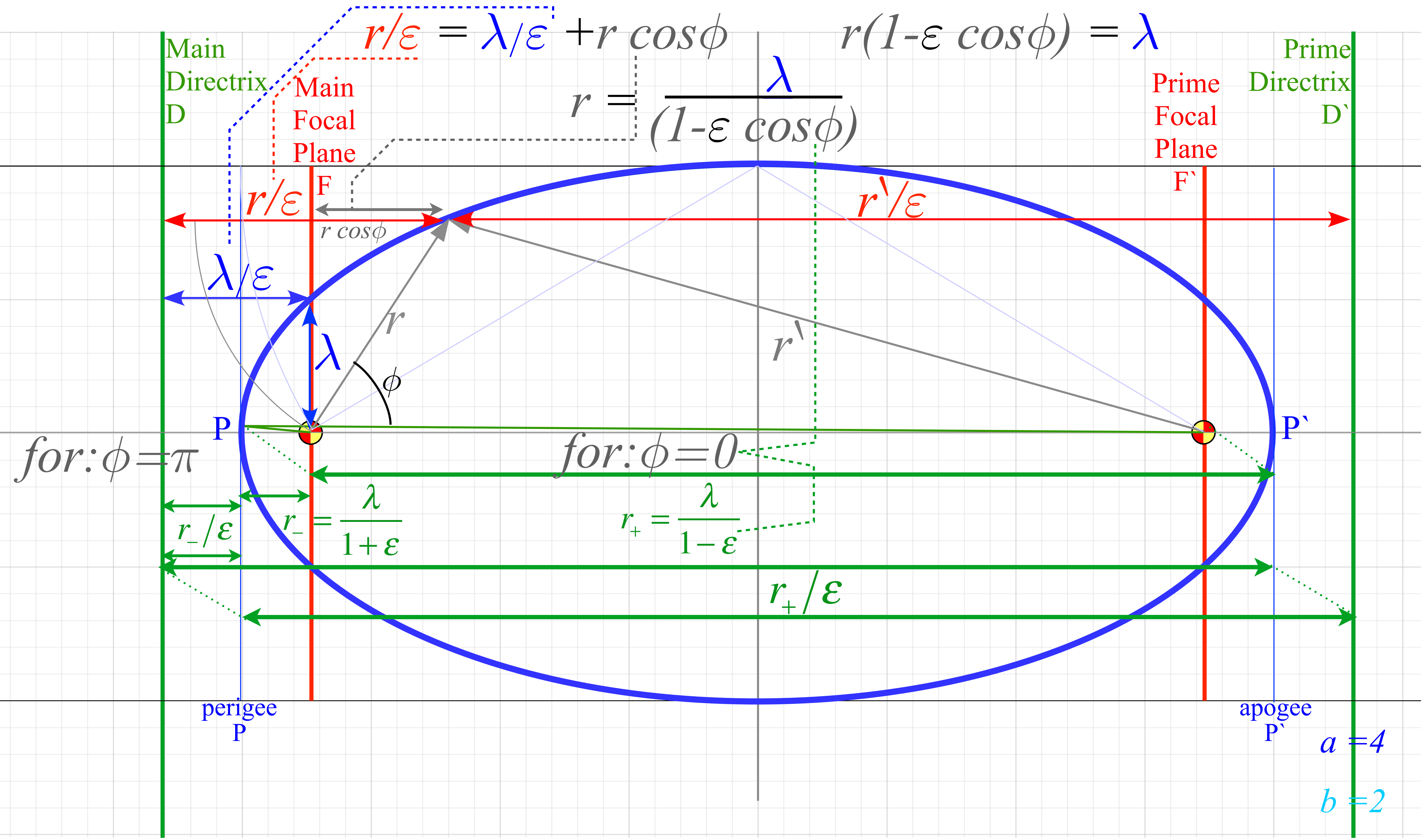
λ = a(1-ε2)

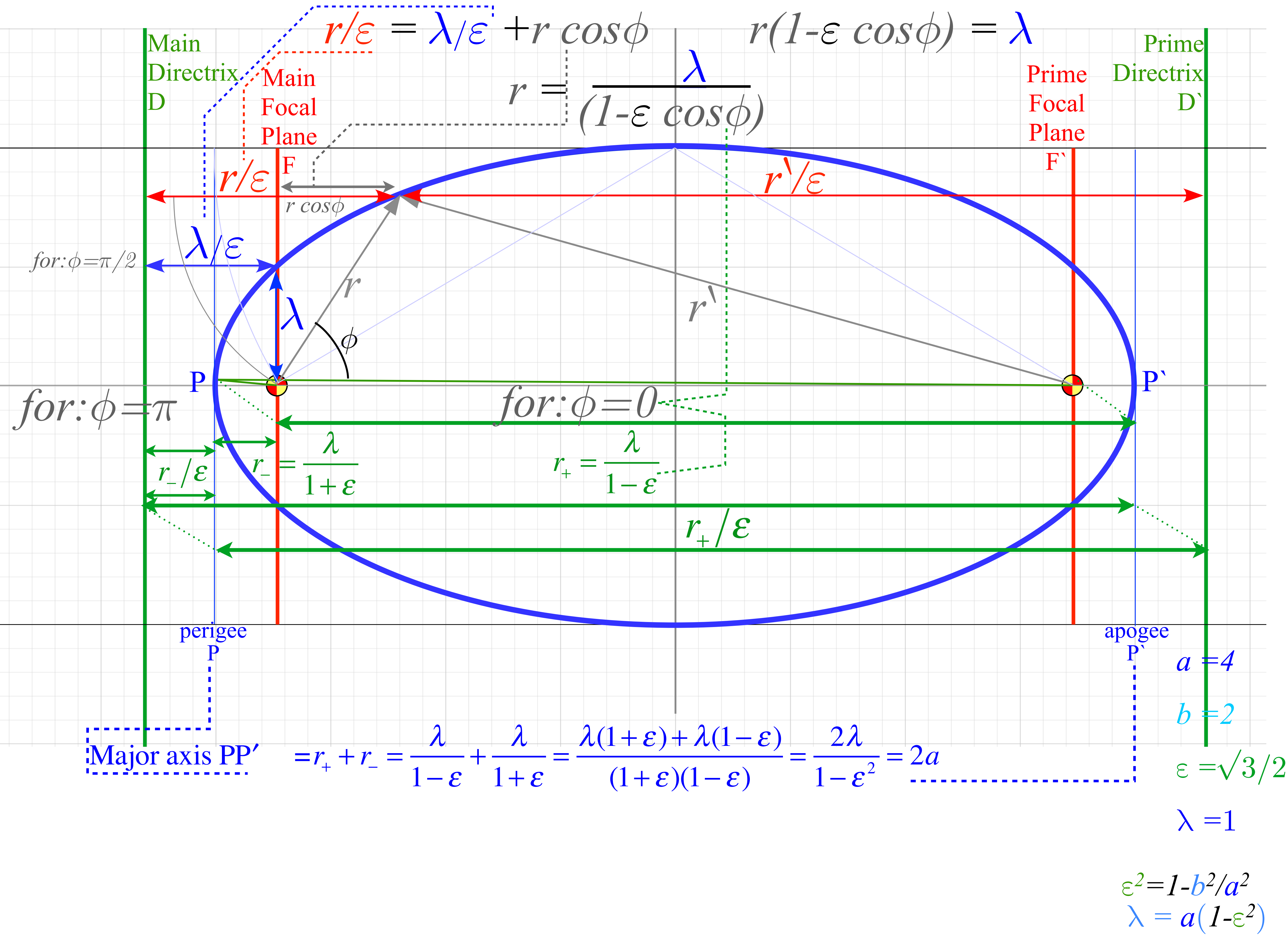


$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$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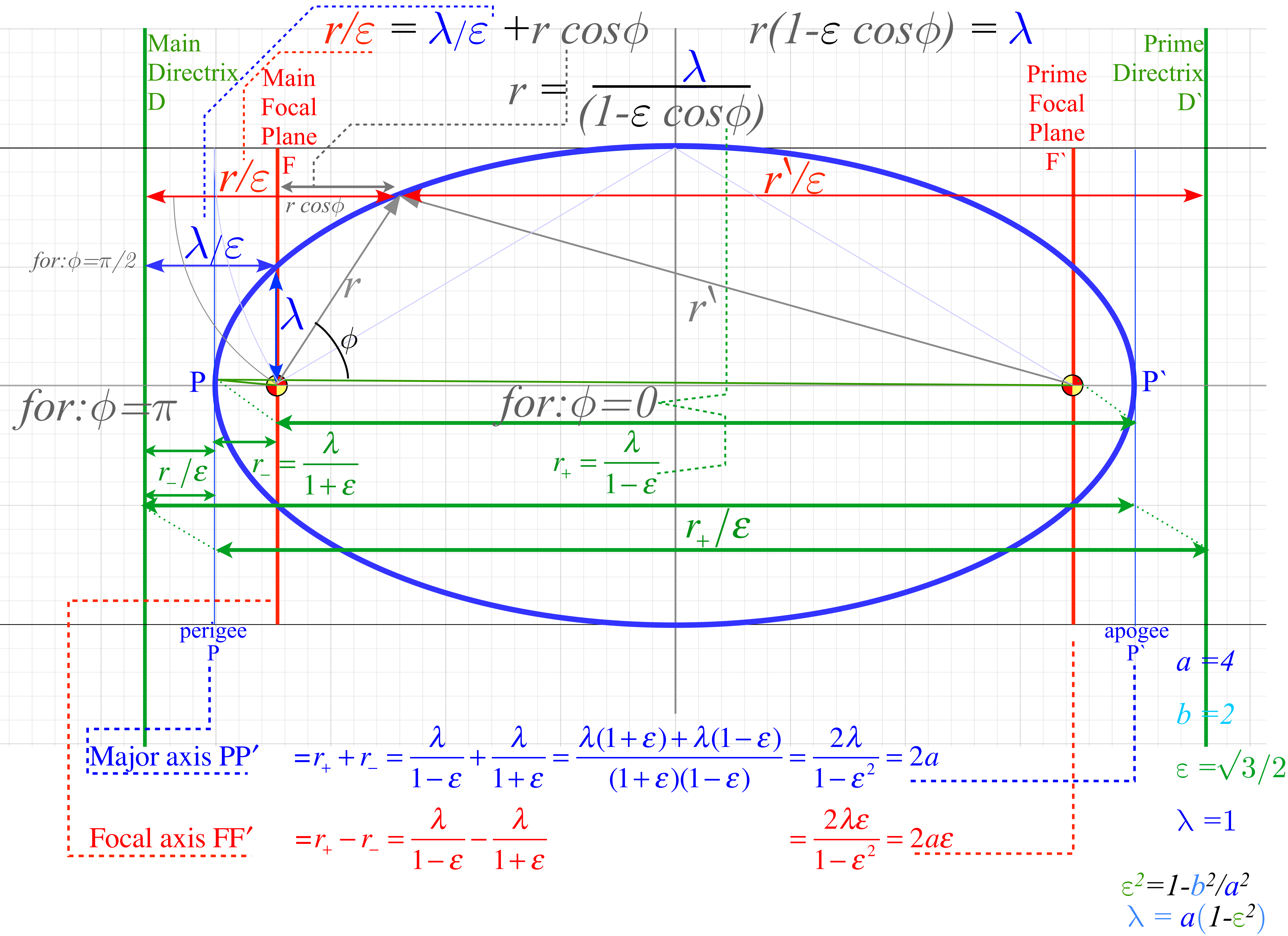
Main Focal Plane
Prime Focal Plane
Prime Directrix D`
F
F`
r/ε
r`/ε
r cosφ
λ/ε
for:φ=π/2
λ
r
r`
φ
P
P`
for:φ=π
for:φ=0
r_- = λ/(1+ε)
r_+ = λ/(1−ε)
r_-/ε
r_+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
Major axis PP′ = r_+ + r_- = λ/(1−ε) + λ/(1+ε) = (λ(1+ε)+λ(1−ε))/((1+ε)(1−ε)) = 2λ/(1−ε²) = 2a
ε²=1-b²/a²
λ = a(1-ε²)



