Lecture 29
Tue. 12.06.2015

Formerly Lect. 23 for Unit 3

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

Some Ways to do constraint analysis

Way 1. Simple constraint insertion
Way 2. GCC constraint webs
- *Find covariant force equations*
- *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

Way 3. OCC constraint webs
- *Sketch of atomic-Stark orbit parabolic OCC analysis*
  - *Classical Hamiltonian separability*

Way 4. Lagrange multipliers
- *Lagrange multiplier as eigenvalues*
- *Multiple multipliers*
- *"Non-Holonomic" multipliers*

Cycloid-like curves for rolling constraints

Quickest intra-planetary subways

Some Ways to do constraint analysis

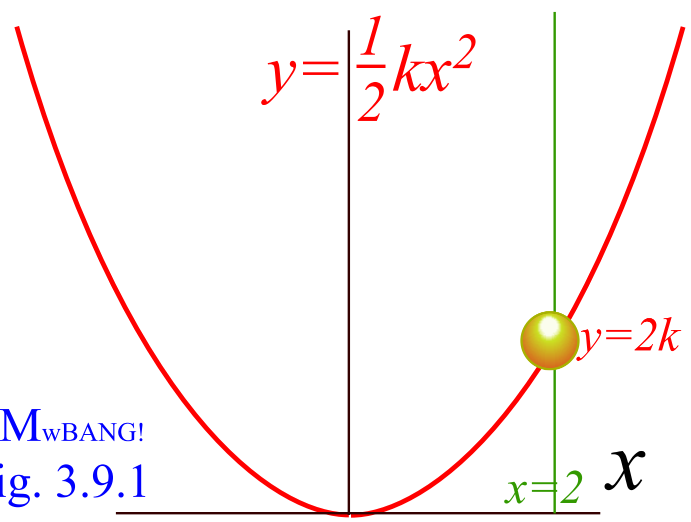
→ *Way 1. Simple constraint insertion*

Way 2. GCC constraint webs

- *Find covariant force equations*
- *Compare covariant vs. contravariant forces*

Ways to analyze a particle m constrained to parabola $y=\frac{1}{2}kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



CMwBANG!
Fig. 3.9.1

[Web Simulation](#)
[OscillatorPE - Parabolic](#)

Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

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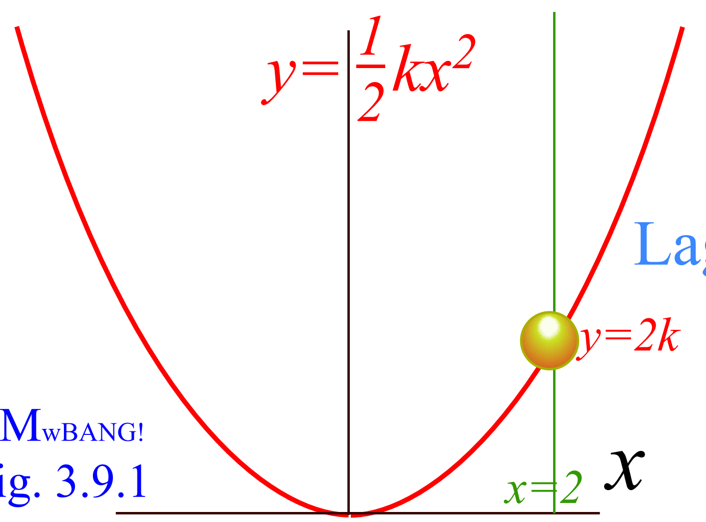

CMwBANG!
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Web Simulation
OscillatorPE - Parabolic

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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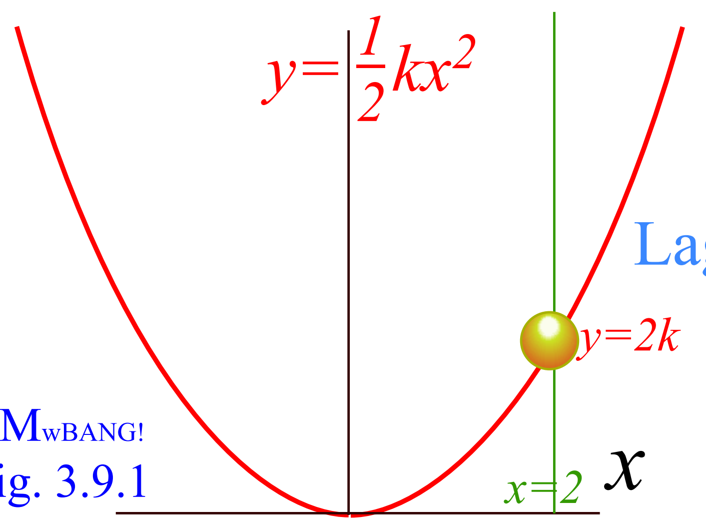

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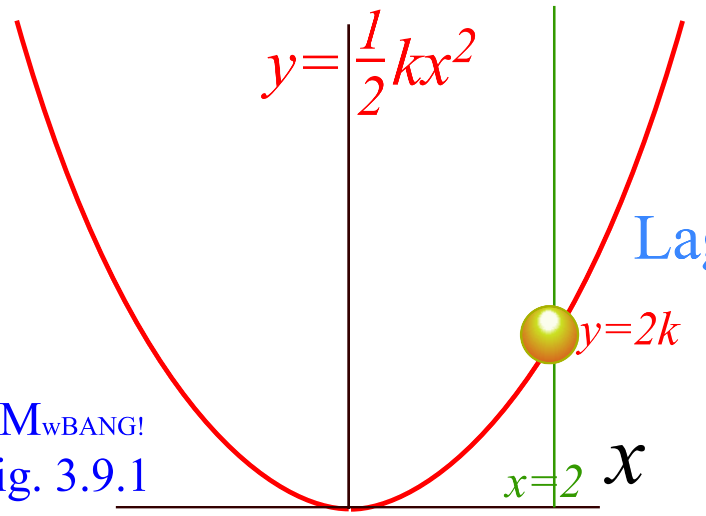

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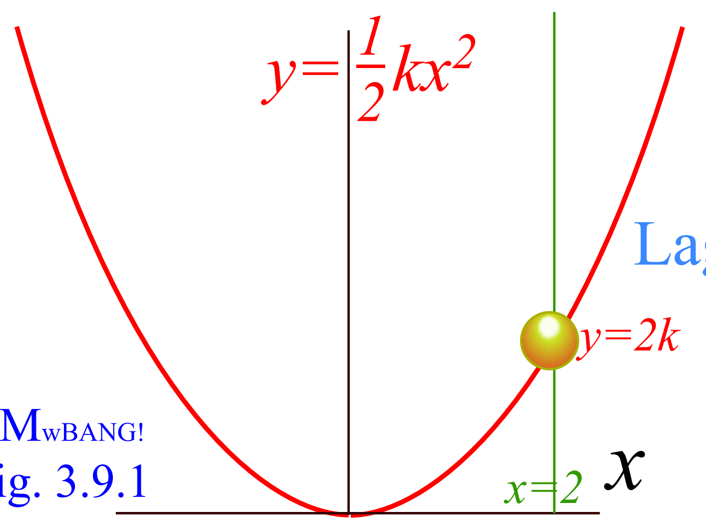

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$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

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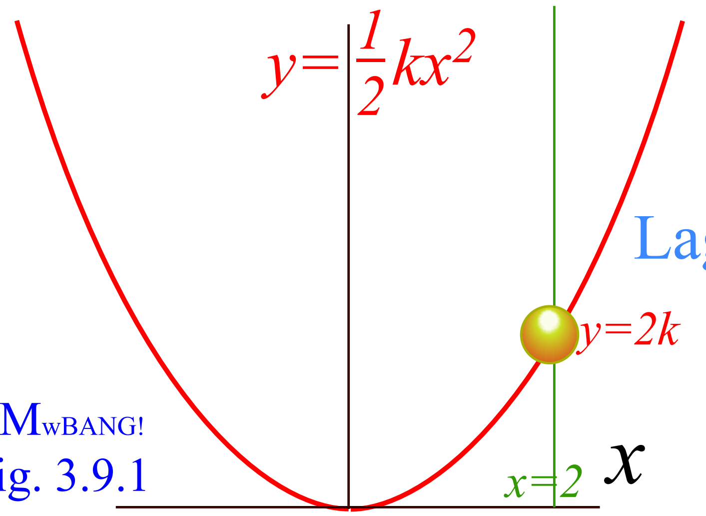

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}$$

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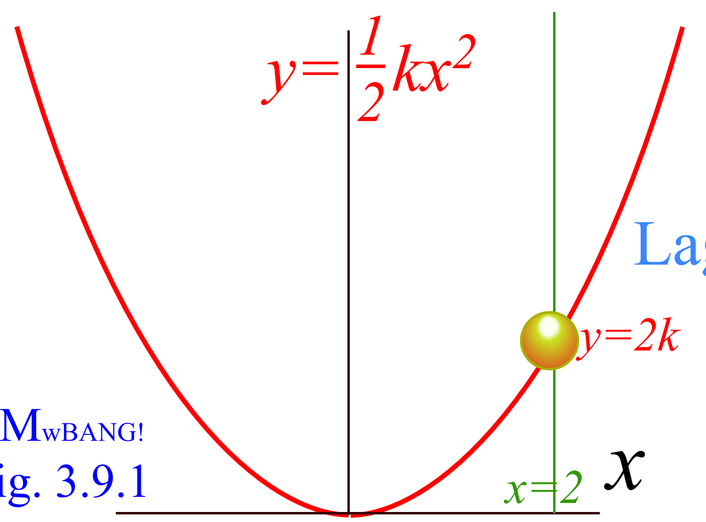

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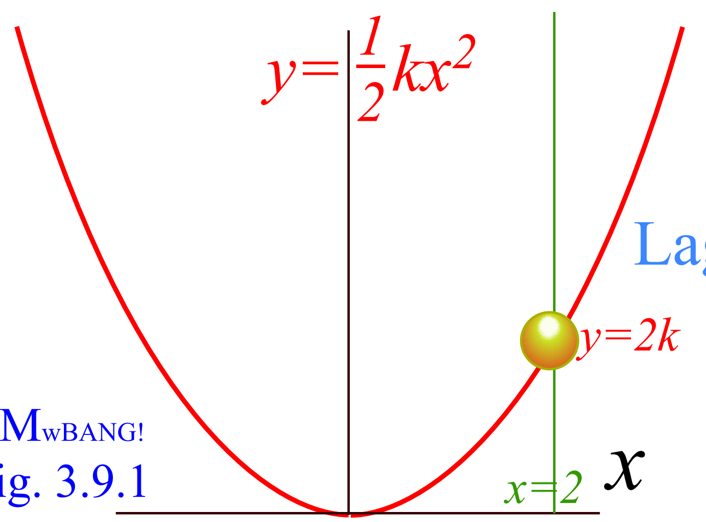

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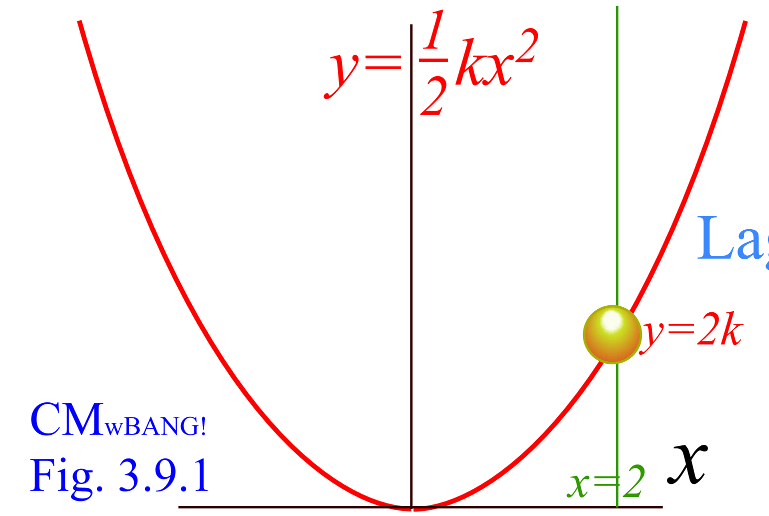
$$\dot{p}_x=m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx$$

Ways to analyze a particle m constrained to parabola $y=\frac{1}{2}kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

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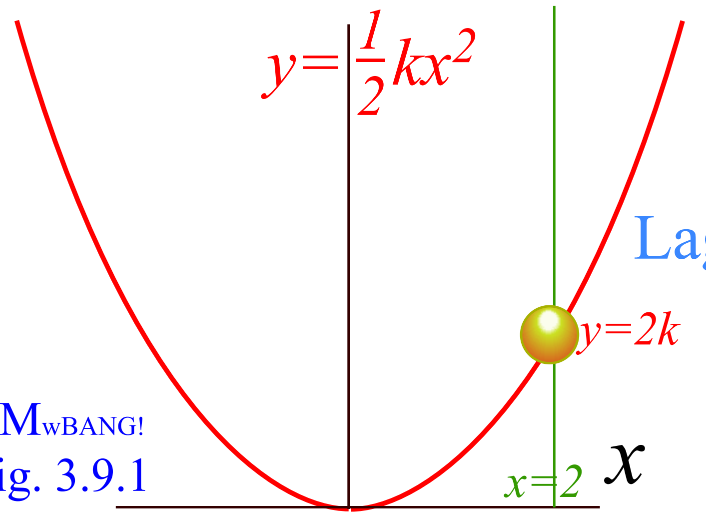

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$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 = \frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x} = -mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

$$\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx$$

Web Simulation OscillatorPE - Parabolic

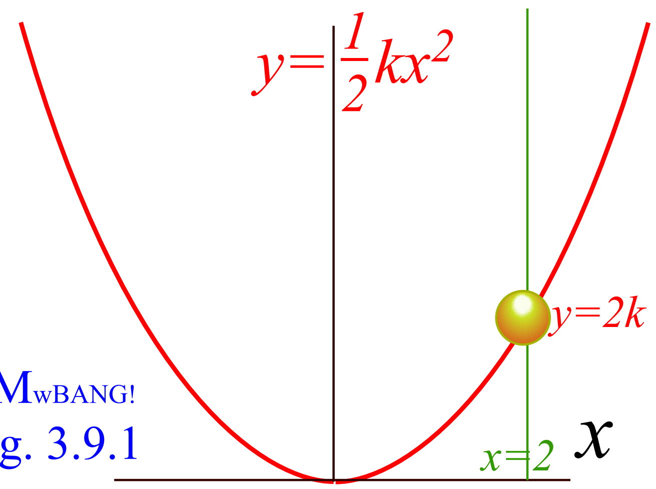
Some Ways to do constraint analysis

Way 1. Simple constraint insertion

→ *Way 2. GCC constraint webs*

- *Find covariant force equations*
- *Compare covariant vs. contravariant forces*

(a) Constrained motion

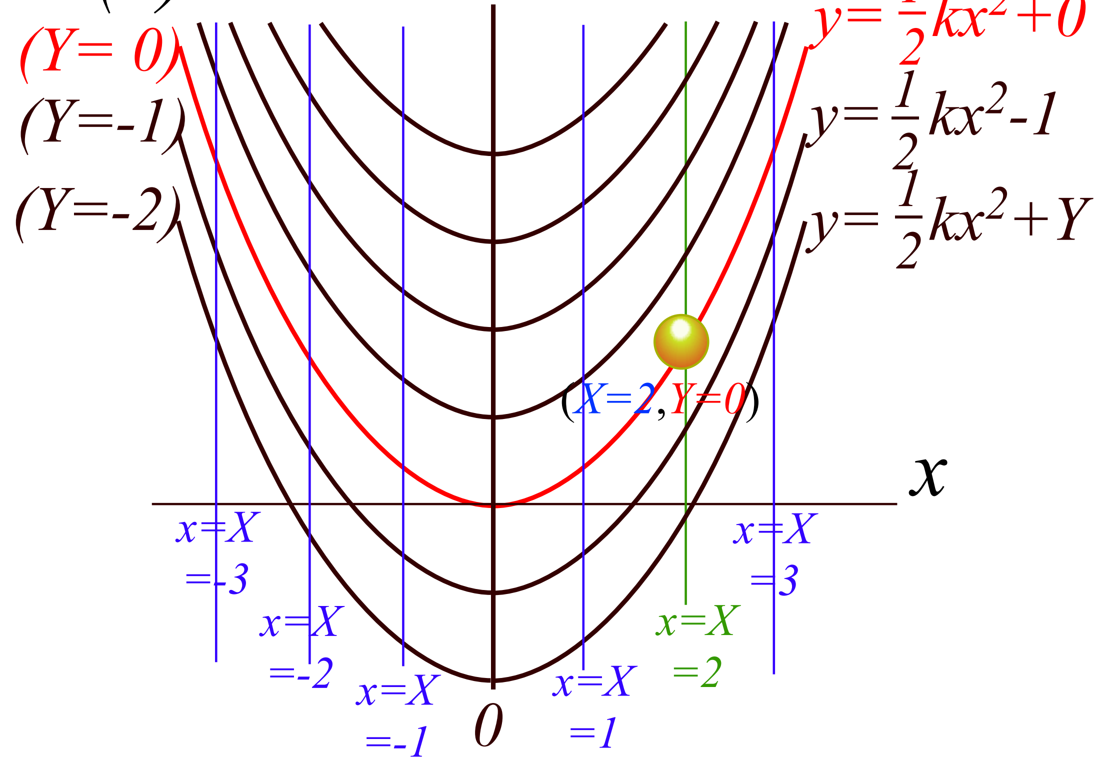


CMwBANG!
Fig. 3.9.1

$$x = X$$
$$y = \frac{1}{2}kx^2 + Y$$

Cartesian (x,y) transform to GCC (X,Y)

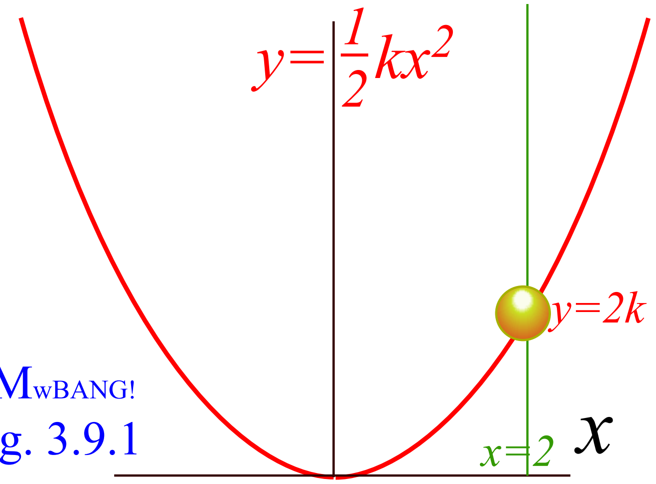
(b) GCC constraint web



Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

Web Simulation - OscillatorPE
Parabolic w/grid

(a) Constrained motion



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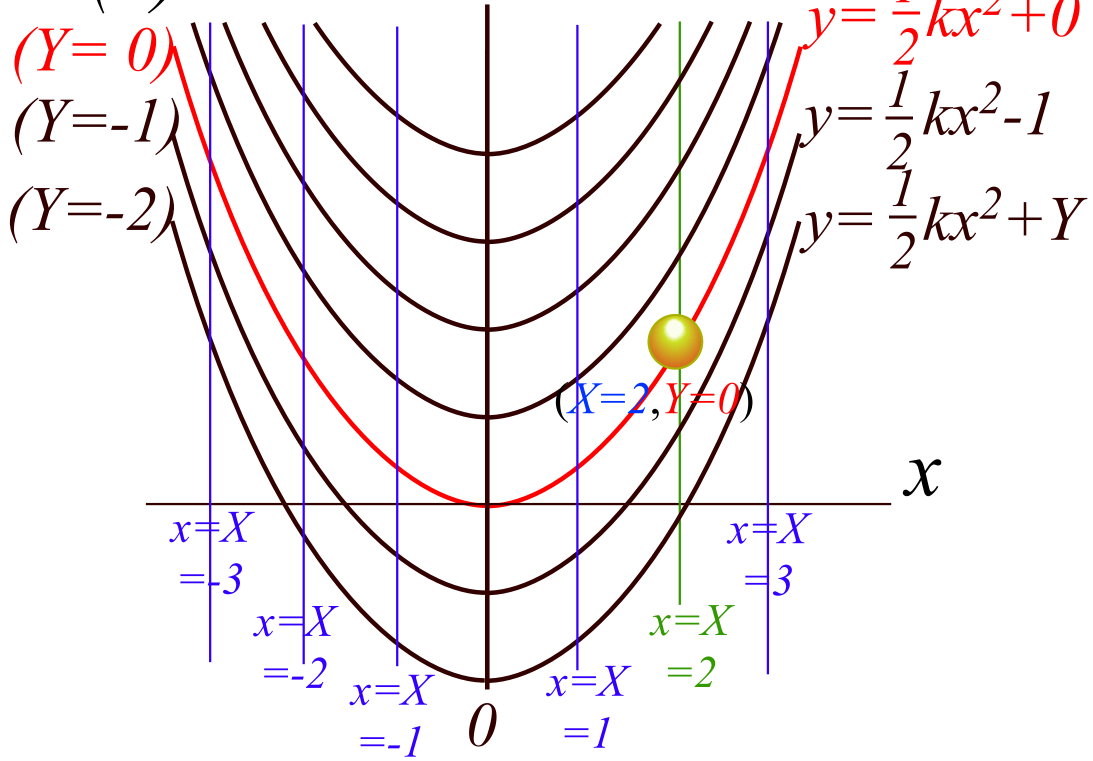


Incorporate the constraint curve $y=½kx^2$ into any matching GCC web.

$x=q^1=X$ $\qquad$ $y=½kx^2 + q^2 = kX^2/2+Y$

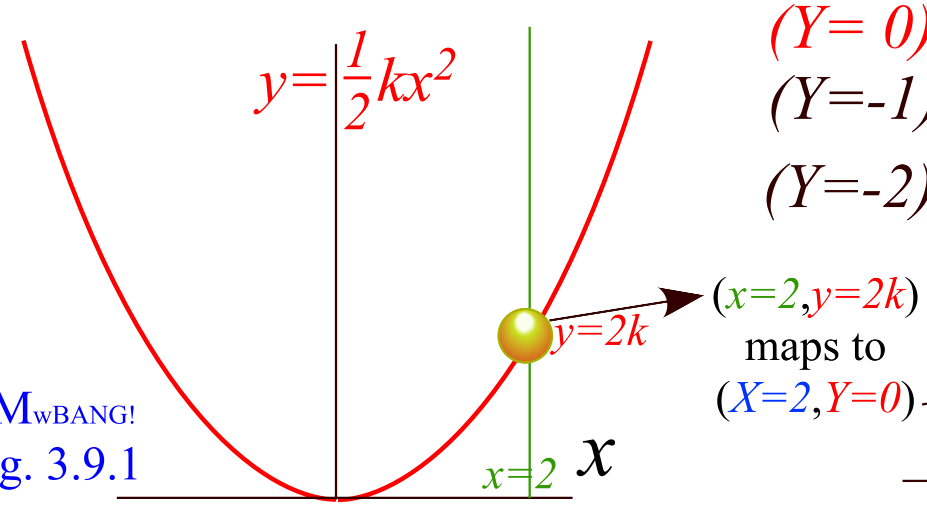
we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}^{Indices}$

Web Simulation - OscillatorPE Parabolic w/grid

Way 2. GCC constraint webs.

(a) Constrained motion



CMwBANG!
Fig. 3.9.1

$x = X$
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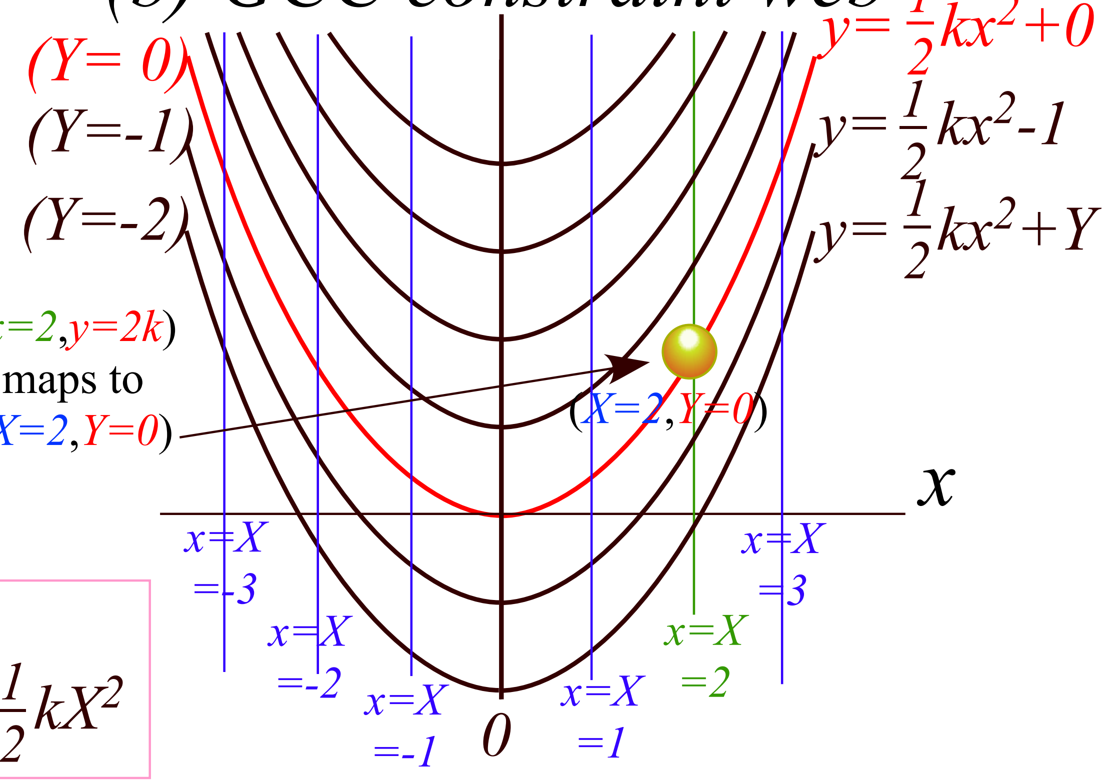
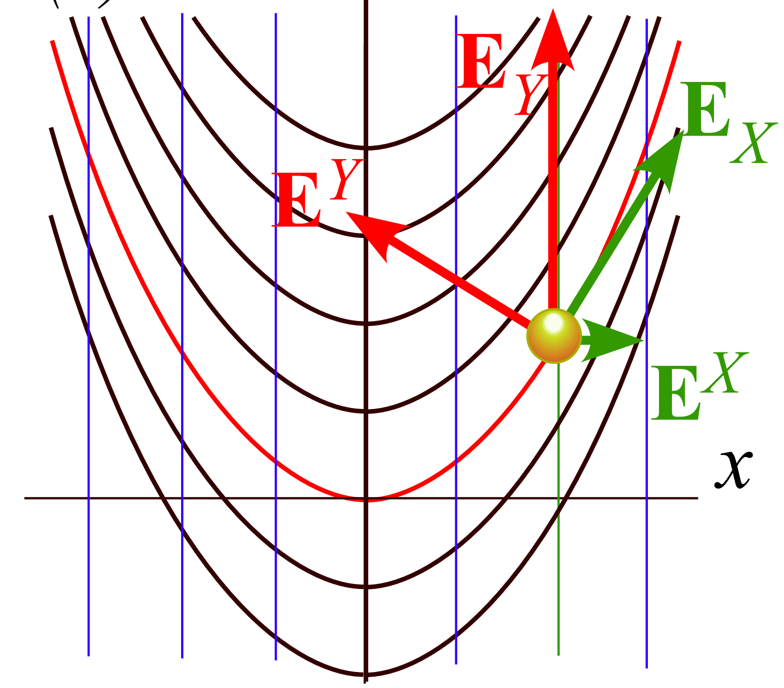
Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y = y - \frac{1}{2}kX^2$

(b) GCC constraint web



(c) GCC **E**-vectors



we define shorthand:

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Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$x = q^1 = X \qquad y = \frac{1}{2}kx^2 + q^2 = kX^2/2 + Y$

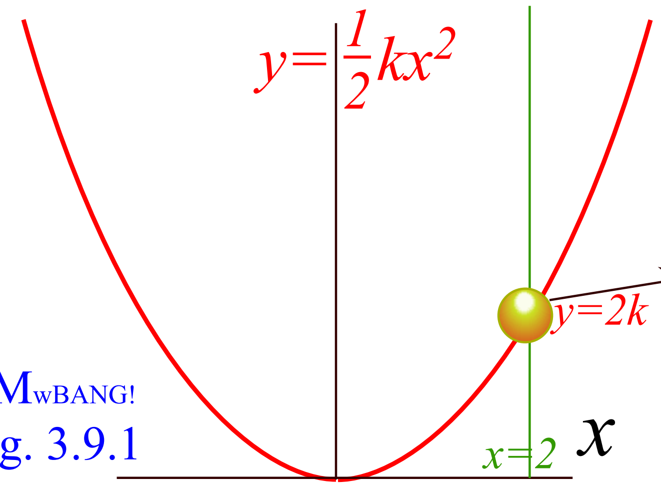
Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Web Simulation - OscillatorPE
Parabolic w/grid & basis vectors

Way 2. GCC constraint webs.

(a) Constrained motion



CMwBANG!
Fig. 3.9.1

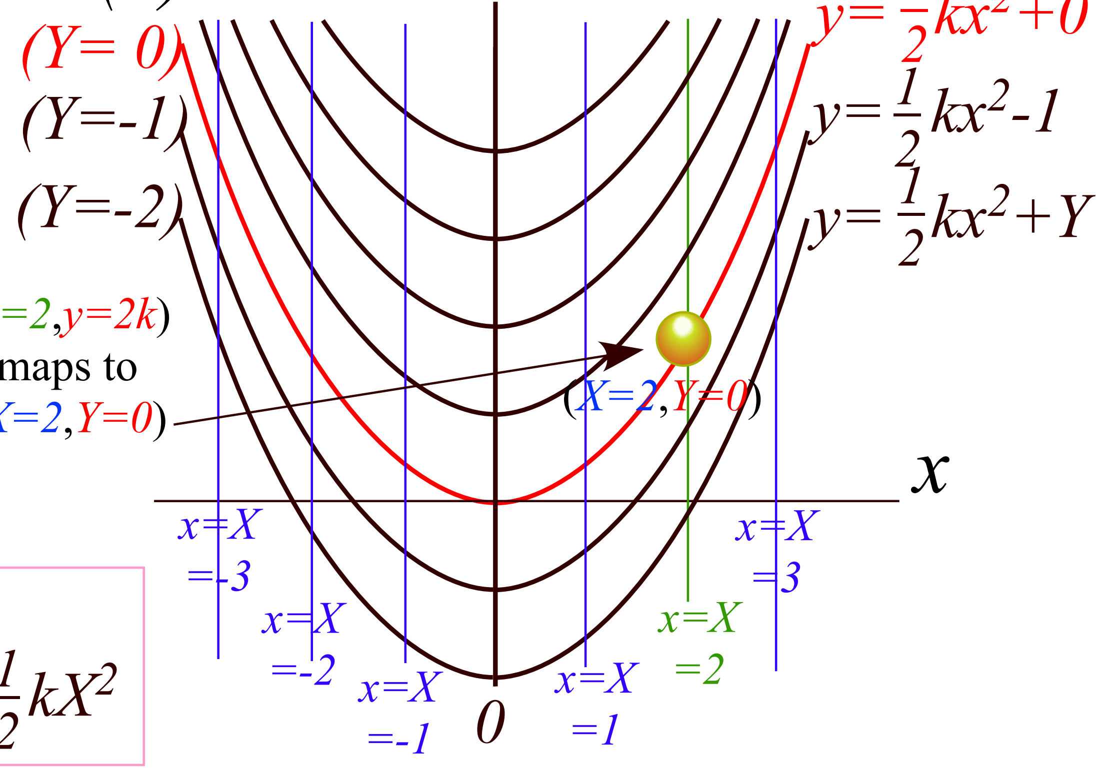
$(x=2, y=2k)$ maps to $(X=2, Y=0)$

$$x = X$$
$$y = \frac{1}{2}kx^2 + Y$$

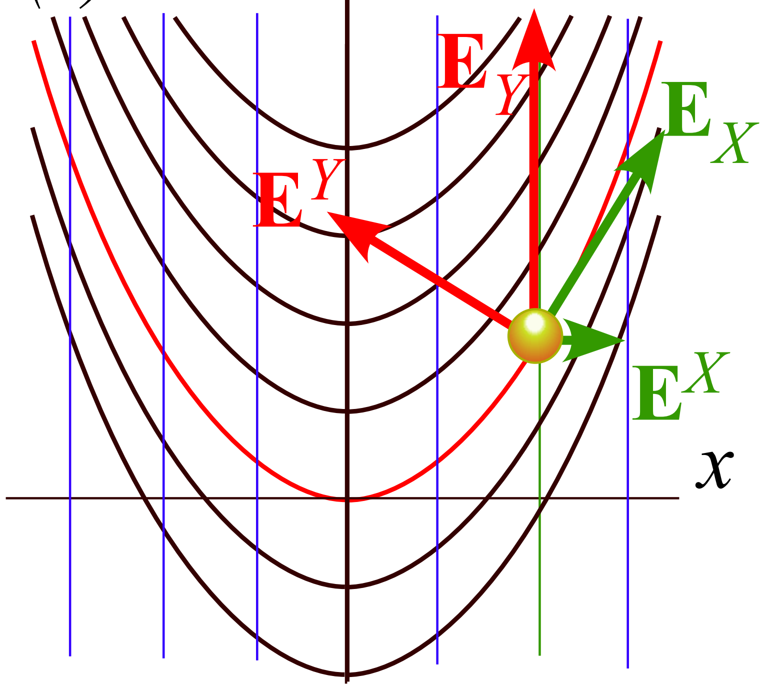
Cartesian (x,y) transform to GCC (X,Y)

$$X = x$$
$$Y = y - \frac{1}{2}kX^2$$

(b) GCC constraint web



(c) GCC E-vectors



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Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$$x = q^1 = X \qquad y = \tfrac{1}{2}kx^2 + q^2 = kX^2/2 + Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

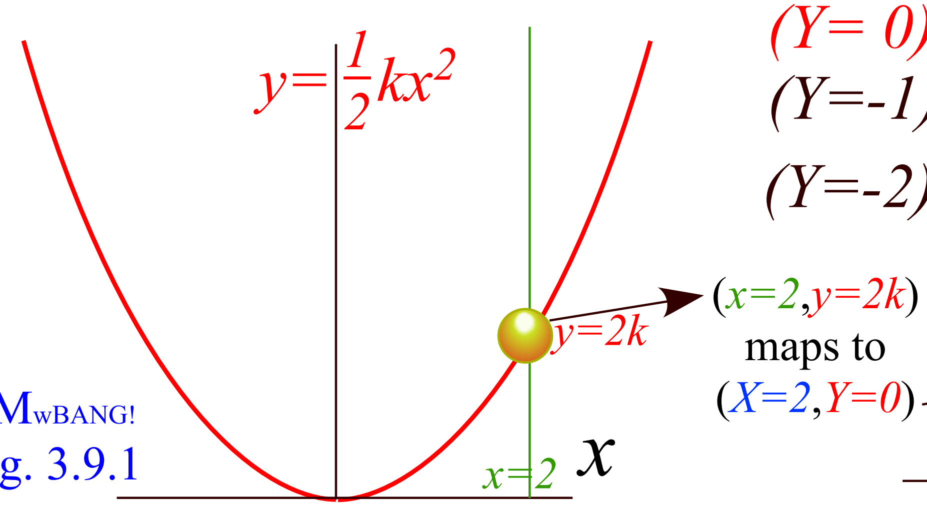
$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \; \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix}$$

Web Simulation - OscillatorPE
Parabolic w/grid & basis vectors

(a) Constrained motion



CMwBANG! Fig. 3.9.1

$x = X$

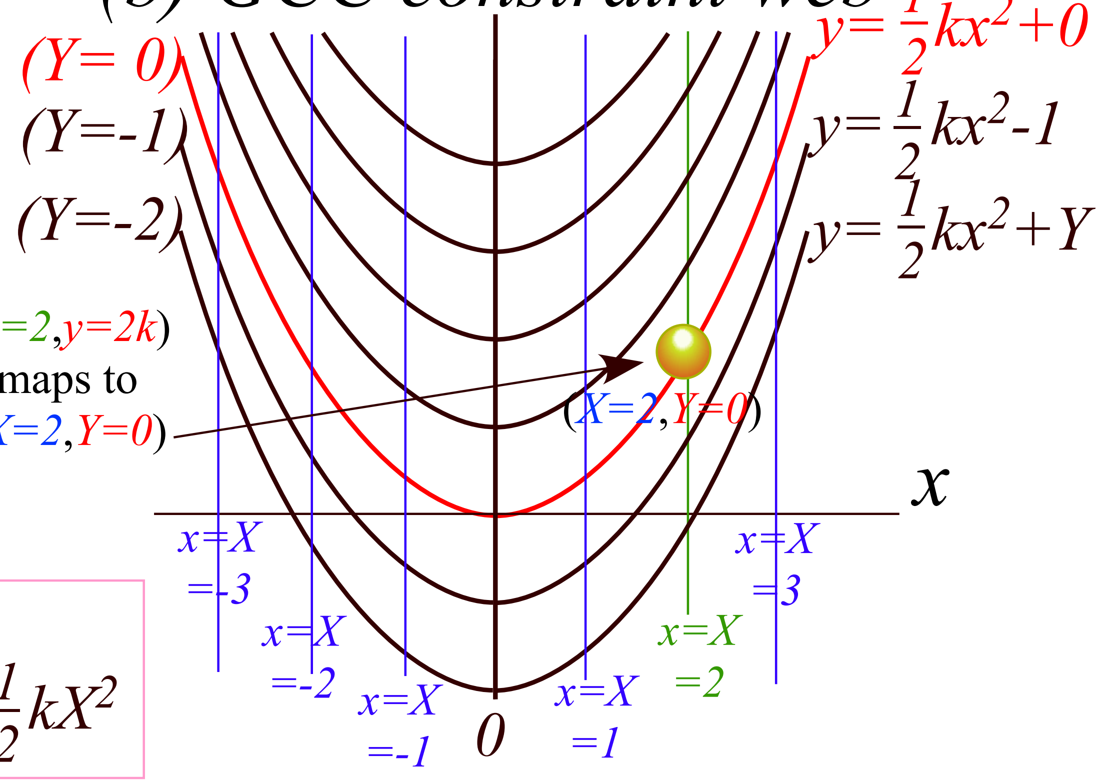
$y=\frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

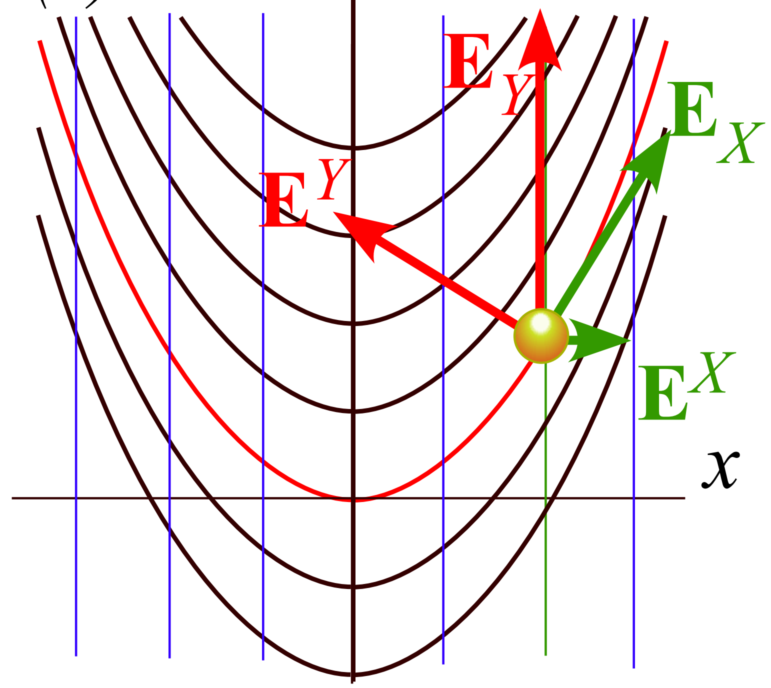
$X = x$

$Y = y - \frac{1}{2}kX^2$

(b) GCC constraint web



(c) GCC **E**-vectors



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Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$x=q^1=X$ $\qquad y=\frac{1}{2}kx^2+q^2= kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

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Contravariant $\mathbf{E}^k$ in rows of Kajobian K

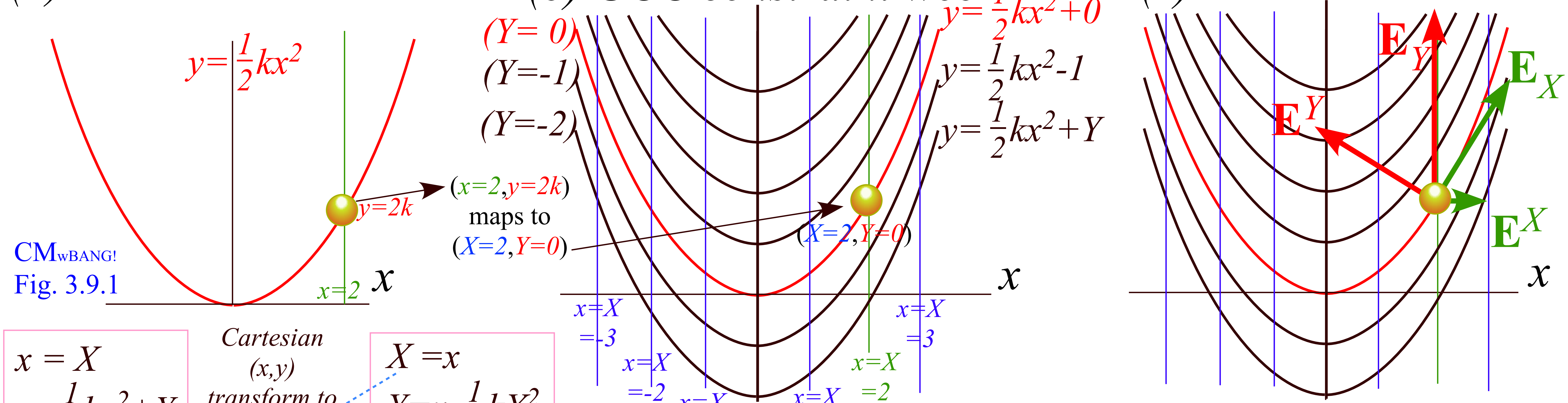
$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \mathbf{E}^X=\begin{pmatrix} 1 & 0 \end{pmatrix} \quad \mathbf{E}^Y=\begin{pmatrix} -kx & 1 \end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}$ $\qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

Web Simulation - OscillatorPE

Parabolic w/grid & basis vectors

(a) Constrained motion (b) GCC constraint web (c) GCC E-vectors



$x = X$
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Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y=y-\frac{1}{2}kX^2$

we define shorthand:
$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Incorporate the constraint curve $y=½kx^2$ into any matching GCC web.

