

Lecture 28
Tue. 12.07.2015

Multi-particle and Rotational Dynamics

(Ch. 2-7 of Unit 6 12.12.14)

2-Particle orbits

Ptolemaic or LAB view and reduced mass

Copernican or COM view and reduced coupling

2-Particle orbits and scattering: LAB-vs.-COM frame views

Ruler & compass construction (or not)

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity ω -ellipsoid vs. angular momentum \mathbf{L} -ellipsoid

Lagrangian ω -equations vs. Hamiltonian momentum \mathbf{L} -equation

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)

Symmetric, asymmetric, and spherical-top dynamics (Constant \mathbf{L})

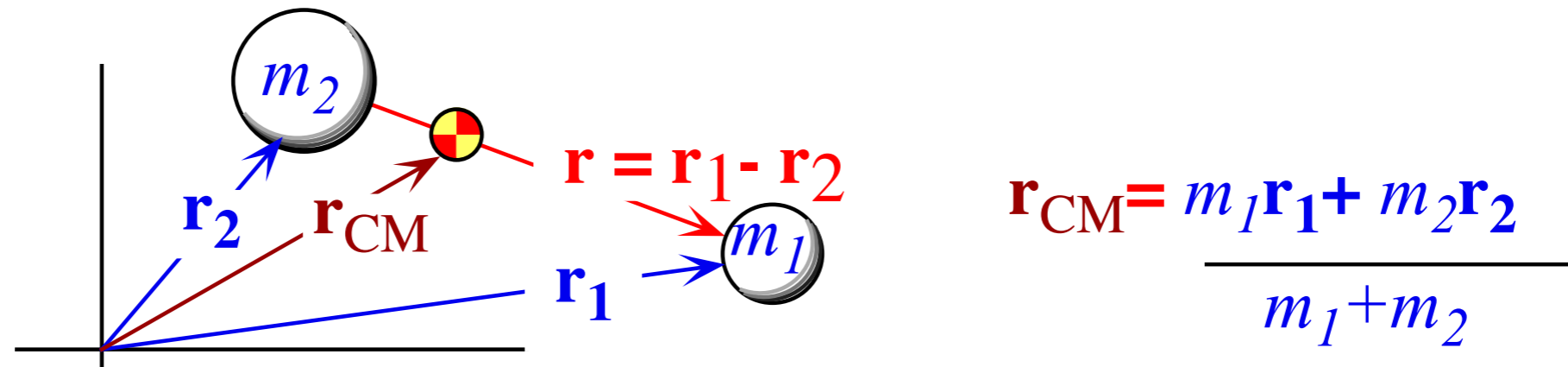
BOD-frame cone rolling on LAB frame cone

Deformable spherical rotor RES and semi-classical rotational states and spectra

Cycloidal geometry of flying levers

Practical poolhall application

2-Particle orbits and center-of-mass (CM) coordinate frame



Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* \mathbf{r}_{CM}

$$\bar{\mathbf{r}} = \mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{\text{CM}} + \frac{m_2 \mathbf{r}}{m_1 + m_2}, \quad \mathbf{r}_2 = \mathbf{r}_{\text{CM}} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

- ➔ *Ptolemaic or LAB view and reduced mass*
- Copernican or COM view and reduced coupling*

Reduced mass: Ptolemaic views

Radial inter-particle force \mathbf{F}_{12} is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = F(r)\mathbf{e}_r = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

\mathbf{F}_{12} acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

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Re-scaled force: A Copernican view

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r},$$

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Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

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2-Particle orbits

Ptolemaic view and reduced mass

➔ *Copernican view and reduced coupling*

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(Here we get "reduced" coupling constants)

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$$F(r) = \frac{k}{r^2} \text{ becomes: } F\left(\frac{m_1}{\mu}r_1\right) = \frac{\mu^2}{m_1^2} \frac{k}{r_1^2}$$

(Coulomb)

$$k \rightarrow k_1 = k \mu^2 / m_1^2, \quad k \rightarrow k_2 = k \mu^2 / m_2^2$$

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Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ $\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$

Sum $\mathbf{F}_{12} + \mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\text{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12} - \mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using **reduced mass:** $\mu = \frac{m_2 m_1}{m_1 + m_2}$ $\ddot{\mathbf{r}}_{\text{CM}} = \mathbf{0}$

$$\left[m_1\ddot{\mathbf{r}}_1 \right] - \left[m_2\ddot{\mathbf{r}}_2 \right] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\left[m_1\ddot{\mathbf{r}}_{\text{CM}} + \frac{m_1 m_2 \ddot{\mathbf{r}}}{m_1 + m_2} \right] - \left[m_2\ddot{\mathbf{r}}_{\text{CM}} + \frac{m_2 m_1 \ddot{\mathbf{r}}}{m_1 + m_2} \right] = \frac{2F(r)}{r}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2 \left(1 - \frac{m_2}{m_1} \dots \right) \quad (m_1 \gg m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1 \left(1 - \frac{m_1}{m_2} \dots \right) \quad (m_2 \gg m_1)$$

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e}_r = \mathbf{F}(r)$$

Re-scaled force: A Copernican view

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \frac{m_2 \mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1} \mathbf{r}$$

$$\mathbf{r}_2 = \frac{-m_1 \mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2} \mathbf{r}$$

(Here we get "reduced" coupling constants)

each particle keeps its original mass m_1 or m_2 , but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F\left(\frac{m_1}{\mu}r_1\right)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F\left(\frac{m_2}{\mu}r_2\right)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$$F(r) = \frac{k}{r^2} \text{ becomes: } F\left(\frac{m_1}{\mu}r_1\right) = \frac{\mu^2}{m_1^2} \frac{k}{r_1^2}$$

(Coulomb)

$$k \rightarrow k_1 = k \mu^2 / m_1^2, \quad k \rightarrow k_2 = k \mu^2 / m_2^2$$

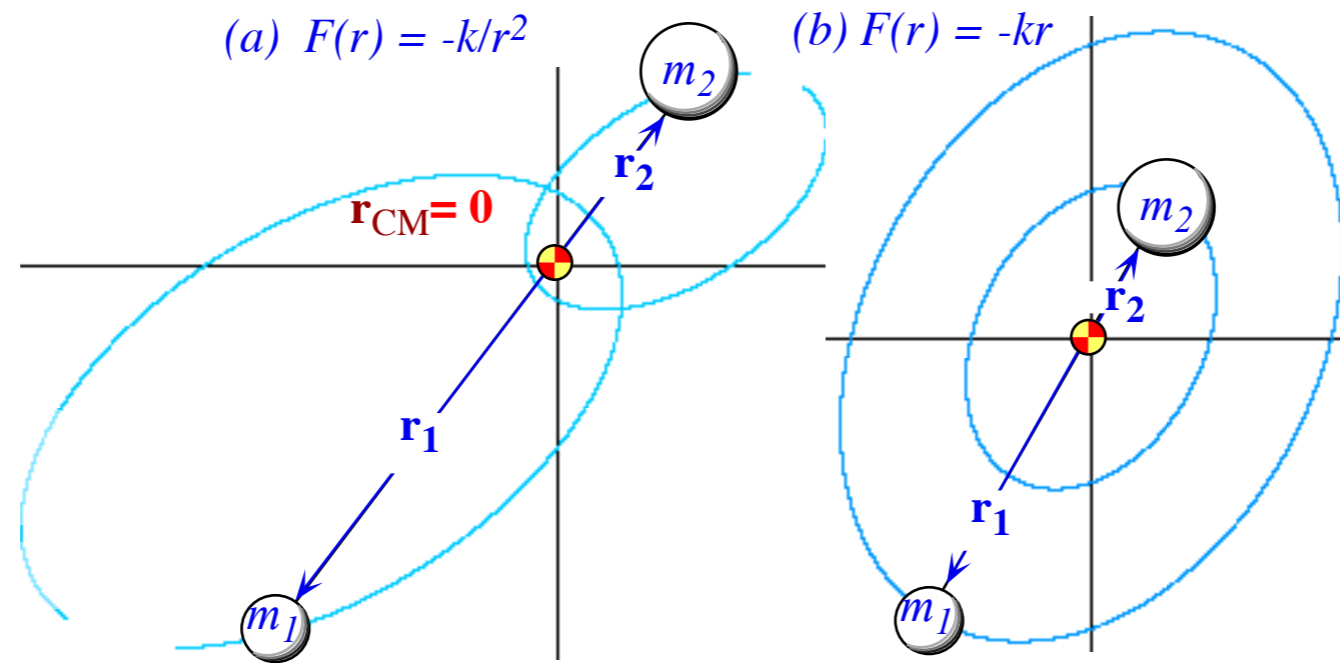
$$F(r) = -kr \text{ becomes: } F\left(\frac{m_1}{\mu}r_1\right) = -\frac{m_1}{\mu} k r_1$$

(Harmonic Oscillator)

$$k \rightarrow k_1 = k m_1 / \mu, \quad k \rightarrow k_2 = k m_2 / \mu$$

2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



[CoulIt Web Simulations](#)
[Coulombic Orbit \(CM Frame\)](#)
[Coulombic Orbit \(Lab Frame\)](#)

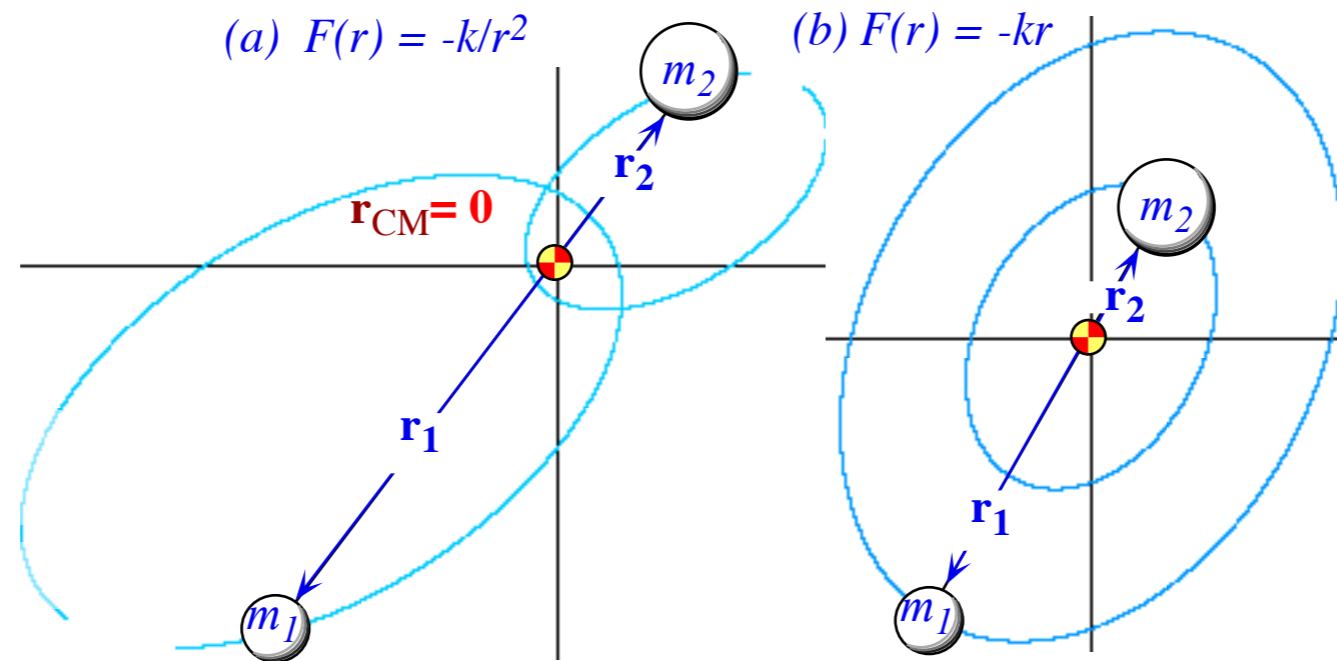
[CoulIt Web Simulations](#)
[Hooke Orbit \(CM Frame\)](#)
[Hooke Orbit \(Lab Frame\)](#)

Two particles are in synchronous motion around fixed CM origin.

Orbit periods are identical to each other.

Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



[CoulIt Web Simulations](#)
[Coulombic Orbit \(CM Frame\)](#)
[Coulombic Orbit \(Lab Frame\)](#)

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[Hooke Orbit \(CM Frame\)](#)
[Hooke Orbit \(Lab Frame\)](#)

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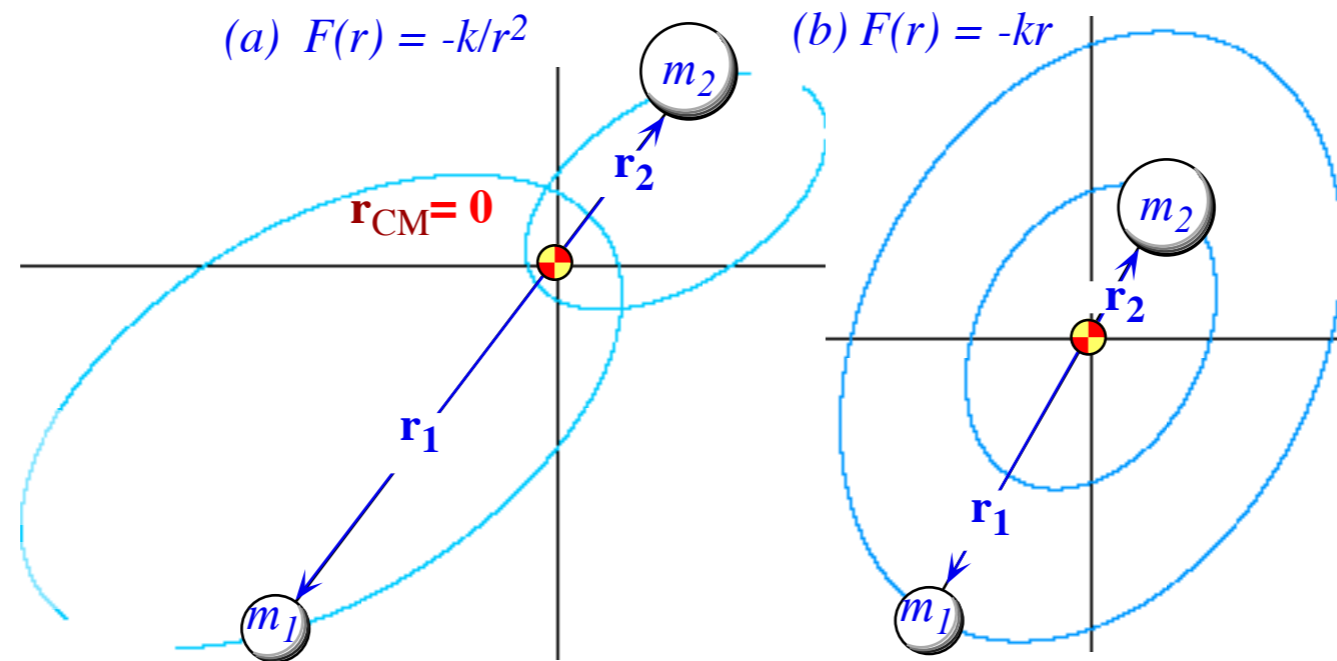
Orbit periods are identical to each other.

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Orbits differ in size of axes (a_1, b_1) and (a_2, b_2)

Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

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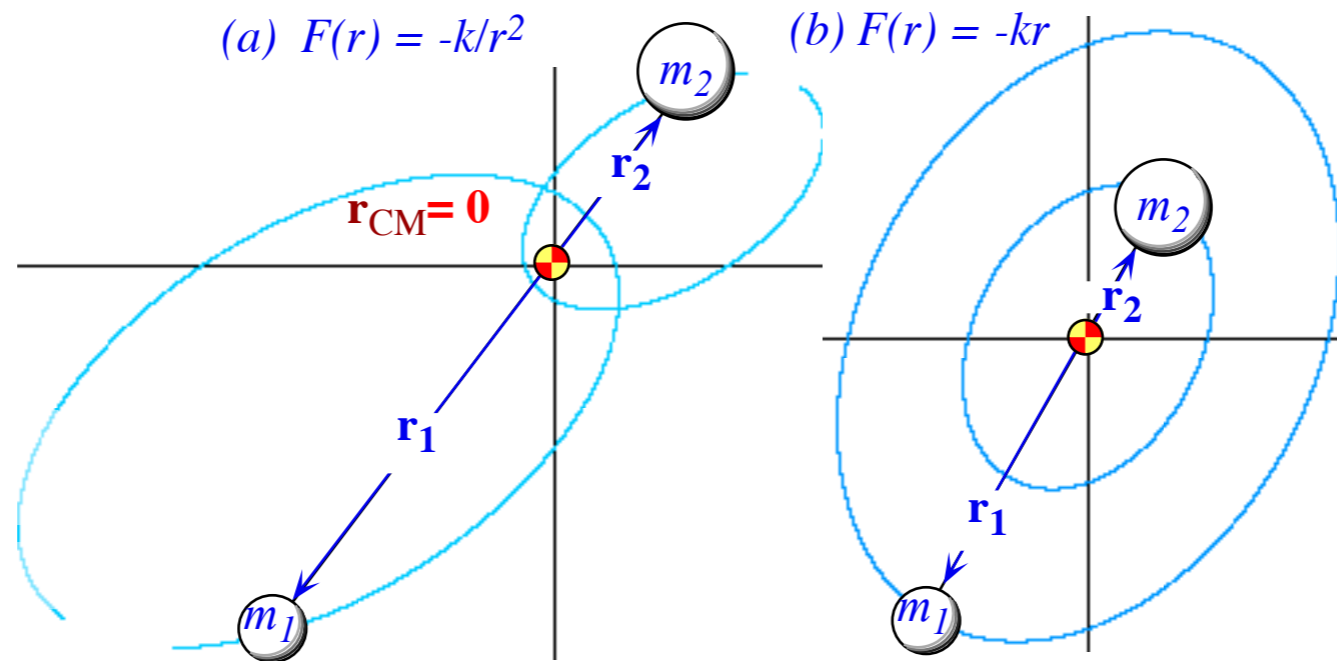
Orbit axial dimensions (a_k, b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a \mu,$$

$$b_1 m_1 = b_2 m_2 = b \mu$$

$$\lambda_1 m_1 = \lambda_2 m_2 = \lambda \mu$$

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



[CoulIt Web Simulations](#)
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Harmonic oscillator periods

and Coulomb orbit periods

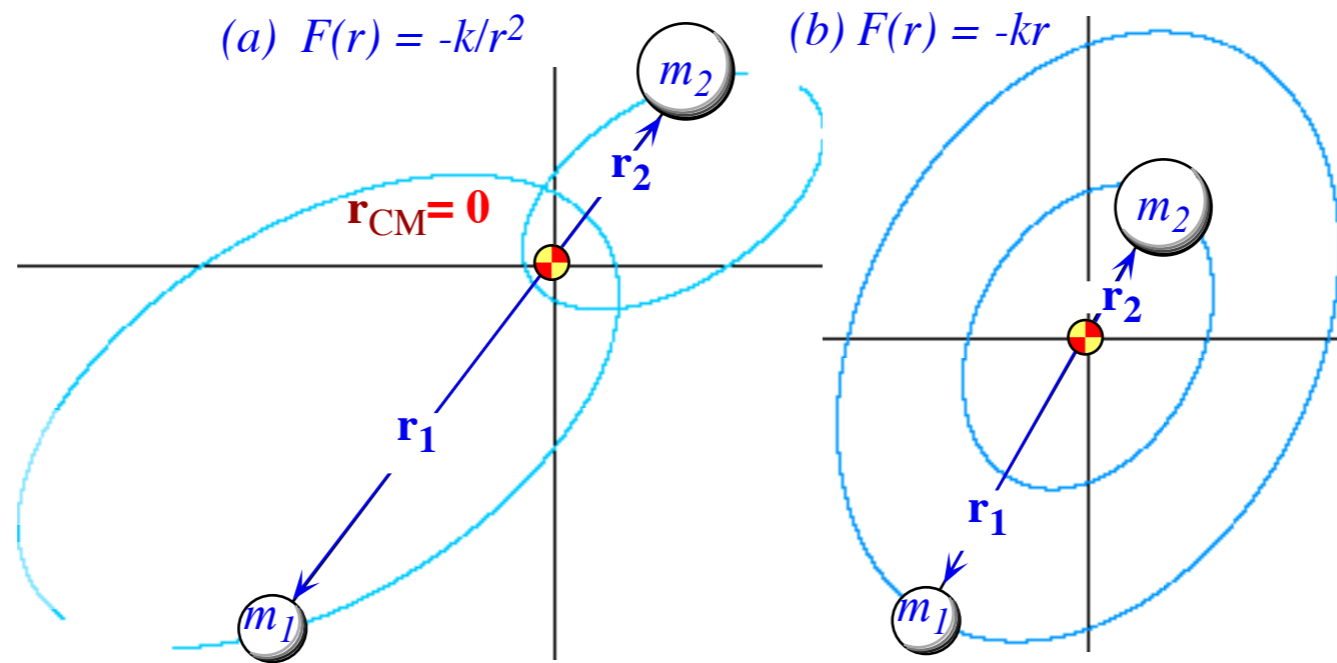
and eccentricity must match

$$T_{IHO} = 2\pi \sqrt{\frac{\mu}{k}} = 2\pi \sqrt{\frac{m_1}{k_1}} = 2\pi \sqrt{\frac{m_2}{k_2}}$$

$$T_{Coul} = 2\pi \sqrt{\frac{\mu a^3}{k}} = 2\pi \sqrt{\frac{m_1 a_1^3}{k_1}} = 2\pi \sqrt{\frac{m_2 a_2^3}{k_2}}$$

$$\epsilon_1 = \epsilon_2 = \epsilon$$

Examples of Coulomb and harmonic oscillator 2-particle “Copernican” orbits in CM system.



[CoulIt Web Simulations](#)
[Coulombic Orbit \(CM Frame\)](#)
[Coulombic Orbit \(Lab Frame\)](#)

[CoulIt Web Simulations](#)
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Three Coulomb orbit energy values satisfy the same proportion relation as their axes

$$E_1 m_1 = E_2 m_2 = E \mu, \quad \text{where: } |E_1| = \frac{|k_1|}{2a_1}, \quad |E_2| = \frac{|k_2|}{2a_2}, \quad |E| = \frac{|k|}{2a}.$$

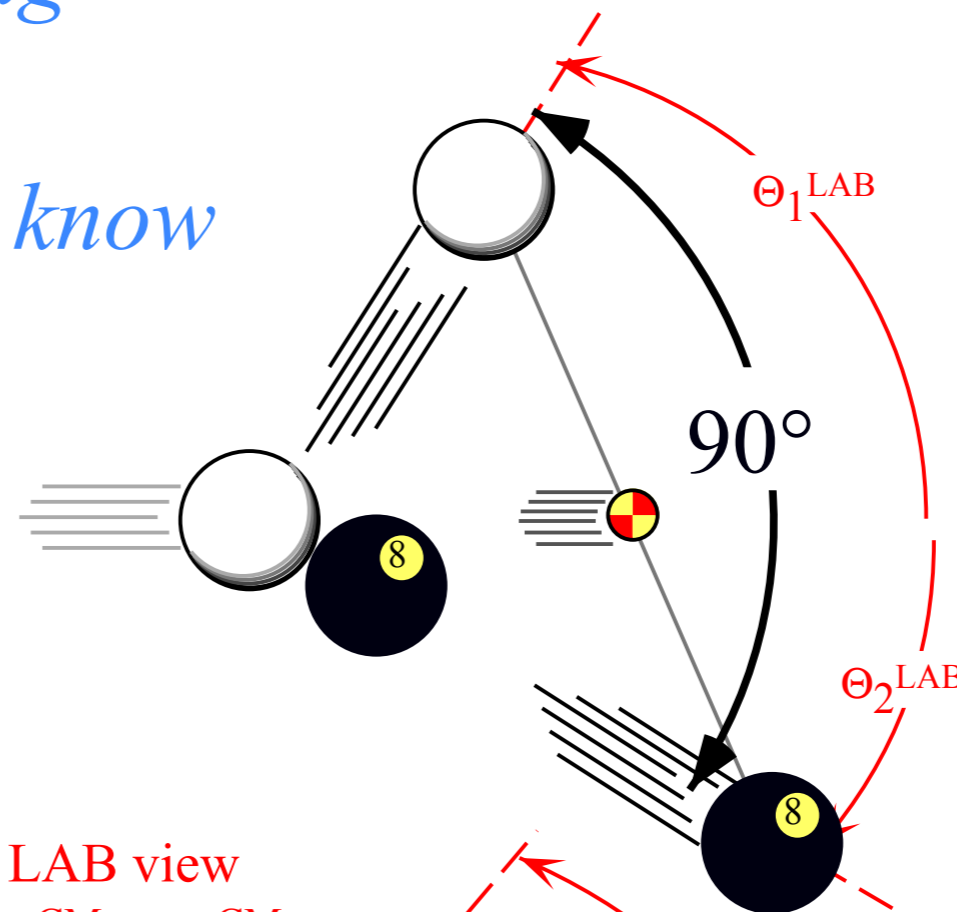
Energy values and axes satisfy similar sum relations

$$E_1 + E_2 = \frac{m_1}{\mu} E + \frac{m_2}{\mu} E = E, \quad \text{and: } a_1 + a_2 = \frac{m_1}{\mu} a + \frac{m_2}{\mu} a = a$$

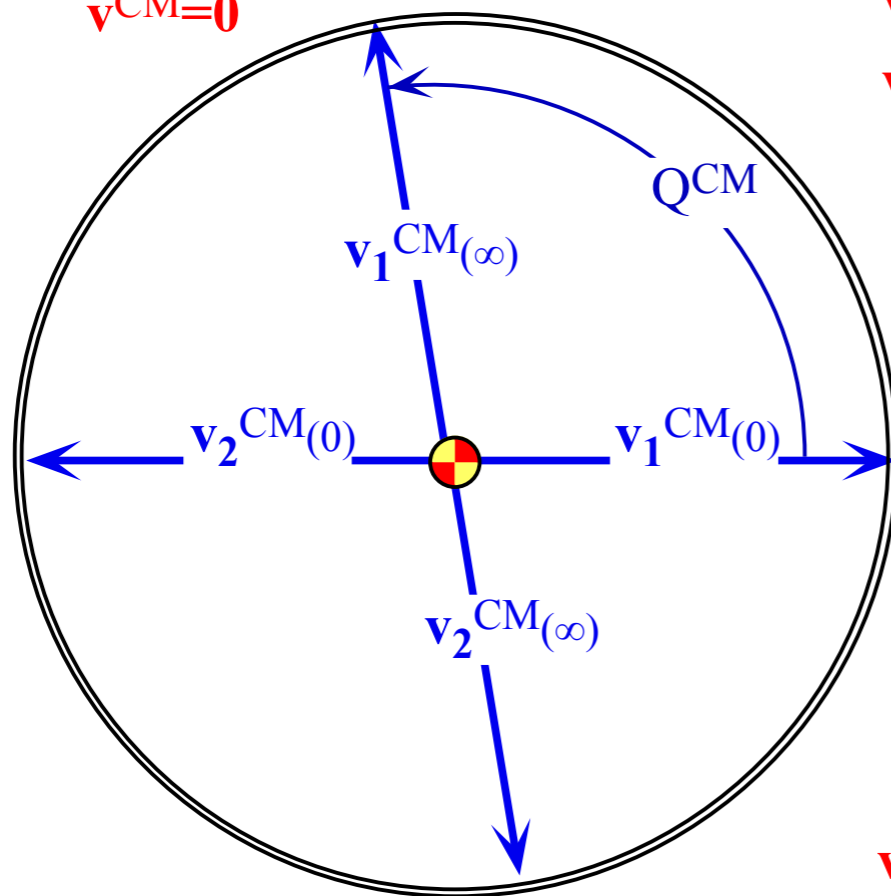
A common type of scattering

$(m_1=m_2)$

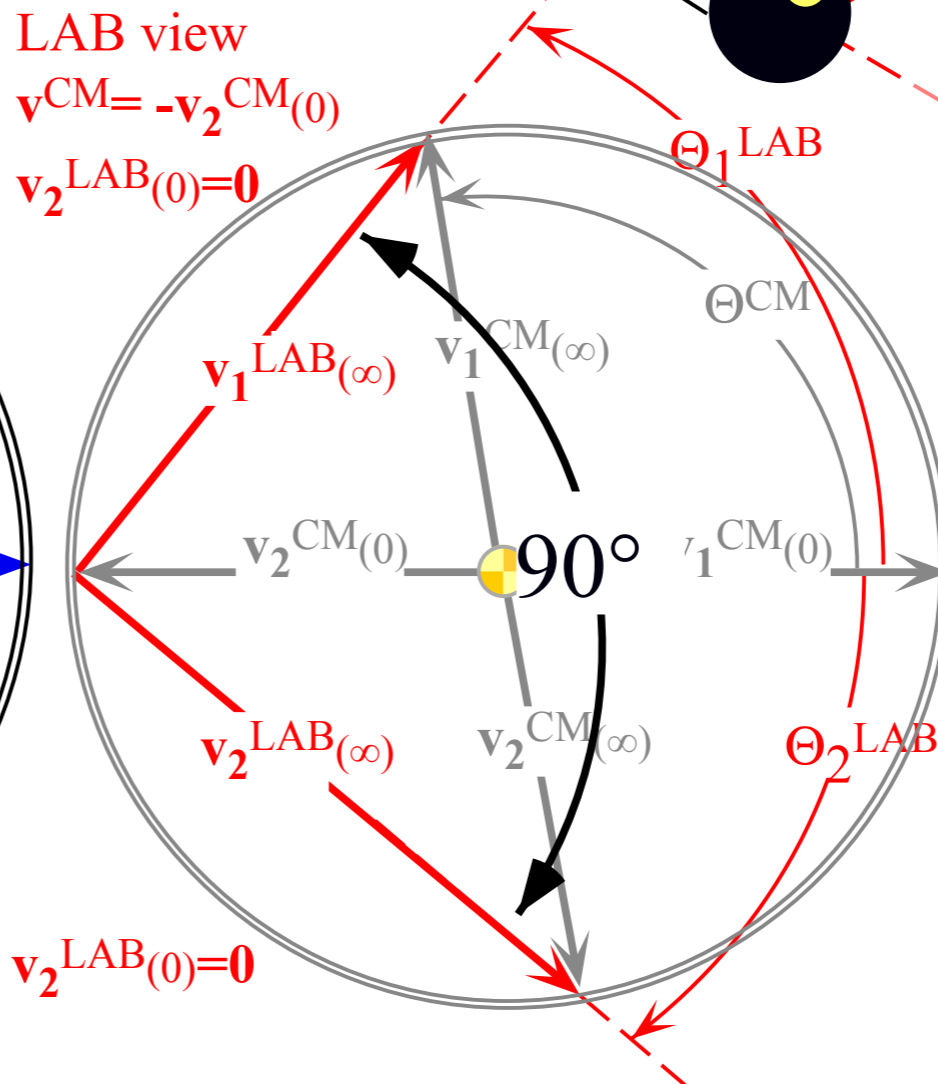
...that every pool shark should know



CM view
 $v^{CM}=0$

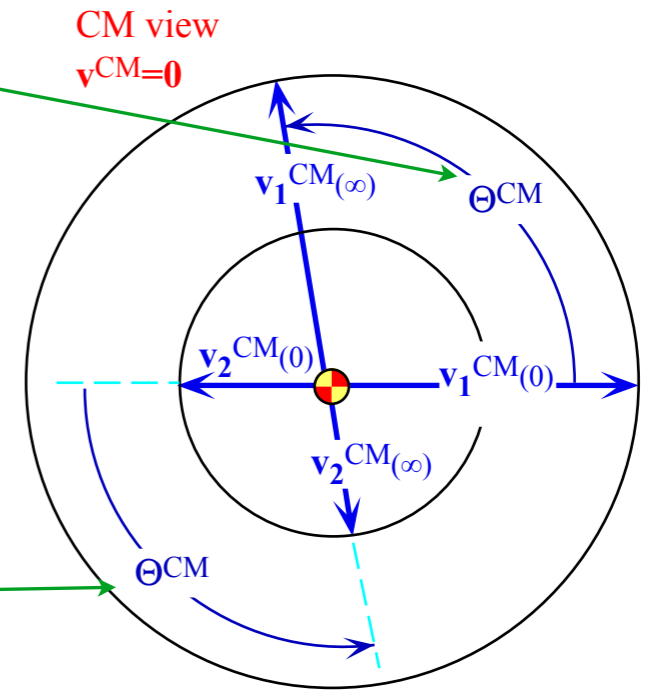
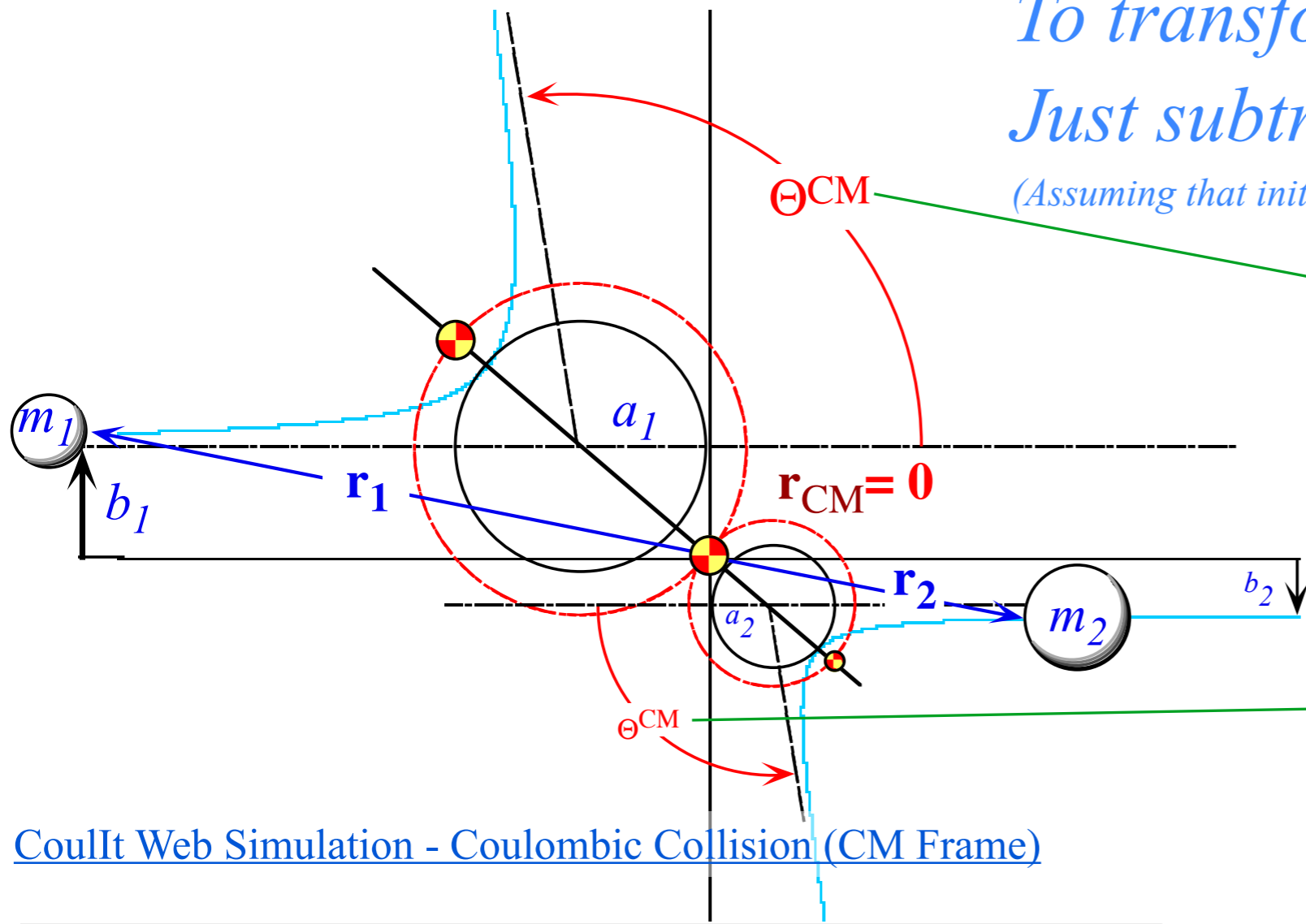


LAB view
 $v^{CM} = -v_2^{CM(0)}$
 $v_2^{LAB(0)} = 0$

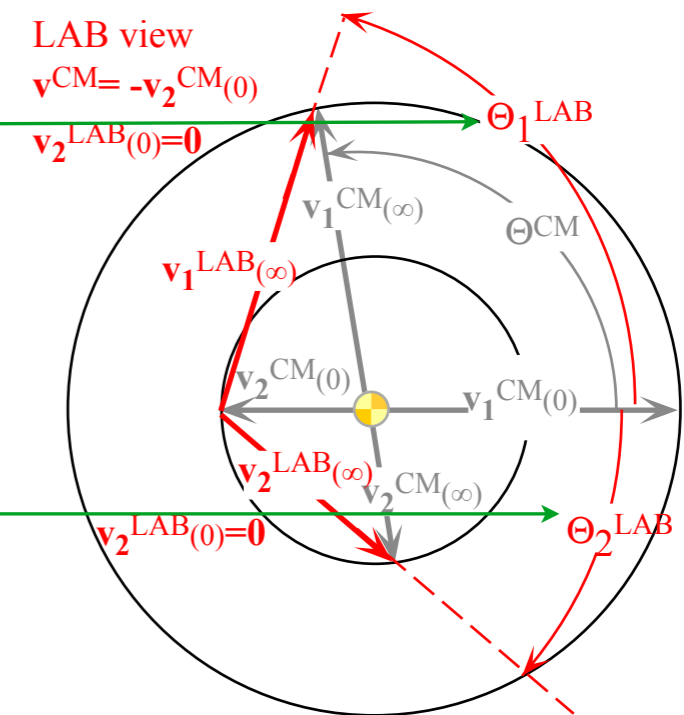
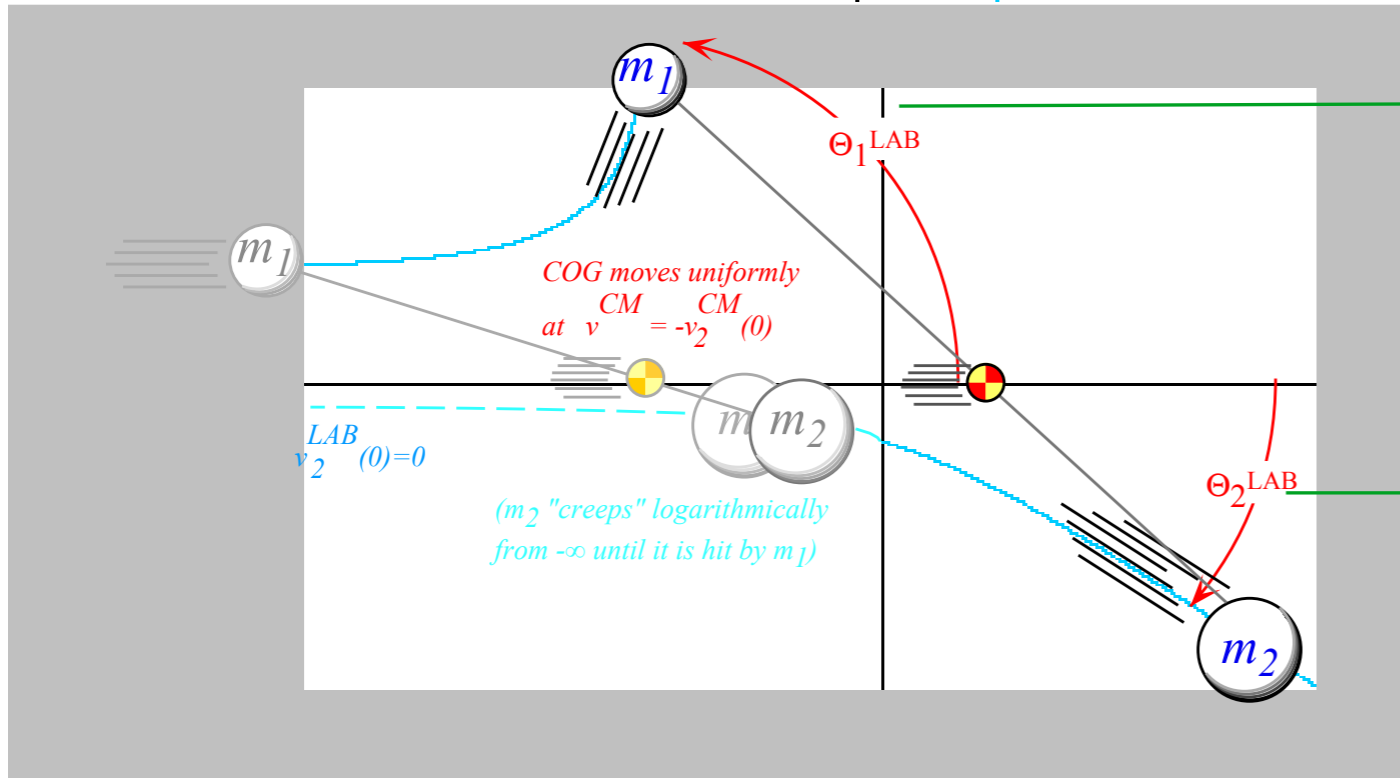