r/ε = λ/ε + r cosϕ
r(1-ε cosϕ) = λ
r = λ / (1-ε cosϕ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r cosϕ
r`/ε
λ/ε
for:ϕ=π/2
λ
r
ϕ
r`
P
P`
for:ϕ=π
for:ϕ=0
r_/ε
r_ = λ/(1+ε)
r+ = λ/(1−ε)
r+/ε
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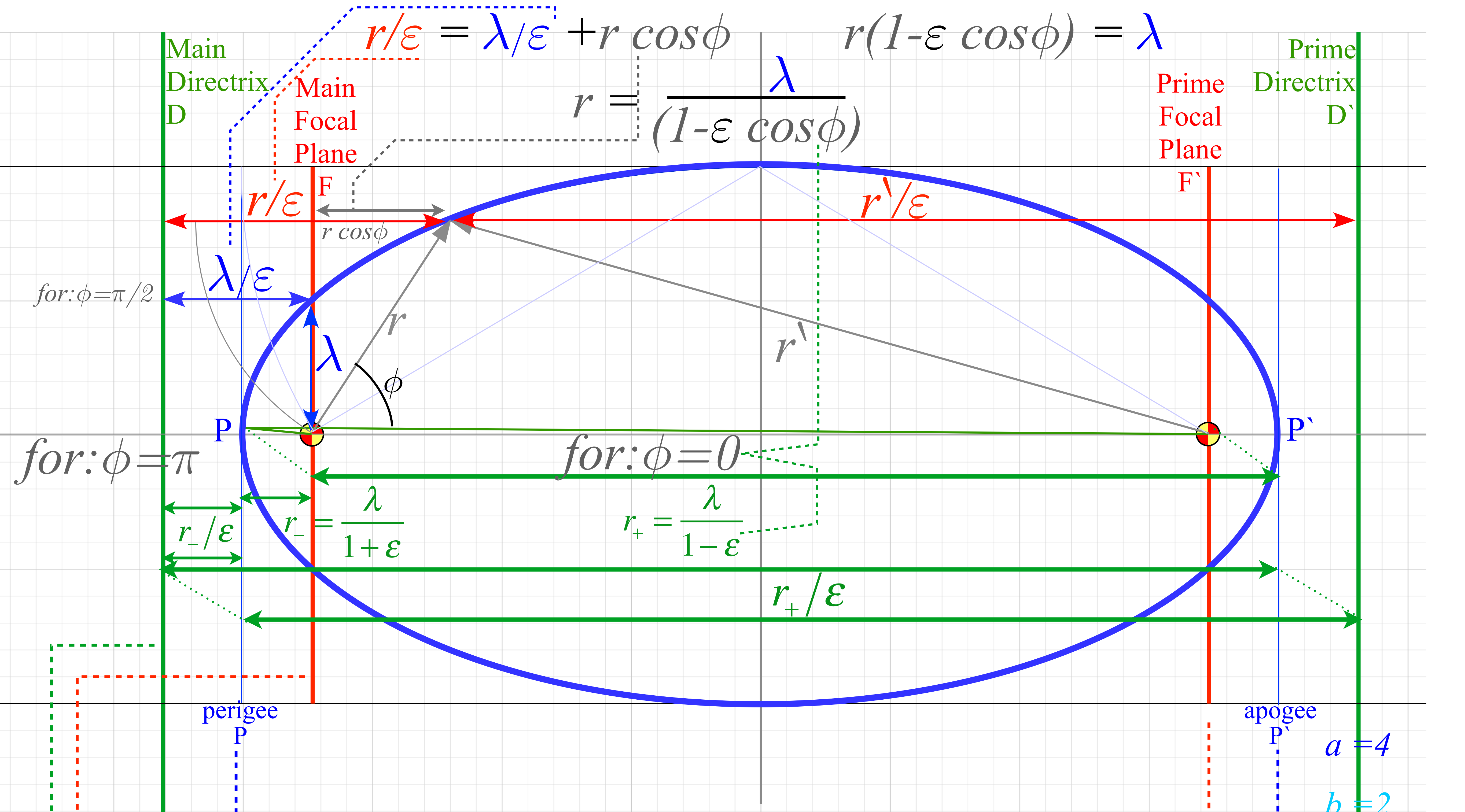
Major axis PP′ = r+ + r_ = λ/(1−ε) + λ/(1+ε) = (λ(1+ε)+λ(1−ε))/((1+ε)(1−ε)) = 2λ/(1−ε²) = 2a
Focal axis FF′ = r+ − r_ = λ/(1−ε) − λ/(1+ε) = 2λε/(1−ε²) = 2aε



$$\text{Major axis } PP' = r_+ + r_- = \frac{\lambda}{1-\varepsilon} + \frac{\lambda}{1+\varepsilon} = \frac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \frac{2\lambda}{1-\varepsilon^2} = 2a$$

$\varepsilon = \sqrt{3}/2$

$$\text{Focal axis } FF' = r_+ - r_- = \frac{\lambda}{1-\varepsilon} - \frac{\lambda}{1+\varepsilon} = \frac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$$

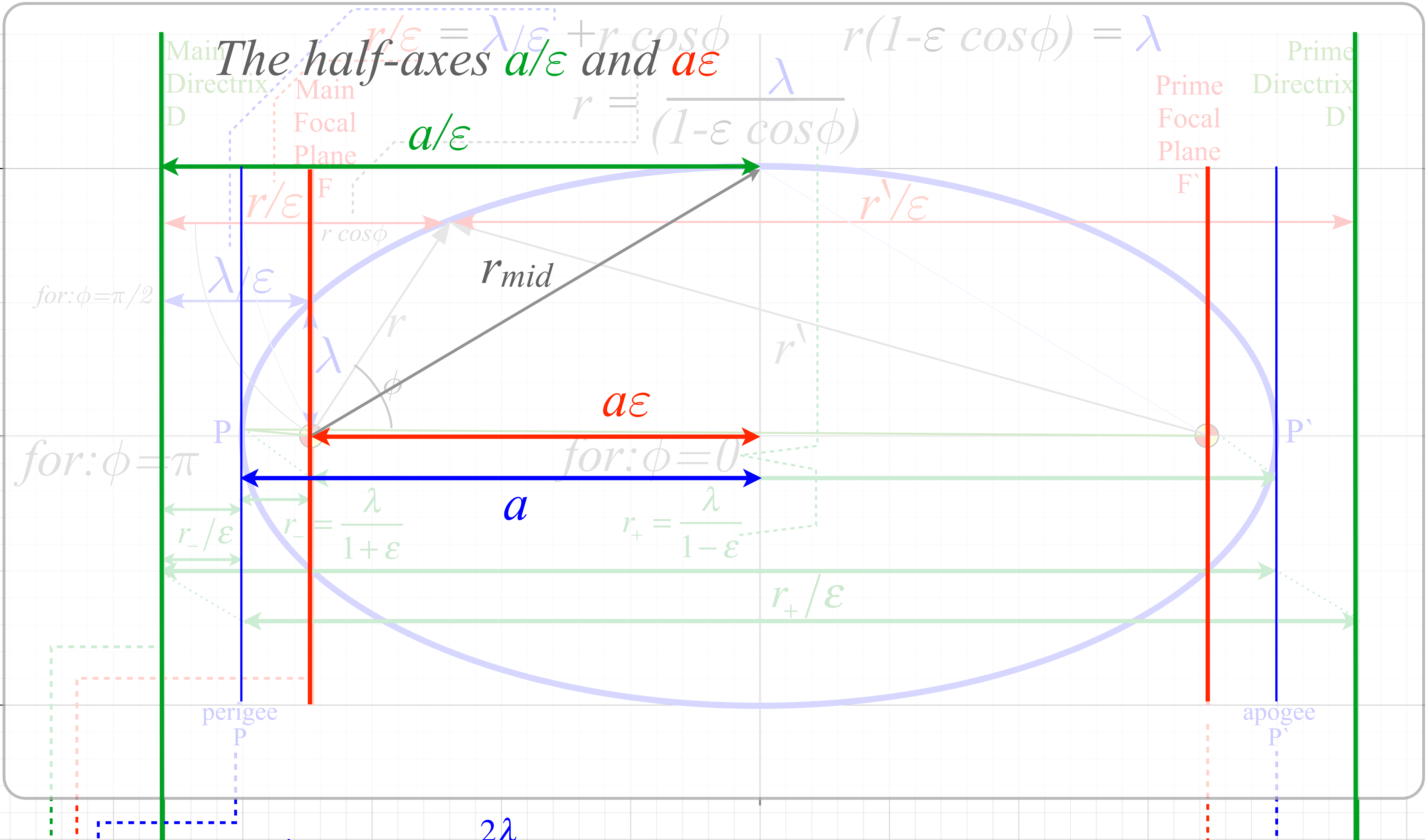
$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$$\text{Directrix axis } DD' = \frac{r_+}{\varepsilon} + \frac{r_-}{\varepsilon} = \frac{2a}{\varepsilon}$$

$\lambda = a(1-\varepsilon^2)$

The half-axes a/ε and $a\varepsilon$

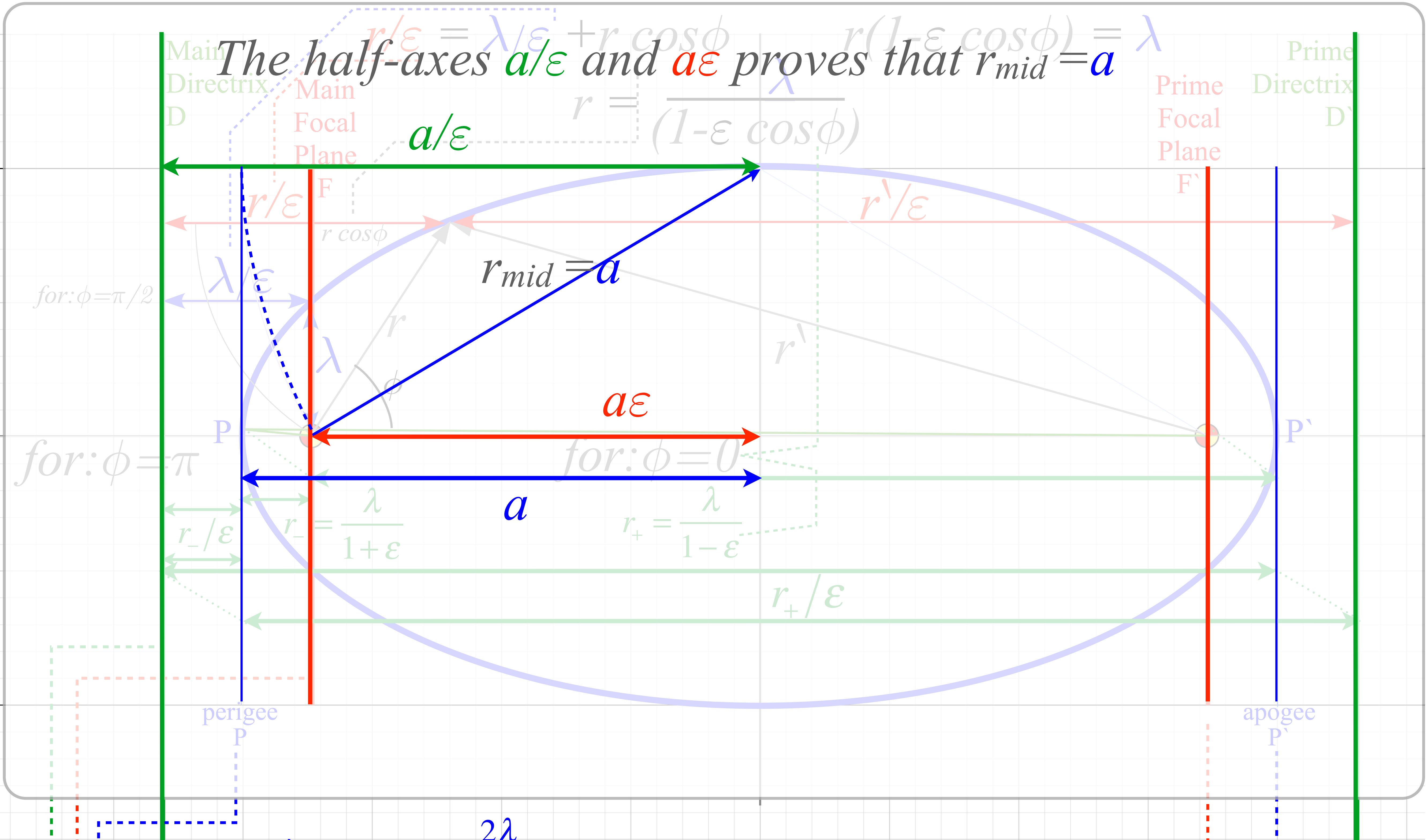


Major axis PP′ $=r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $=r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′$=\dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$



Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

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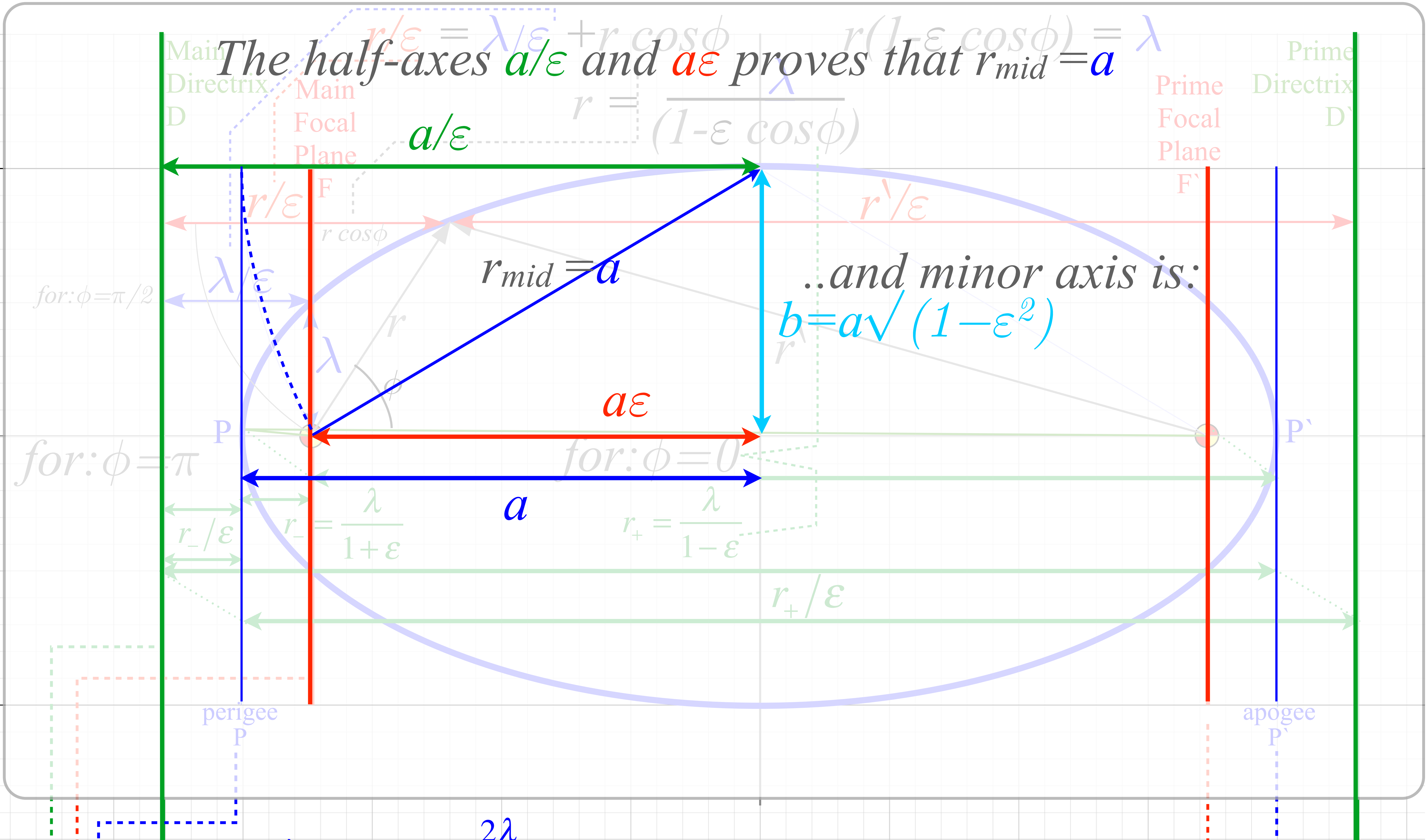

Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

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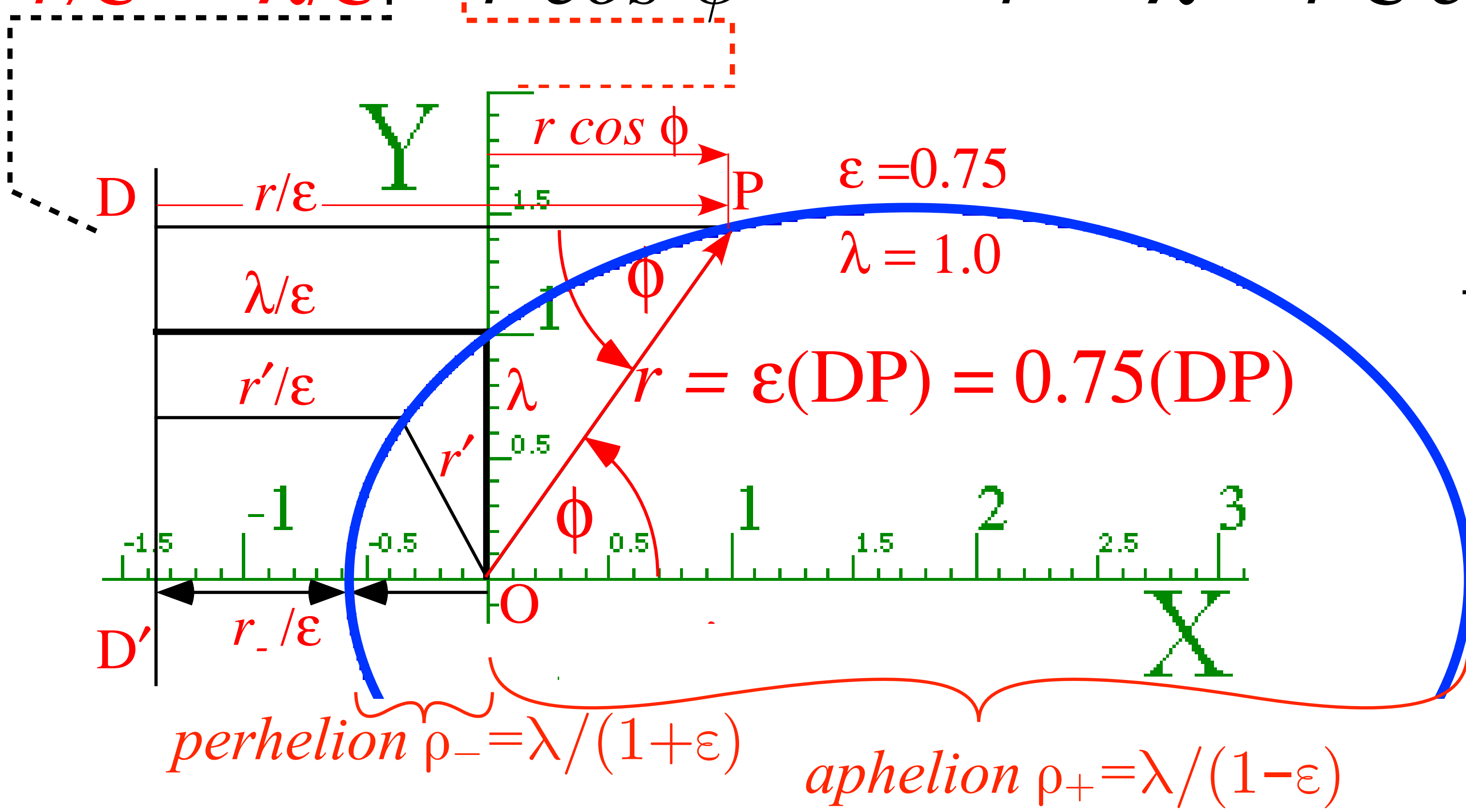
Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



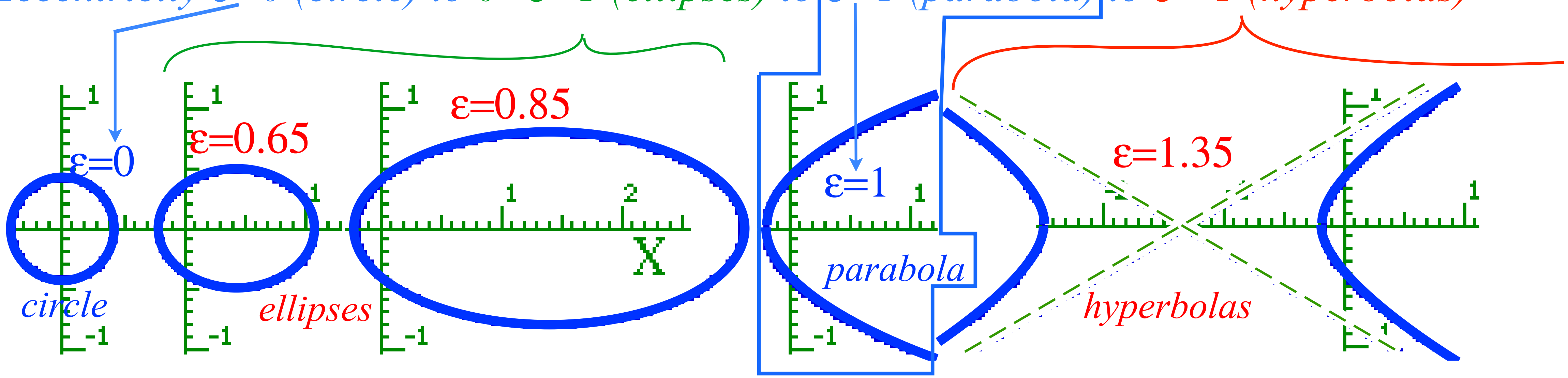
$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Becoming more and more eccentric...

Singular Case!

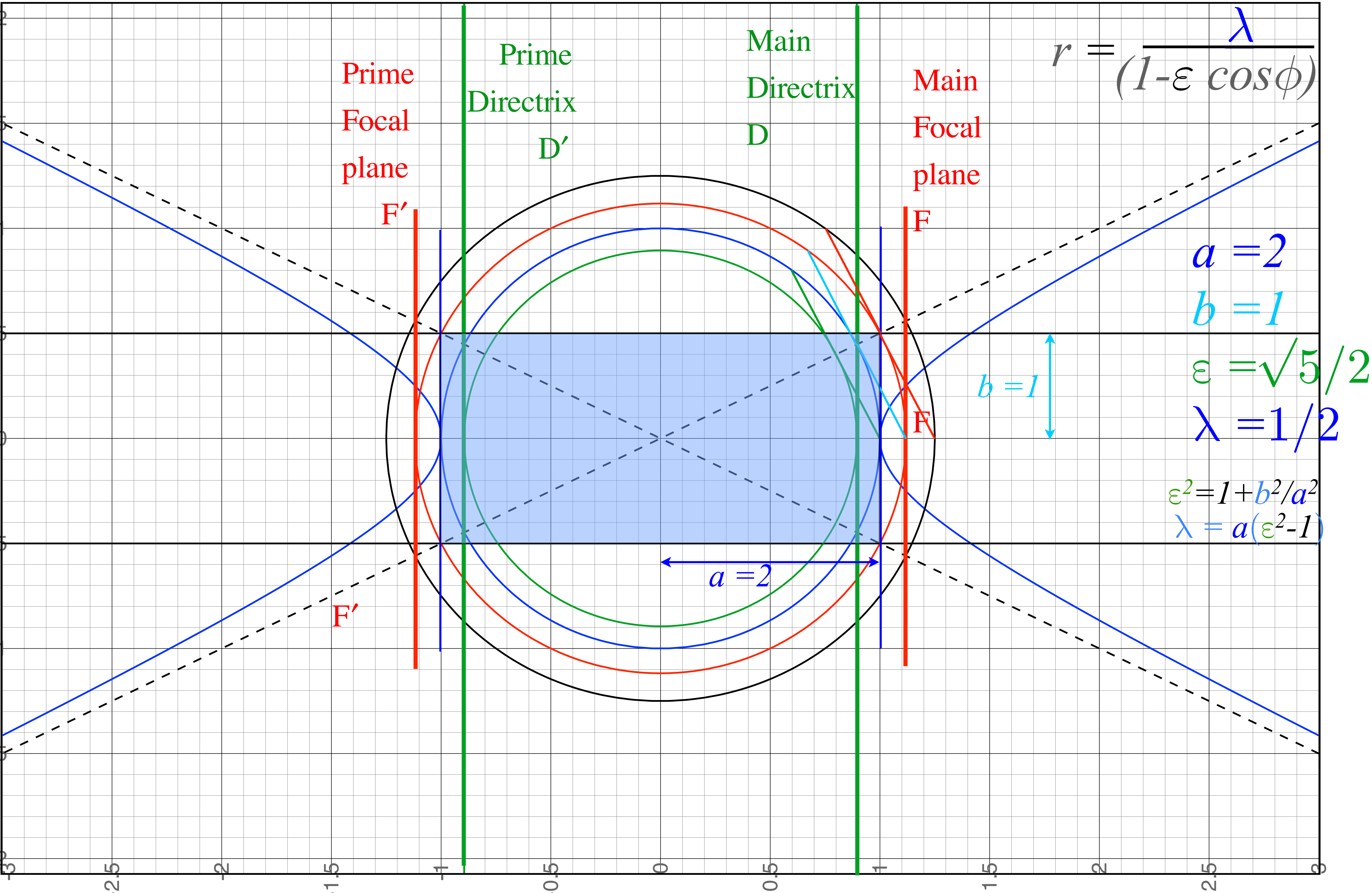
Eccentricity ε=0 (circle) to 0<ε<1 (ellipses) to ε=1 (parabola) to ε >1 (hyperbolas)