$x=q^1=X$ $\qquad y=½kx^2+q^2=kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}$$

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Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} \qquad \begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$$

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Web Simulation - OscillatorPE
Parabolic w/grid & basis vectors

Way 2. GCC constraint webs.

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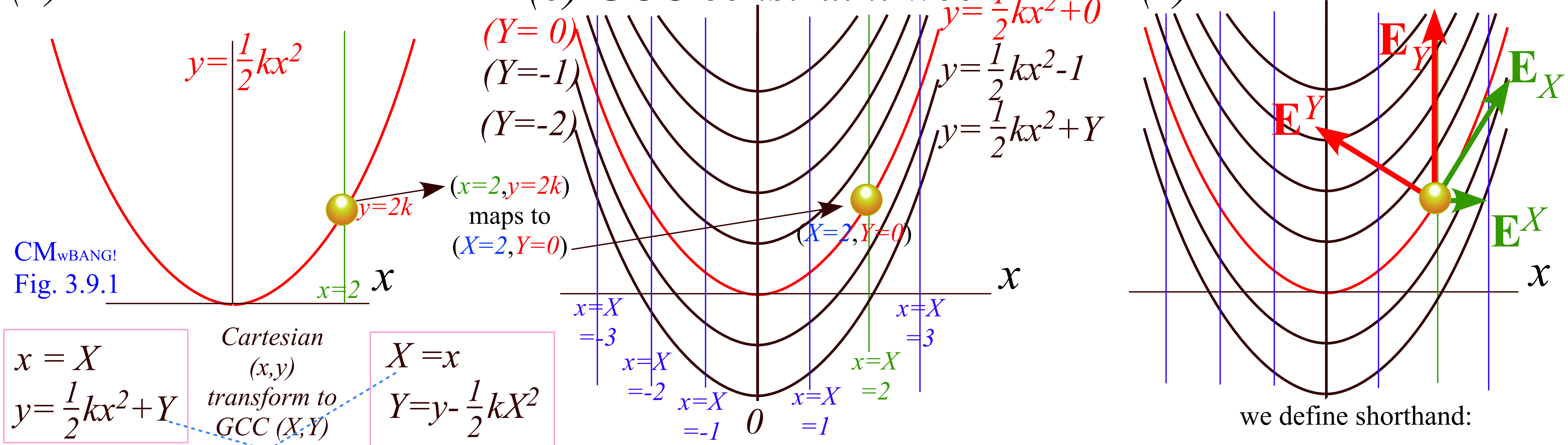

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Web Simulation - OscillatorPE
Parabolic w/grid & basis vectors

Way 2. GCC constraint webs.

(a) Constrained motion

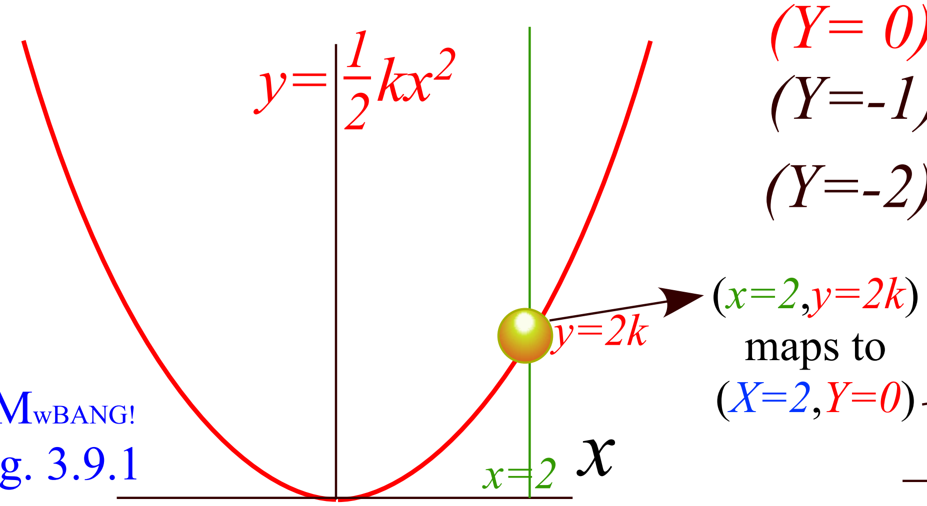

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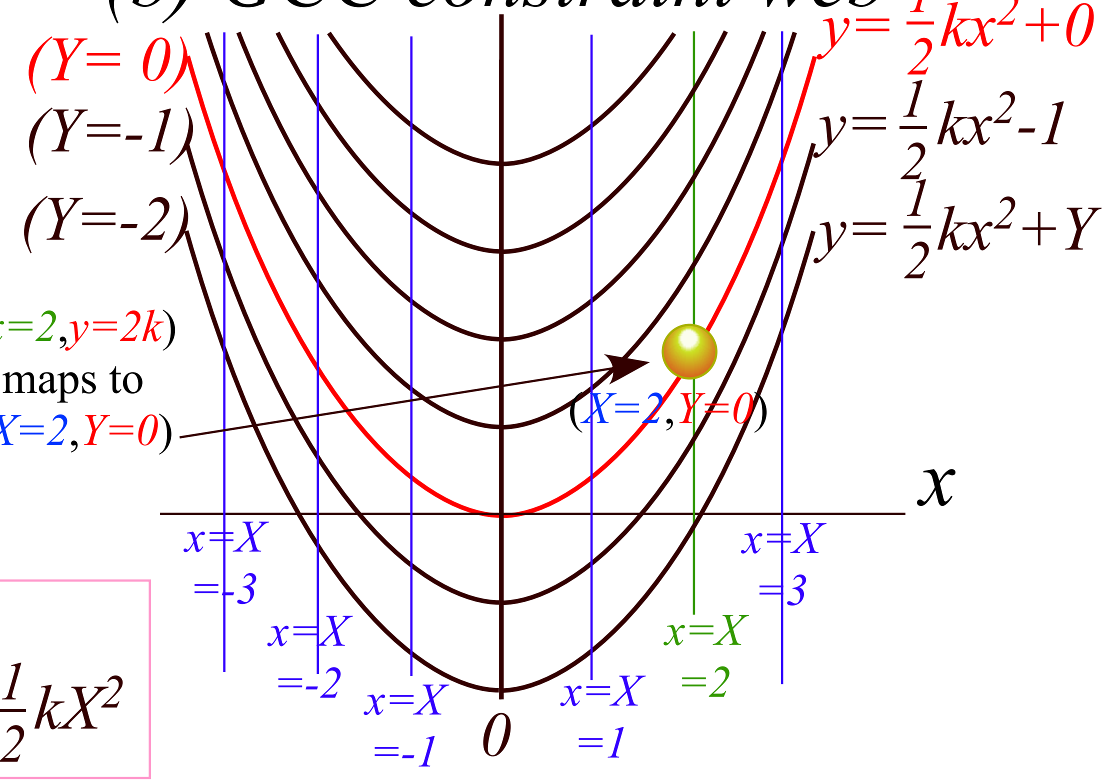
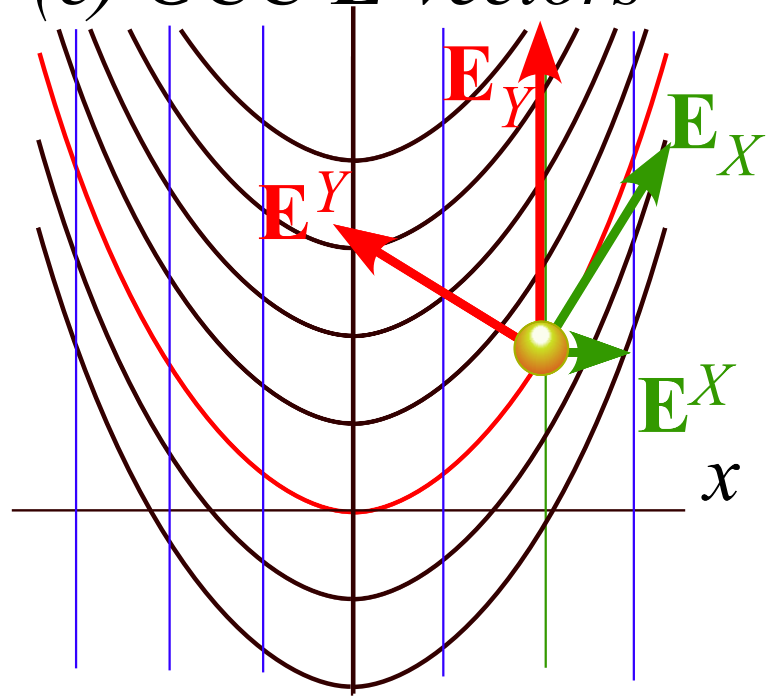
Fig. 3.9.1

(b) GCC constraint web



(c) GCC E-vectors



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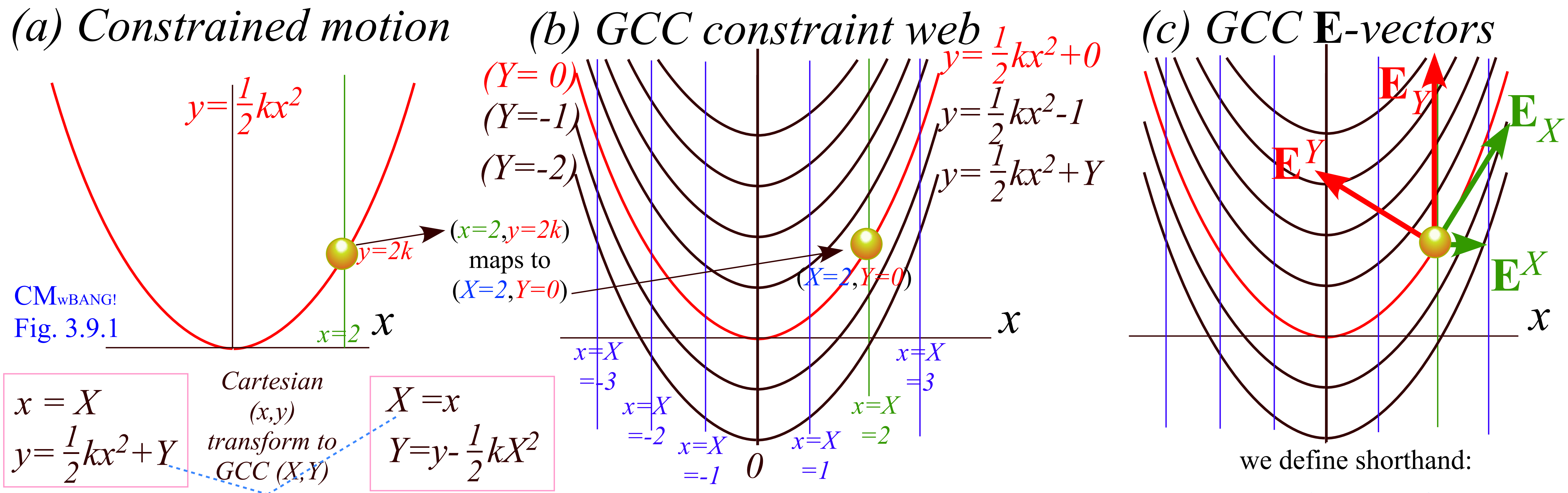
Way 2. GCC constraint webs.



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...and Lagrangian:
$$L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2+kX\dot{X}\dot{Y}+\frac{1}{2}\dot{Y}^2-gY-\frac{gk}{2}X^2\right]$$

$V=mgy=mg(Y+kX^2/2)$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

→ *Find covariant force equations*

Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

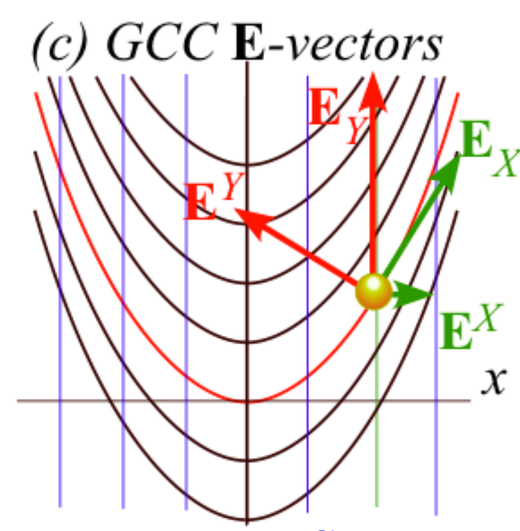

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Fig. 3.9.1

Web Simulation - OscillatorPE
Parabolic w/grid & basis vectors

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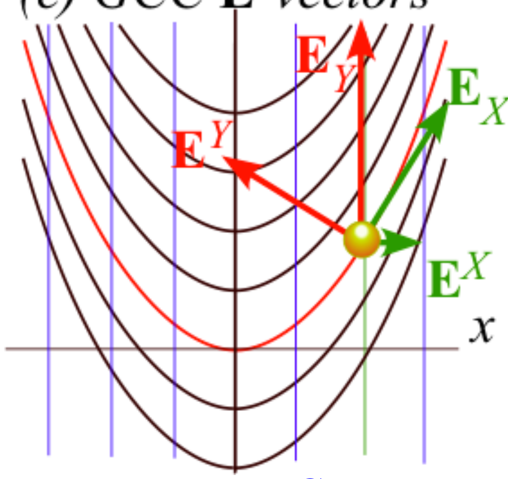
(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$



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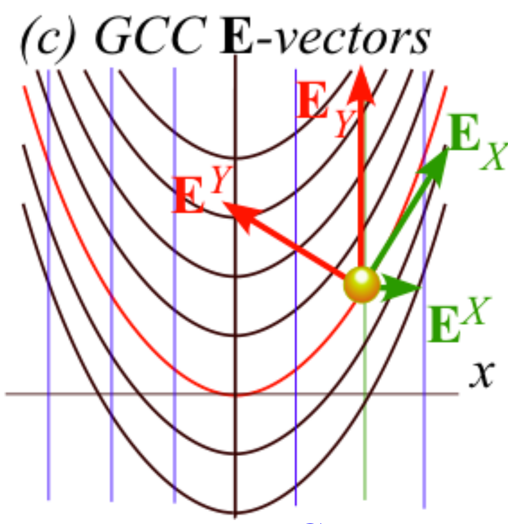

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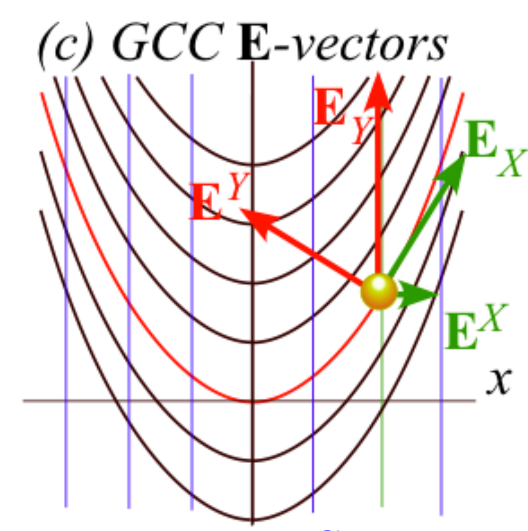

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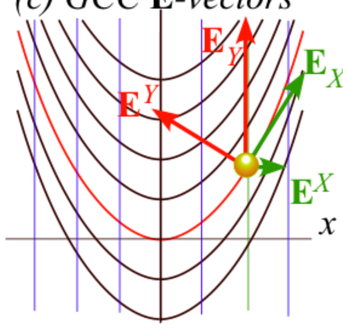

CMwBANG!
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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

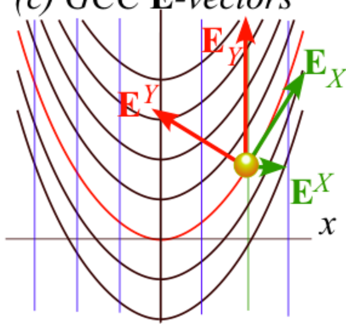
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov} = 0 = F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



CMwBANG!
Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

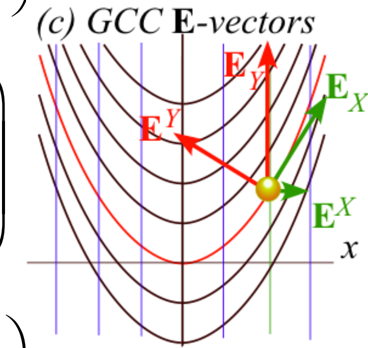
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov} = 0 = F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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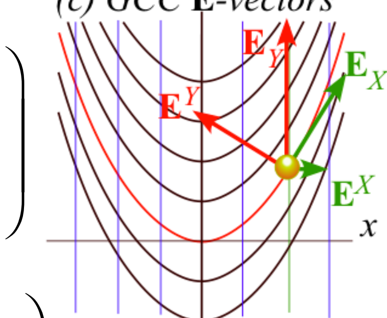
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

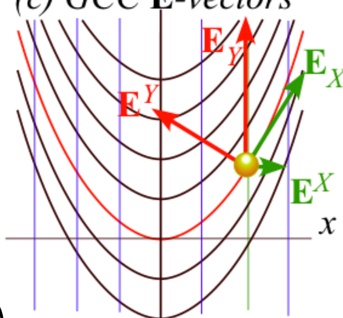
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB})

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1^{st} Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2^{nd} Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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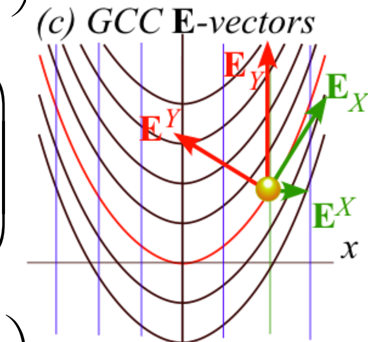
$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint Y=const.)

(inverse of γ_{AB}) (inverse of γ_{AB})

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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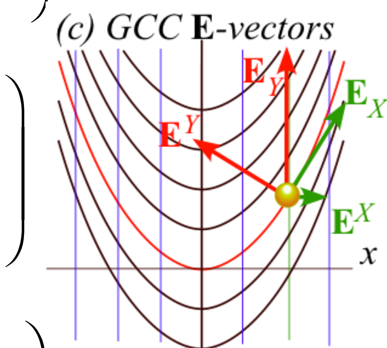
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB}) (cancelations)

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2 + g) \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$\ddot{x} = 0 = \ddot{X}$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

- *Find covariant force equations*
- → *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

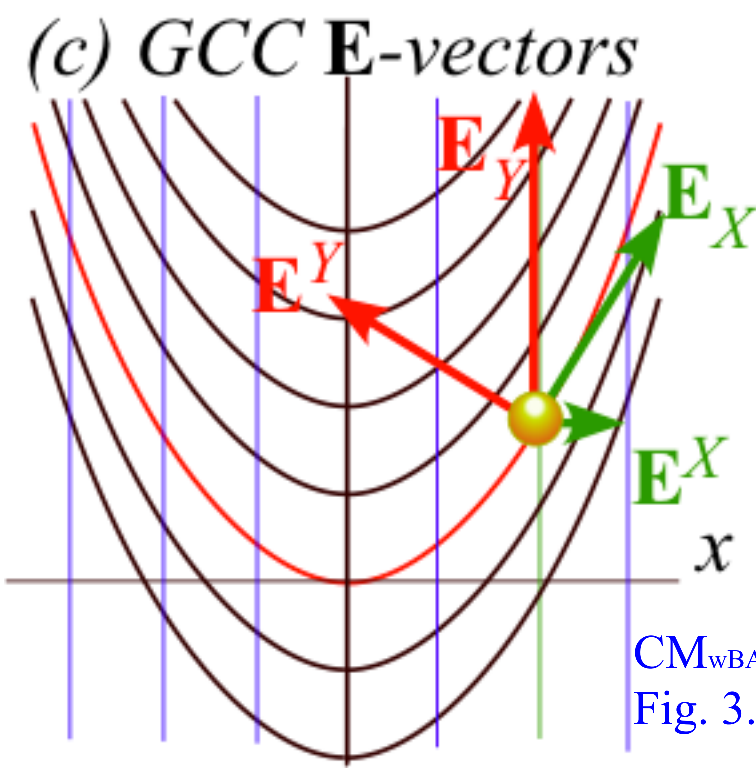
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

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$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

Frictional force components are contravariant

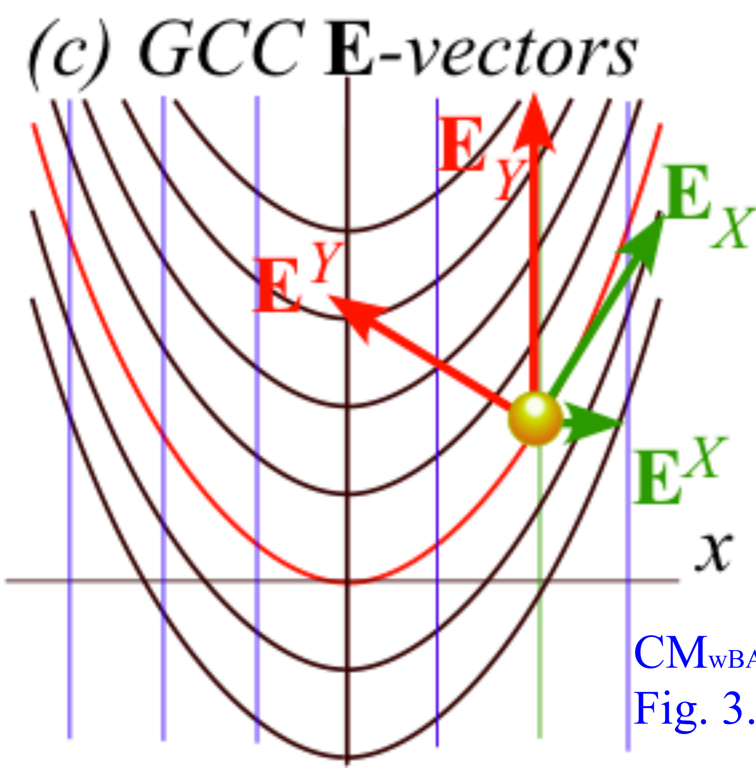
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

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*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

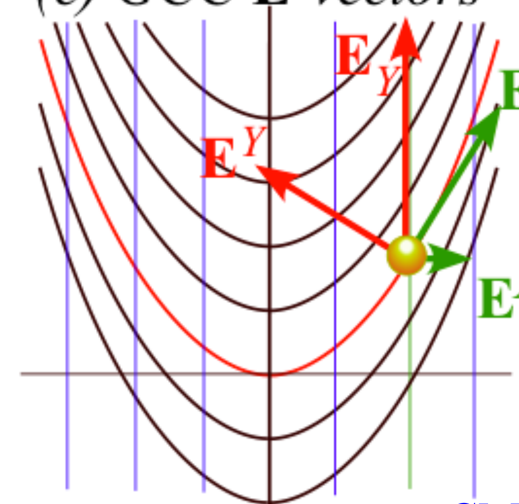
Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E**-vectors



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass m by parabola $Y=const.$ is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
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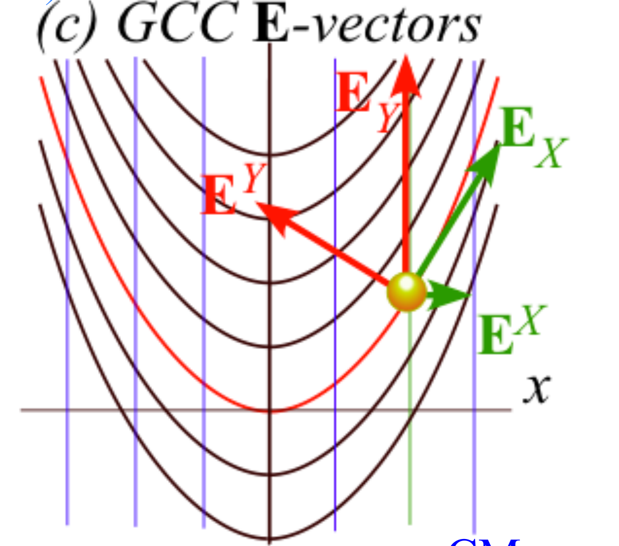
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

FINALLY ! We get the Way 1. solution of p.12

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

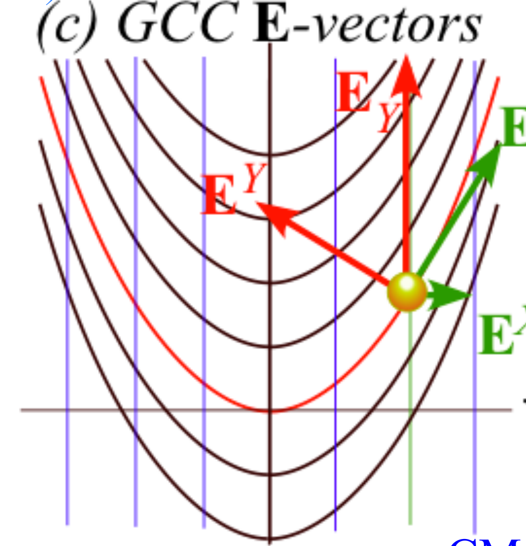
Frictional or driving forces have *contravariant* components F^A_{con}

$$\mathbf{F} = F^X_{con}\mathbf{E}_X + F^Y_{con}\mathbf{E}_Y = F^X_{con}\frac{\partial \mathbf{r}}{\partial X} + F^Y_{con}\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass m by parabola Y=*const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X}+0+k^2X\dot{X}^2+gkX \\ kX\ddot{X}+0+k\dot{X}^2+g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2+gkX}{1+k^2X^2} = -\frac{k\dot{X}^2+g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y$$
$$= m(kX\ddot{X}+0+k\dot{X}^2+g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X}+kX\ddot{Y}+k^2X\dot{X}^2+gkX \\ kX\ddot{X}+\ddot{Y}+k\dot{X}^2+g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

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Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

$$= 0 \cdot \nabla X + F_Y^{cov}\nabla Y$$

$$= 0 \cdot \mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix} (1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ F_Y^{cov} \end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov} \mathbf{E}^Y$$

$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$

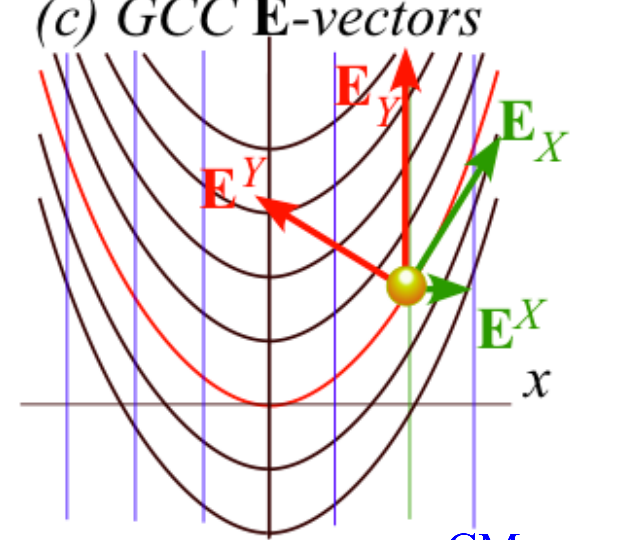
$$= m\left(\frac{-kX(k\dot{X}^2 + g)}{1+k^2X^2} + \frac{(k\dot{X}^2 + g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix} -kX \\ 1 \end{pmatrix}$$

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

Recall: $x \equiv X$

$$\ddot{X} \equiv \ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

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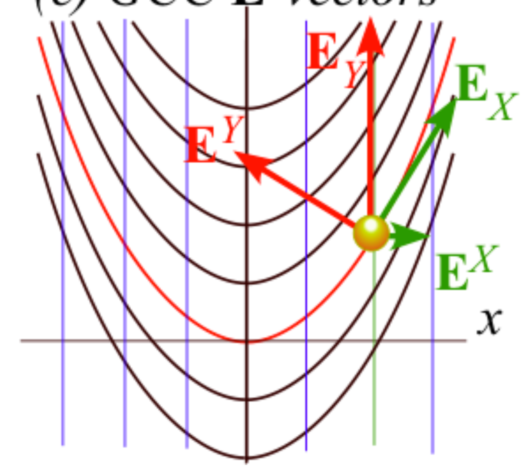
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$\begin{pmatrix}F_x \\ F_y\end{pmatrix} = \left(=\begin{pmatrix}0 \\ mk\dot{X}^2 + mg\end{pmatrix}\right)_{at:X=0}$$

Centripetal force mkv^2+mg *(what roller-coaster rider feels at bottom)*

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

$$-g = \ddot{y} = \tfrac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right)$$
$$= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\,(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0)$$

Other Ways to do constraint analysis

→ *Way 3. OCC constraint webs*

Sketch of atomic-Stark orbit parabolic OCC analysis

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

"Non-Holonomic" multipliers

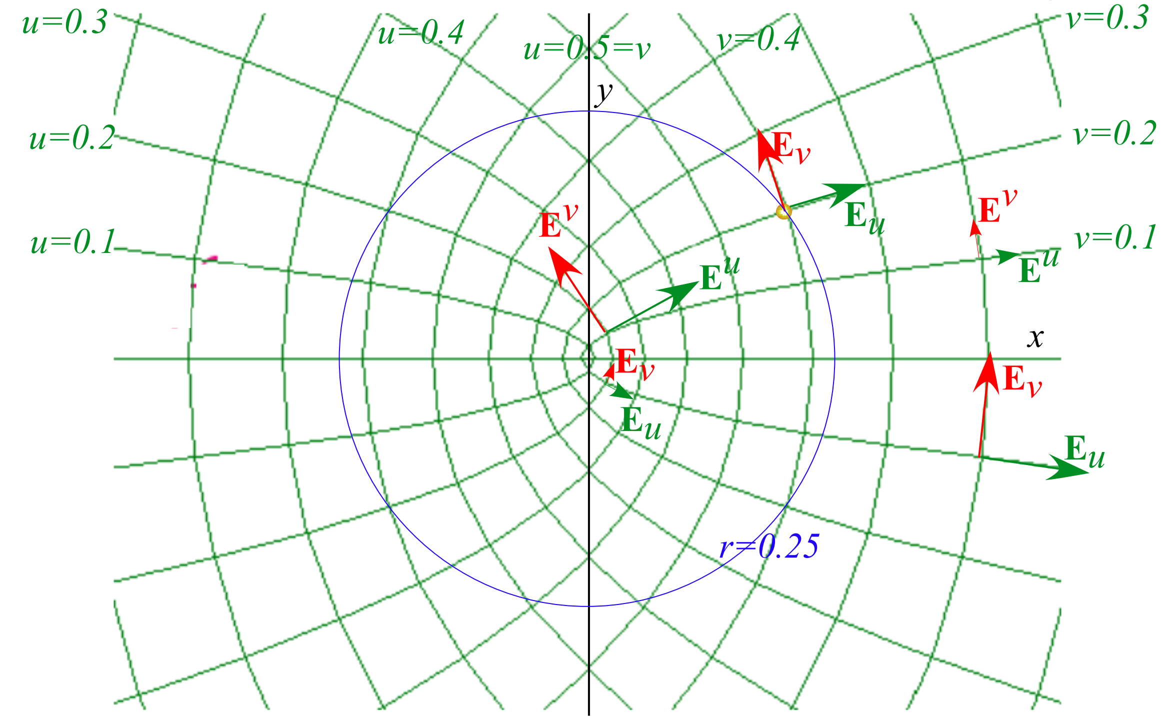
Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$

CMwBANG! Fig. 3.9.2



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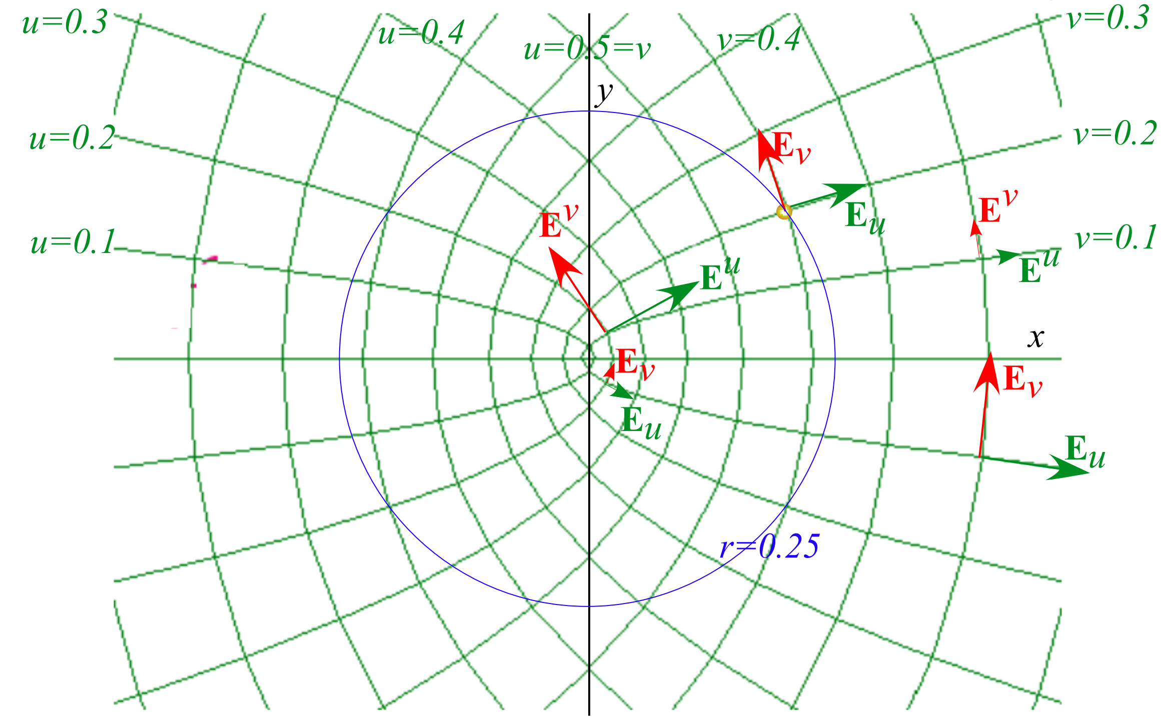
$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$

