Lecture 2
Thur 8.25.2016

Analysis of 1D 2-Body Collisions

(Ch. 2 to Ch. 4 of Unit 1)

Review of elastic Kinetic Energy ellipse geometry

The X2 Superball pen launcher
- *Perfectly elastic "ka-bong" velocity amplification effects (Faux-Flubber)*

Geometry of X2 launcher bouncing in box (gravity-free)
- *Independent Bounce Model (IBM)*
- *Geometric optimization and range-of-motion calculation(s)*
- *Integration of (V_1,V_2) data to space-time plots ($y_1(t)$,t) and ($y_2(t)$,t) plots*
- *Integration of (V_1,V_2) data to space-space plots (y_1, y_2)* *Examples (M_1=7, M_2=1) and (M_1=49, M_2=1)*

Multiple collisions calculated by matrix operator products
- *Matrix or tensor algebra of 1-D 2-body collisions*

Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
  - *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*
- *Reflections in the clothing store: "It's all done with mirrors!"*
- *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*
  - *Group multiplication and product table*
  - *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*
    - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$ and ($M_1=100$, $M_2=1$)*

Review of elastic Kinetic Energy ellipse geometry

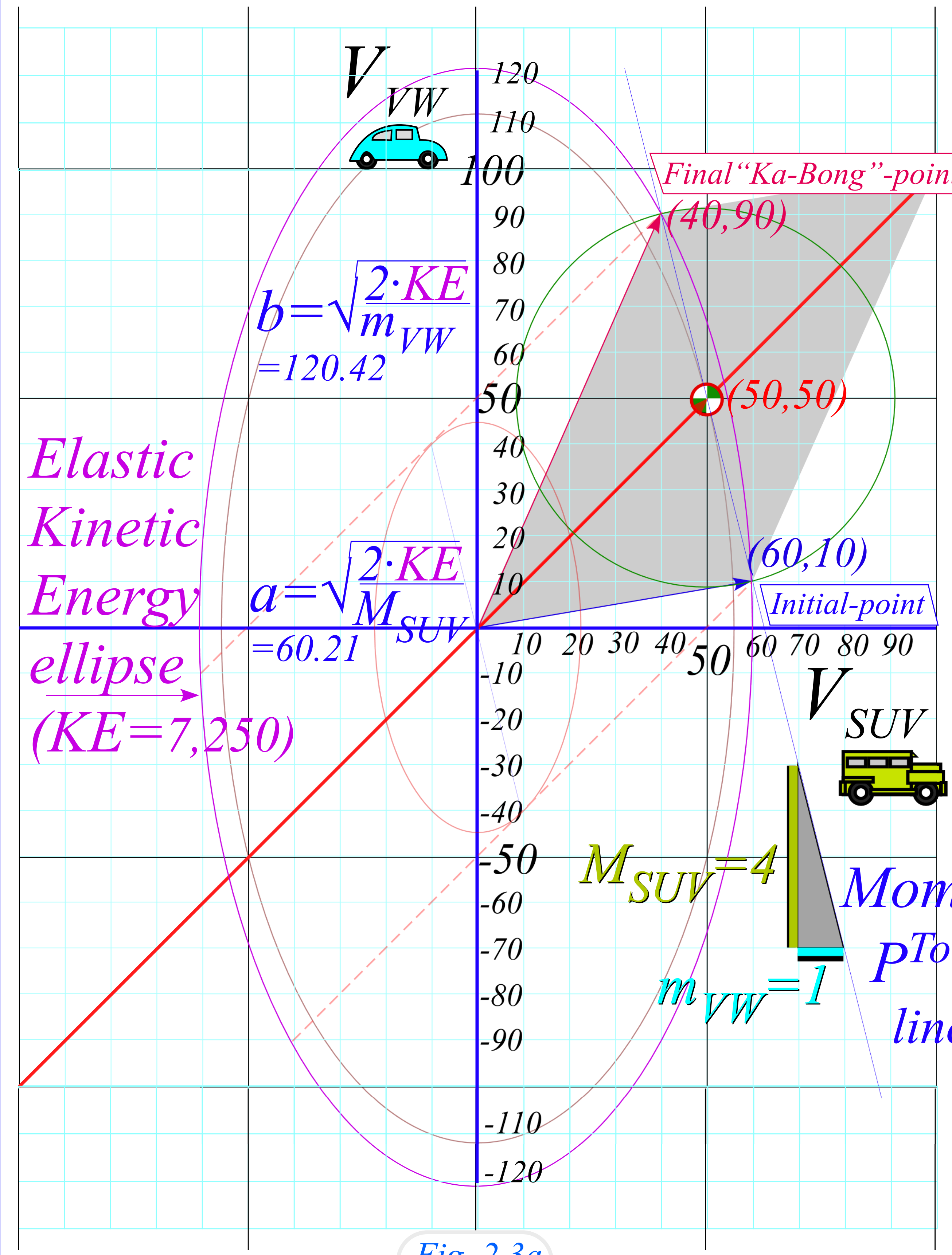


Fig. 2.3a
(Unit 1)

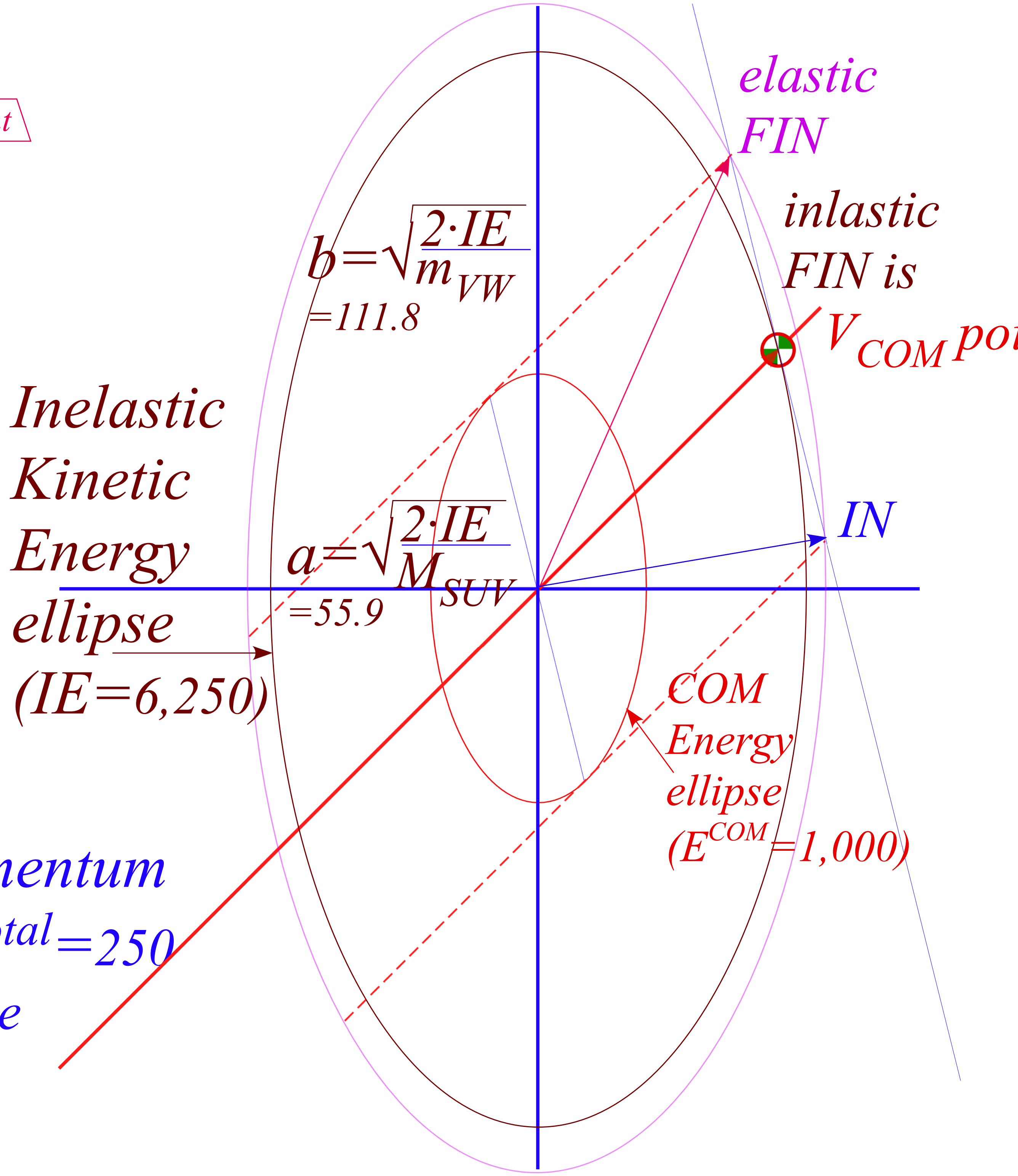


Fig. 2.3b
(Unit 1)

The X-2 Pen launcher and Superball Collision Simulator*

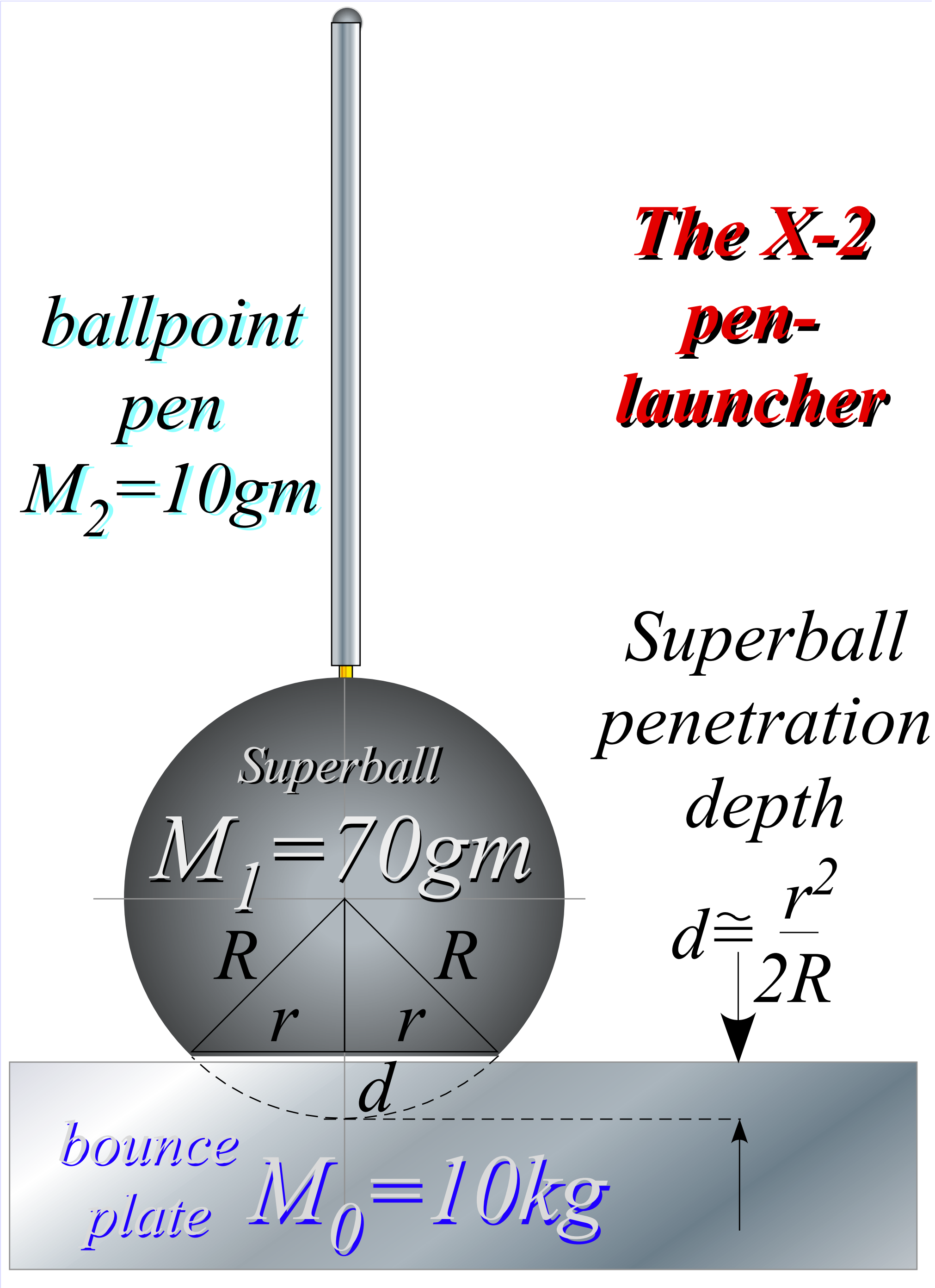
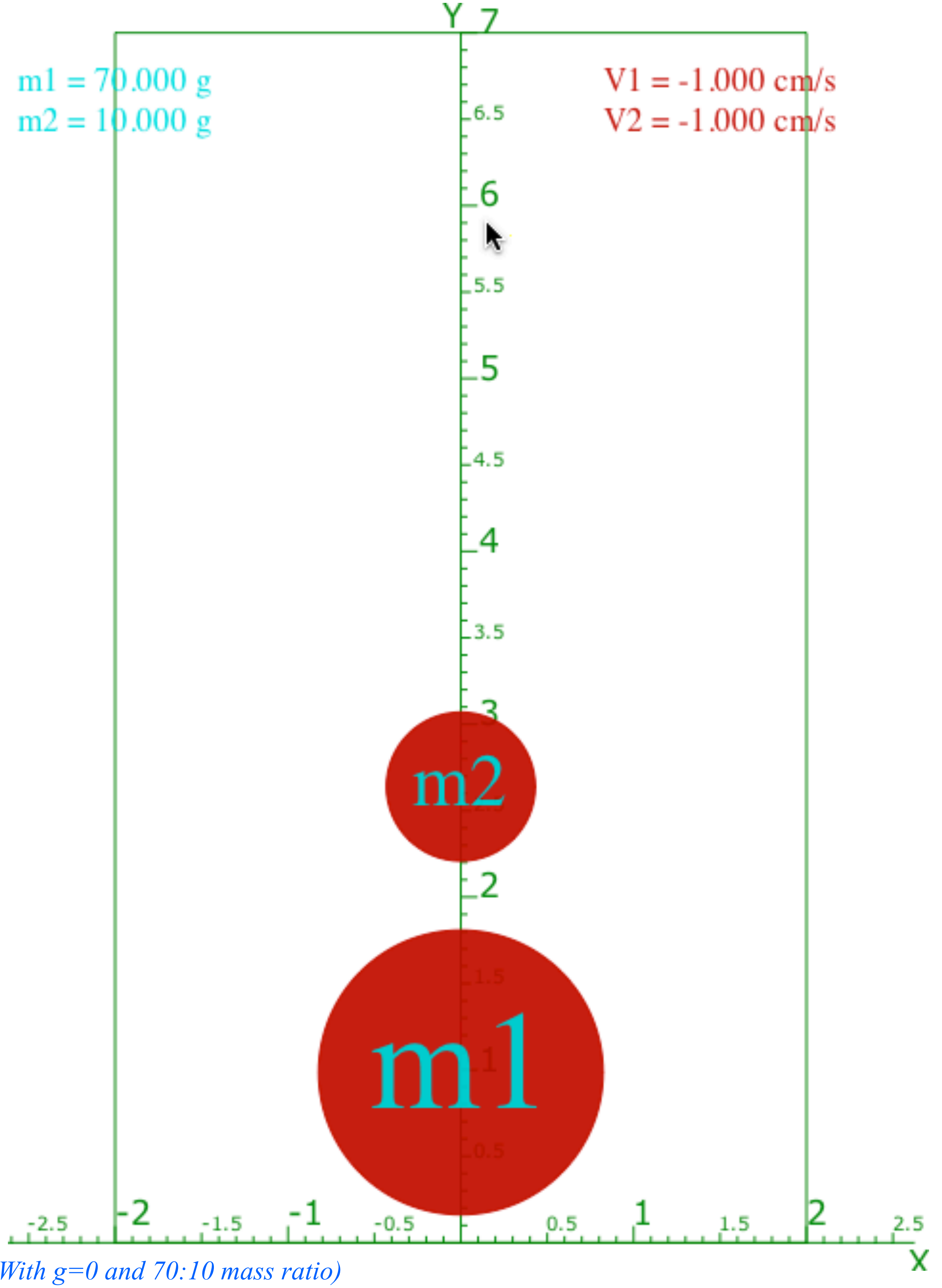


Fig. 3.1 (Unit 1)



m1 = 70.000 g
m2 = 10.000 g

V1 = -1.000 cm/s
V2 = -1.000 cm/s

(With g=0 and 70:10 mass ratio)

*Launch Generic Superball Collision Web Simulator http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300