[BounceIt Web Simulations](#)
[Hard Collision \(CM Frame\)](#)
[Hard Collision \(Lab Frame\)](#)

To transform CM to LAB frame
 Just subtract $v_2^{CM}(0)$ from all
 (Assuming that initial $v_2^{LAB}(0)$ is zero so $v_2^{CM}(0)$ is CM velocity in LAB)



[CoulIt Web Simulation - Coulombic Collision \(CM Frame\)](#)



[CoulIt Web Simulation - Coulombic Collision \(LAB Frame\)](#)

Geometrical Aspects of Classical Coulomb Scattering

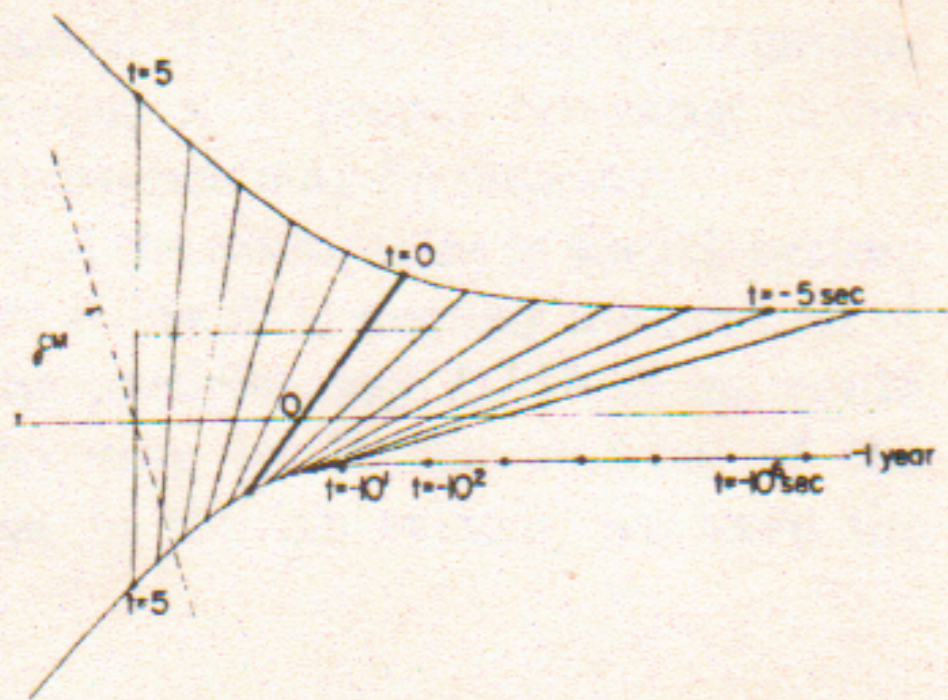


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

The trouble with the Coulomb field is...

$$\int t^{-1} dt = \ln t + C$$

$$\begin{aligned} v_2^{\text{LAB}}(t) &= \int (|F|/m_2) dt \\ &\cong \int k dt / m_2 [v_1^{\text{CM}}(\text{initial})t]^2 \\ &\cong [-k/m_2 v_1^{\text{CM}}(\text{initial})^2] t^{-1} \end{aligned}$$

1856 / December 1972

From: Geometric aspects of classical Coulomb scattering
American Journal of Physics 40, 1852-1856 (1972)
 Class project when I taught Jr. CM at Georgia Tech
 (Just 5 students)

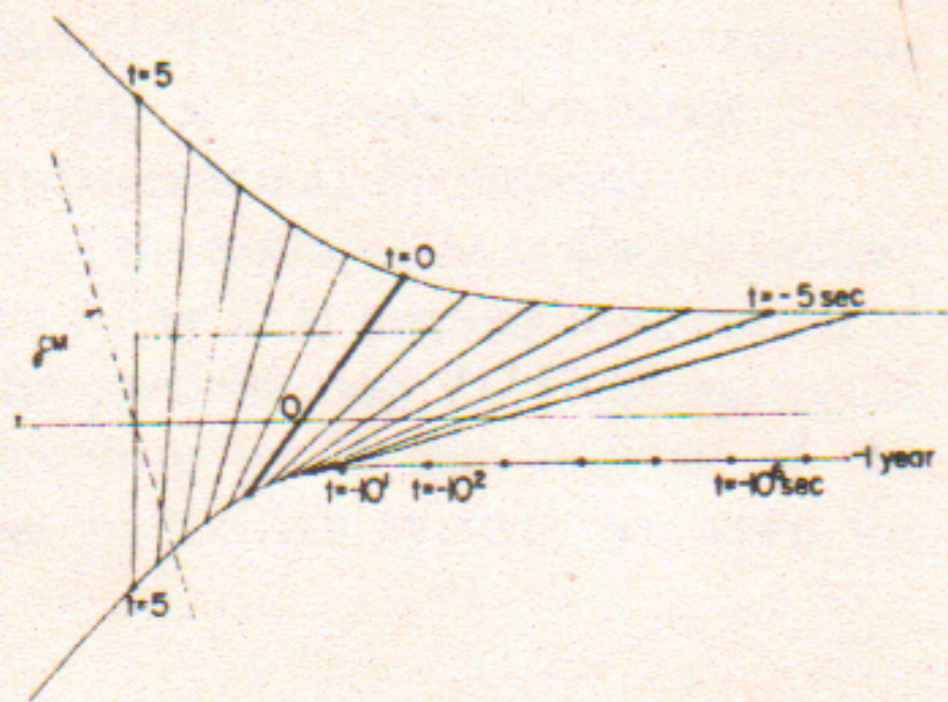


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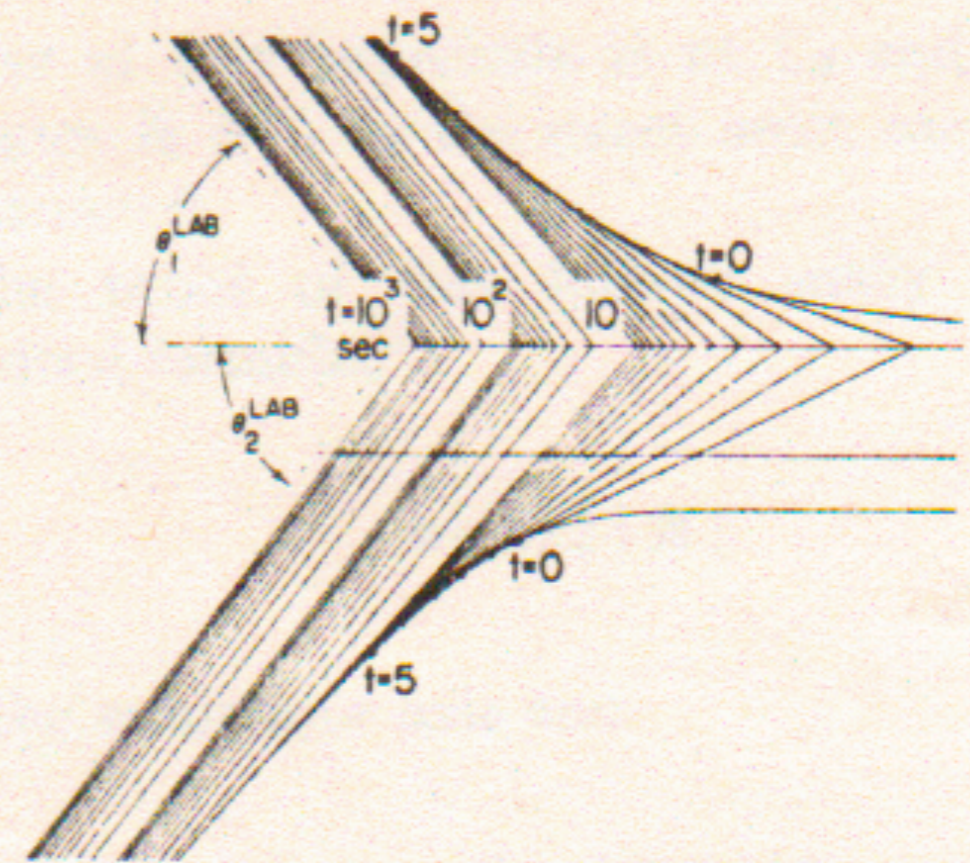


FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t = 10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04° , respectively.

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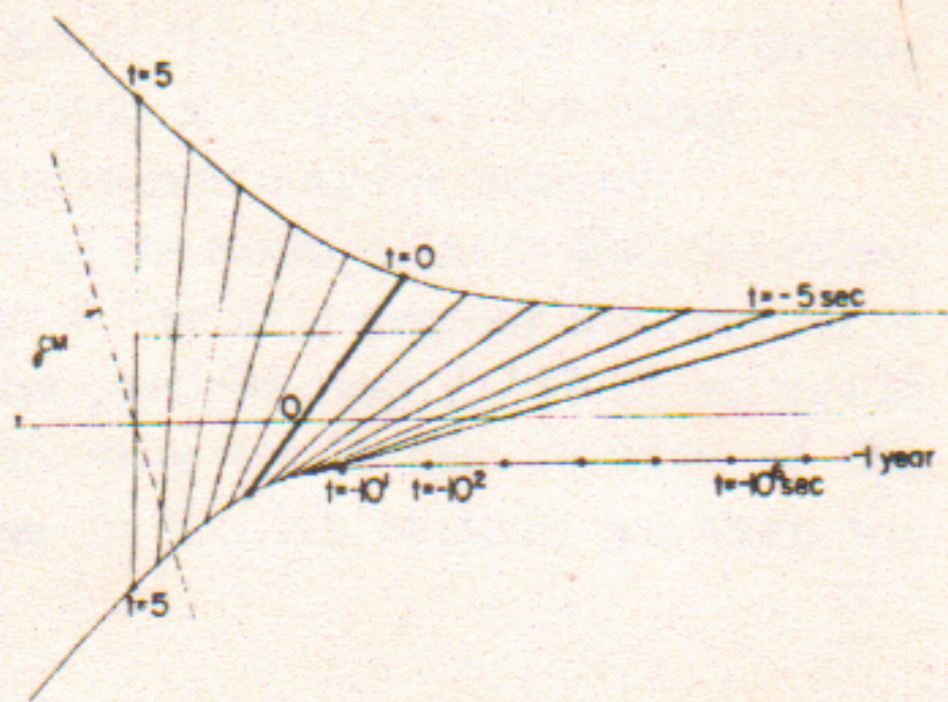


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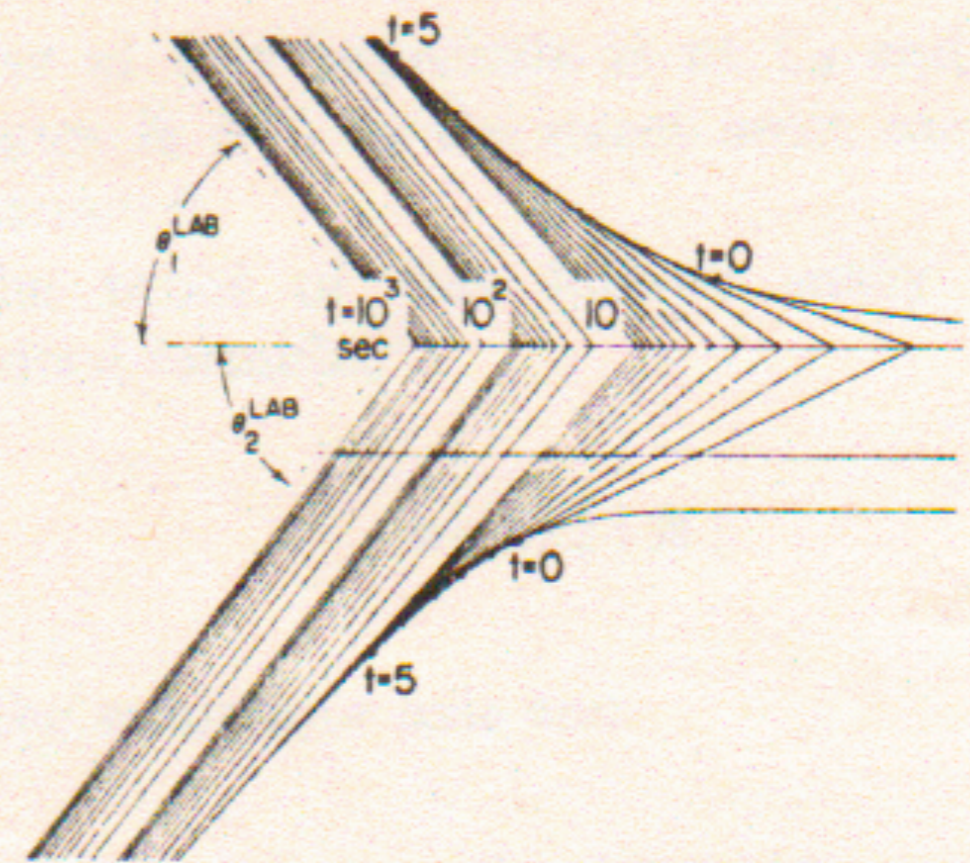


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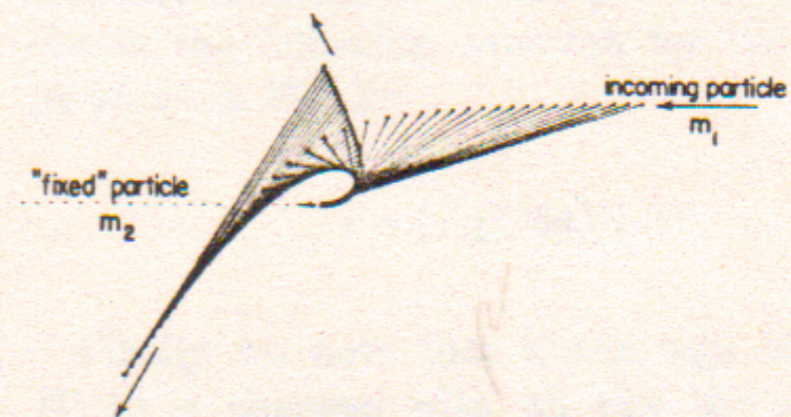


FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

➔ *Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$*
How to make my boomerang come back
The gyrocompass and mechanical spin analogy

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$

Angular momentum vector \mathbf{L}_j of a mass m_j is its linear momentum \mathbf{p}_j times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

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The sum-total angular momentum is

$$\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^3 \mathbf{L}_j = \sum_{j=1}^3 \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$$

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$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_{j=1}^3 \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{total} \\ &= \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^3 \mathbf{r}_j \times \left(\sum_{k=1(k \neq j)}^3 \mathbf{F}_{jk}^{constraint} \right) \end{aligned}$$

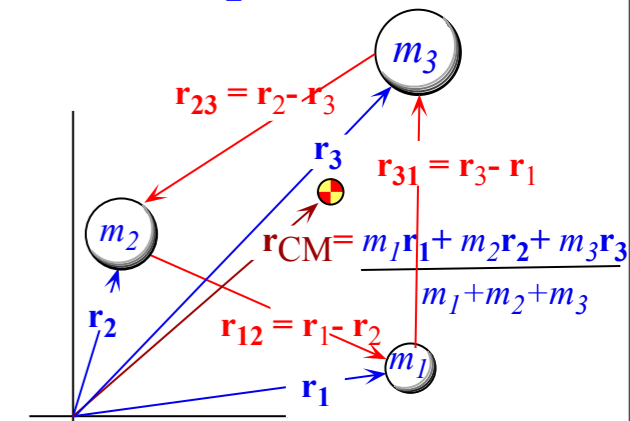


Fig. 6.4.1 Three-particle coordinate vectors

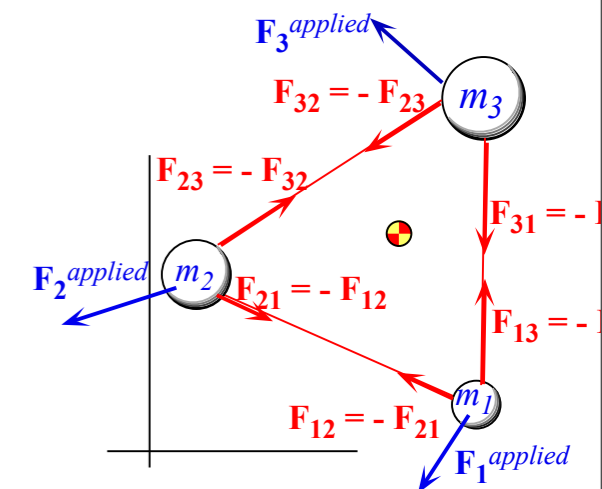


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

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Internal constraint or coupling force terms appear at first to be a nuisance.

$$\begin{aligned} \sum_{j=1}^3 \sum_{k=1(k \neq j)}^3 \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}) + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}) + \mathbf{r}_3 \times (\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}) \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12}^{constraint} + (\mathbf{r}_1 - \mathbf{r}_3) \times \mathbf{F}_{13}^{constraint} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23}^{constraint} = \mathbf{0} \end{aligned}$$

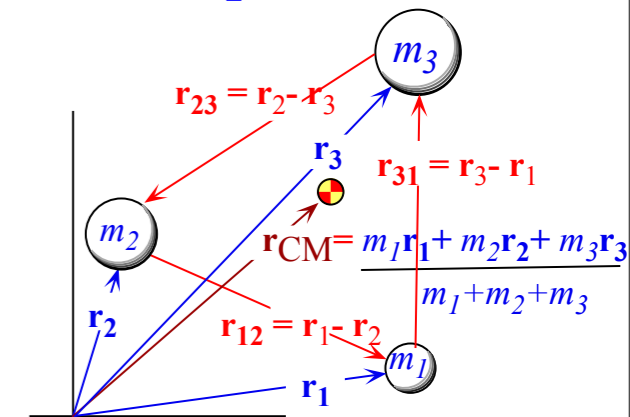


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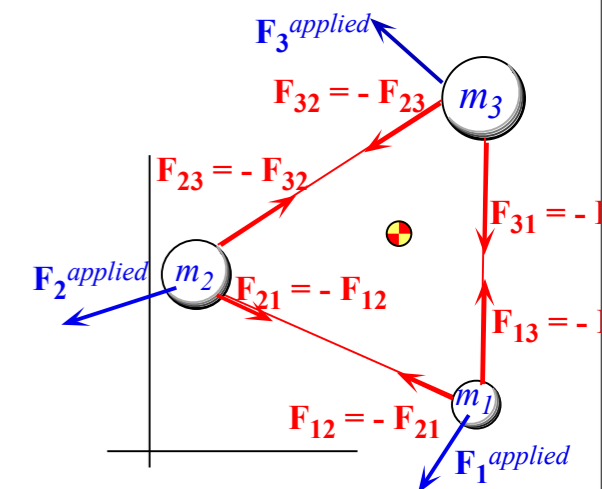


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However, they vanish if coupling forces act along lines connecting the masses.

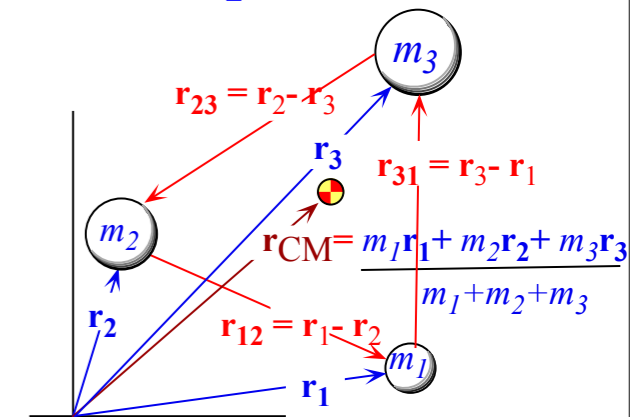


Fig. 6.4.1 Three-particle coordinate vectors

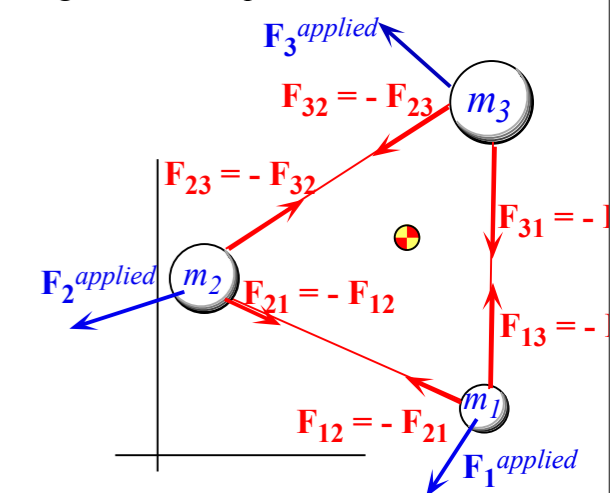


Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F} = d\mathbf{p}/dt$ equations: $\mathbf{N} = d\mathbf{L}/dt$

Angular momentum vector \mathbf{L}_j of a mass m_j is its linear momentum \mathbf{p}_j times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j = \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

The sum-total angular momentum is

$$\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^3 \mathbf{L}_j = \sum_{j=1}^3 \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$$

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$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_{j=1}^3 \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{total} \\ &= \sum_{j=1}^3 \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^3 \mathbf{r}_j \times \left(\sum_{k=1(k \neq j)}^3 \mathbf{F}_{jk}^{constraint} \right) \end{aligned}$$

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\begin{aligned} \sum_{j=1}^3 \sum_{k=1(k \neq j)}^3 \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}) + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}) + \mathbf{r}_3 \times (\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}) \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12}^{constraint} + (\mathbf{r}_1 - \mathbf{r}_3) \times \mathbf{F}_{13}^{constraint} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23}^{constraint} = \mathbf{0} \end{aligned}$$

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The results are the *rotational Newton's equation*.

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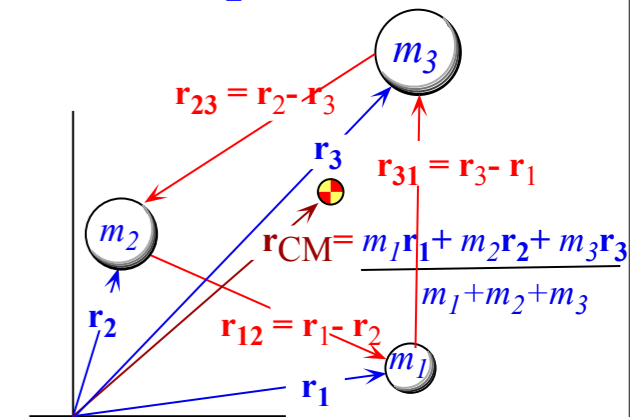


Fig. 6.4.1 Three-particle coordinate vectors

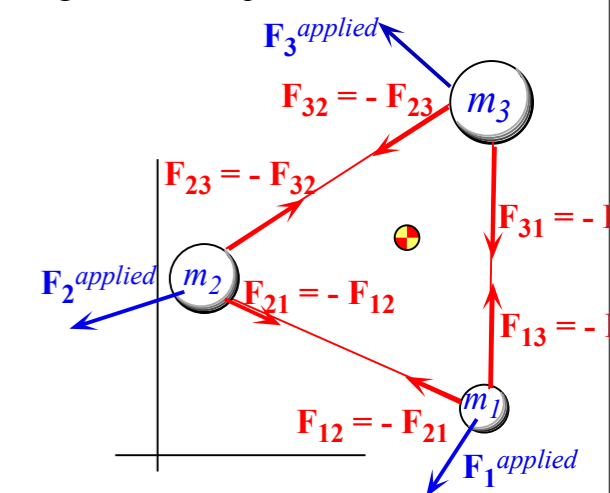


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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

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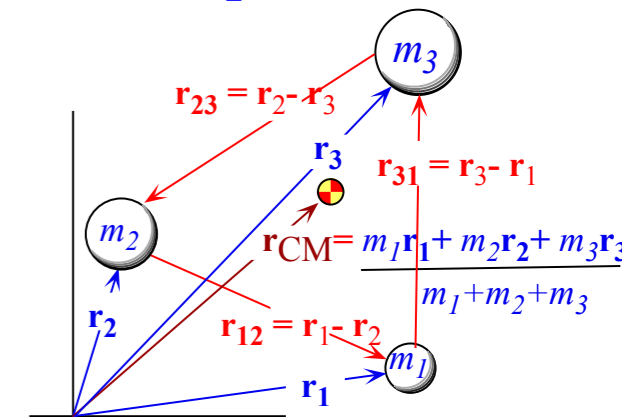


Fig. 6.4.1 Three-particle coordinate vectors

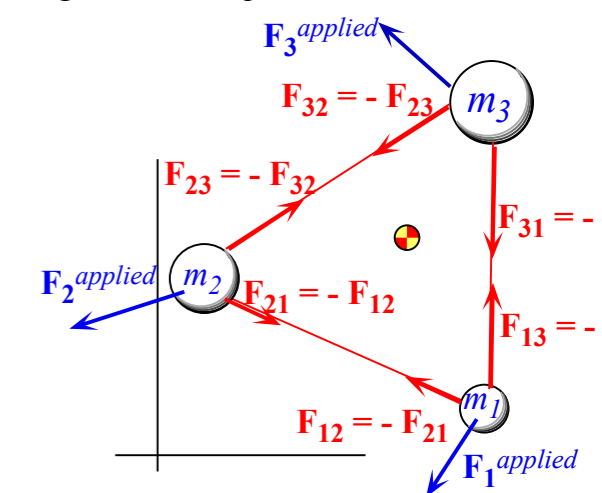


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$$\frac{d\mathbf{P}}{dt} = \mathbf{F}, \text{ where: } \mathbf{F} = \sum_{j=1}^3 \mathbf{F}_j^{applied}$$

Remaining $3N-6$ equations consist of normal mode or GCC equations of some kind.

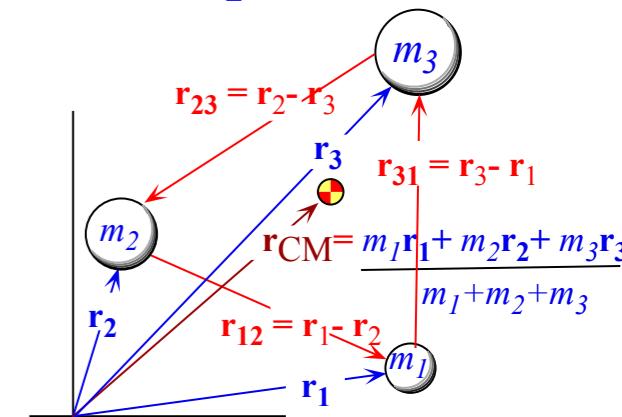


Fig. 6.4.1 Three-particle coordinate vectors

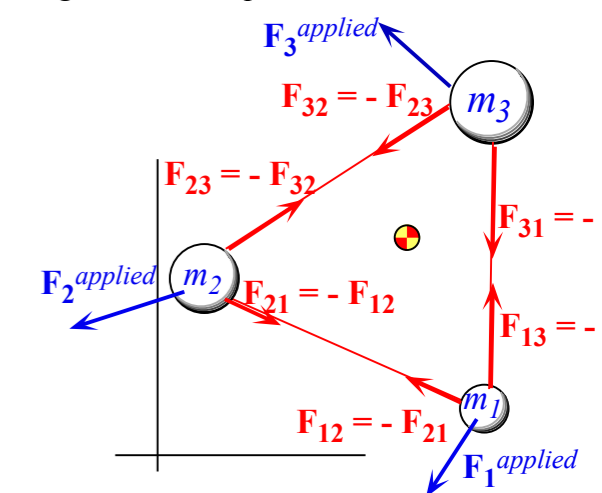


Fig. 6.4.2 Three-particle force vectors

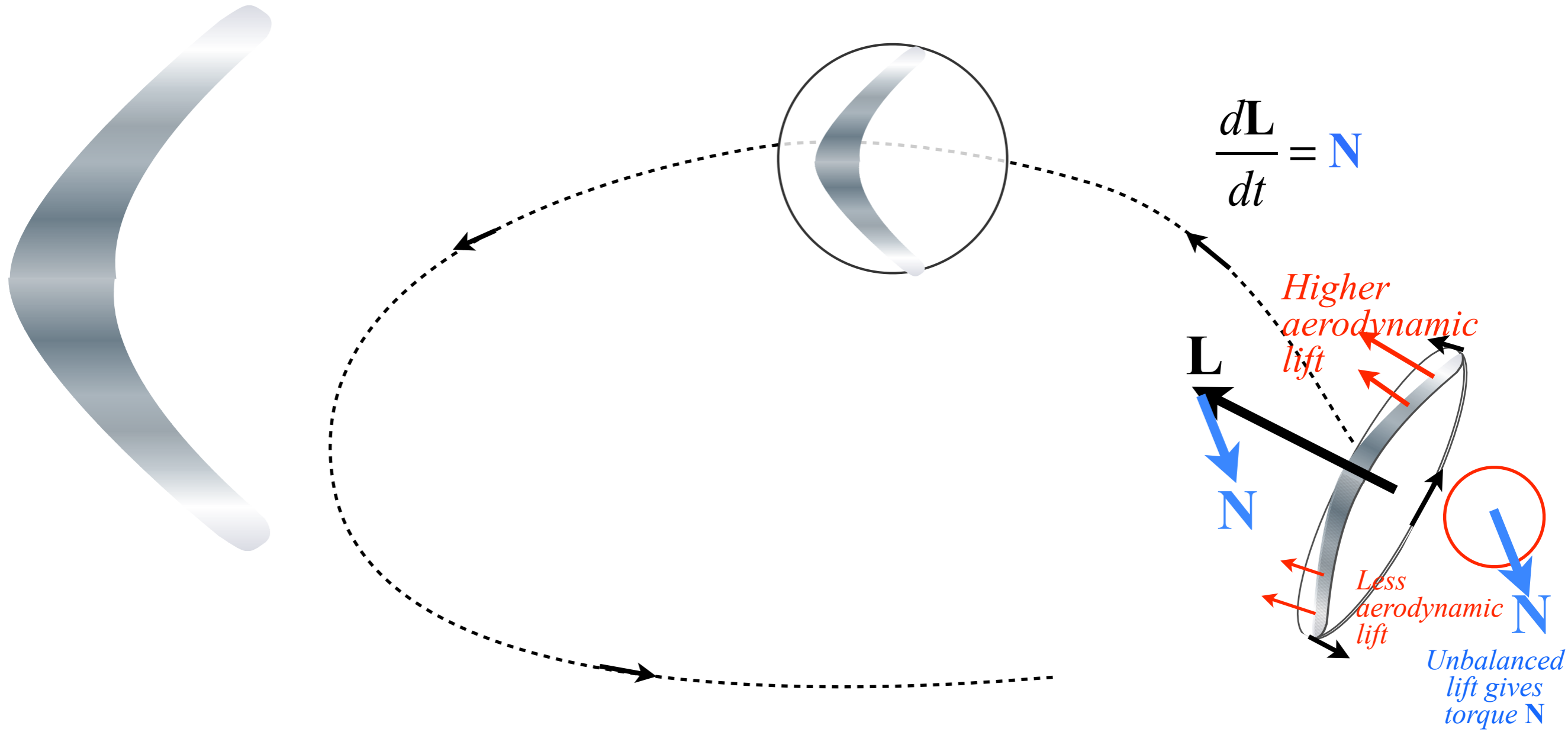
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$



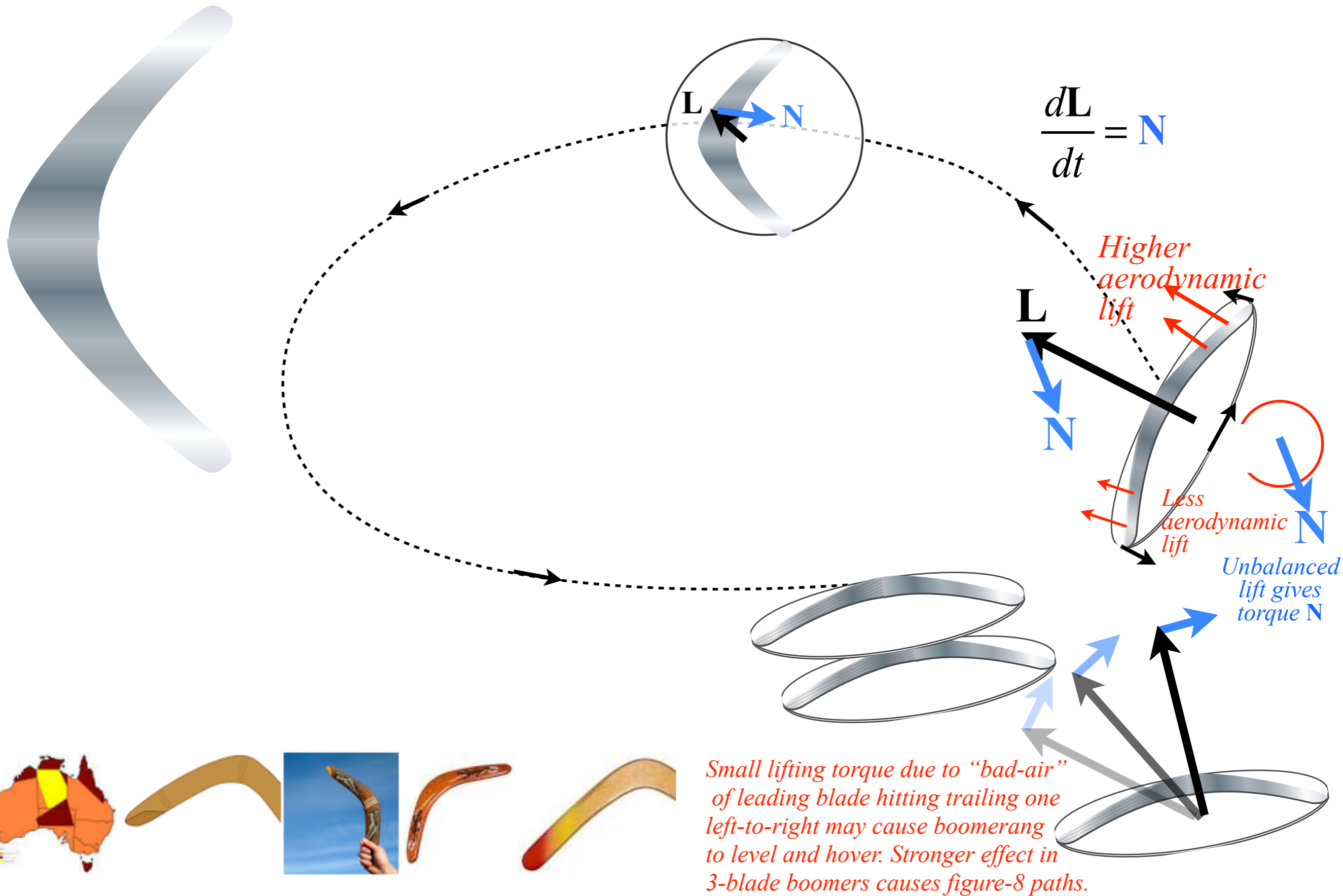
How to make my boomerang come back

The gyrocompass and mechanical spin analogy

The Australian Boomerang (that comes back!)



The Australian Boomerang (that comes back and hovers down!)



The Australian Boomerang (that comes back and hovers down!)

Charlie Drake's famous 1961 song:

My boomerang won't come back!

My boomerang won't come back!

My boomerang won't come back!