CMwBANG! Fig. 3.9.2



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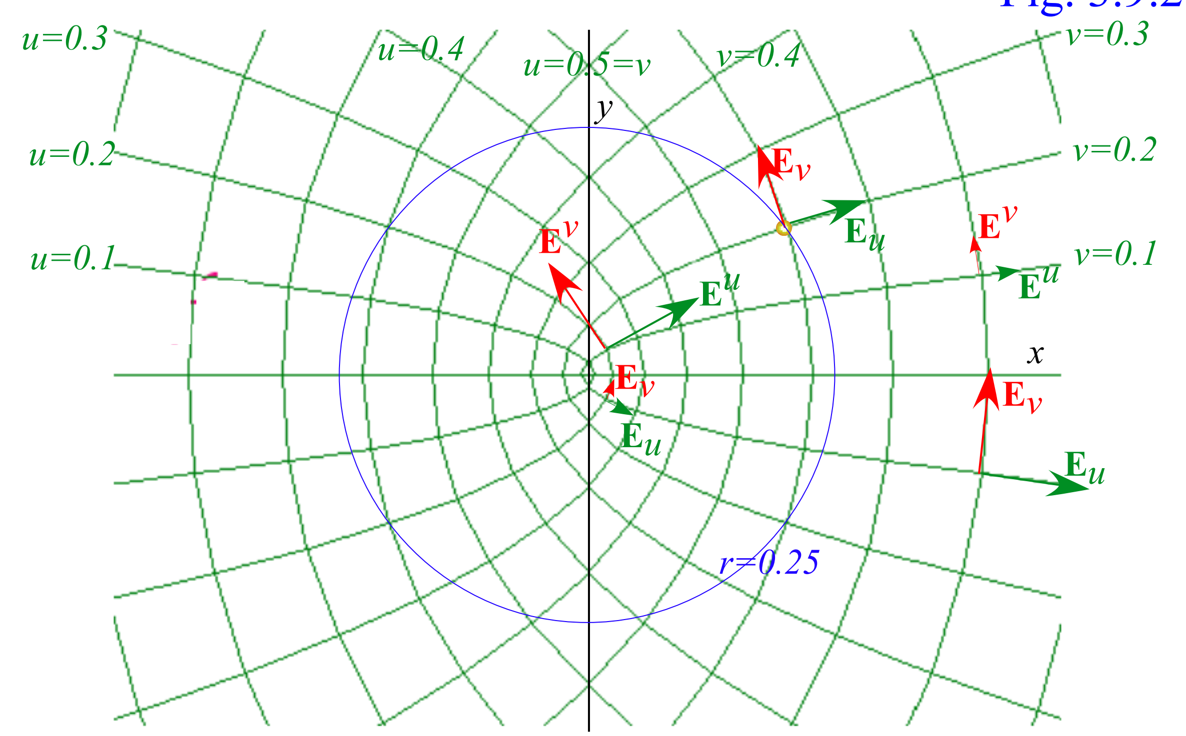
$$x = u^2 - v^2$$

$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

CMwBANG! Fig. 3.9.2



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CMwBANG!
Fig. 3.9.2

$$x = u^2 - v^2$$

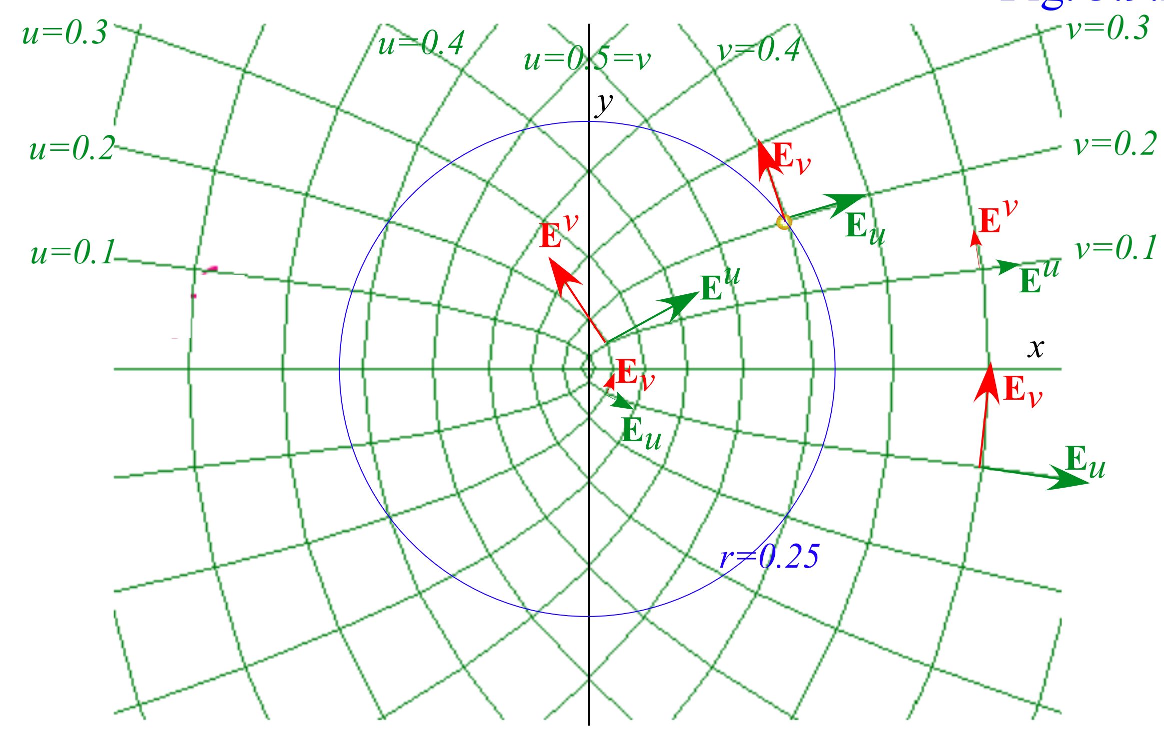
$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2 + y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2 + y^2} - x$$

$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4v^2u^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



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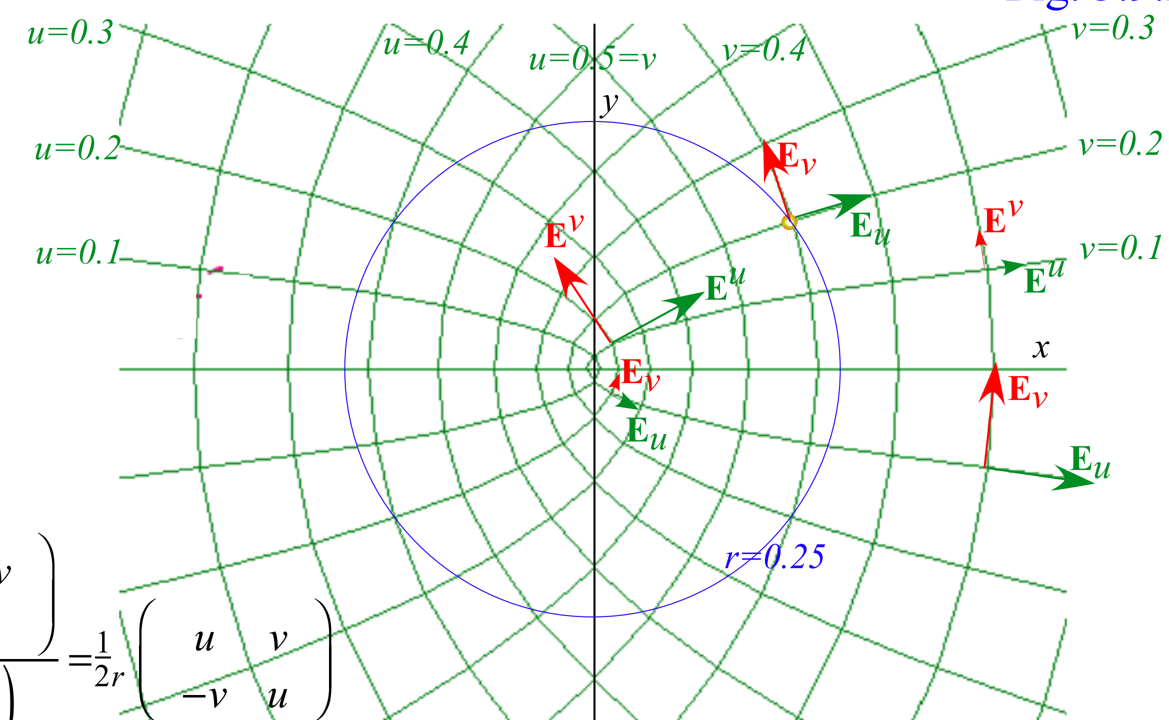
CMwBANG! Fig. 3.9.2



$$x=u^2-v^2$$

$$y=2uv \qquad 2u^2=r+x=\sqrt{x^2+y^2}+x$$

$$r=u^2+v^2 \qquad 2v^2=r-x=\sqrt{x^2+y^2}-x$$

$$y^2=4u^2v^2=4u^2(u^2-x)$$

$$y^2=4u^2v^2=4v^2(v^2+x)$$

Gives confocal parabolics

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

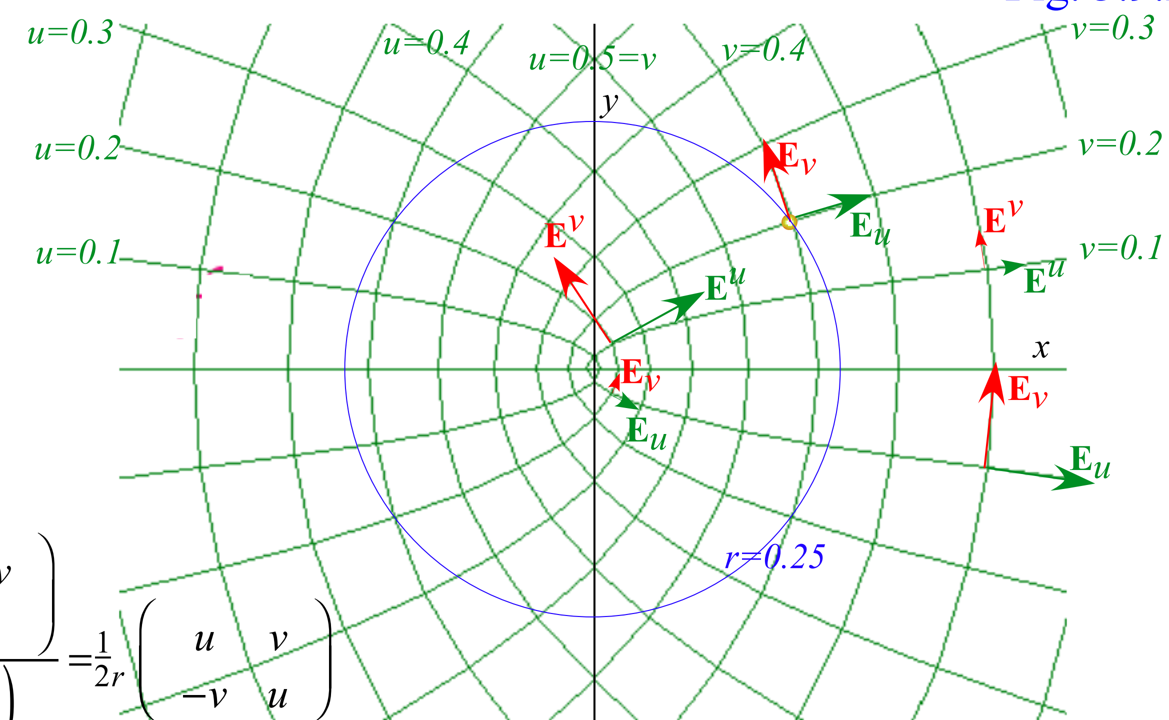
Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

CMwBANG!
Fig. 3.9.2



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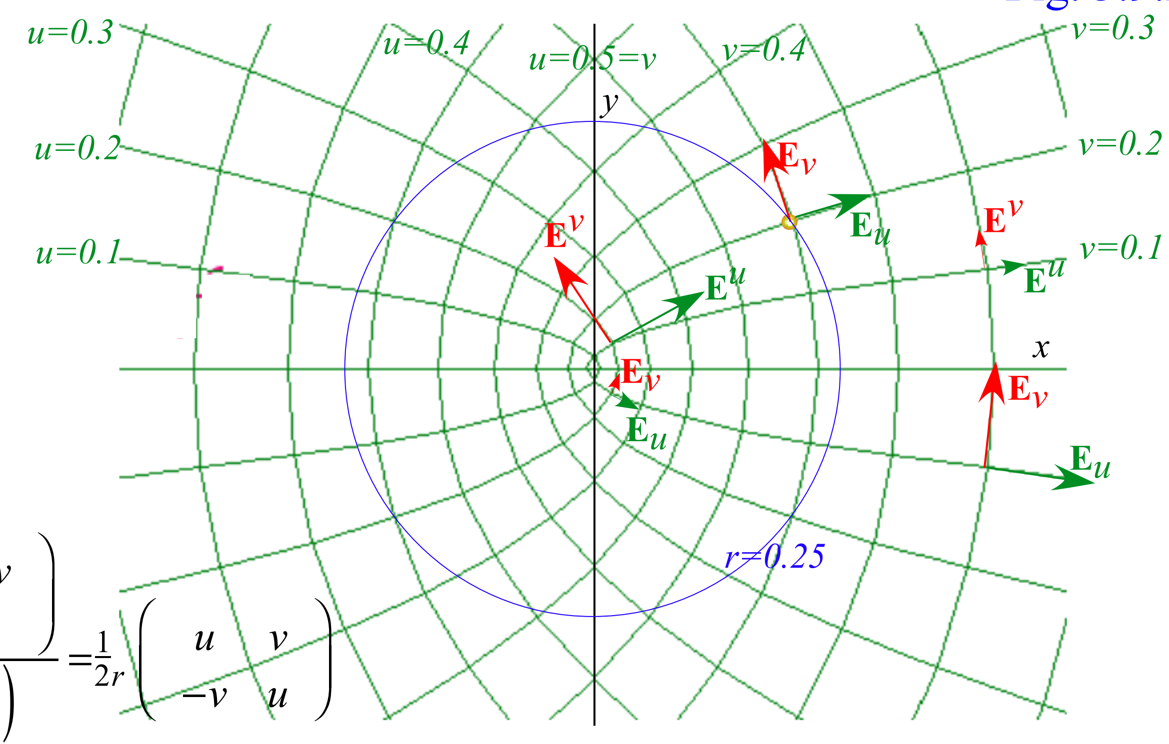

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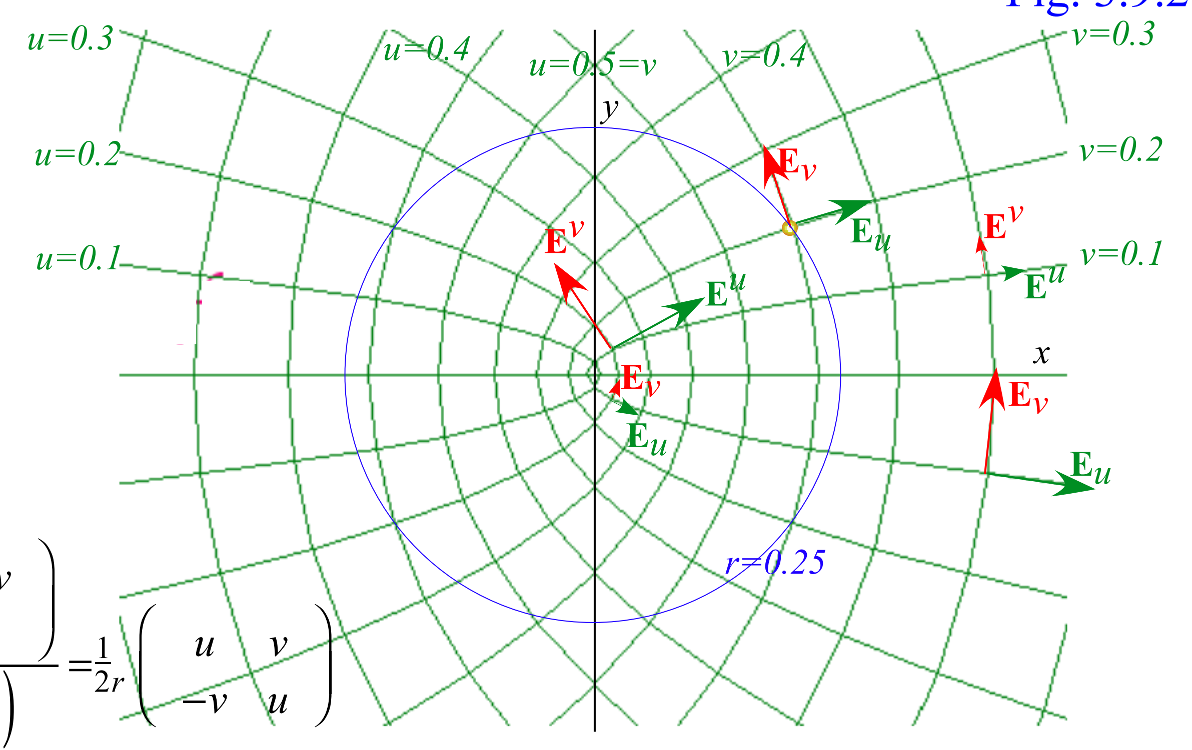

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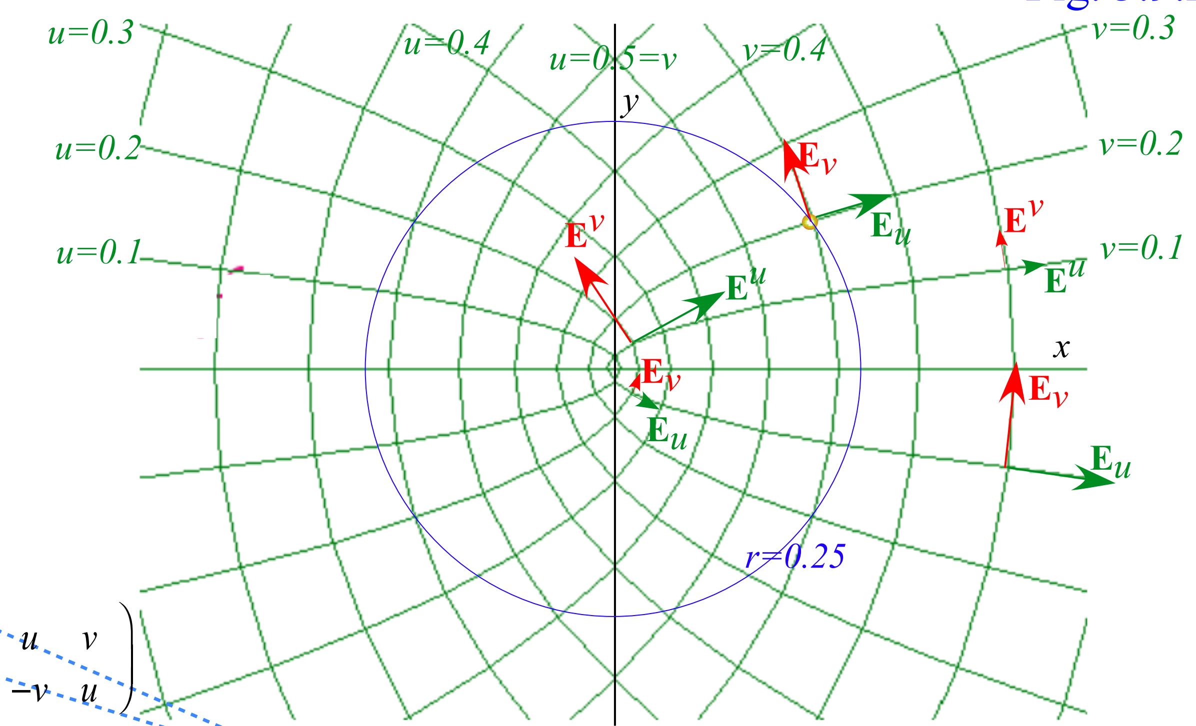

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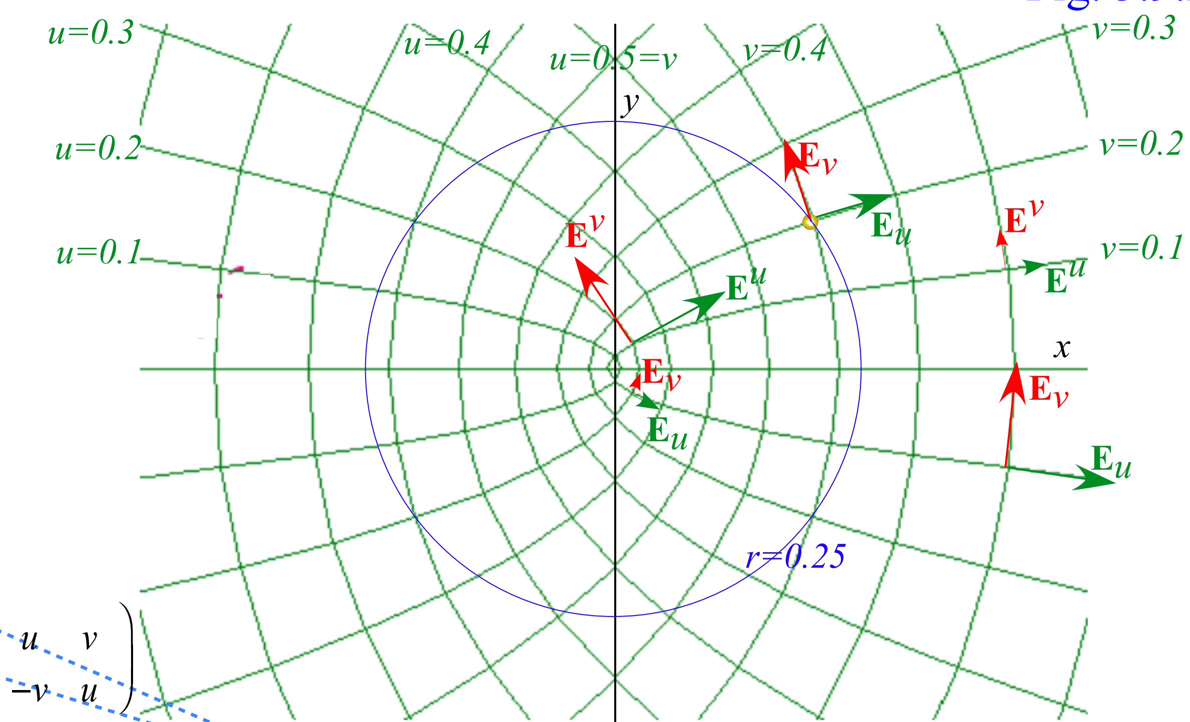

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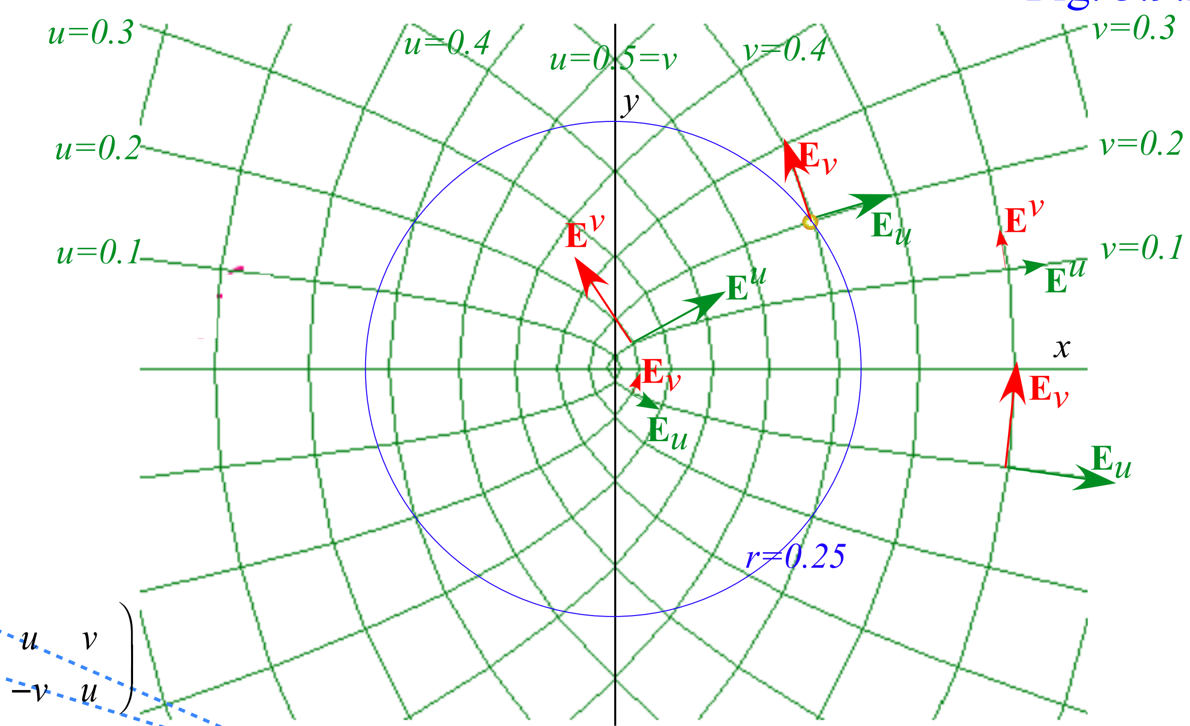

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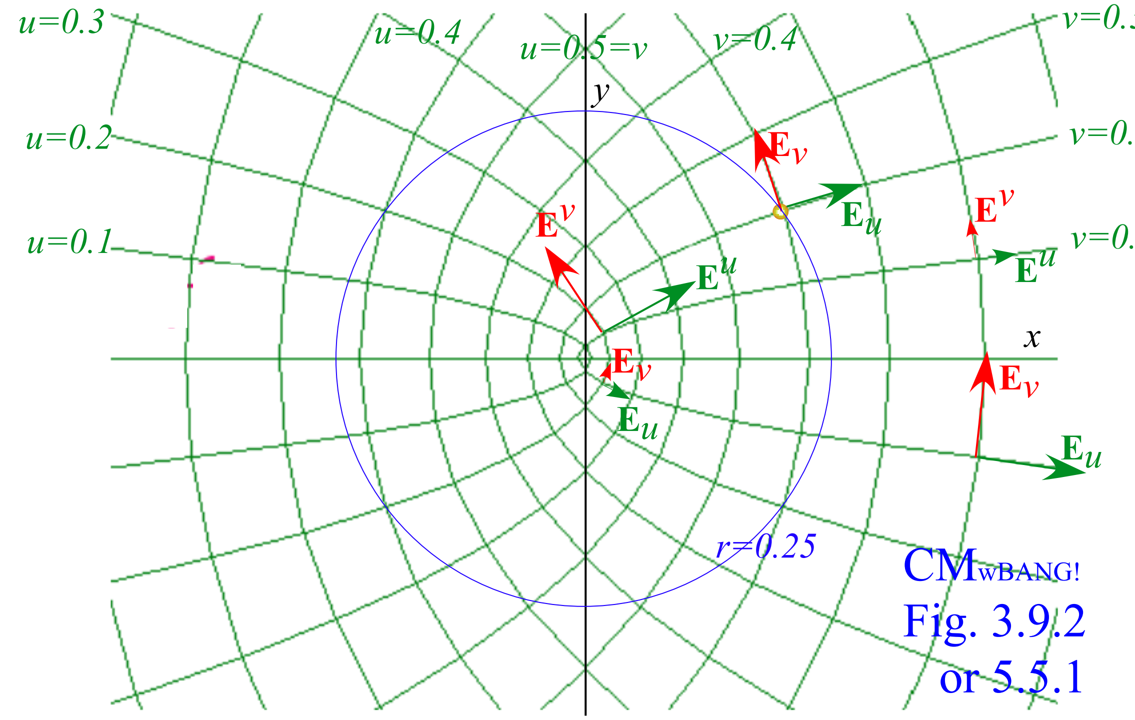

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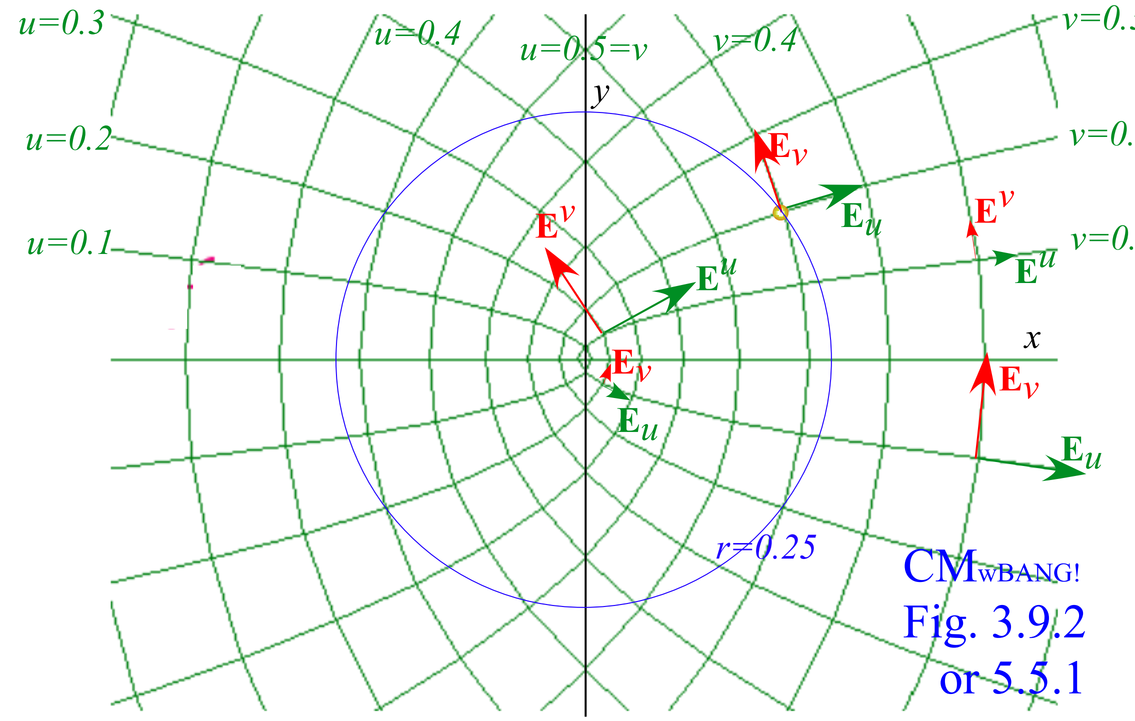

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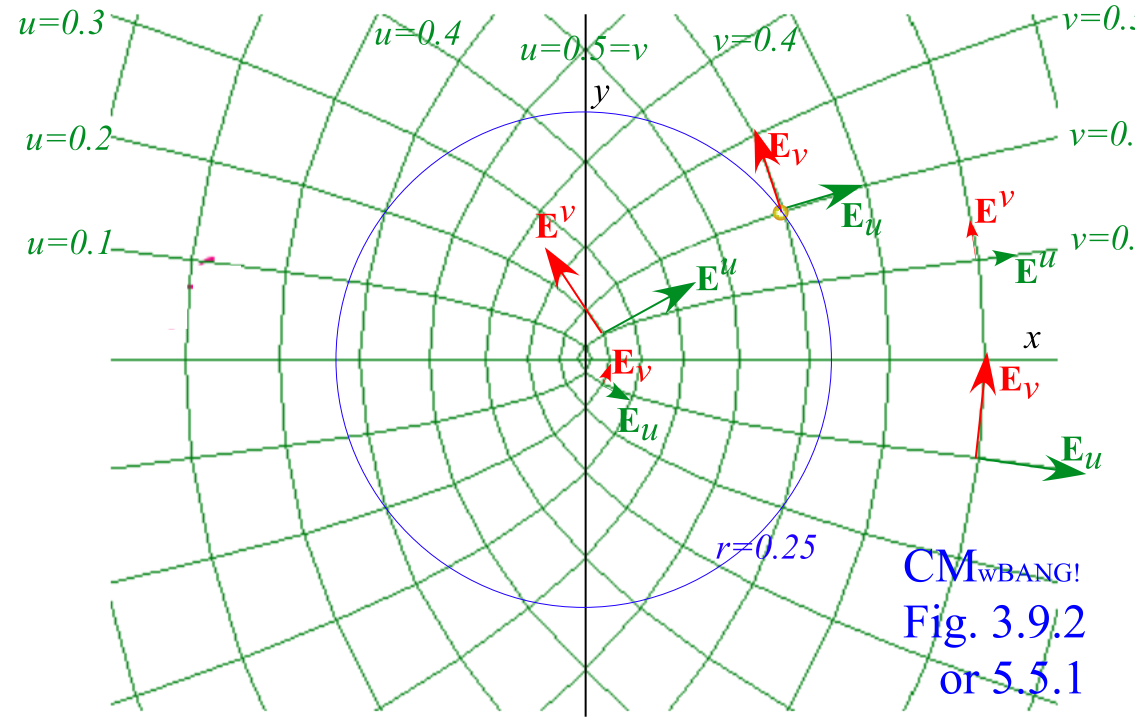

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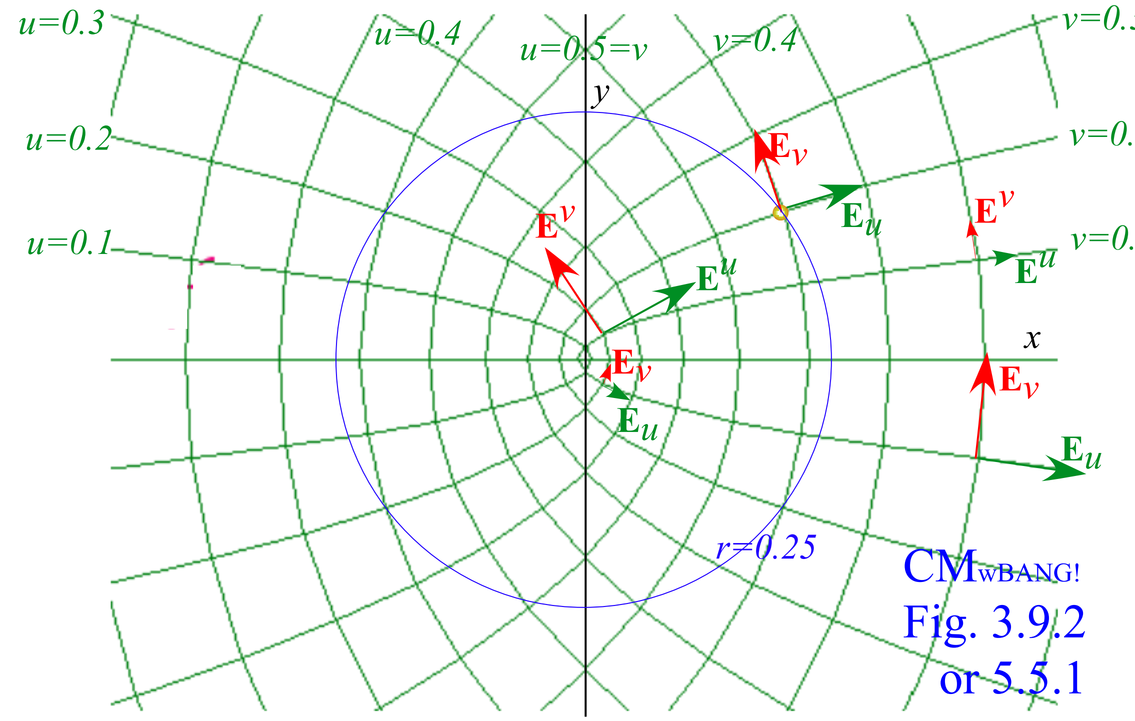

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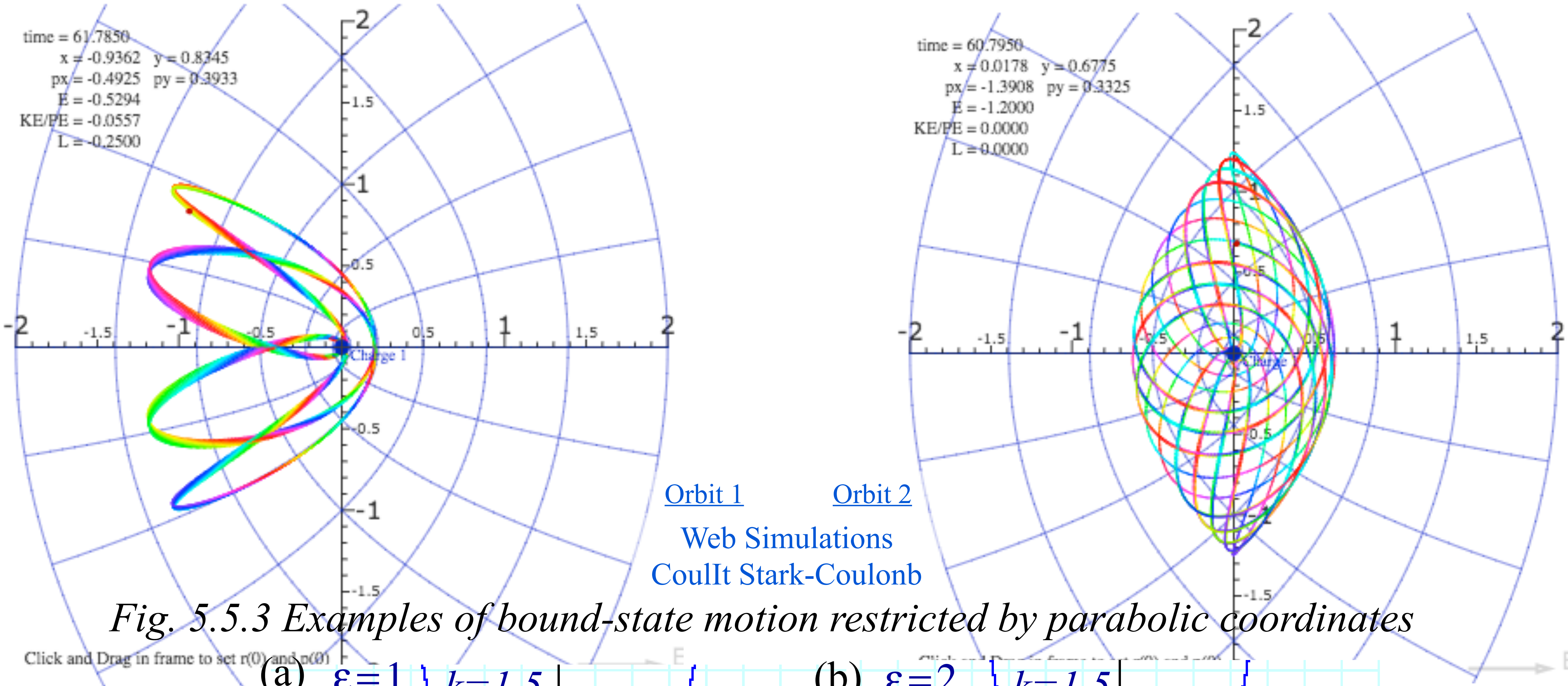
Zero Stark-field *(ε=0)* gives h_u or h_v harmonic oscillation if *E<0*. It's unstable or anharmonic otherwise.