The X-2 Pen launcher and Superball Collision Simulator*

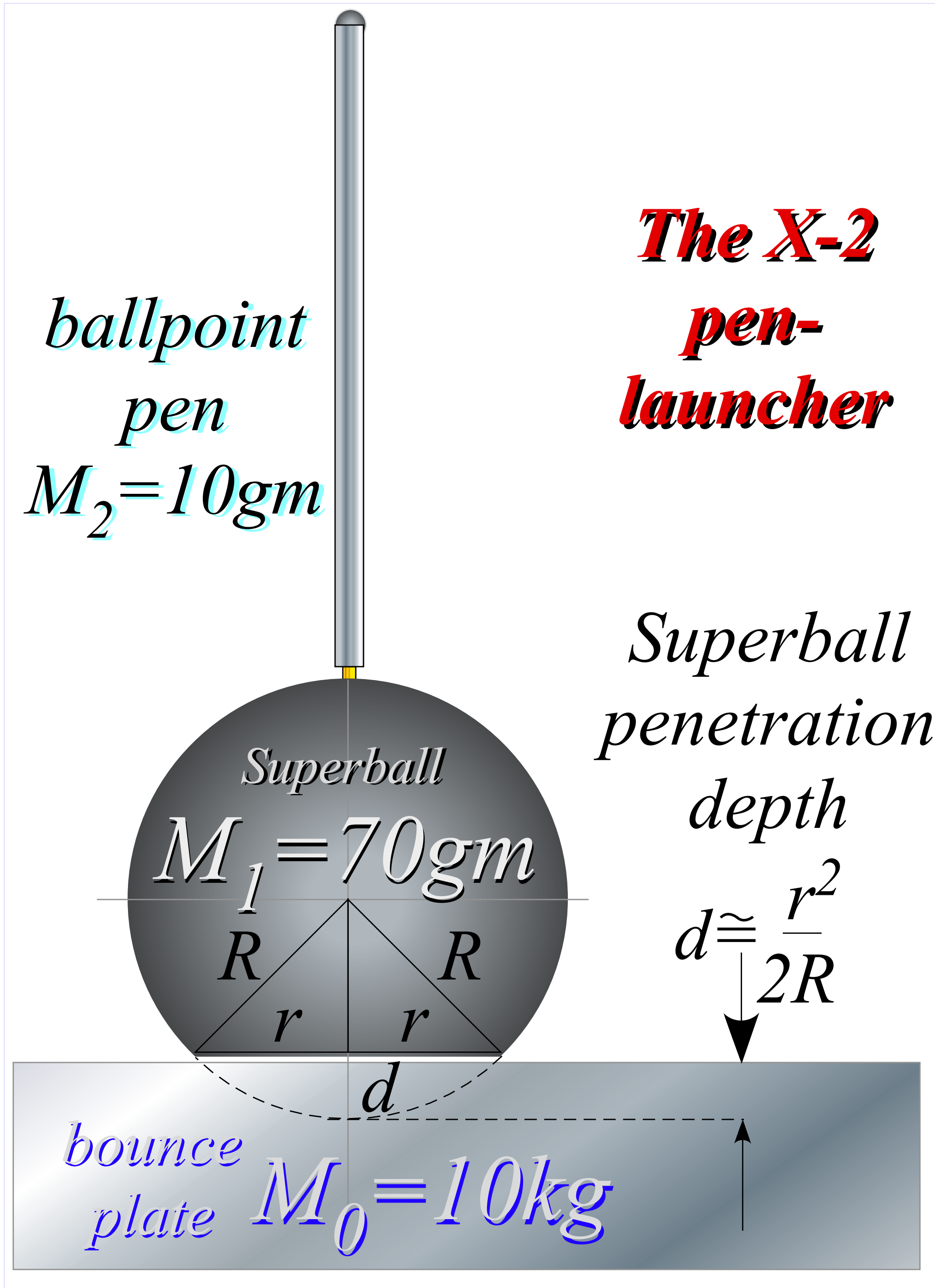
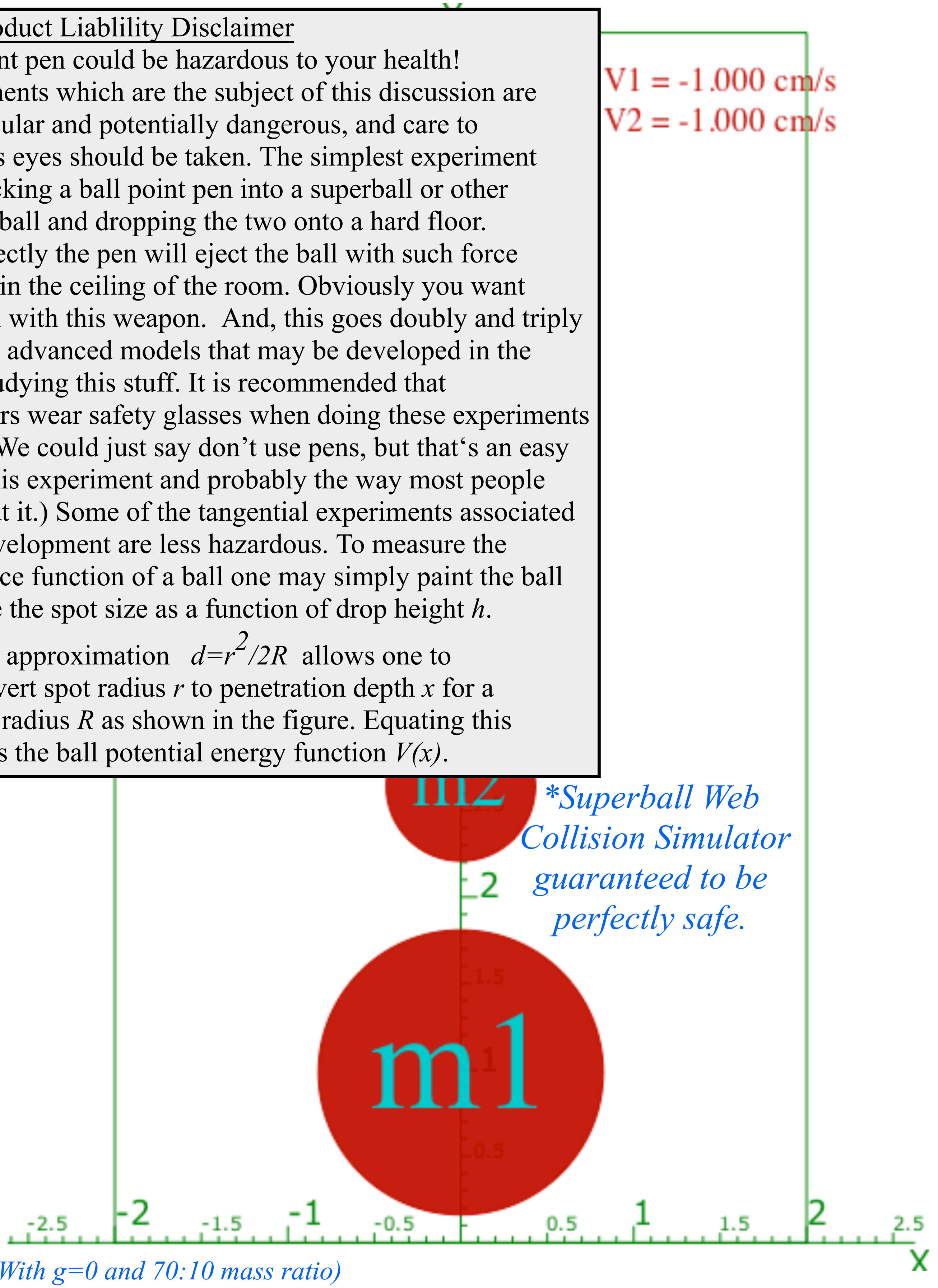


Fig. 3.1
(Unit 1)

Caution: Product Liablility Disclaimer

This ballpoint pen could be hazardous to your health! The experiments which are the subject of this discussion are both spectacular and potentially dangerous, and care to protect one’s eyes should be taken. The simplest experiment involves sticking a ball point pen into a superball or other hard rubber ball and dropping the two onto a hard floor. If done correctly the pen will eject the ball with such force it may stick in the ceiling of the room. Obviously you want to be careful with this weapon. And, this goes doubly and triply for the more advanced models that may be developed in the course of studying this stuff. It is recommended that experimenters wear safety glasses when doing these experiments with pens. (We could just say don’t use pens, but that‘s an easy way to do this experiment and probably the way most people will go about it.) Some of the tangential experiments associated with this development are less hazardous. To measure the potential force function of a ball one may simply paint the ball and measure the spot size as a function of drop height *h*.

The saggital approximation $d=r^2/2R$ allows one to quickly convert spot radius r to penetration depth x for a superball of radius R as shown in the figure. Equating this to Mgh gives the ball potential energy function $V(x)$.

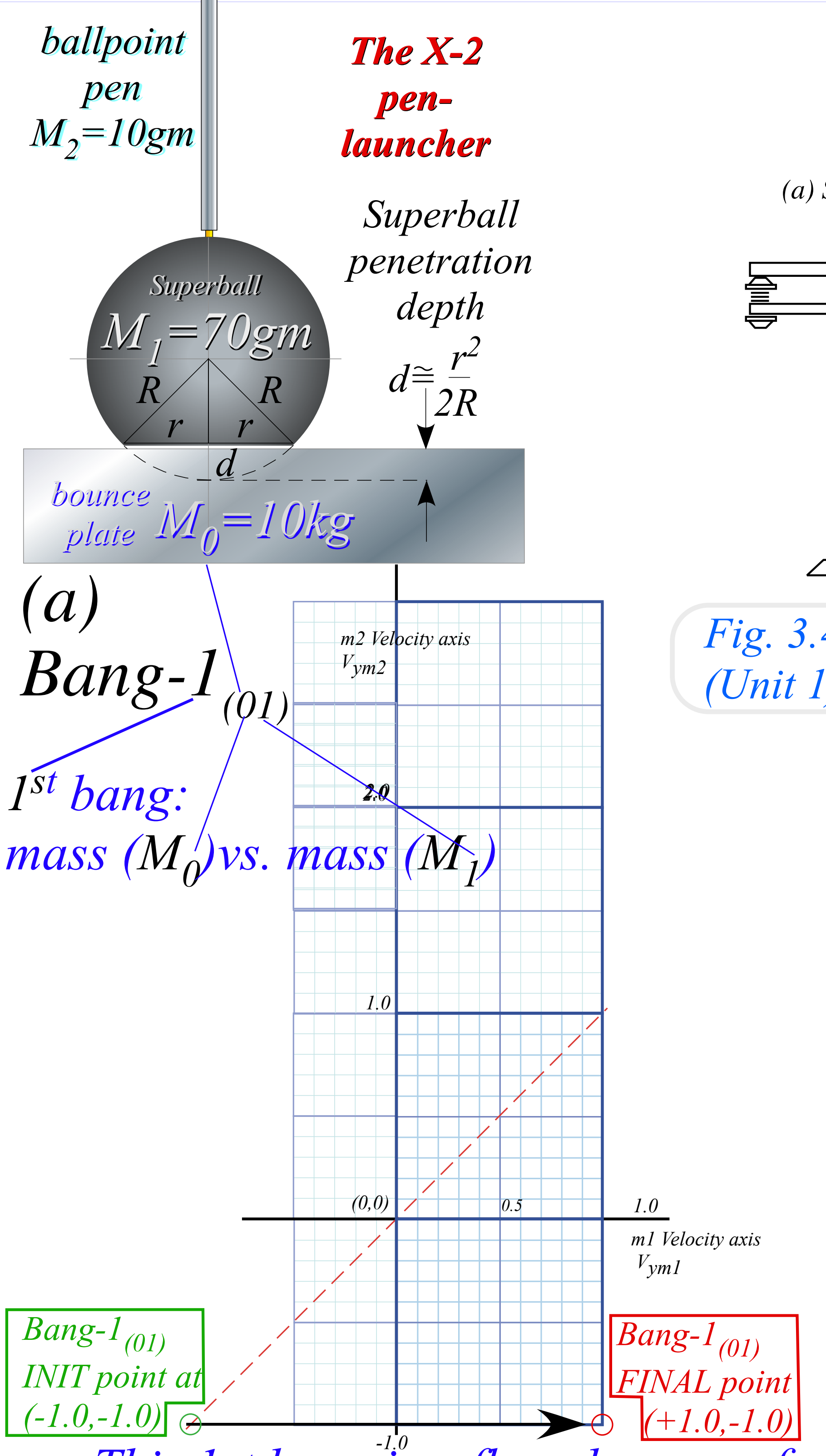


*Superball Web Collision Simulator guaranteed to be perfectly safe.

(With g=0 and 70:10 mass ratio)

*Launch Generic Superball Collision Web Simulator http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300

The X-2 pen-launcher



(a)

Bang-$1_{(01)}$

1^{st} bang:
mass (M_0) vs. mass (M_1)

This 1st bang is a floor-bounce of
M_1 off very massive plate/Earth M_0

Fig. 3.3
(Unit 1)

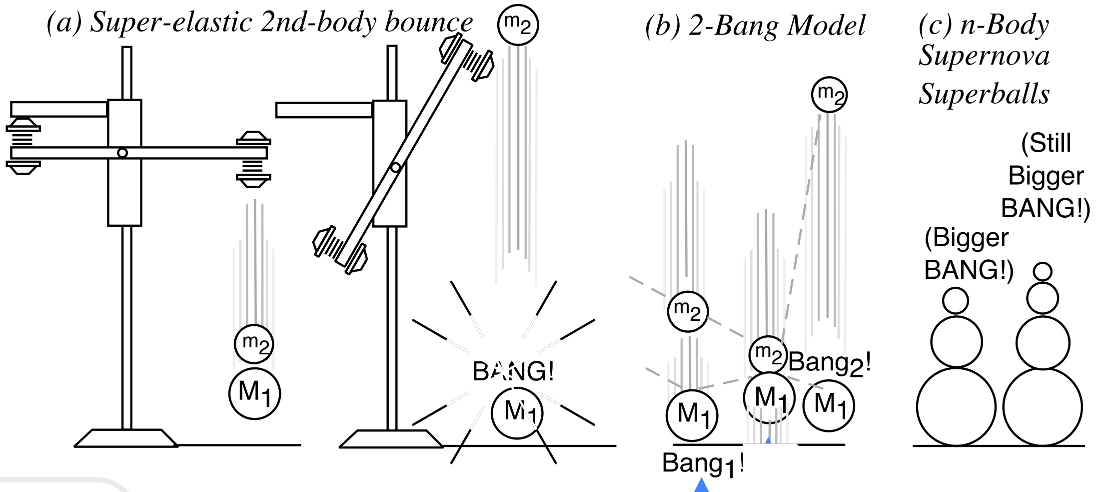


Fig. 3.4
(Unit 1)

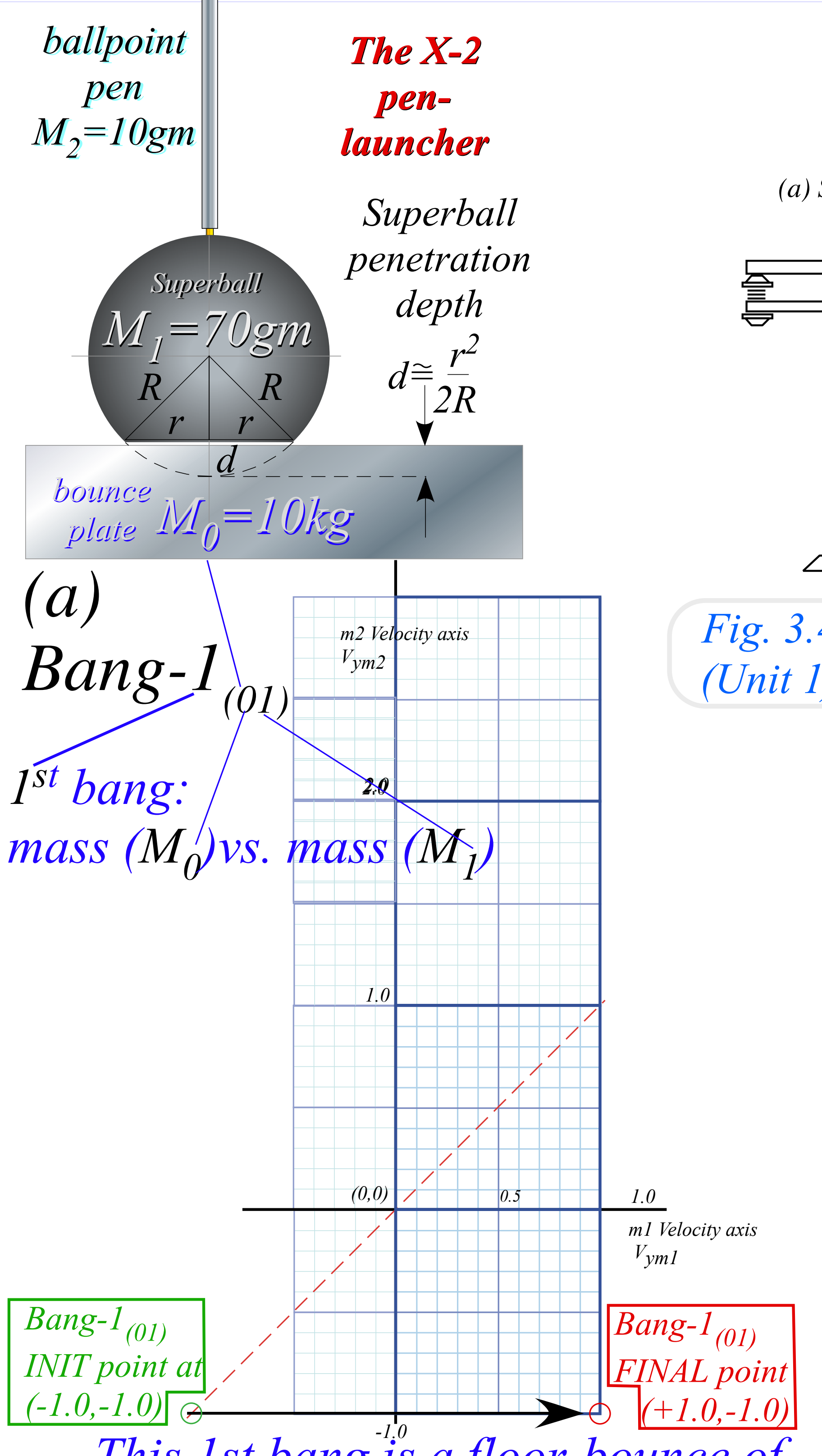
1st bang:
M_1 off floor

(With g=0 and 70:10 mass ratio)

*Launch Generic Superball Collision Web Simulator http://www.uark.edu/ua/modphys/markup/BounceItWeb.html

The X-2 pen-launcher

Superball penetration depth



(a) Bang-1$_{(01)}$

1^{st} bang: mass (M_0) vs. mass (M_1)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 3.3 (Unit 1)

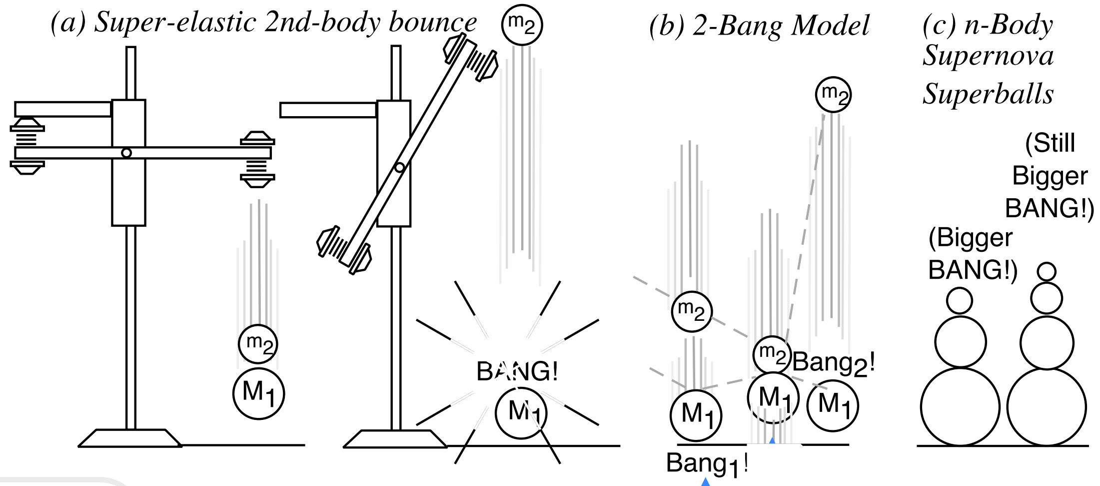
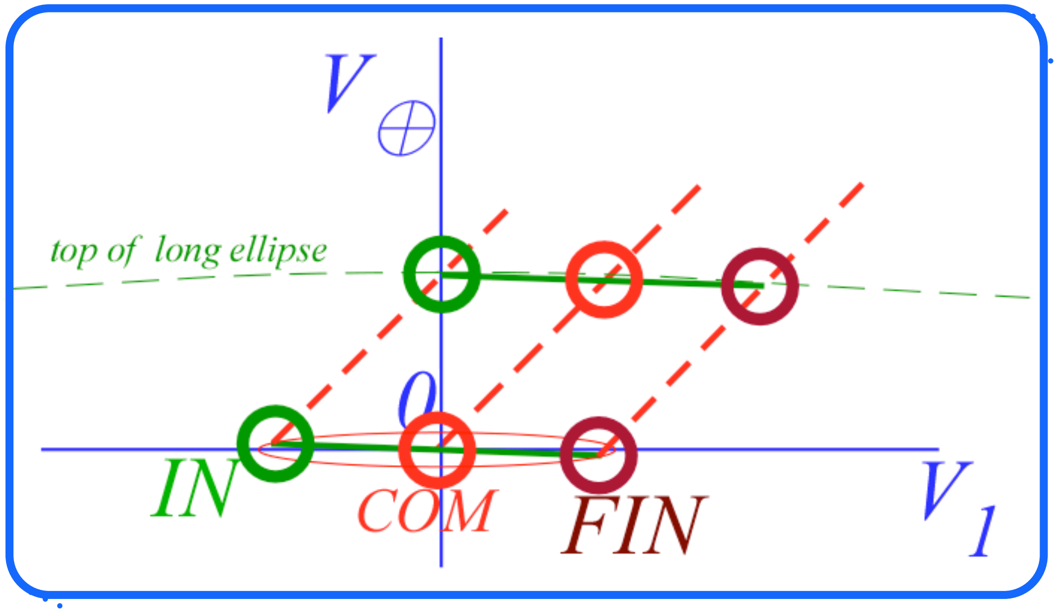
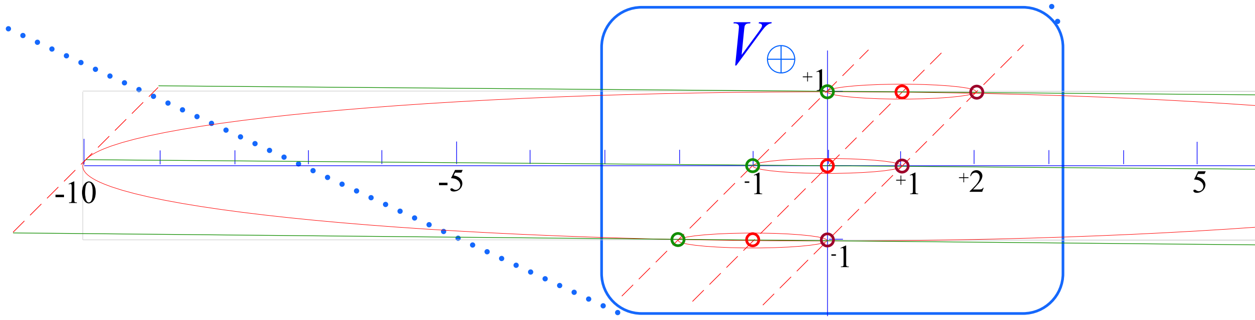