I've waved the thing all over the place

Practiced til' I was blue in the face*

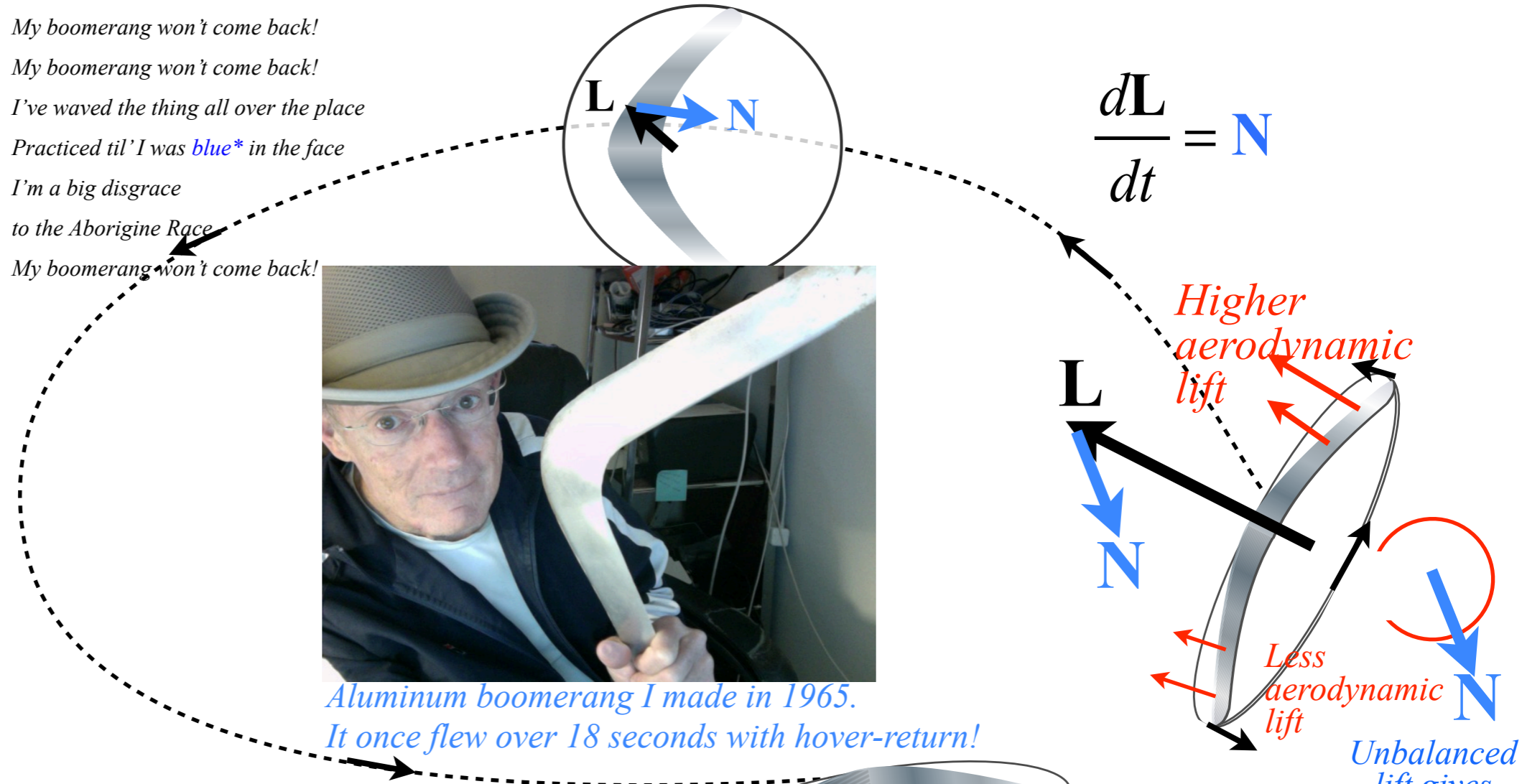
I'm a big disgrace

to the Aborigine Race

My boomerang won't come back!

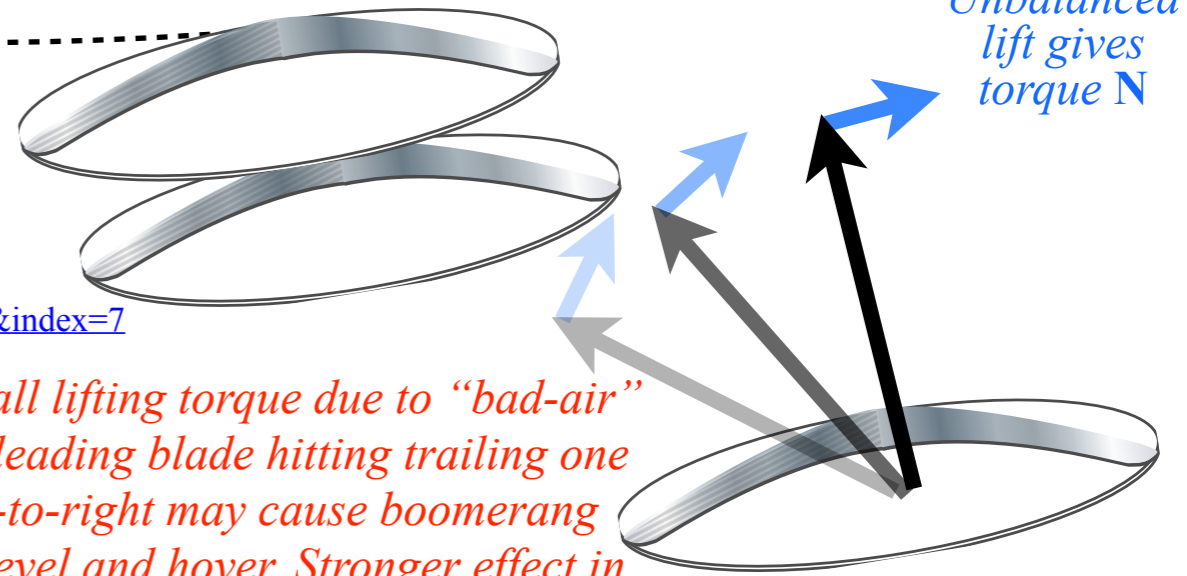


*blue later replaced black



Aluminum boomerang I made in 1965.


It once flew over 18 seconds with hover-return!



https://www.youtube.com/watch?v=EXJR5NWM_xI&list=PLGwmGldCxzLxbPIFVG8Z89WZIBuT4m0li&index=7

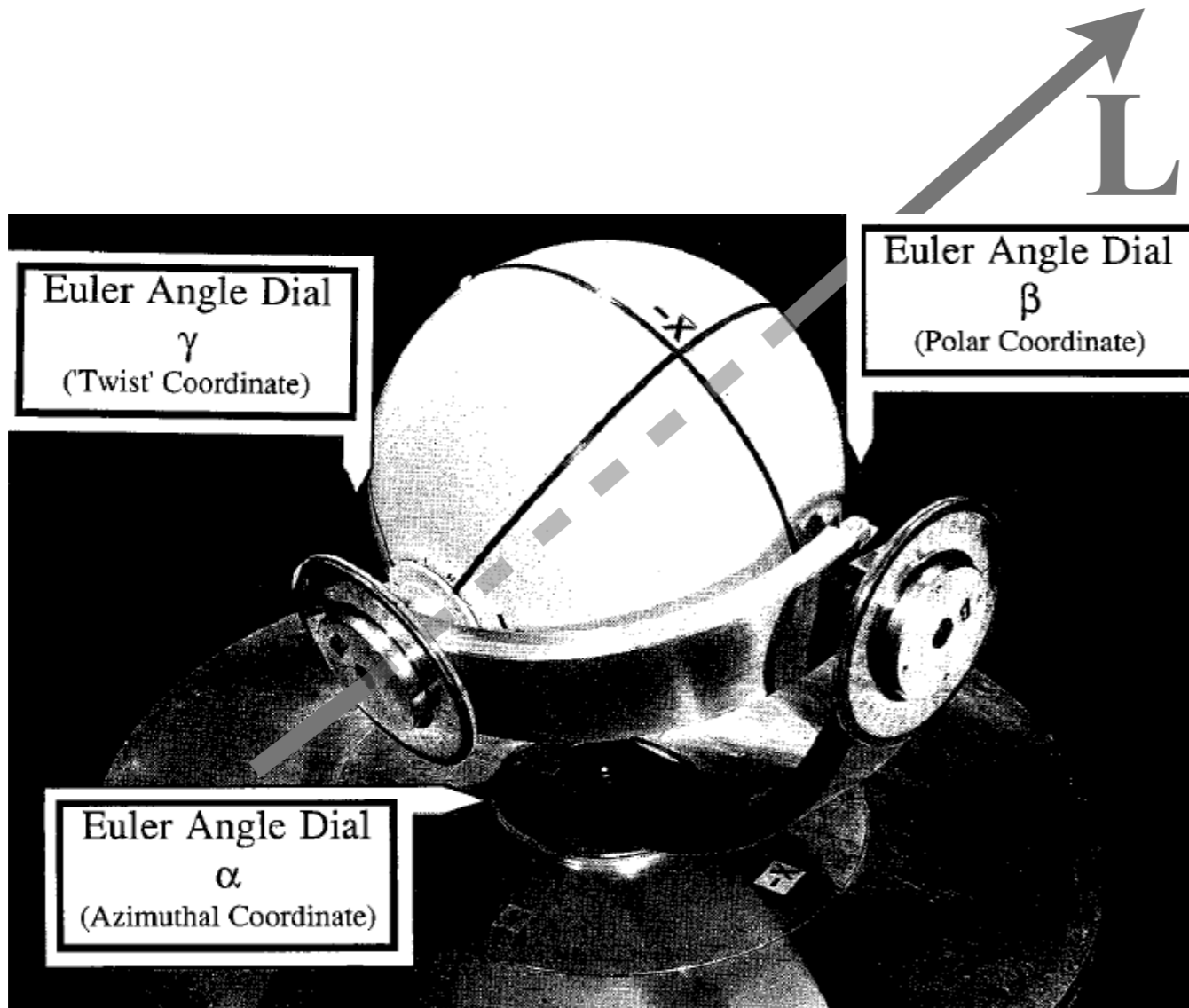


Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ equations: $\mathbf{N}=d\mathbf{L}/dt$
How to make my boomerang come back
 *The gyrocompass and mechanical spin analogy*

The gyrocompass and mechanical spin analogy

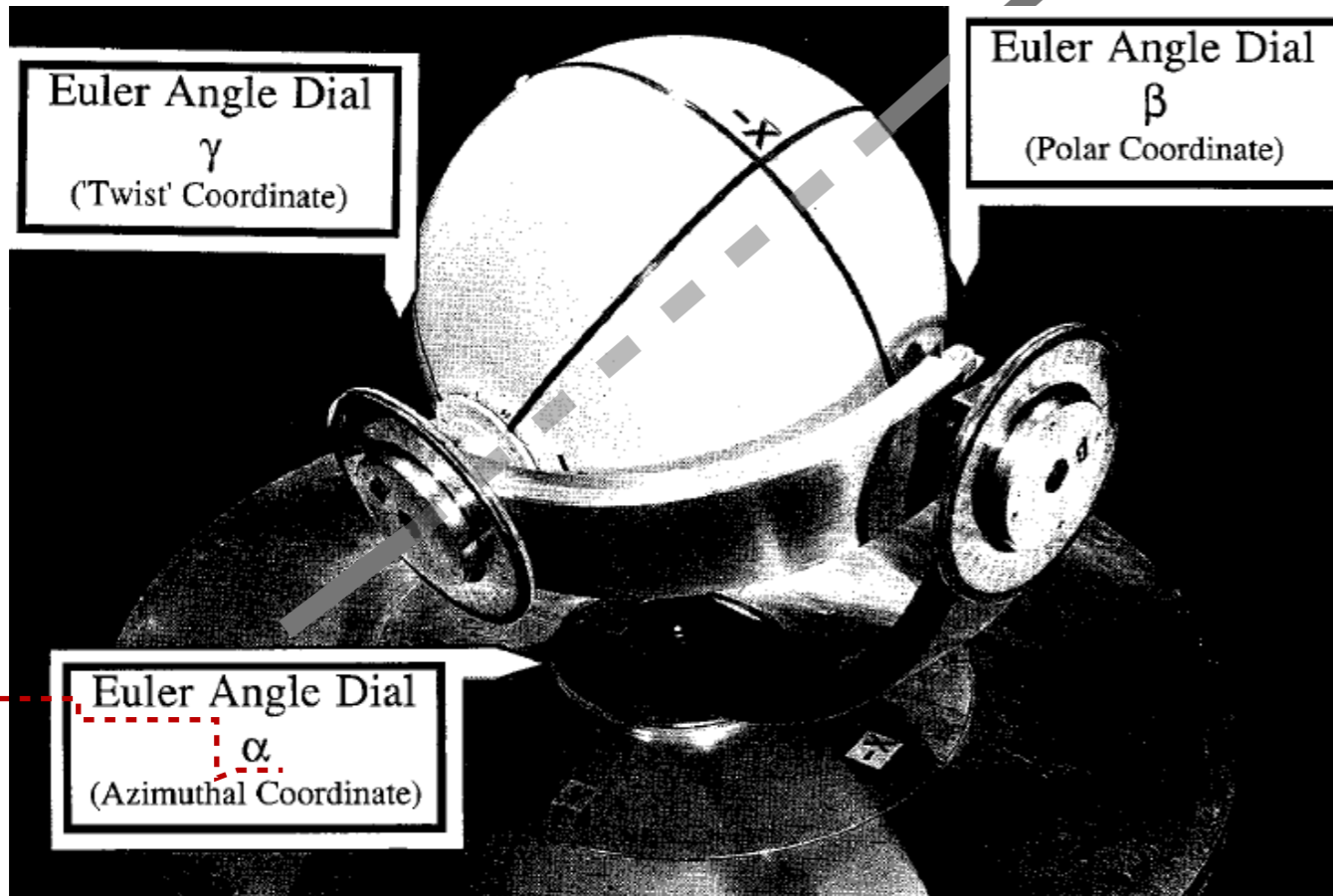
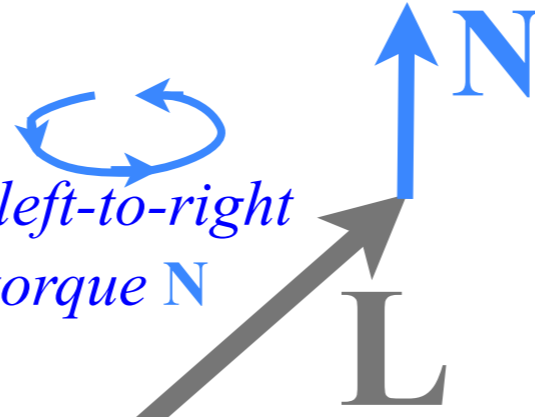
Suppose Euler ball has right-hand rotation with angular momentum \mathbf{L}



The gyrocompass and mechanical spin analogy

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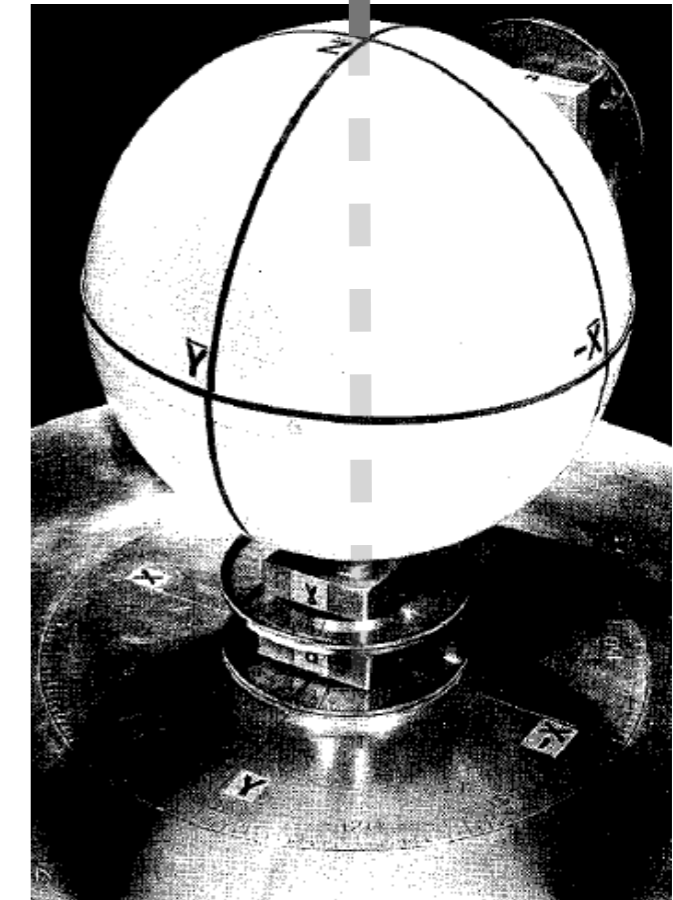
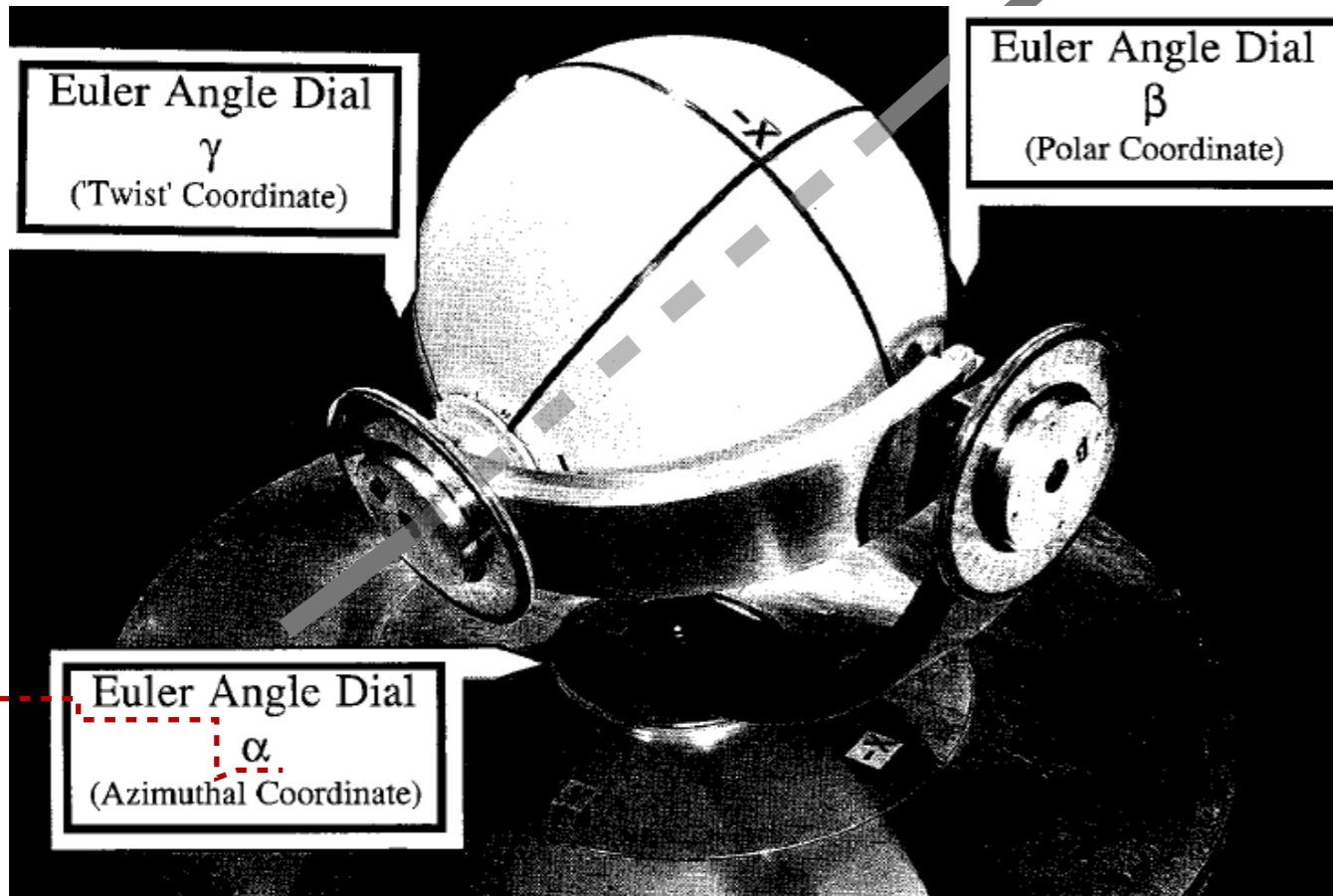
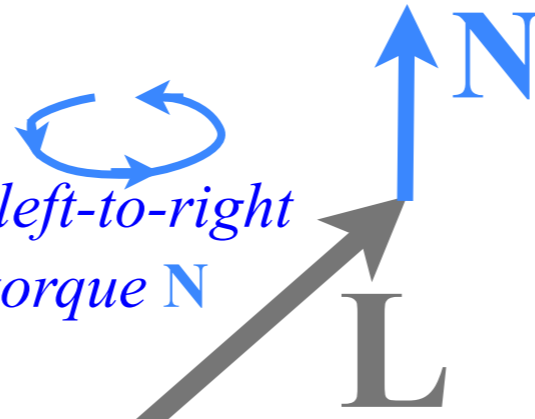
If the α -dial for z -rotation is turning left-to-right this applies righthand "thumbs-up" torque \mathbf{N}



The gyrocompass and mechanical spin analogy

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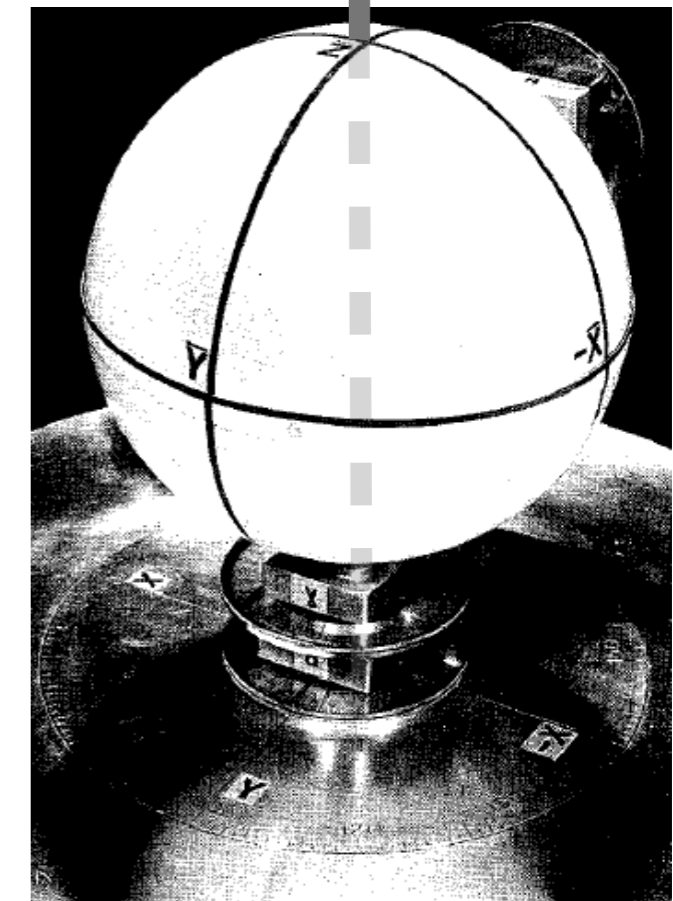
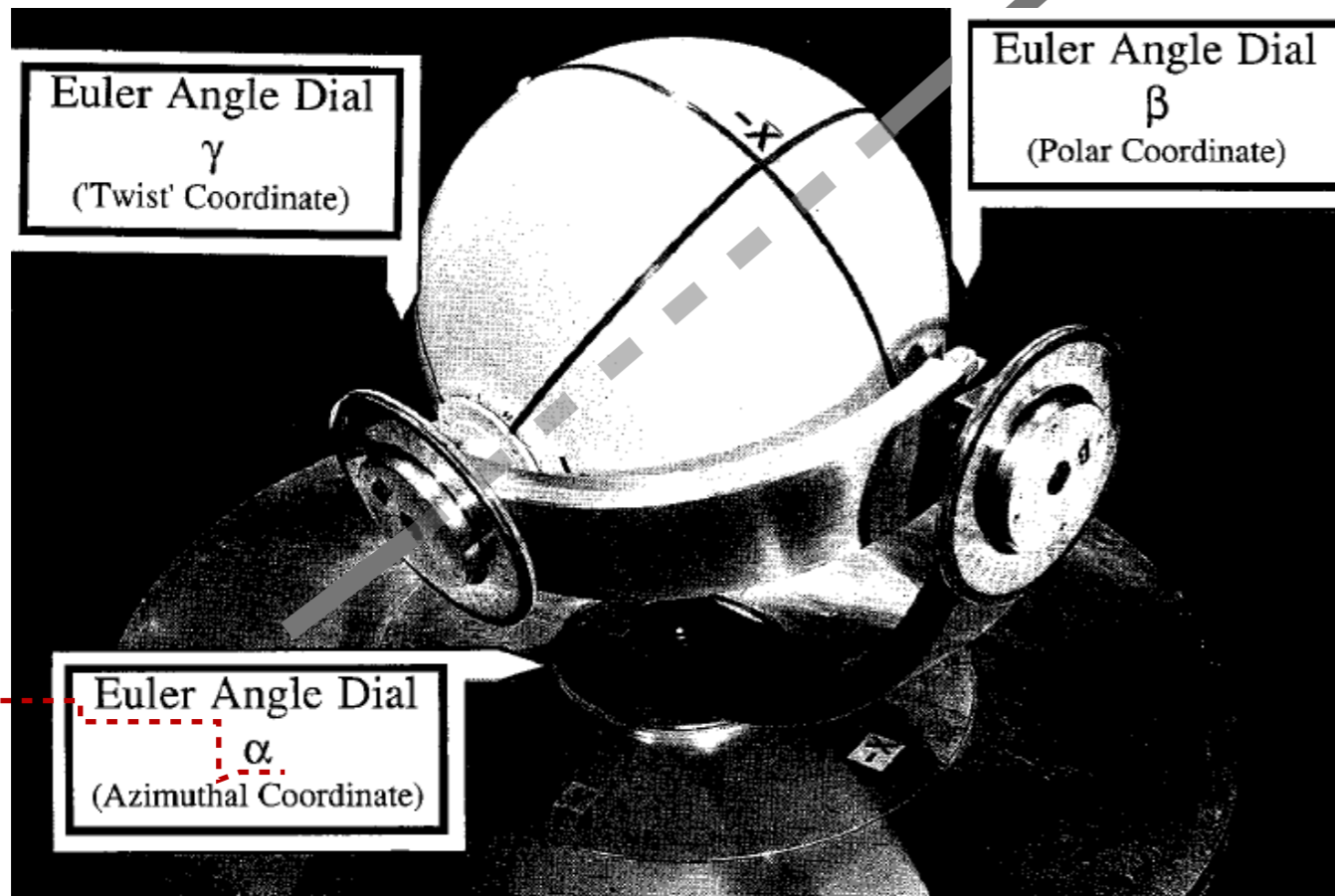
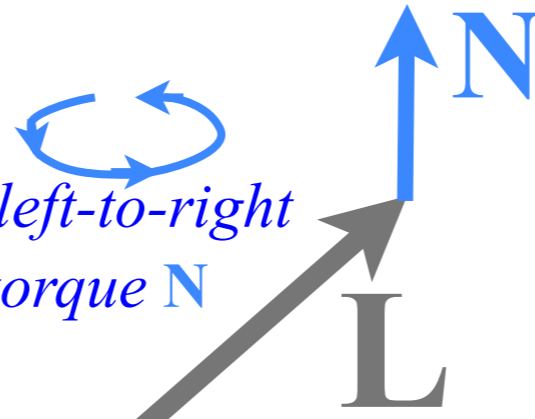


Then the ball tends to line-up with z -axis (and may go past z , then come back, etc. in a precessional or “hunting” motion)

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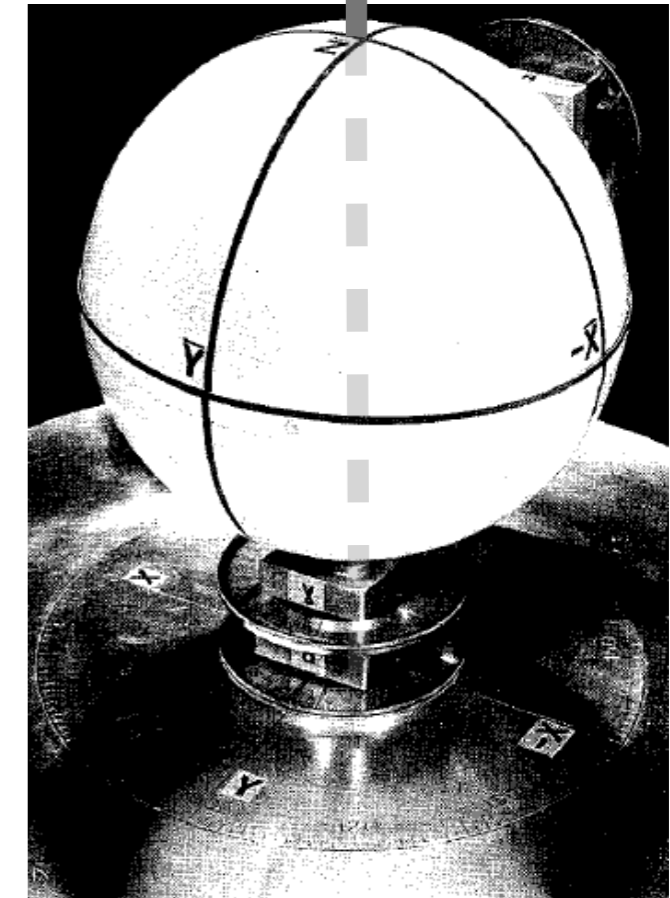
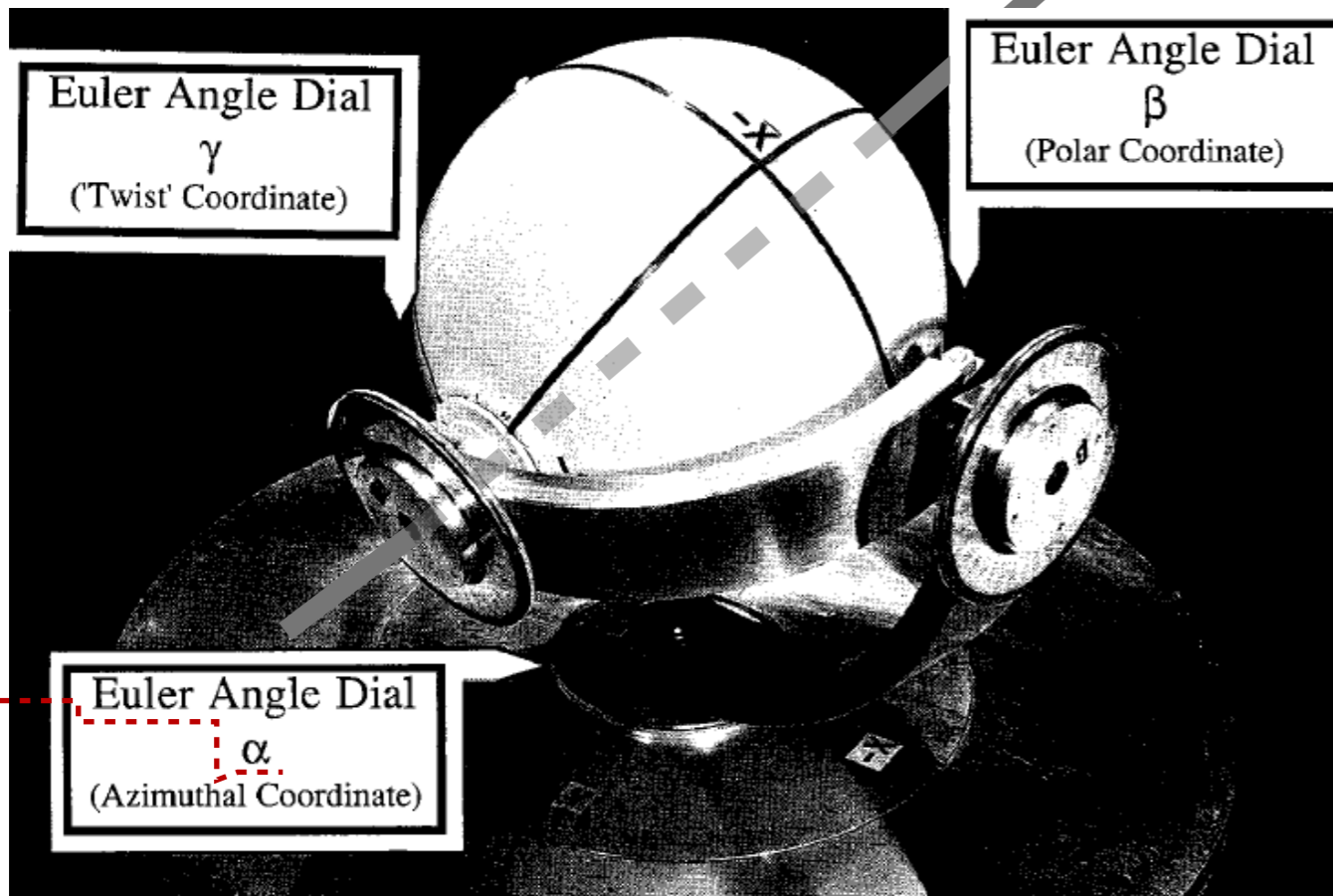
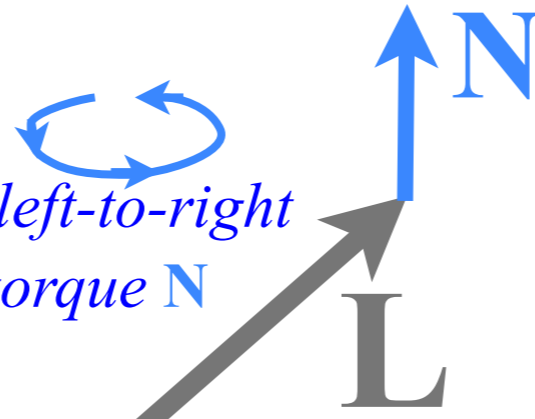
A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.

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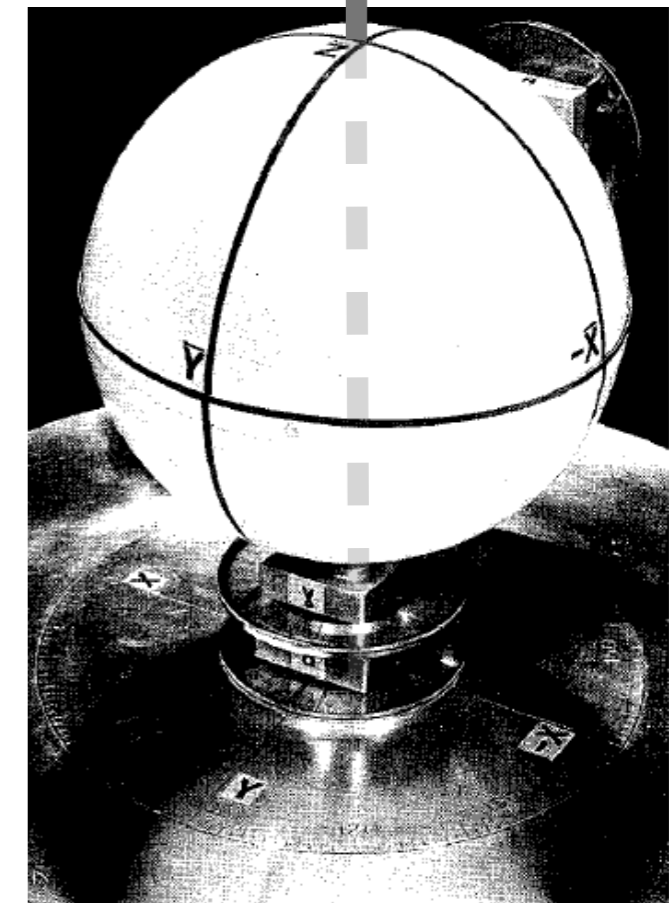
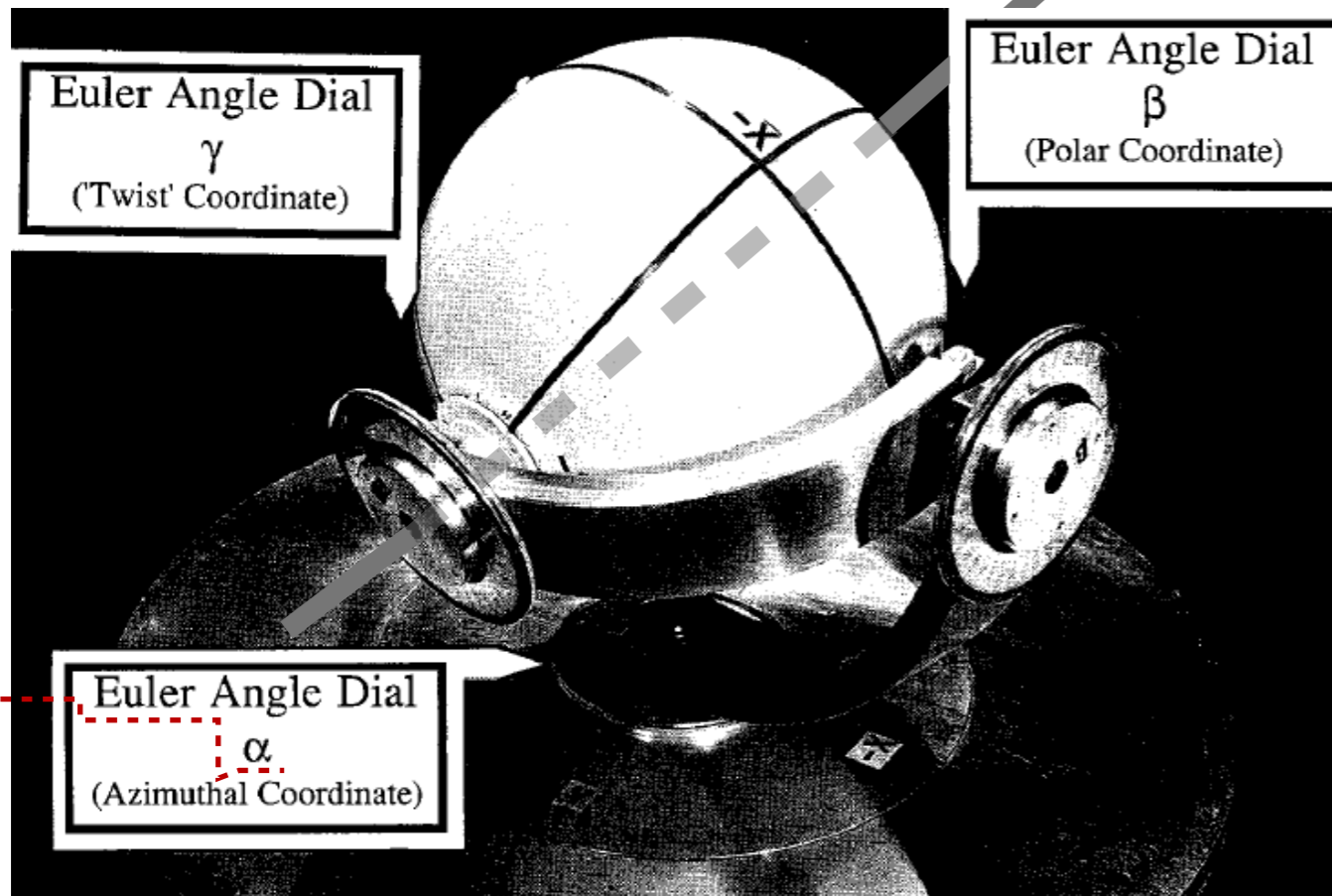
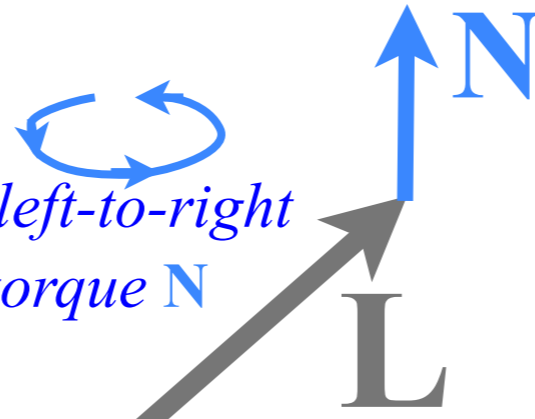
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This is analogous to the tendency for spin magnetic moments to align (or precess about) the B -direction of a magnetic field
Recall \mathbf{S} -precession discussion in CMwB Unit 4 Ch.4 and Lect.26

The gyrocompass and mechanical spin analogy

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General Rule: Gyros tend to “line-up” so they are rotating with whatever is most closely coupled to them.