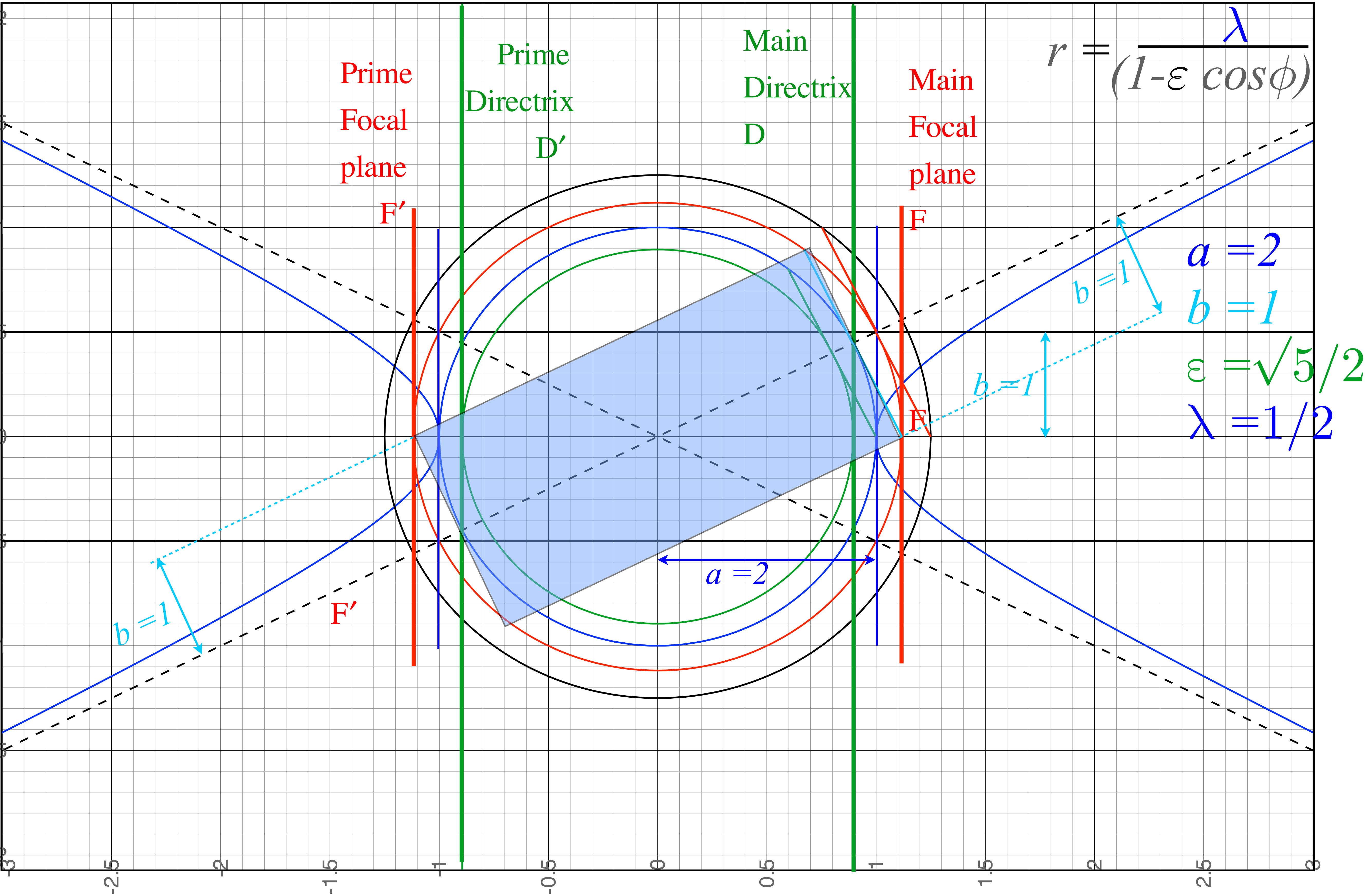


Geometry and Symmetry of Coulomb orbits

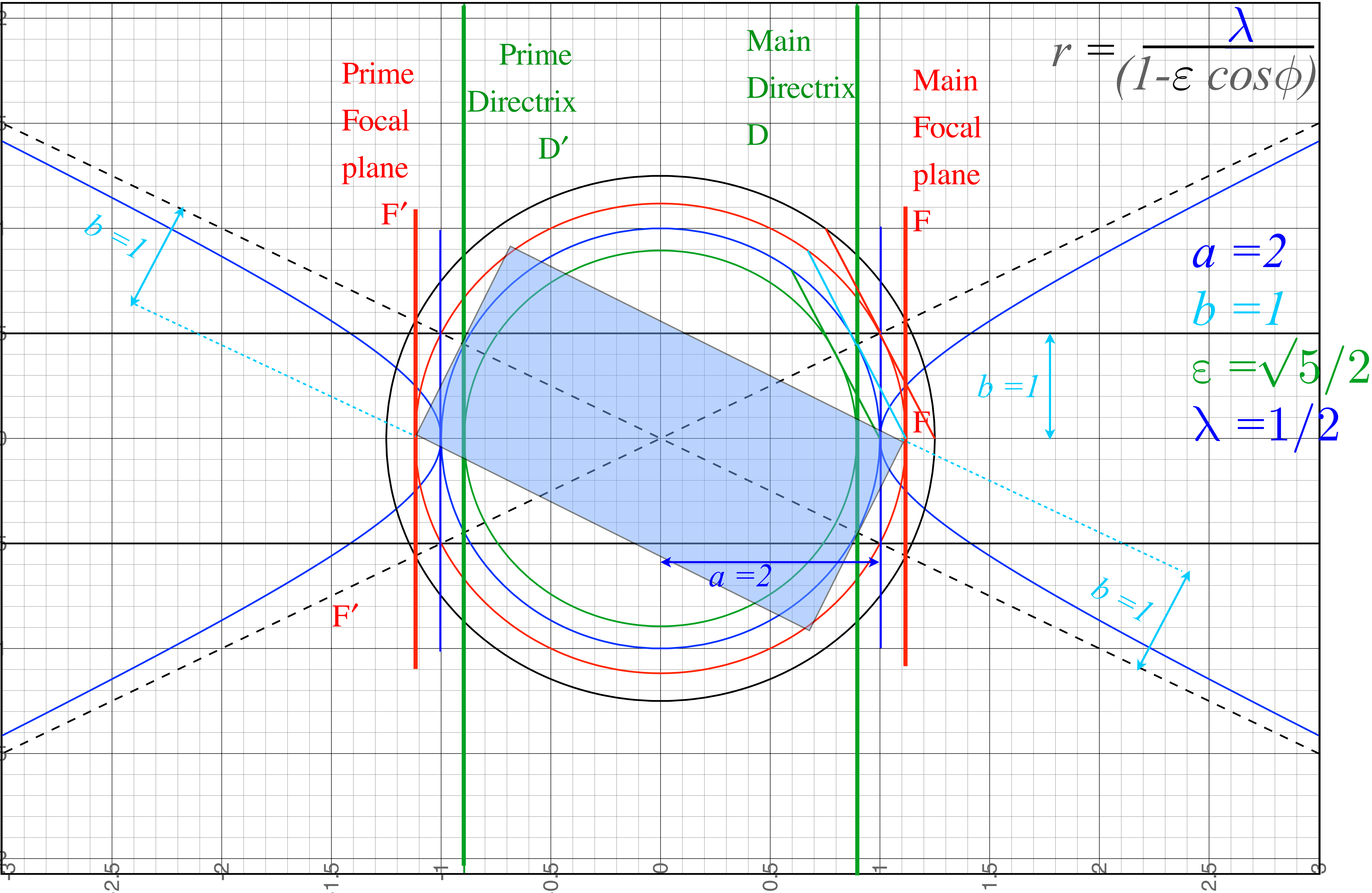
Detailed elliptic geometry

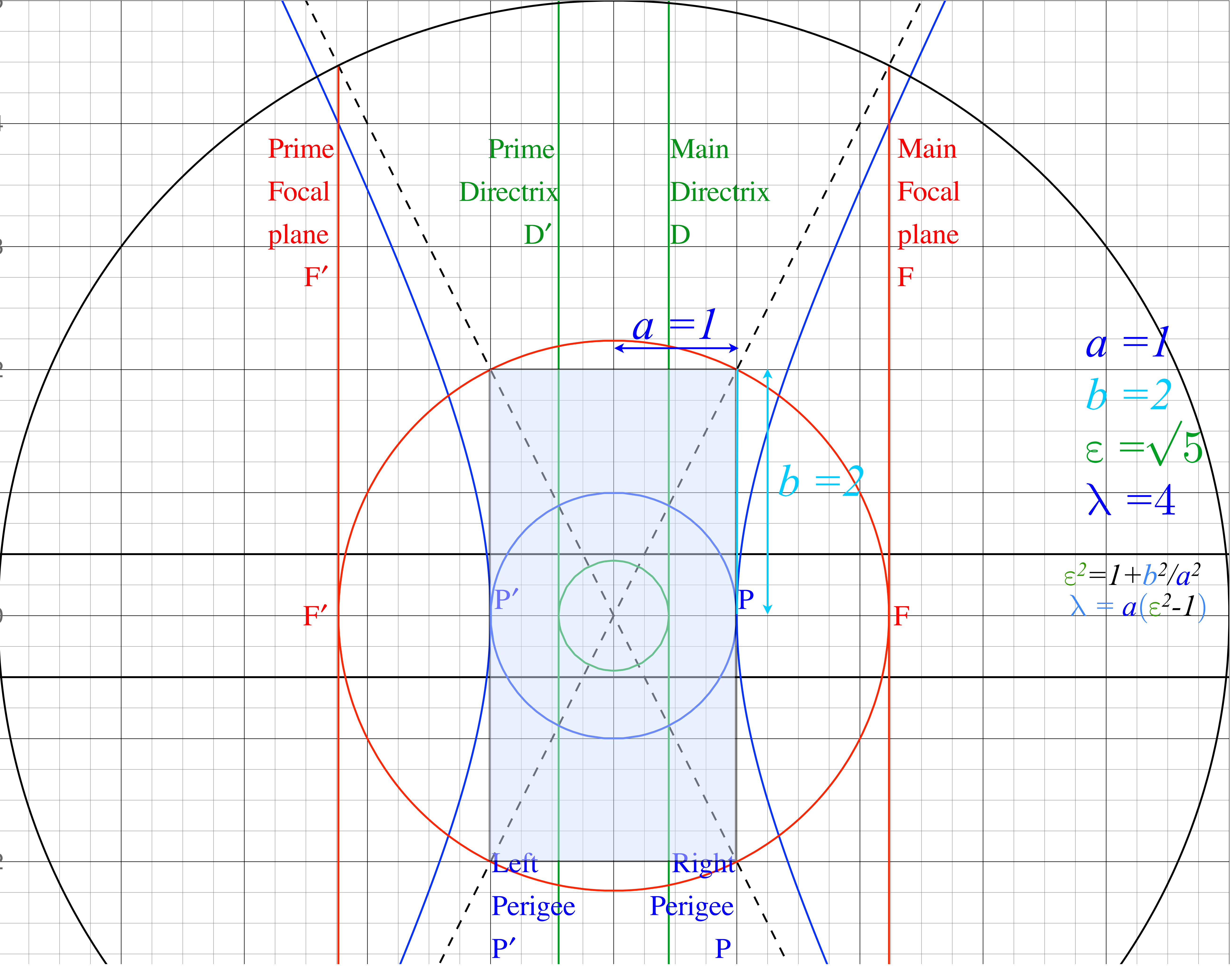
➤ *Detailed hyperbolic geometry*


Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
r = λ / (1-ε cosφ)
a =2
b =1
ε =√5/2
λ =1/2
b =1
F
ε2=1+b2/a2
λ = a(ε2-1)
a =2
F′
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