Fig. 3.4 (Unit 1)

1st bang: M_1 off floor





(With g=0 and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With g and 70:35 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300

(a) 1st bang of M_1 off

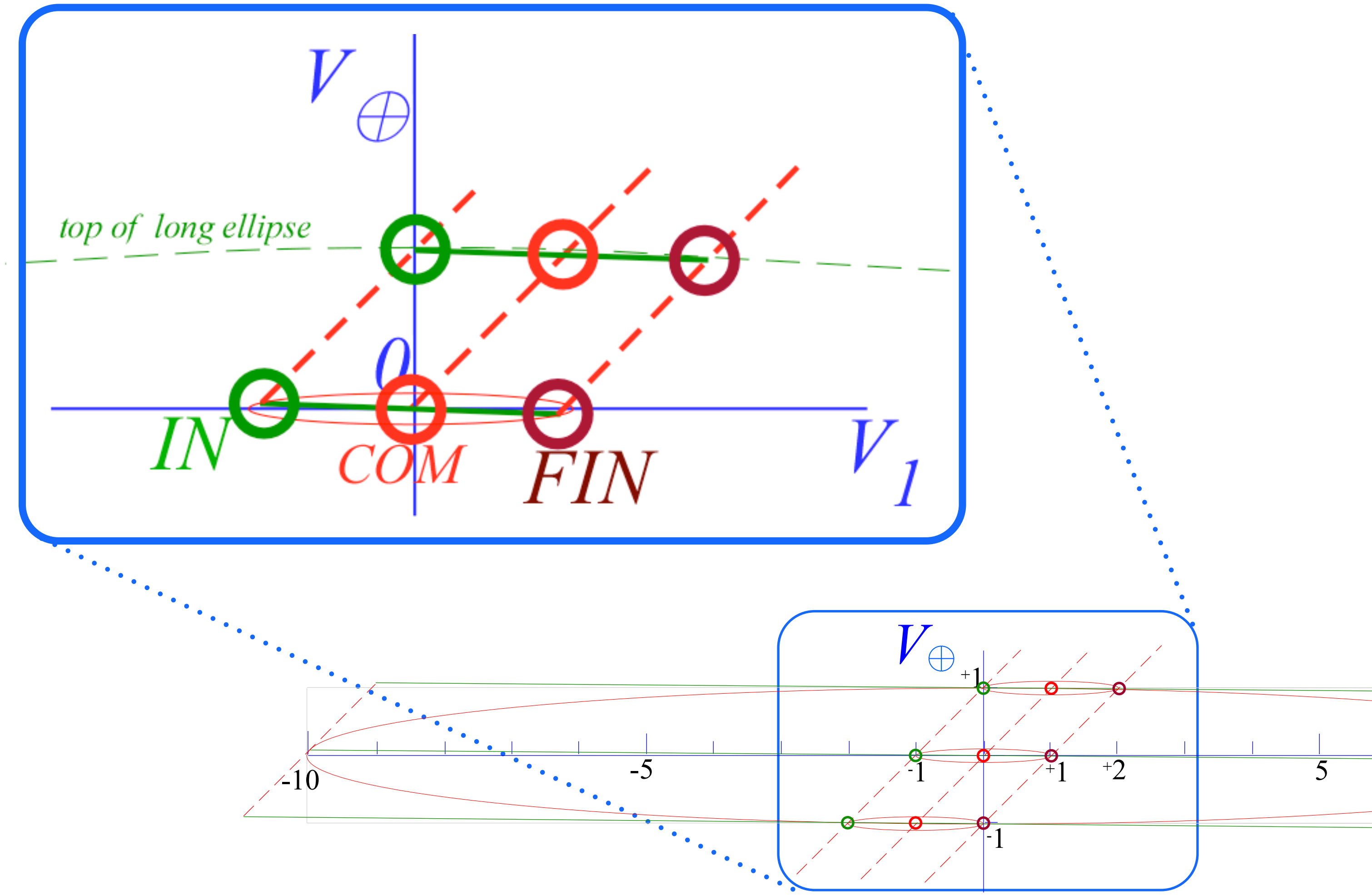
floor plate $M_\oplus = 100\ M_1$ along

$(V_1, V_\oplus)$-momentum line of slope

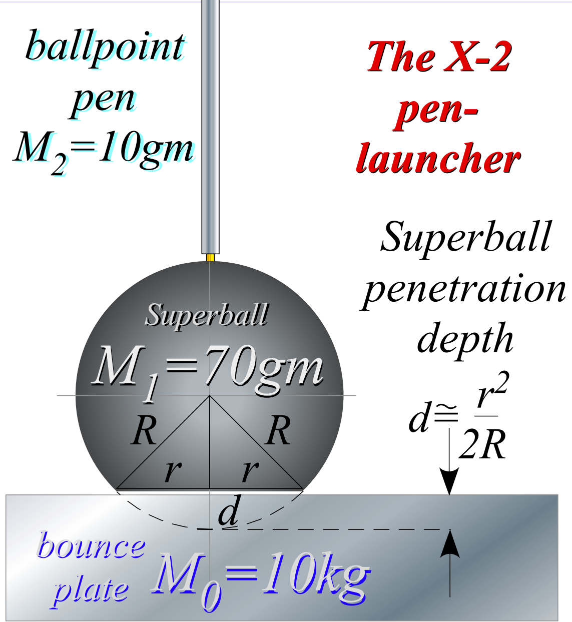
$-M_1/M_\oplus = -1/100$

from IN-end to COM to FIN-end

of ($a/b = \sqrt{M_\oplus}/\sqrt{M_1} = 10$) ellipse



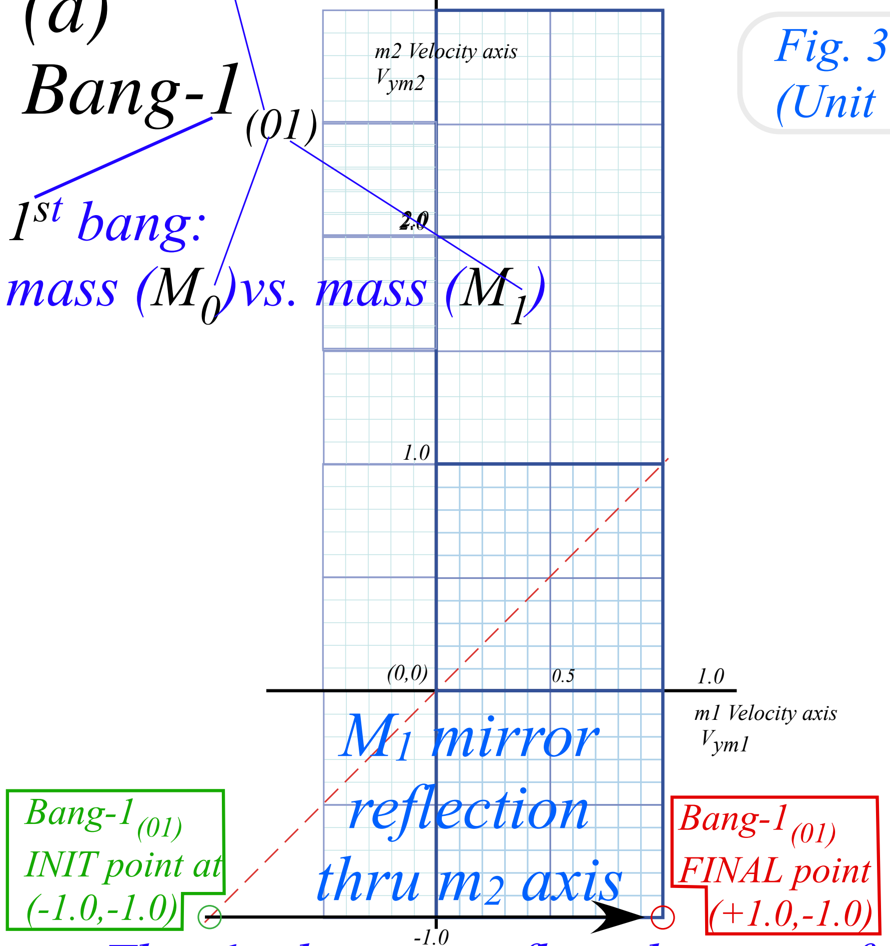
The X-2 pen-launcher



(a)

Bang-1$_{(01)}$

1^{st} bang: mass (M_0) vs. mass (M_1)



M_1 mirror reflection thru m_2 axis

Bang-1$_{(01)}$ INIT point at (-1.0,-1.0)

Bang-1$_{(01)}$ FINAL point (+1.0,-1.0)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 3.3 (Unit 1)

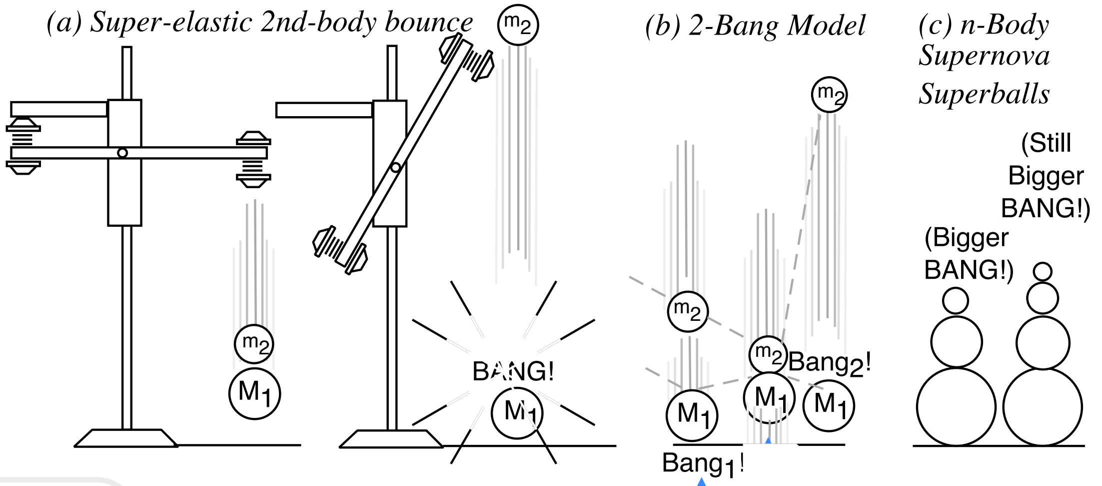


Fig. 3.4 (Unit 1)

1st bang: M_1 off floor

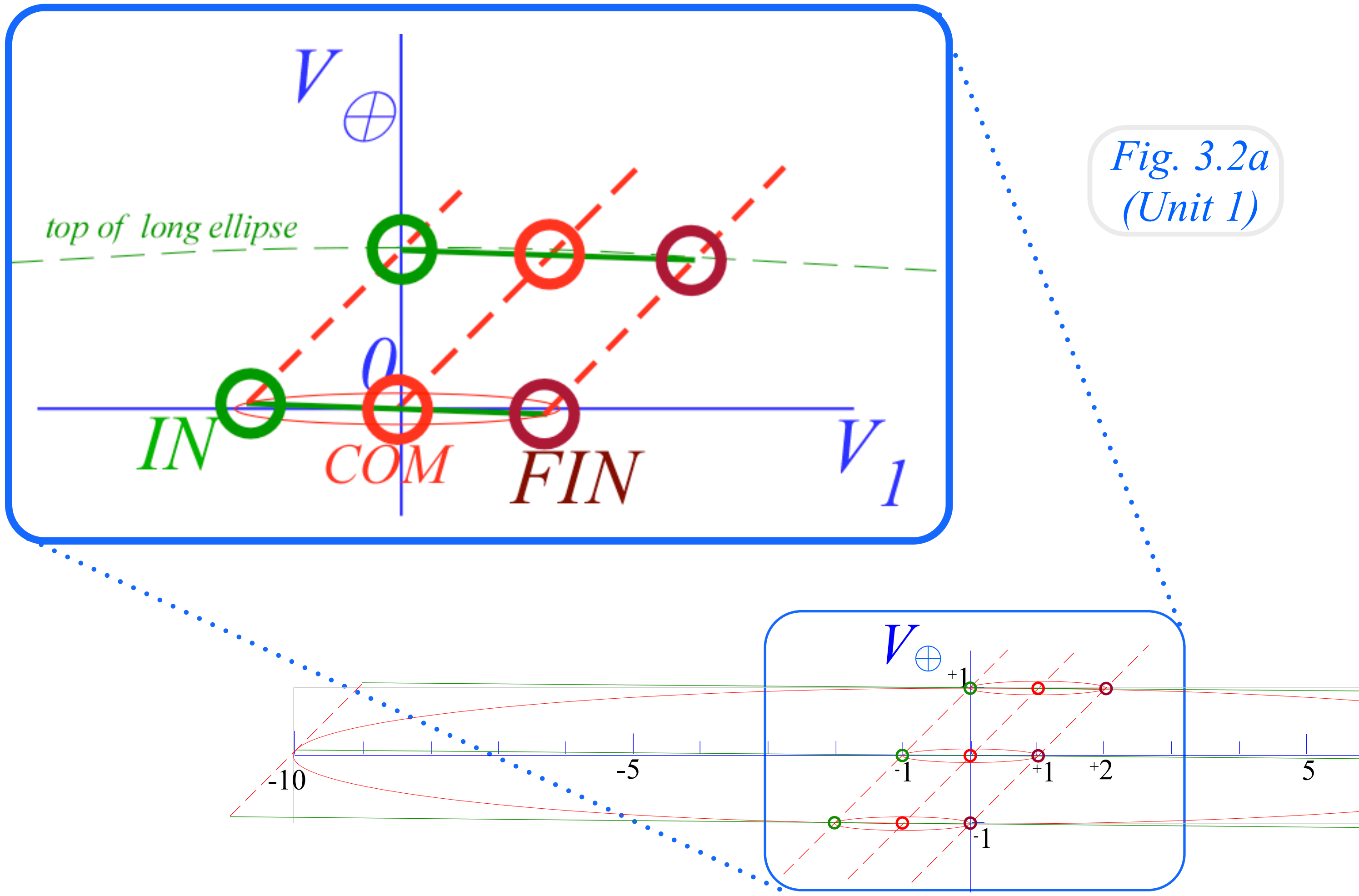
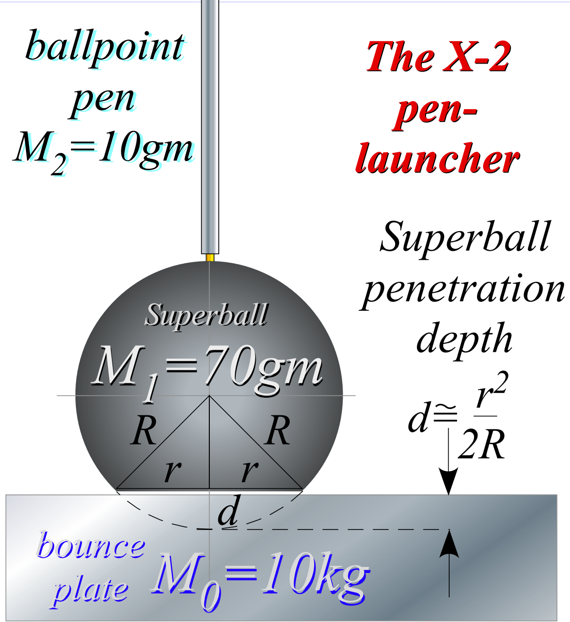


Fig. 3.2a (Unit 1)

(With g=0 and 70:10 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

The X-2 pen-launcher



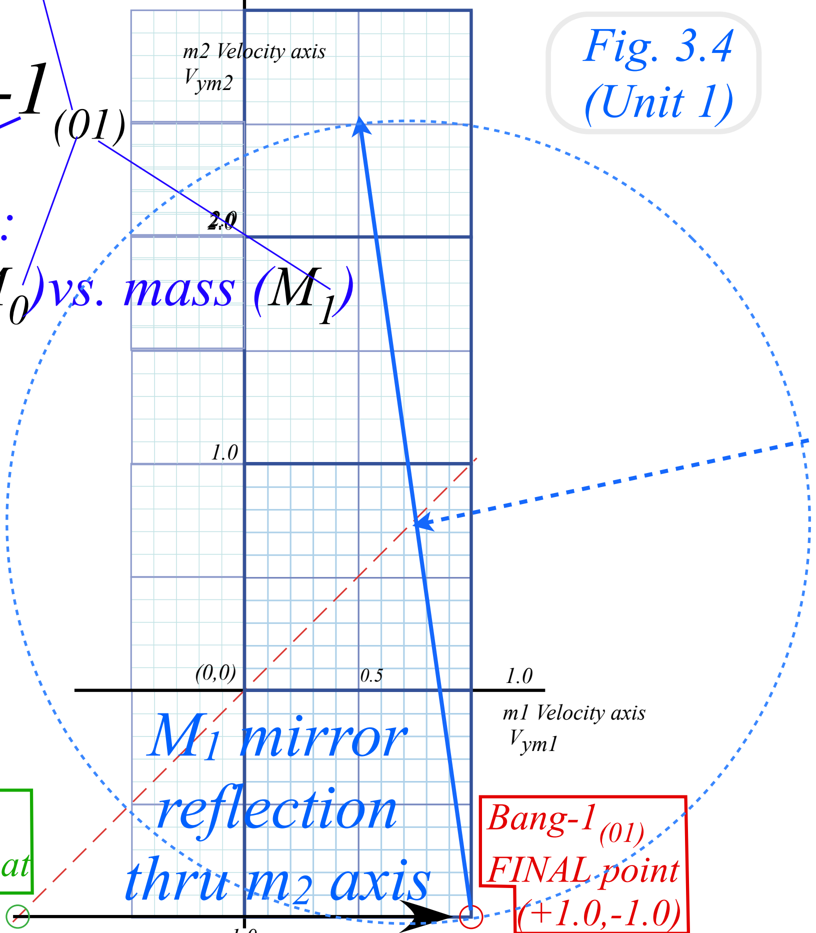
Superball penetration depth

$d \cong \frac{r^2}{2R}$

(a)