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Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity ω -ellipsoid vs. angular momentum \mathbf{L} -ellipsoid

Lagrangian ω -equations vs. Hamiltonian momentum \mathbf{L} -equation

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^N \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^N m_j \mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \quad \text{with} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

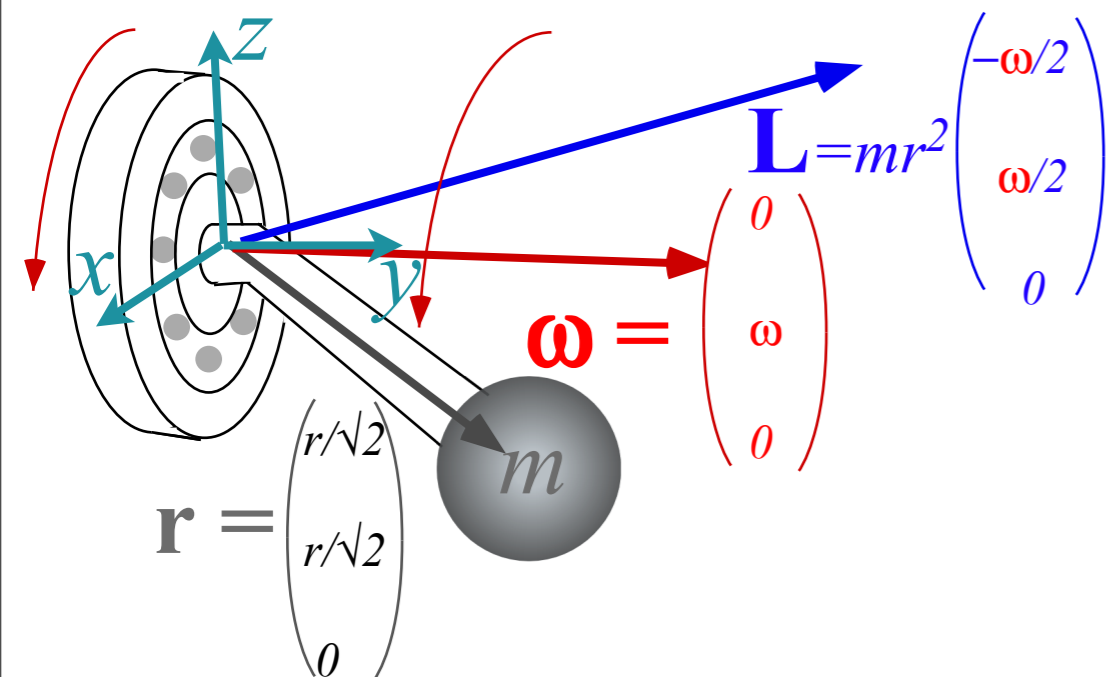


Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

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This produces the *rotational inertia tensor* \mathbf{I} :
$$\vec{\mathbf{I}} = \sum_{j=1}^N \vec{\mathbf{I}}_j = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right]$$

in the $\boldsymbol{\omega}$ -to- \mathbf{L} relation:
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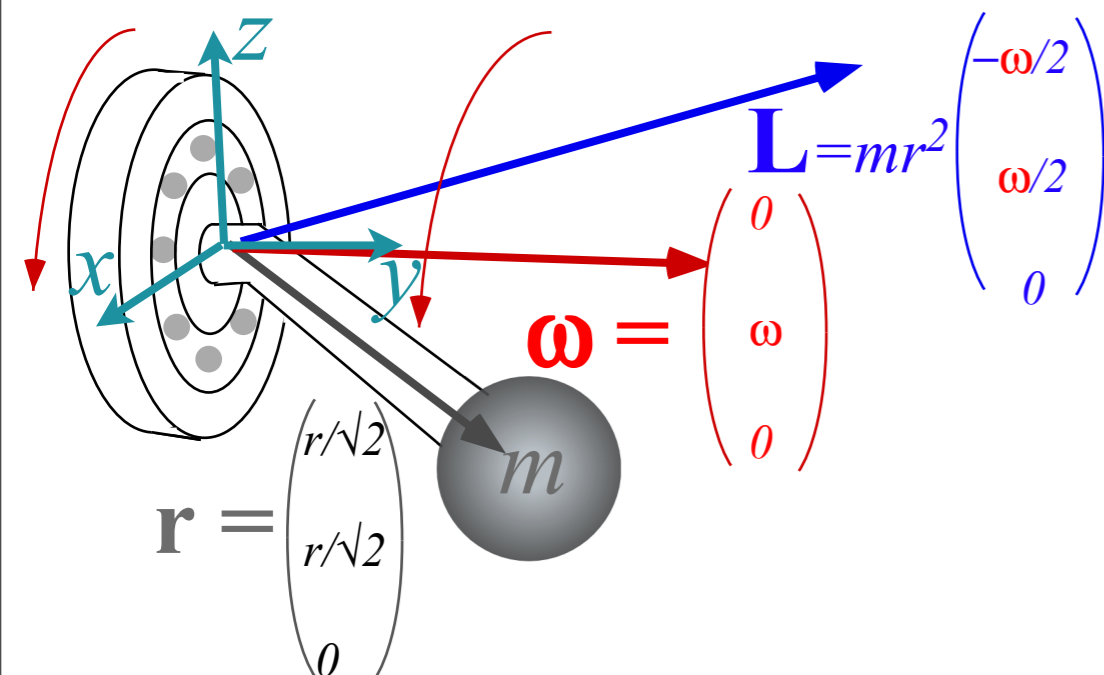


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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \langle \tilde{\mathbf{I}} \rangle = \sum_{j=1}^N \langle \tilde{\mathbf{I}}_j \rangle = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

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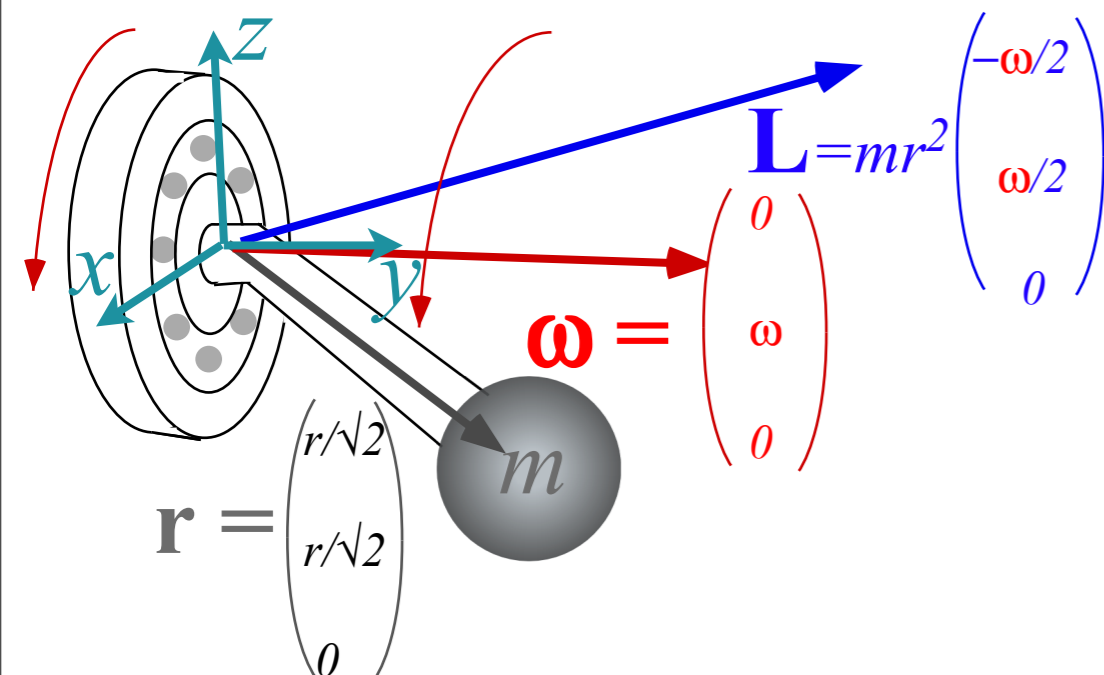


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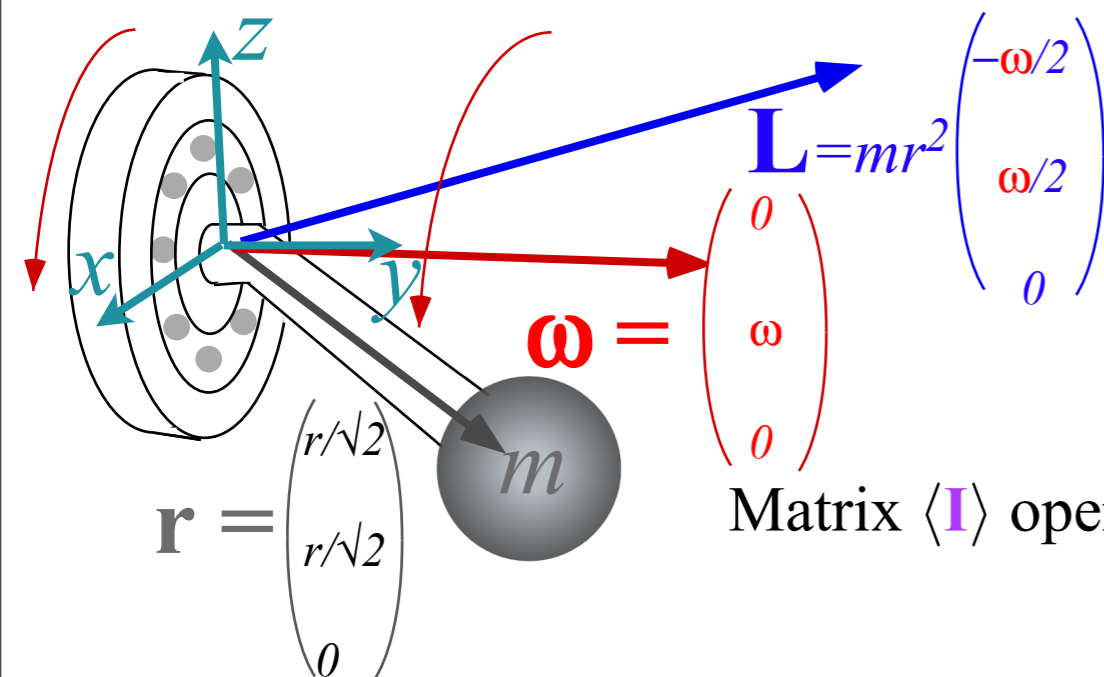
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$$\tilde{\mathbf{I}} = \sum_{j=1}^N \tilde{\mathbf{I}}_j = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right]$$

in the $\boldsymbol{\omega}$ -to- \mathbf{L} relation:
$$\mathbf{L} = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega}) \mathbf{r}_j \right] = \sum_{j=1}^N m_j \left[(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - \mathbf{r}_j \mathbf{r}_j \right] \cdot \boldsymbol{\omega} = \tilde{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \langle \tilde{\mathbf{I}} \rangle = \sum_{j=1}^N \langle \tilde{\mathbf{I}}_j \rangle = \sum_{j=1}^N m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle \mathbf{I} \rangle$ of inertia is:

$$\langle \tilde{\mathbf{I}} \rangle = mr^2 \begin{pmatrix} (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})(1/\sqrt{2}) & -(1/\sqrt{2})0 \\ -(1/\sqrt{2})(1/\sqrt{2}) & (1/\sqrt{2})^2 + 0 & -(1/\sqrt{2})0 \\ -0(1/\sqrt{2}) & -0(1/\sqrt{2}) & (1/\sqrt{2})^2 + (1/\sqrt{2})^2 \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum \mathbf{L}

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^N \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^N m_j \mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \quad \text{with} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

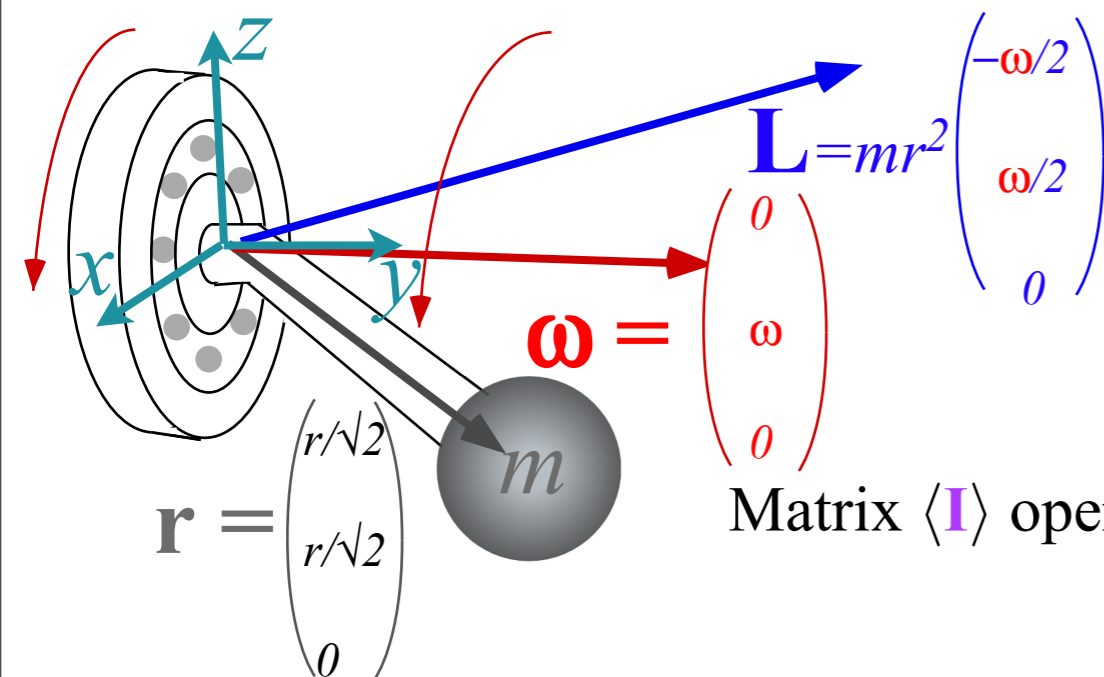
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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

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Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum \mathbf{L}

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N -body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* \mathbf{L} relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \quad \text{and} \quad \mathbf{L} = \sum_{j=1}^N \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^N m_j \mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \quad \text{with} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

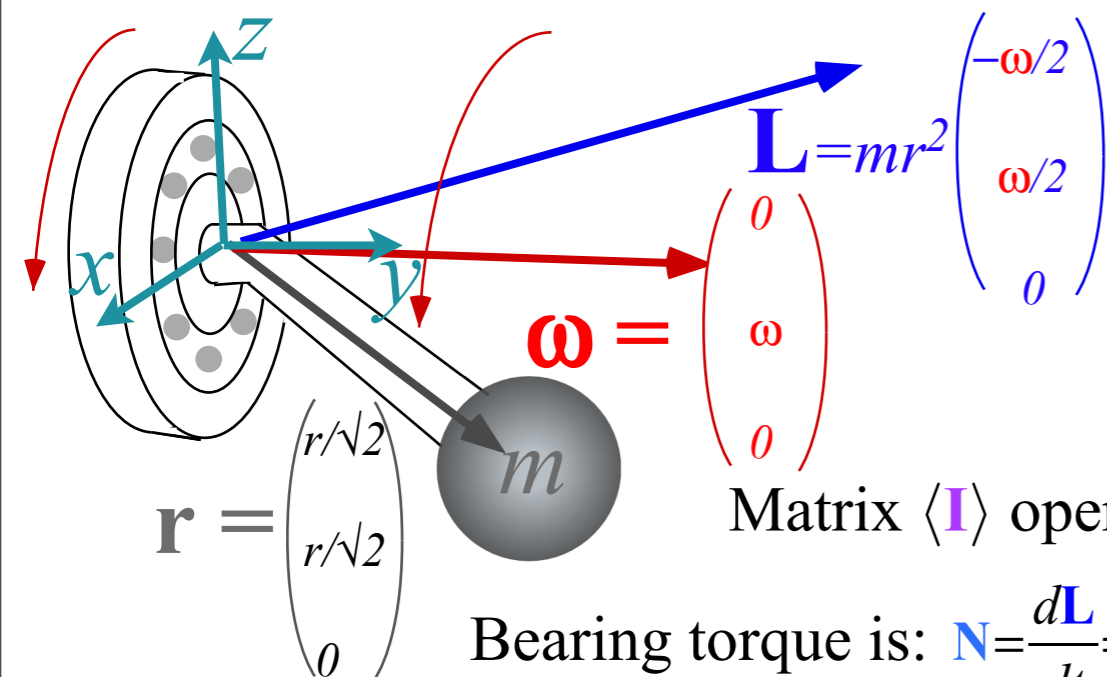
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Matrix form of the $\boldsymbol{\omega}$ -to- \mathbf{L} relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

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Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



Instantaneous matrix $\langle \mathbf{I} \rangle$ of inertia is:

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Matrix $\langle \mathbf{I} \rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum \mathbf{L}

Bearing torque is: $\mathbf{N} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2 \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2 \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix \mathbf{I}

$$T = \frac{1}{2} \sum_{j=1}^3 m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j = \frac{1}{2} \sum_{j=1}^3 m_j (\boldsymbol{\omega} \times \mathbf{r}_j) \cdot (\boldsymbol{\omega} \times \mathbf{r}_j)$$

Levi-Civita identity
 $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

$$T = \frac{1}{2} \sum_{j=1}^3 m_j [(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_j \cdot \mathbf{r}_j) - (\boldsymbol{\omega} \cdot \mathbf{r}_j)(\mathbf{r}_j \cdot \boldsymbol{\omega})]$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \sum_{j=1}^3 m_j [(\mathbf{r}_j \cdot \mathbf{r}_j) \mathbf{1} - (\mathbf{r}_j)(\mathbf{r}_j)] \cdot \boldsymbol{\omega}$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \bar{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T = \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \langle \omega | x \rangle & \langle \omega | y \rangle & \langle \omega | z \rangle \end{pmatrix} \begin{pmatrix} \langle x | \mathbf{I} | x \rangle & \langle x | \mathbf{I} | y \rangle & \langle x | \mathbf{I} | z \rangle \\ \langle y | \mathbf{I} | x \rangle & \langle y | \mathbf{I} | y \rangle & \langle y | \mathbf{I} | z \rangle \\ \langle z | \mathbf{I} | x \rangle & \langle z | \mathbf{I} | y \rangle & \langle z | \mathbf{I} | z \rangle \end{pmatrix} \begin{pmatrix} \langle x | \omega \rangle \\ \langle y | \omega \rangle \\ \langle z | \omega \rangle \end{pmatrix} \quad (\text{Dirac notation})$$

$$= \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \sum_{j=1}^3 m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Simplifies in *principle inertial axes* $\{X, Y, Z\}$ or *body eigen-axes*

$$T = \frac{1}{2} \begin{pmatrix} \omega_X & \omega_Y & \omega_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix} \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \omega_X & \omega_Y & \omega_Z \end{pmatrix} \begin{pmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{pmatrix} \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix} = \frac{I_{XX} \omega_X^2}{2} + \frac{I_{YY} \omega_Y^2}{2} + \frac{I_{ZZ} \omega_Z^2}{2}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

$$\mathbf{L} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega}, \quad \text{generally implies:} \quad \boldsymbol{\omega} = \vec{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum \mathbf{L} , or both at once. once

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \vec{\mathbf{I}} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{L} \cdot \vec{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

$$\begin{aligned} T &= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}} \end{aligned}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

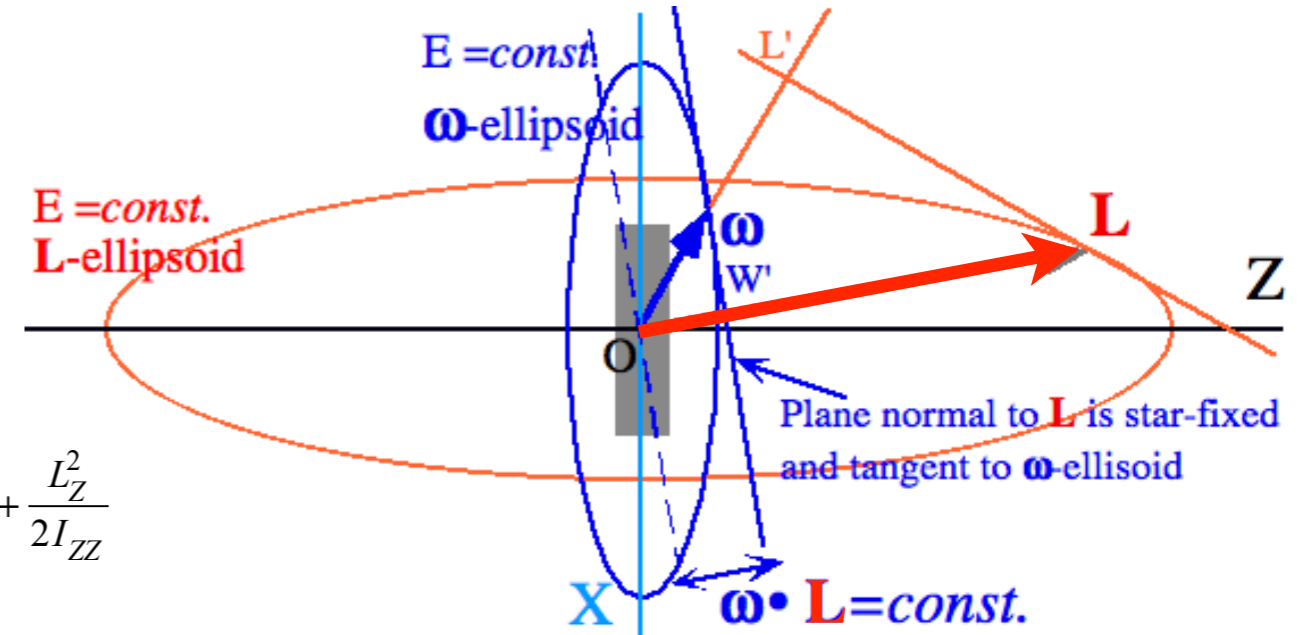
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Hamiltonian form is the equation of the *angular momentum or L-ellipsoid*
 Lagrangian form is the equation of the *angular velocity or omega-ellipsoid*

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

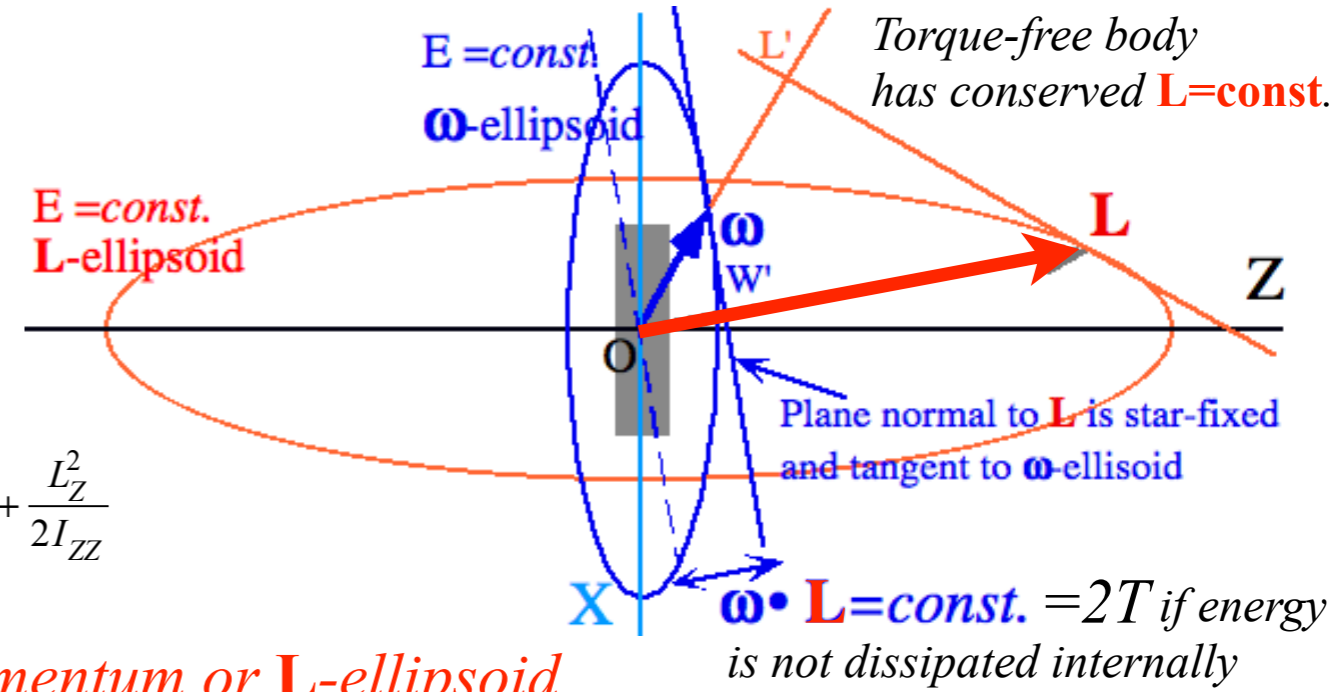
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Lagrangian form is the equation of the *angular velocity or ω-ellipsoid* $\boldsymbol{\omega}$ is generally not conserved unless it is aligned to \mathbf{L} or body has symmetry

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

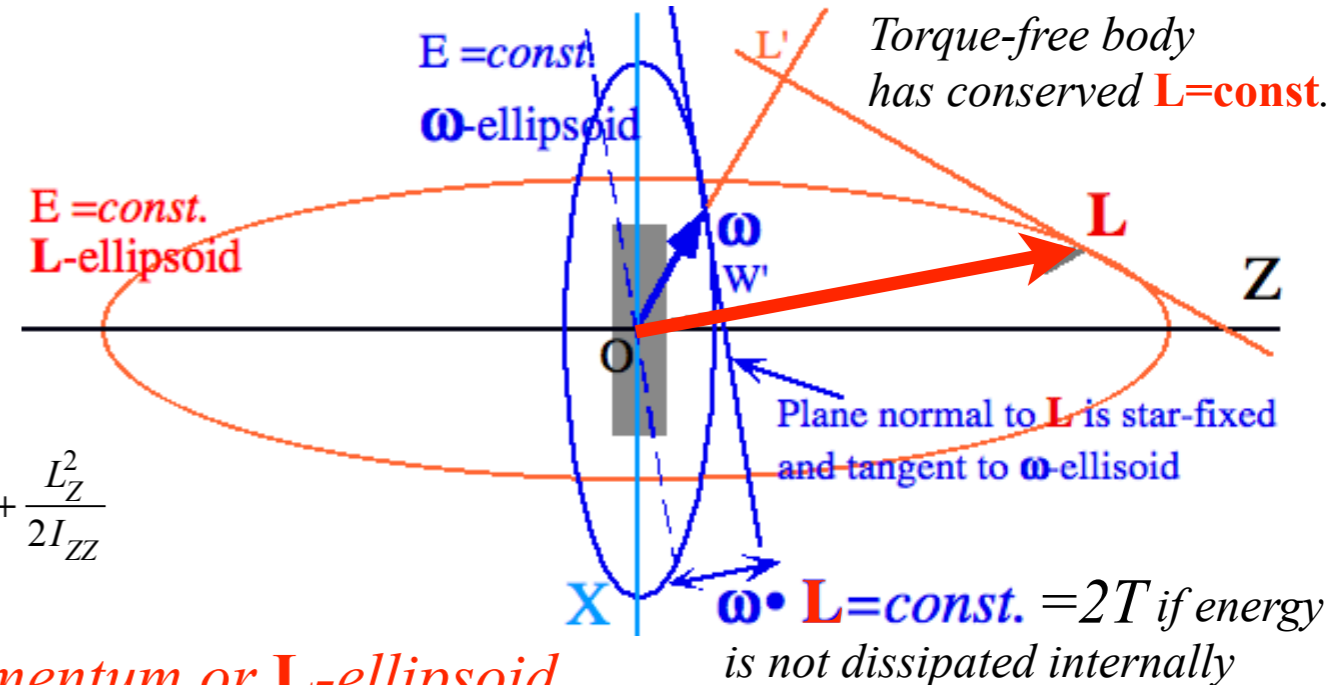
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Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

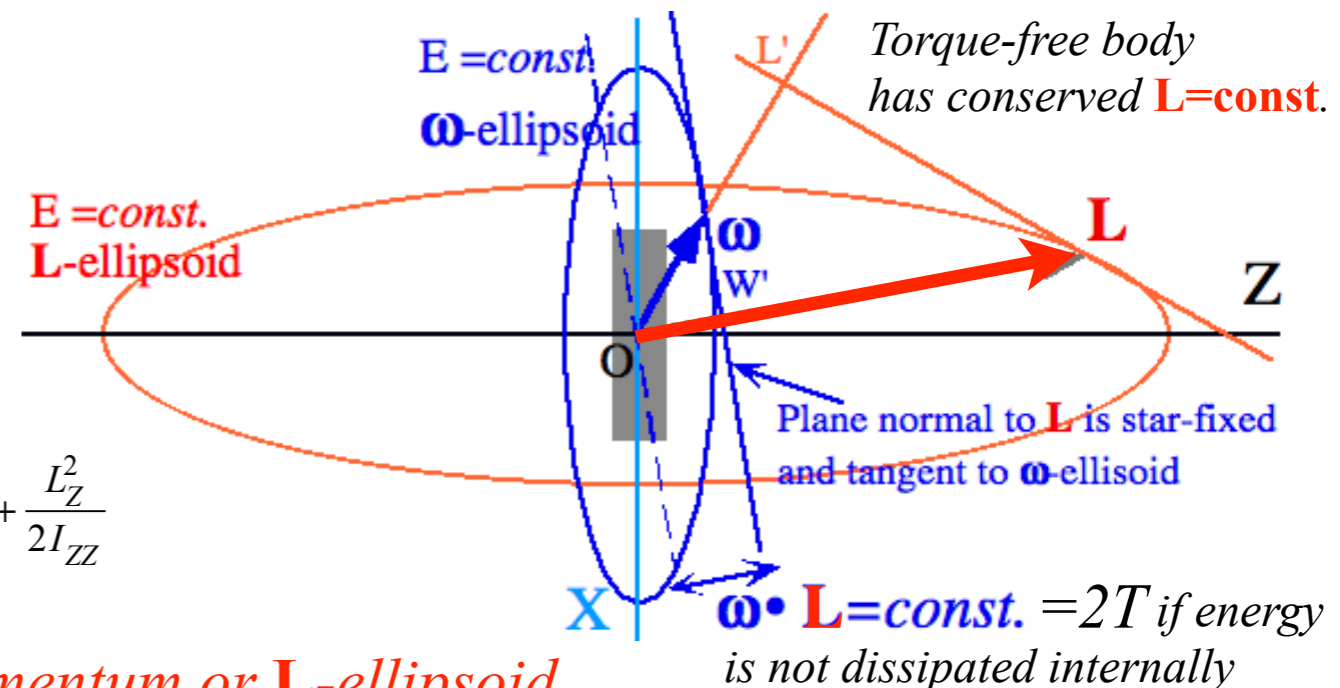
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Hamilton's 1st equations: $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \cdot \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}}{2} = \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

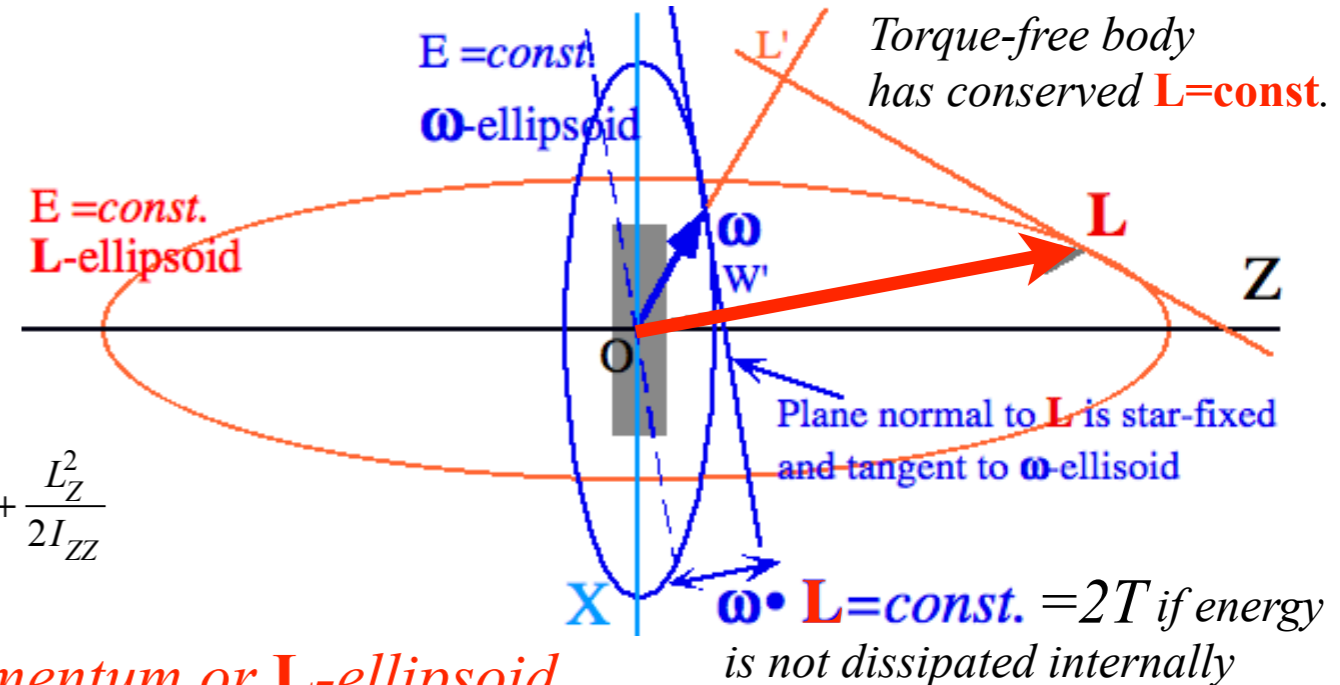
$$\mathbf{L} = \tilde{\mathbf{I}} \cdot \boldsymbol{\omega}, \quad \text{generally implies:} \quad \boldsymbol{\omega} = \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum \mathbf{L} , or both at once. once

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{L} \cdot \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

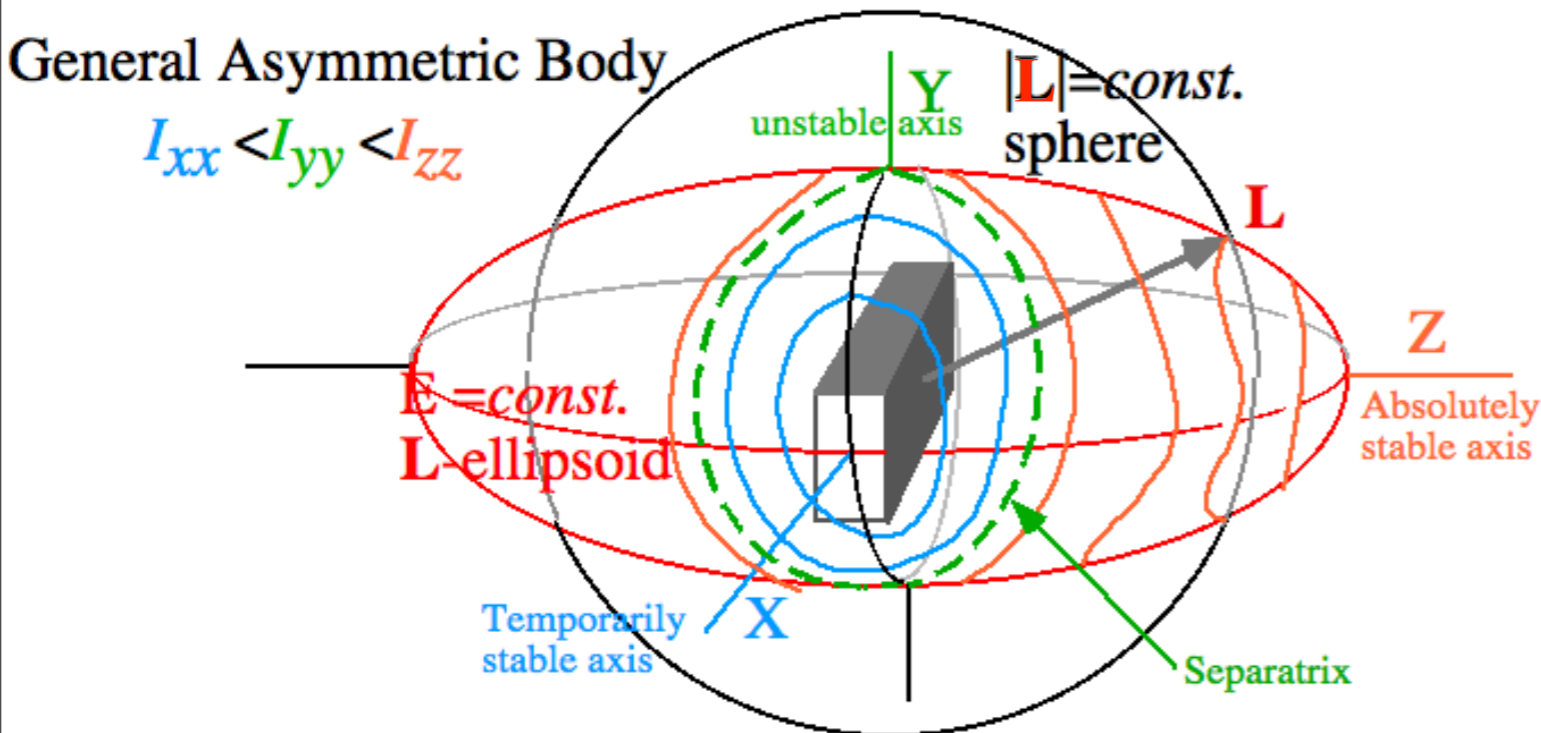
$$T = \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or L-ellipsoid*

Lagrangian form is the equation of the *angular velocity or omega-ellipsoid* $\boldsymbol{\omega}$ is generally not conserved unless it is aligned to \mathbf{L} or body has symmetry



Canonical momentum: $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ (where: $L = T$)

$$\mathbf{L} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \frac{\partial}{\partial \boldsymbol{\omega}} \frac{\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}}{2} = \mathbf{I} \cdot \boldsymbol{\omega}$$

Hamilton's 1st equations: $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \cdot \mathbf{I}^{-1} \cdot \mathbf{L}}{2} = \mathbf{I}^{-1} \cdot \mathbf{L}$$