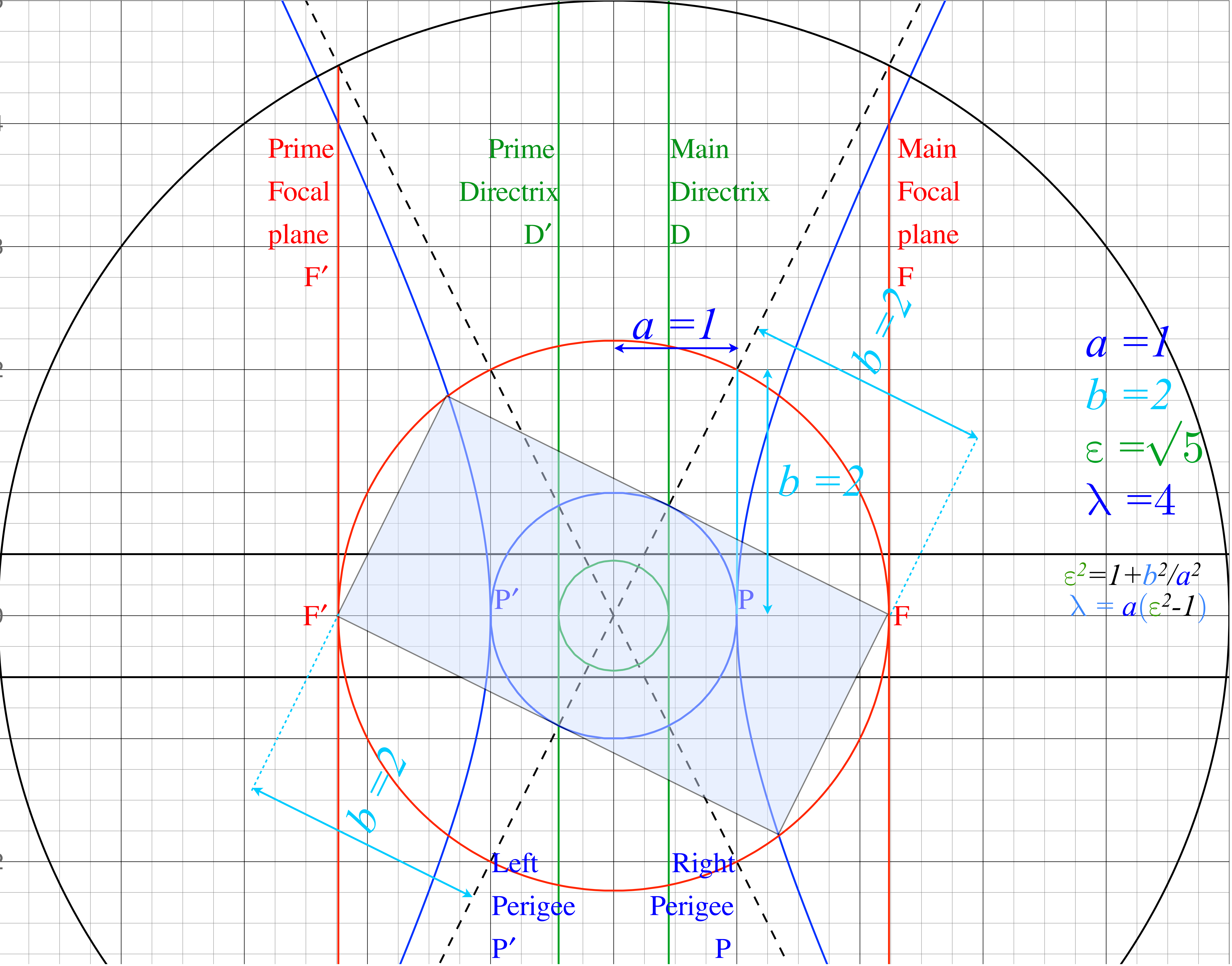


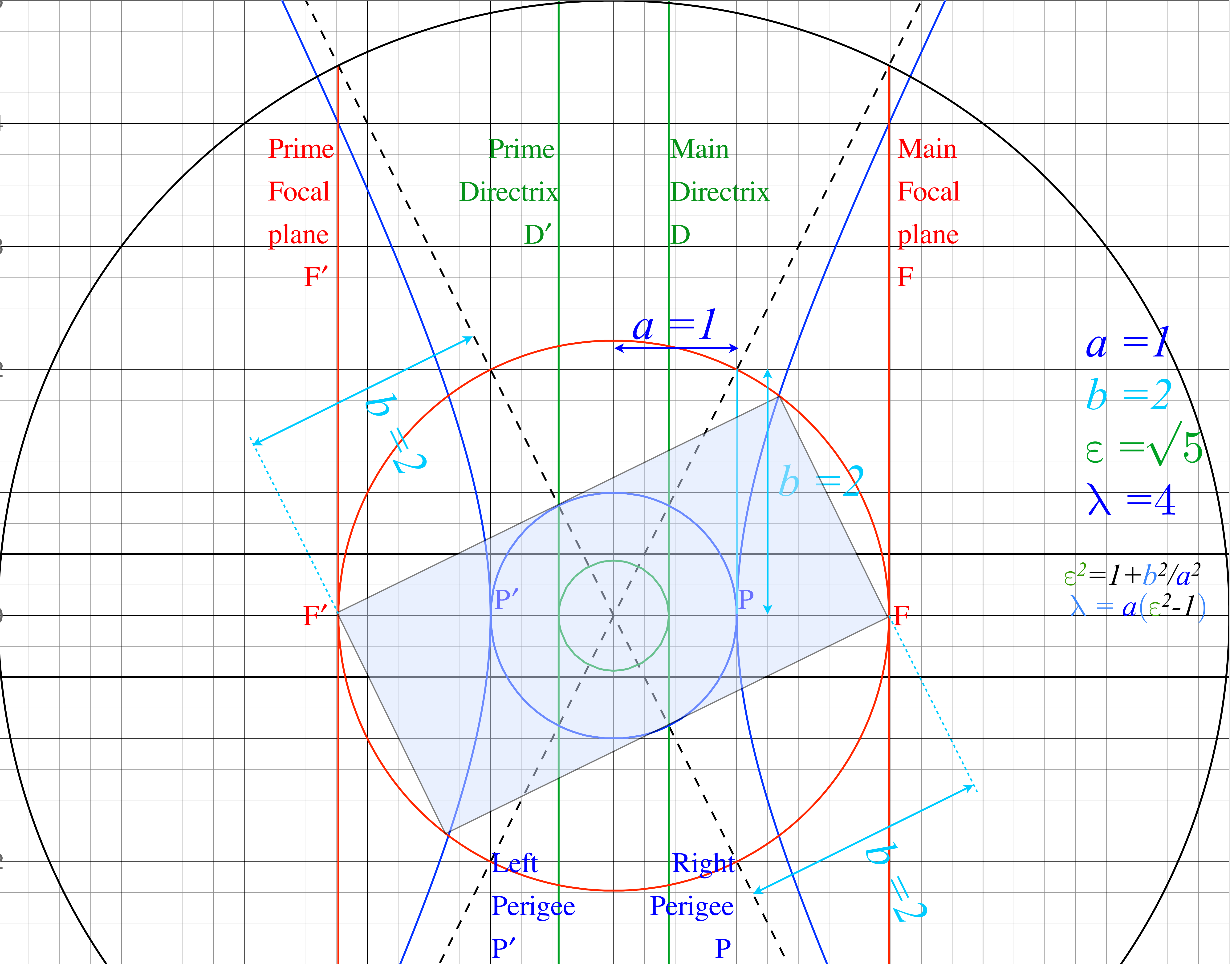
Prime Focal plane
F′
Prime Directrix D′
Main Directrix D
Main Focal plane
F
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
a =2
b =1
ε =√5/2
λ =1/2
b =1
b =1
b =1
F
F′
a =2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

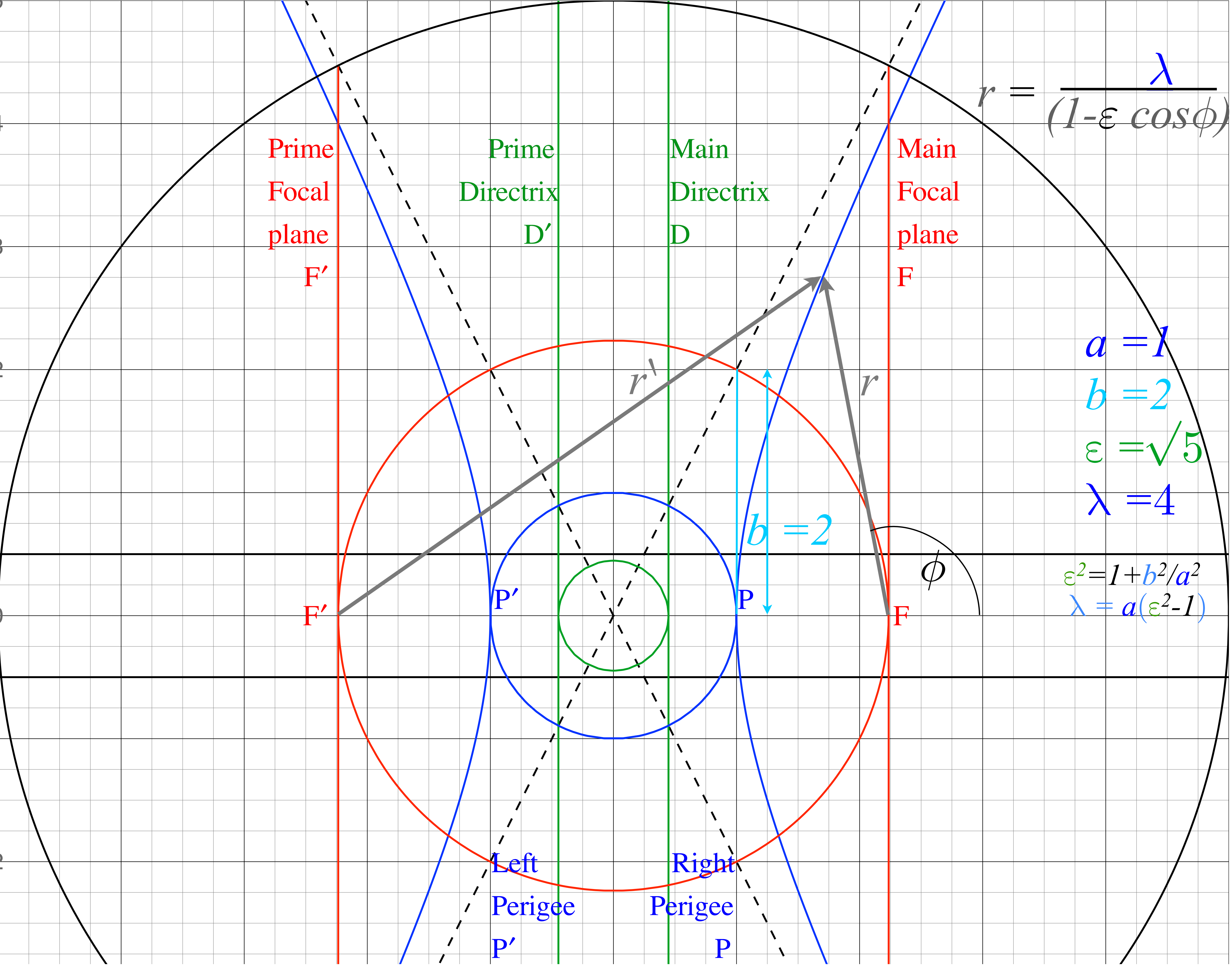

Prime Focal plane
F′
Prime Directrix D′
Main Directrix D
Main Focal plane
F
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
$a = 2$
$b = 1$
$\varepsilon = \sqrt{5}/2$
$\lambda = 1/2$
$b = 1$
$b = 1$
$b = 1$
F
$a = 2$
F′
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3



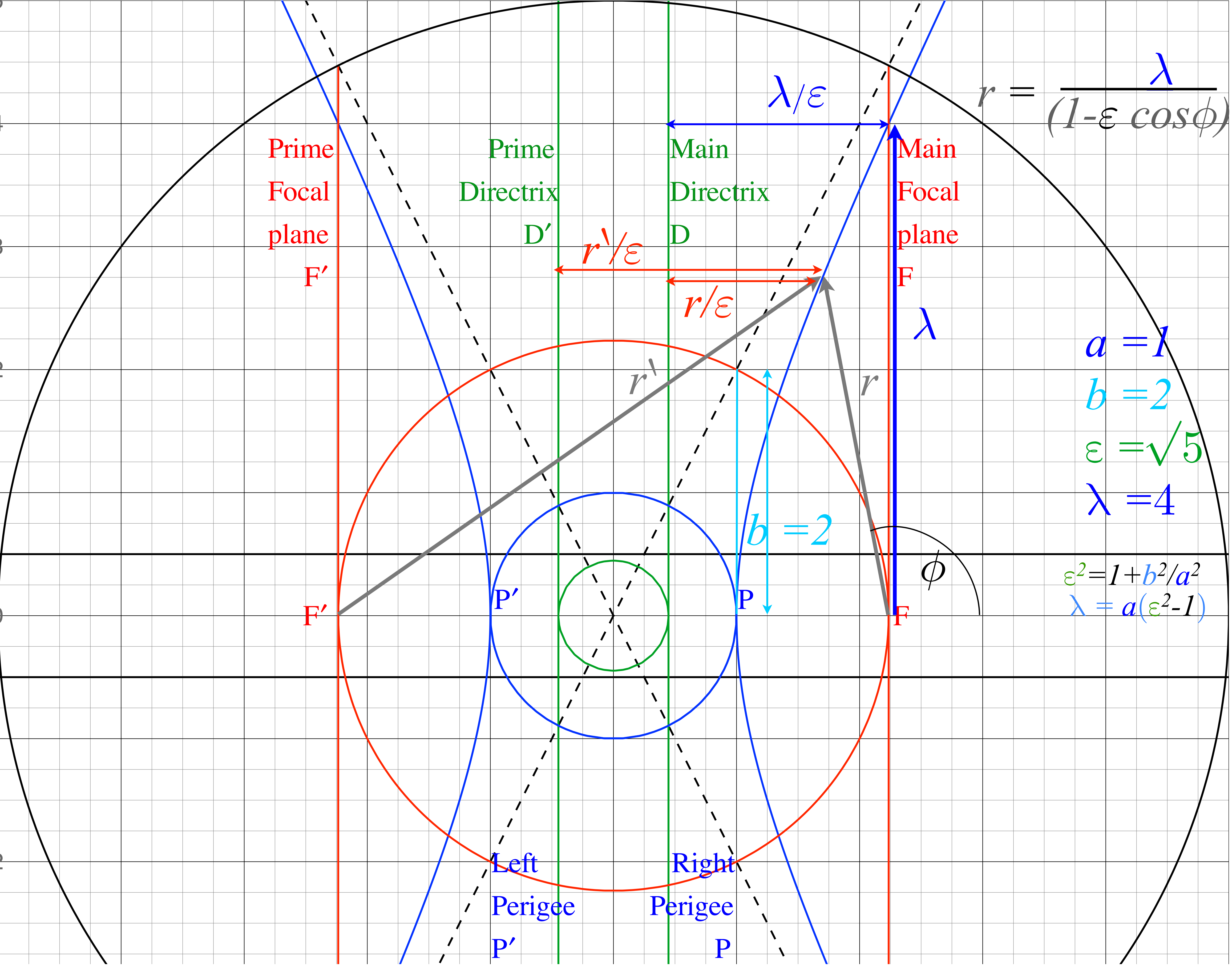
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
a =1
b =2
ε =√5
λ =4
b =2
ε²=1+b²/a²
λ = a(ε²-1)
P′
P
F′
F
Left
Perigee
P′
Right
Perigee
P


Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
b =2
b =2
b =2
b =2
a =1
b =2
ε =√5
λ =4
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
Left
Perigee
P′
Right
Perigee
P