Bang-1 (01)

1^{st} bang:
mass (M_0) vs. mass (M_1)



M_1 mirror reflection thru m_2 axis

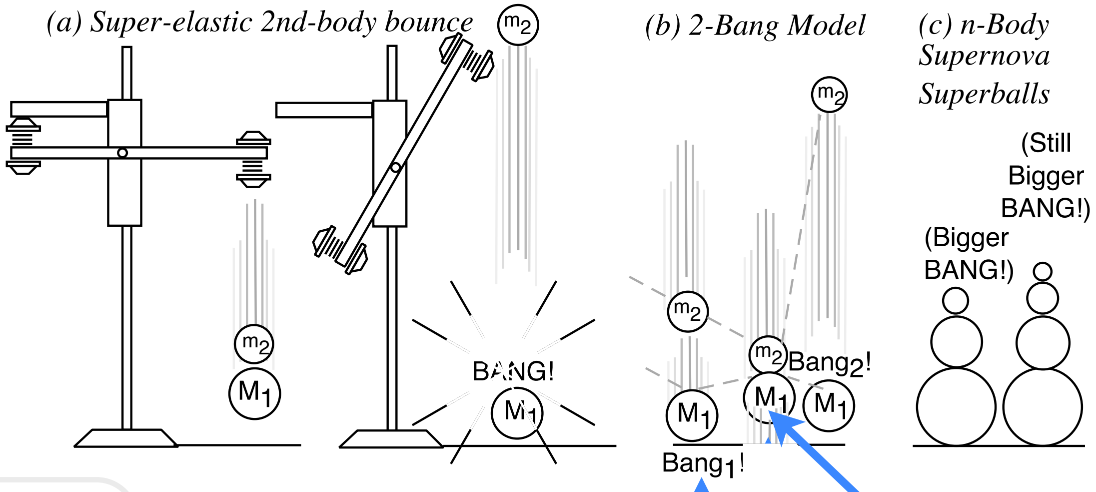
Bang-1 (01) INIT point at (-1.0,-1.0)

Bang-1 (01) FINAL point (+1.0,-1.0)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 3.4 (Unit 1)

Fig. 3.3 (Unit 1)

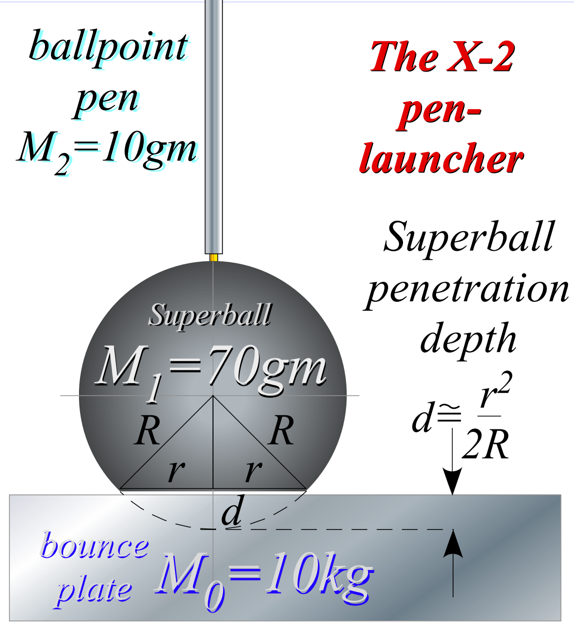


1st bang: M_1 off floor

2nd bang: m_2 off M_1

(With g=0 and 70:10 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With g and 70:35 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300



The X-2 pen-launcher

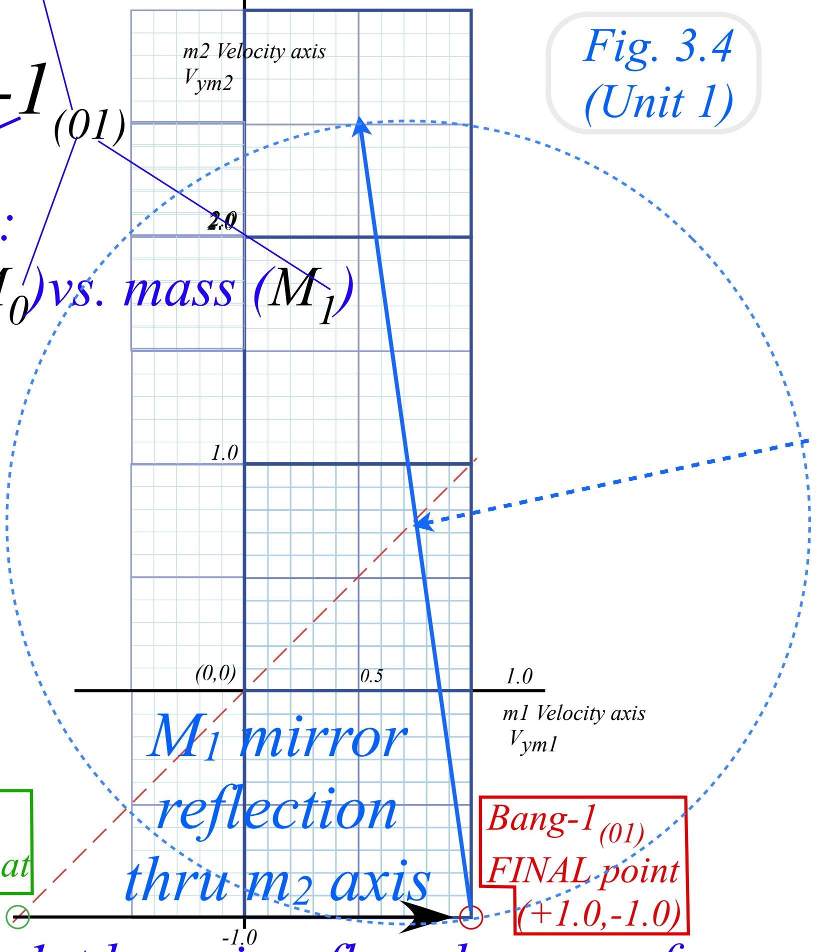
Superball penetration depth

$d \cong \frac{r^2}{2R}$

(a)

Bang-$1_{(01)}$

1^{st} bang:
mass (M_0) vs. mass (M_1)



Bang-$1_{(01)}$
INIT point at
(-1.0,-1.0)

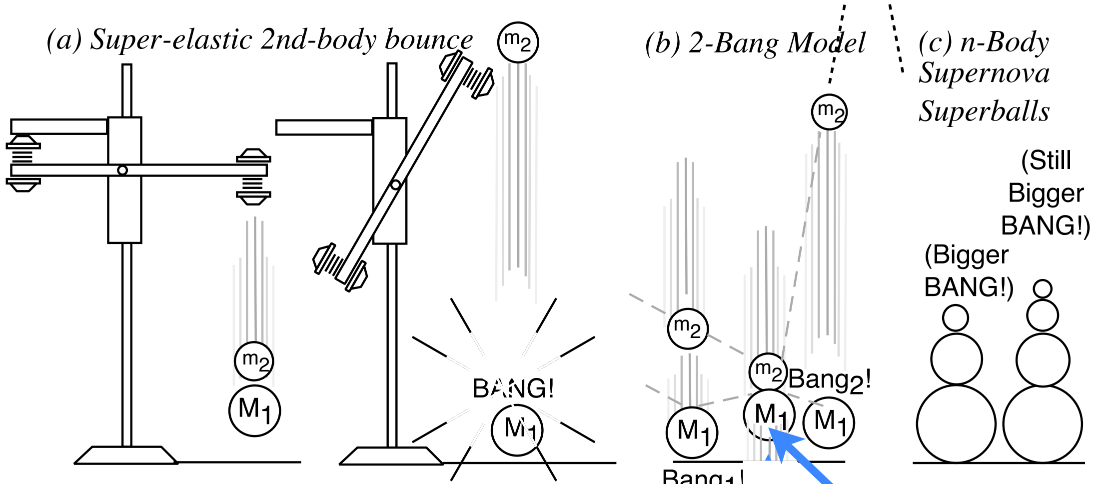
M_1 mirror reflection thru m_2 axis

Bang-$1_{(01)}$
FINAL point
(+1.0,-1.0)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 3.4
(Unit 1)

Fig. 3.3
(Unit 1)



3rd bang:
m_2 off ceiling

1st bang:
M_1 off floor

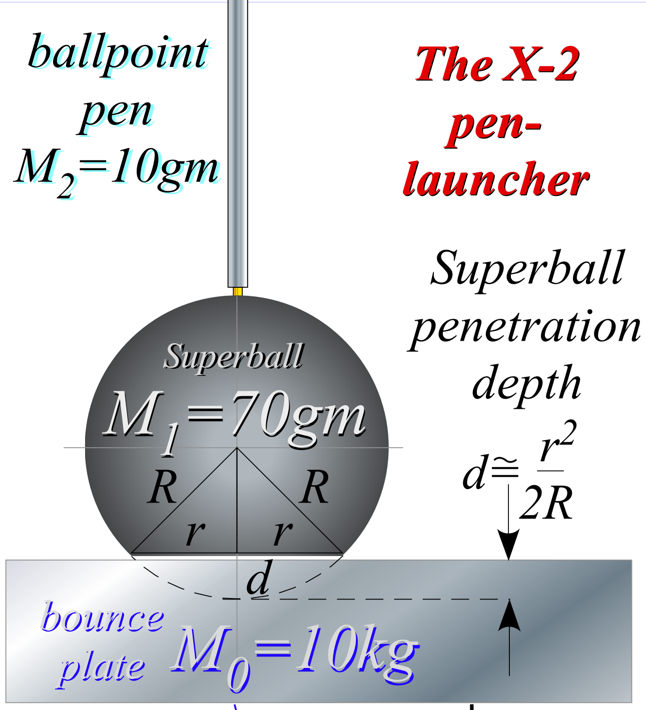
2nd bang:
m_2 off M_1

(With g=0 and 70:10 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With g and 70:35 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300

Geometry of X2 launcher bouncing in box *(gravity-free)*

→ *Independent Bounce Model (IBM)*
Geometric optimization and range-of-motion calculation(t)
Integration of (V_1,V_2) *data to space-time plots* $(y_1(t),t)$ *and* $(y_2(t),t)$ *plots*
Integration of (V_1,V_2) *data to space-space plots* (y_1, y_2)



The X-2 pen-launcher

Superball penetration depth

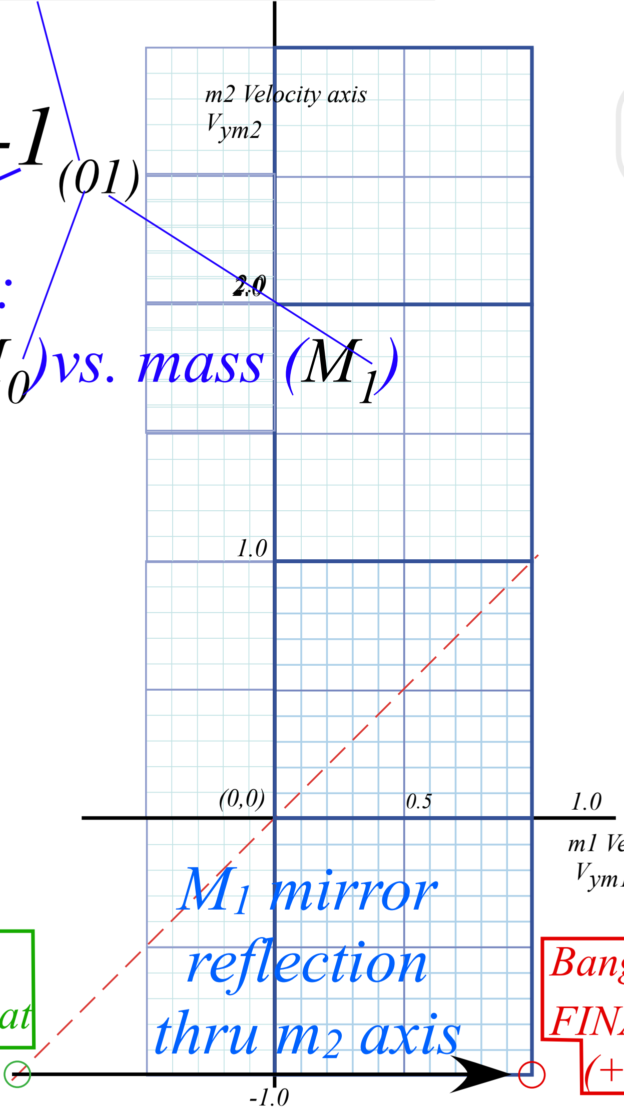
$d \cong \frac{r^2}{2R}$

(a)

Bang-1$_{(01)}$

1^{st} bang:
mass (M_0)vs. mass (M_1)

Fig. 3.4
(Unit 1)



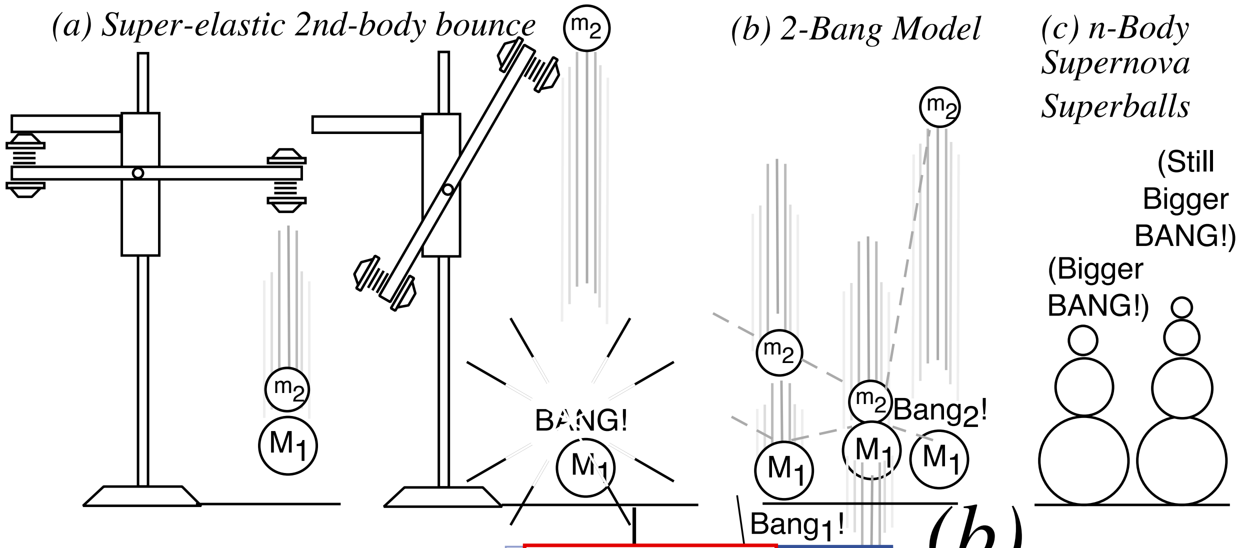
M_1 mirror reflection thru m_2 axis

Bang-1$_{(01)}$
INIT point at
(-1.0,-1.0)

Bang-1$_{(01)}$
FINAL point
(+1.0,-1.0)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 3.3
(Unit 1)

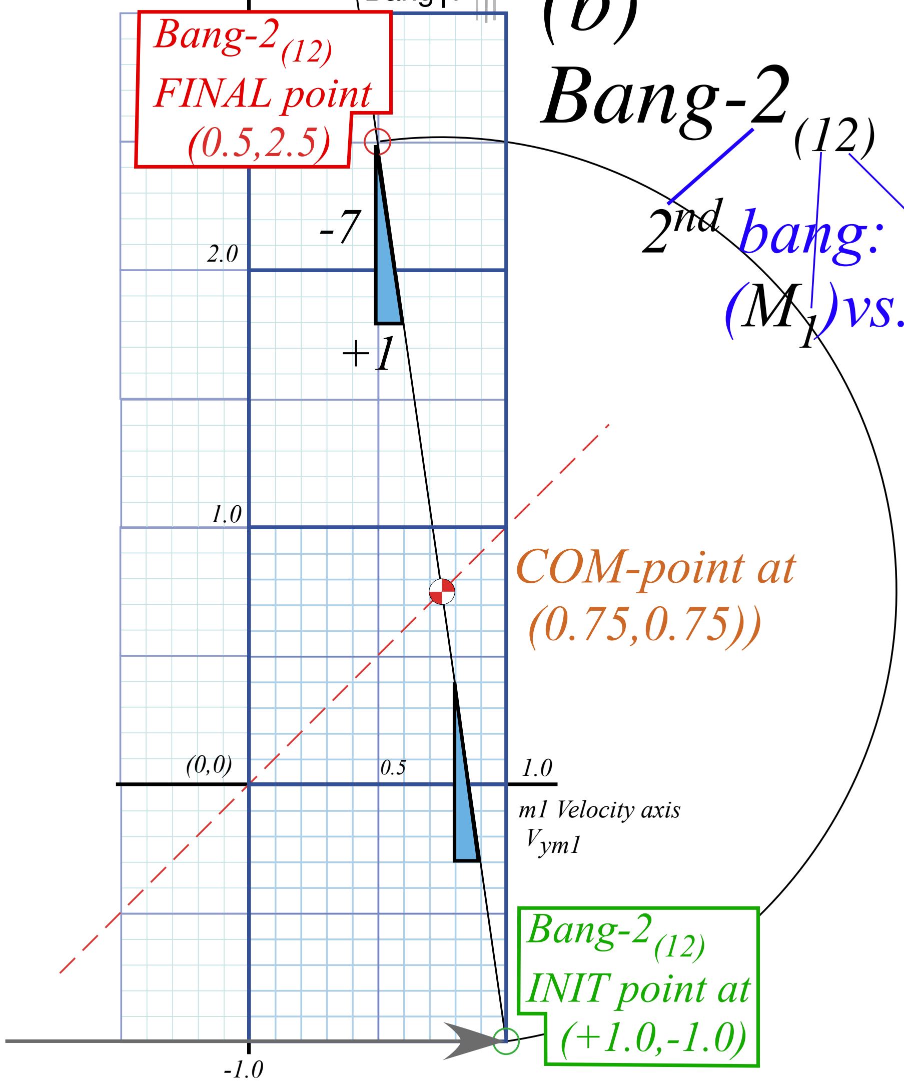


(b)

Bang-2$_{(12)}$

2^{nd} bang:
(M_1)vs.(M_2)

Bang-2$_{(12)}$
FINAL point
(0.5,2.5)