Kinetic energy in terms of momentum \mathbf{L} and rotational Hamiltonian

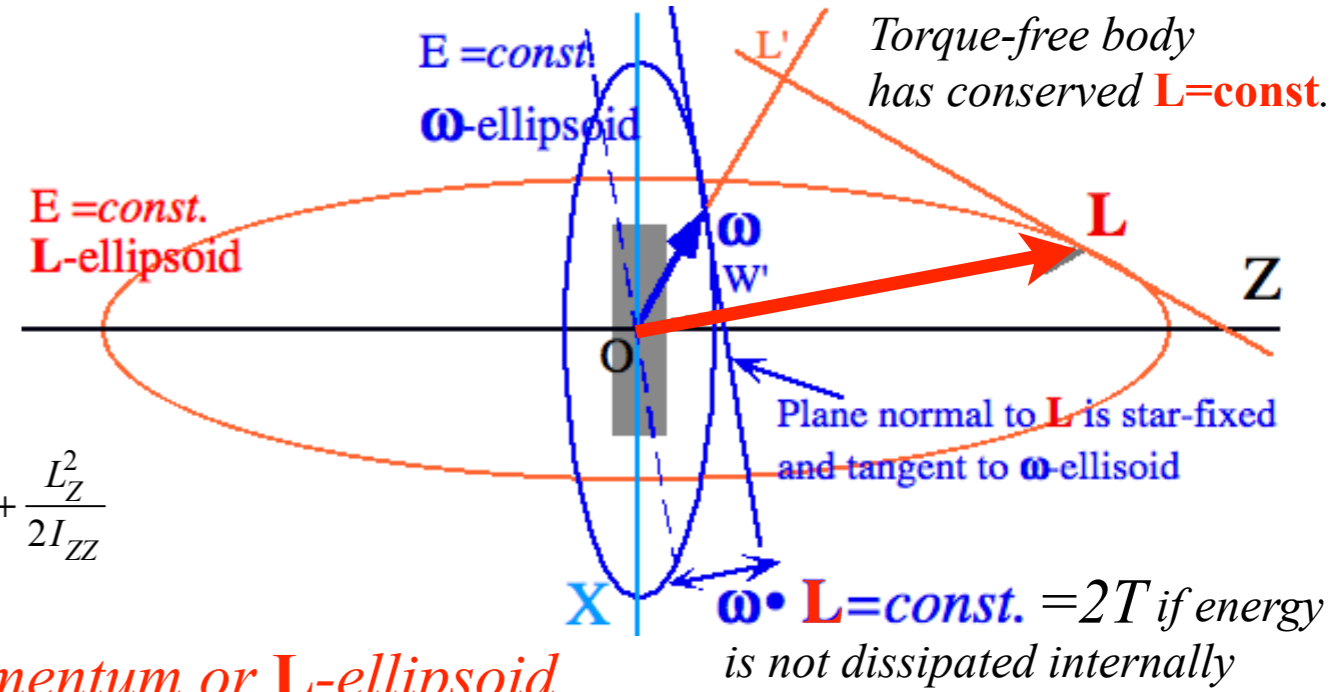
$$\mathbf{L} = \tilde{\mathbf{I}} \cdot \boldsymbol{\omega}, \quad \text{generally implies:} \quad \boldsymbol{\omega} = \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

Express kinetic energy T in terms of angular velocity $\boldsymbol{\omega}$, momentum \mathbf{L} , or both at once. once

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{L} \cdot \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

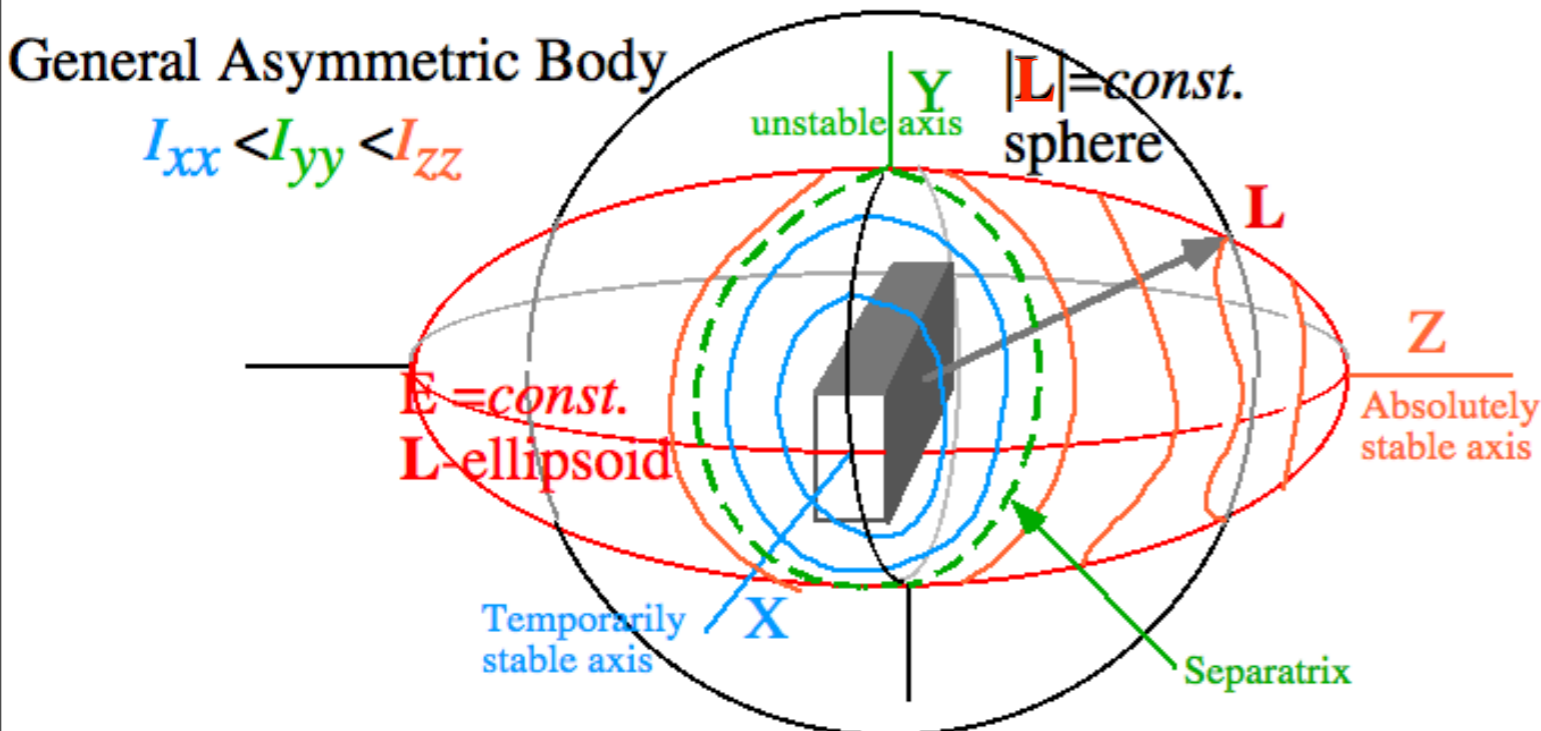
$$T = \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix} \begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$



Hamiltonian form is the equation of the *angular momentum or L-ellipsoid*

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Hamilton's 1st equations: $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}$ (where: $H = T$)

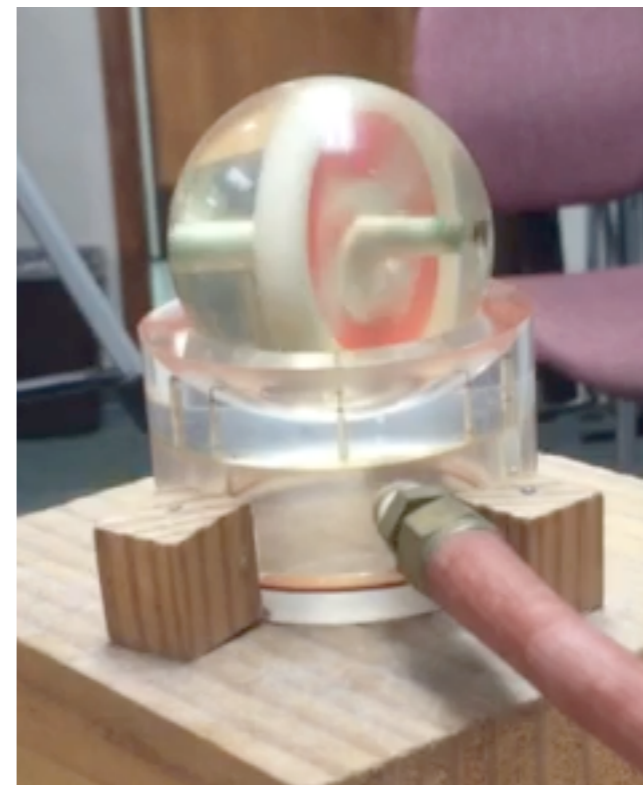
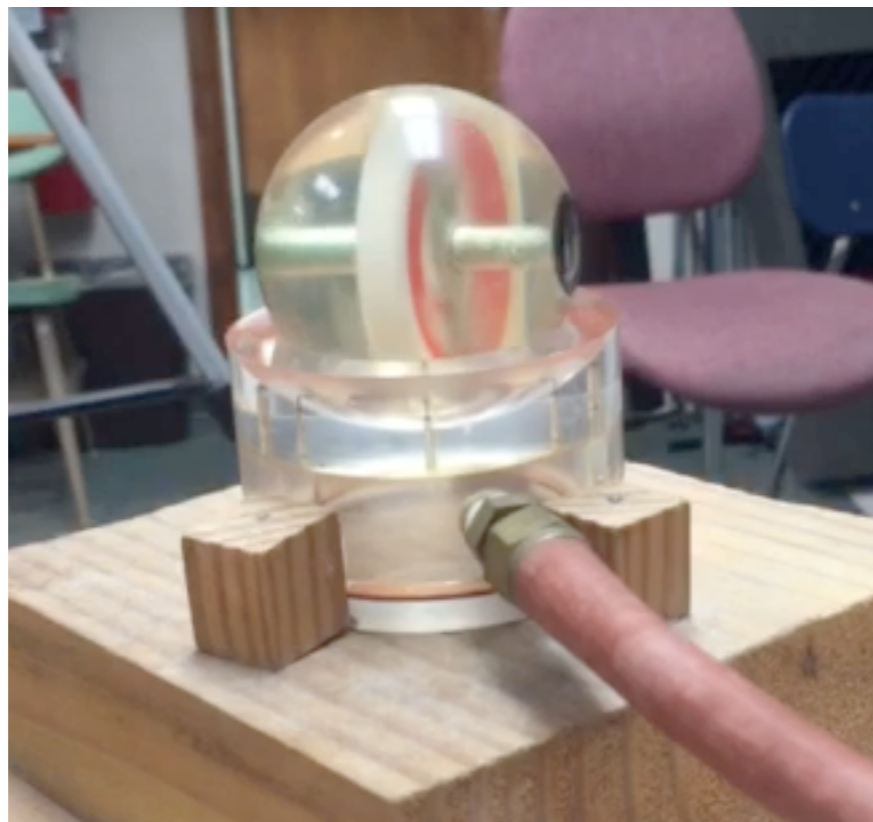
$$\boldsymbol{\omega} = \frac{\partial H}{\partial \mathbf{L}} = \nabla_{\mathbf{L}} H = \frac{\partial}{\partial \mathbf{L}} \frac{\mathbf{L} \cdot \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}}{2} = \tilde{\mathbf{I}}^{-1} \cdot \mathbf{L}$$

In body frame momentum \mathbf{L} moves along intersection of \mathbf{L} -ellipsoid and \mathbf{L} -sphere (Length $|\mathbf{L}|$ is constant in any classical frame.)



Rotational Energy Surfaces (RES)

*Symmetric, asymmetric, and spherical-top dynamics (Constant **L**)
BOD-frame cone rolling on **LAB** frame cone*



Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)

Rotational Energy Surface (RES) is quadratic multipole function plotted radially

Constant Energy Surface (CES) is asymmetric ellipsoid of constant E

$$E = \frac{J_x^2}{2I_x} + \frac{J_y^2}{2I_y} + \frac{J_z^2}{2I_z} \text{ with } J = \text{const.}$$

$$E = \frac{J_x^2}{2I_x} + \frac{J_y^2}{2I_y} + \frac{J_z^2}{2I_z} = \text{const.}$$

Here notation L or \mathbf{L} for angular momentum is replaced by J or \mathbf{J}

$$= J^2 \left(\frac{\sin^2\theta \cos^2\phi}{2I_x} + \frac{\sin^2\theta \sin^2\phi}{2I_y} + \frac{\cos^2\theta}{2I_z} \right)$$

or: $\frac{J_x^2}{2EI_x} + \frac{J_y^2}{2EI_y} + \frac{J_z^2}{2EI_z} = 1$

(a) RE surface

(b) CE surface

(c) RES intersecting CES

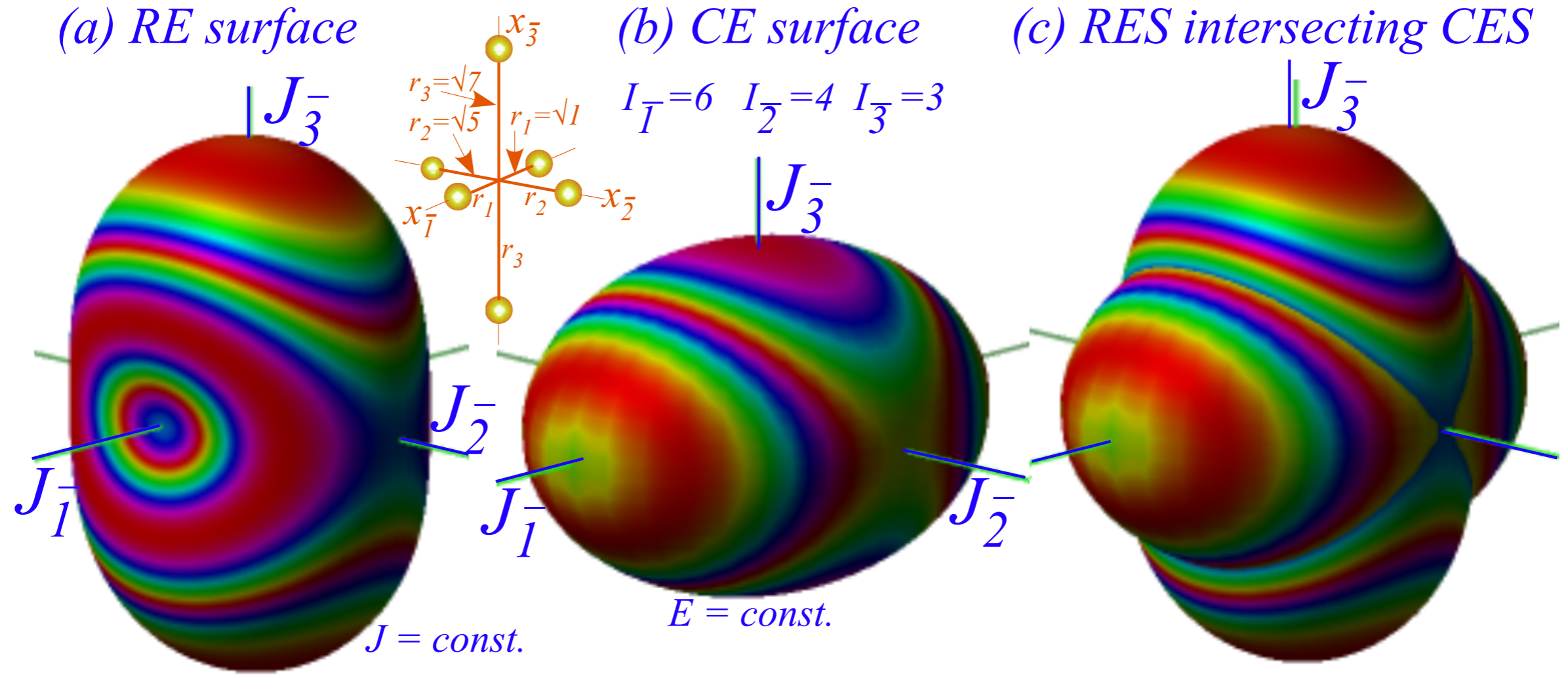


Fig. 6.8.1 Rigid rotor surfaces (a) RES polynomial, (b) CES ellipsoid, and (c) RES and CES intersected.

RES and CES for nearly-symmetric prolate rotors and nearly-symmetric oblate rotors

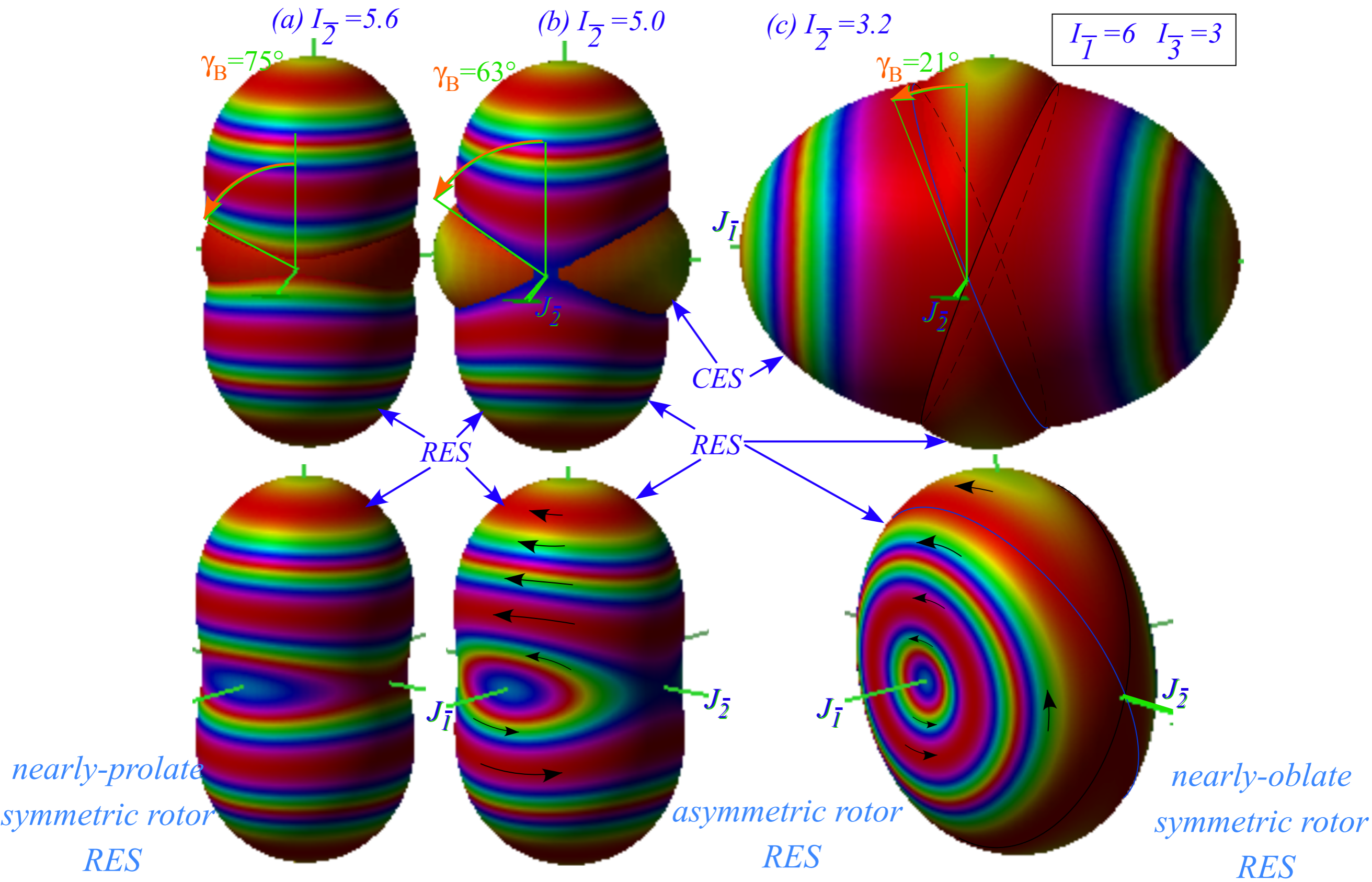
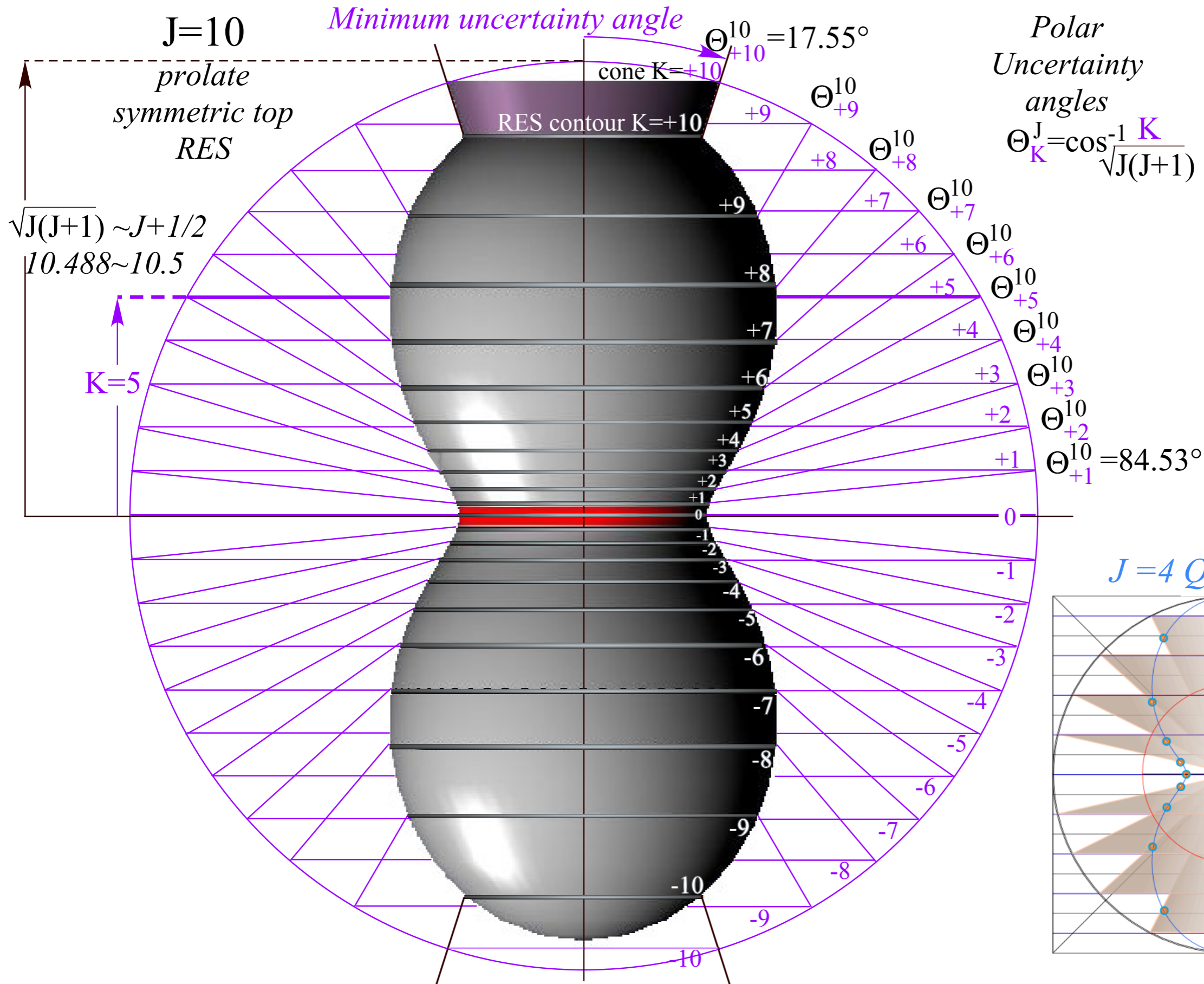
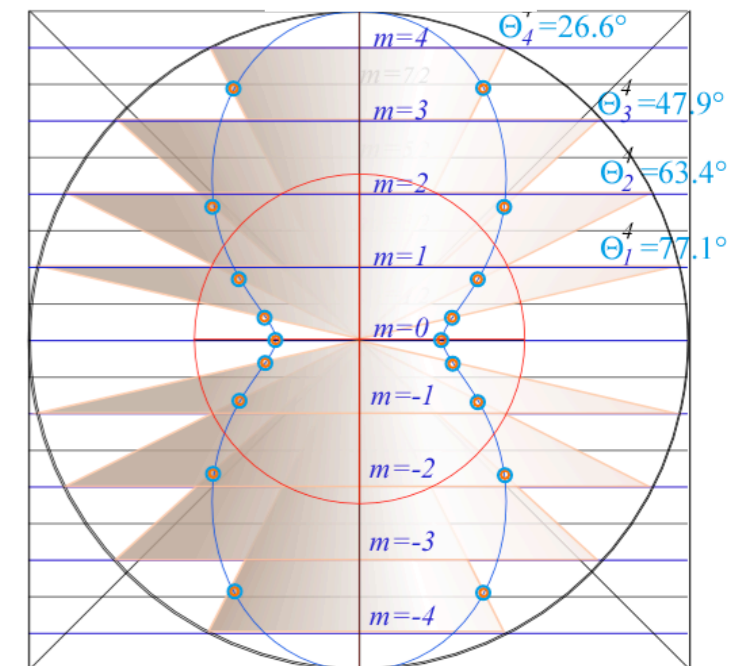


Fig. 6.8.2 Fixed-J- RES with CES at separatrix $E=J^2 / 2I_2$ as I_2 varies. (a) $I_2=5.6$ and $\gamma_B=75.5^\circ$ (Nearly prolate low-E CES), (b) $I_2=5.0$ and $\gamma_B=63.4^\circ$, (c) $I_2=3.2$ and $\gamma_B=20.7^\circ$ (Nearly oblate high-E CES).

RES for symmetric prolate rotor locates $J=10$ quantum ($-J < K < J$) levels (at RES-quantum cone intersections)

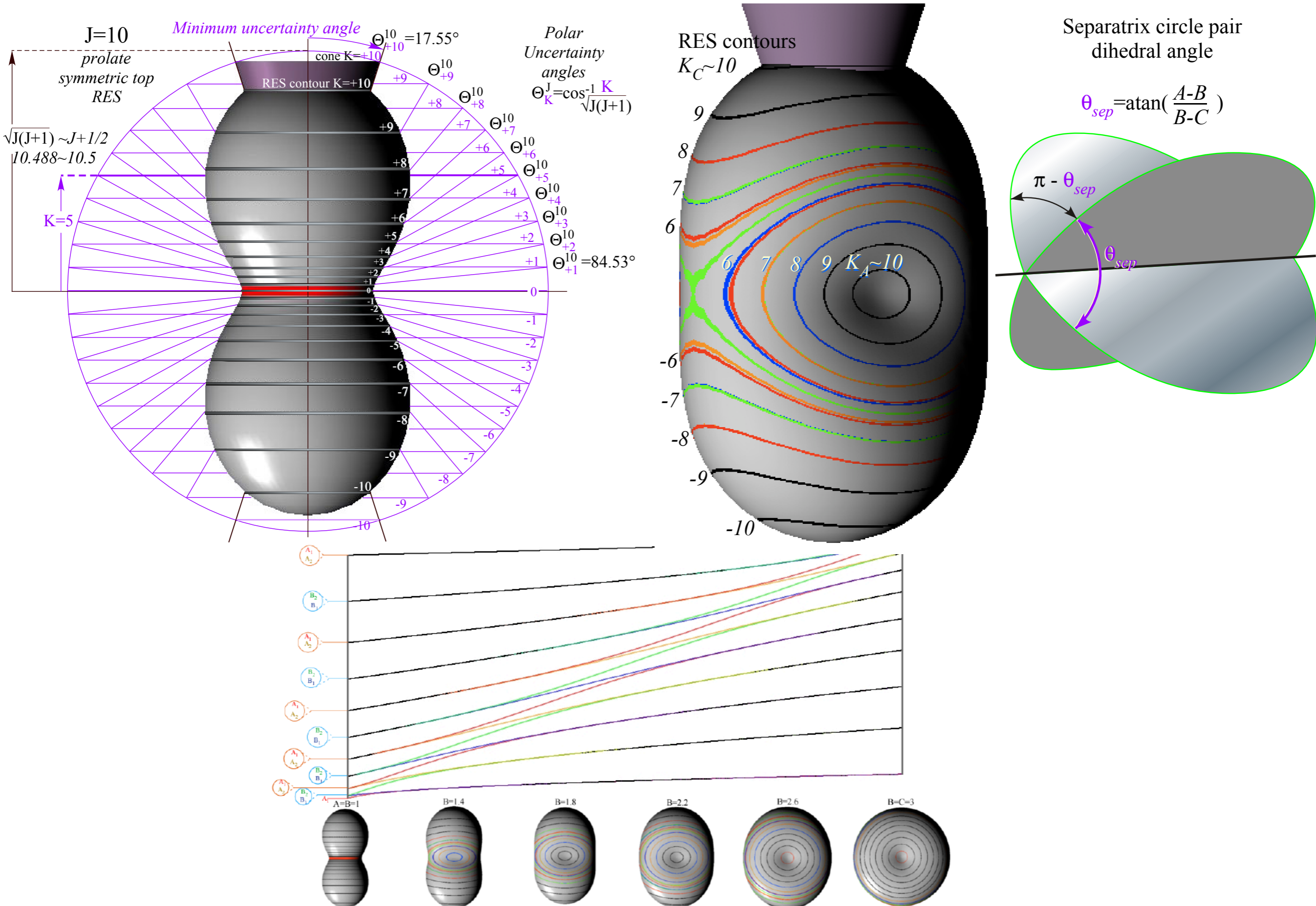


J=4 Quantum cones



[Link to pdf of: W. G. Harter and J.C. Mitchell, International Journal of Molecular Science, 14, 714-806 \(2013\) Fig. 1-5 p.730](#)

RES for symmetric and asymmetric rotor approximates $J=10$ ($-J < K < J$) levels (near RES-quantum cone levels)

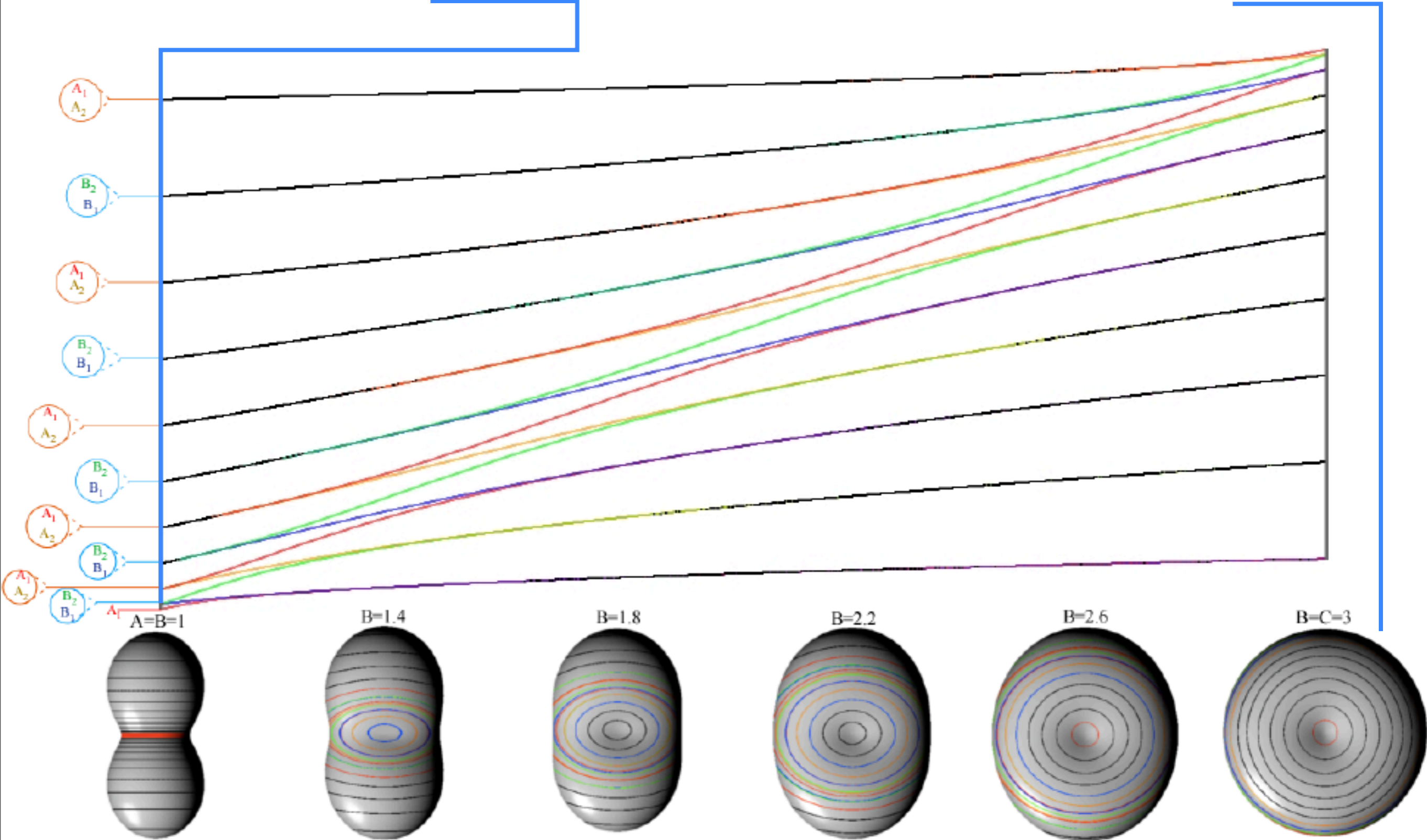


[Link to pdf of: W. G. Harter and J C. Mitchell, International Journal of Molecular Science, 14, 714-806 \(2013\) Fig. 1-5 p.730](#)

RES for symmetric prolate rotor locates $J = 10$ quantum ($-J < K < J$) levels (at RES-quantum cone intersections)

$$E = A\mathbf{J}_x^2 + B\mathbf{J}_y^2 + C\mathbf{J}_z^2 \quad \text{with } J = \text{const.}$$

Spectra varies as symmetric prolate RES changes through a range of asymmetric RES to oblate RES



[Link to pdf of: W. G. Harter and J. C. Mitchell, International Journal of Molecular Science, 14, 714-806 \(2013\) Fig. 1-5 p.730](#)

Asymmetric-top dynamics (Constant L)

1. NASA Space station video



<https://youtu.be/1n-HMSCDYtM>

For those physicist who are brave of heart, make note the video's comments

2. UAF lab air-supported asymmetric top video



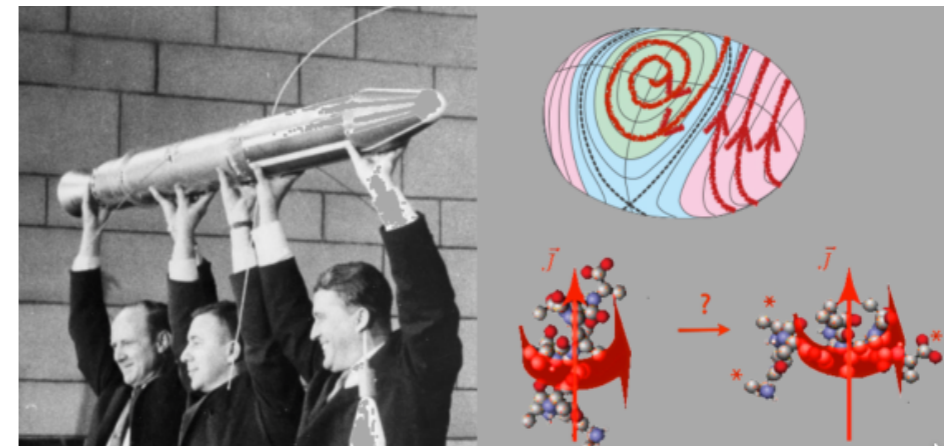
<https://youtu.be/HWjGvCaqx5g>

3. NASA-Rotating Solid Bodies in Microgravity (2008)



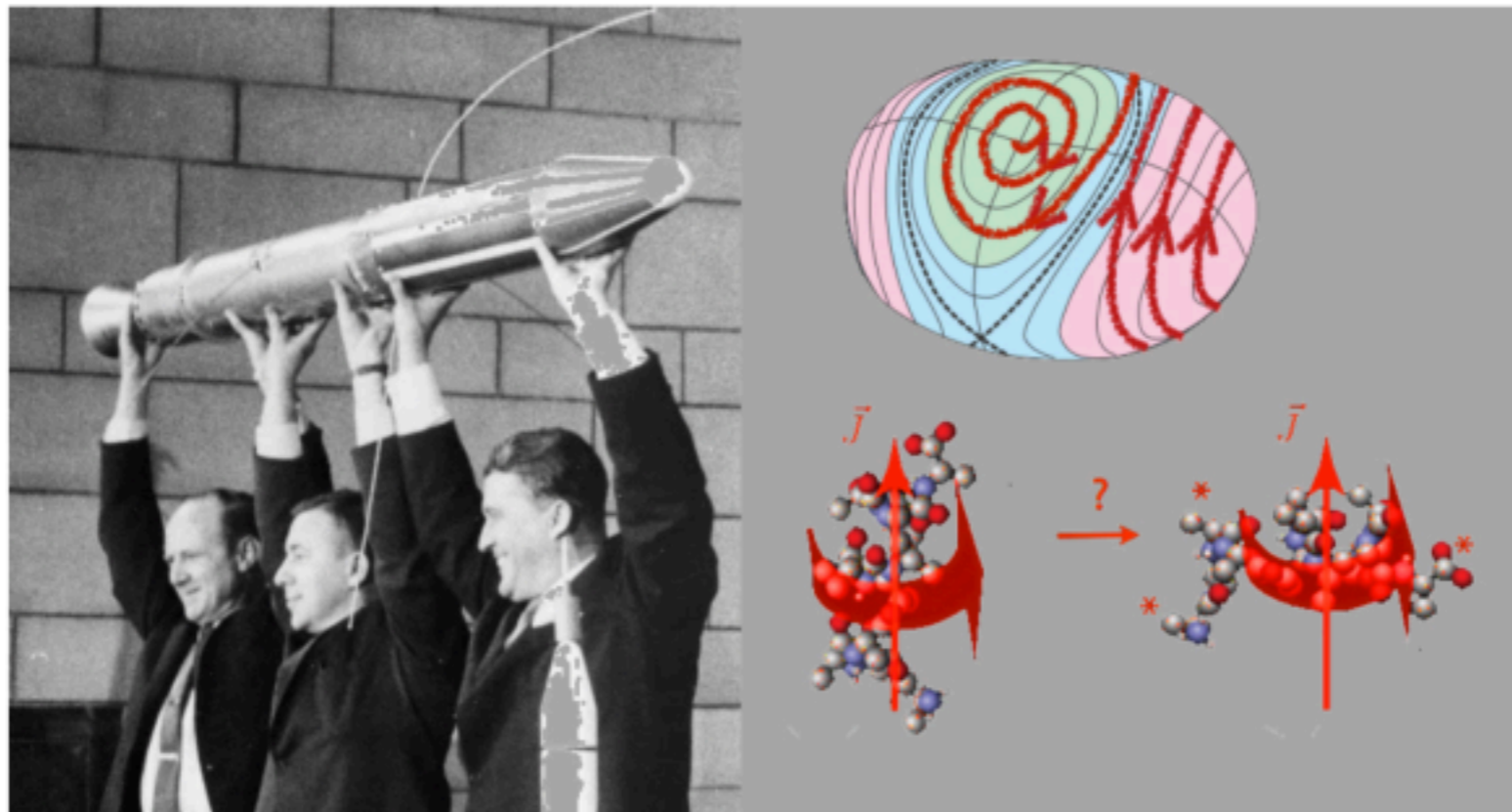
<https://www.youtube.com/watch?v=BPMjcN-sBJ4>

4. Early NASA-JPL satellite blunder (1958)



Continued \Rightarrow

4. Early NASA-JPL satellite blunder (1958)



*From text in preparation
by Rick Heller on semi-
classical dynamics of
polyatomic molecules*

Figure 10.3: NASA-JPL early blunder. Rockets are not rigid bodies, especially with floppy whip antennas attached. The Explorer 1 satellite was the first one launched successfully by the United States. Seen in the left panel are James van Allen (center), William Pickering (left), and Werner von Braun, with a full-size model of the satellite, just after it was successfully orbited in 1958. As this press conference took place, the satellite was busily tumbling out of control. Van Allen soon realized that the intermittent signal from the satellite was due to tumbling. Fortunately, enough antennas were bristling from the satellite that it still gave much useful data, resulting in discovery of the van Allen radiation belts. The tumbling took place because friction due to slight wobbling is converted to heat, lowering the rotational energy, but not changing the angular momentum. The only way for this to happen is for the satellite to start rotating around a lower energy axis, until it bottoms out in end and over and tumbling at the lowest possible rotational energy for the given angular momentum. The author thanks Prof. William Harter for pointing out the existence and the physics of this story.