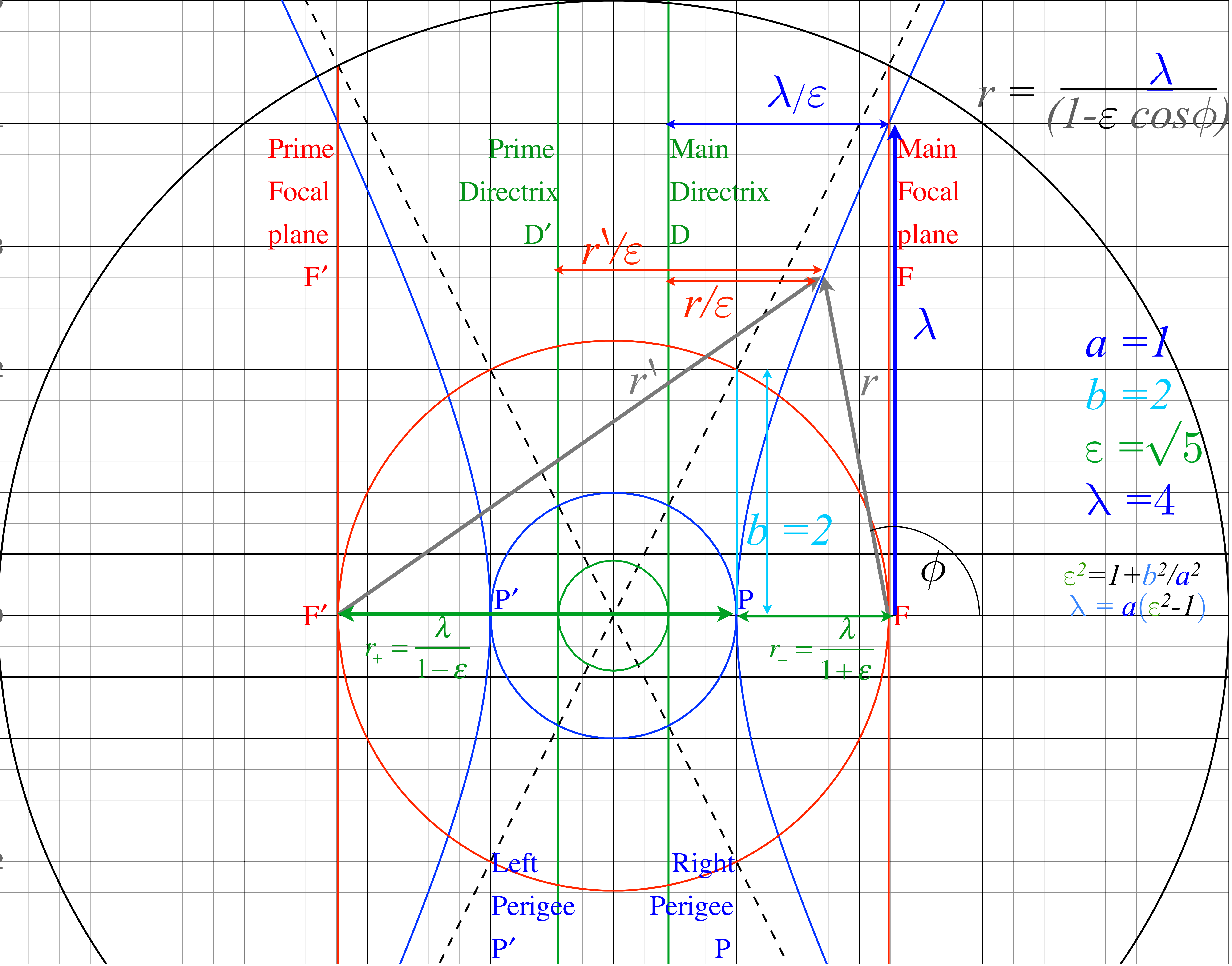

Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
b =2
b =2
a =1
b =2
ε =√5
λ =4
ε²=1+b²/a²
λ = a(ε²-1)
P′
P
F′
F
b =2
Left
Perigee
P′
Right
Perigee
P



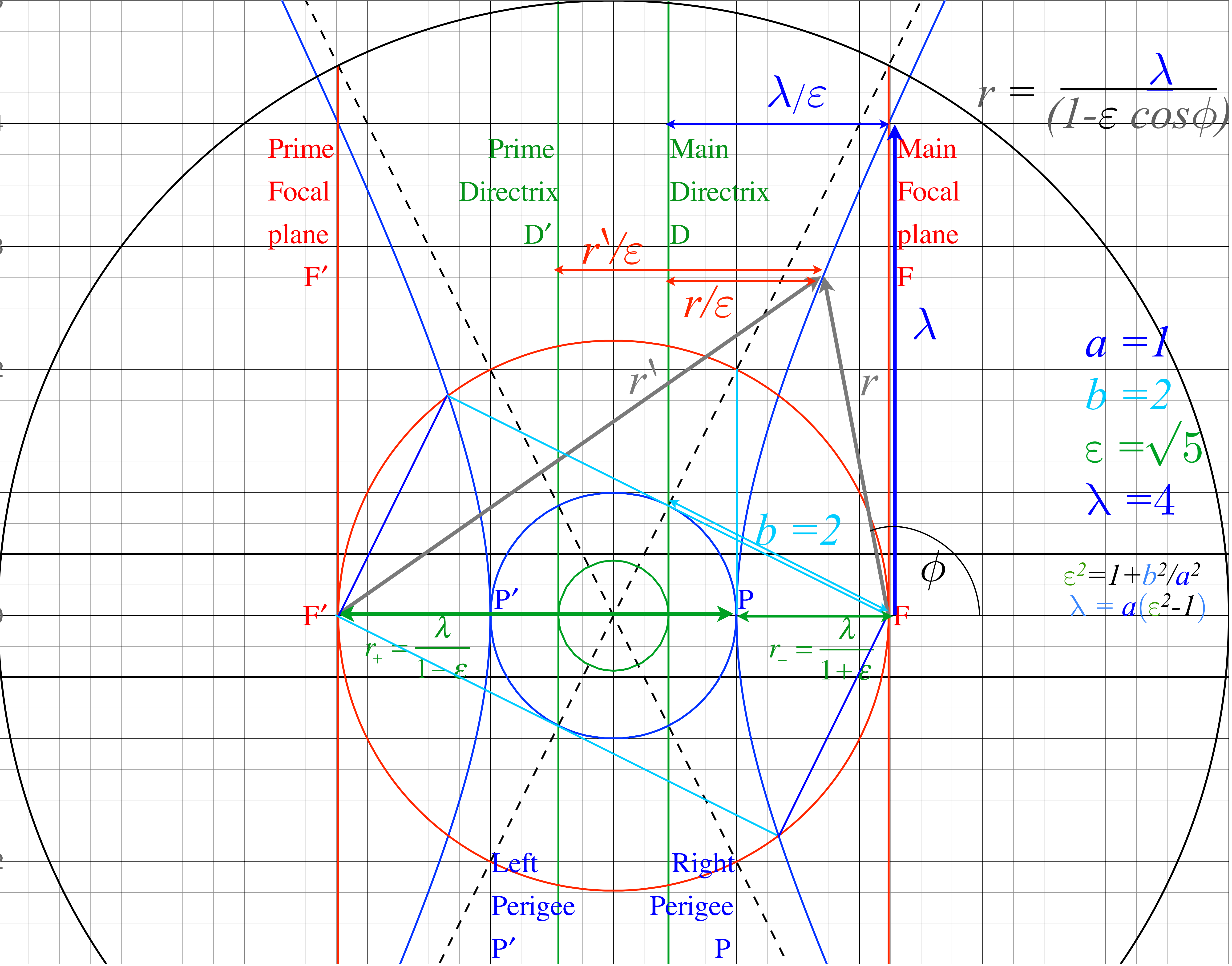
r = λ / (1-ε cosφ)
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
a = 1
b = 2
ε = √5
λ = 4
r′
r
b = 2
φ
ε² = 1 + b²/a²
λ = a(ε²-1)
F′
P′
P
F
Left Perigee P′
Right Perigee P



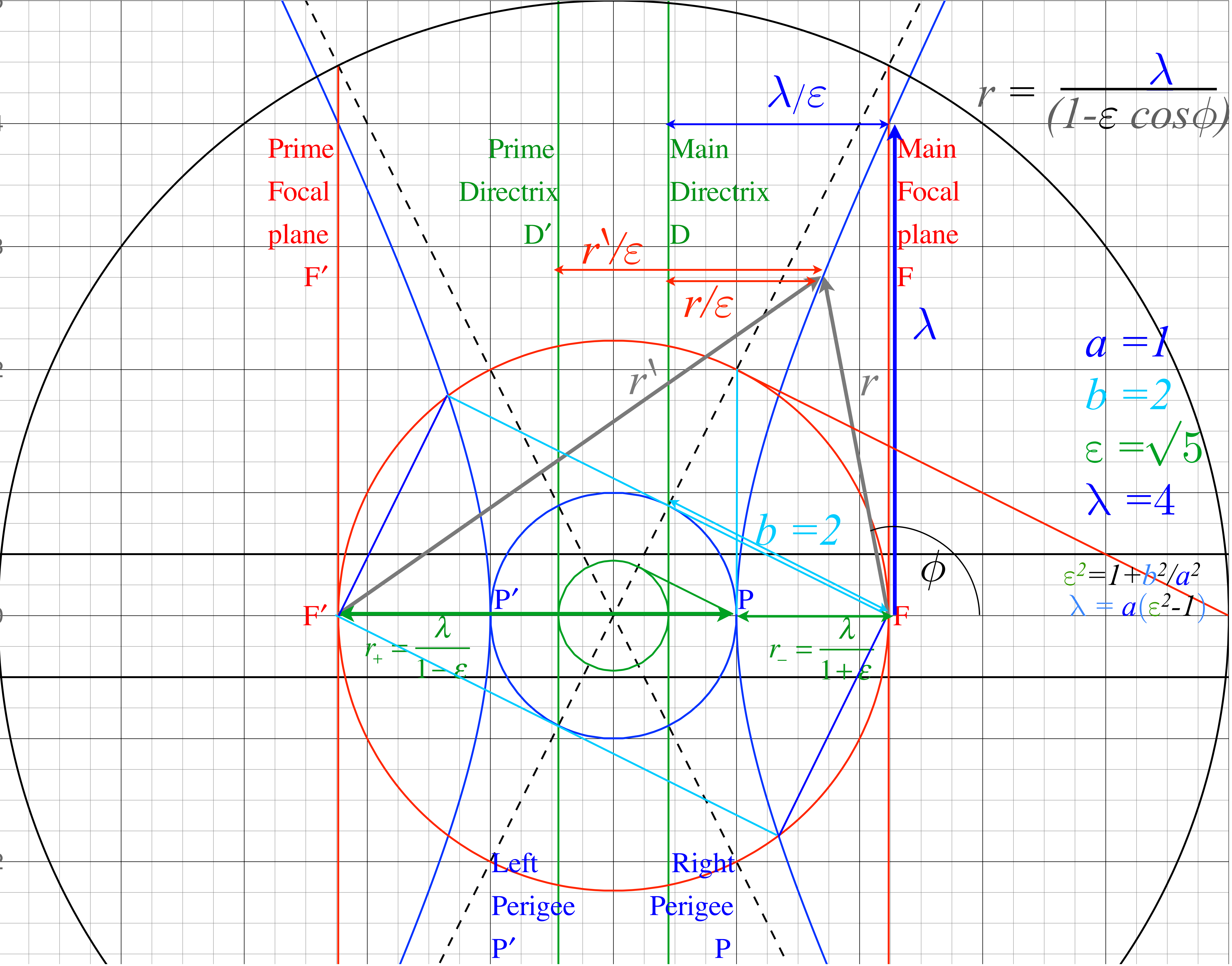
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left Perigee P′
Right Perigee P