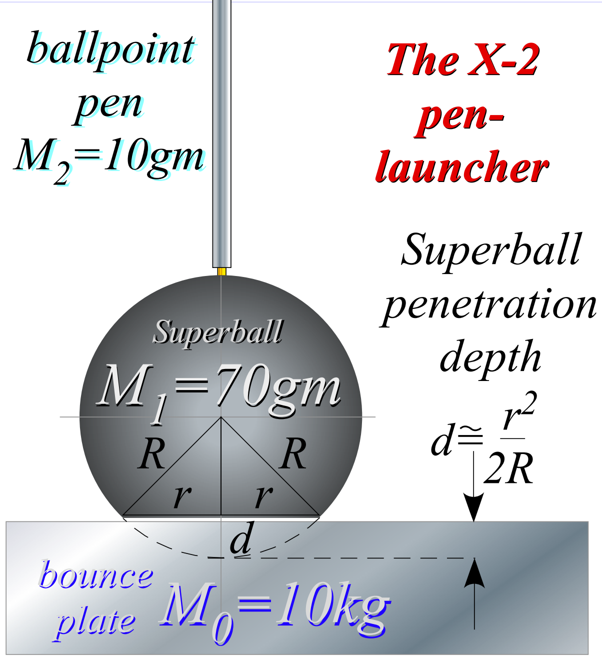
COM-point at
(0.75,0.75))

Bang-2$_{(12)}$
INIT point at
(+1.0,-1.0)

Geometry of X2 launcher bouncing in box *(gravity-free)*

- *Independent Bounce Model (IBM)*
- *Geometric optimization and range-of-motion calculation(s)*
- *Integration of (V_1,V_2) data to space-time plots ($y_1(t)$,t) and ($y_2(t)$,t) plots*
- *Integration of (V_1,V_2) data to space-space plots (y_1, y_2)*

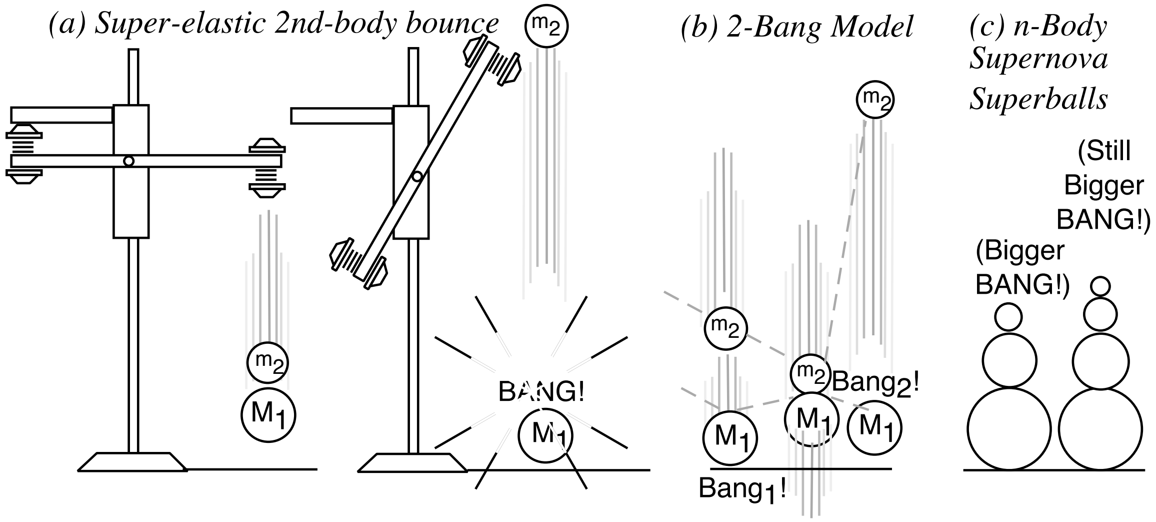
The X-2 pen-launcher

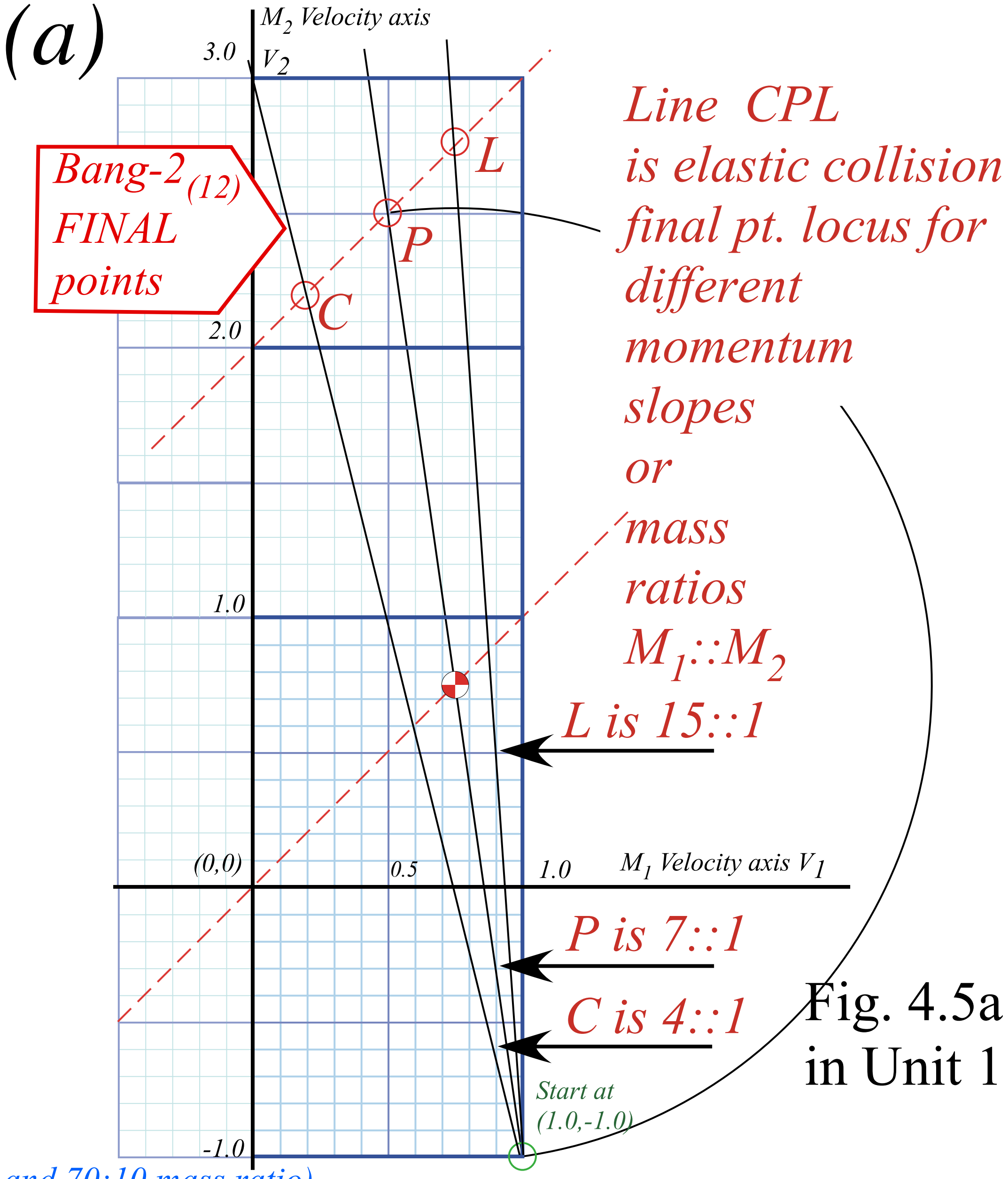


Superball penetration depth

$$d \cong \frac{r^2}{2R}$$

Fig. 3.3 (Unit 1)





Line CPL is elastic collision final pt. locus for different momentum slopes or mass ratios M_1::M_2

L is 15::1

P is 7::1

C is 4::1

Fig. 4.5a in Unit 1

(With g=0 and 70:10 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With g and 70:35 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=6300

The X-2 pen-launcher

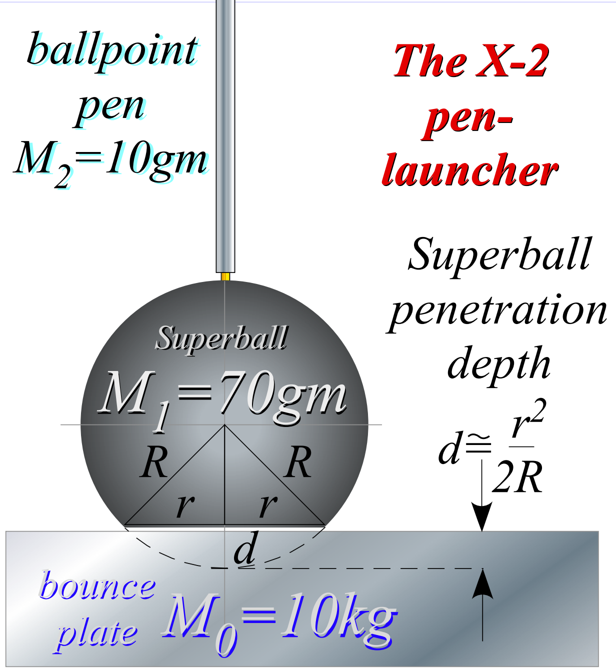
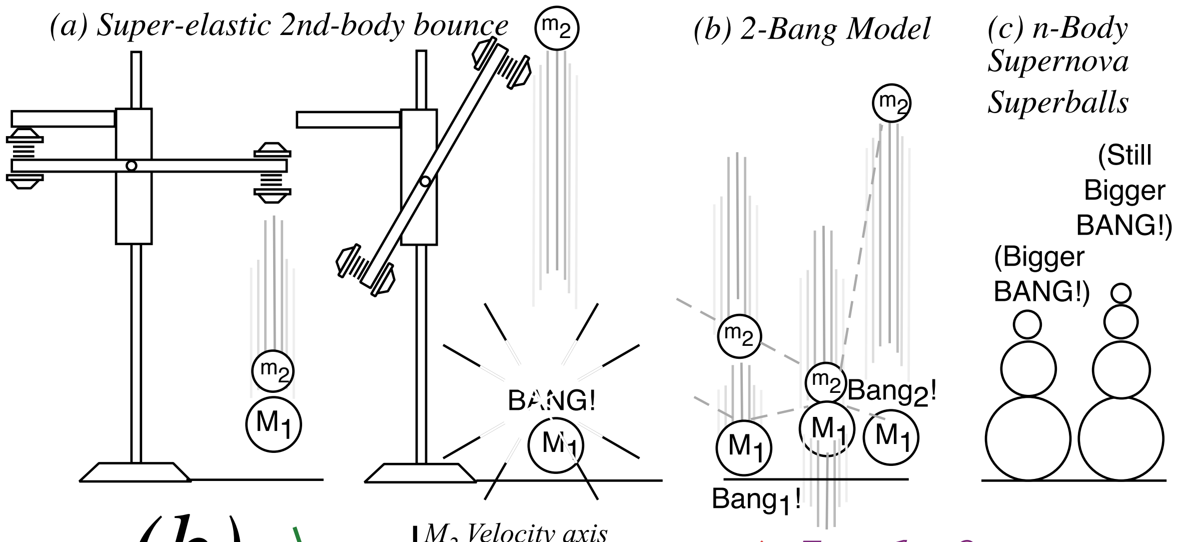
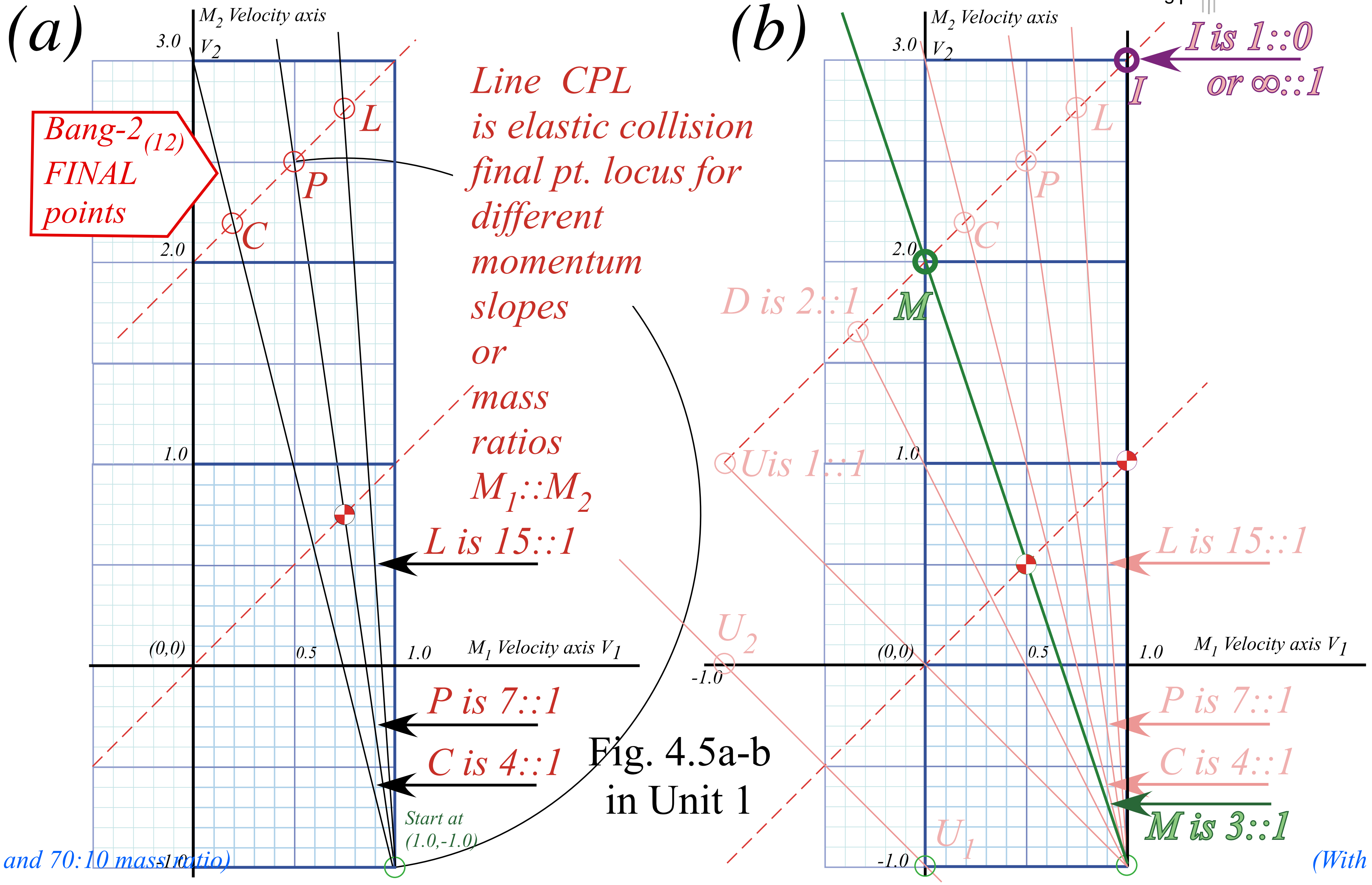


Fig. 3.3 (Unit 1)





Line CPL is elastic collision final pt. locus for different momentum slopes or mass ratios M_1::M_2

Fig. 4.5a-b in Unit 1

(With g=0 and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With g and 70:35 mass ratio)

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Geometry of X2 launcher bouncing in box (gravity-free)

Independent Bounce Model (IBM)

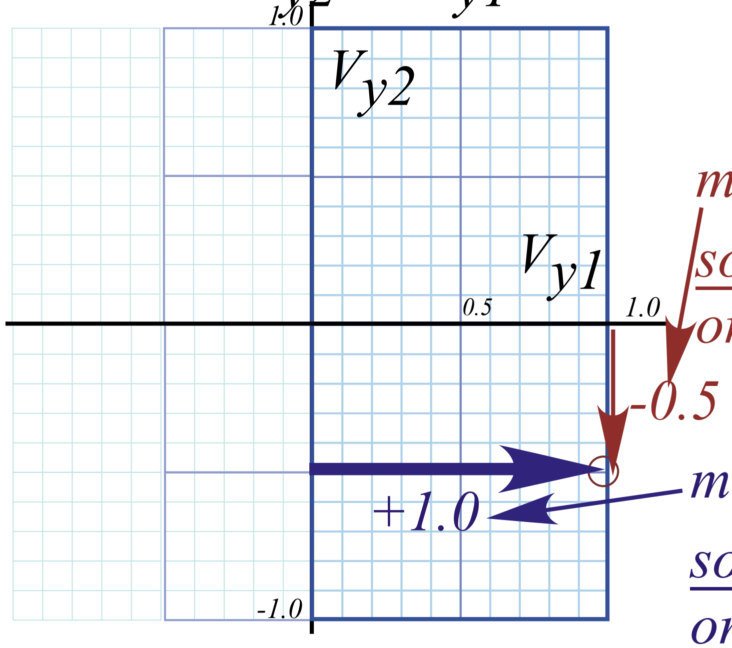
Geometric optimization and range-of-motion calculation(s)

→ *Integration of* (V_1,V_2) *data to space-time plots* $(y_1(t),t)$ *and* $(y_2(t),t)$ *plots*

Integration of (V_1,V_2) *data to space-space plots* (y_1, y_2) Examples $(M_1=7, M_2=1)$ and $(M_1=49, M_2=1)$

Geometric "Integration" (Converting Velocity data to Spacetime)

Velocity V_{y2} vs. V_{y1} Plot

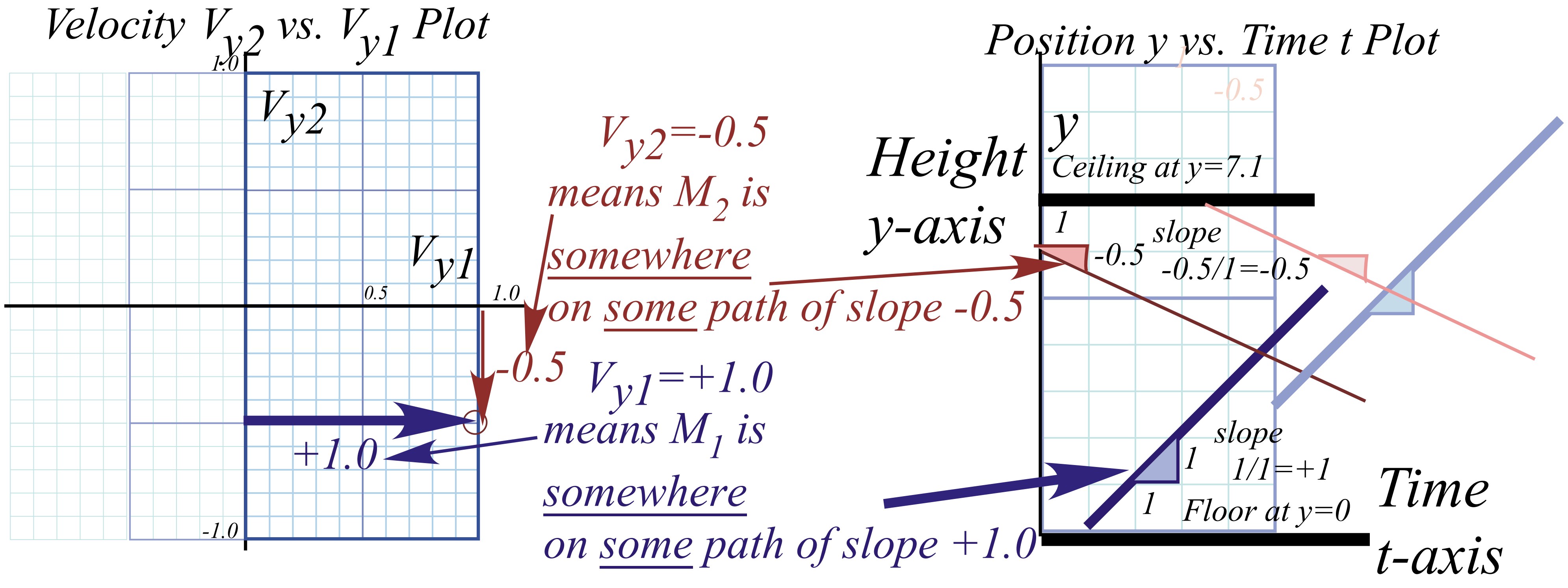


Position y vs. Time t Plot

V_{y2}=-0.5
means M_2 is
<u>somewhere</u>
on <u>some</u> path of slope -0.5

V_{y1}=+1.0
means M_1 is
<u>somewhere</u>
on <u>some</u> path of slope +1.0

Geometric “Integration” (Converting Velocity data to Spacetime)