$$\dot{p}_u = -\frac{\partial h_u}{\partial u} = -8Eu + 16\varepsilon u^3 \qquad \dot{u} = \frac{\partial h_u}{\partial p_u} = p_u/m \qquad \dot{p}_v = -\frac{\partial h_v}{\partial v} = -8Ev - 16\varepsilon v^3 \qquad \dot{v} = \frac{\partial h_v}{\partial p_v} = p_v/m$$

Stark orbit parabolic OCC analysis



Orbit 1 Orbit 2

Web Simulations

CoulIt Stark-Coulonb

Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

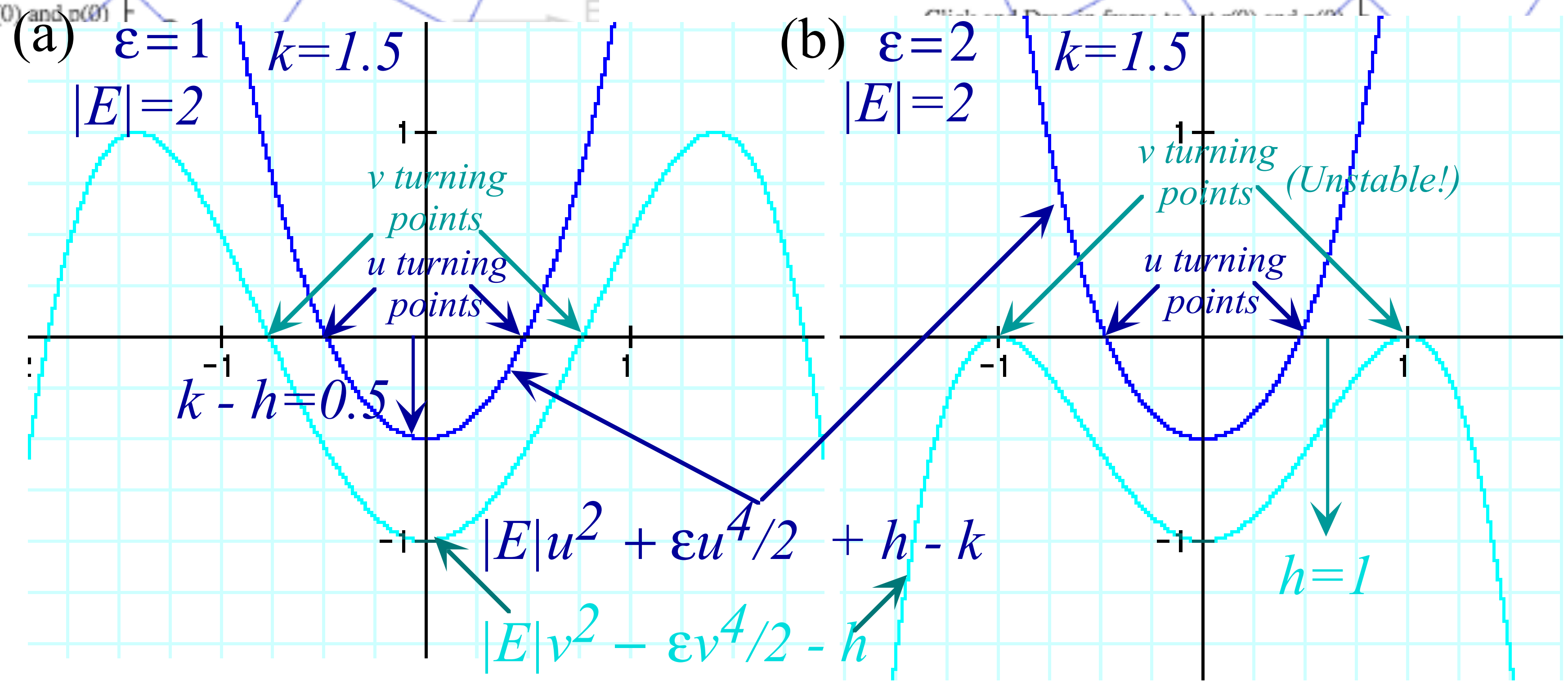
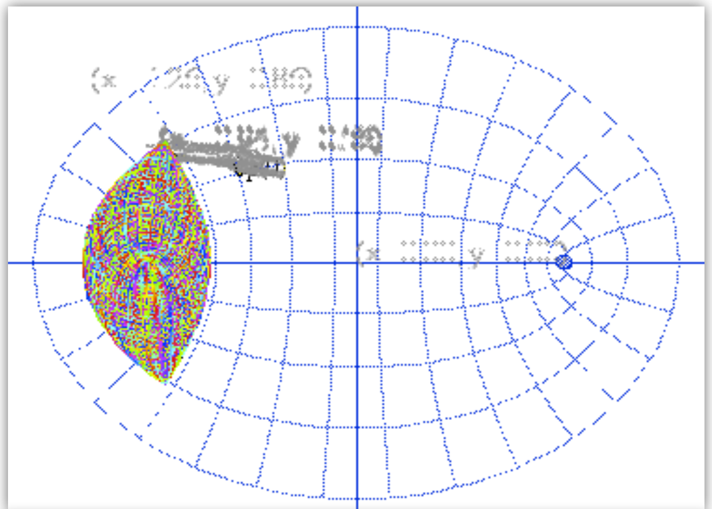
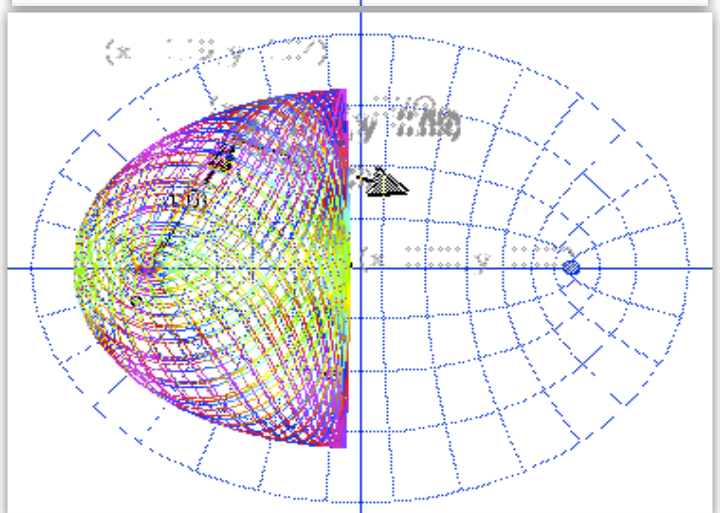
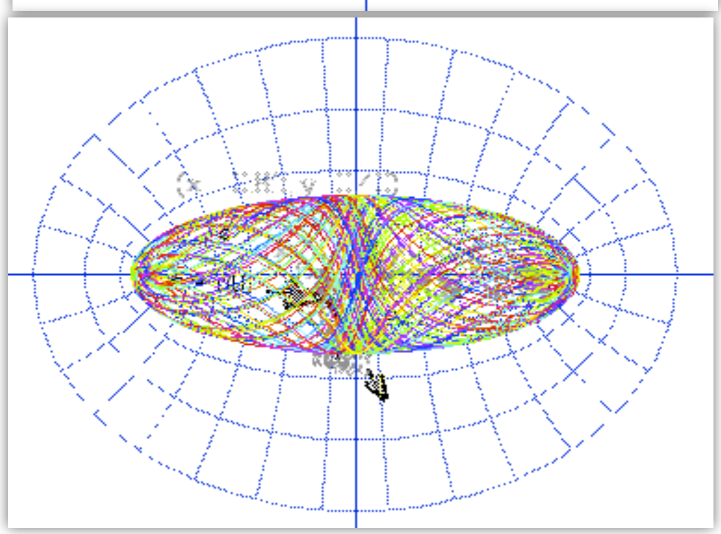


Fig. 5.5.2 Effective potentials for parabolic coordinates

Hs+-ion orbit elliptic-hyperbolic OCC bound trajectories

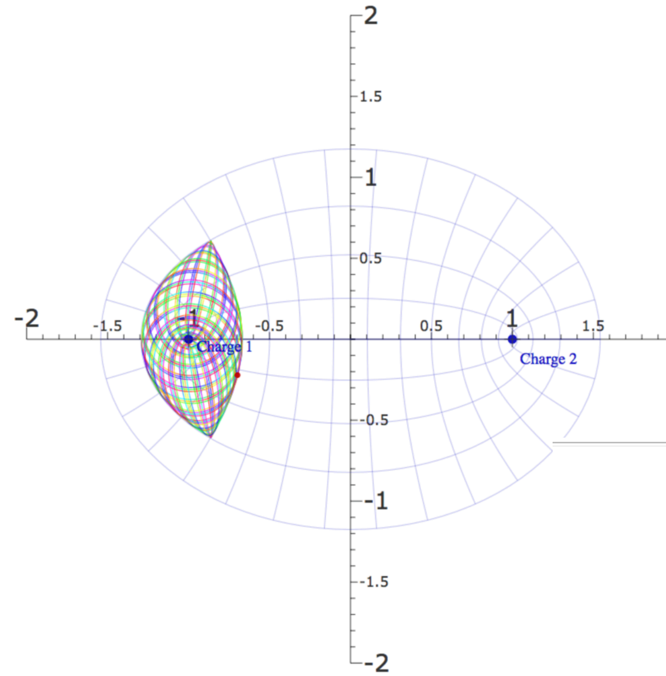




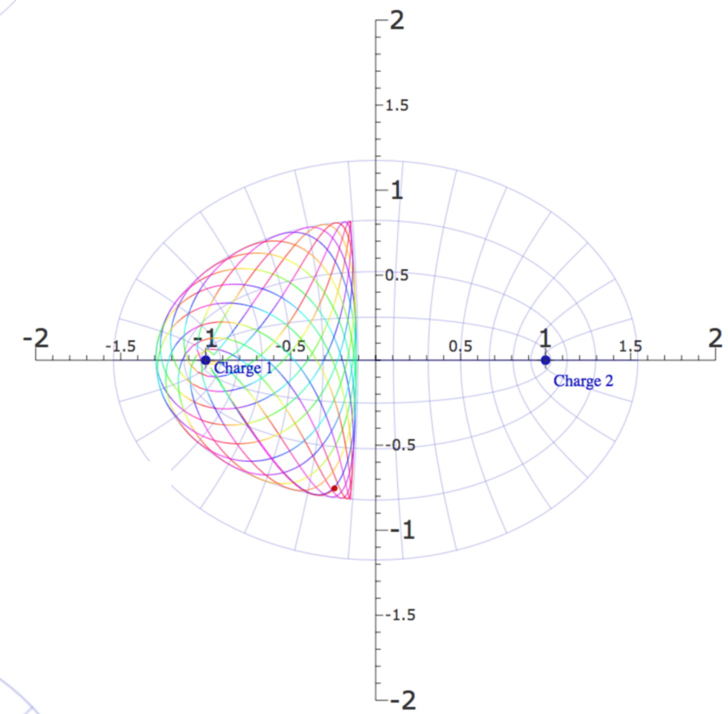


Web Simulations
CoulIt H_2^+

Orbit 1: Localized on C_1

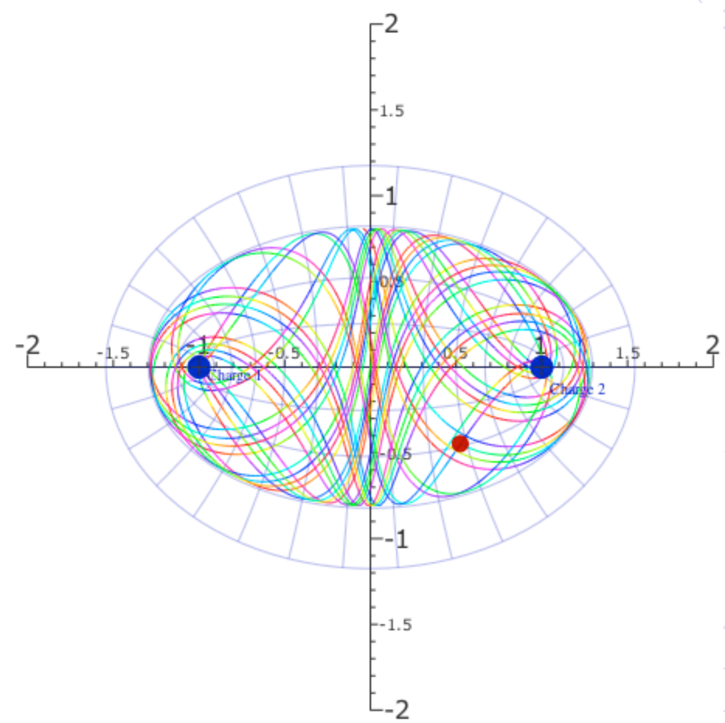


Orbit 2: Less localized on C_1

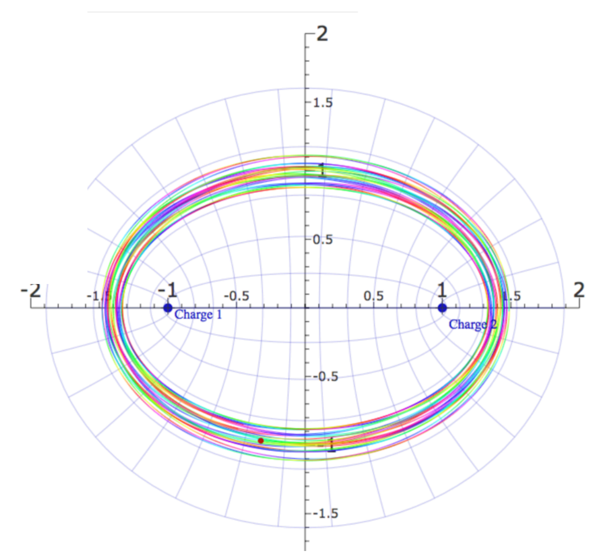


Orbit 3a: Sharing C_1 and C_2

Orbit 3b: Sharing C_1 and C_2



Orbit 4: Quasi-Stable Elliptical



CMwBANG!
Fig. 5.5.4

Other Ways to do constraint analysis

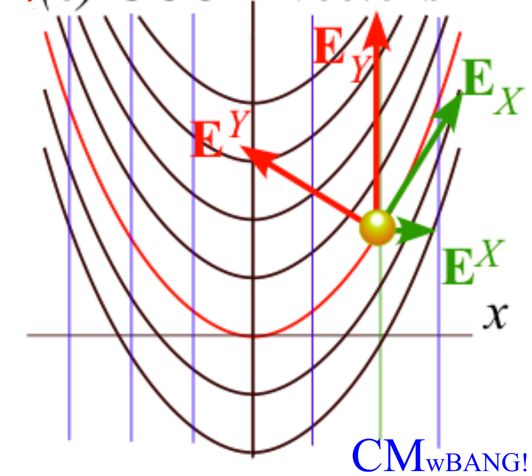
- *Way 3. OCC constraint webs*
    - *Sketch of atomic-Stark orbit parabolic OCC analysis*
        - *Classical Hamiltonian separability*
- → *Way 4. Lagrange multipliers*
    - *Lagrange multiplier as eigenvalues*
    - *Multiple multipliers*
    - *“Non-Holonomic” multipliers*

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)



CMwBANG! Fig. 3.9.1

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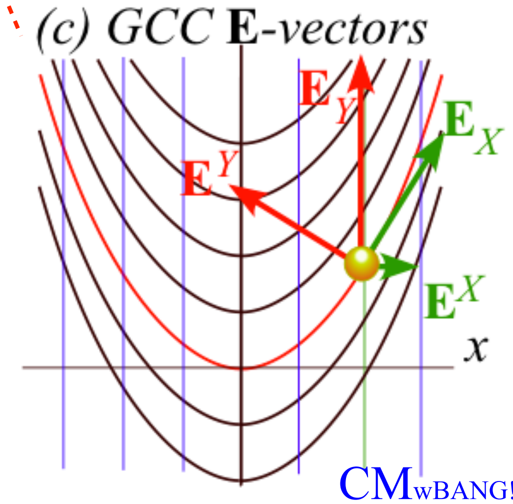

CMwBANG!
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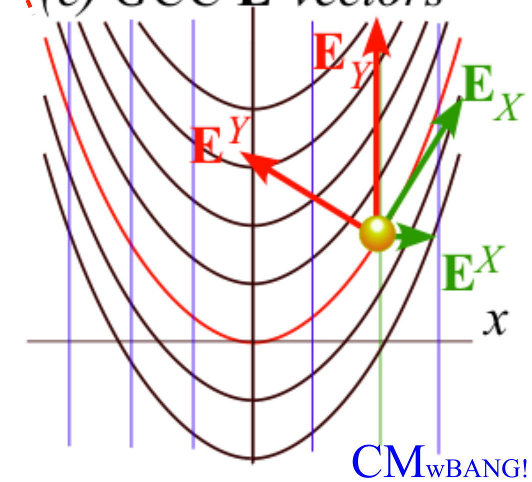

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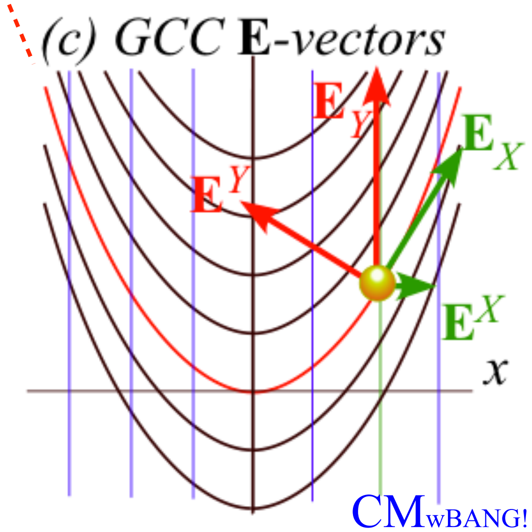

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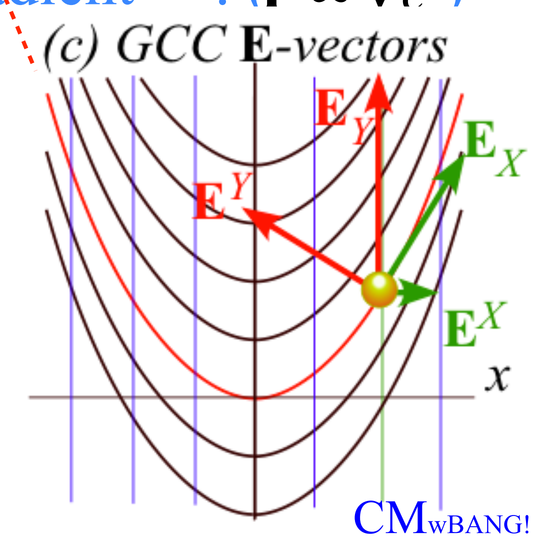

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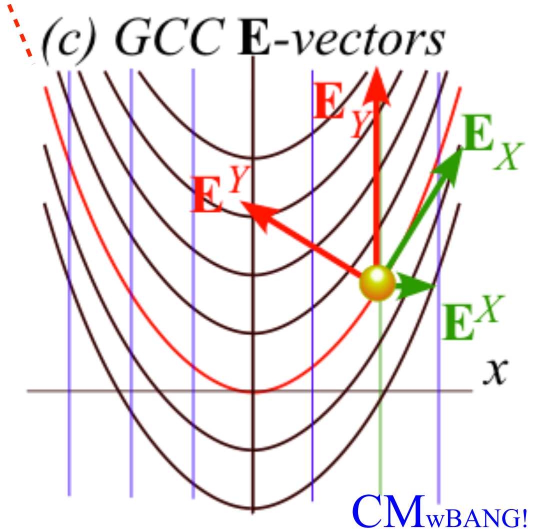

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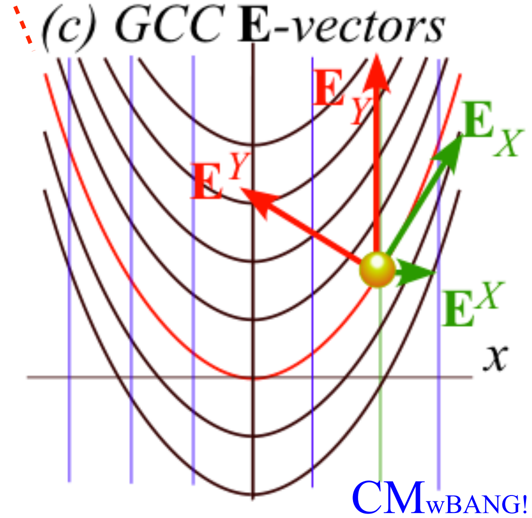

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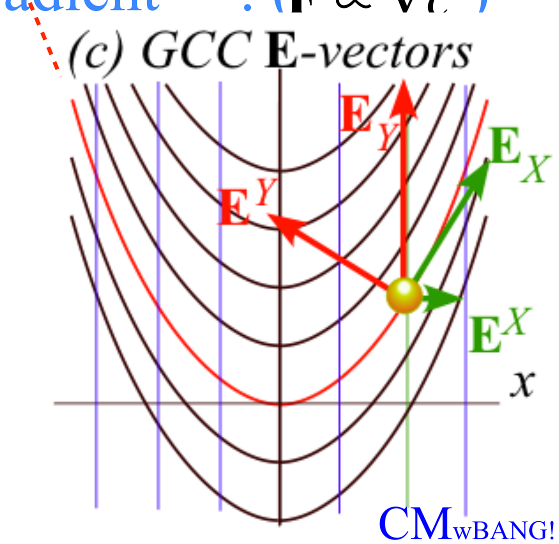

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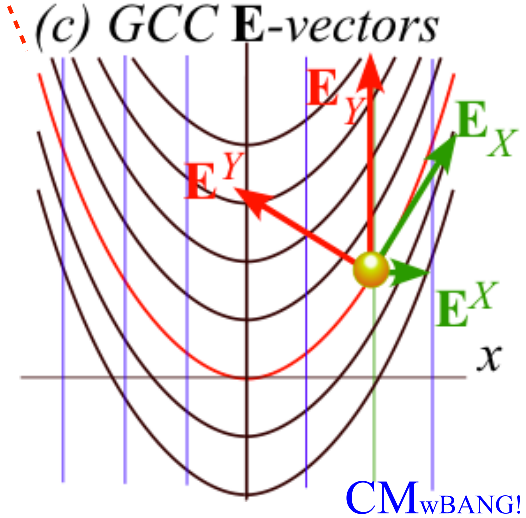

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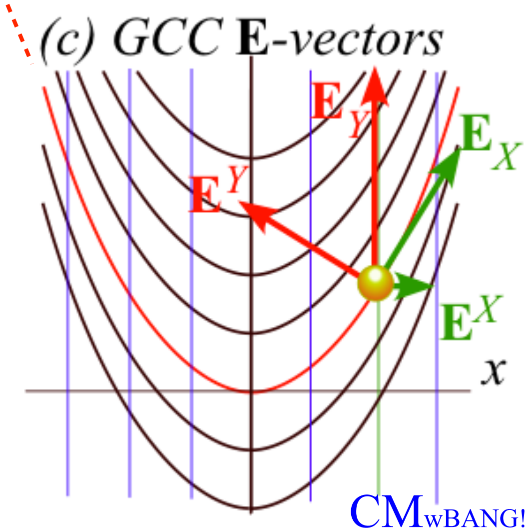

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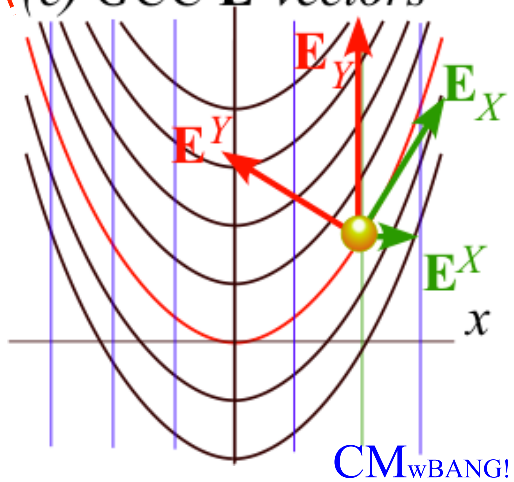

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$$\lambda = m(-k\dot{x}^2 - kx\ddot{x} - g)$$

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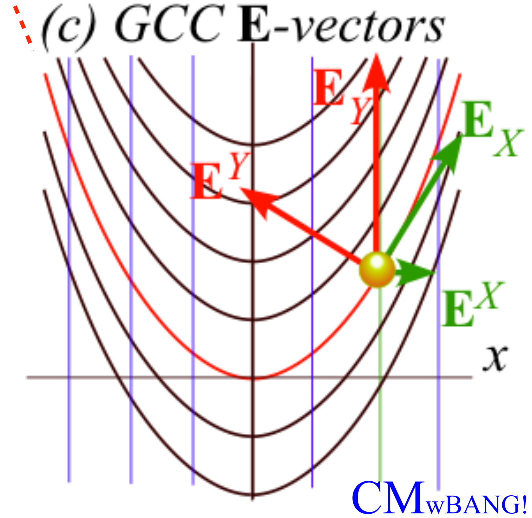

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$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx$$

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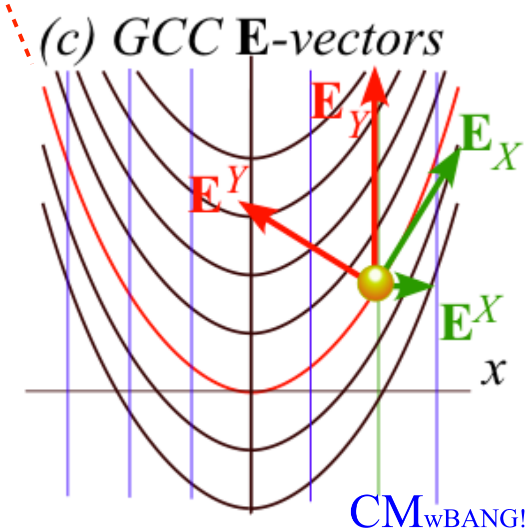

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$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

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$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=\frac{1}{2}kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained x-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

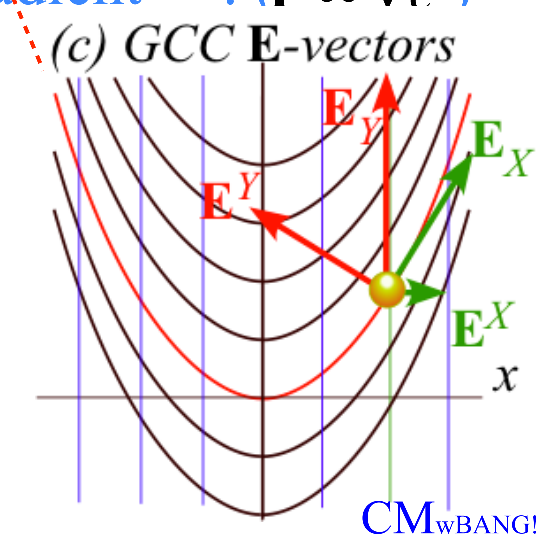
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$ *(Back to "Stupid-Parabolic" GCC)*

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla (\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

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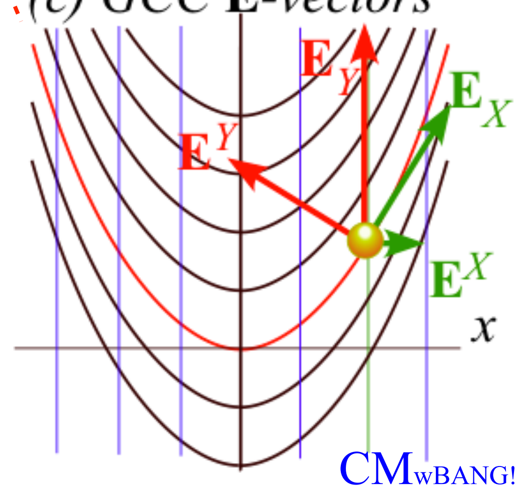

CMwBANG! Fig. 3.9.1

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$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

(Same equation as on p.12)

$$\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Sketch of atomic-Stark orbit parabolic OCC analysis

Classical Hamiltonian separability

Way 4. Lagrange multipliers

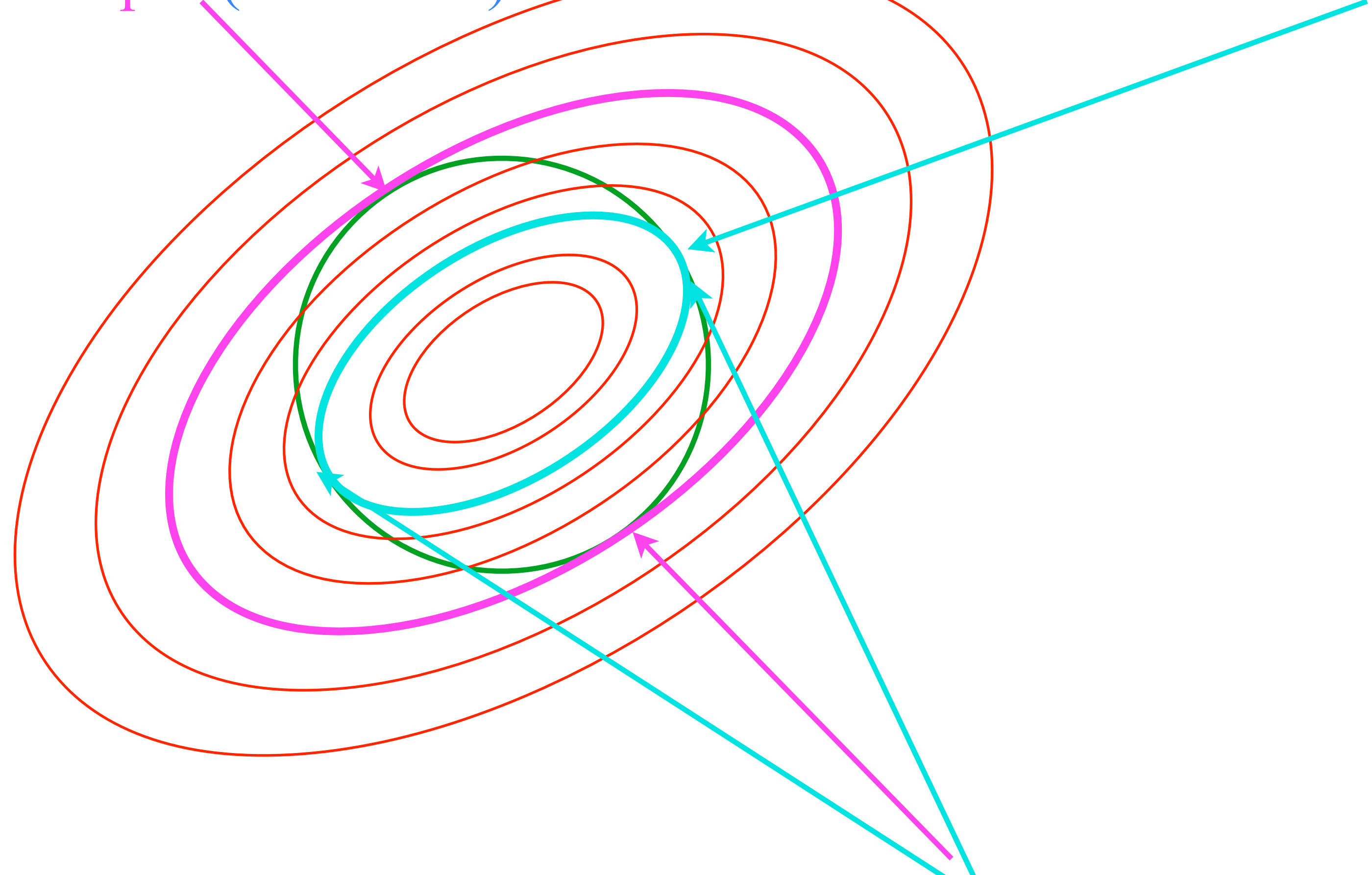
→ *Lagrange multiplier as eigenvalues*

Multiple multipliers

"Non-Holonomic" multipliers

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Extreme cases occur only at *contact points*

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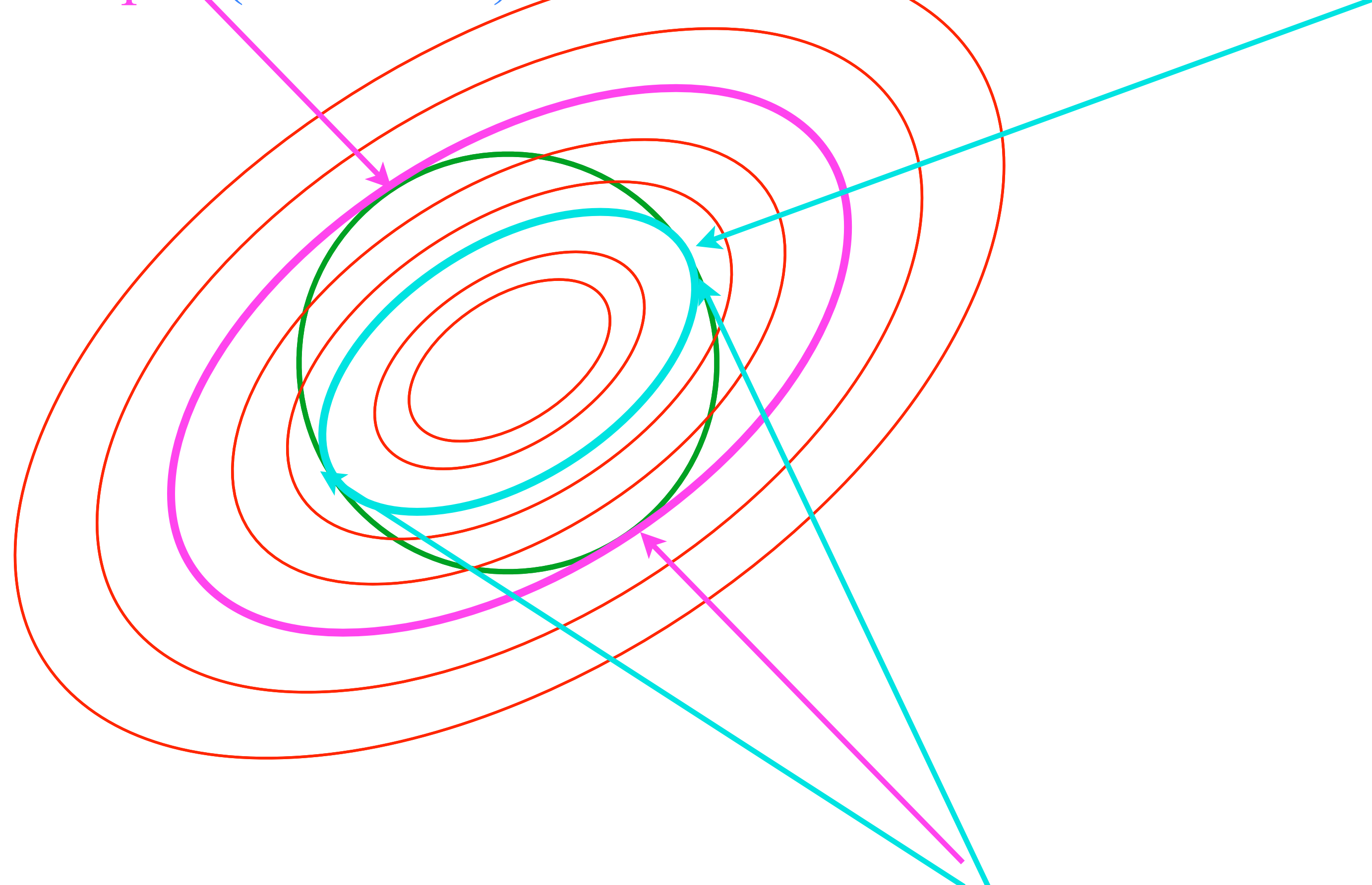

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This amounts to a *λ-eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek "L")

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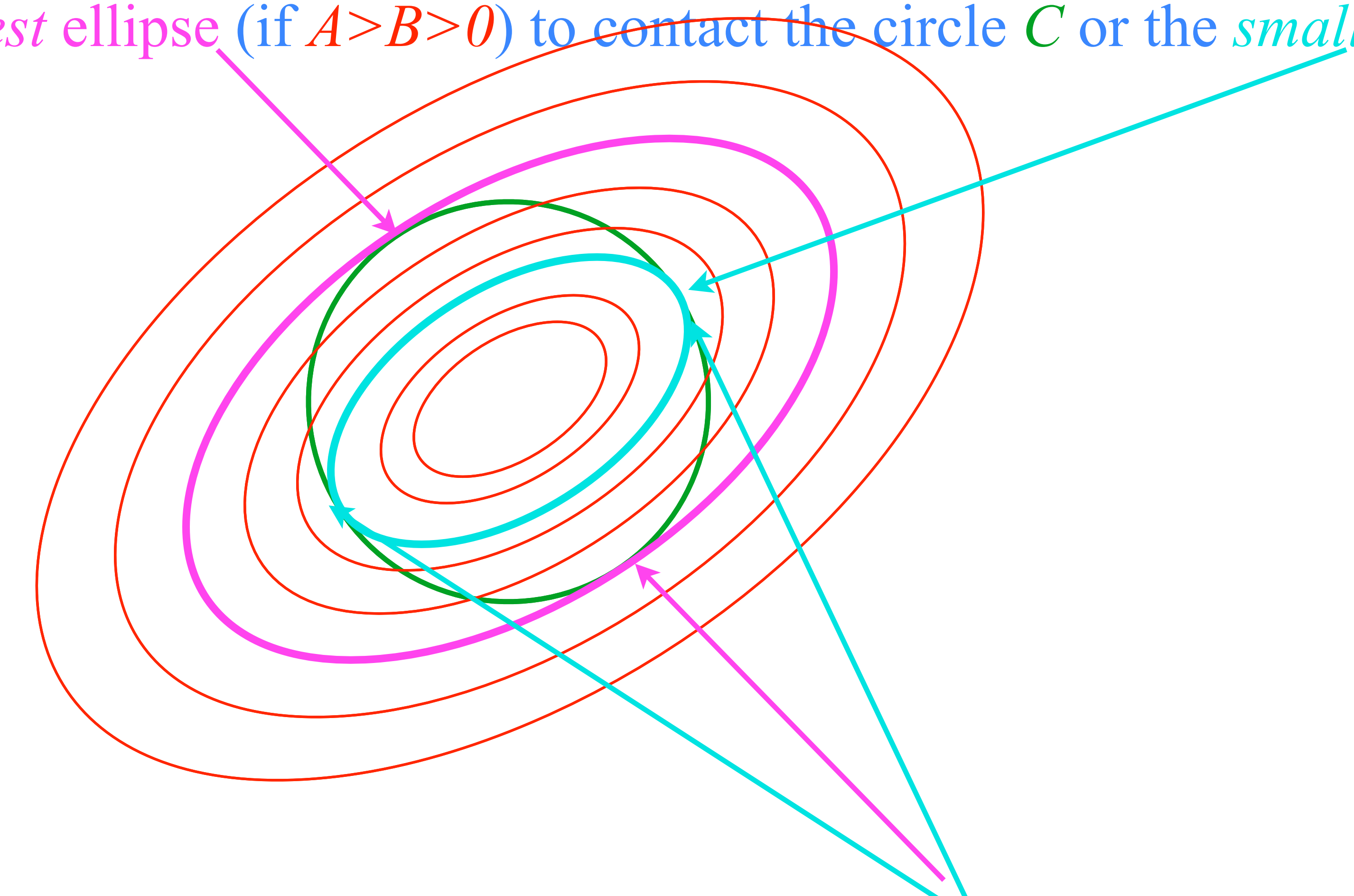