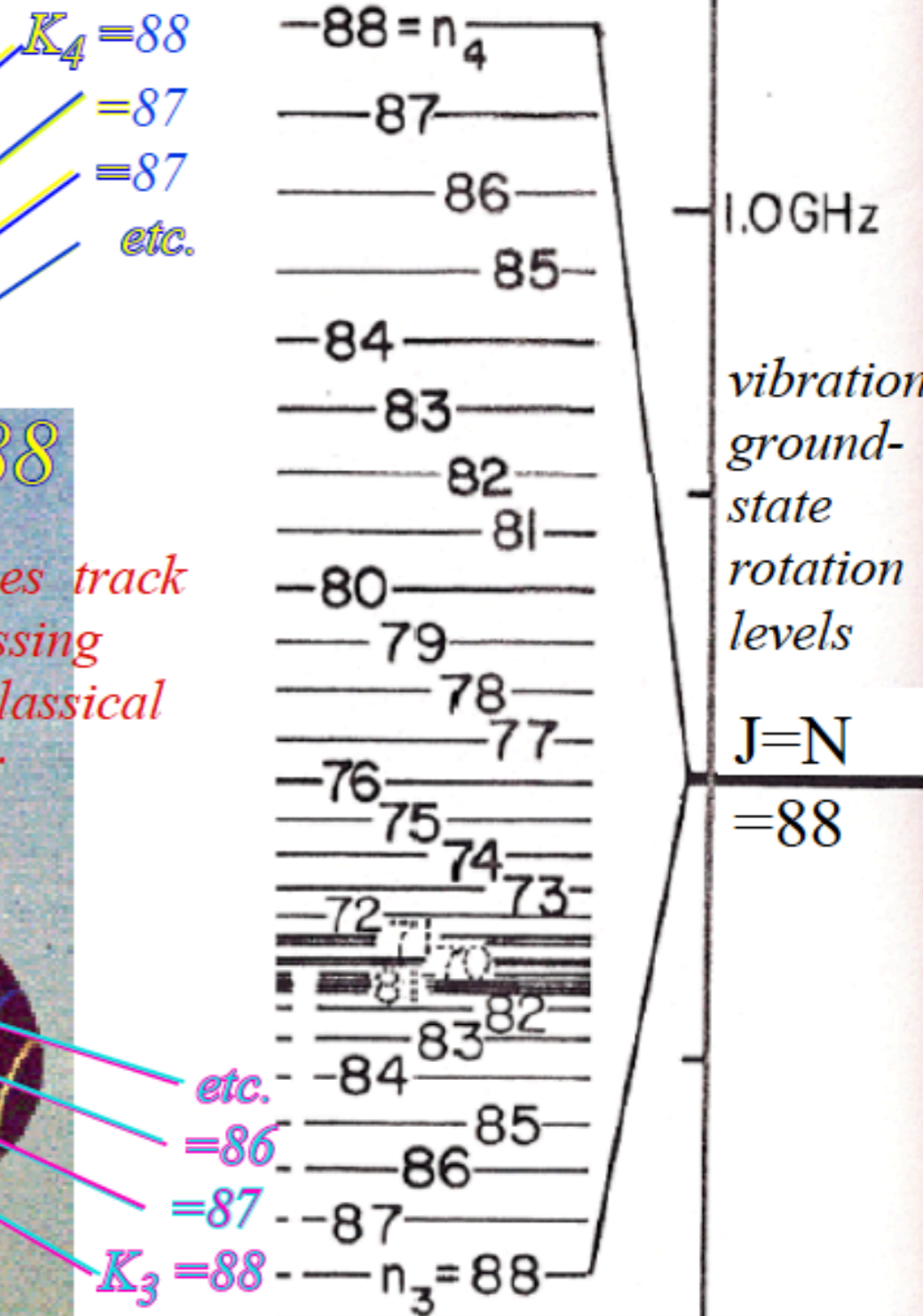
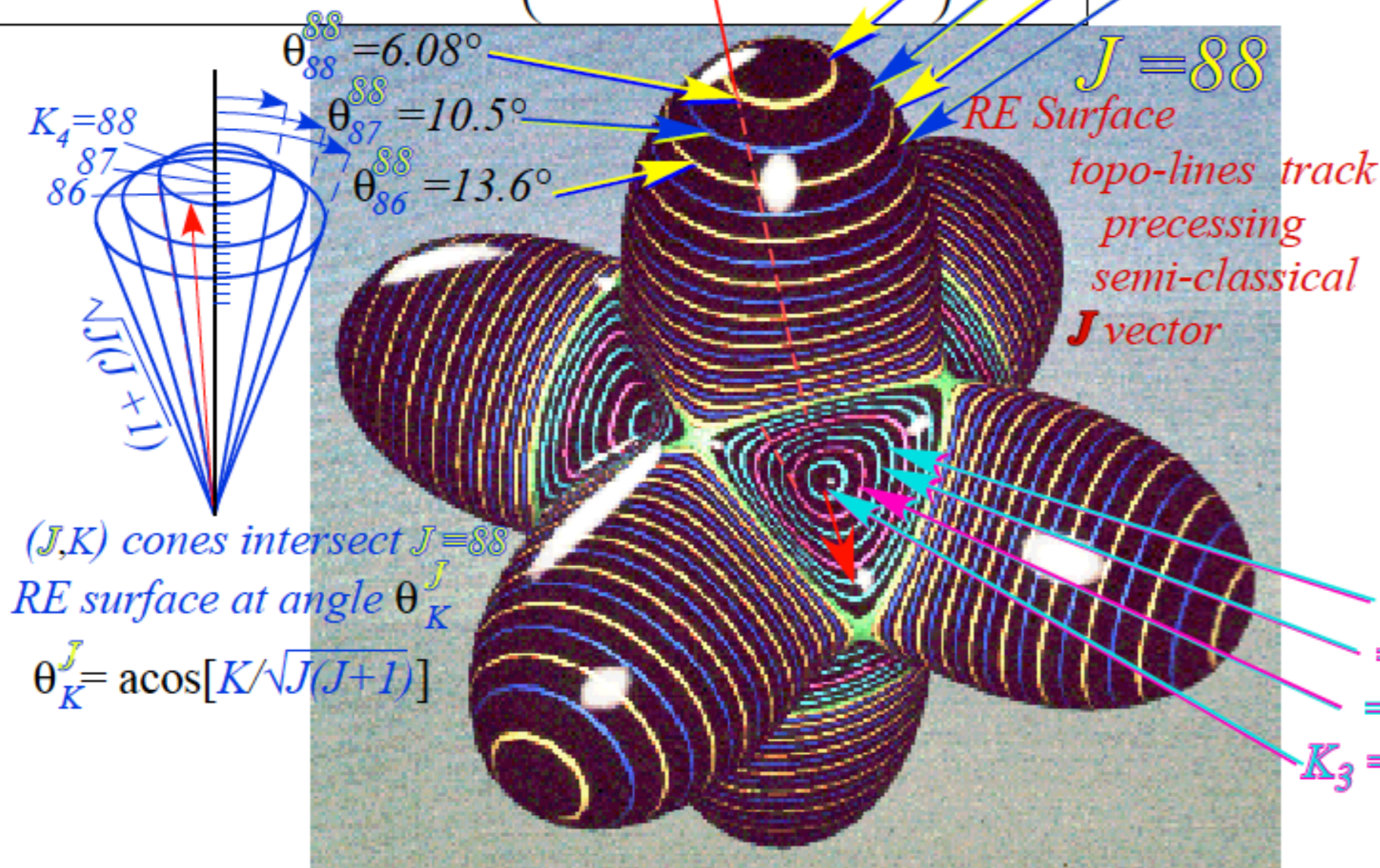
RES for spherical rotor approximates $J=88$ ($-J < K < J$) levels of SF_6

$$\langle H \rangle \sim v_{\text{vib}} + BJ(J+1) + \langle H^{\text{Scalar Coriolis}} \rangle + \langle H^{\text{Tensor Centrifugal}} \rangle + \langle H^{\text{Tensor Coriolis}} \rangle + \langle H^{\text{Nuclear Spin}} \rangle + \dots$$

O_h or T_d Spherical Top: (Hecht CH_4 Hamiltonian 1960)

$$H = B(J_x^2 + J_y^2 + J_z^2) + t_{440} \left(J_x^4 + J_y^4 + J_z^4 - \frac{3}{5} J^4 \right) + \dots$$

$$= BJ^2 + t_{440} \left(T_0^4 + \sqrt{\frac{5}{14}} [T_4^4 + T_{-4}^4] \right) + \dots$$



(next page shows slice)

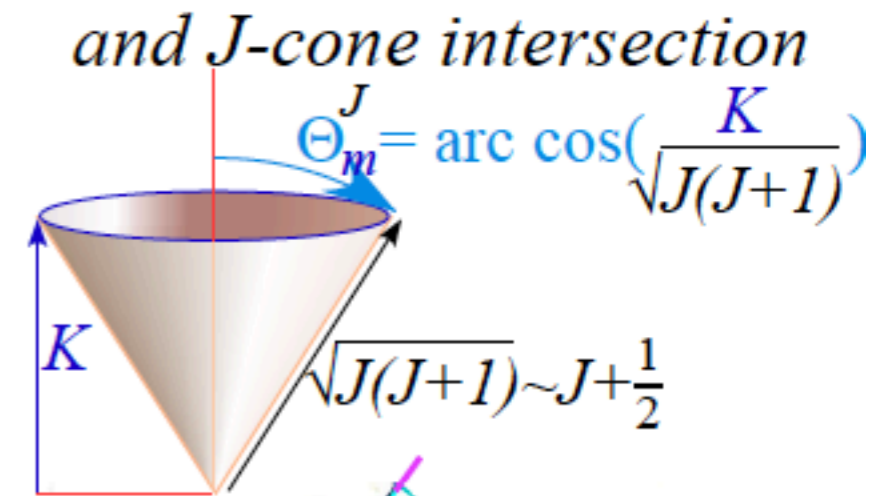
Link to pdf of: [W. G. Harter and J C. Mitchell, International Symposium on Molecular Spectroscopy, OSU Columbus \(2009\)](#)



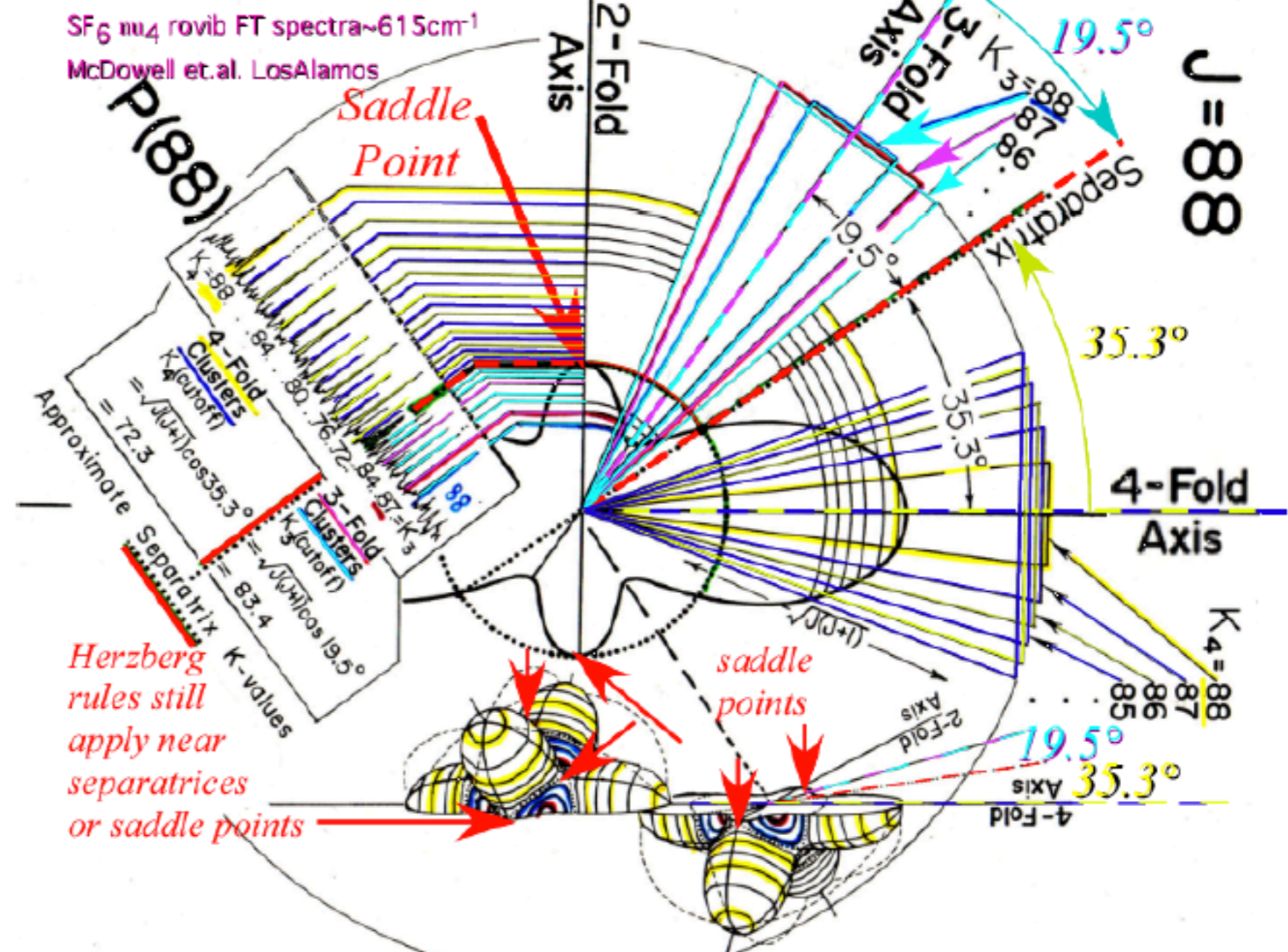
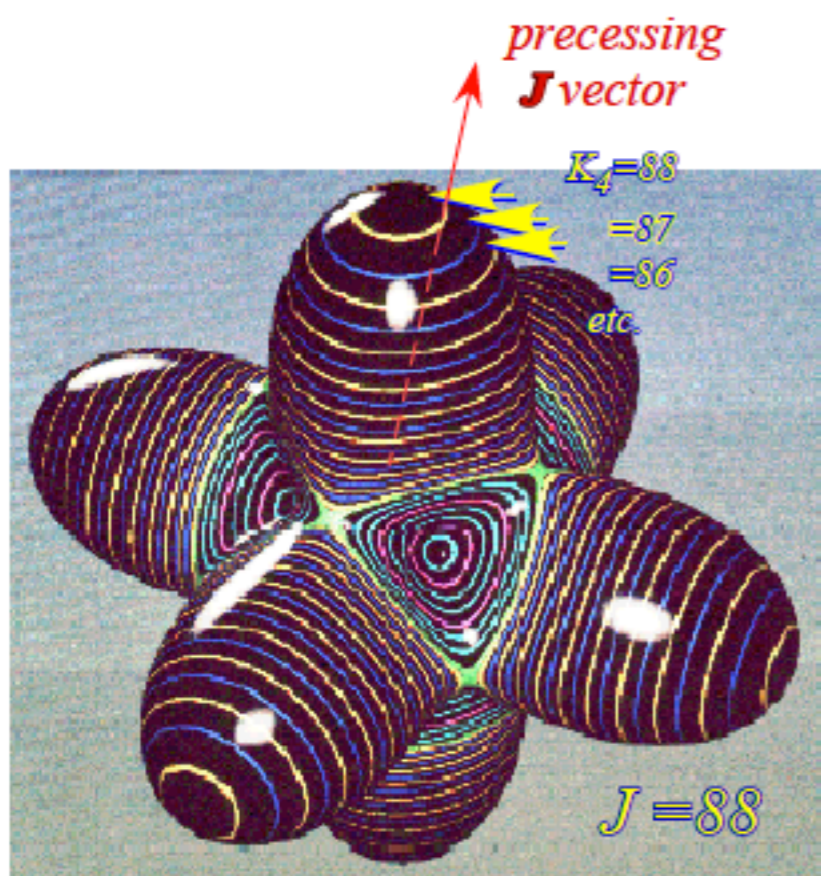
SF₆ Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography and J-cone intersection

$$\mathbf{H} = B(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2) + t_{440} \left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5} J^4 \right) + \dots$$

$$= B\mathbf{J}^2 + t_{440} \left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}} [\mathbf{T}_4^4 + \mathbf{T}_{-4}^4] \right) + \dots$$



Rovibronic Energy (RE) Tensor Surface



[Link to pdf of: W. G. Harter and J C. Mitchell, International Symposium on Molecular Spectroscopy, OSU Columbus \(2009\)](#)

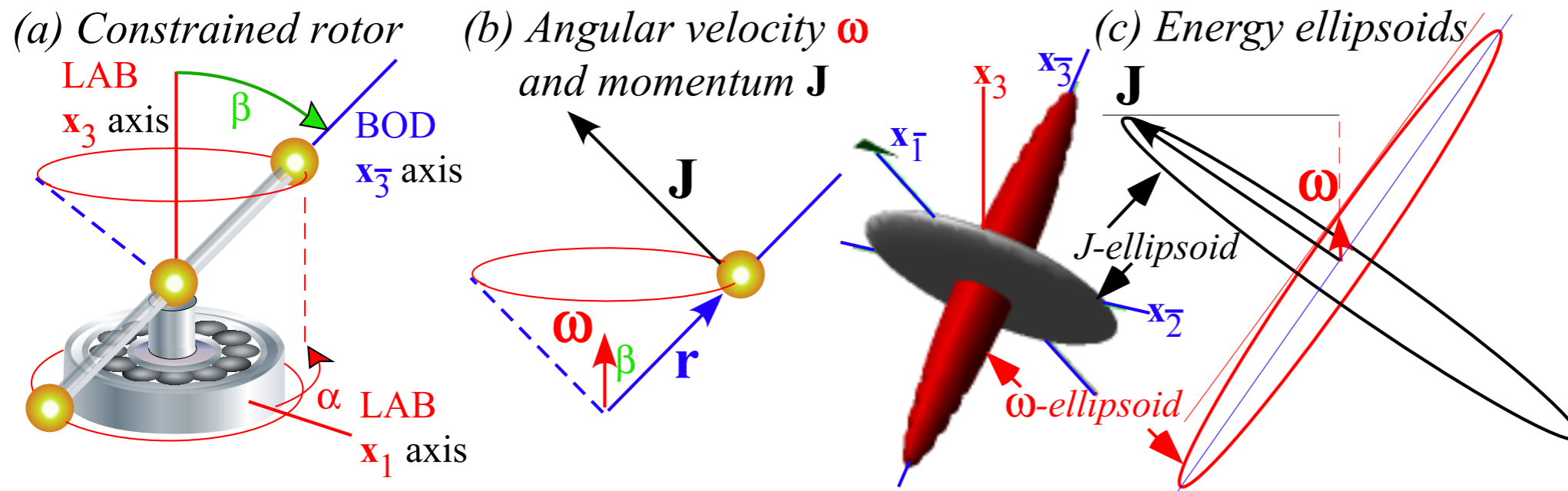


Fig. 6.7.1 Elementary ω -constrained rotor and angular velocity-momentum geometry.

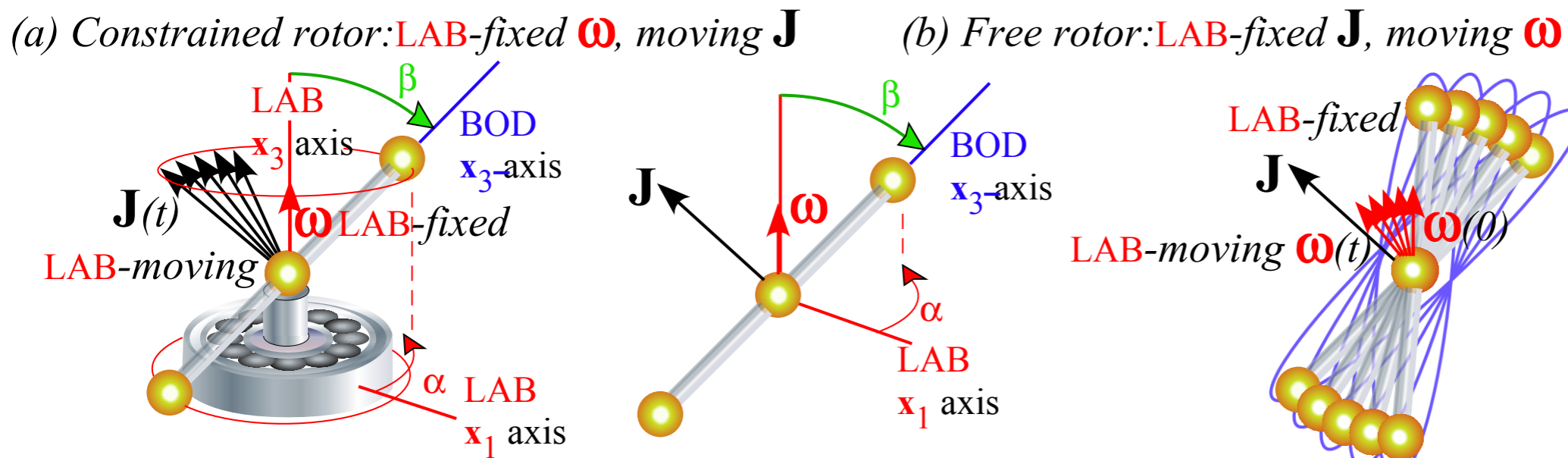


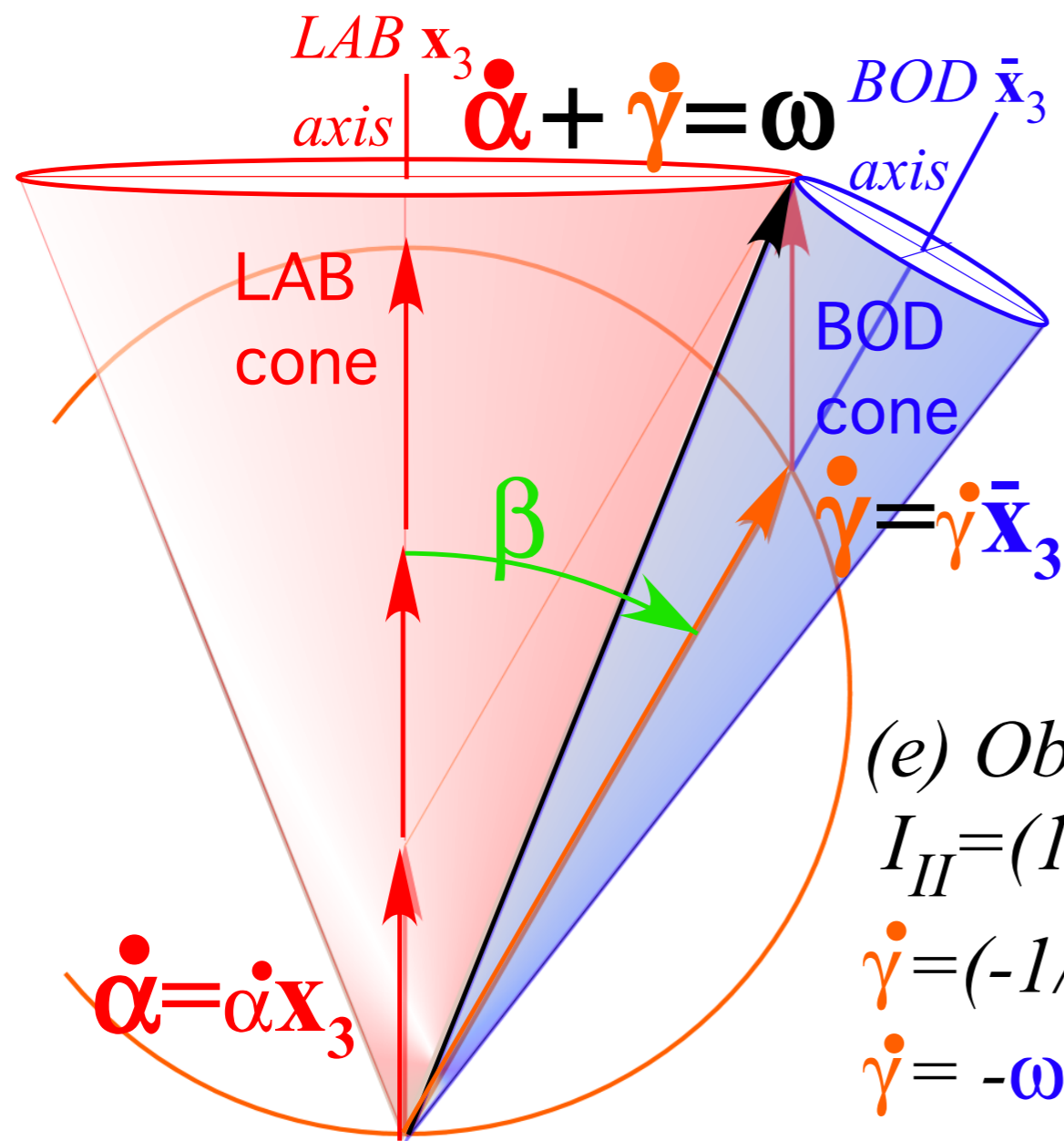
Fig. 6.7.2 Free rotor cut loose from *LAB*-constraining ω -axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...

Prolate tops: (a) $I_{II}=4I_3$

$$\dot{\gamma} = 3\dot{\alpha} \cos\beta$$

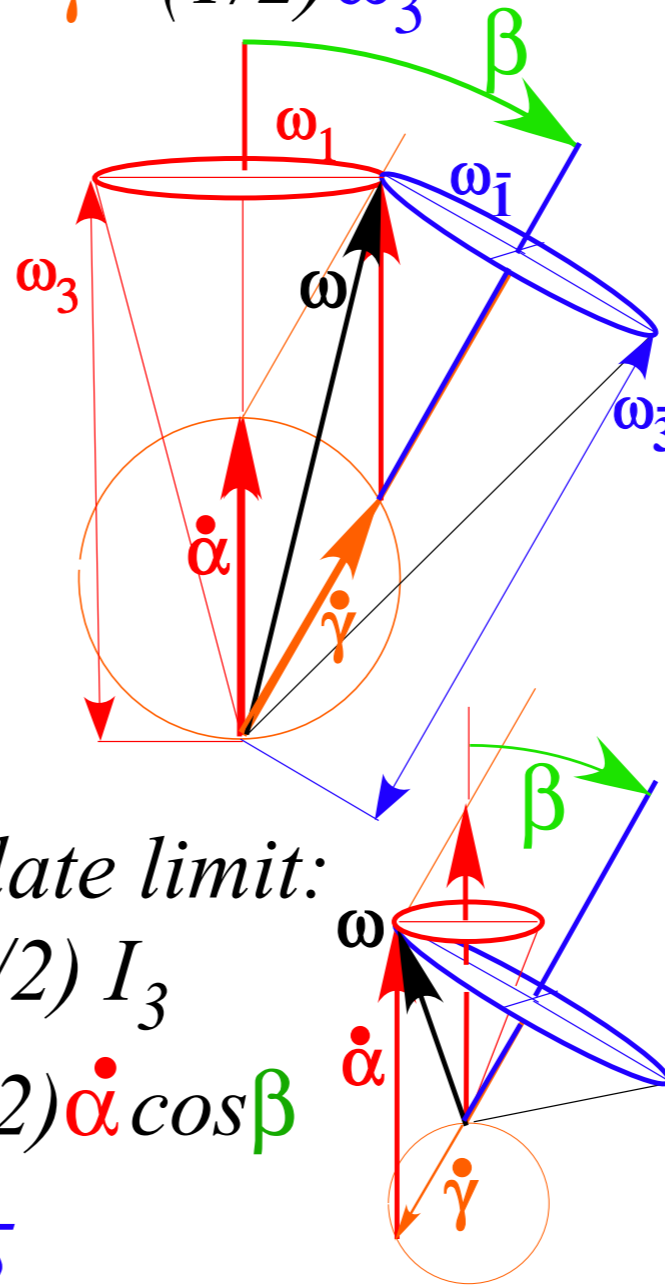
$$\dot{\gamma} = (3/4)\omega_{\bar{3}}$$



(b) $I_{II}=2I_3$

$$\dot{\gamma} = \dot{\alpha} \cos\beta$$

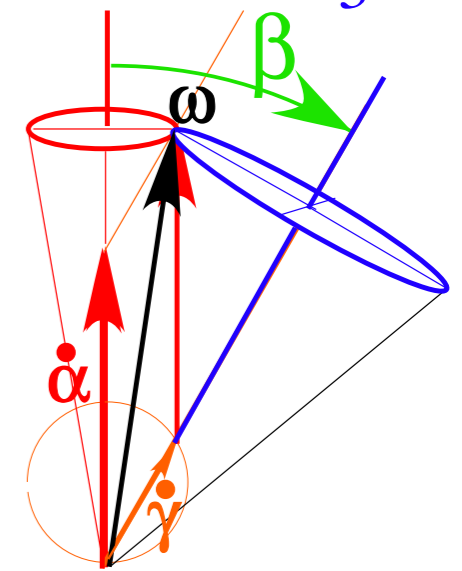
$$\dot{\gamma} = (1/2)\omega_{\bar{3}}$$



(c) $I_{II}=(3/2)I_3$

$$\dot{\gamma} = (1/2)\dot{\alpha} \cos\beta$$

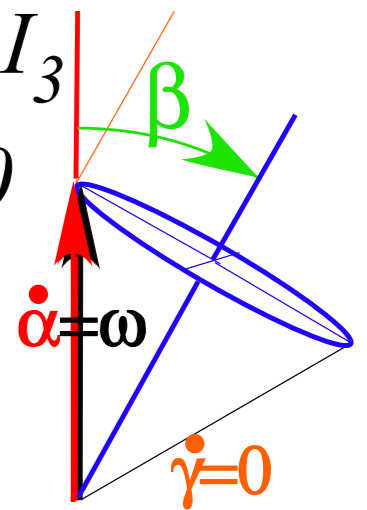
$$\dot{\gamma} = (1/3)\omega_{\bar{3}}$$



(d) Spherical top:

$$I_{II} = I_3$$

$$\dot{\gamma} = 0$$

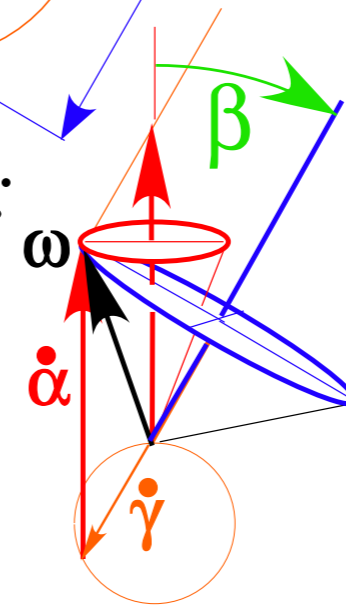


(e) Oblate limit:

$$I_{II} = (1/2)I_3$$

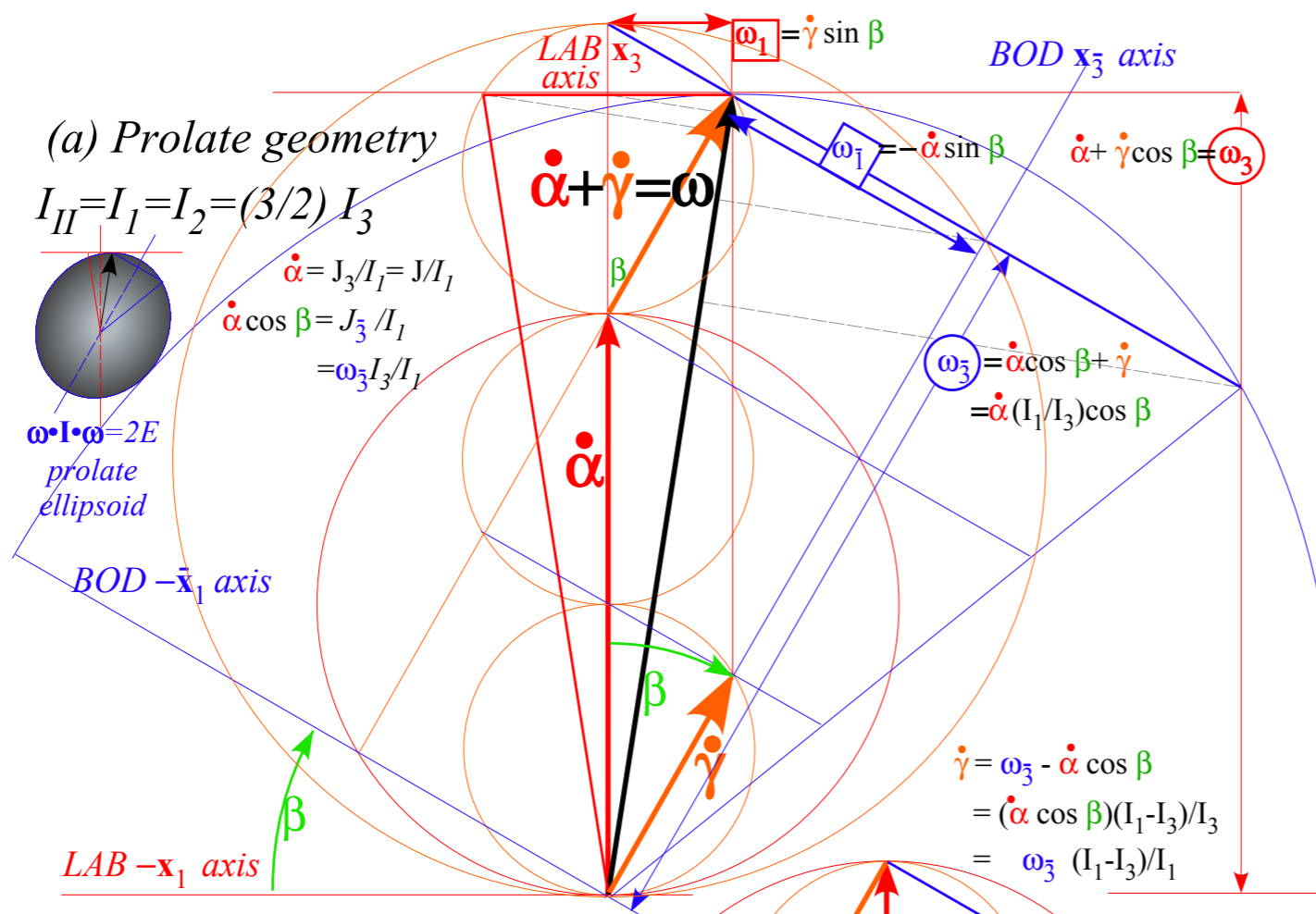
$$\dot{\gamma} = (-1/2)\dot{\alpha} \cos\beta$$

$$\dot{\gamma} = -\omega_{\bar{3}}$$



Blue BOD-frame cones roll (around ω -sticking axis) without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω -cones for $\beta=30^\circ$ and inertial ratios: (a) $I_{II}/I_3 = 3$, (b) 1, (c) $1/2$, (d) 0, (e) $-1/2$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

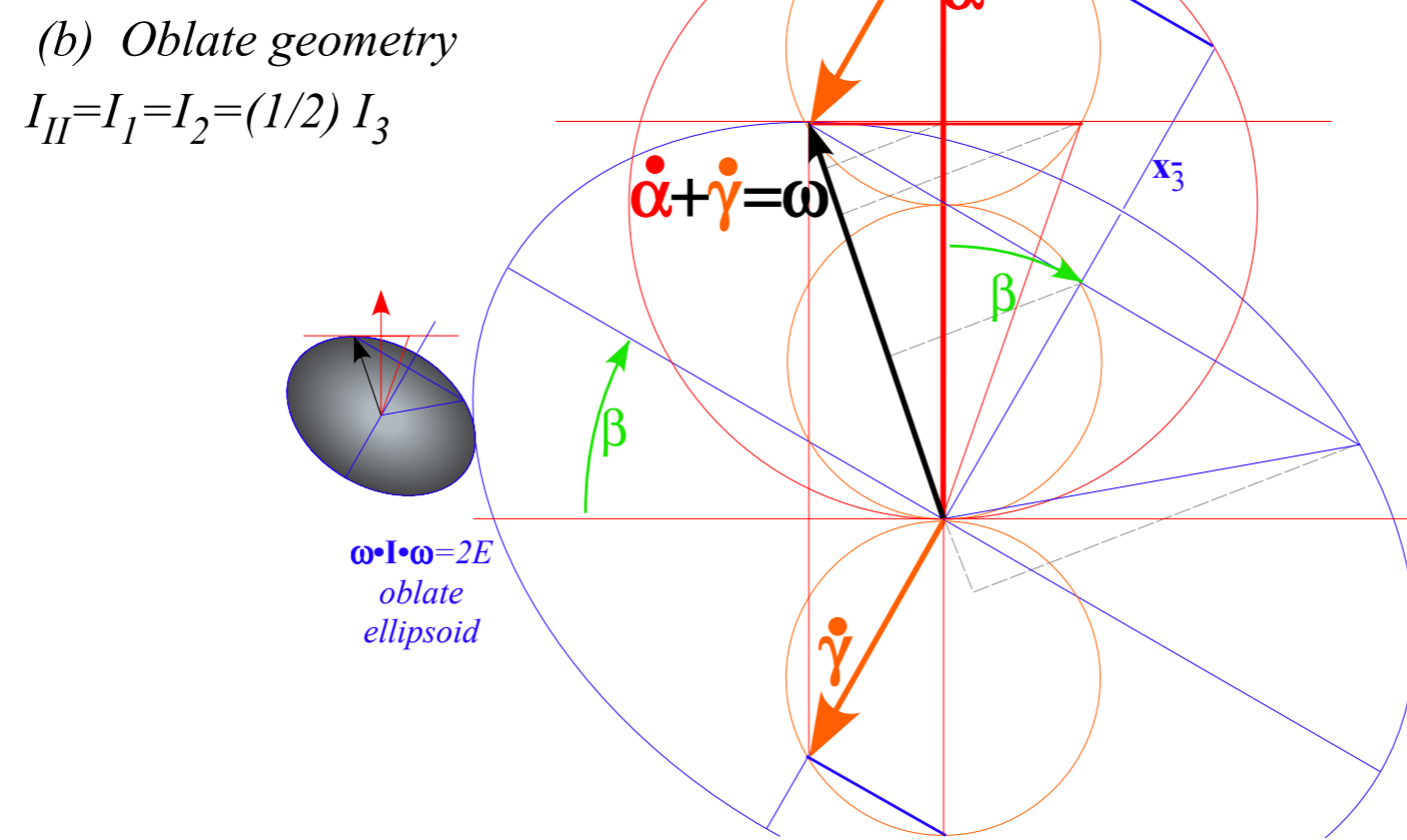


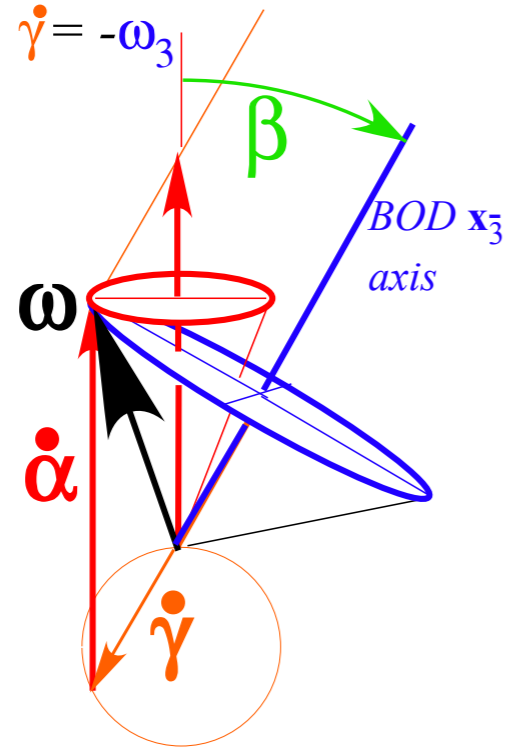
Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case