V_{y2}=-0.5
means M_2 is
somewhere
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V_{y1}=+1.0
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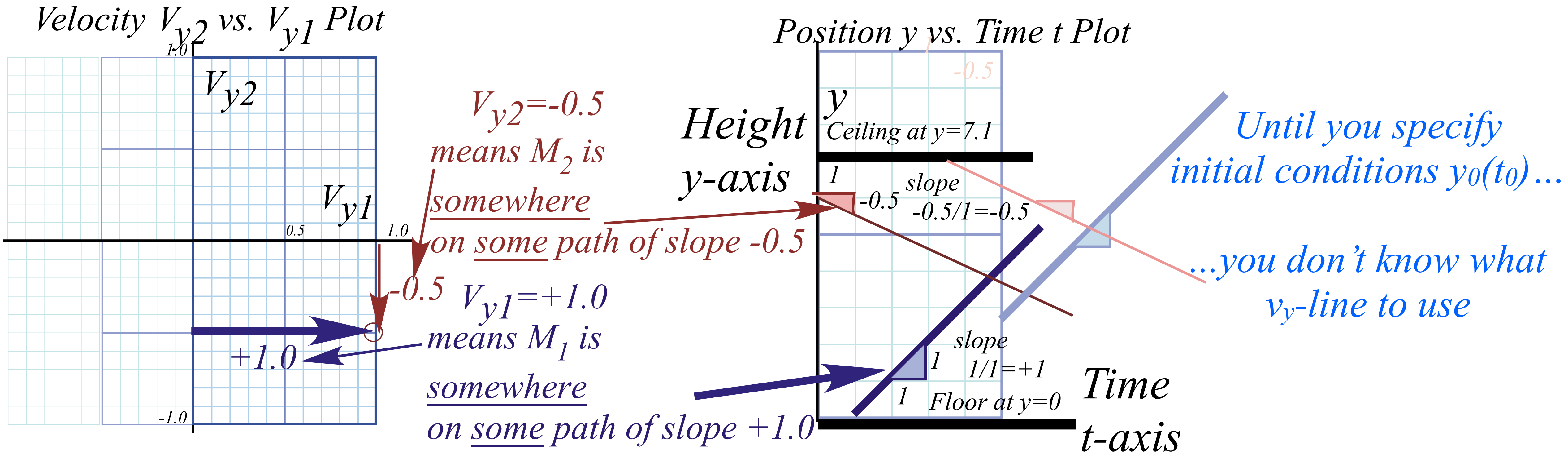
Geometric "Integration" (Converting Velocity data to Spacetime)



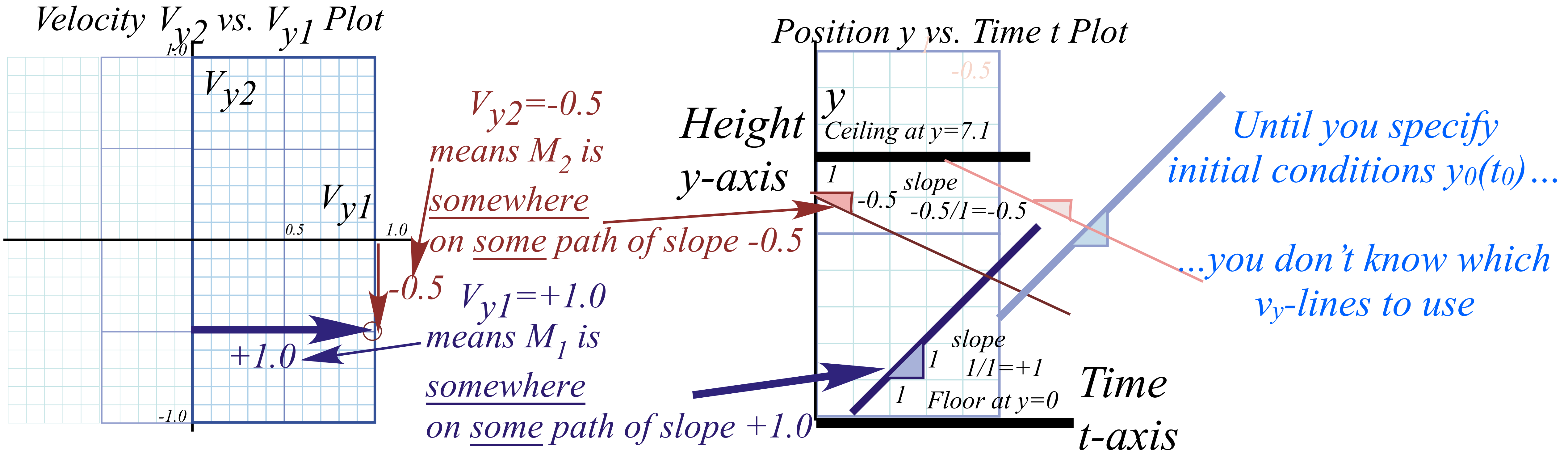
V_{y2}=-0.5 means M_2 is somewhere on some path of slope -0.5

V_{y1}=+1.0 means M_1 is somewhere on some path of slope +1.0

Until you specify initial conditions $y_0(t_0)$…

…you don't know what v_y-line to use

Geometric "Integration" (Converting Velocity data to Spacetime)



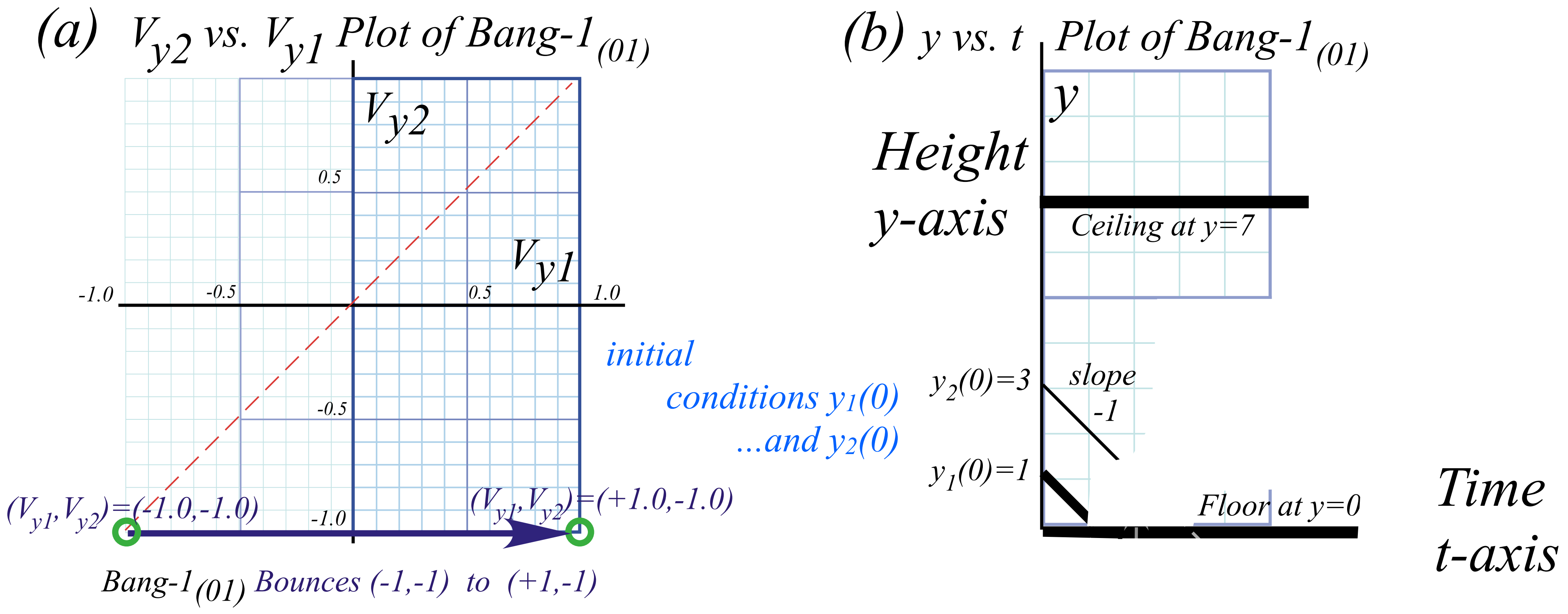
$V_{y2}=-0.5$ means M_2 is somewhere on some path of slope -0.5

$V_{y1}=+1.0$ means M_1 is somewhere on some path of slope +1.0

Until you specify initial conditions $y_0(t_0)$…

…you don't know which v_y-lines to use

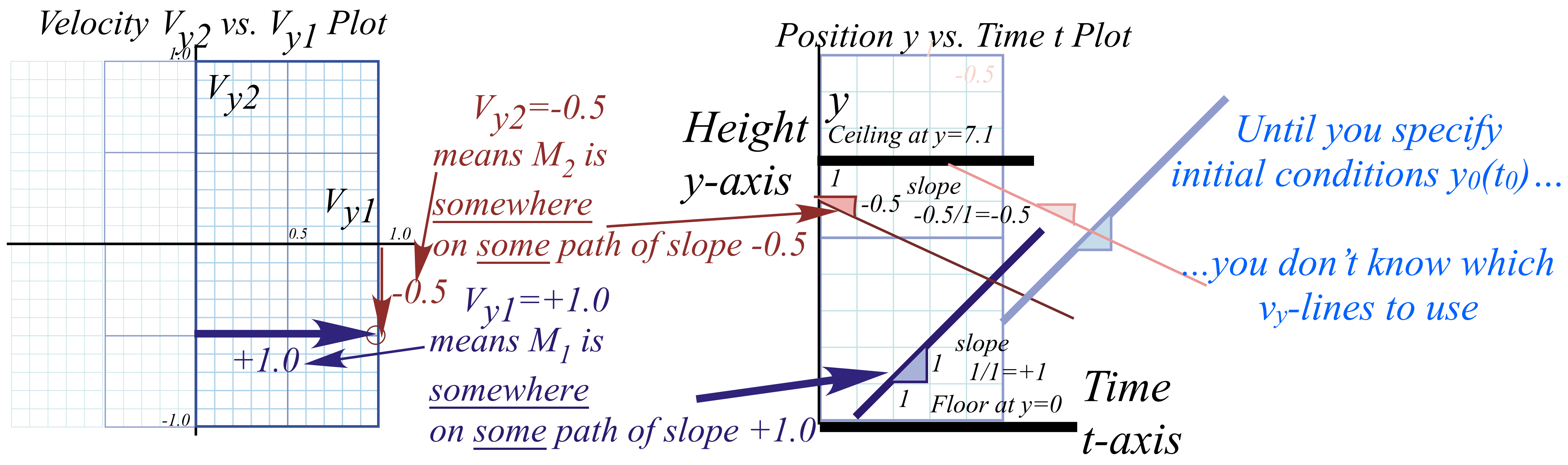
Fig. 4.6a-b
in Unit 1



Bang-$1_{(01)}$ Bounces (-1,-1) to (+1,-1)

initial conditions $y_1(0)$ …and $y_2(0)$

Geometric "Integration" (Converting Velocity data to Spacetime)



V_{y2}=-0.5 means M_2 is somewhere on some path of slope -0.5

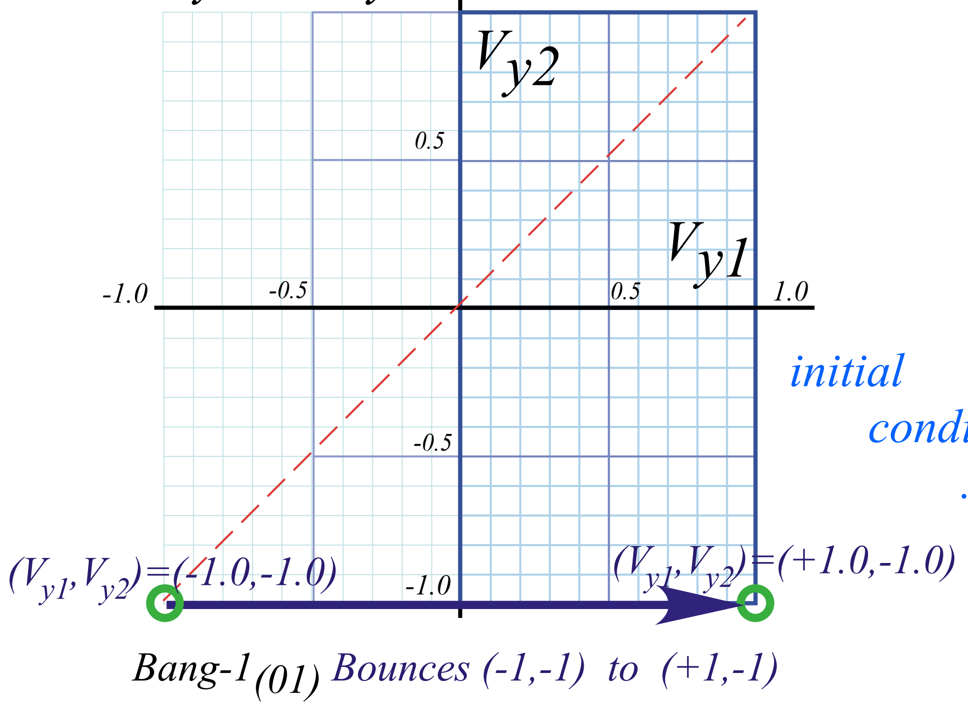
V_{y1}=+1.0 means M_1 is somewhere on some path of slope +1.0

Until you specify initial conditions $y_0(t_0)$…

…you don't know which v_y-lines to use

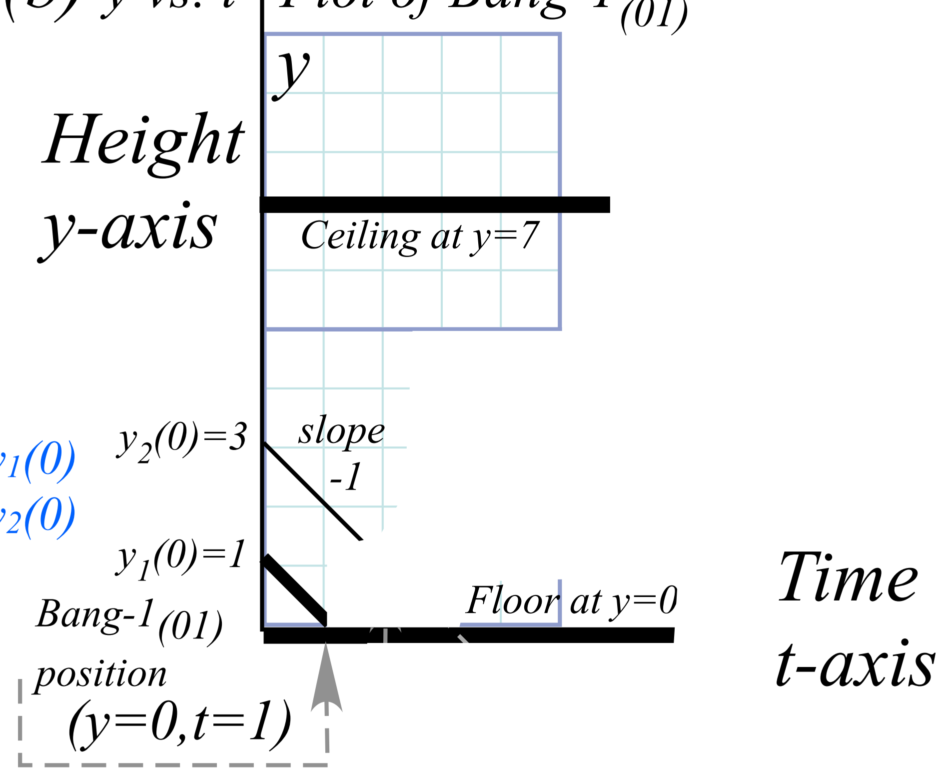
Fig. 4.6a-b
in Unit 1

(a) V_{y2} vs. V_{y1} Plot of Bang-$1_{(01)}$

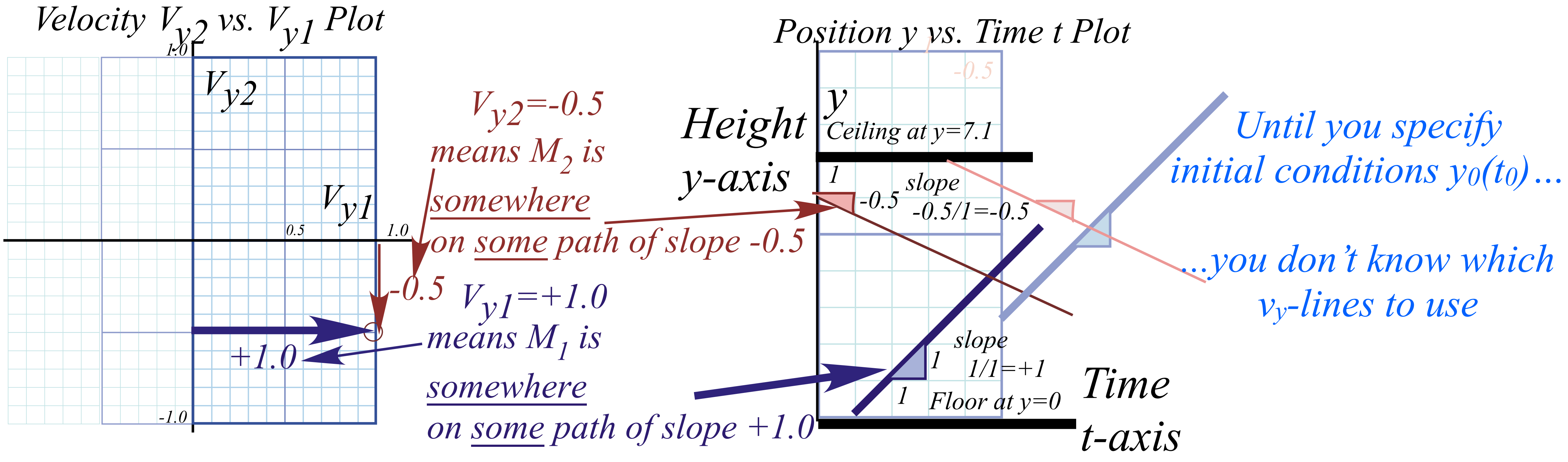


initial conditions $y_1(0)$ …and $y_2(0)$

(b) y vs. t Plot of Bang-$1_{(01)}$



Geometric "Integration" (Converting Velocity data to Spacetime)