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Eigenvalues λ are ***extreme*** matrix "**own**"-values $\langle\psi|M|\psi\rangle$ subject ***Norm-constraint*** $\langle\psi|\psi\rangle=1$

***Eigen* - LEO Online German Dictionary**

Other Ways to do constraint analysis

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Sketch of atomic-Stark orbit parabolic OCC analysis

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Lagrange multiplier as eigenvalues

→ *Multiple multipliers*

"Non-Holonomic" multipliers

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{}{\partial q^k}\frac{\partial c}{} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

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Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

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Two or more constraints $c^1(q^k) = const., c^2(q^k) = const., \cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

→ *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

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Constrained equations of motion

$$\dot{p}_1 - \frac{\partial L}{\partial q^1} = \lambda_1 \frac{\partial c^1}{\partial q^1} + \lambda_2 \frac{\partial c^2}{\partial q^1} + \ldots$$

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If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.
I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)
A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

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Force components $F_k^\gamma = \frac{\partial c^\gamma}{\partial q^k} = C_k^\gamma$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

Integral constraint differentials

$$\frac{\partial F_k^\gamma}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F_j^\gamma}{\partial q^k}$$

General differential constraint relations

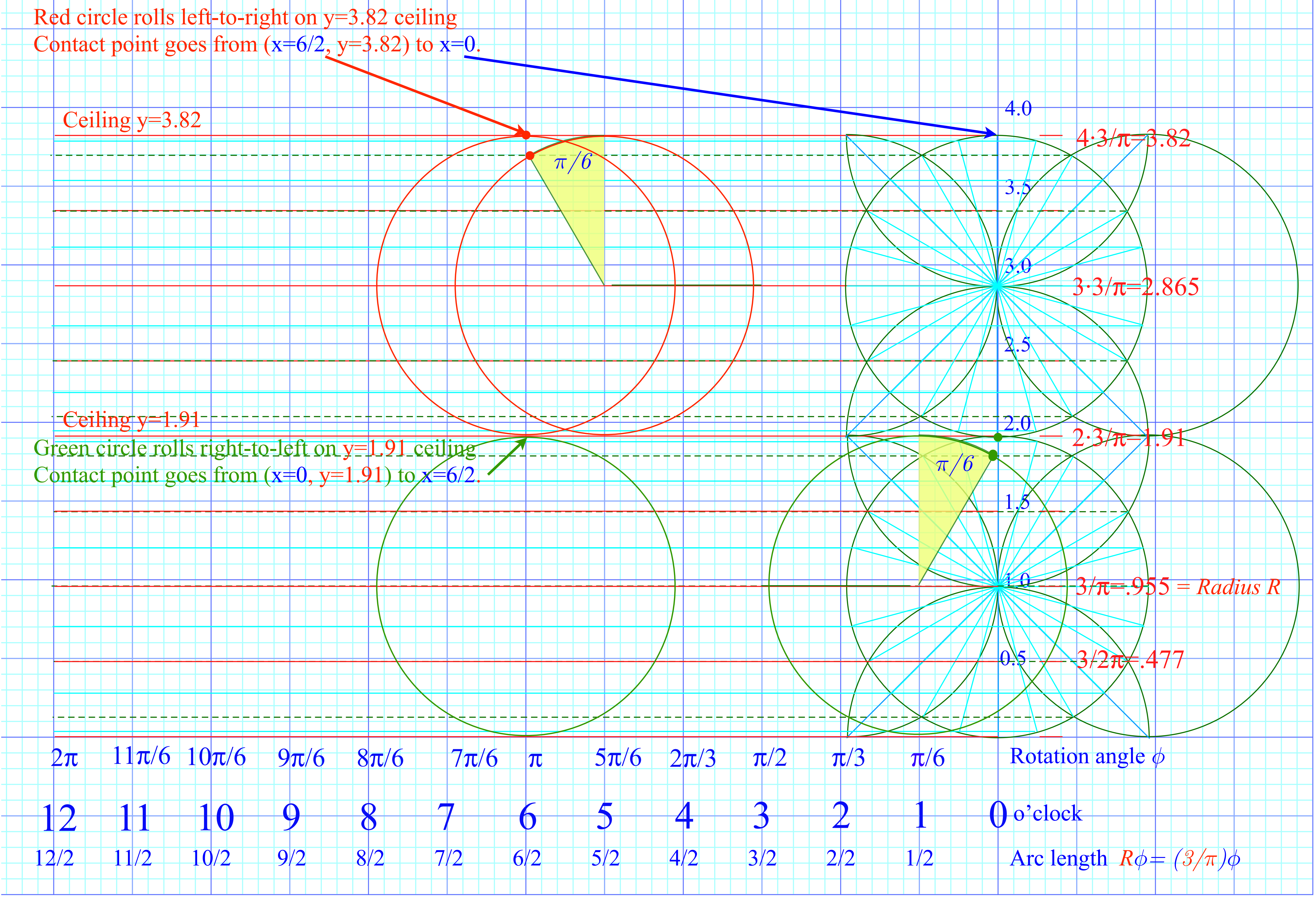
$$\frac{\partial C_k^\gamma}{\partial q^j} \quad \begin{matrix} \textit{may or} \\ \textit{may not be} \end{matrix} \quad \frac{\partial C_j^\gamma}{\partial q^k}$$

Cycloid-like curves for rolling constraints

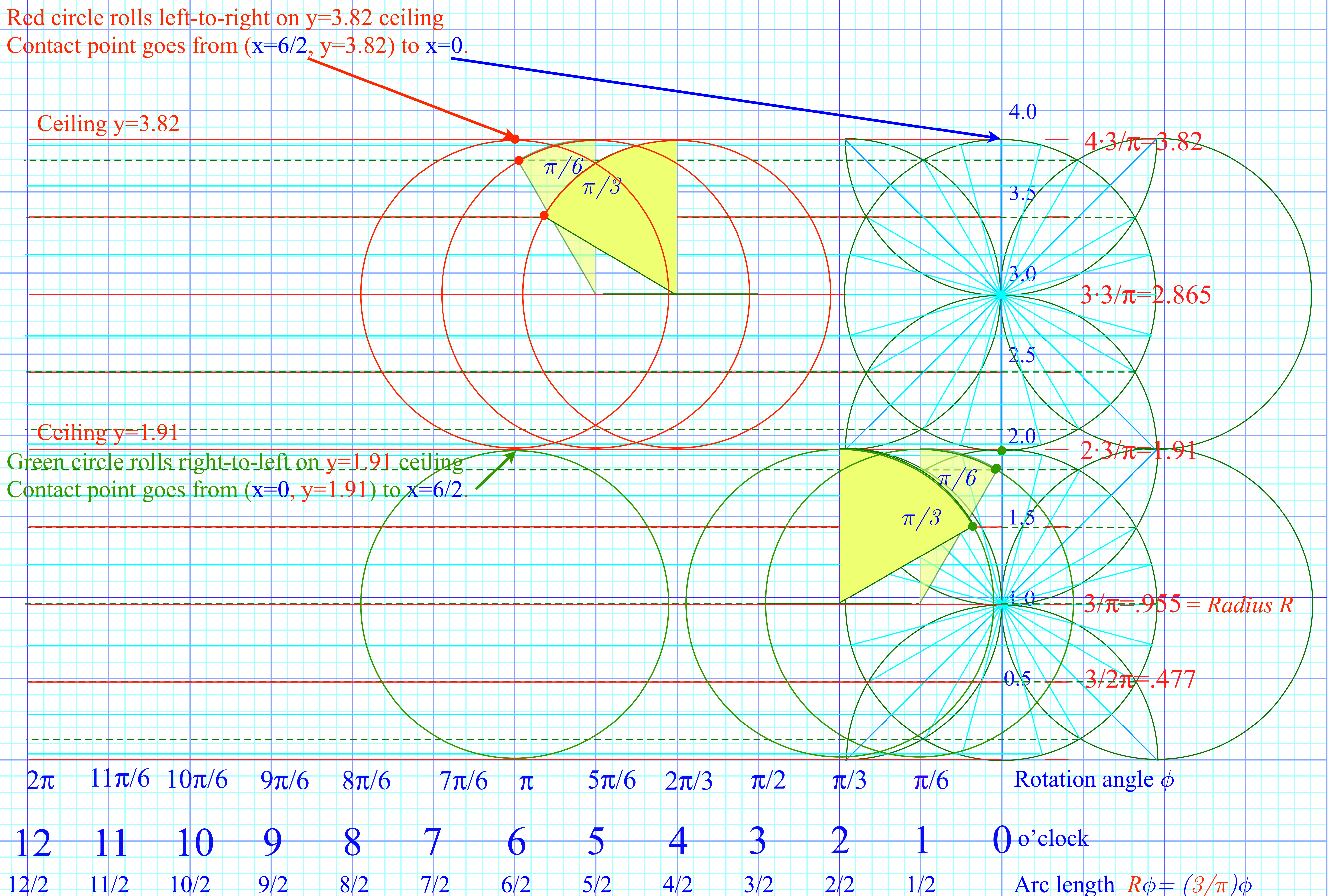
Cycloid-like curves for rolling constraints

First: A regular cycloid construction

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled -out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$



Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled -out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$



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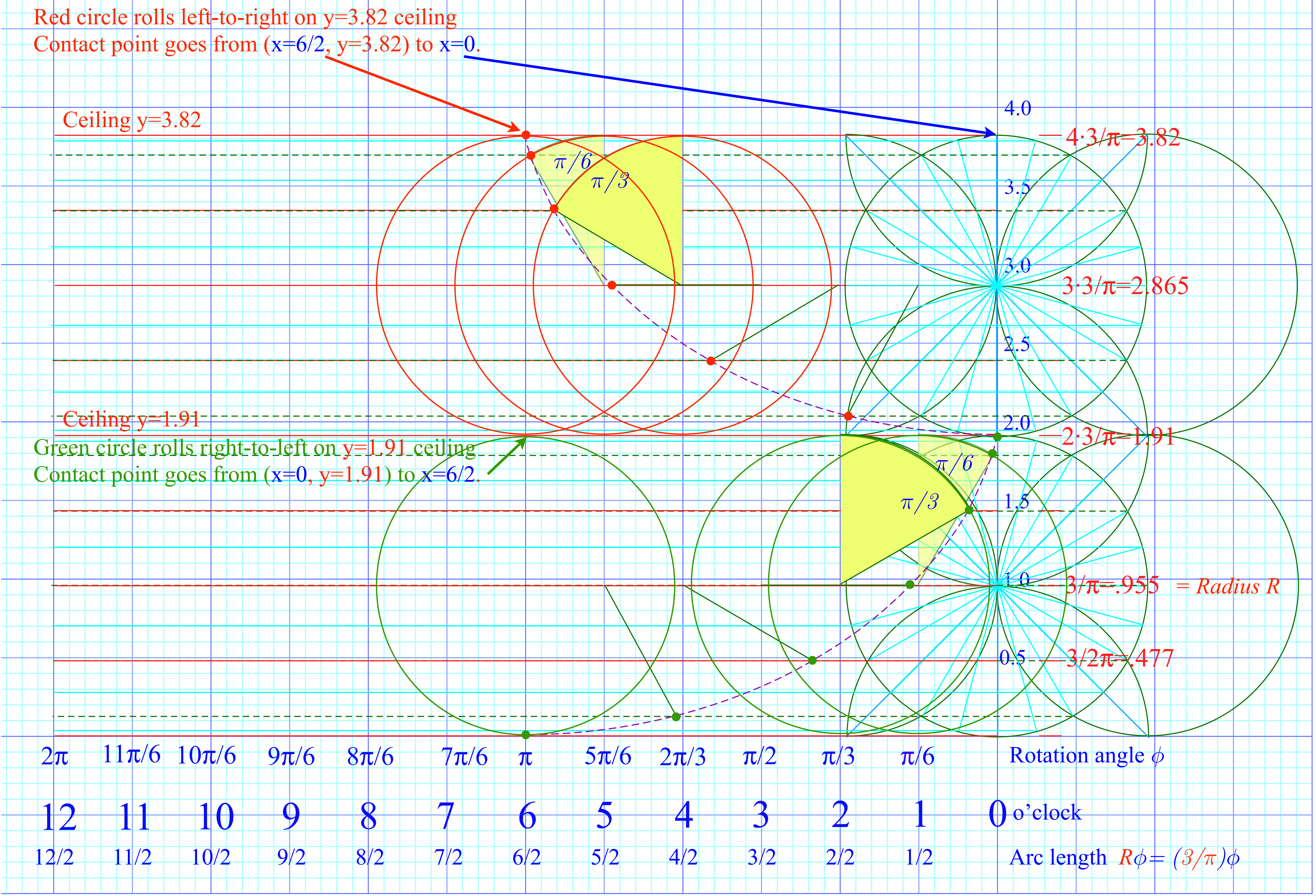
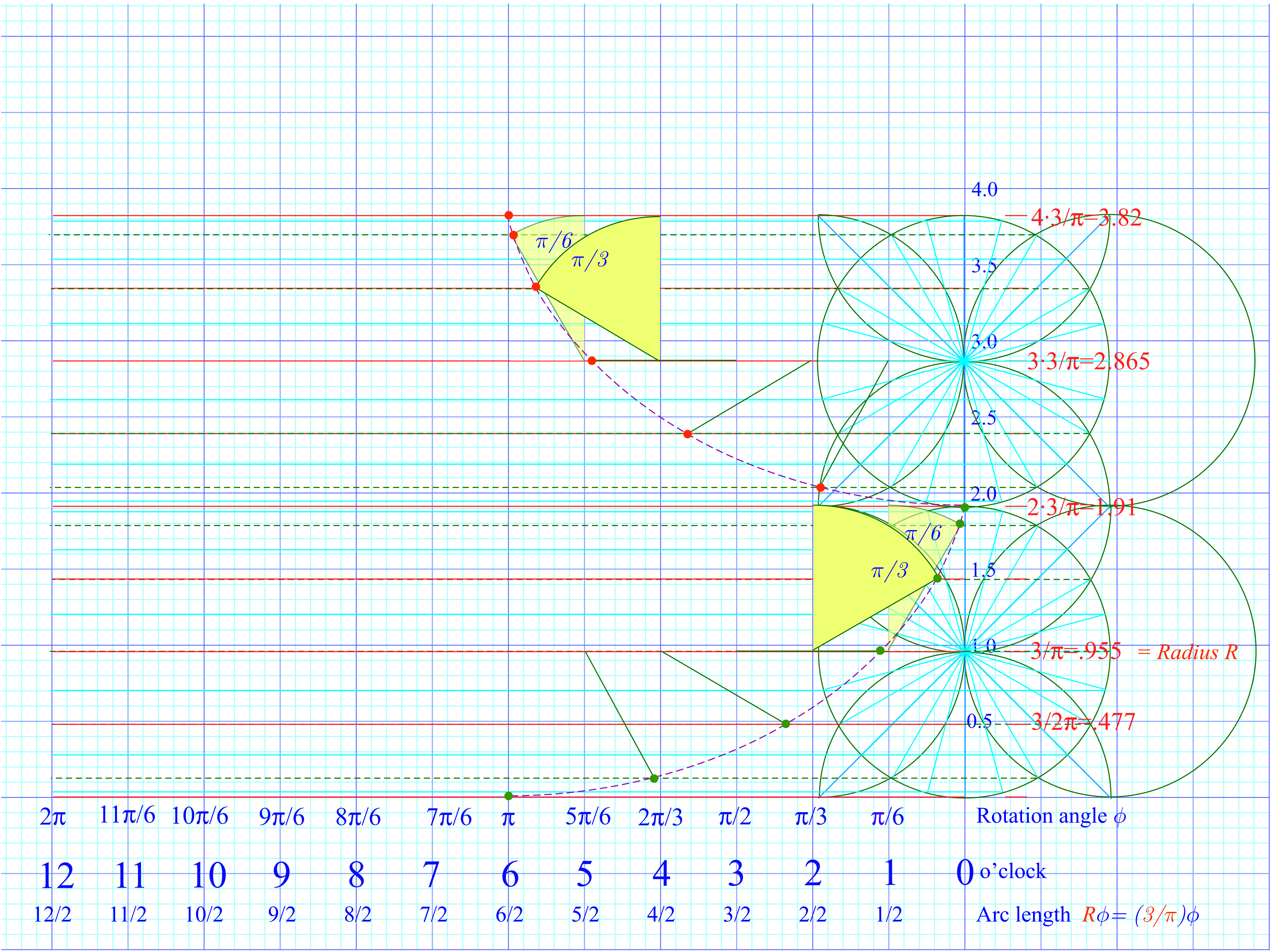
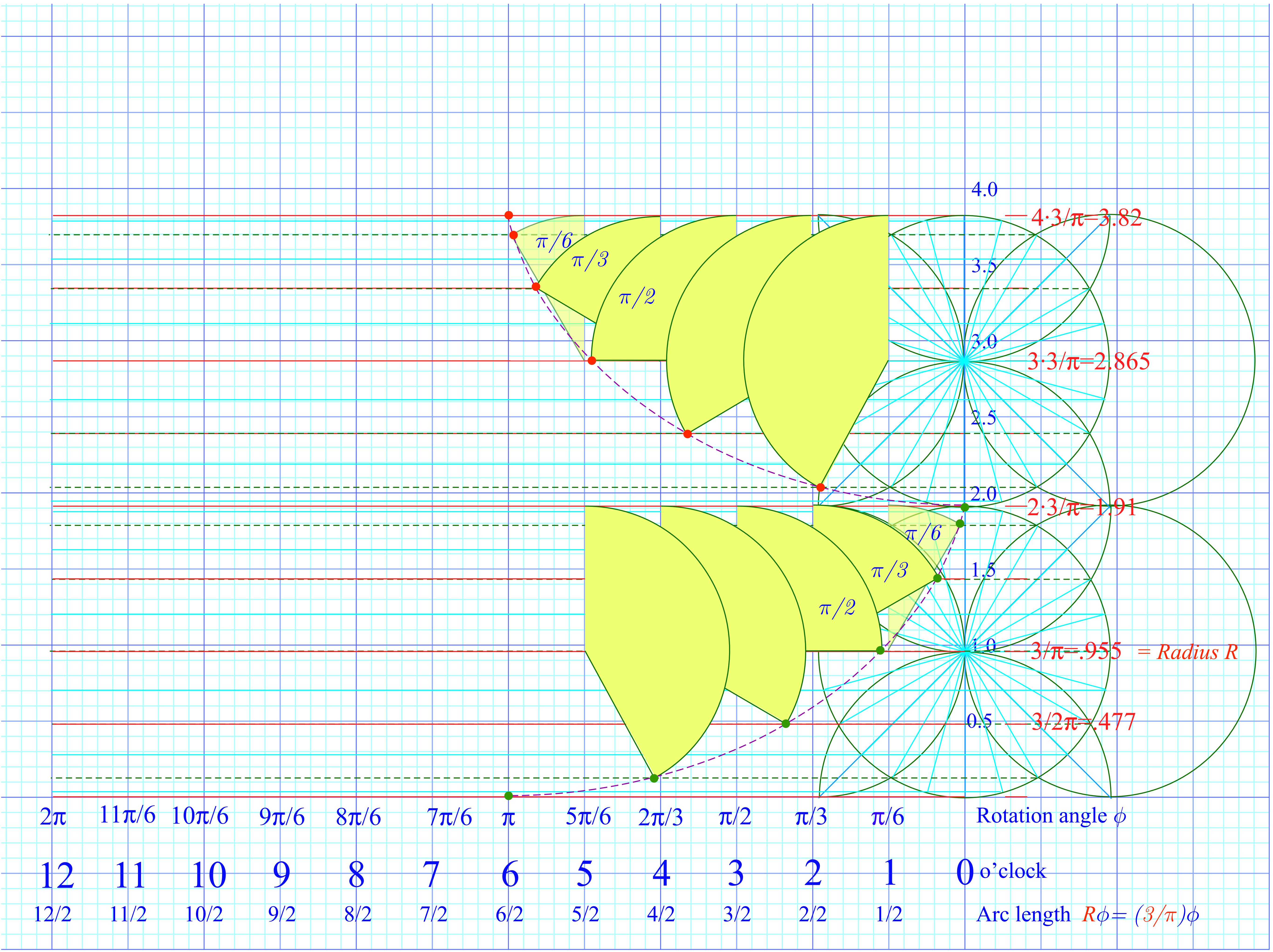
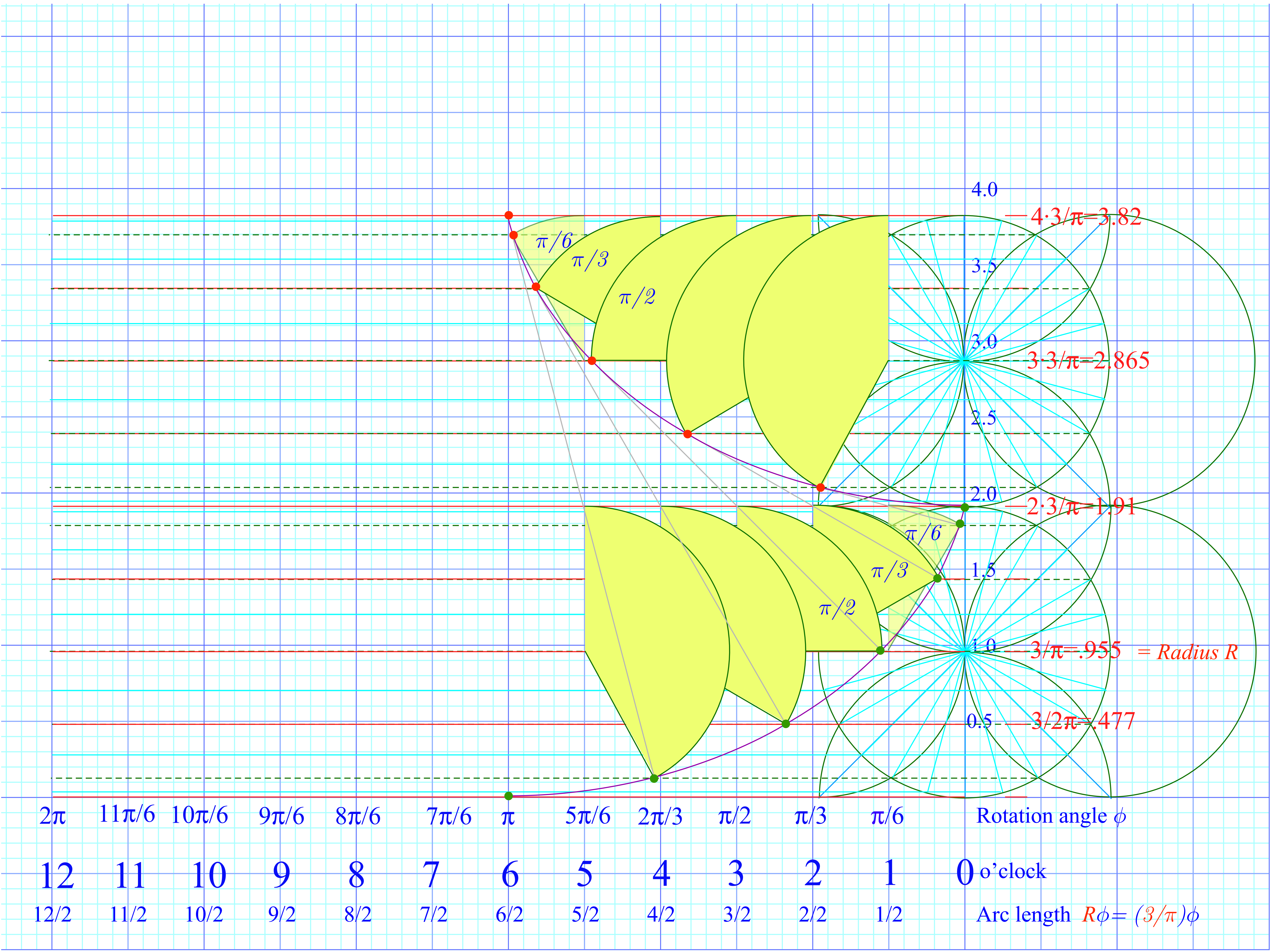


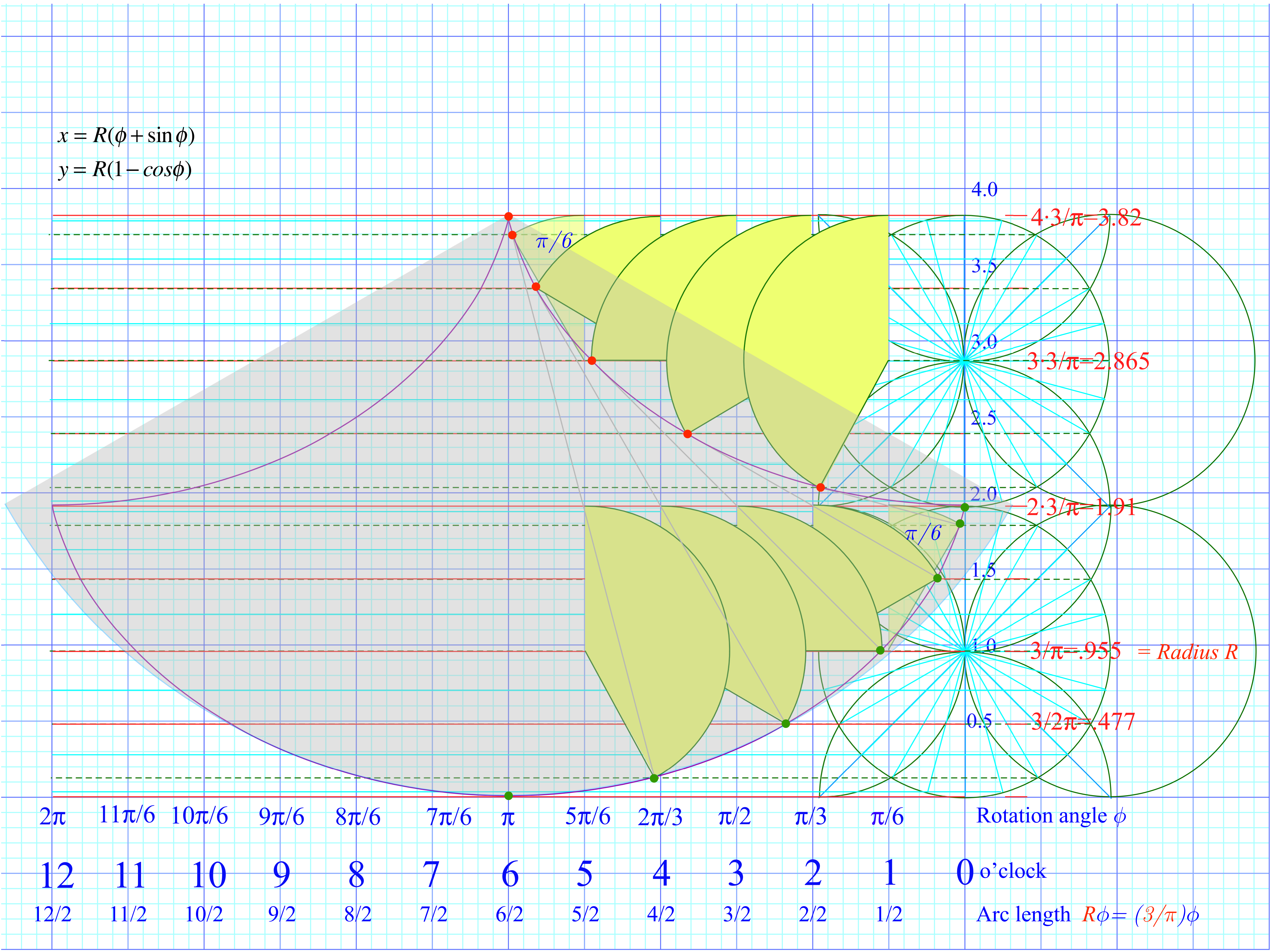


4.0
4·3/π=3.82
π/6
π/3
3.5
3.0
3·3/π=2.865
2.5
2.0
2·3/π=1.91
π/6
π/3
1.5
1.0
3/π=.955 = Radius R
0.5
3/2π=.477
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle ϕ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rϕ= (3/π)ϕ



4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rφ= (3/π)φ


4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/2
π/6
π/3
π/2
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle φ
12
11
10
9
8
7
6
5
4
3
2
1
0 o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rφ= (3/π)φ

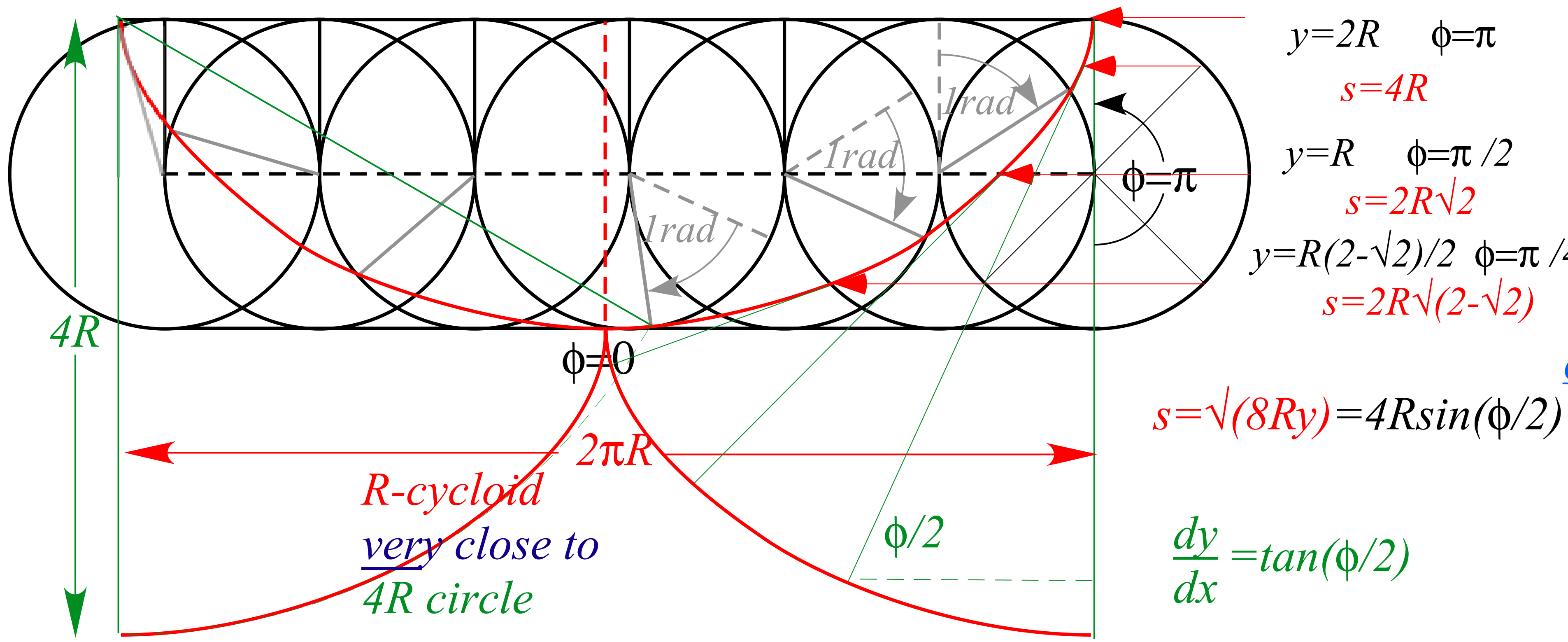

x = R(ϕ + sin ϕ)
y = R(1 − cosϕ)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955
= Radius R
3/2π=.477
π/6
π/6
2π
11π/6
10π/6
9π/6
8π/6
7π/6
π
5π/6
2π/3
π/2
π/3
π/6
Rotation angle ϕ
12
11
10
9
8
7
6
5
4
3
2
1
0
o'clock
12/2
11/2
10/2
9/2
8/2
7/2
6/2
5/2
4/2
3/2
2/2
1/2
Arc length Rϕ = (3/π)ϕ

$$ds^2 = dx^2 + dy^2 = 2R^2(1+\cos\phi)d\phi^2 = 4R^2\cos^2\frac{\phi}{2}d\phi^2$$

$$x = R(\phi + \sin\phi) \qquad dx = R(1+\cos\phi)d\phi$$

$$y = R(1 - \cos\phi) \qquad dy = R\sin\phi\, d\phi$$

$$ds = 2R\cos\frac{\phi}{2}d\phi \quad \text{or: } s = \int ds = 4R\sin\frac{\phi}{2} = 4R\sqrt{\frac{1-\cos\phi}{2}} = \sqrt{8Ry} = 4R(\text{if}: y = 2R)$$



[Web Simulation - OscillatorPE](#)

[Cycloidally constrained pendulum](#)

Cycloid Lagrangian $L = mR^2(1+\cos\phi)\dot{\phi}^2 - mgR(1-\cos\phi)$ gives: $p_\phi = 2mR^2(1+\cos\phi)\dot{\phi}$

and equation of motion

$$\ddot{\phi} = \frac{(R\dot{\phi}^2 - g)\sin\phi}{2R(1+\cos\phi)} = (R\dot{\phi}^2 - g)\frac{2\sin\phi/2\cos\phi/2}{4R\cos^2\phi/2} = \frac{(R\dot{\phi}^2 - g)}{2R}\tan\frac{\phi}{2} \qquad \text{Note: } \tan\frac{\phi}{2} \xrightarrow[\phi\to\pm\pi]{} \pm\infty$$

Time diff.eq.: $\dot{s}^2 = 2gy_0 - 2gy = 2g\frac{s_0^2 - s^2}{8R}$ integrates to: $t = \int dt = \sqrt{\frac{4R}{g}}\int\frac{ds}{\sqrt{s_0^2 - s^2}} = \sqrt{\frac{4R}{g}}\sin^{-1}\frac{s}{s_0}$ *+const.*

Arc length oscillates: $s = s_0\sin(\omega t - const.)$ at frequency $\omega = \sqrt{\frac{g}{4R}}$ of an ℓ*=4R* pendulum.

The rolling φ-angle time behavior $s = 4R\sin\frac{\phi}{2} = s_0\sin(\omega t - const.)$ is: $\frac{\phi}{2} = \sin^{-1}\left[\frac{s_0}{4R}\sin(\omega t - const.)\right]$

If initial value s_0 is maximum s_0*=4R* then $\phi(t) = 2\omega t - const.$ has constant angular velocity $\dot{\phi} = 2\omega$ for −π/2<φ<π/2.