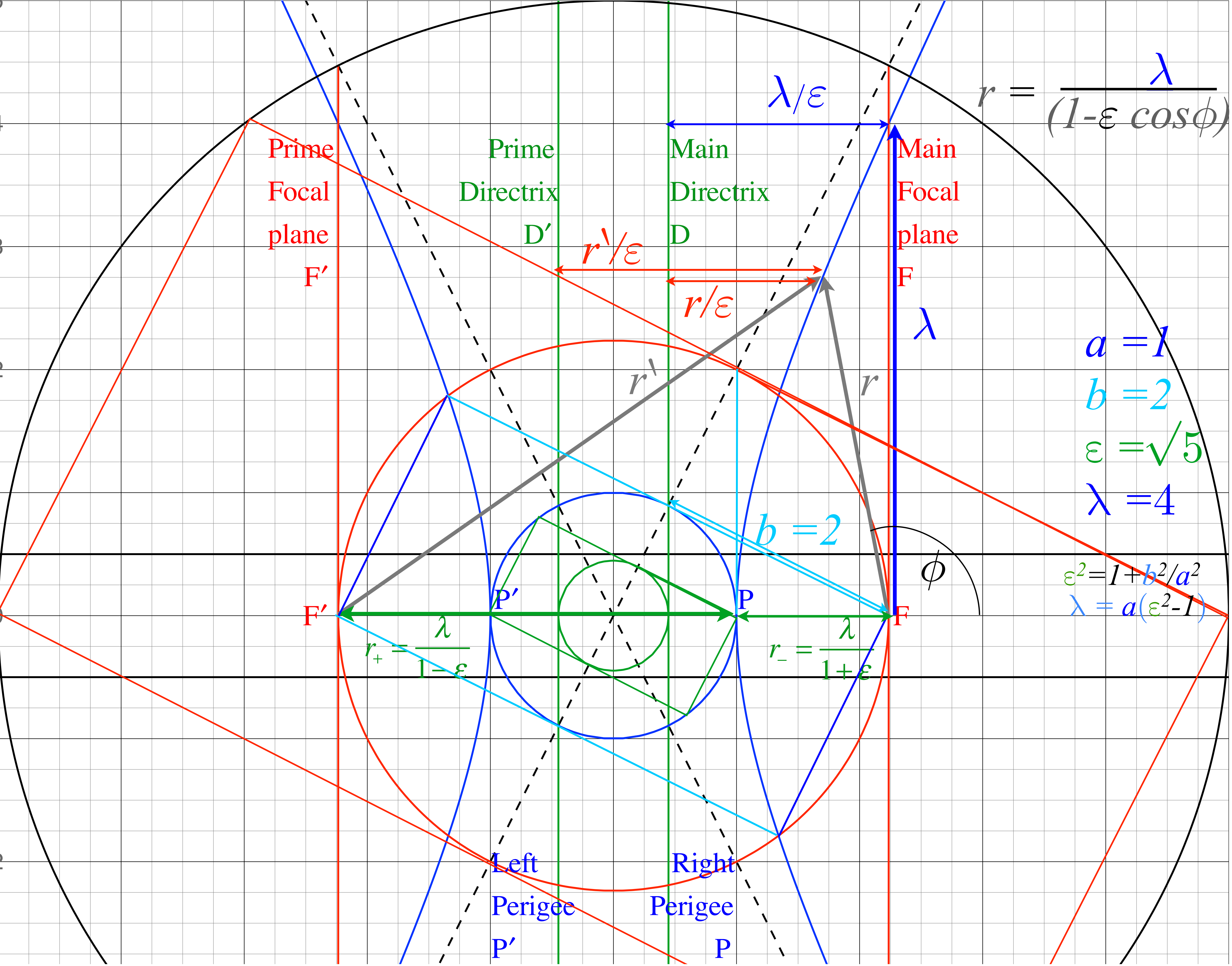
$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



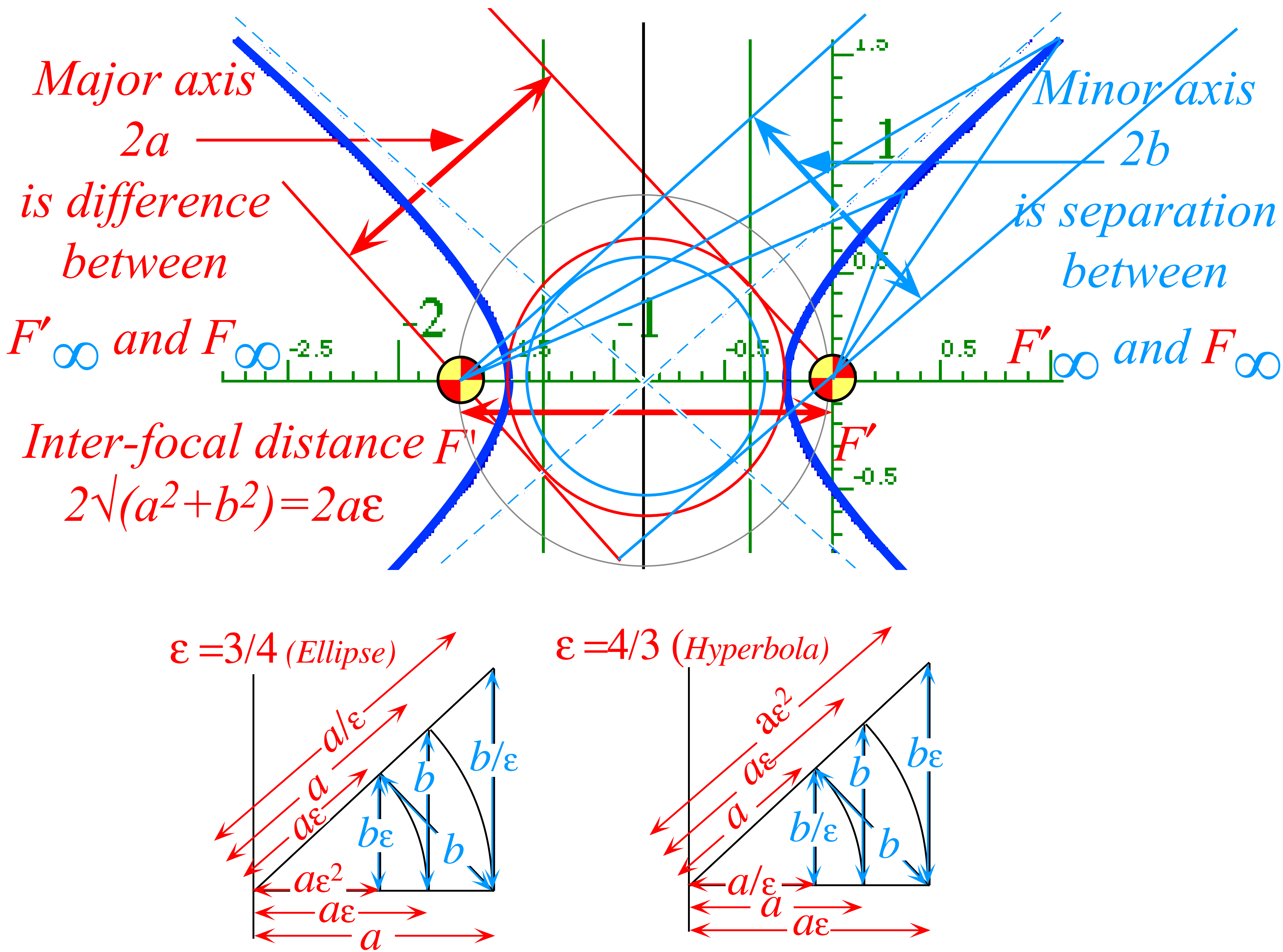
$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



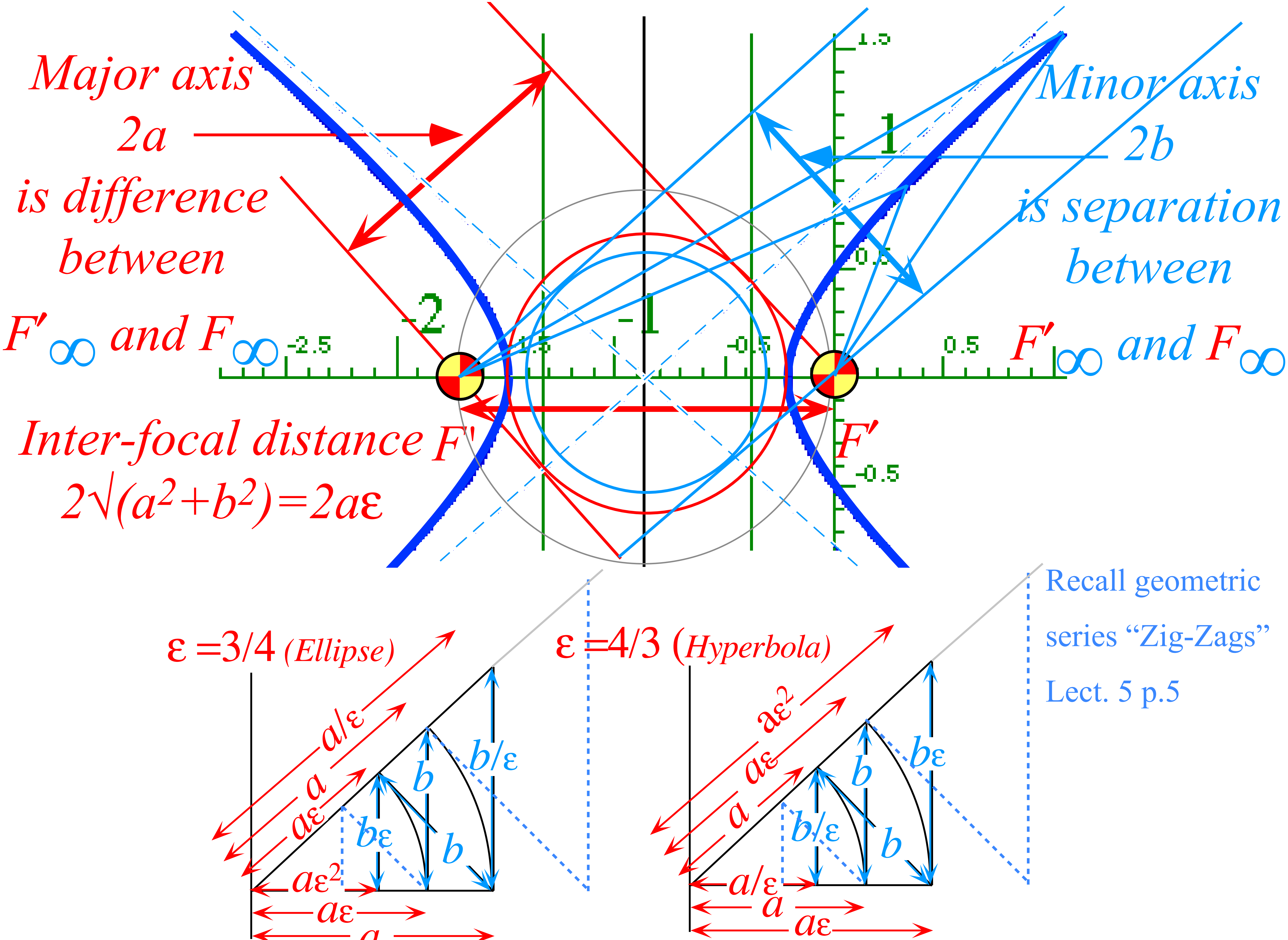
$r = \dfrac{\lambda}{(1\text{-}\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r′/ε
r/ε
λ
r′
r
a = 1
b = 2
ε = √5
λ = 4
b = 2
ϕ
ε² = 1 + b²/a²
λ = a(ε² - 1)
F′
P′
P
F
$r_+ = \dfrac{\lambda}{1-\varepsilon}$
$r_- = \dfrac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



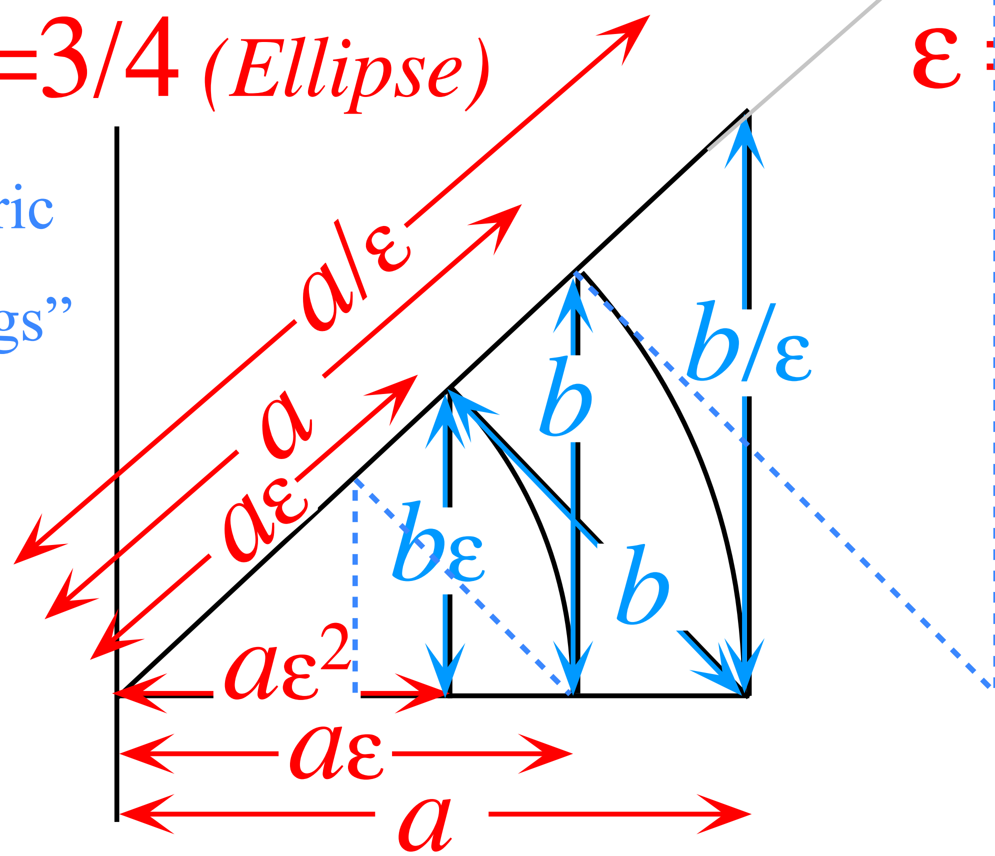
$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P