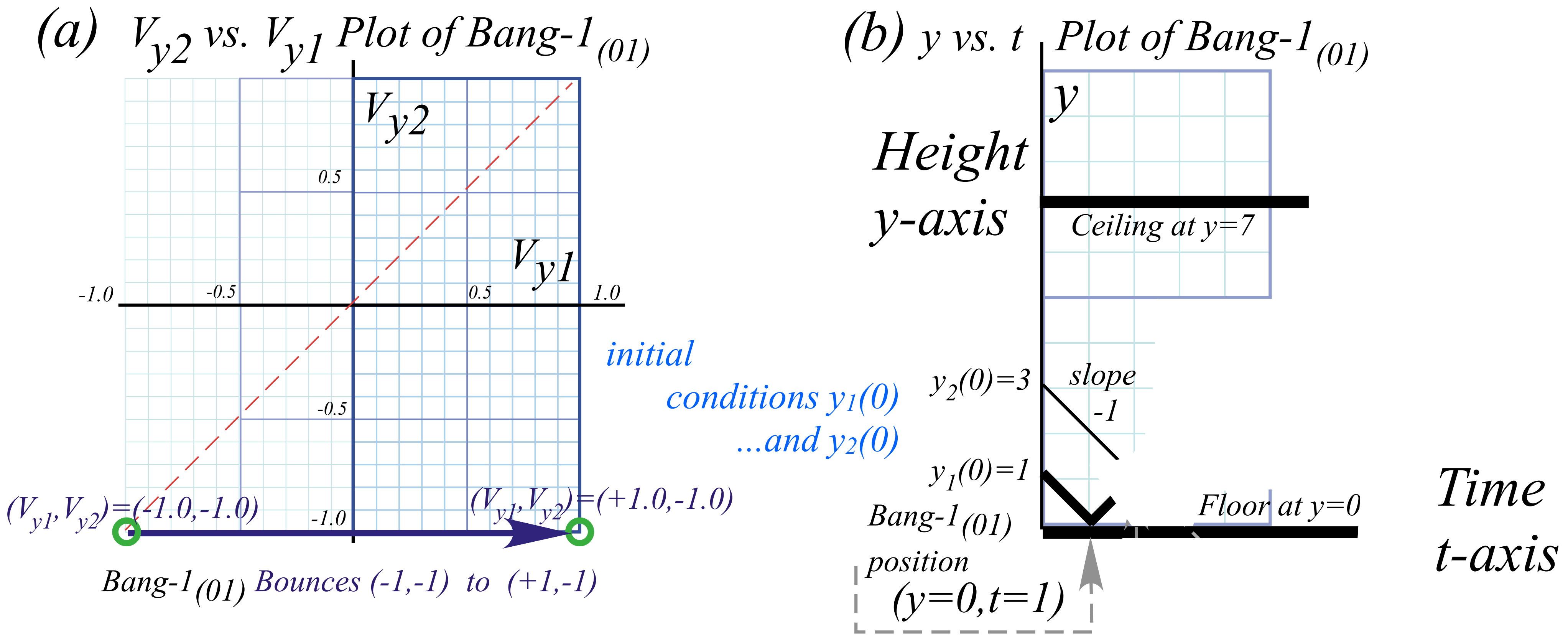
V_{y2}=-0.5 means M_2 is somewhere on some path of slope -0.5

V_{y1}=+1.0 means M_1 is somewhere on some path of slope +1.0

Until you specify initial conditions $y_0(t_0)$…

…you don't know which v_y-lines to use

Fig. 4.6a-b
in Unit 1



initial conditions $y_1(0)$ …and $y_2(0)$

Geometric "Integration" (Converting Velocity data to Spacetime)

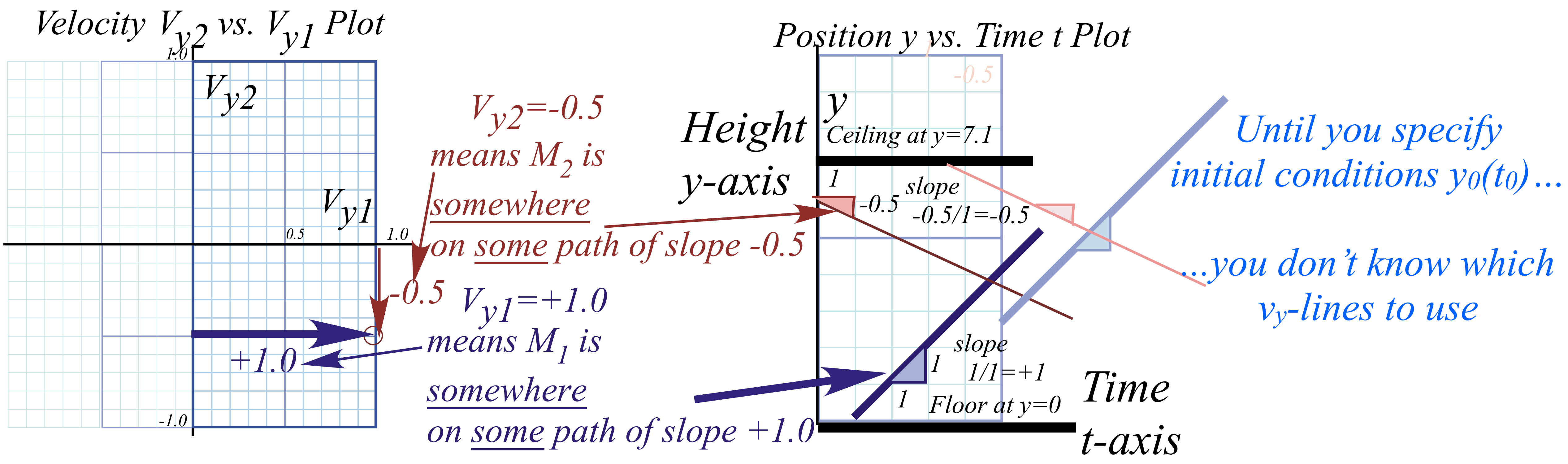


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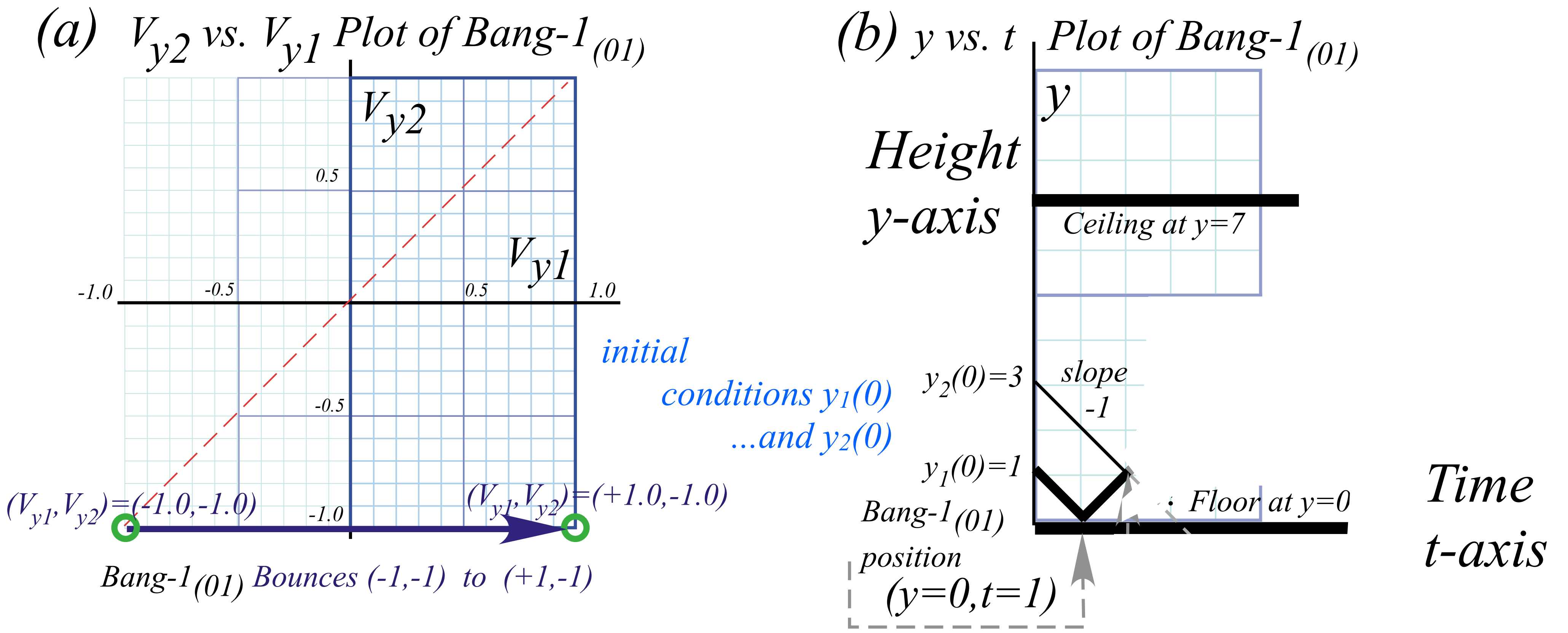
Fig. 4.6a-b
in Unit 1



initial conditions $y_1(0)$ …and $y_2(0)$

Geometric "Integration" (Converting Velocity data to Spacetime)

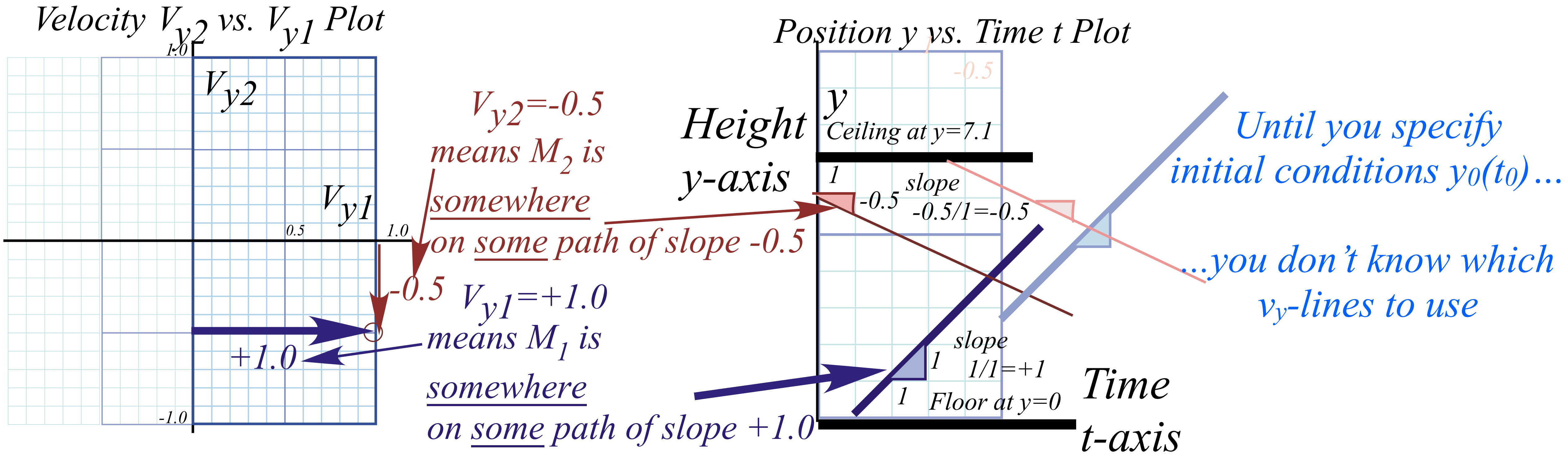


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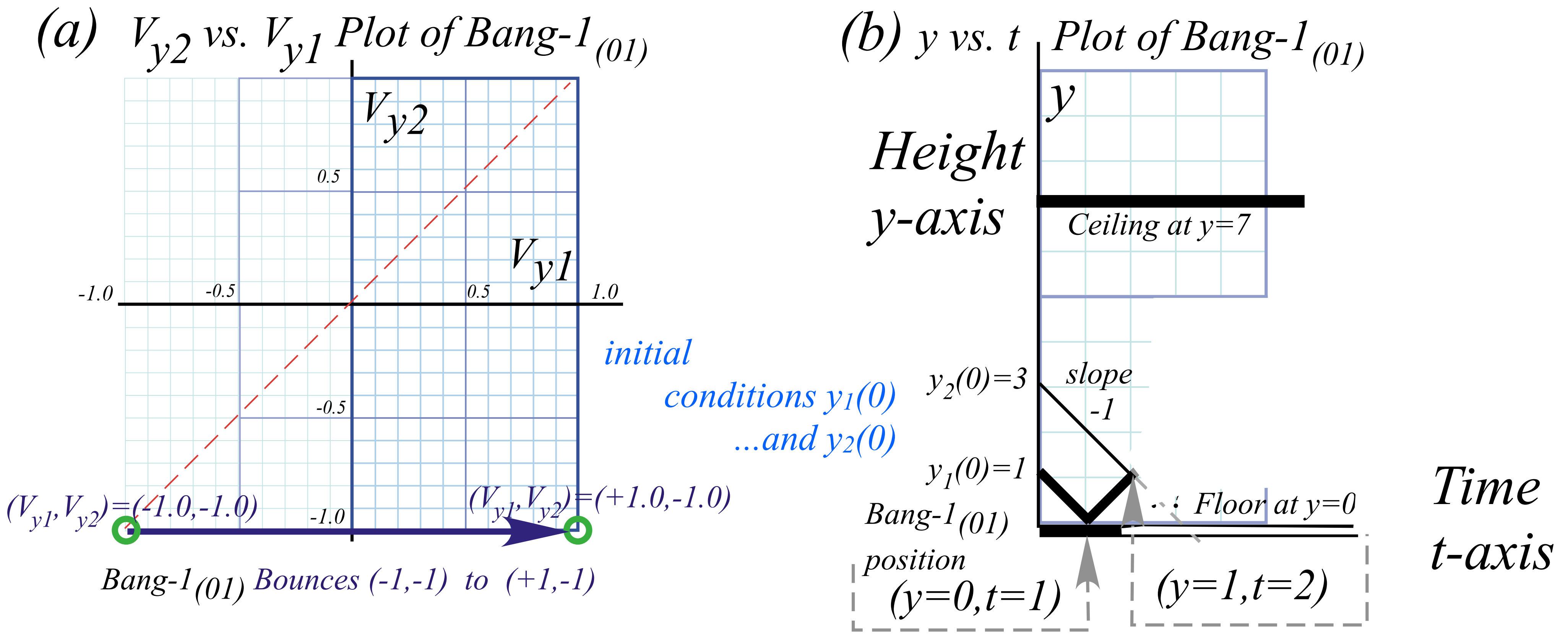
Fig. 4.6a-b
in Unit 1



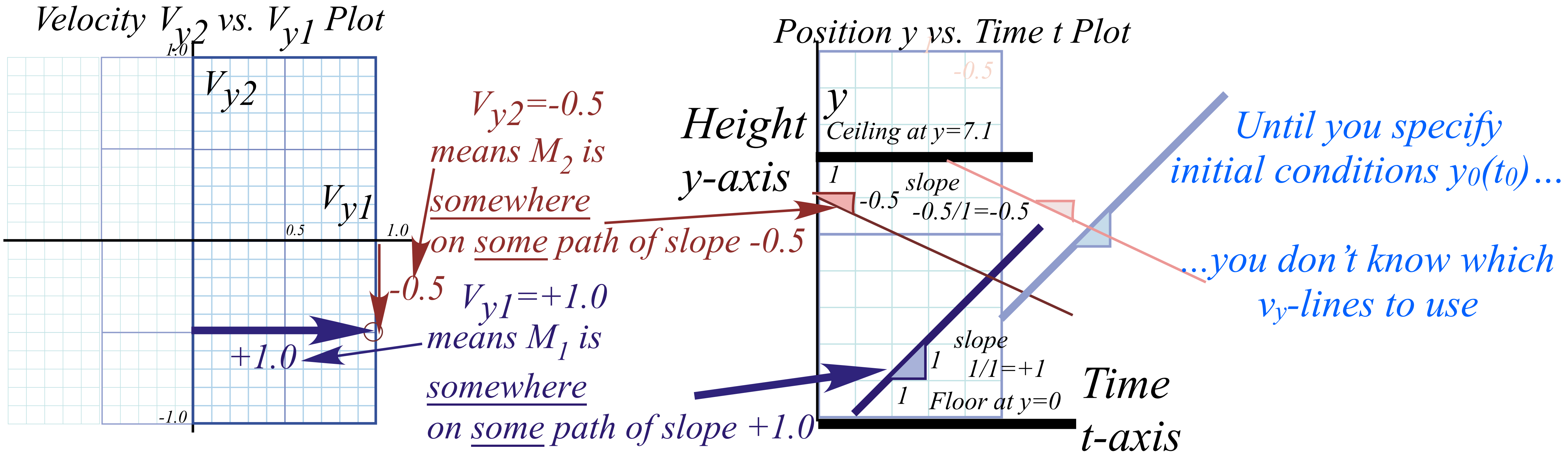
Bang-$1_{(01)}$ Bounces (-1,-1) to (+1,-1)

initial conditions $y_1(0)$ …and $y_2(0)$

Bang-$1_{(01)}$ position (y=0,t=1)

(y=1,t=2)

Geometric "Integration" (Converting Velocity data to Spacetime)



V_{y2}=-0.5 means M_2 is somewhere on some path of slope -0.5

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Until you specify initial conditions $y_0(t_0)$…