Cycloid-like curves for rolling constraints

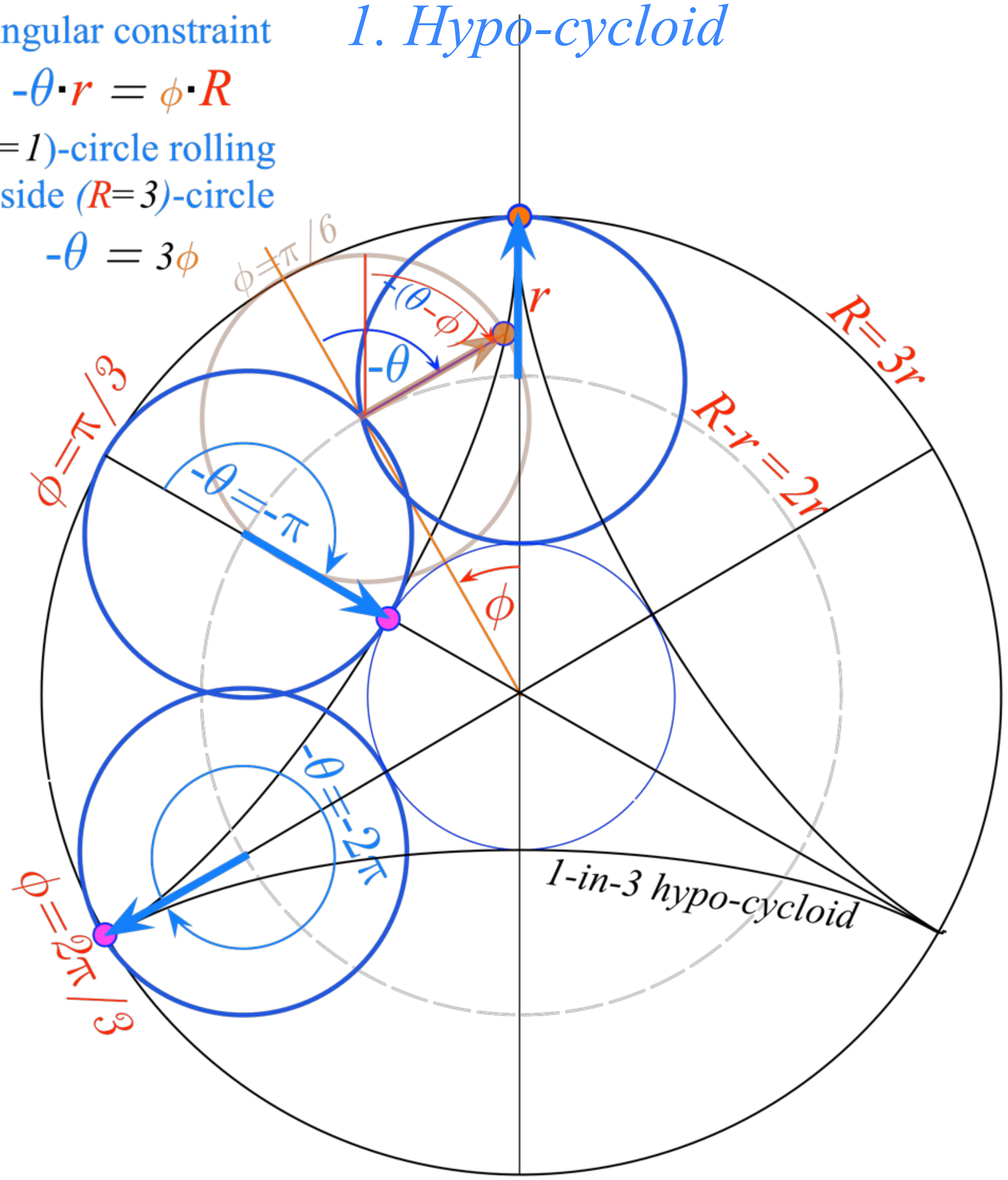
Angular constraint

$-\theta \cdot r = \phi \cdot R$

(r=1)-circle rolling inside (R=3)-circle

$-\theta = 3\phi$

1. Hypo-cycloid



Web Simulation - OscillatorPE

Hypocycloidally constrained motion

Under construction

Cycloid-like curves for rolling constraints

Angular constraint

$-\theta \cdot r = \phi \cdot R$

(r=1)-circle rolling inside (R=3)-circle

$-\theta = 3\phi$

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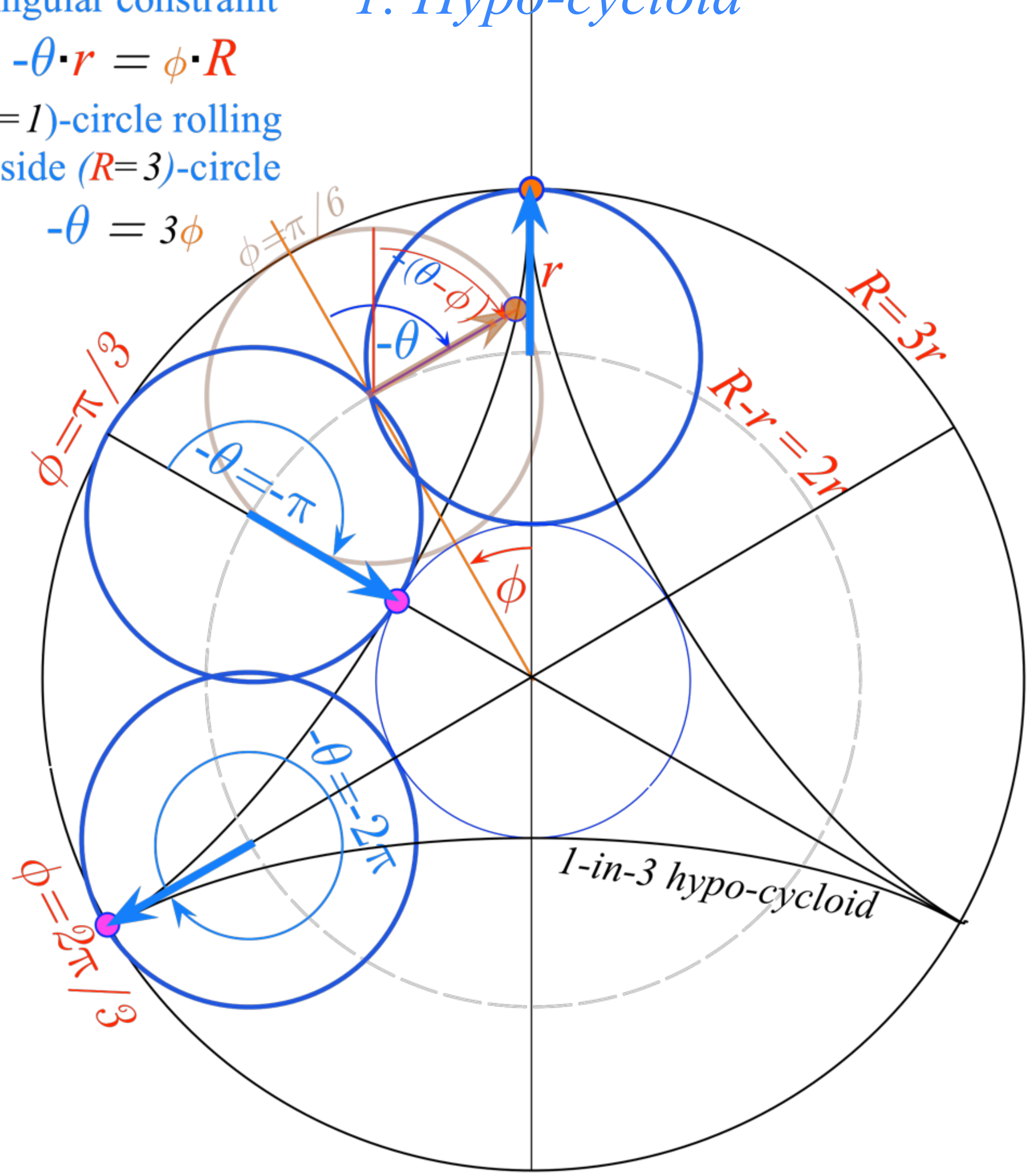

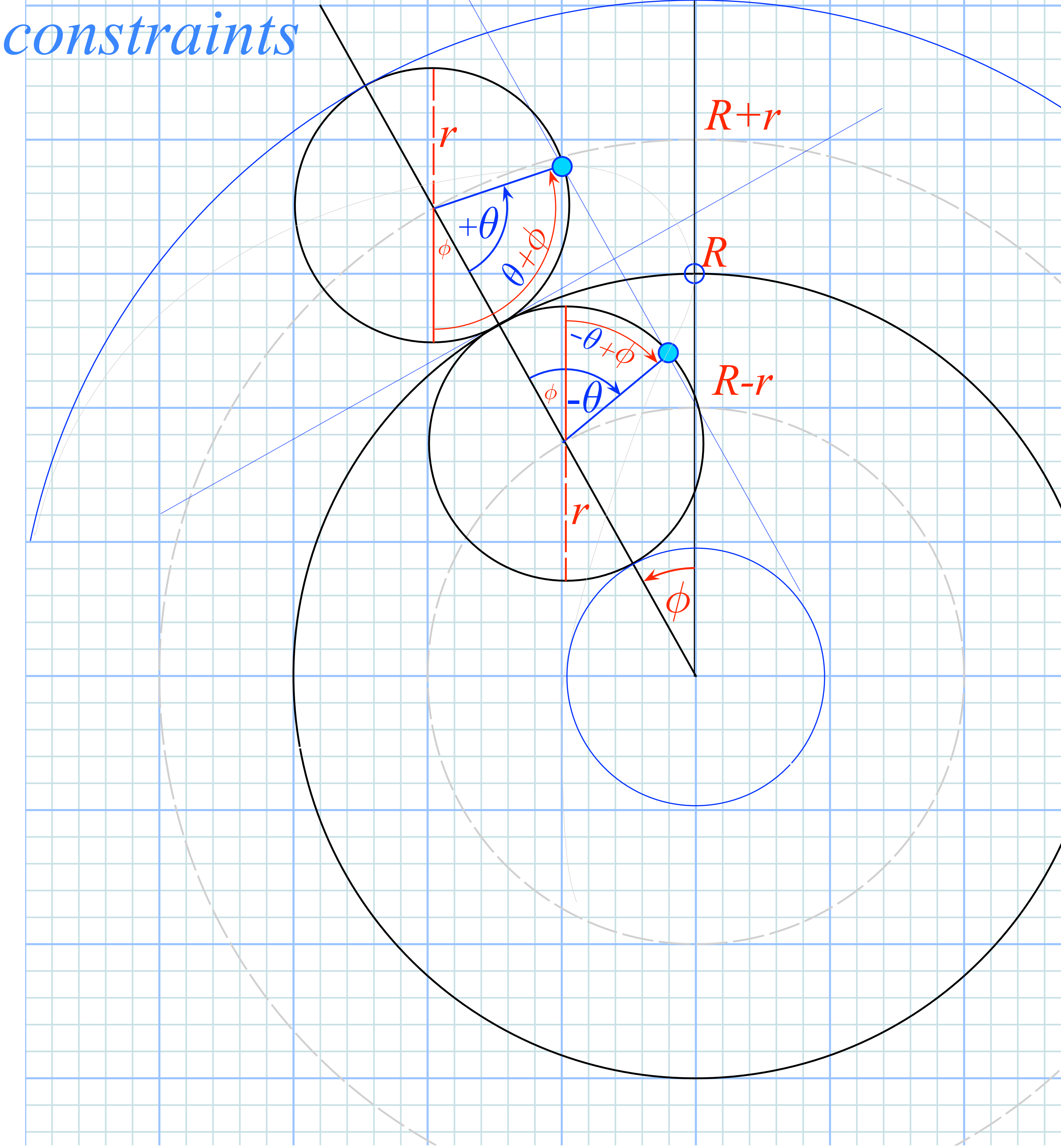

Web Simulation - OscillatorPE
Hypocycloidally constrained motion

Cycloid-like curves for rolling constraints

Angular constraint

$-\theta \cdot r = \phi \cdot R$

$(r=1)$-circle rolling inside $(R=3)$-circle

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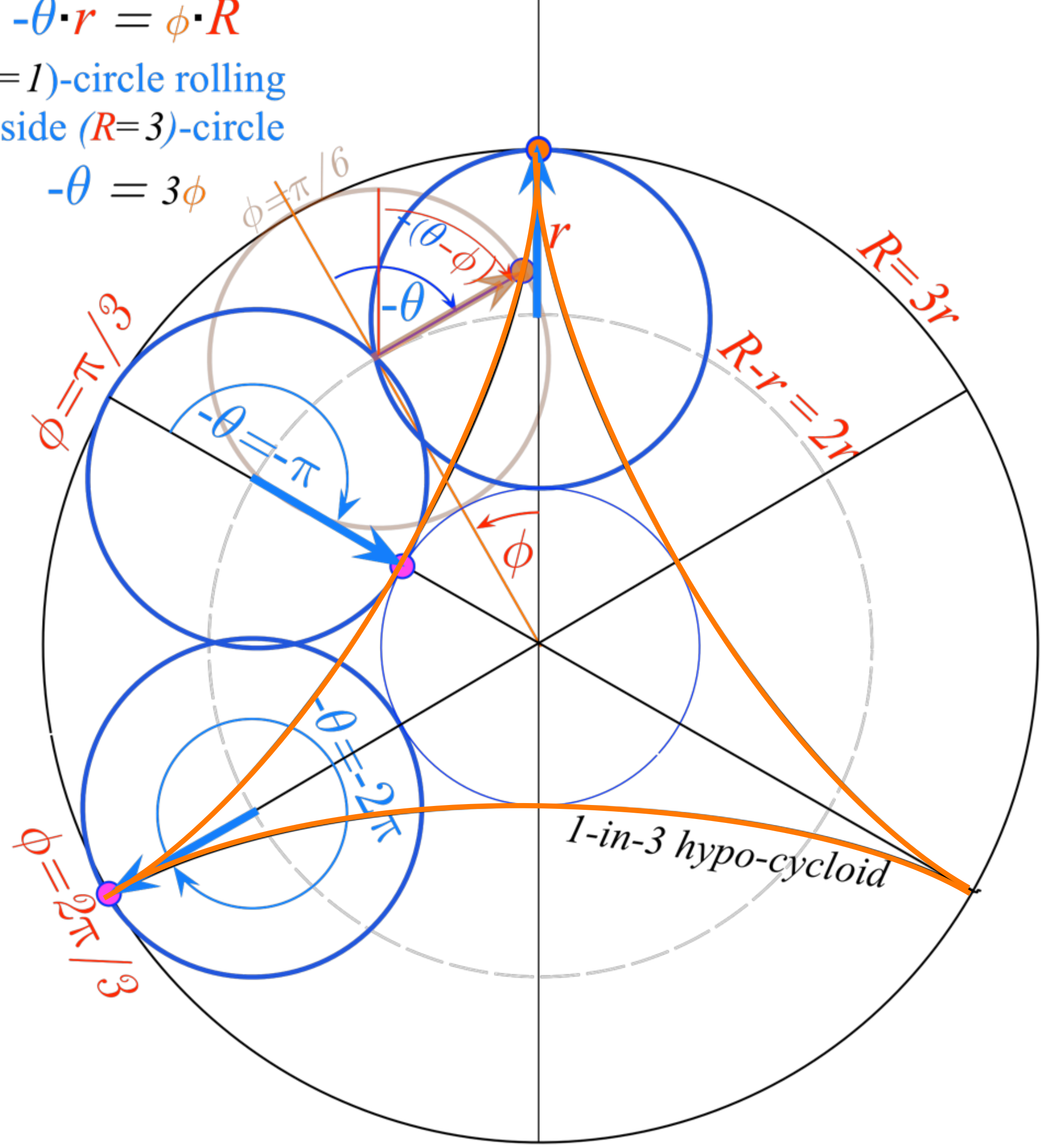
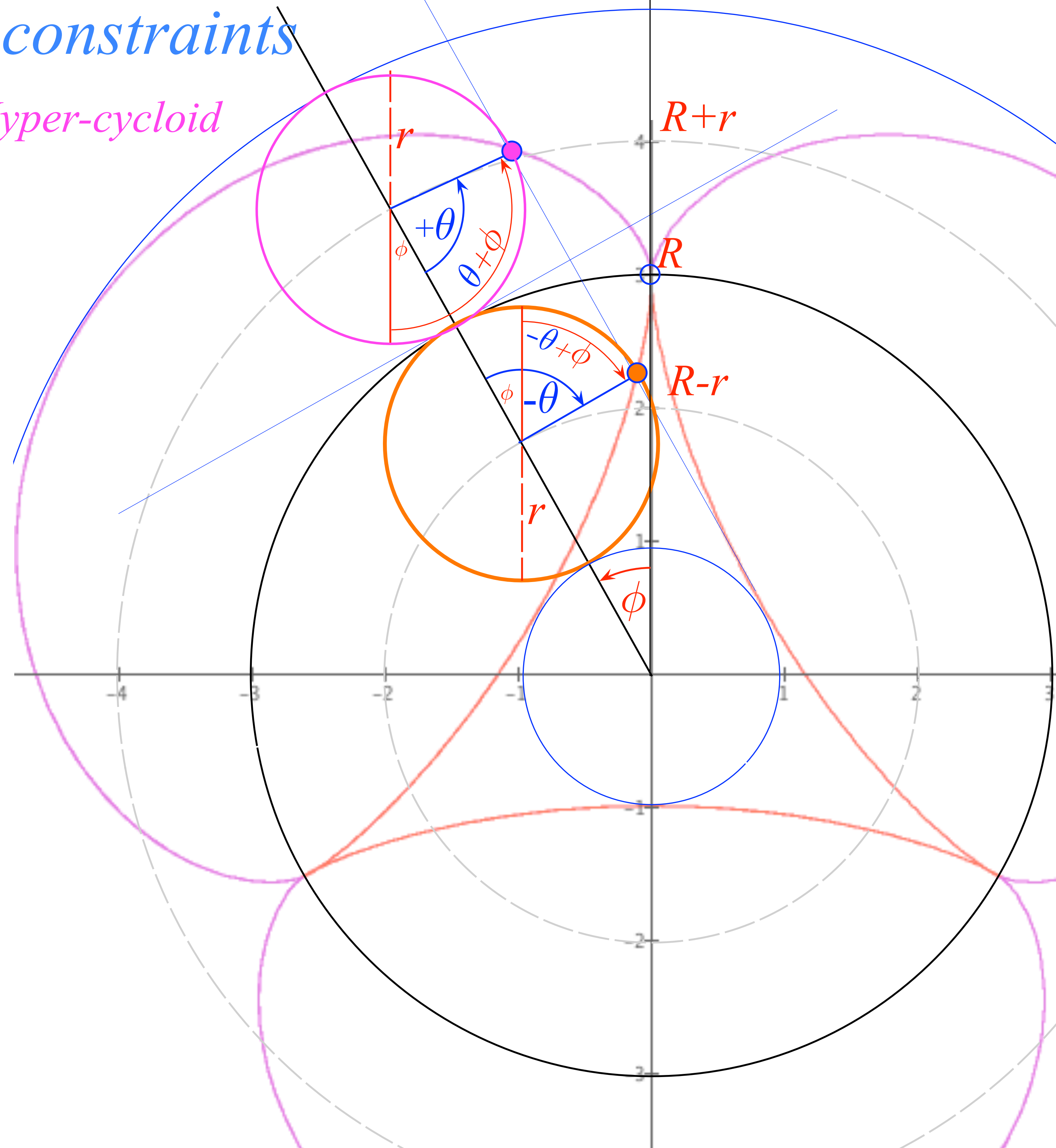


2. Hyper-cycloid



[Web Simulation - OscillatorPE](#)

[Hypocycloidally constrained motion](#)

Under construction

Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

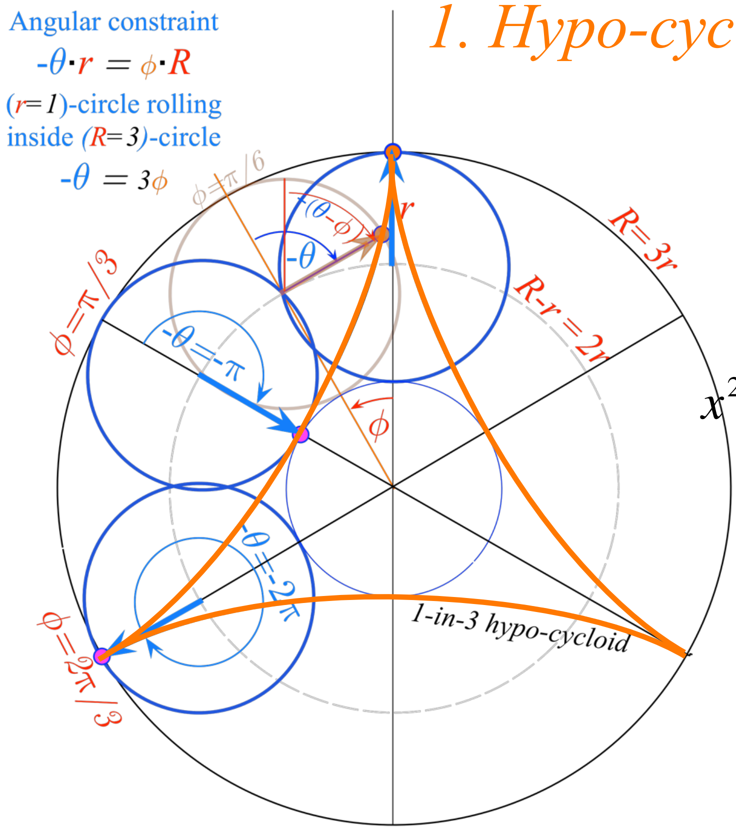
$$y = (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$

Hyper-cycloid constrained by: $\theta r = R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R+r)\sin\phi + r\sin(\theta+\phi) = r\left[-\left(\frac{R}{r}+1\right)\sin\phi + \sin\left(\frac{R}{r}+1\right)\phi\right]$$

$$y = (R+r)\cos\phi - r\cos(\theta+\phi) = r\left[\left(\frac{R}{r}+1\right)\cos\phi - \cos\left(\frac{R}{r}+1\right)\phi\right]$$

Cycloid-like curves for rolling constraints



1. Hypo-cycloid

2. Hyper-cycloid

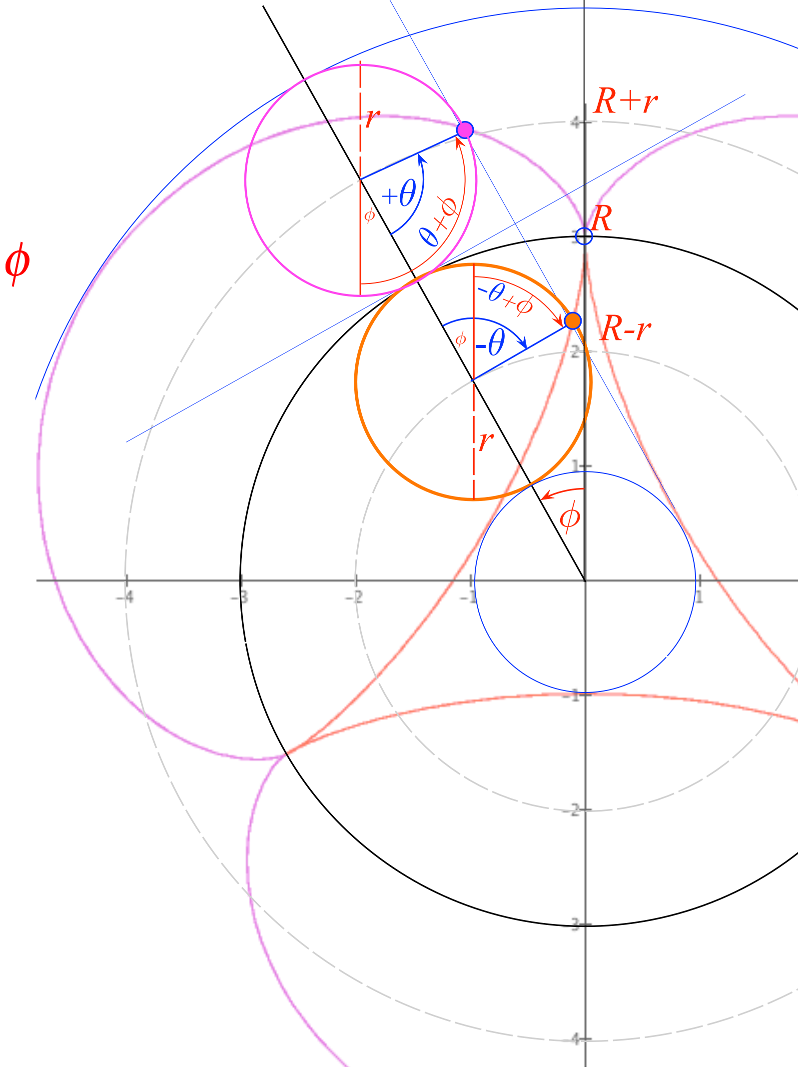
Hyper-cycloid constrained by $A=\frac{R}{r}+1,\ \theta=(A-1)\phi=\frac{R}{r}\phi$

$$x=-A\sin\phi+r\sin A\phi,\qquad y=A\cos\phi-r\cos A\phi.$$

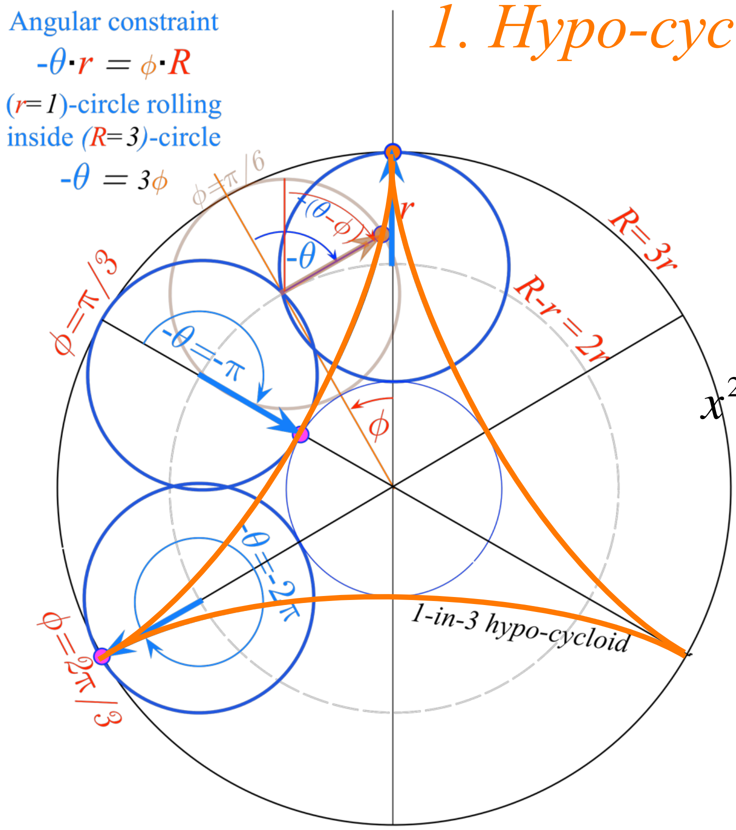
$$x^2/r^2=A^2\sin^2\phi-2A\sin\phi\sin A\phi+\sin^2 A\phi,$$

$$y^2/r^2=A^2\cos^2\phi-2A\cos\phi\cos A\phi+\cos^2 A\phi,$$

$$x^2/r^2+y^2/r^2=A^2-2A(\sin\phi\sin A\phi+\cos\phi\cos A\phi)+1$$



Cycloid-like curves for rolling constraints



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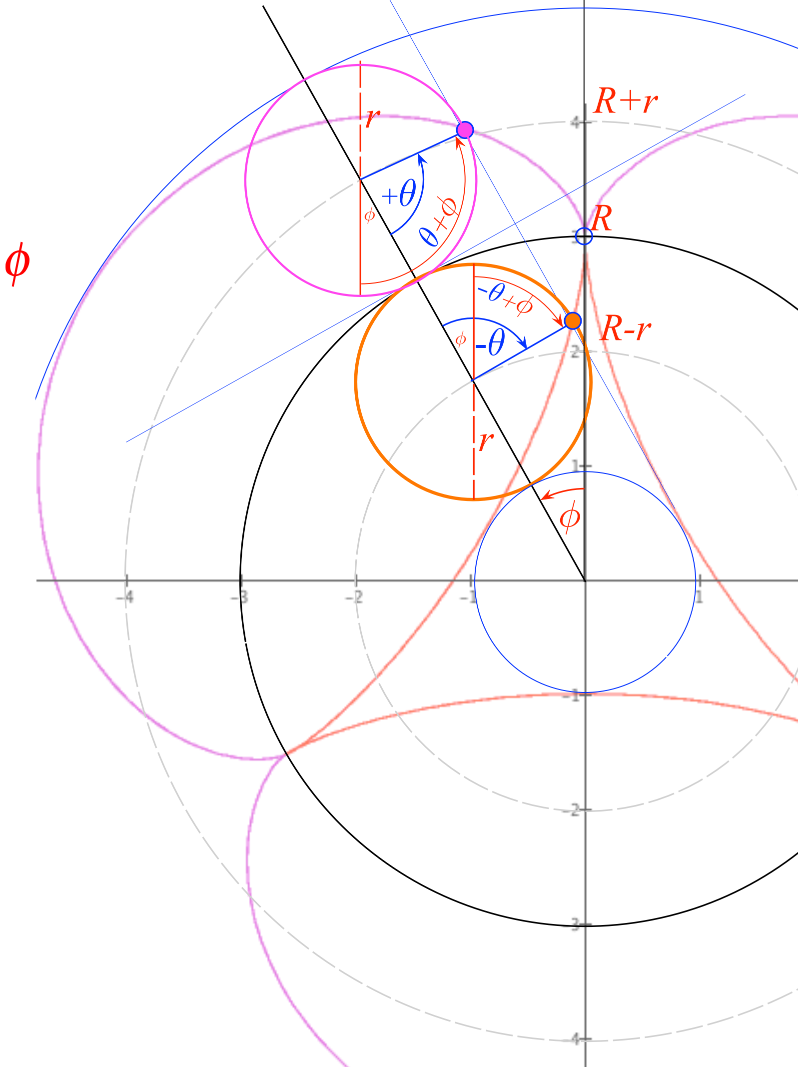
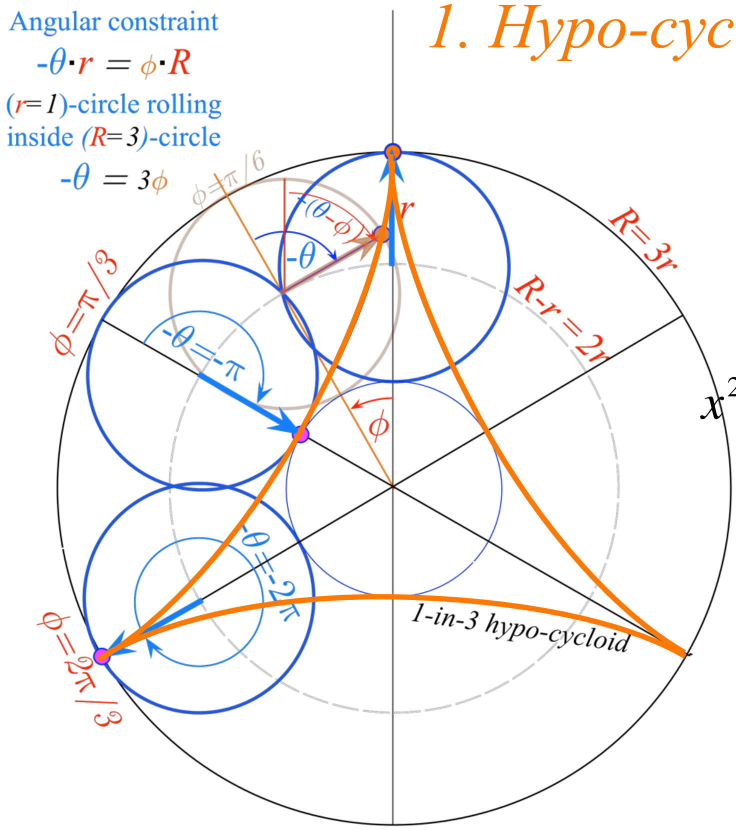
$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$$

Hyper-cycloid radius ρ:



Cycloid-like curves for rolling constraints



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2. Hyper-cycloid

Hyper-cycloid constrained by $A=\frac{R}{r}+1$, $\theta=(A-1)\phi=\frac{R}{r}\phi$

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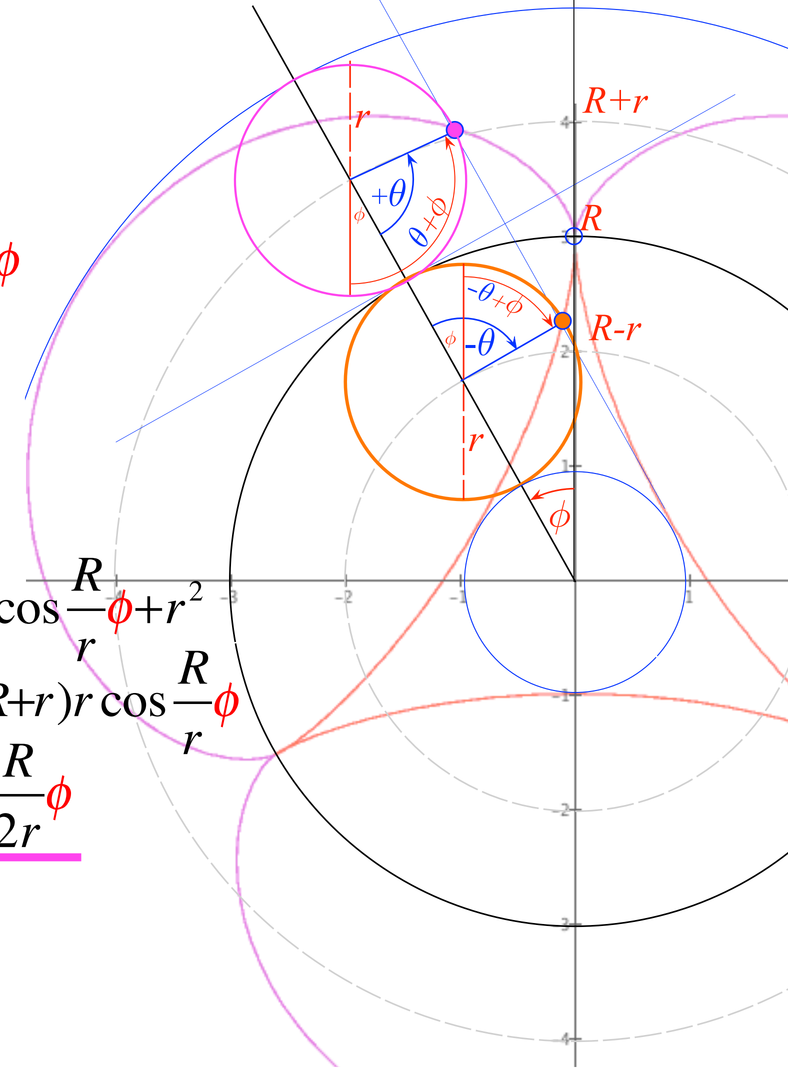
$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$$

Hyper-cycloid radius ρ:

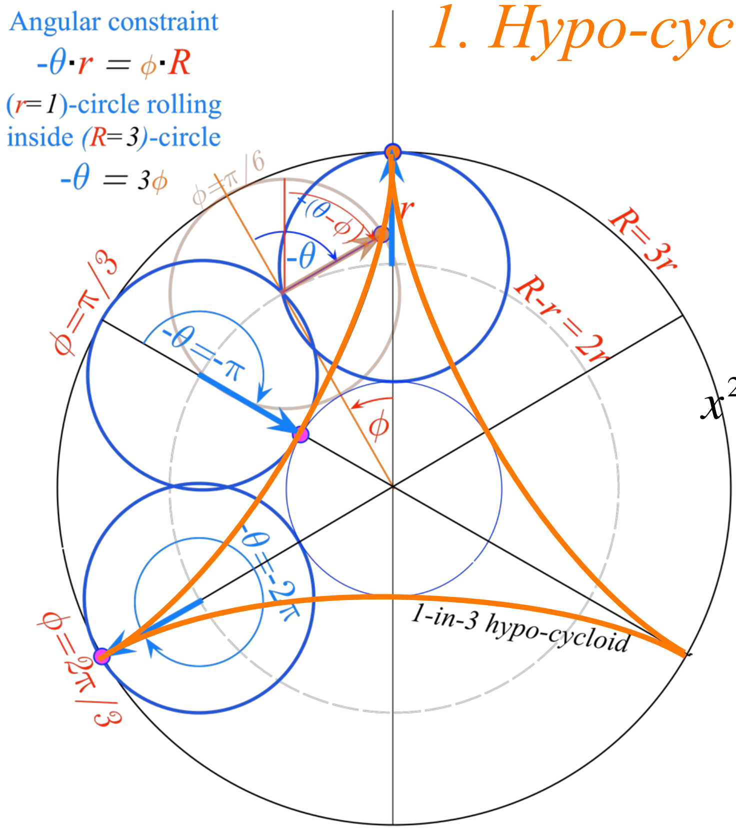
$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

$$=R^2+2(R+r)r-2(R+r)r\cos\frac{R}{r}\phi$$

$$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$$



Cycloid-like curves for rolling constraints



1. Hypo-cycloid

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$x^2/r^2+y^2/r^2=A^2-2A(\sin\phi\sin A\phi+\cos\phi\cos A\phi)+1$

$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$

$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$

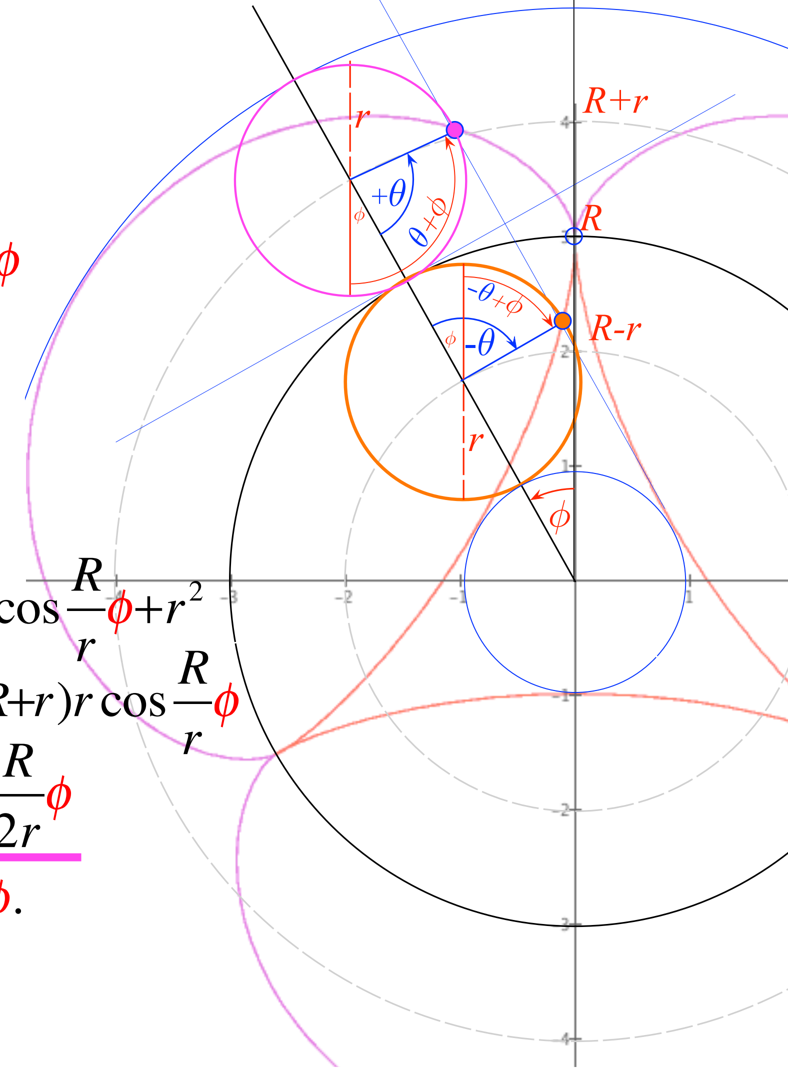
$=R^2+2(R+r)r-2(R+r)r\cos\frac{R}{r}\phi$

Hyper-cycloid radius ρ:

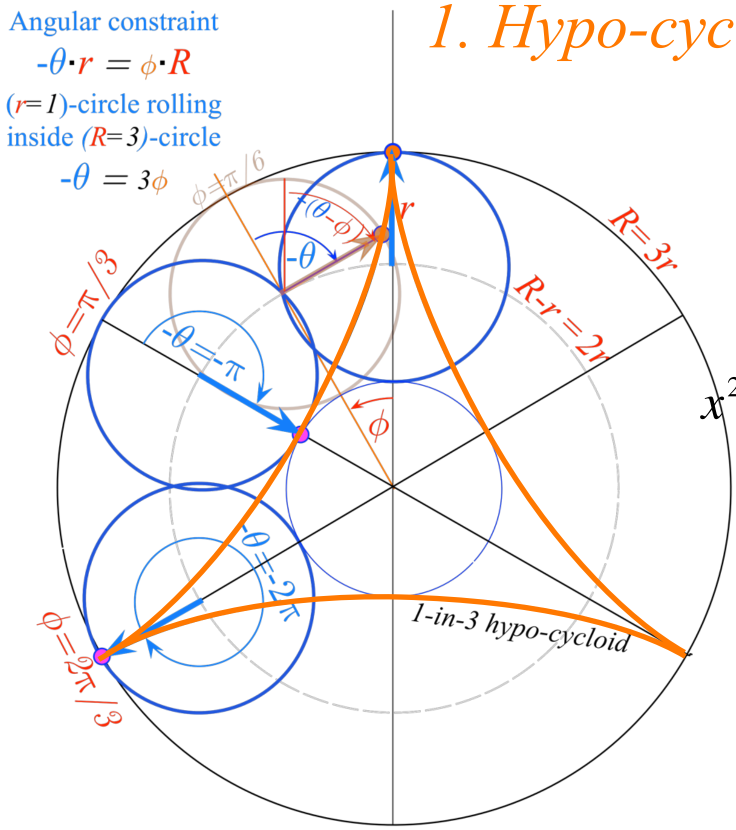
$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$

Hyper-cycloid velocity

$\dot{x}/r=-A\dot{\phi}\cos\phi+A\dot{\phi}\cos A\phi,\qquad \dot{y}/r=-A\dot{\phi}\sin\phi+A\dot{\phi}\sin A\phi.$



Cycloid-like curves for rolling constraints



1. Hypo-cycloid

2. Hyper-cycloid

Hyper-cycloid constrained by $A=\frac{R}{r}+1$, $\theta=(A-1)\phi=\frac{R}{r}\phi$

$$x=-A\sin\phi+r\sin A\phi,\qquad y=A\cos\phi-r\cos A\phi.$$

$$x^2/r^2 = A^2\sin^2\phi-2A\sin\phi\sin A\phi+\sin^2 A\phi,$$

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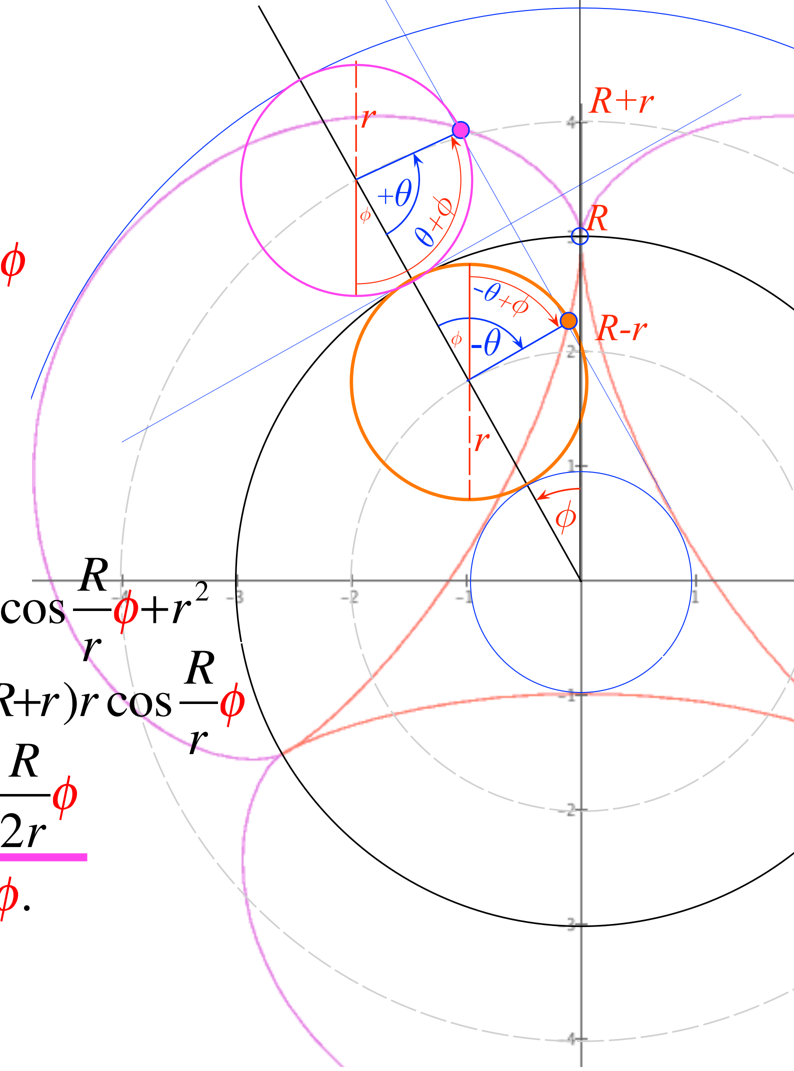
$$\dot{x}/r=-A\dot{\phi}\cos\phi+A\dot{\phi}\cos A\phi,\qquad \dot{y}/r=-A\dot{\phi}\sin\phi+A\dot{\phi}\sin A\phi.$$

$$\dot{x}^2/r^2 = A^2\dot{\phi}^2\cos^2\phi-2A^2\dot{\phi}^2\cos\phi\cos A\phi+A^2\dot{\phi}^2\cos^2 A\phi$$

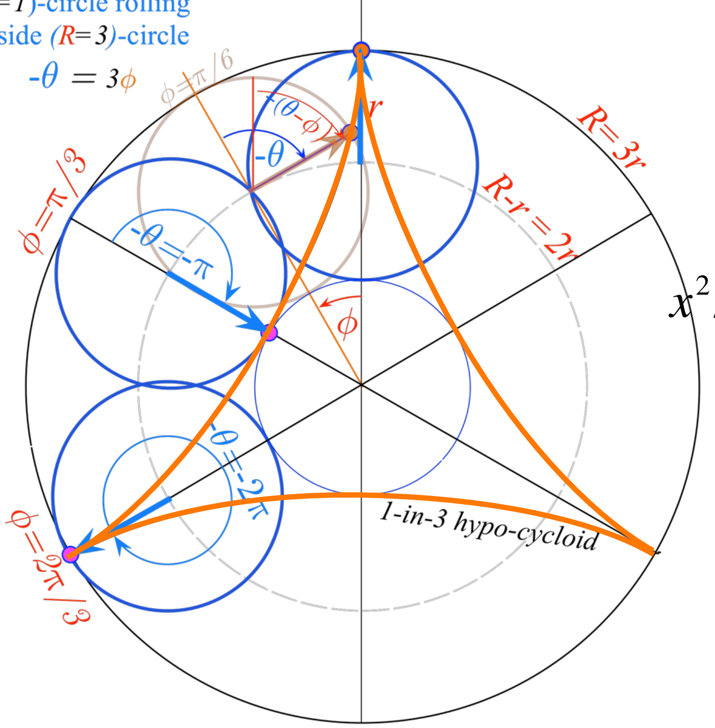
$$\dot{y}^2/r^2 = A^2\dot{\phi}^2\sin^2\phi-2A^2\dot{\phi}^2\sin\phi\sin A\phi+A^2\dot{\phi}^2\sin^2 A\phi$$



Cycloid-like curves for rolling constraints

Angular constraint
$-\theta\cdot r = \phi\cdot R$
(r=1)-circle rolling inside (R=3)-circle
$-\theta = 3\phi$

1. Hypo-cycloid



2. Hyper-cycloid

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$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

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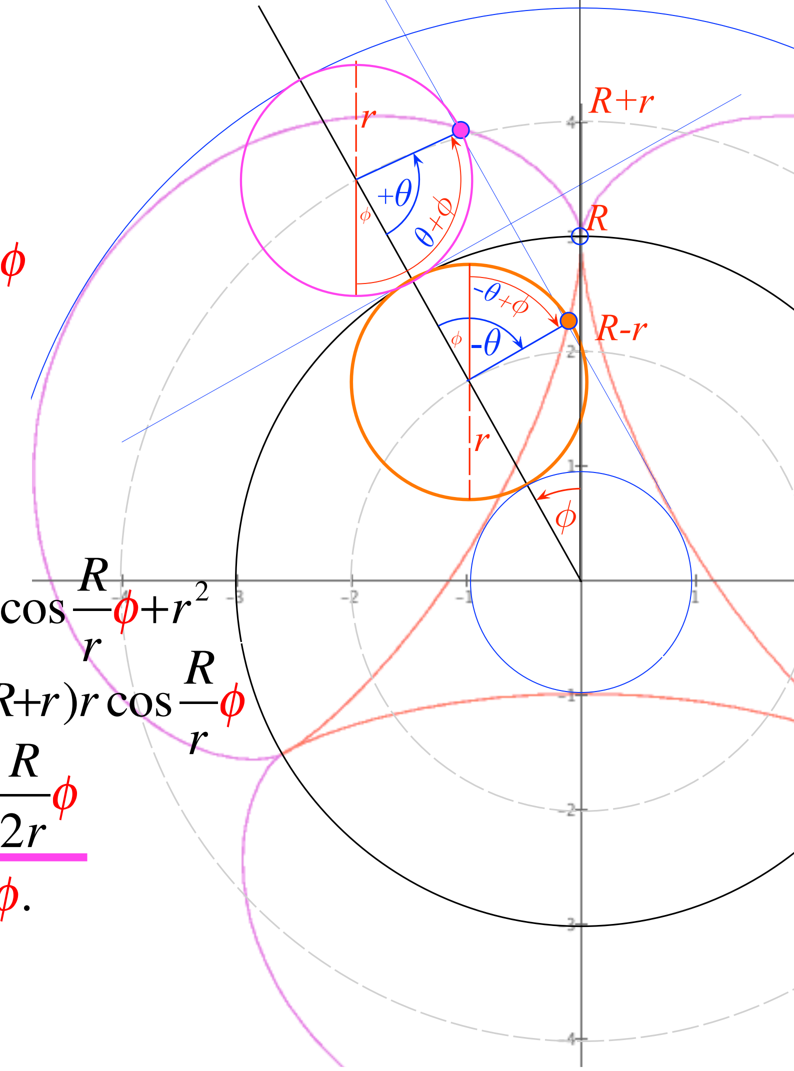
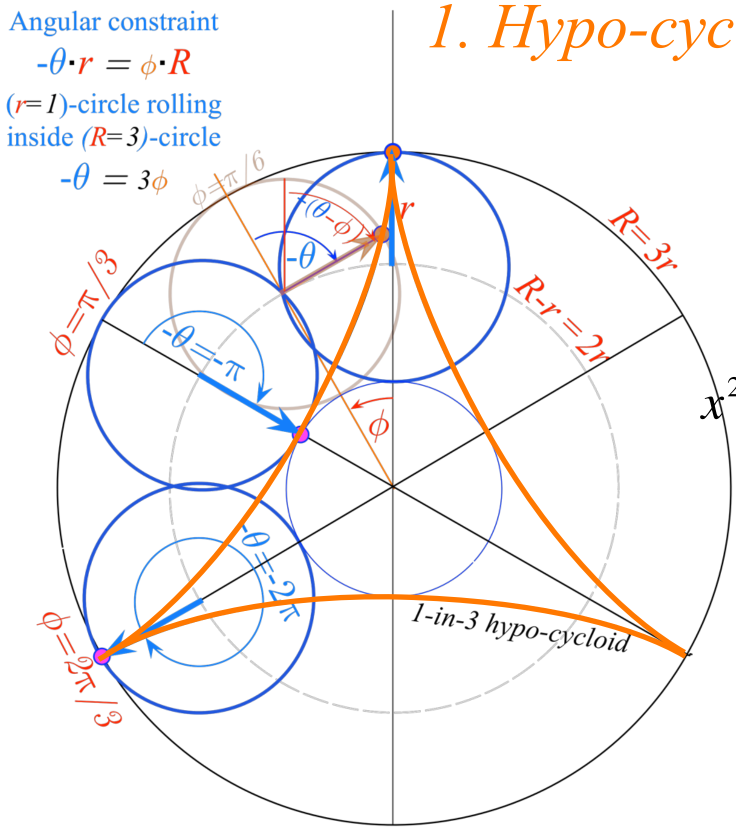
$$\dot{x}^2/r^2+\dot{y}^2/r^2=2A^2\dot{\phi}^2(1-\cos\phi\cos A\phi+\sin\phi\sin A\phi)=2A^2\dot{\phi}^2(1-\cos(A-1)\phi)$$



Cycloid-like curves for rolling constraints

1. Hypo-cycloid



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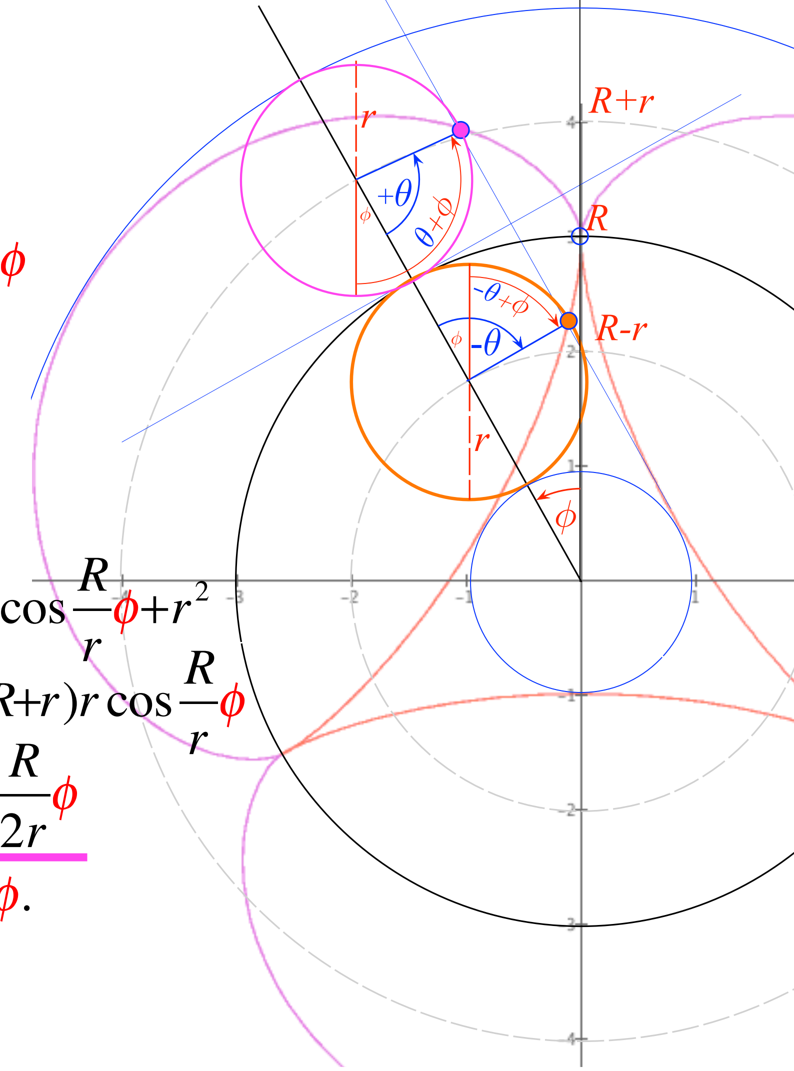
$$\dot{y}^2/r^2=A^2\dot{\phi}^2\sin^2\phi-2A^2\dot{\phi}^2\sin\phi\sin A\phi+A^2\dot{\phi}^2\sin^2 A\phi$$

$$\dot{x}^2/r^2+\dot{y}^2/r^2=2A^2\dot{\phi}^2(1-\cos\phi\cos A\phi+\sin\phi\sin A\phi)=2A^2\dot{\phi}^2(1-\cos(A-1)\phi)$$

$$\dot{\rho}^2=\dot{x}^2+\dot{y}^2=2A^2r^2\dot{\phi}^2(1-\cos(A-1)\phi)$$



Cycloid-like curves for rolling constraints

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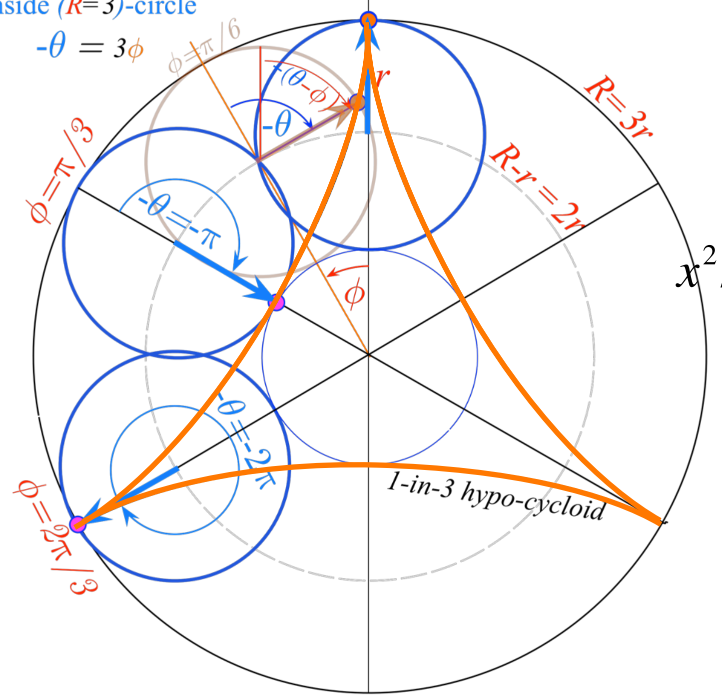

2. Hyper-cycloid

Hyper-cycloid constrained by $A=\frac{R}{r}+1$, $\theta=(A-1)\phi=\frac{R}{r}\phi$

$$x=-A\sin\phi+r\sin A\phi,\qquad y=A\cos\phi-r\cos A\phi.$$

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$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$$

Hyper-cycloid radius ρ:

$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

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$$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$$

Hyper-cycloid velocity

$$\dot{x}/r=-A\dot\phi\cos\phi+A\dot\phi\cos A\phi,\qquad \dot{y}/r=-A\dot\phi\sin\phi+A\dot\phi\sin A\phi.$$

$$\dot{x}^2/r^2=A^2\dot\phi^2\cos^2\phi-2A^2\dot\phi^2\cos\phi\cos A\phi+A^2\dot\phi^2\cos^2 A\phi$$

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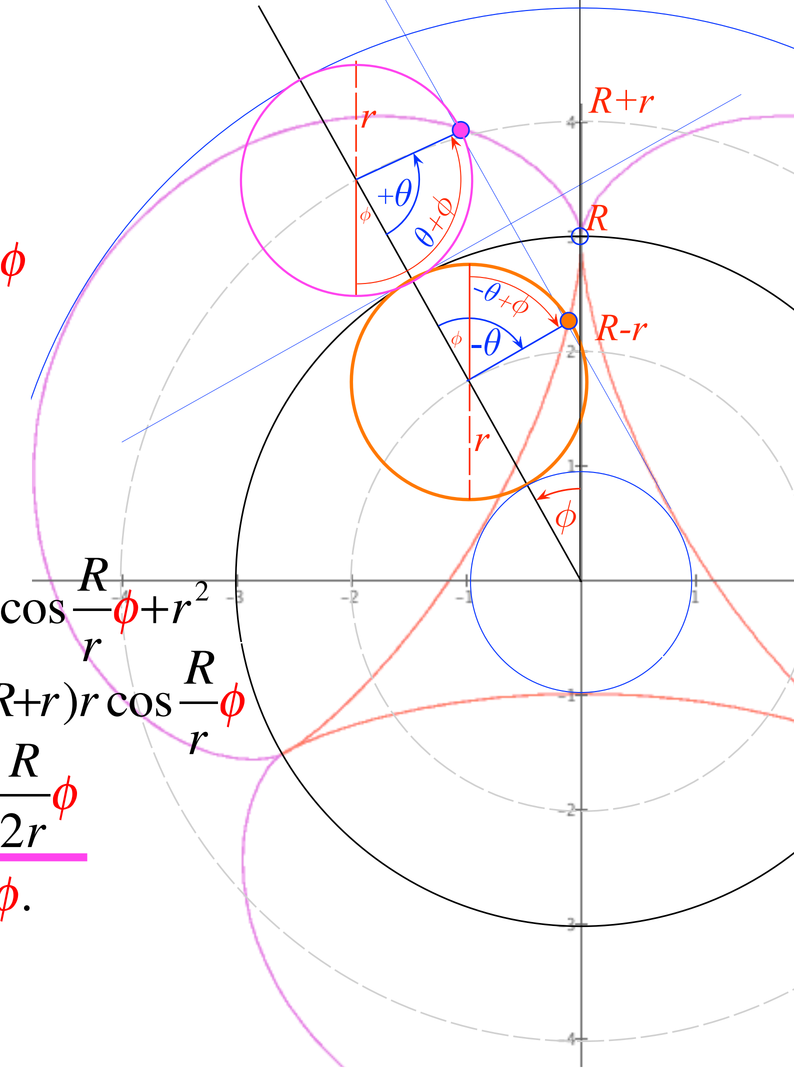
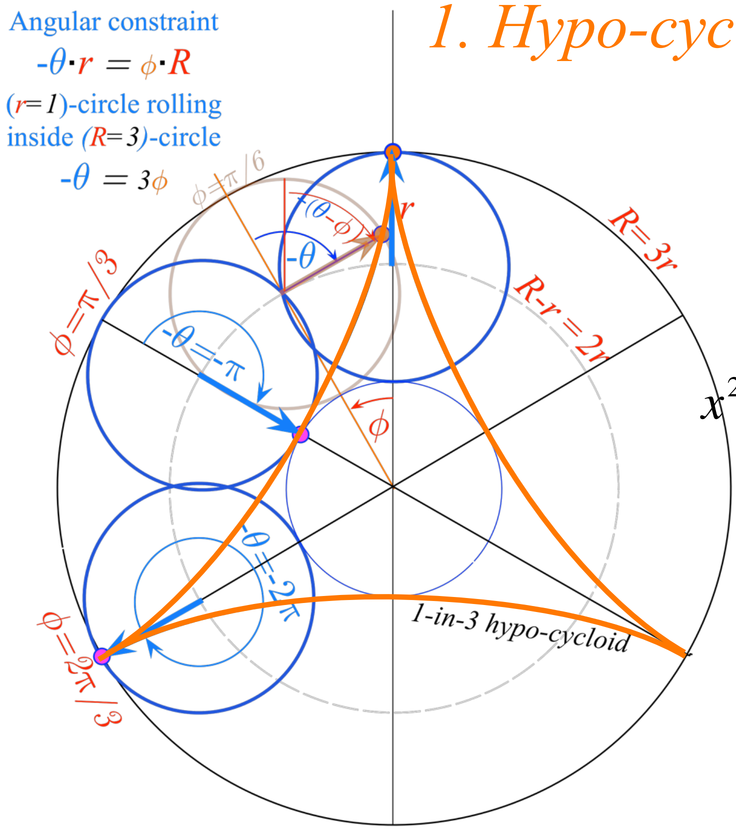
$$\dot\rho^2=\dot{x}^2+\dot{y}^2=2A^2r^2\dot\phi^2(1-\cos(A-1)\phi)=2(R+r)^2\dot\phi^2(1-\cos\frac{R}{r}\phi)$$

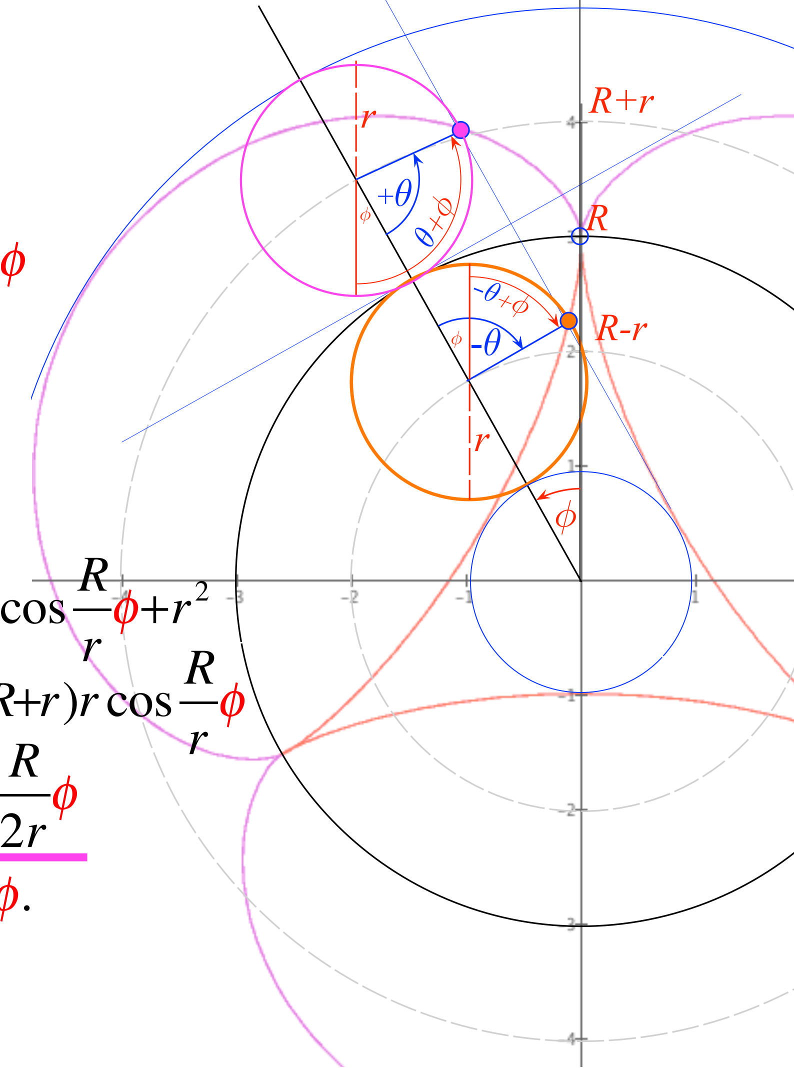


Cycloid-like curves for rolling constraints



1. Hypo-cycloid

2. Hyper-cycloid



Hyper-cycloid constrained by $A=\frac{R}{r}+1,\ \theta=(A-1)\phi=\frac{R}{r}\phi$

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$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$$

Hyper-cycloid radius ρ:

$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

$$=R^2+2(R+r)r-2(R+r)r\cos\frac{R}{r}\phi$$

$$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$$

Hyper-cycloid velocity

$$\dot{x}/r=-A\dot{\phi}\cos\phi+A\dot{\phi}\cos A\phi, \qquad \dot{y}/r=-A\dot{\phi}\sin\phi+A\dot{\phi}\sin A\phi.$$

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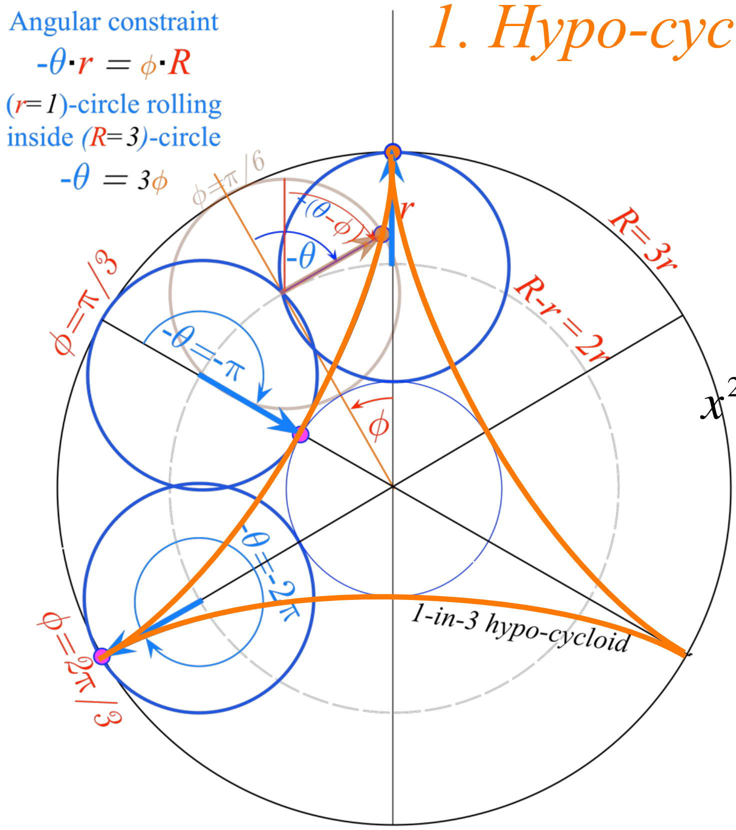
Cycloid-like curves for rolling constraints



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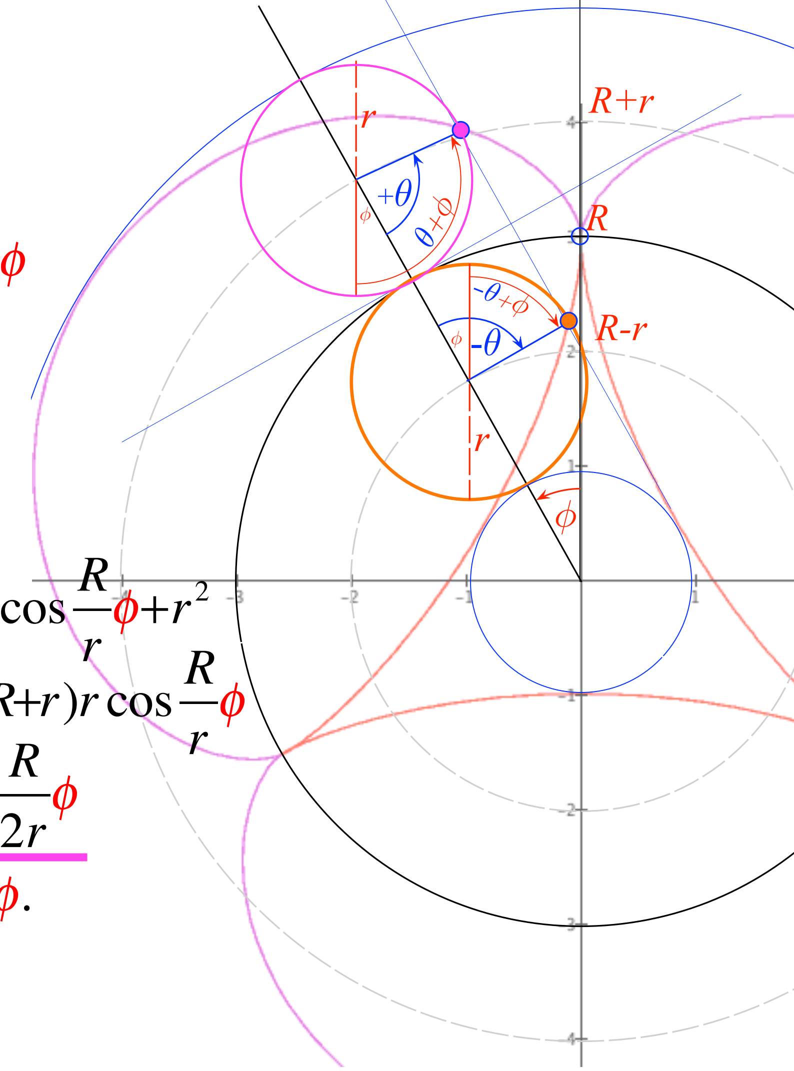

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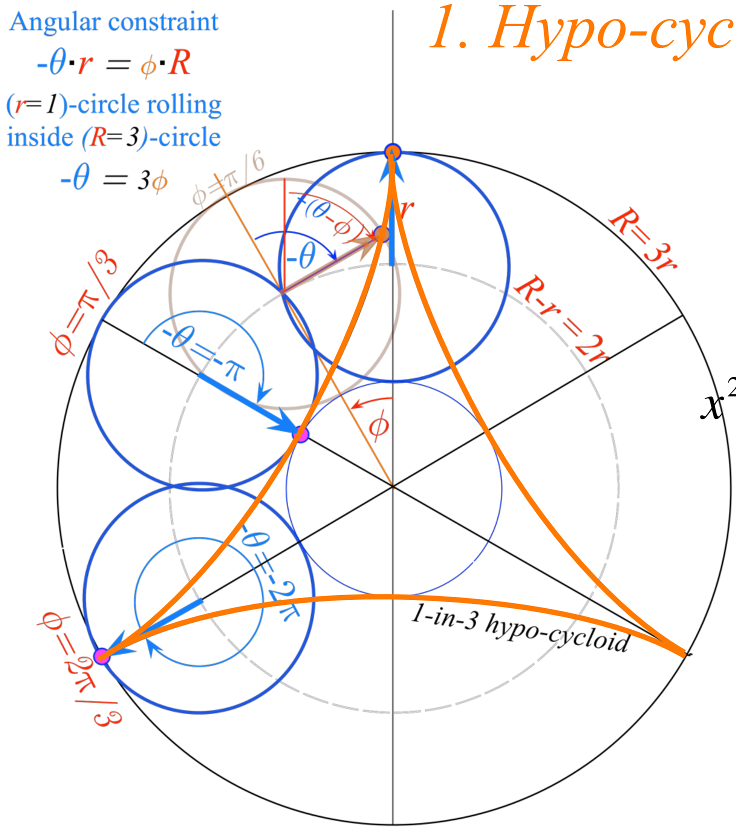
Cycloid-like curves for rolling constraints



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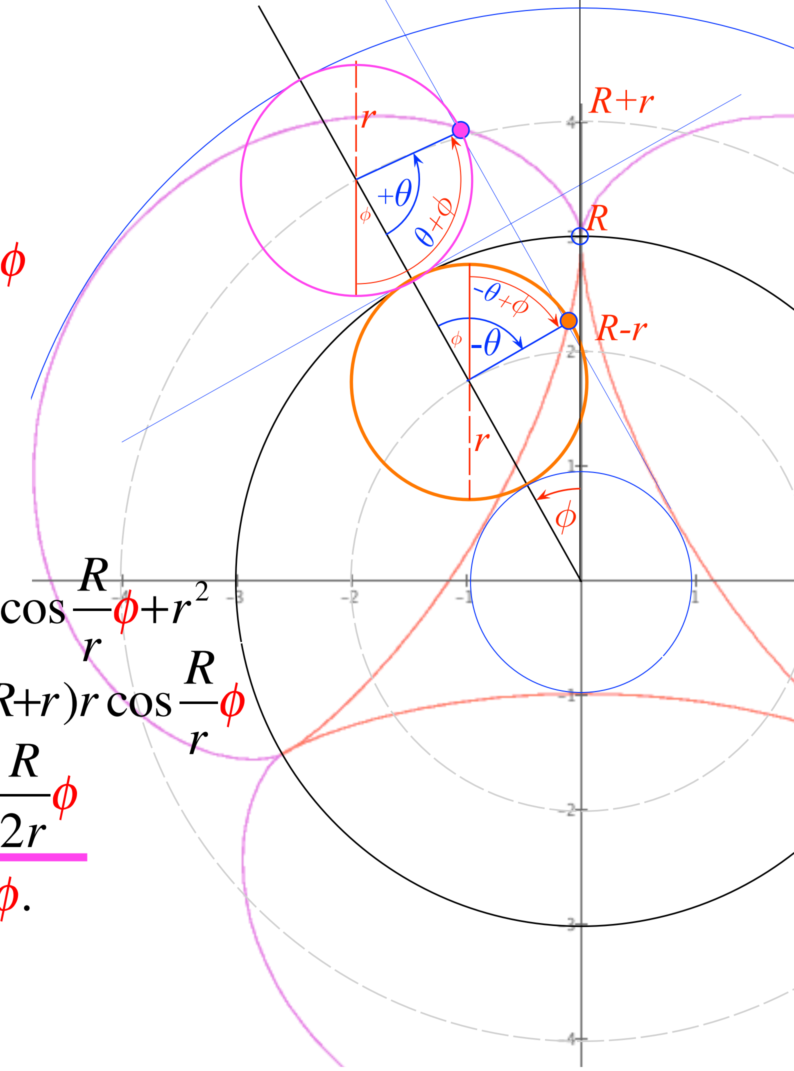

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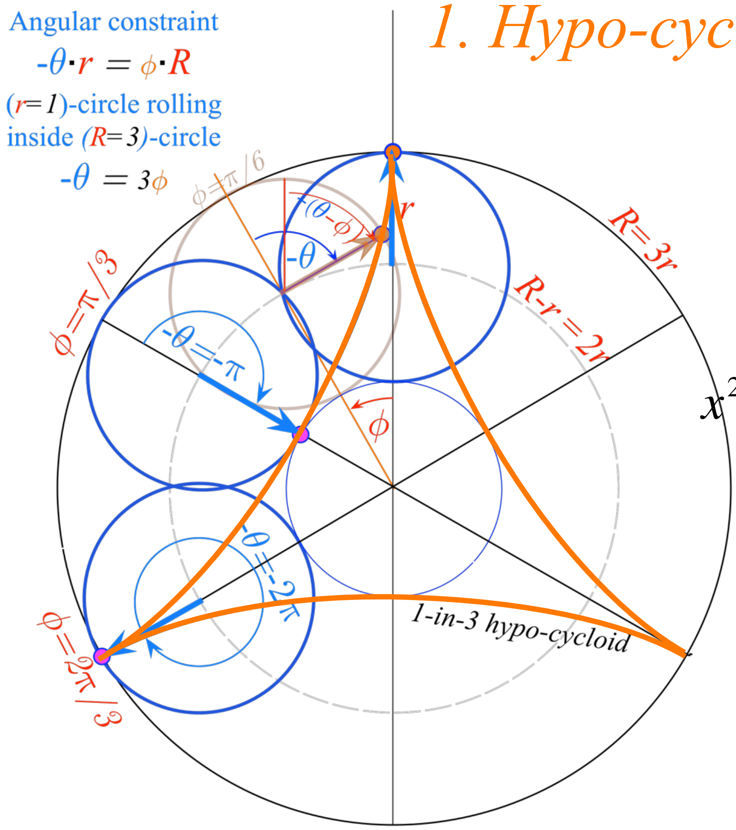
Cycloid-like curves for rolling constraints



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Hyper-cycloid radius ρ:

$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

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Hyper-cycloid velocity

$$\dot{x}/r=-A\dot\phi\cos\phi+A\dot\phi\cos A\phi,\qquad \dot{y}/r=-A\dot\phi\sin\phi+A\dot\phi\sin A\phi.$$