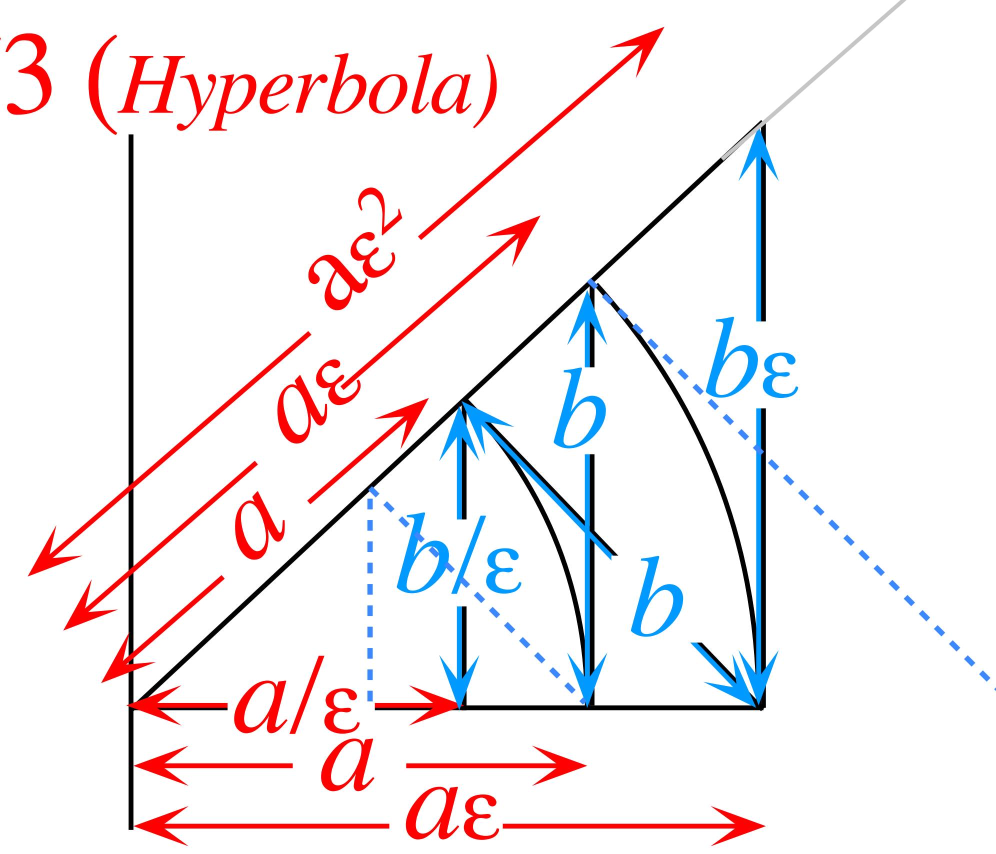


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′F
2√(a²+b²)=2aε
F′
-2.5
-2
-1.5
-1
-0.5
0.5
1.5
1
0.5
-0.5
ε =3/4 (Ellipse)
a/ε
a
aε
b/ε
b
bε
b
aε²
aε
a
ε =4/3 (Hyperbola)
aε²
aε
a
bε
b
b/ε
b
a/ε
a
aε


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′F
2√(a²+b²)=2aε
F′
-2.5
-2
-1.5
-1
-0.5
0.5
1.5
1
0.5
-0.5
ε =3/4 (Ellipse)
a/ε
a
aε
b/ε
b
bε
b
aε²
aε
a
ε =4/3 (Hyperbola)
aε²
aε
a
bε
b
b/ε
b
a/ε
a
aε
Recall geometric
series "Zig-Zags"
Lect. 5 p.5

Recall geometric series “Zig-Zags” Lect. 5 p.5





For the elliptic geometry *(ε<1)*:

$b^2 = a^2 - a^2\varepsilon^2 = a\lambda$,

$b = a\sqrt{(1-\varepsilon^2)} = \sqrt{(a\lambda)}$,

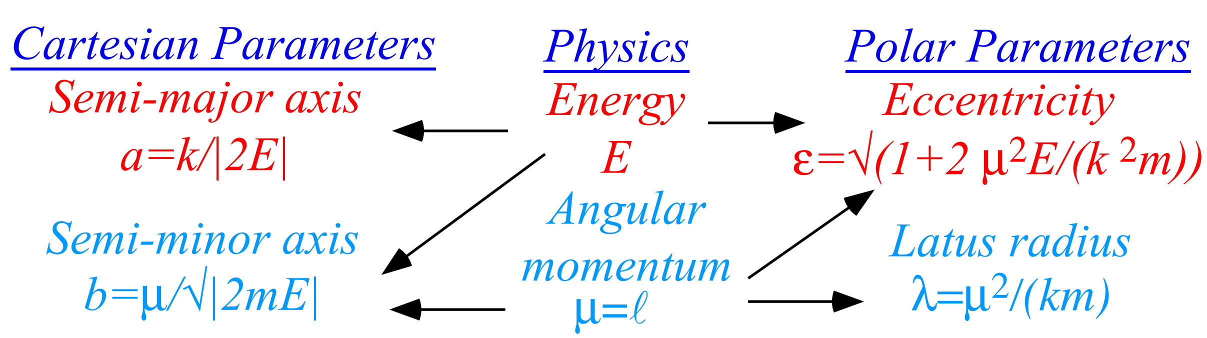
For hyperbolic geometry*(ε>1)*:

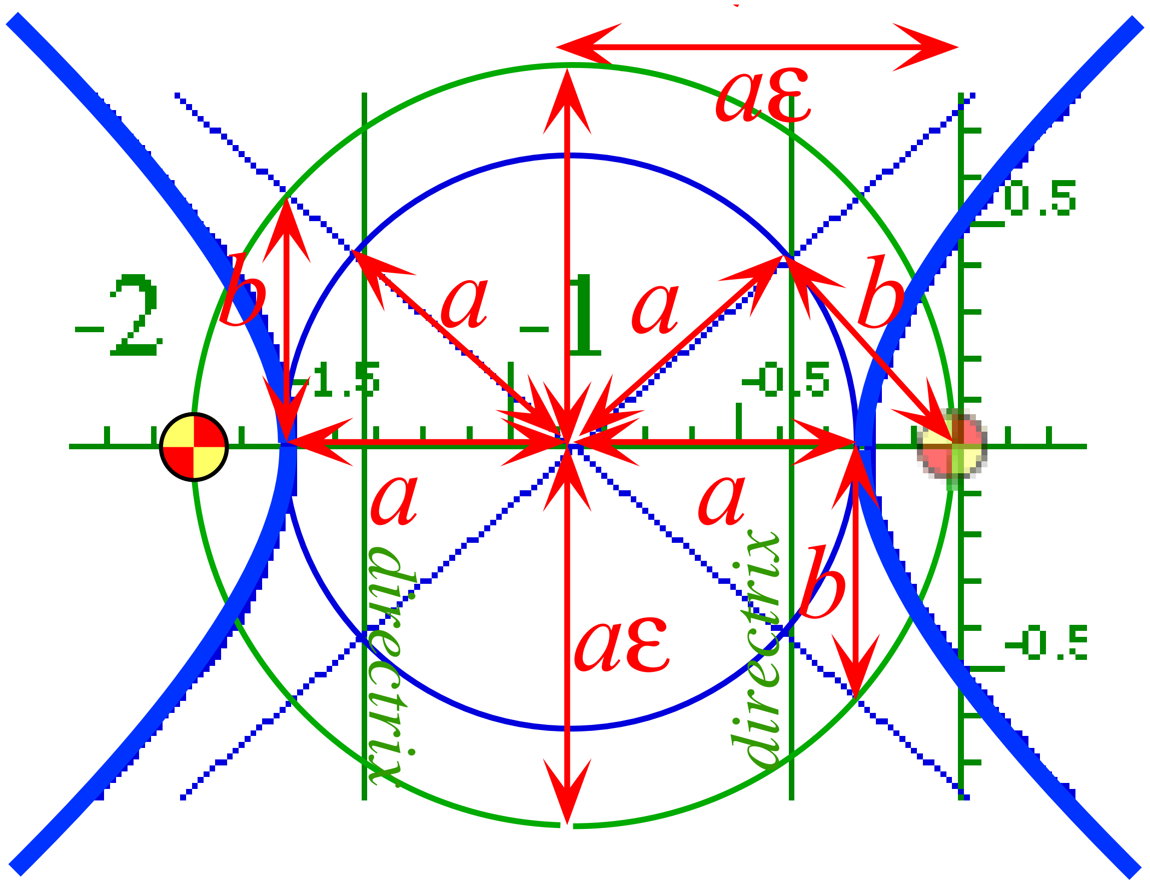
$b^2 = a^2\varepsilon^2 - a^2 = a\lambda$,

$b = a\sqrt{(\varepsilon^2-1)} = \sqrt{(a\lambda)}$.

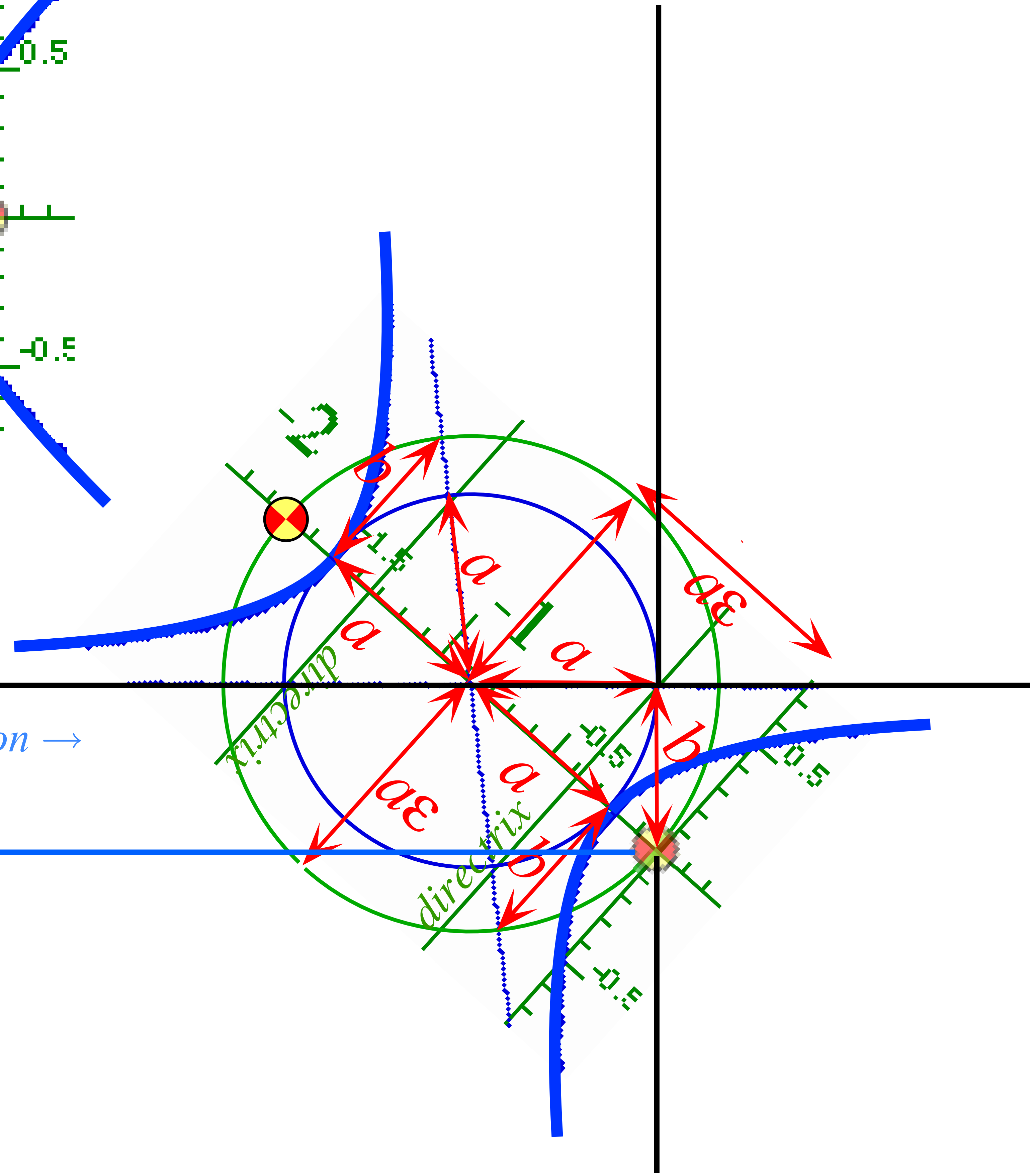
(λ,ε)-(a,b) expressions and their inverses follow.

$a = \lambda/(1-\varepsilon^2)$	$a = \lambda/(\varepsilon^2-1)$
$b^2 = \lambda^2/(1-\varepsilon^2)$	$b^2 = \lambda^2/(\varepsilon^2-1)$
$\lambda = a(1-\varepsilon^2) = b^2/a$	$\lambda = a(\varepsilon^2-1) = b^2/a$
$\varepsilon^2 = 1 - b^2/a^2$	$\varepsilon^2 = 1 + b^2/a^2$





Rutherford scattering geometry...



Alpha-particle beam direction →

Gold nuclear target →