Oblate limit:

$$I_{II} = (1/2) I_3$$

$$\dot{\gamma} = (-1/2) \dot{\alpha} \cos \beta$$

$$\dot{\gamma} = -\omega_3$$

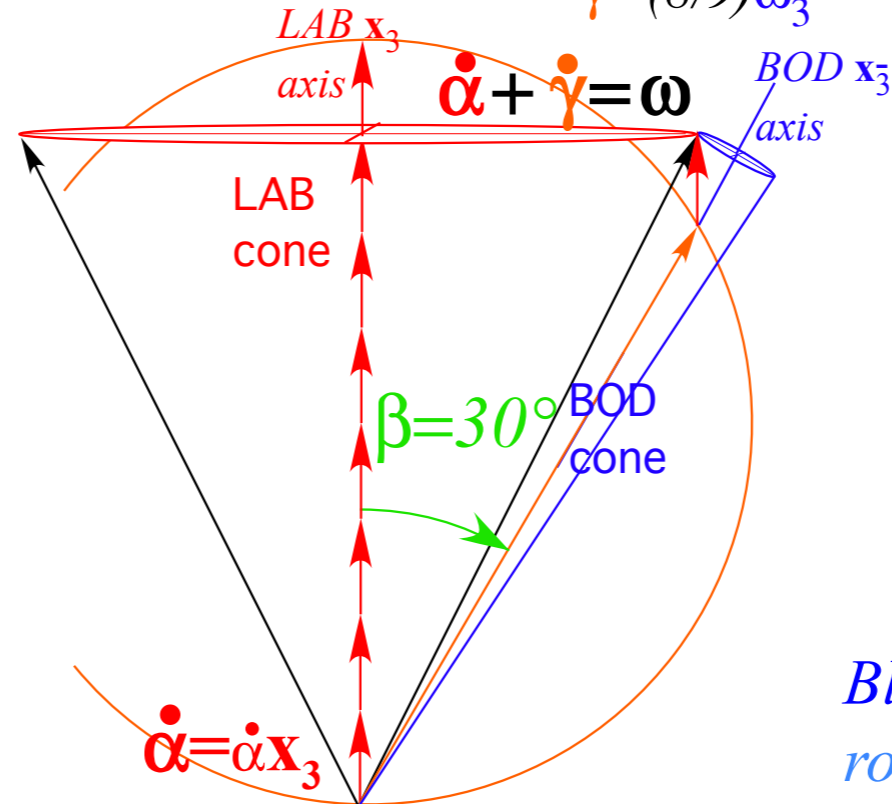


$$\begin{aligned} \dot{\gamma} &= \omega_3 - \dot{\alpha} \cos \beta \\ &= (\dot{\alpha} \cos \beta)(I_1 - I_3)/I_3 \\ &= \omega_3 (I_1 - I_3)/I_1 \end{aligned}$$

Very prolate top: $I_{II} = 9I_3$

$$\dot{\gamma} = 8\dot{\alpha} \cos \beta$$

$$\dot{\gamma} = (8/9)\omega_3$$



Blue BOD-frame cones roll without slipping on red LAB-frame cone

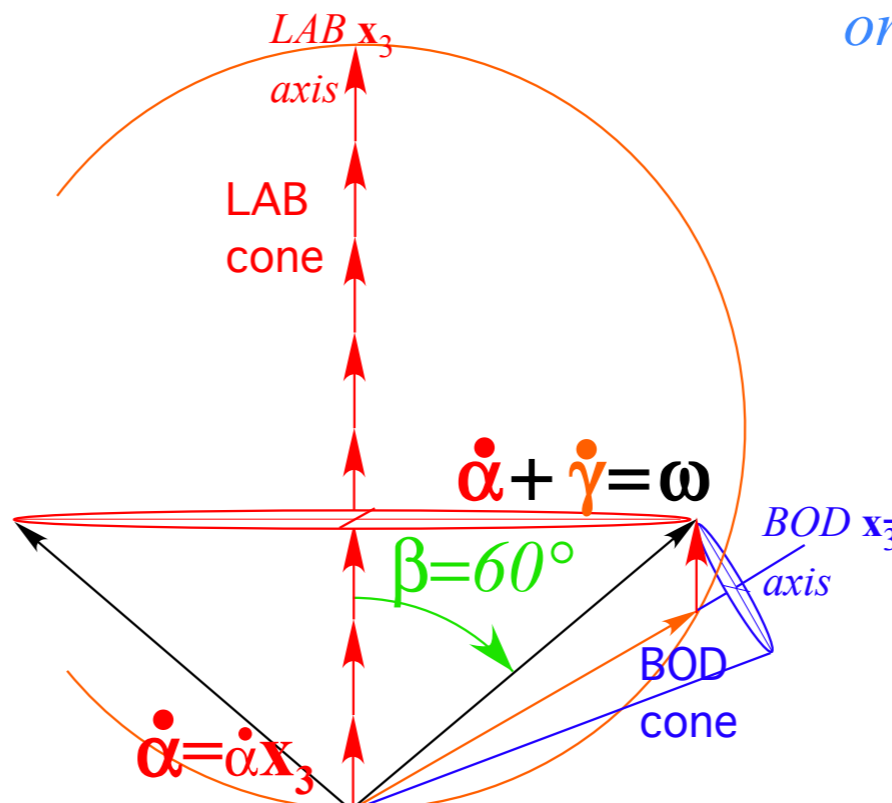
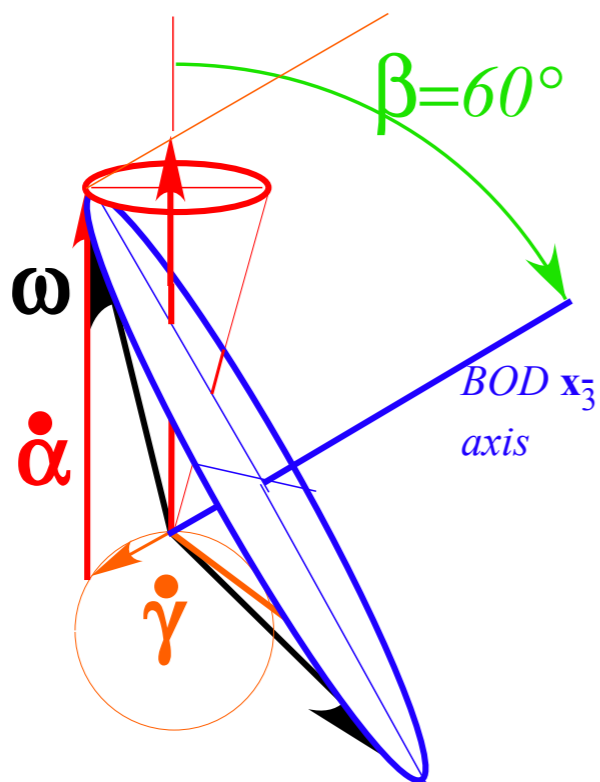


Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.

 *Cycloidal geometry of flying levers*
Practical poolhall application

If you hammer a stick at a point h meters from its center
 you give it some linear momentum Π
 and some angular momentum $\Lambda = h \cdot \Pi$

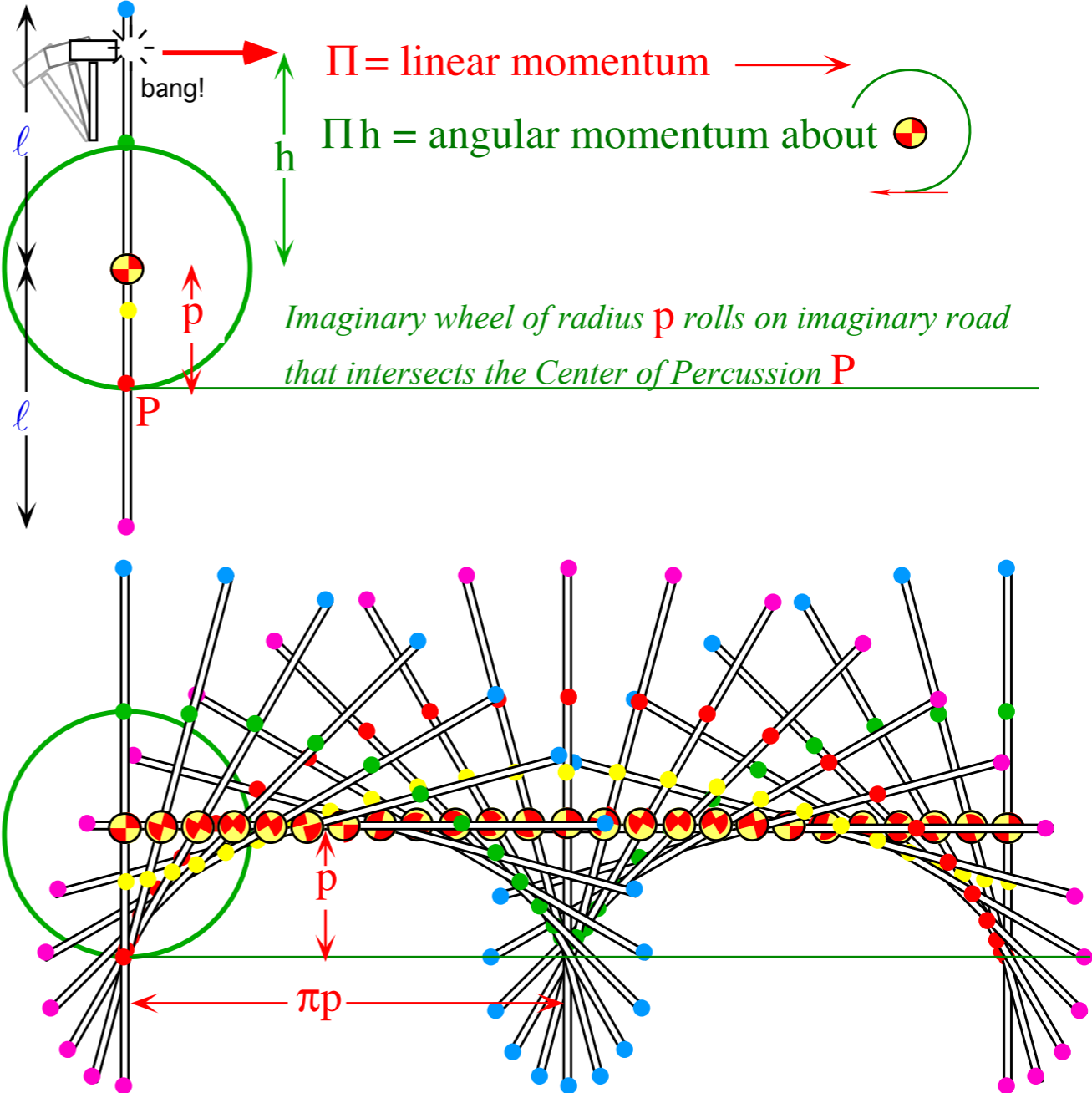


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
 you give it some linear momentum Π
 and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
 is angular momentum Λ divided by
 moment of inertia $I = M \ell^2/3$ of the stick.

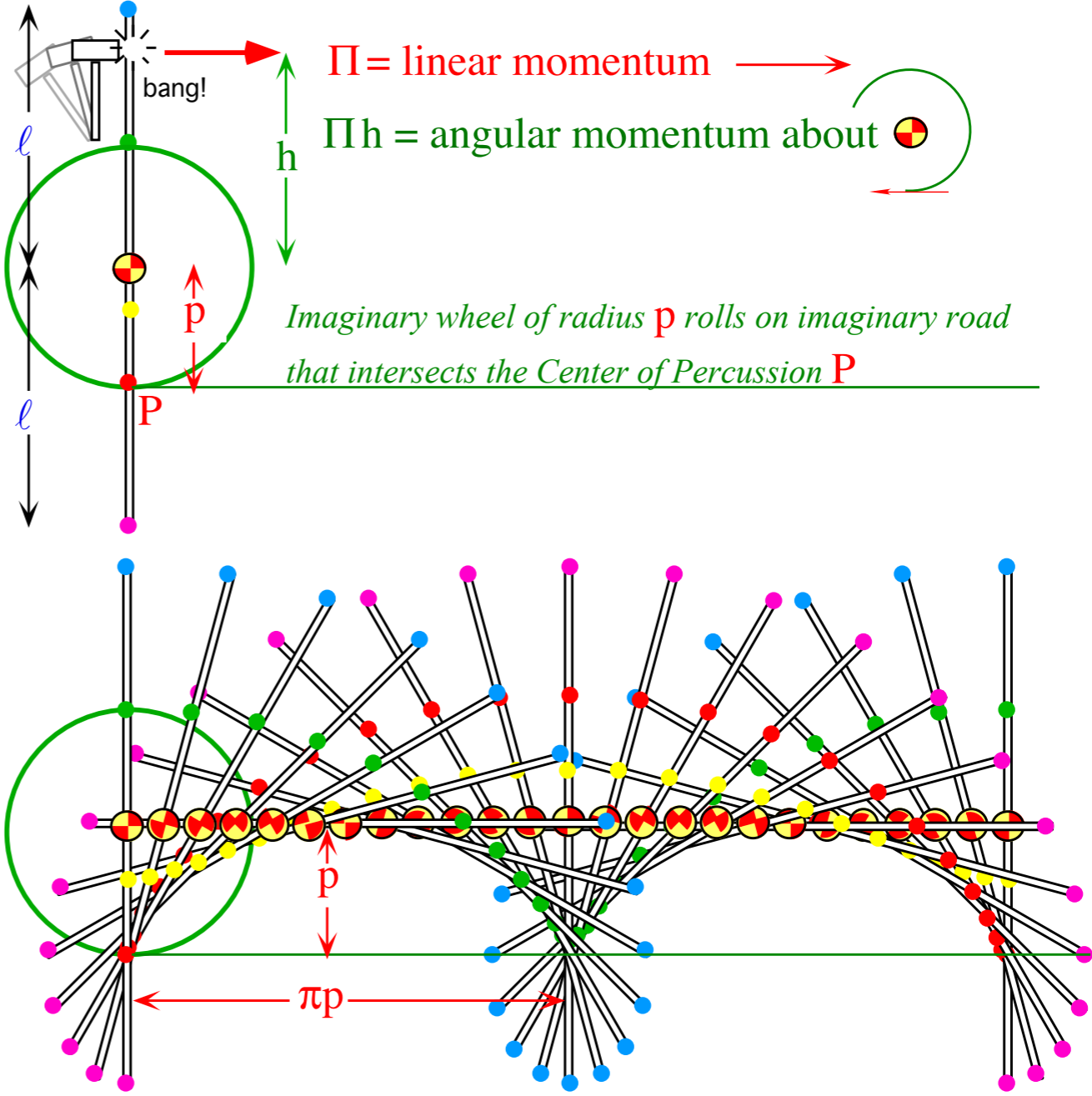


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Resulting angular velocity ω about the center
 is angular momentum Λ divided by
 moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \Lambda / I \quad (=3\Lambda / (M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi / (M \ell^2) \text{ for stick})$$

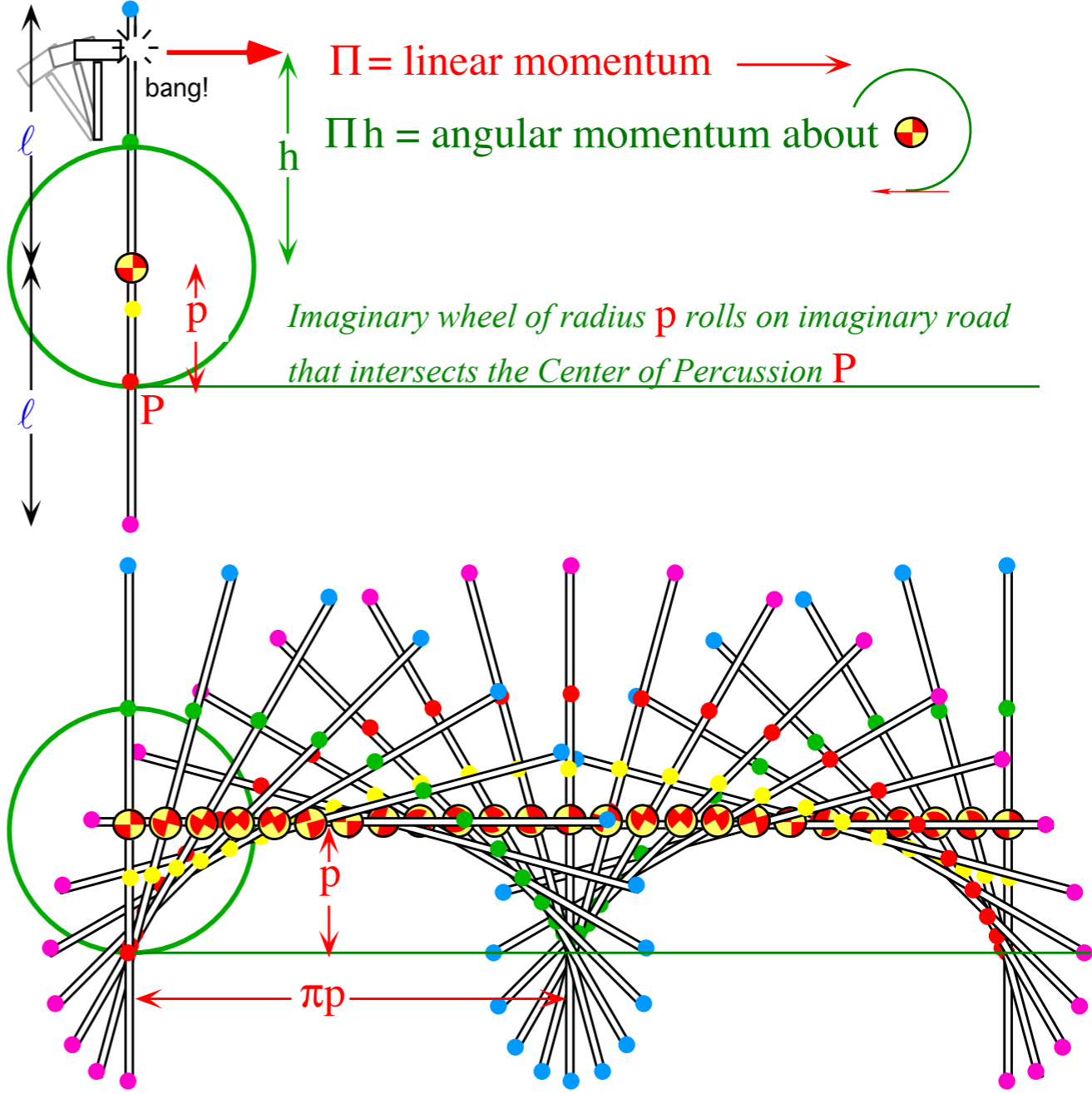


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

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Resulting angular velocity ω about the center
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$$\omega = \Lambda / I \quad (=3\Lambda / (M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi / (M \ell^2) \text{ for stick})$$

One point P , or *center of percussion (CoP)*, is
 on the wheel where speed $p\omega$ due to rotation
 just cancels translational speed V_{Center} of stick.

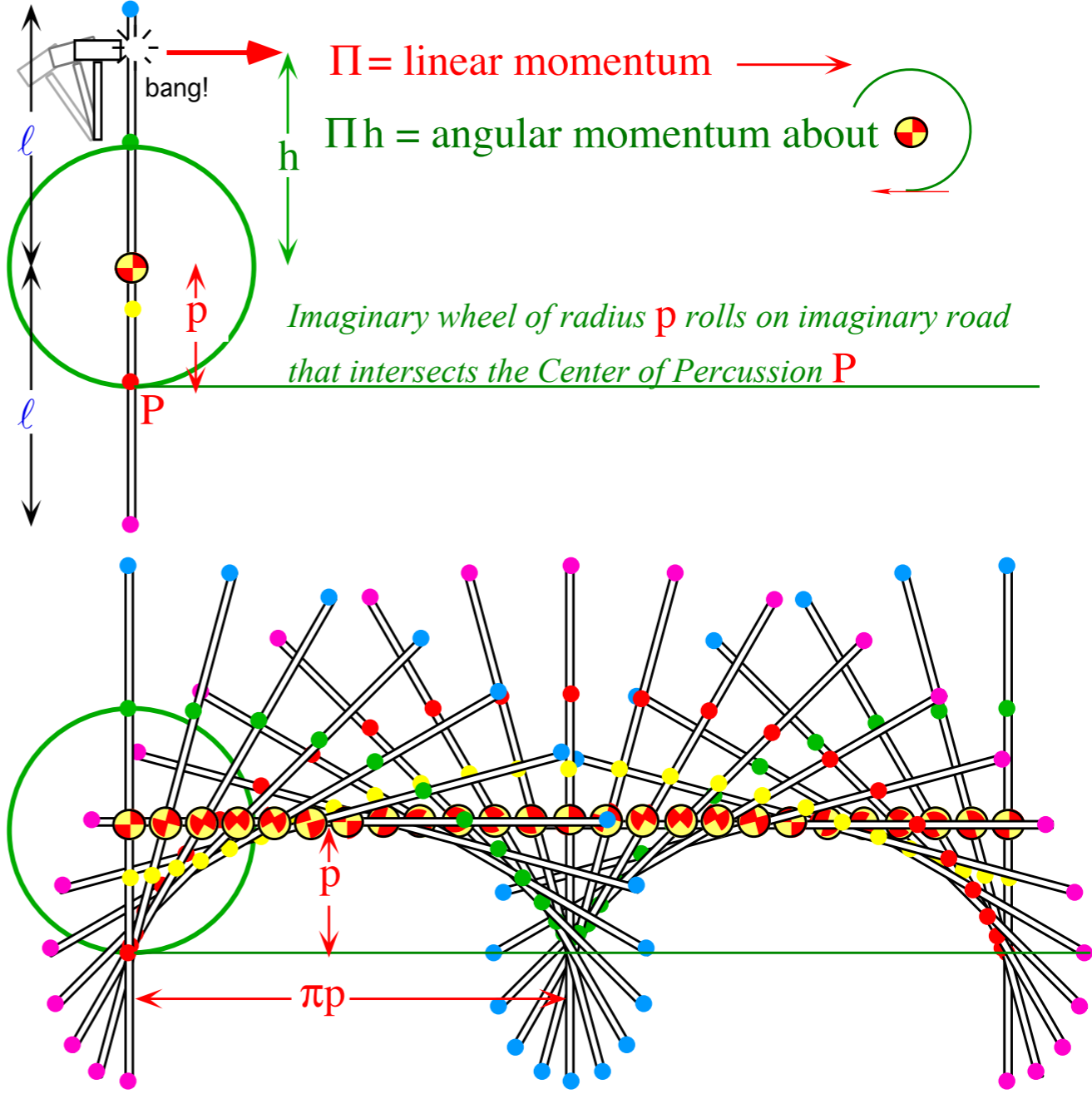


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One point P , or *center of percussion (CoP)*, is
 on the wheel where speed $p\omega$ due to rotation
 just cancels translational speed V_{Center} of stick.

$$\Pi / M = V_{Center} = |p\omega| = p \cdot h\Pi / I$$

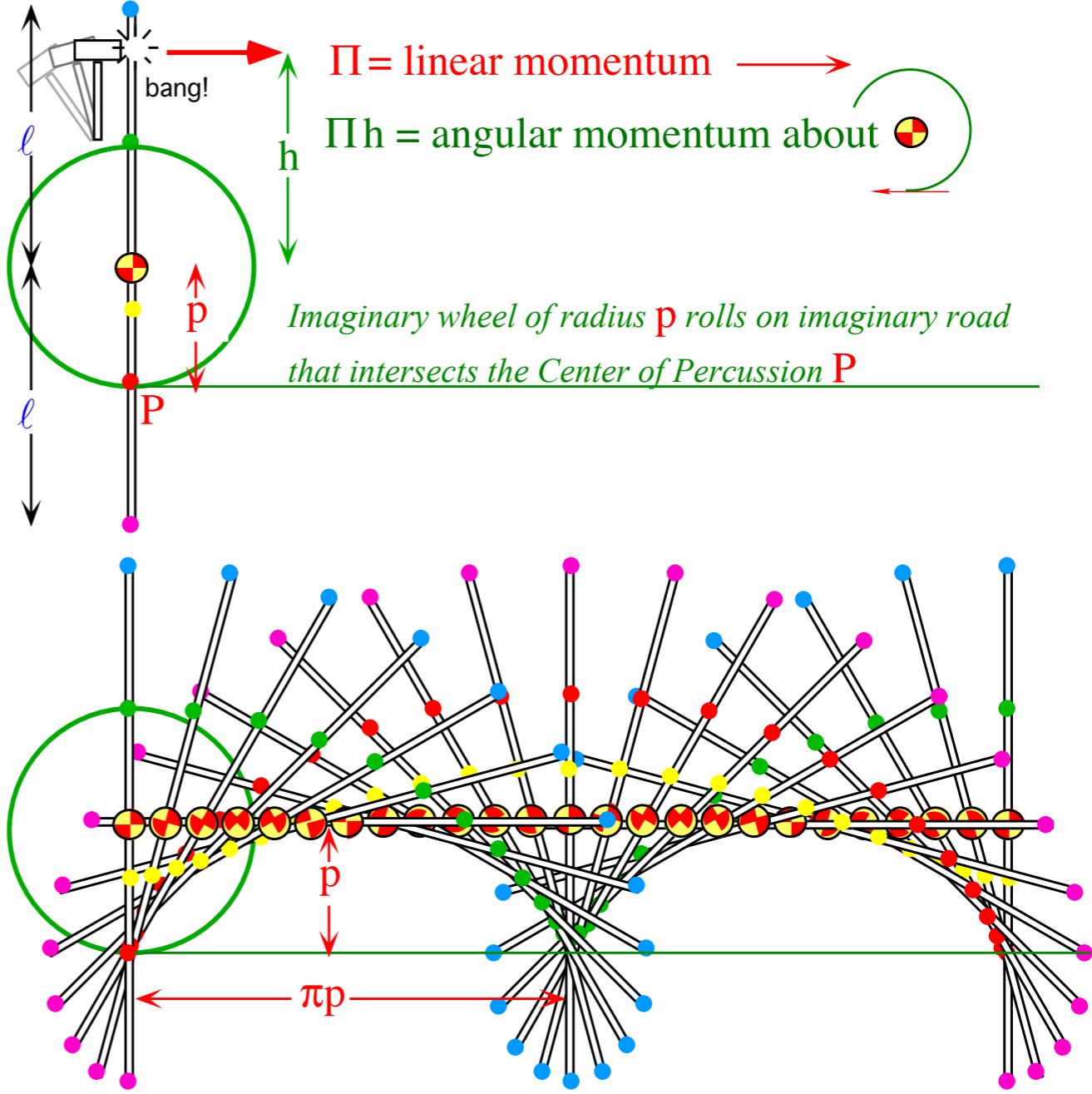


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
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 moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \Lambda / I \quad (=3\Lambda / (M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi / (M \ell^2) \text{ for stick})$$

One point P , or *center of percussion (CoP)*, is
 on the wheel where speed $p\omega$ due to rotation
 just cancels translational speed V_{Center} of stick.

$$\Pi / M = V_{Center} = |p\omega| = p \cdot h\Pi / I$$

$$I / M = \quad = \quad = p \cdot h$$

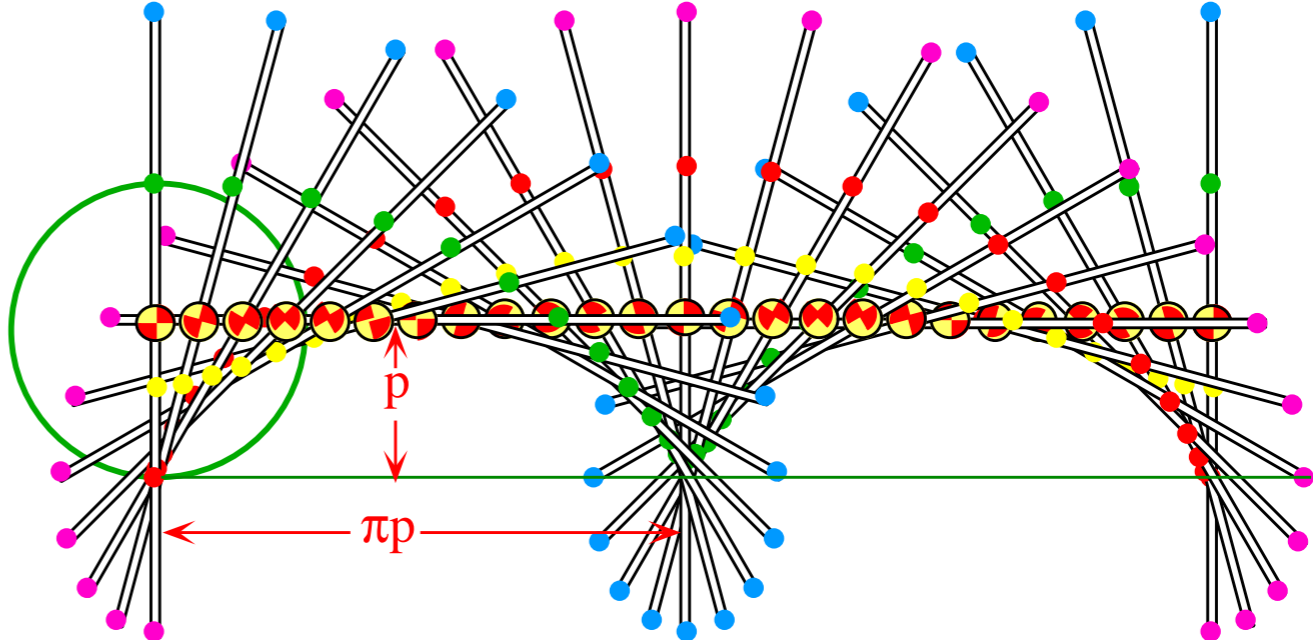
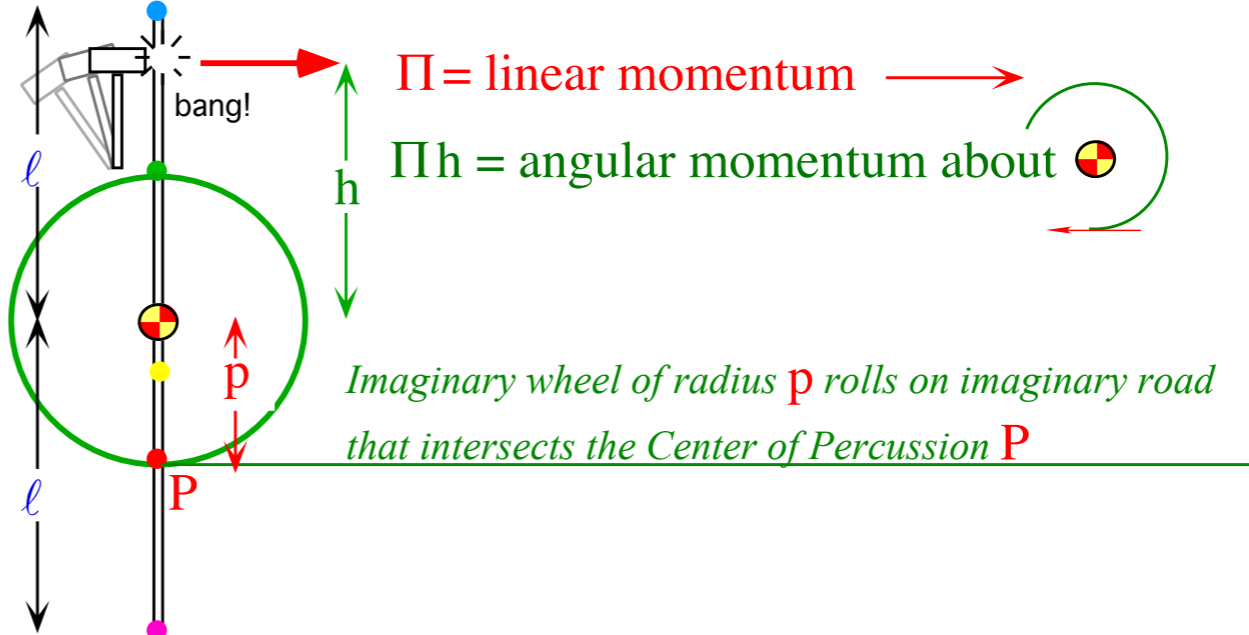


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
 you give it some linear momentum Π
 and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center
 is angular momentum Λ divided by
 moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \Lambda / I \quad (=3\Lambda / (M \ell^2) \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi / (M \ell^2) \text{ for stick})$$

One point P , or *center of percussion (CoP)*, is
 on the wheel where speed $p\omega$ due to rotation
 just cancels translational speed V_{Center} of stick.

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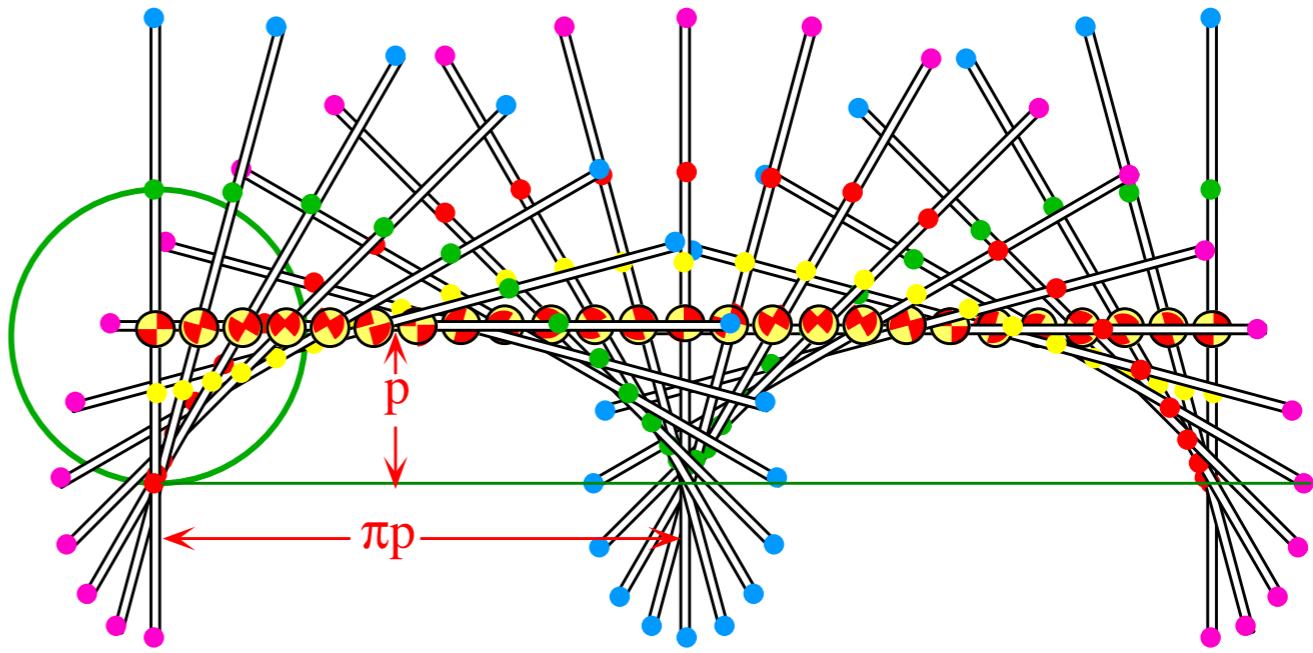
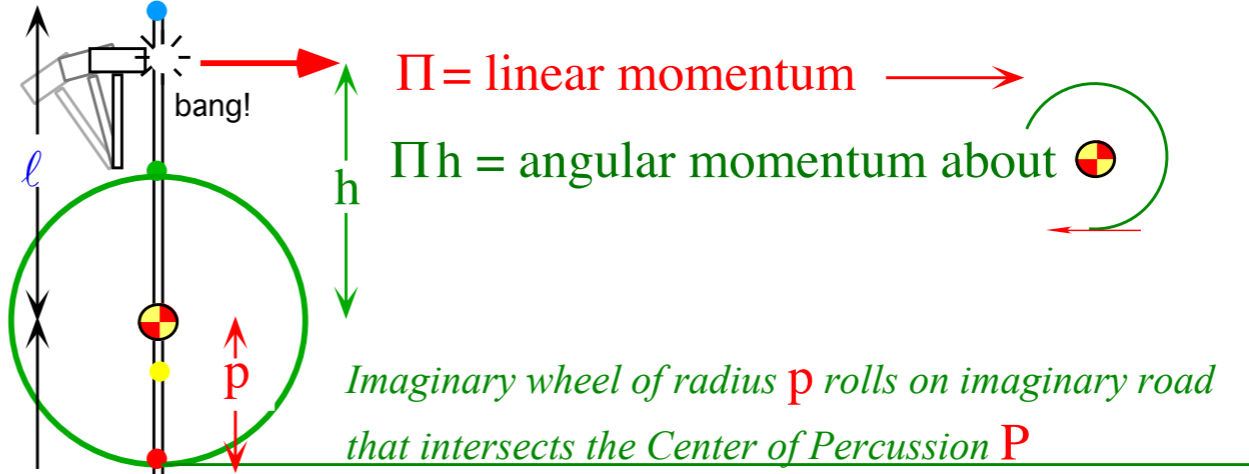


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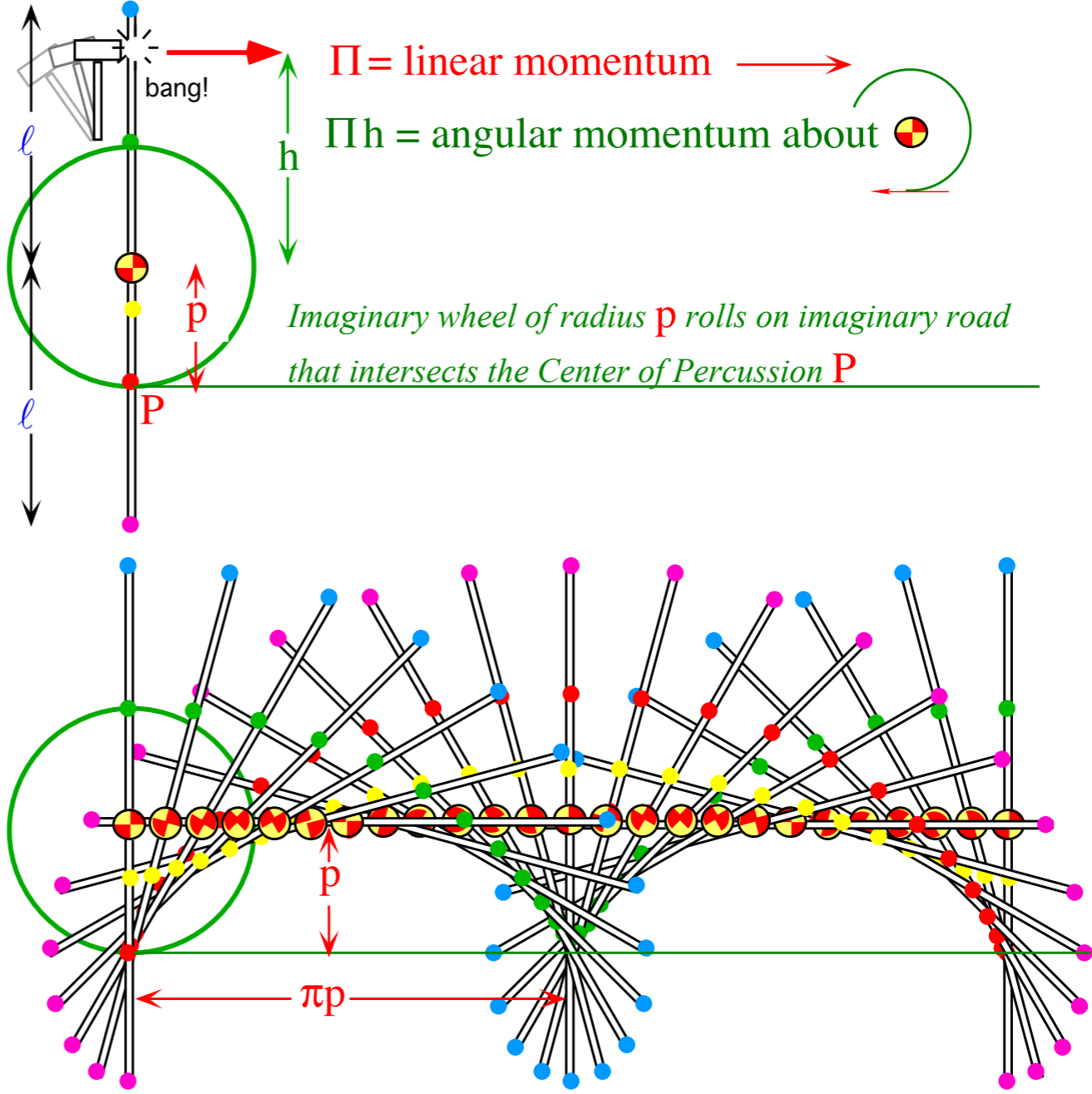


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The *percussion radius* $p = \ell^2/3h$ is of the **CoP** point that has no velocity just after hammer hits at h .