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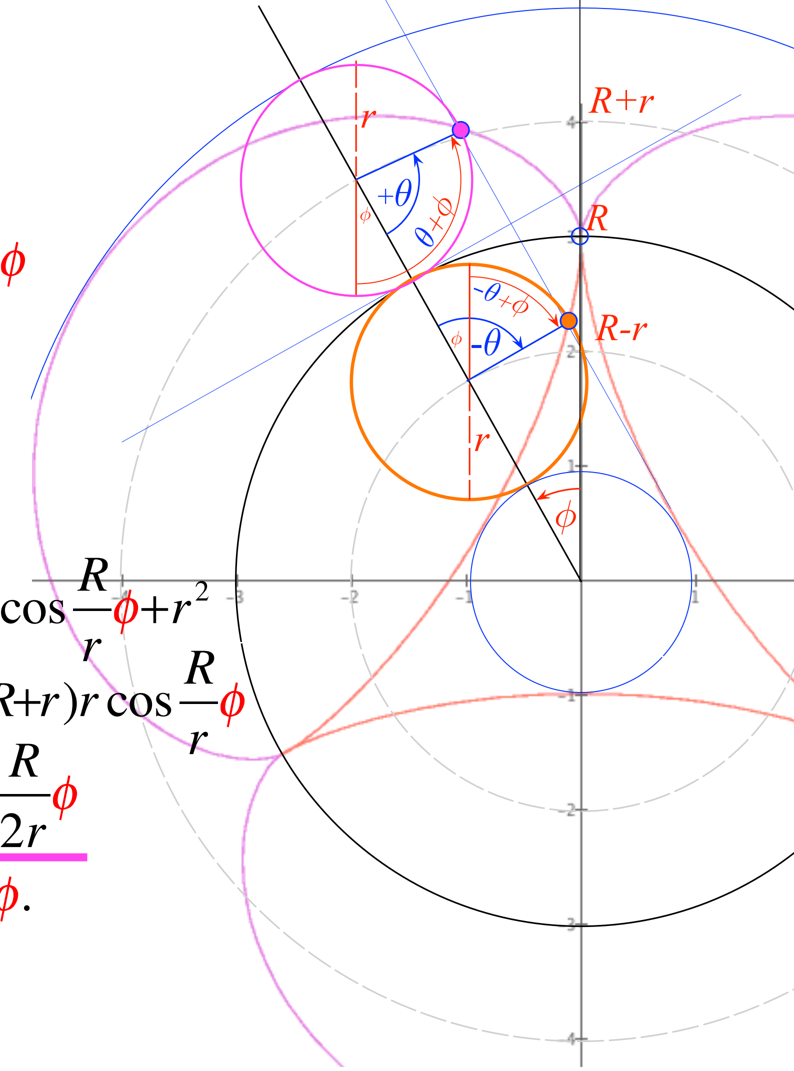

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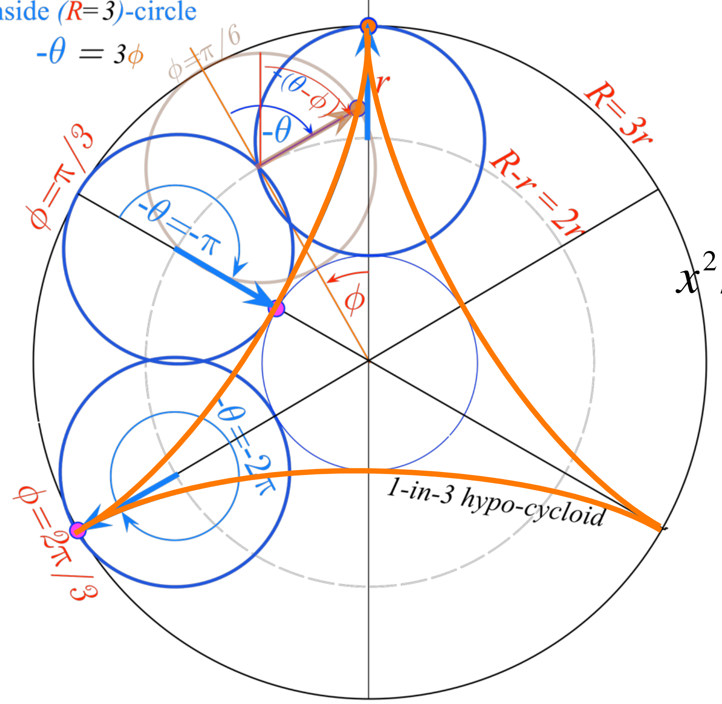
Then at any time t : $\dot\rho_t^2-\omega_\odot^2\rho_t^2=-\omega_\odot^2R^2=const.$ (Constant total energy)

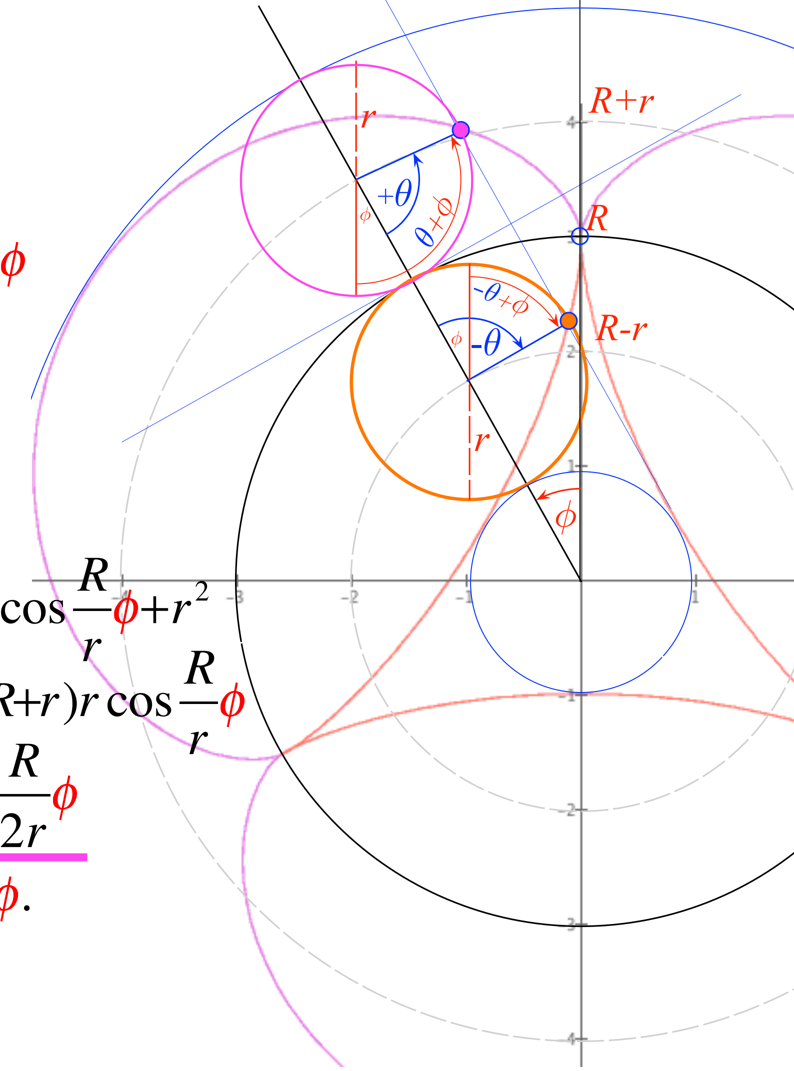
Cycloid-like curves for rolling constraints

Angular constraint
$-\theta\cdot r = \phi\cdot R$
(r=1)-circle rolling inside (R=3)-circle
$-\theta = 3\phi$

1. Hypo-cycloid



2. Hyper-cycloid



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$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1 \qquad \rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

Hyper-cycloid radius ρ:

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Hyper-cycloid velocity

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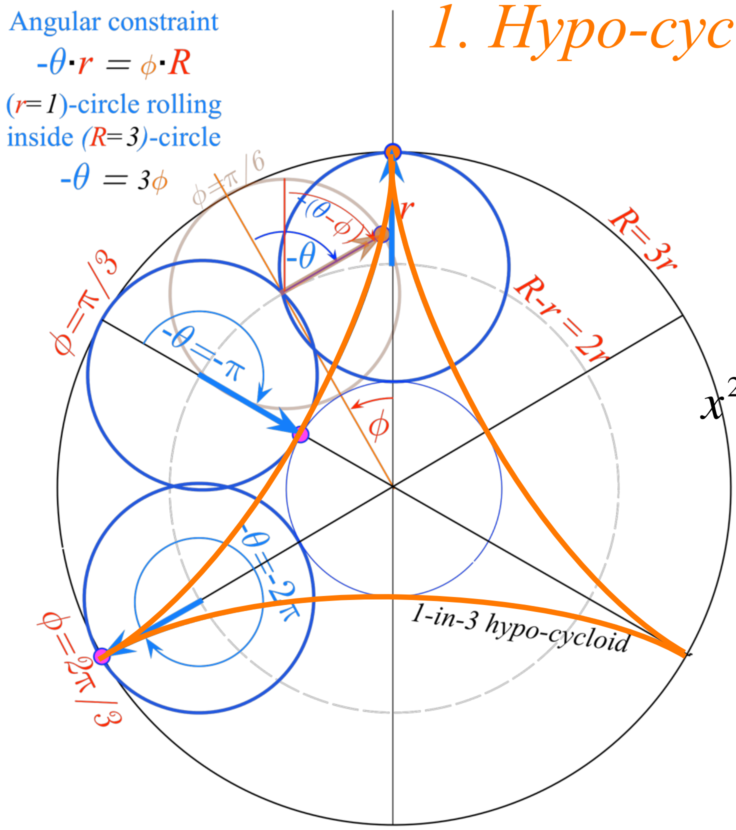
$$4(R+r)^2\dot{\phi}^2\sin^2\frac{R}{2r}\phi-\omega_\odot^2\left[R^2+4(R+r)r\sin^2\frac{R}{2r}\phi\right]=-\omega_\odot^2R^2=const.$$

Cycloid-like curves for rolling constraints



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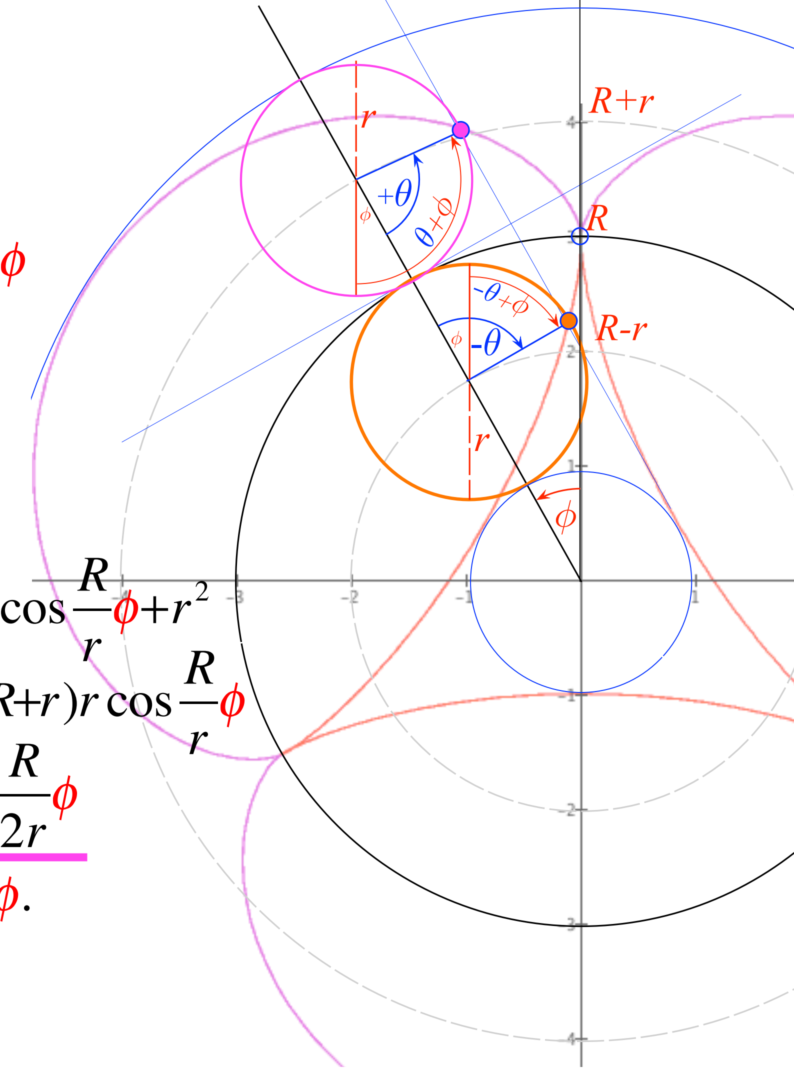

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Cycloid-like curves for rolling constraints

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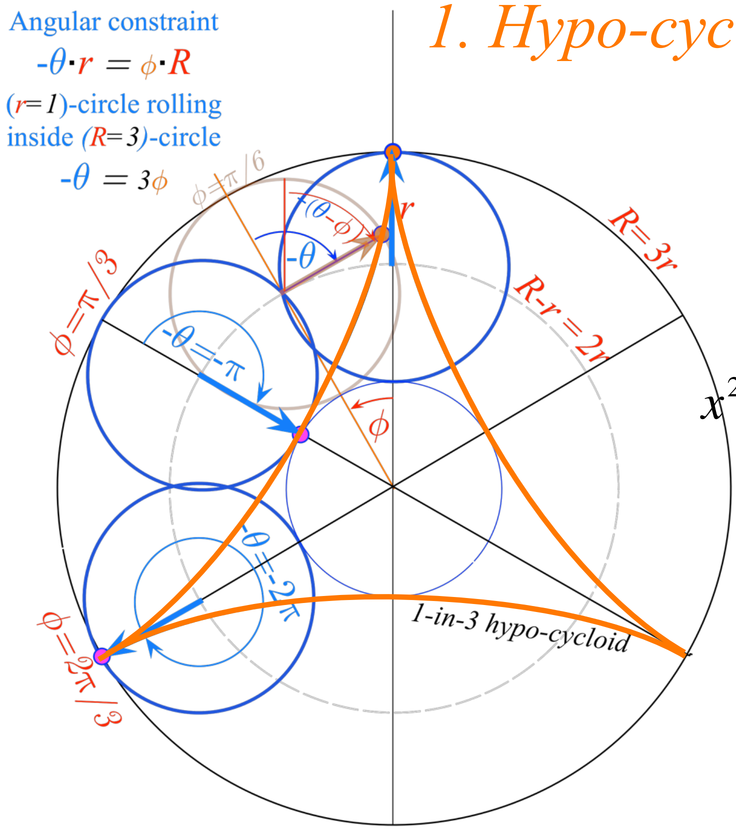

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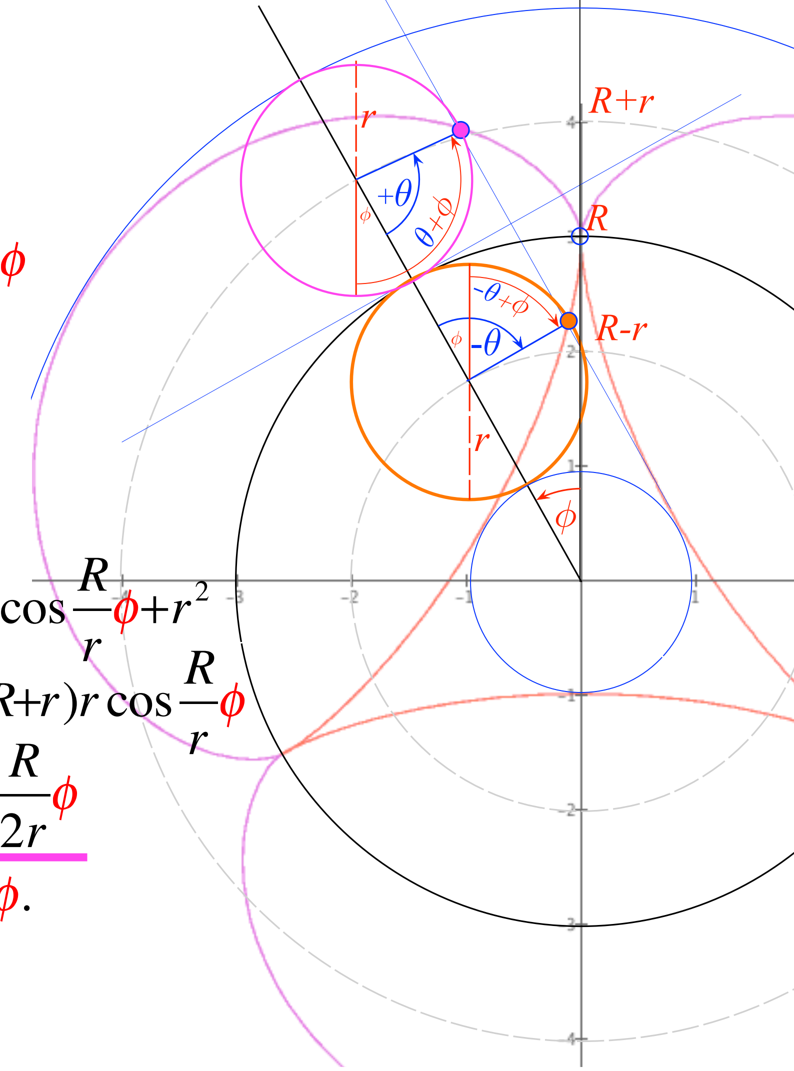

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Start with 100% Potential Energy ($\dot{\rho}_0=0$) at $\rho_0=R$: $\frac{2E_0}{m}=\dot{\rho}_0^2-\omega_\odot^2\rho_0^2=-\omega_\odot^2R^2=const.$

Then at any time t : $\dot{\rho}_t^2-\omega_\odot^2\rho_t^2=-\omega_\odot^2R^2=const.$ (Constant total energy)

$$4(R+r)^2\dot{\phi}^2\sin^2\frac{R}{2r}\phi-\omega_\odot^2\left[\cancel{R^2}+4(R+r)r\sin^2\frac{R}{2r}\phi\right]=-\cancel{\omega_\odot^2R^2}=const.$$

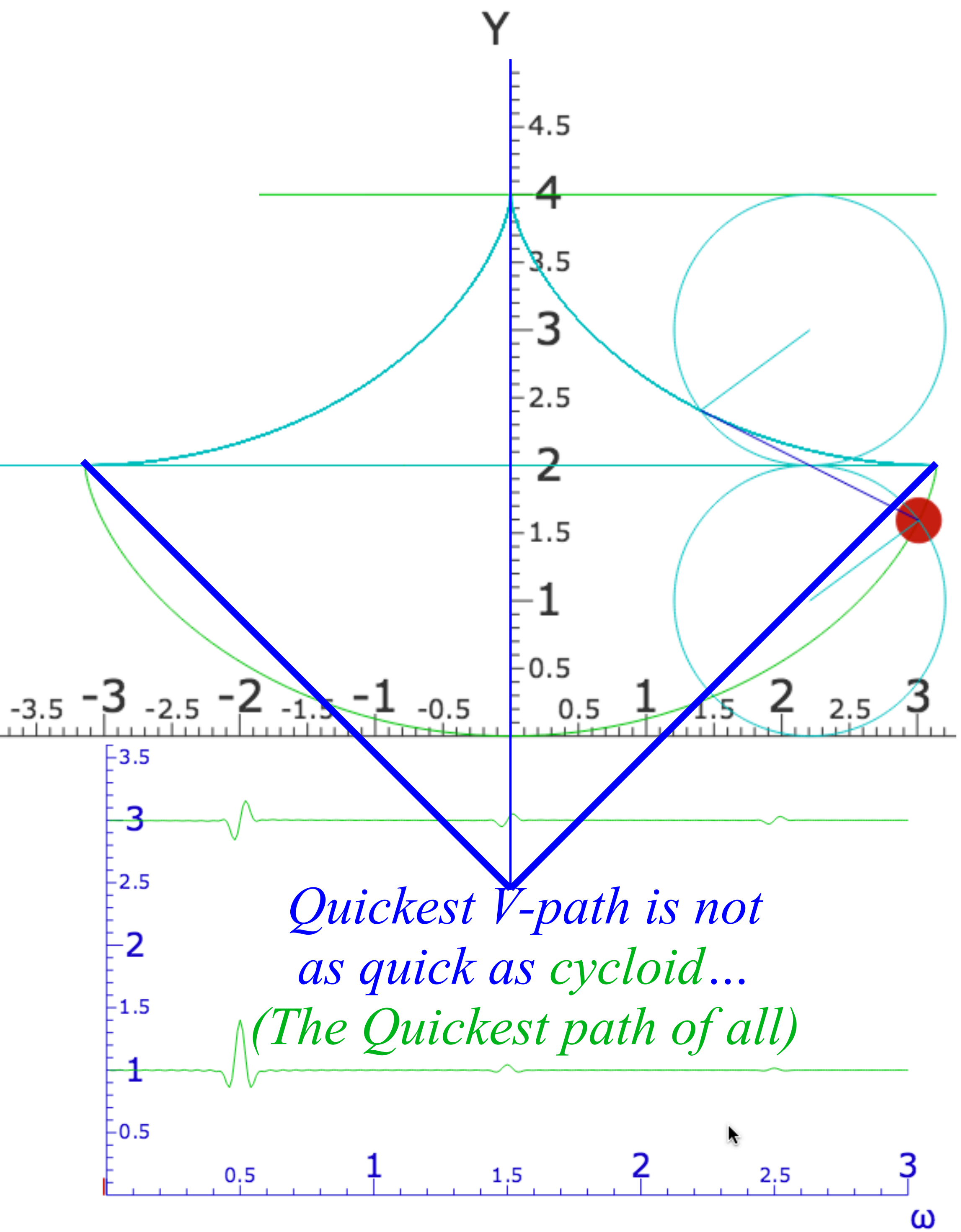
$$(R+r)\dot{\phi}^2 \qquad -\omega_\odot^2\left[r\right]=0$$

Results in hyper-circle orbiting at constant $\dot{\phi}=\omega_\odot\sqrt{\frac{r}{R+r}}=\frac{\omega_\odot}{\sqrt{A}}$

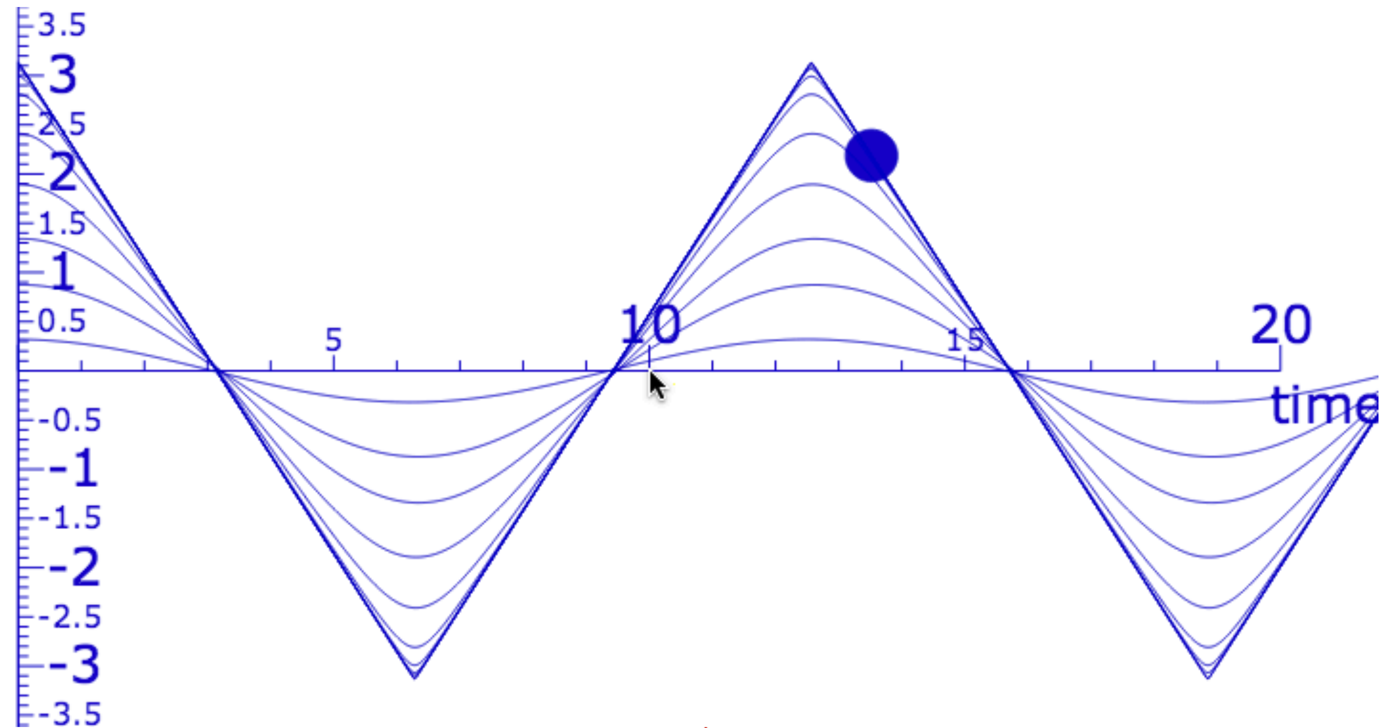
...and turning at constant $\dot{\theta}=\frac{R}{r}\dot{\phi}=\omega_\odot\frac{R}{\sqrt{r(R+r)}}$

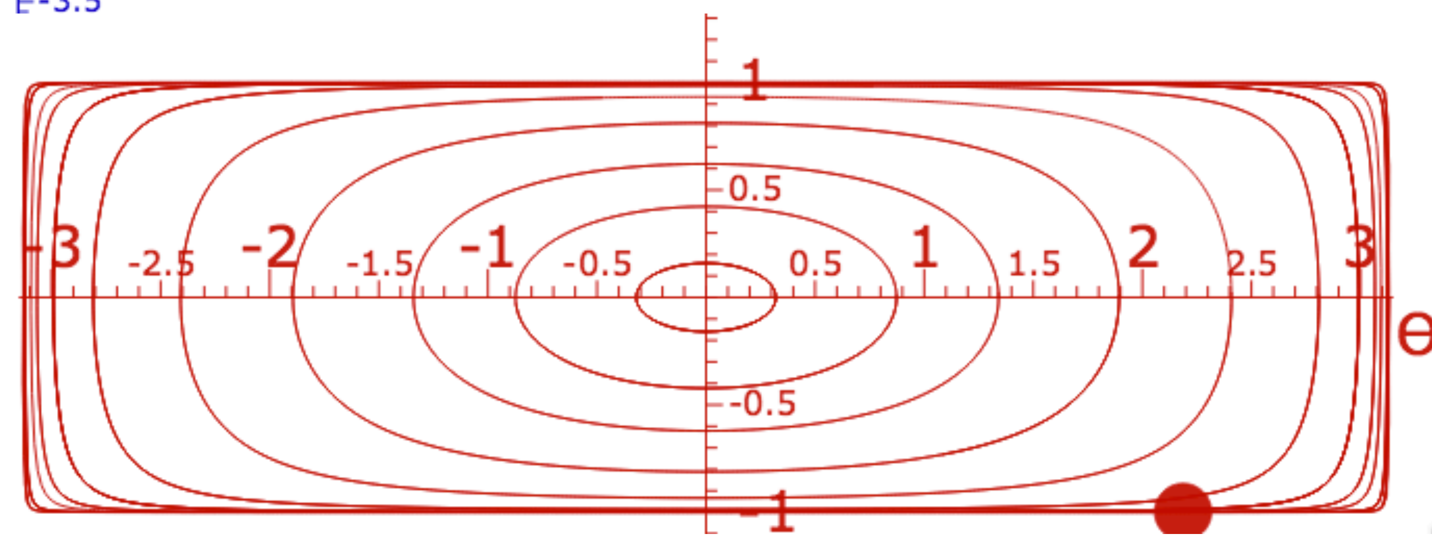
Cycloid-like curves for rolling constraints

Quickest intra-planetary subways



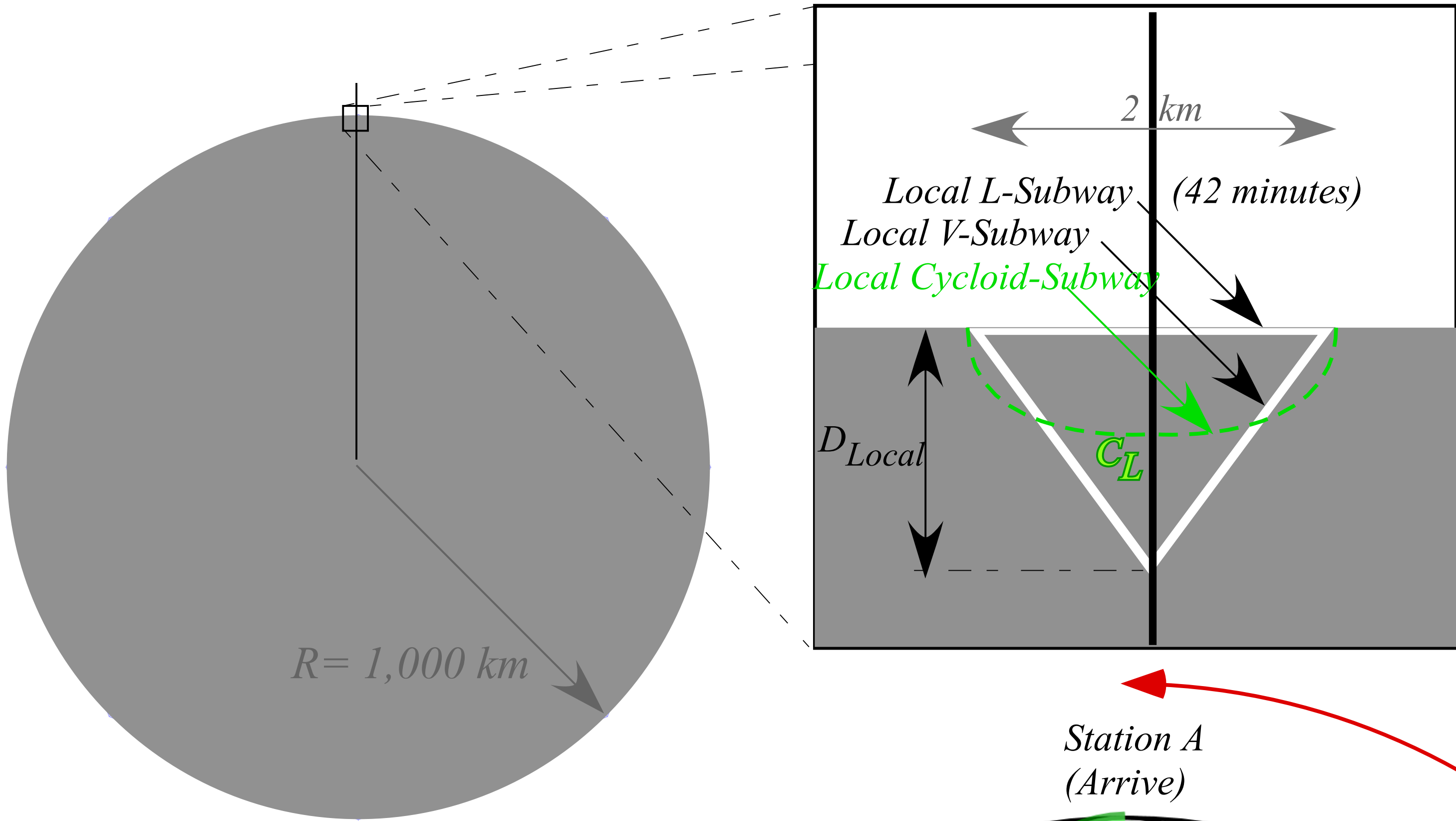
Quickest V-path is not
as quick as cycloid…
(The Quickest path of all)

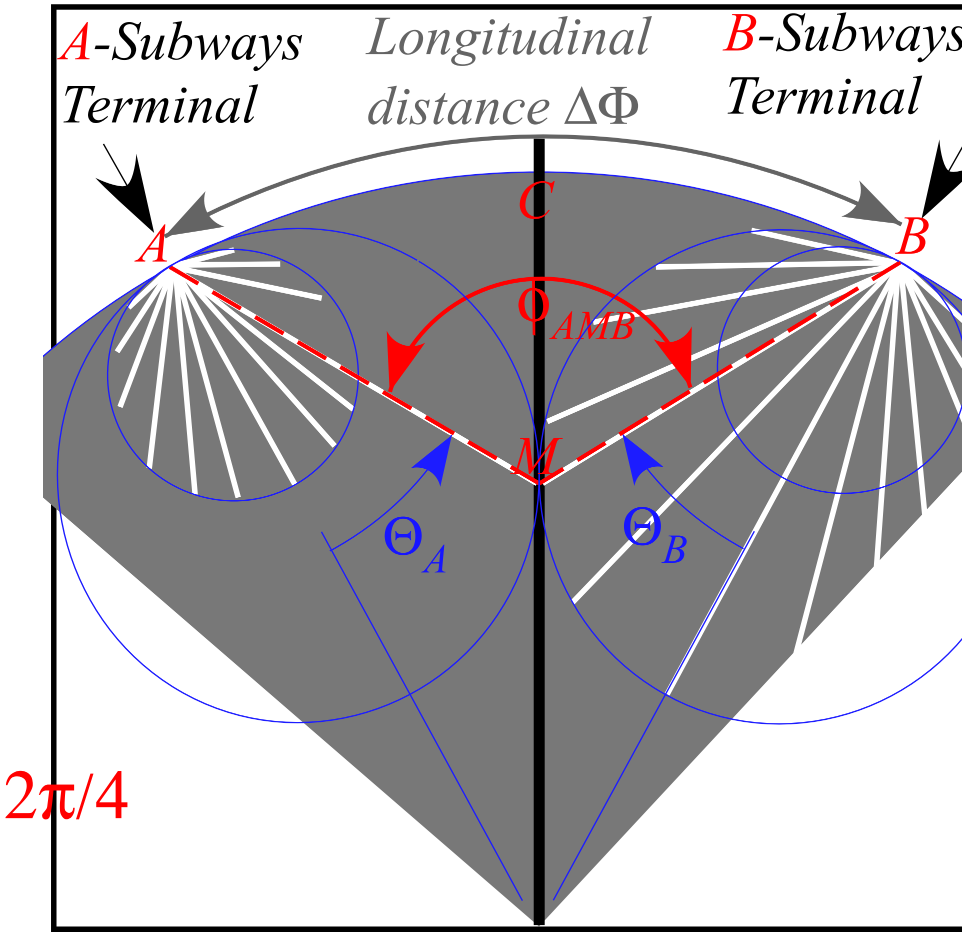


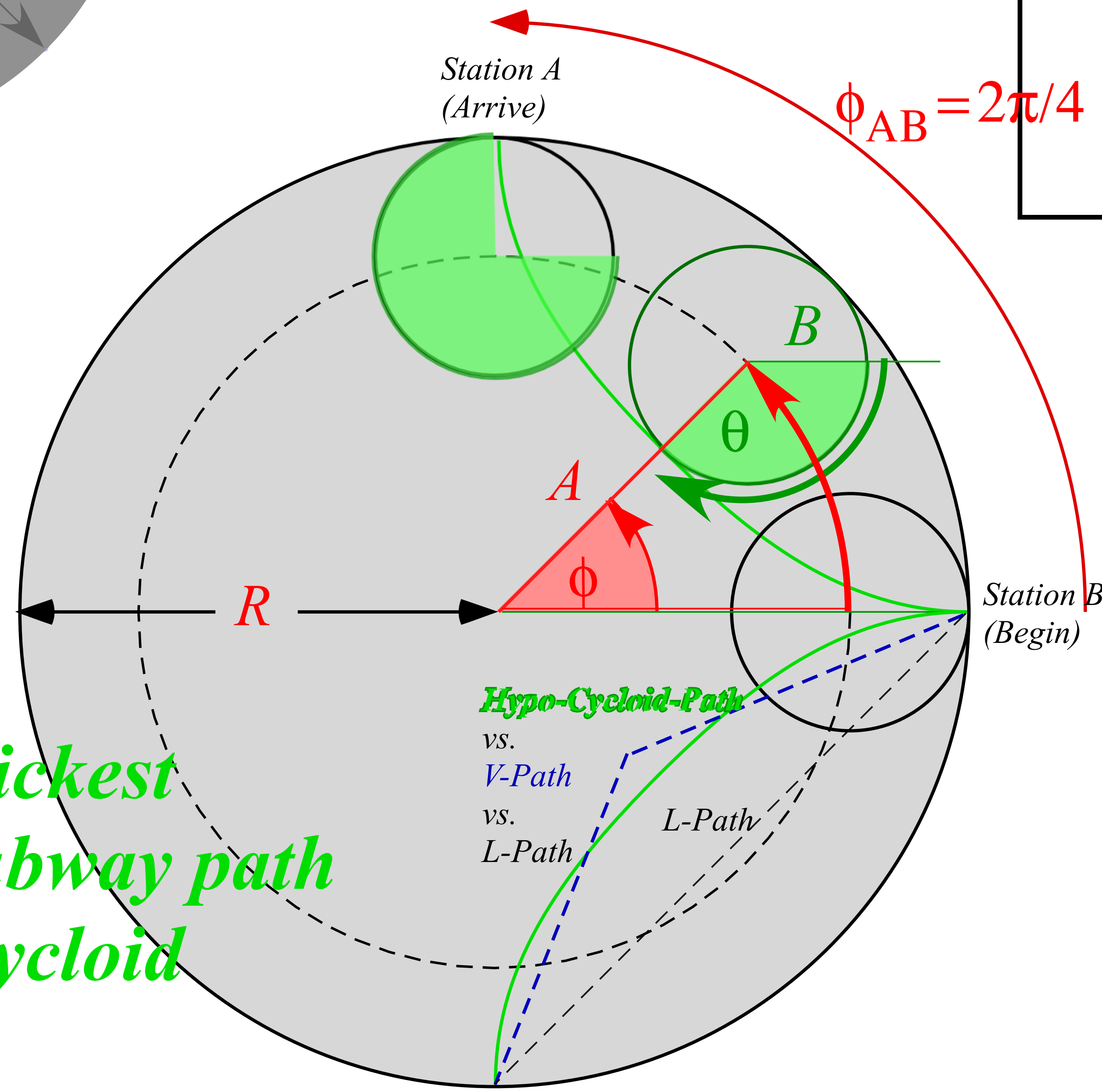


[*Web Simulation - OscillatorPE*](#)
[*Cycloidally constrained pendulum*](#)

http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html







The very quickest planetary subway path is a Hypo-Cycloid