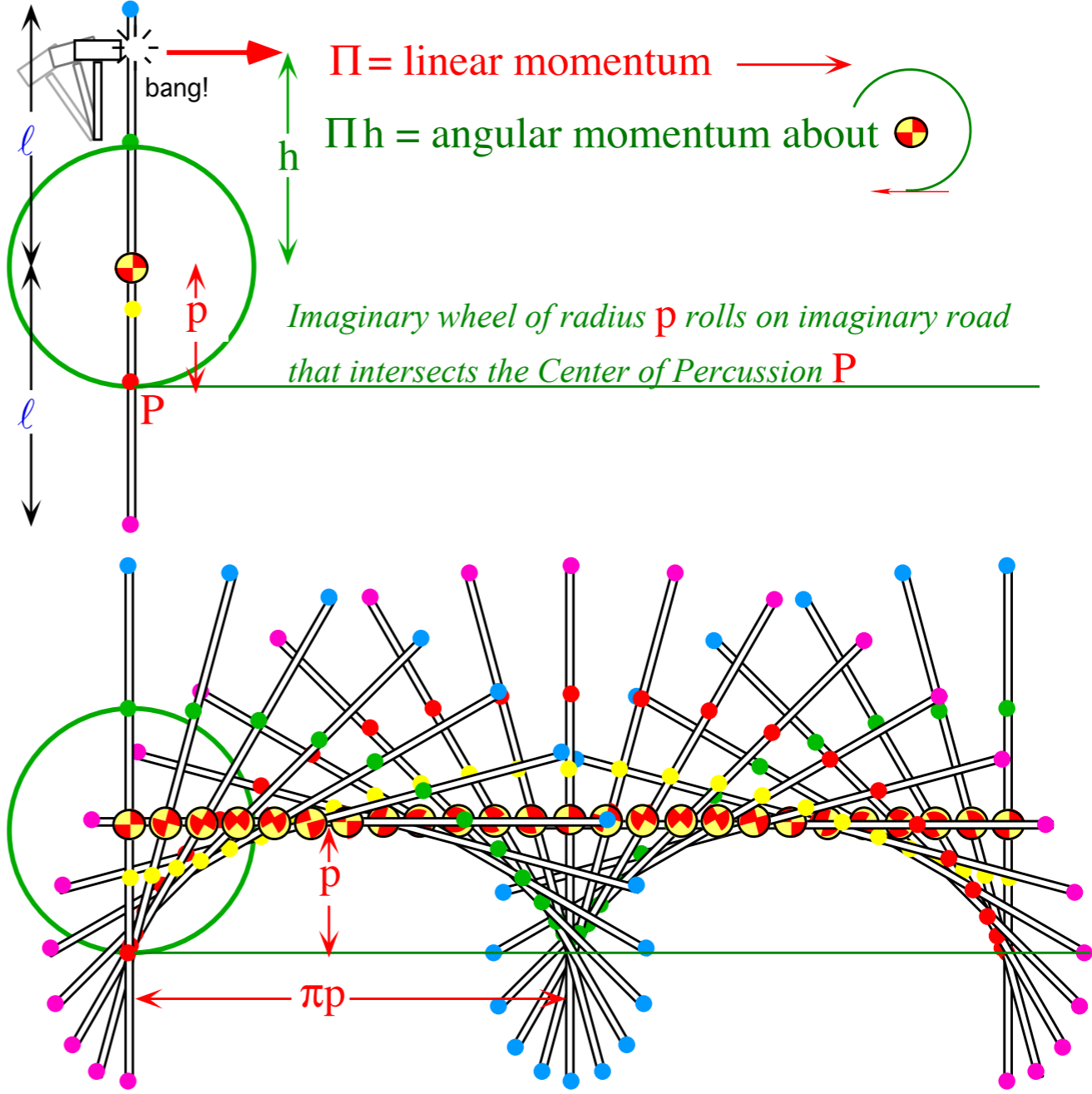


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

Separation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or “Bowl-Bowling”

Cycloidal ruler&compass geometry

Cycloid as brachistichrone

Cycloid as tautochrone

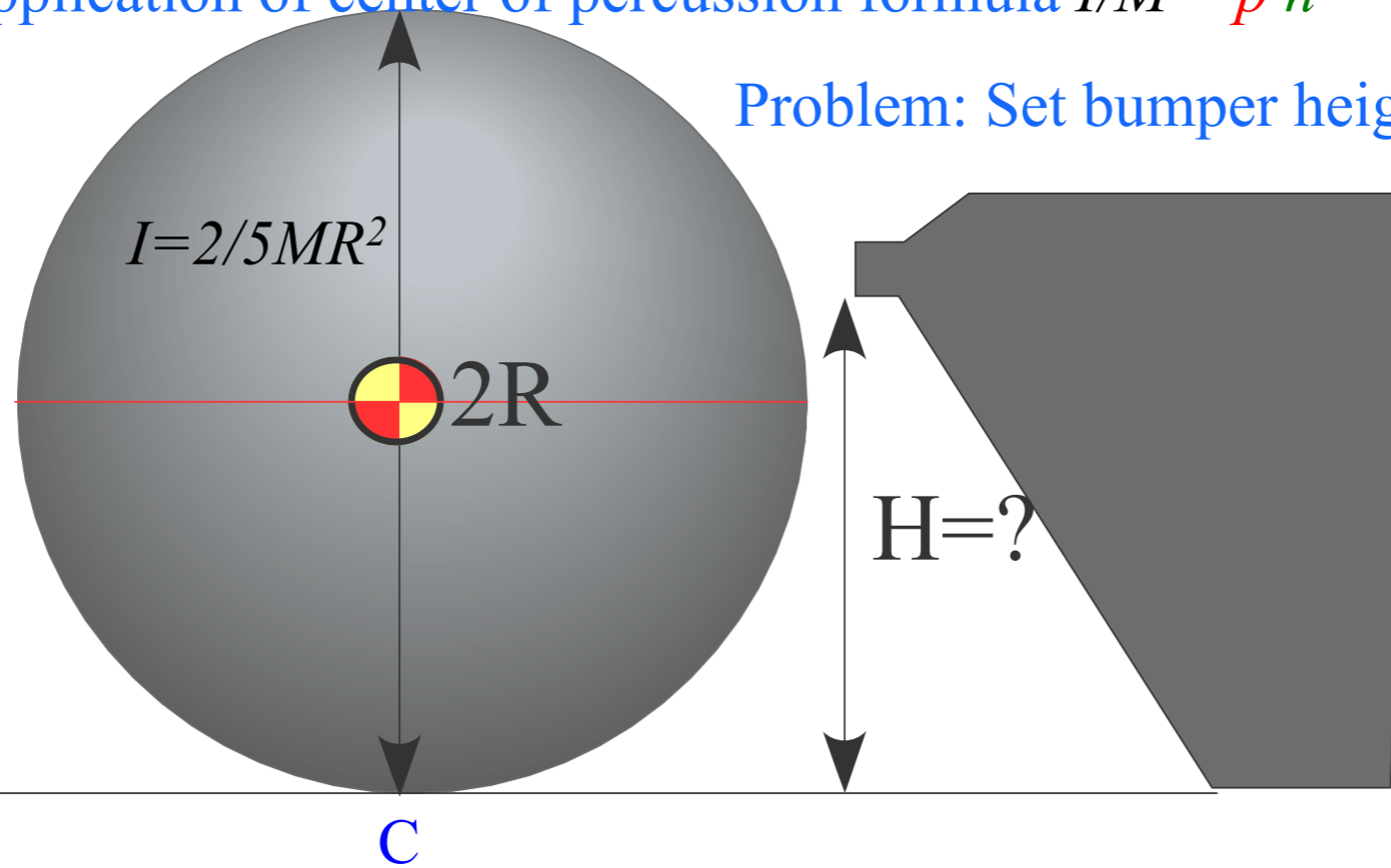
Cycloidulum vs Pendulum

Cycloidal geometry of flying levers

 *Practical poolhall application*

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

center of percussion P
above contact point C

$$I = \frac{2}{5}MR^2$$

$2R$

P

h

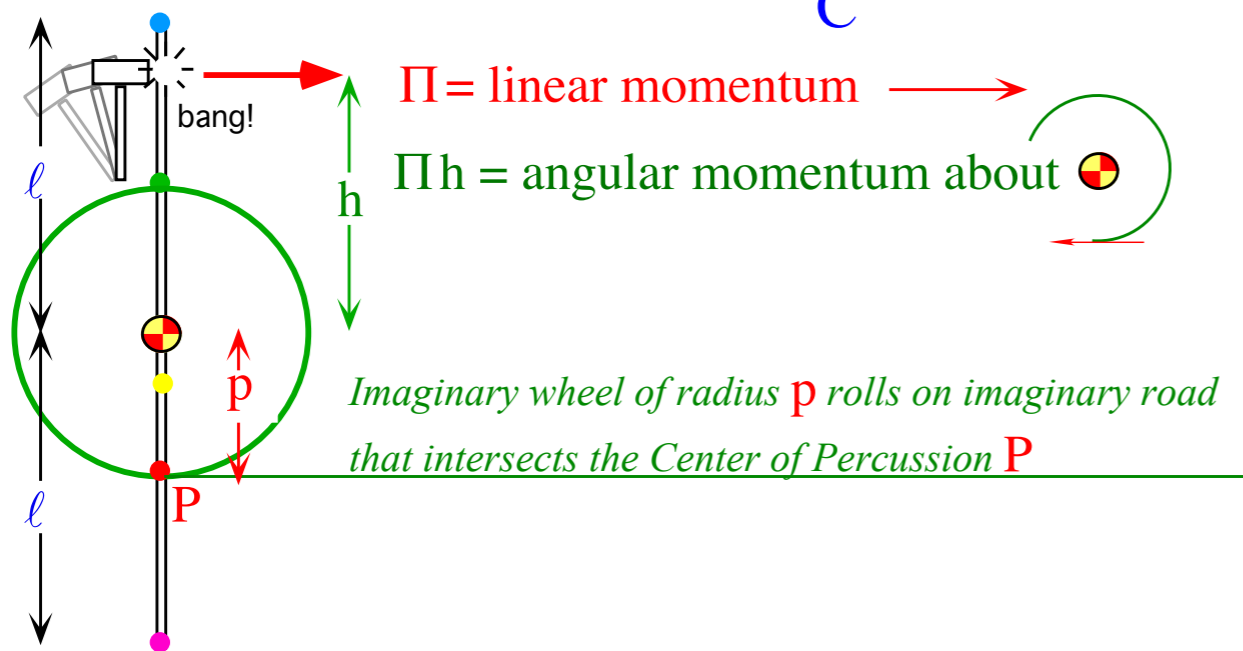
p

$H = ?$

Where should bumper height H be set to make ball contact point C at the center of percussion P ?

C

$$I/M = p \cdot h$$

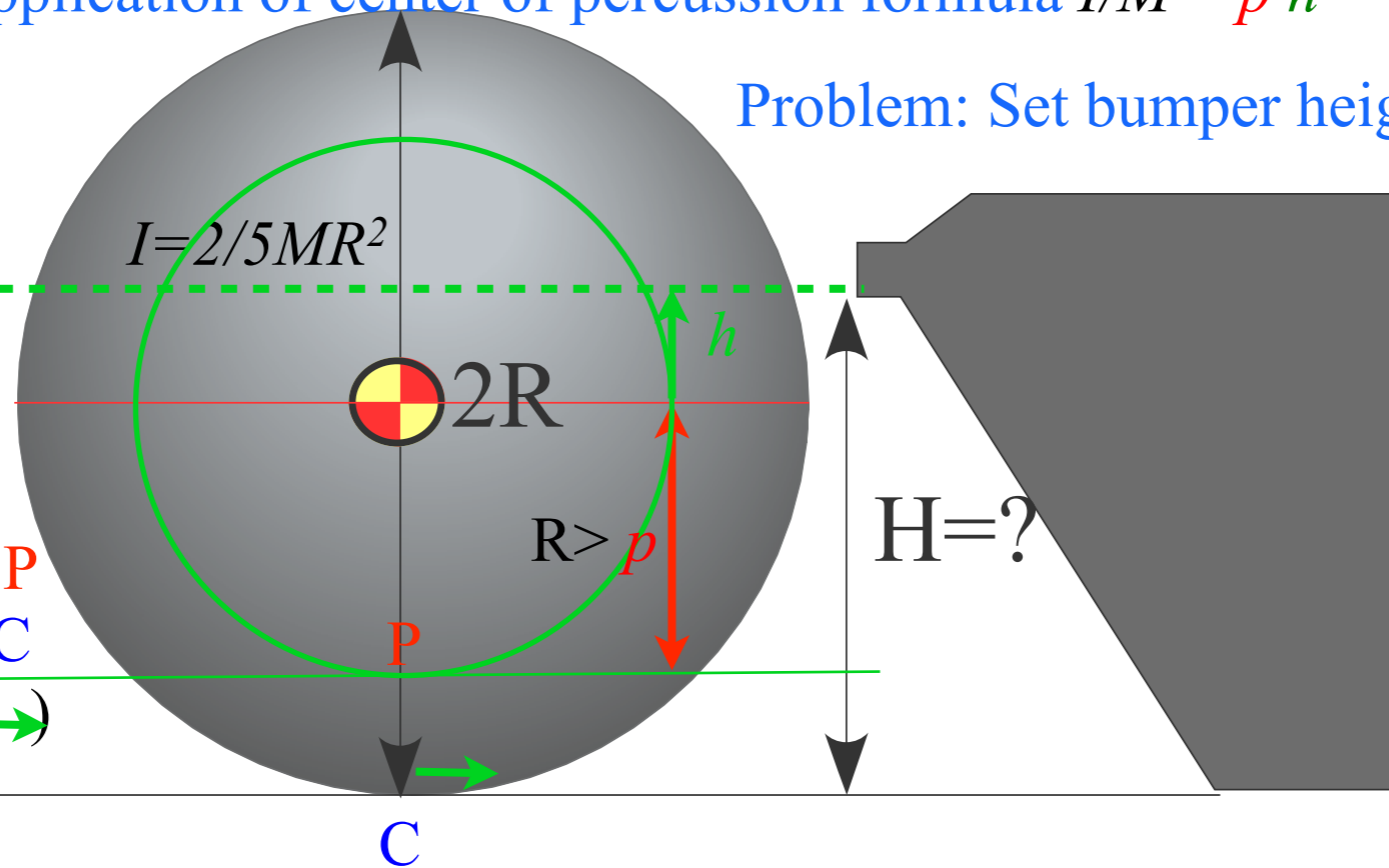


Practical poolhall application of center of percussion formula $I/M = p \cdot h$

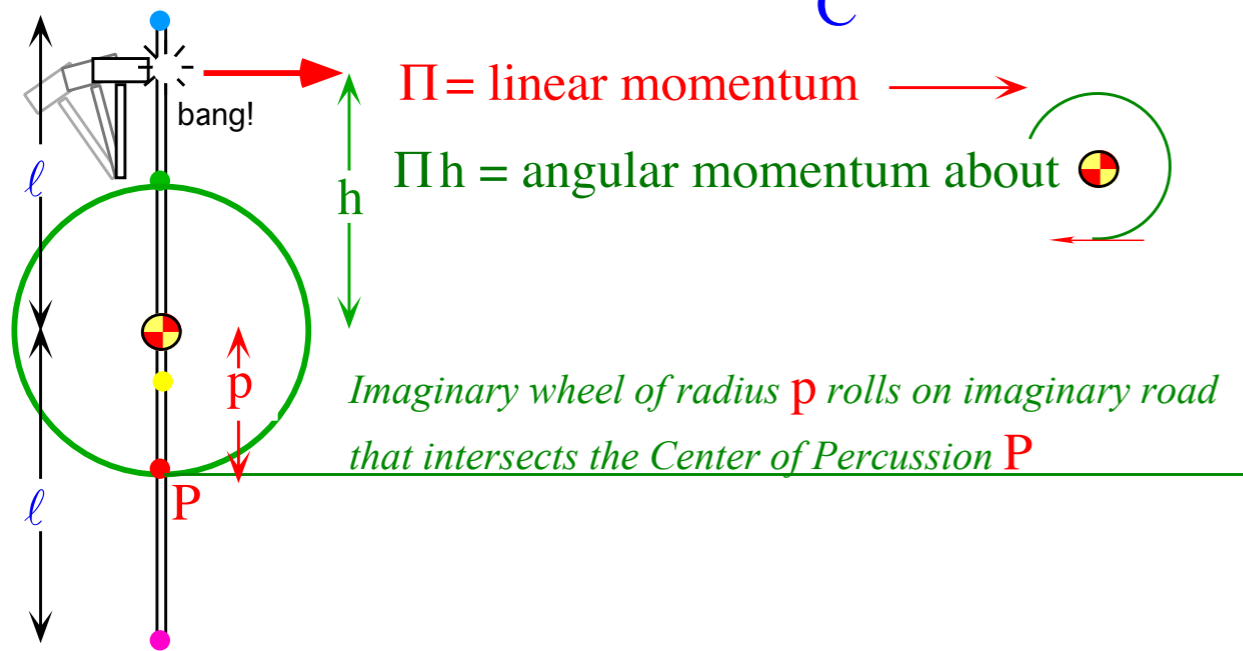
Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P ?

center of percussion P
above contact point C
(Ball skids to right \rightarrow)



$$I/M = p \cdot h$$



Practical poolhall application of center of percussion formula $I/M = p \cdot h$

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Where should bumper height H be set to make ball contact point C at the center of percussion P ?

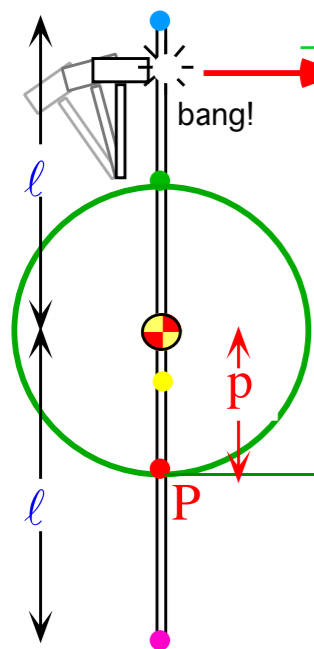
center of percussion P
below contact point C
(Ball skids to left \leftarrow)

$$I = \frac{2}{5}MR^2$$

$2R$

$$R < p$$

$H = ?$



$\Pi =$ linear momentum \rightarrow

$\Pi h =$ angular momentum about

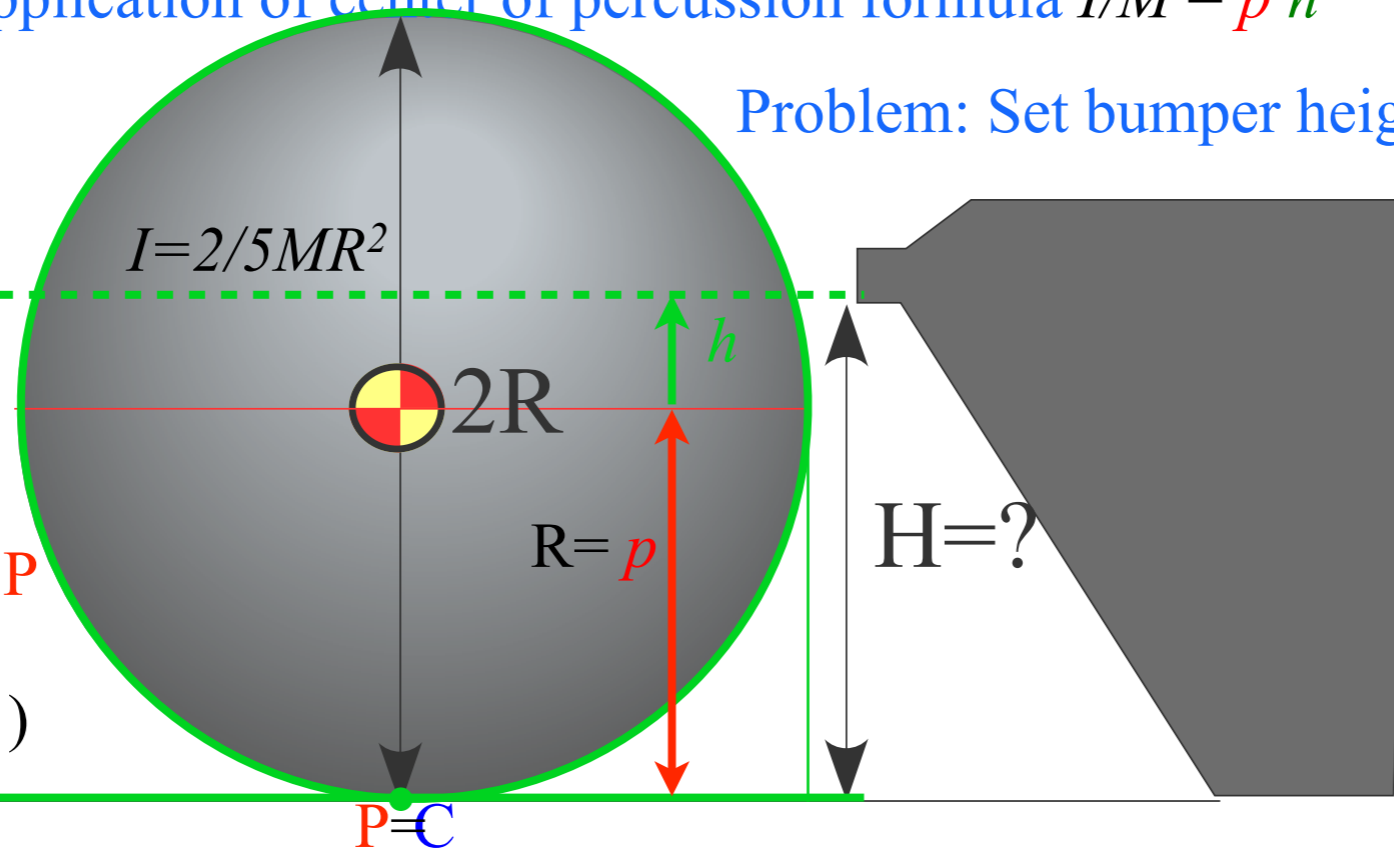
$$I/M = p \cdot h$$

Imaginary wheel of radius p rolls on imaginary road that intersects the Center of Percussion P

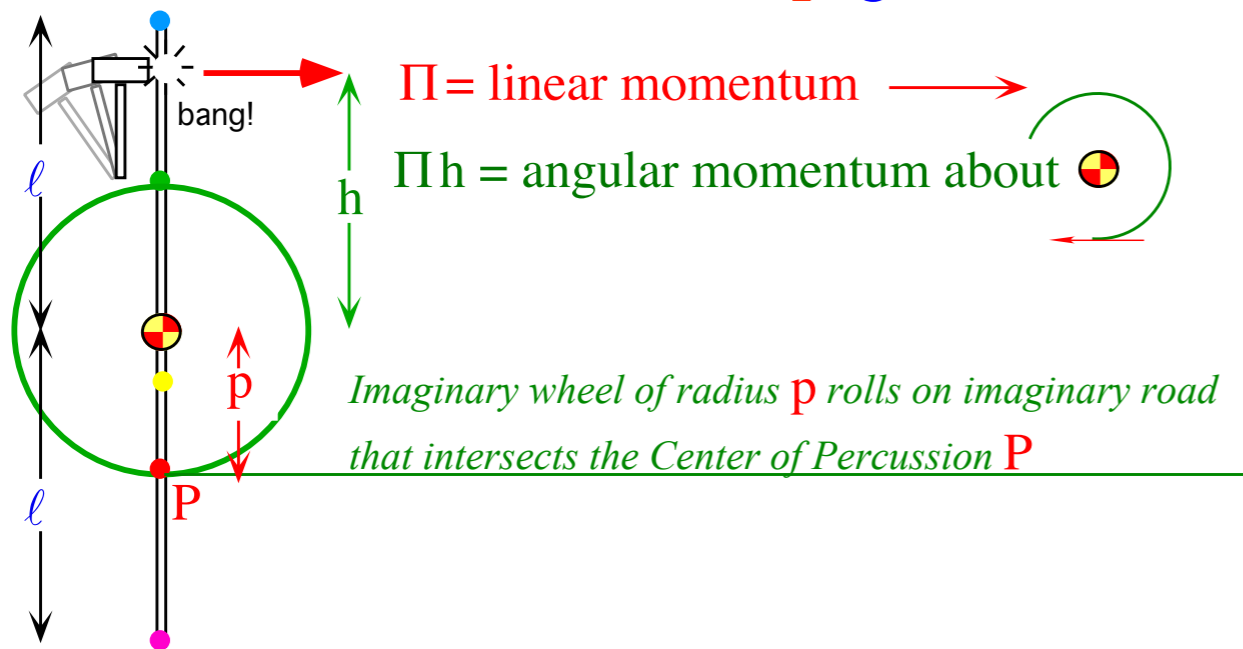
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

center of percussion P
at contact point C
(Ball does not skid •)



Where should bumper height H be set to make ball contact point C at the center of percussion P ?



$$I/M = p \cdot h$$

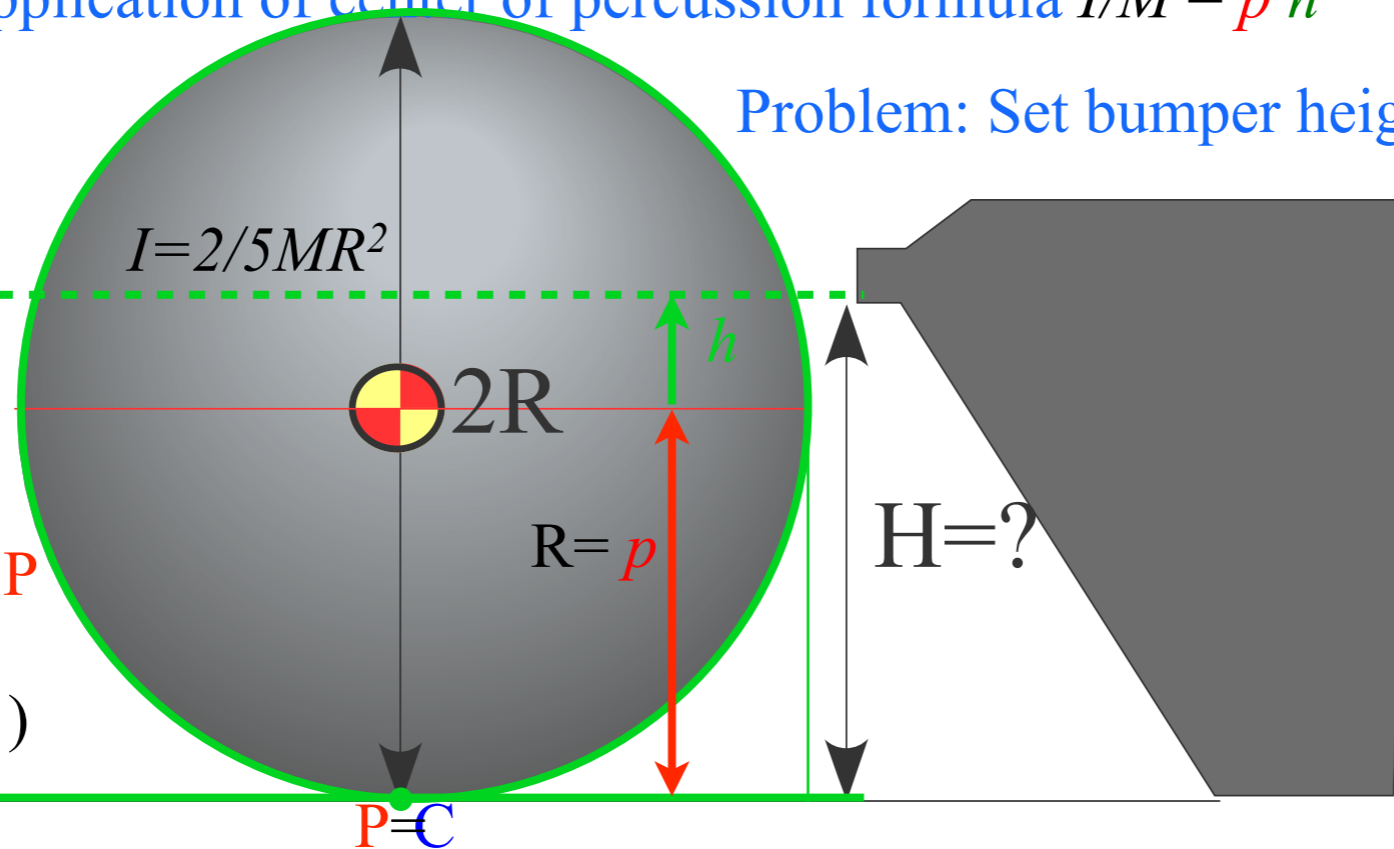
$$h = I/Mp = I/MR$$

(For $R = p$)

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.

center of percussion P
at contact point C
(Ball does not skid •)



Where should bumper height H be set to make ball contact point C at the center of percussion P ?

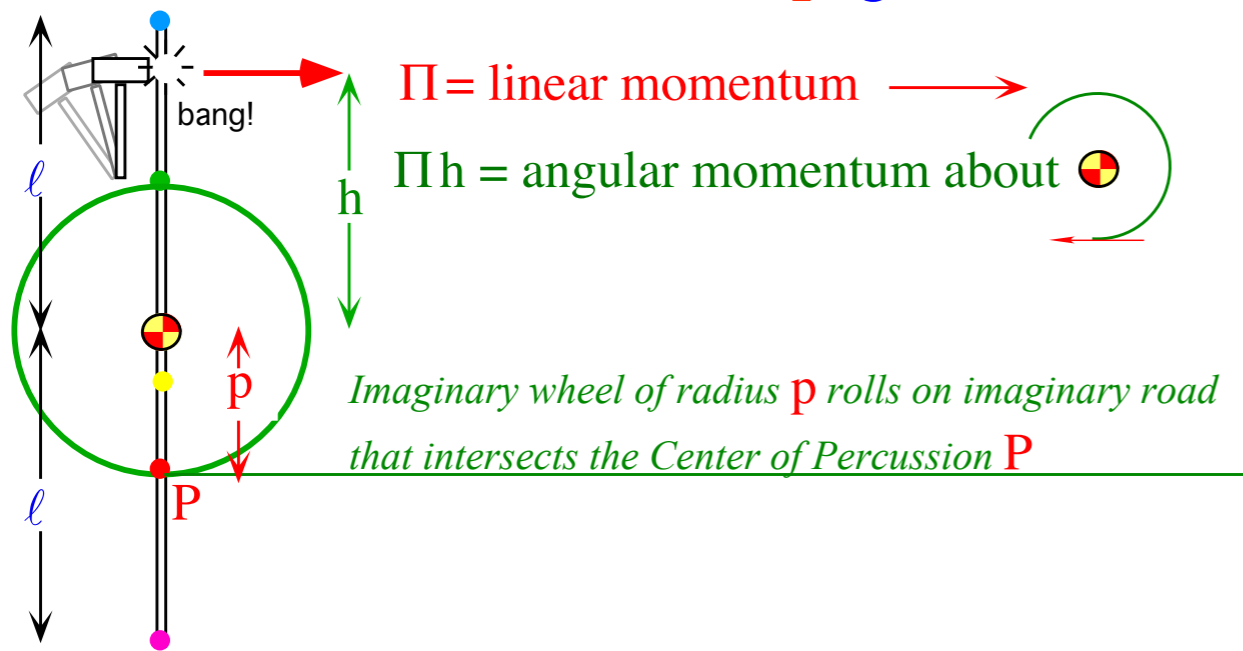
$$I/M = p \cdot h$$

$$h = I/Mp = I/MR \quad (\text{For } R=p)$$

$$= 2/5MR^2/MR$$

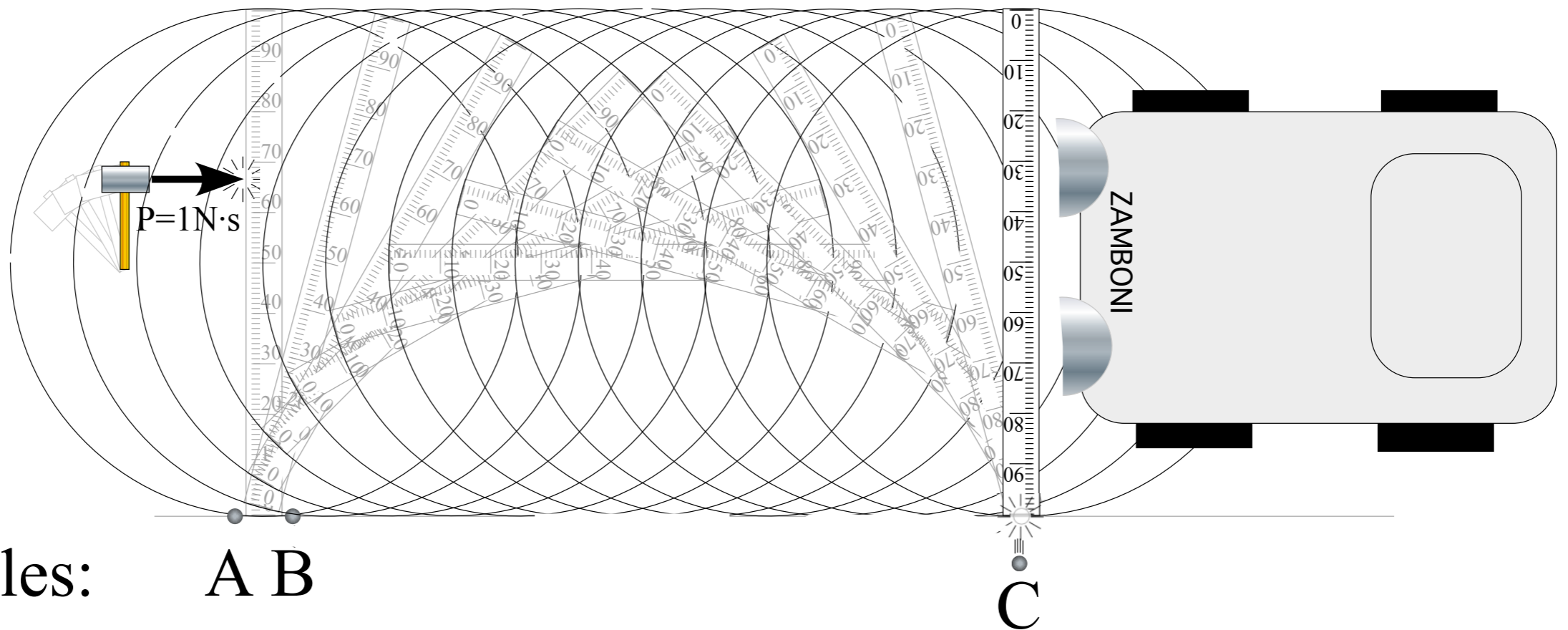
$$= 2/5R$$

For: $H = R + h = 7/10(2R)$ ball does not skid.



The Zamboni-Ice-Shot problem

(Assumes frictionless ice rink)



Marbles: A B

Where on a meter-stick do you hit it
so as to not disturb marbles A or B
and...

...knock marble C down as shown.