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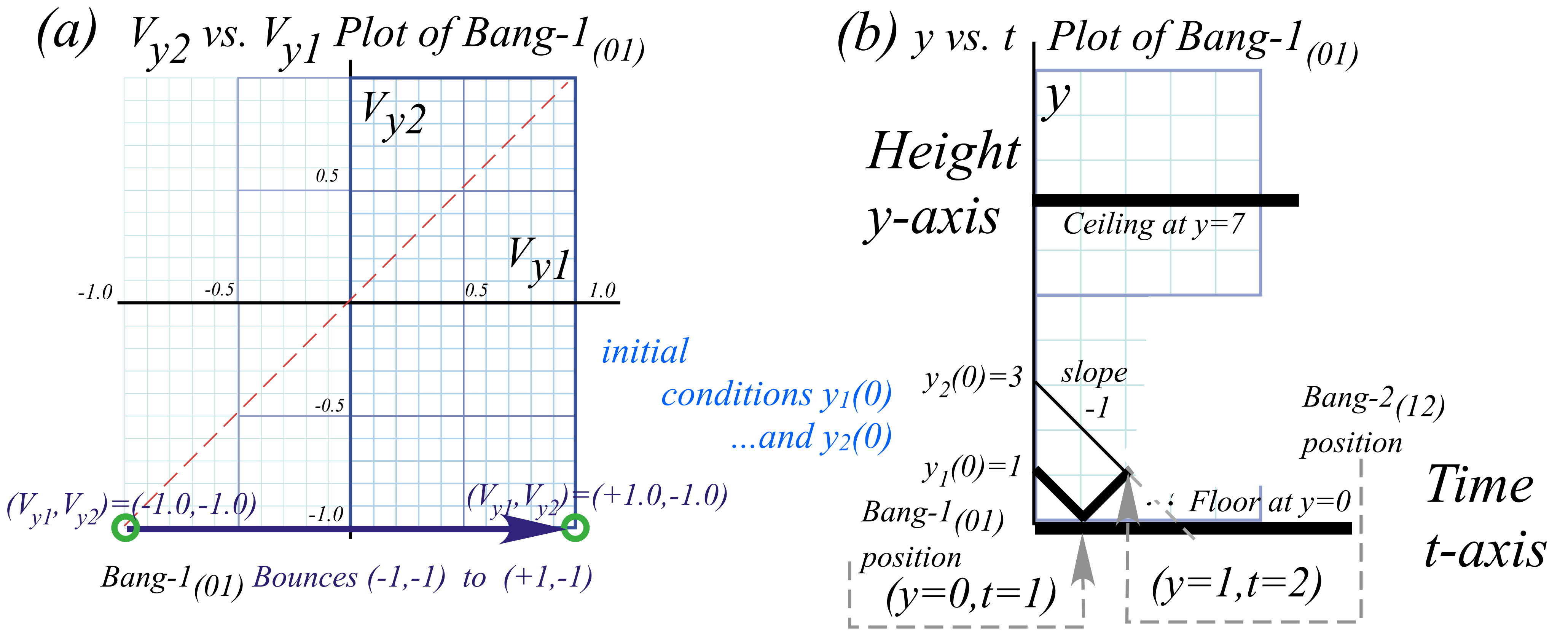
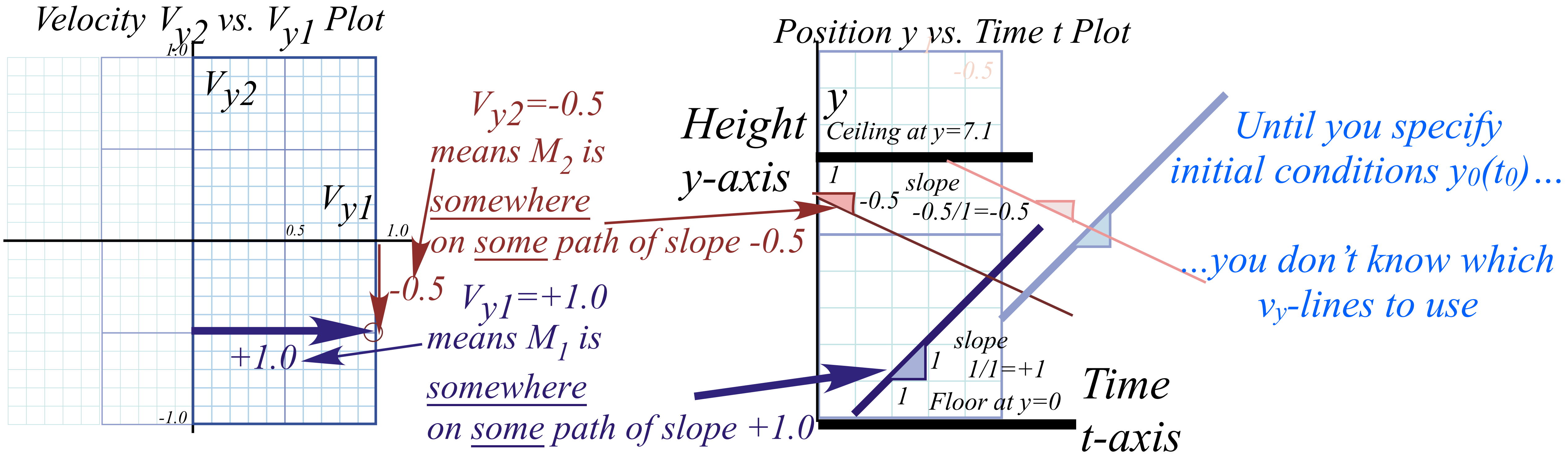
Fig. 4.6a-b
in Unit 1



initial conditions $y_1(0)$ …and $y_2(0)$

Geometric "Integration" (Converting Velocity data to Spacetime)



V_{y2}=-0.5 means M_2 is somewhere on some path of slope -0.5

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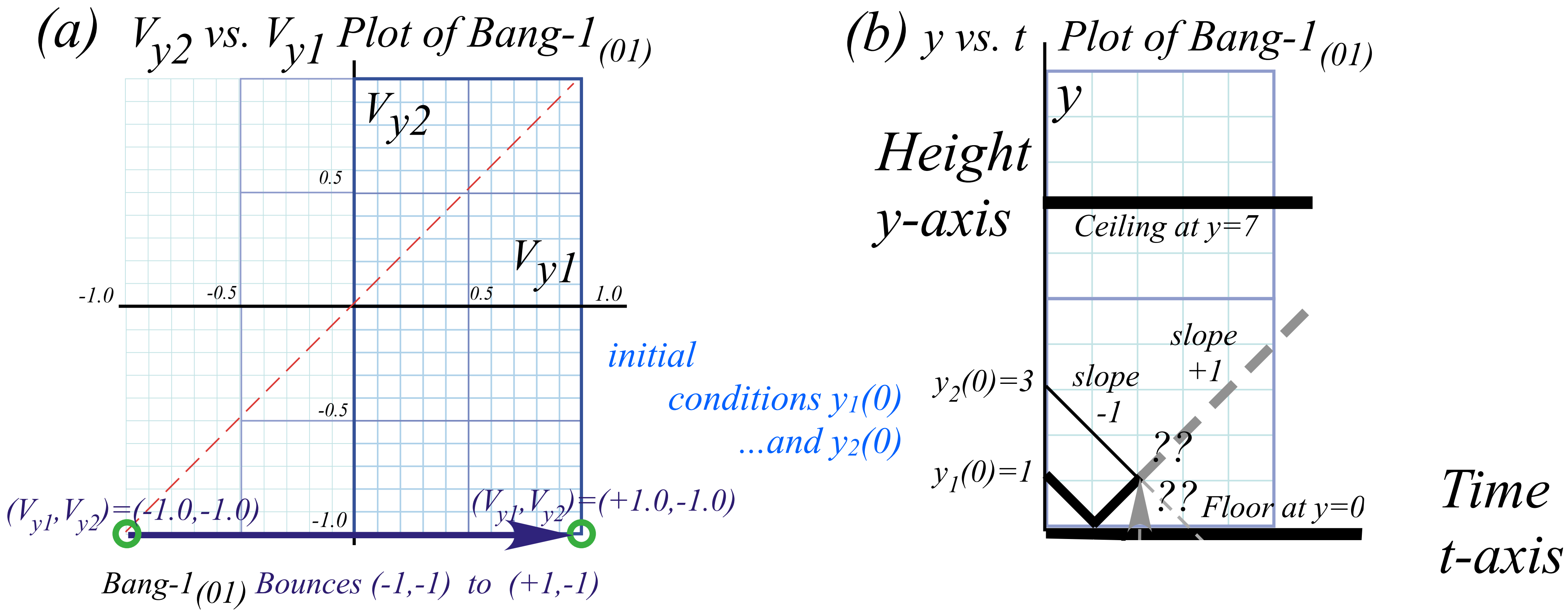
Until you specify initial conditions $y_0(t_0)$…

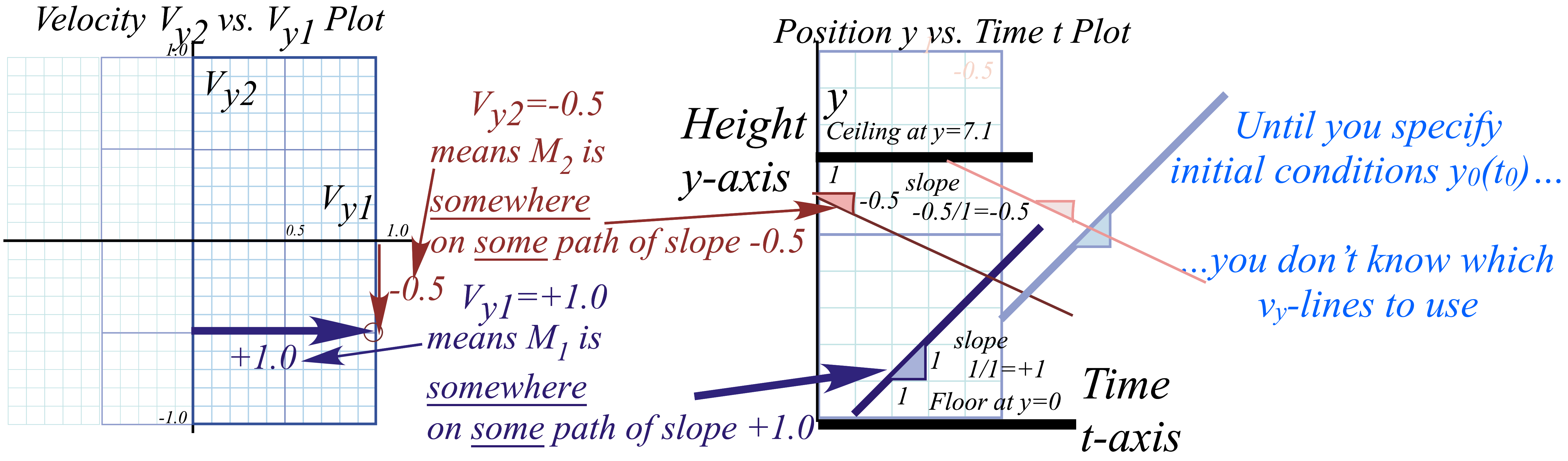
…you don't know which v_y-lines to use

Fig. 4.6a-b
in Unit 1



initial conditions $y_1(0)$ …and $y_2(0)$

Geometric "Integration" (Converting Velocity data to Spacetime)



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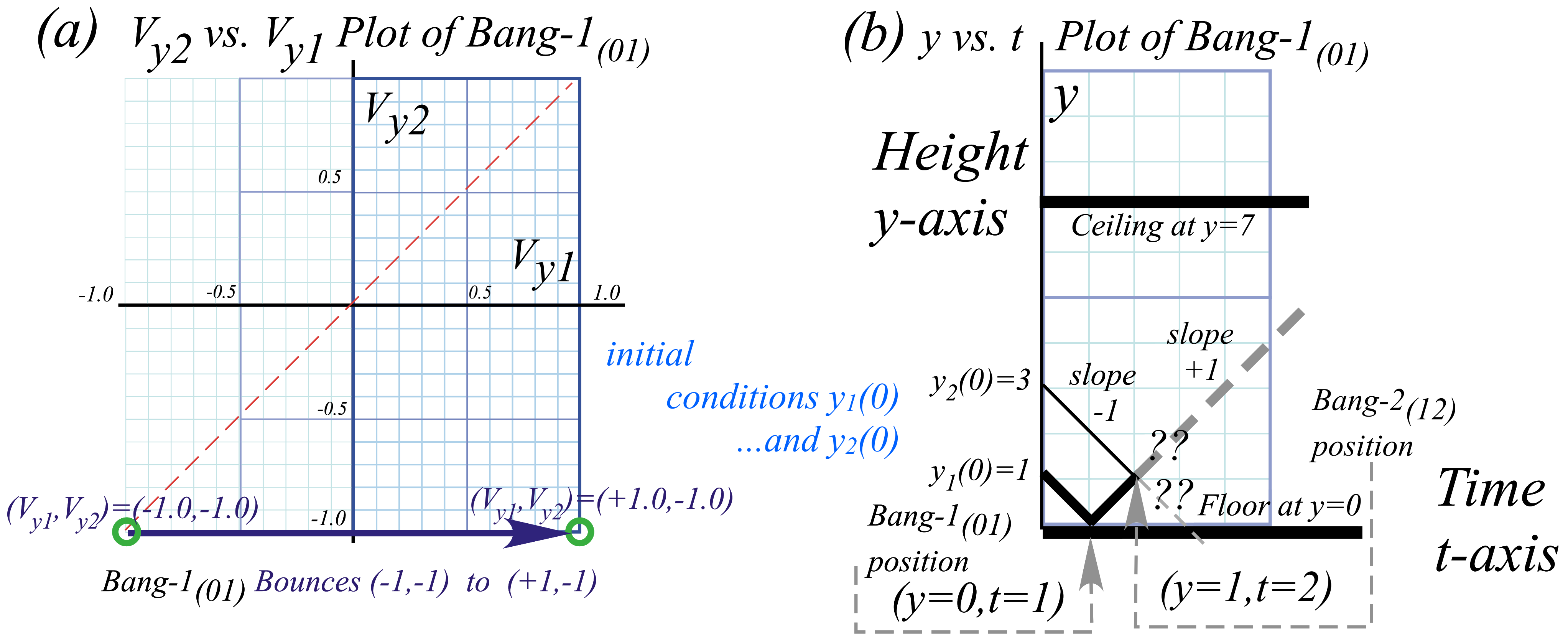
Until you specify initial conditions $y_0(t_0)$…

…you don't know which v_y-lines to use

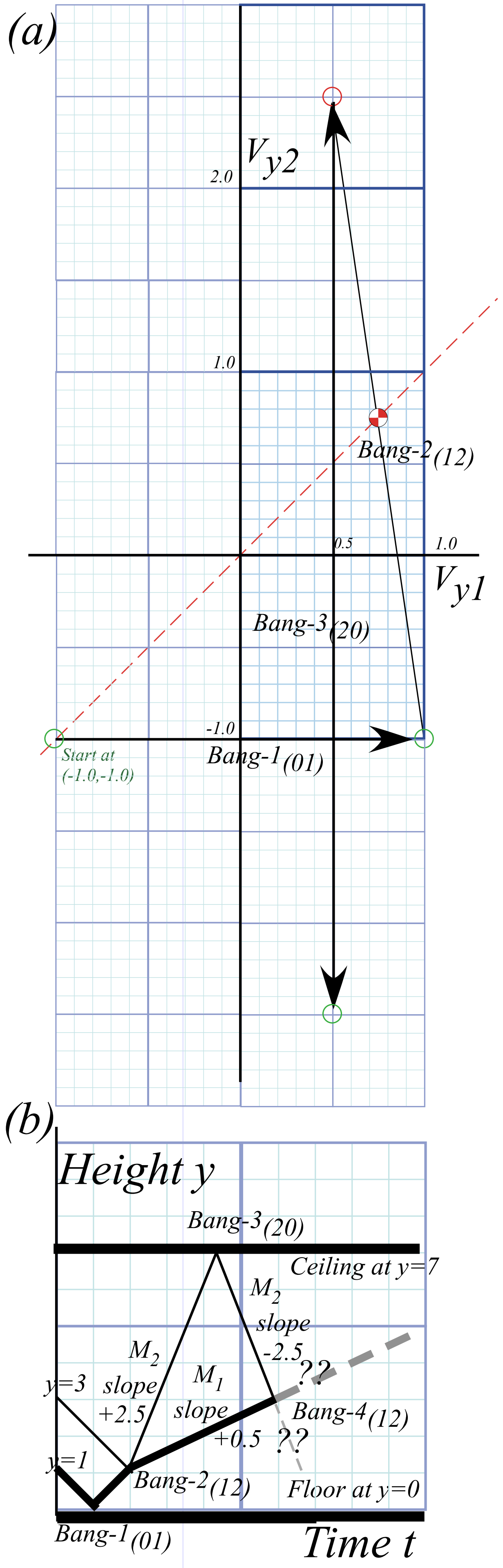
Fig. 4.6a-b
in Unit 1



initial conditions $y_1(0)$ …and $y_2(0)$

Bang-$1_{(01)}$ Bounces (-1,-1) to (+1,-1)

Geometric "Integration" (Converting Velocity data to Spacetime)



Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2}M_1V_1^2 + \frac{1}{2}M_2V_2^2 = \frac{1}{2} + \frac{7}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

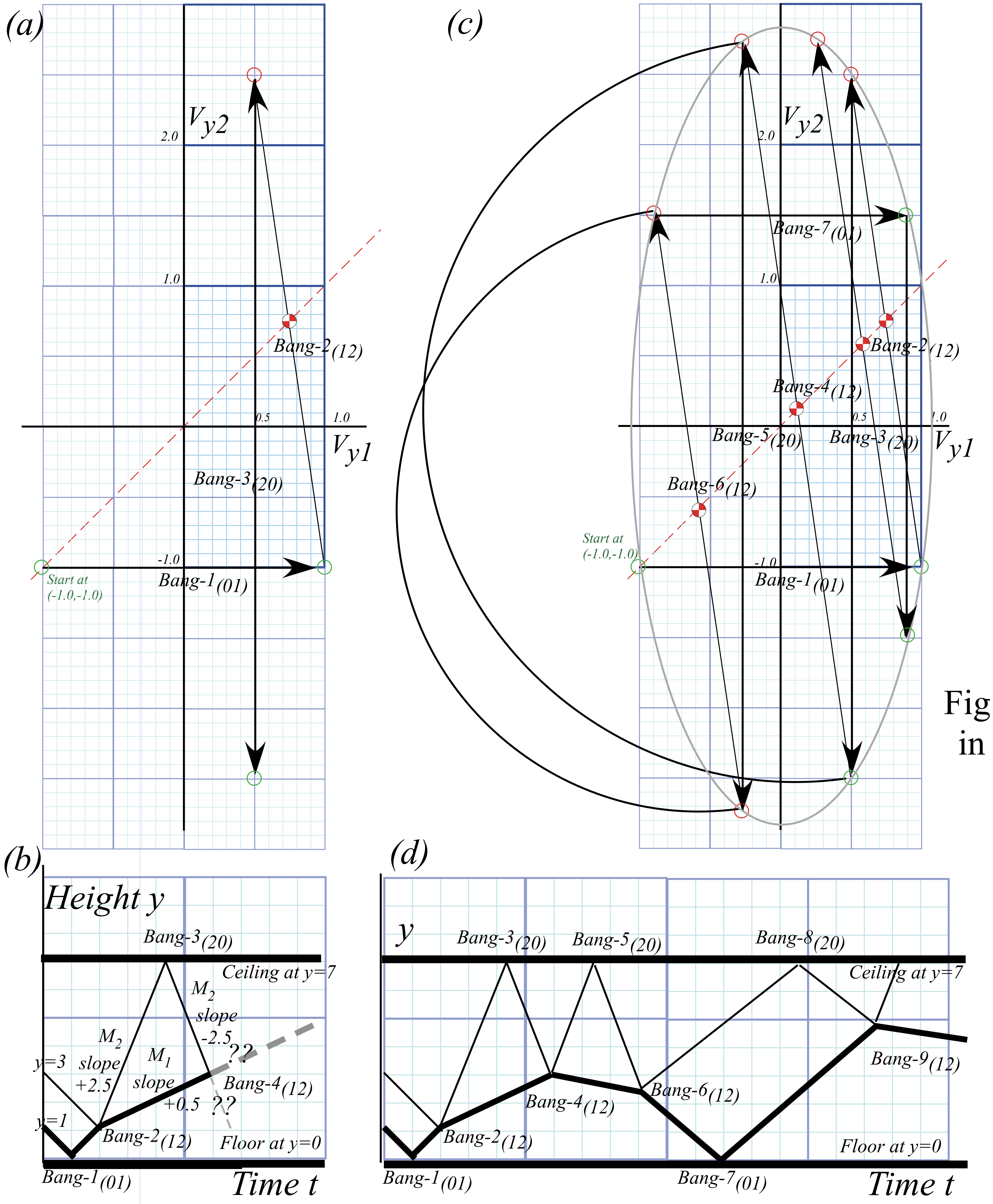


Fig. 4.7a-d
in Unit 1

Geometric "Integration" (Converting Velocity data to Spacetime)

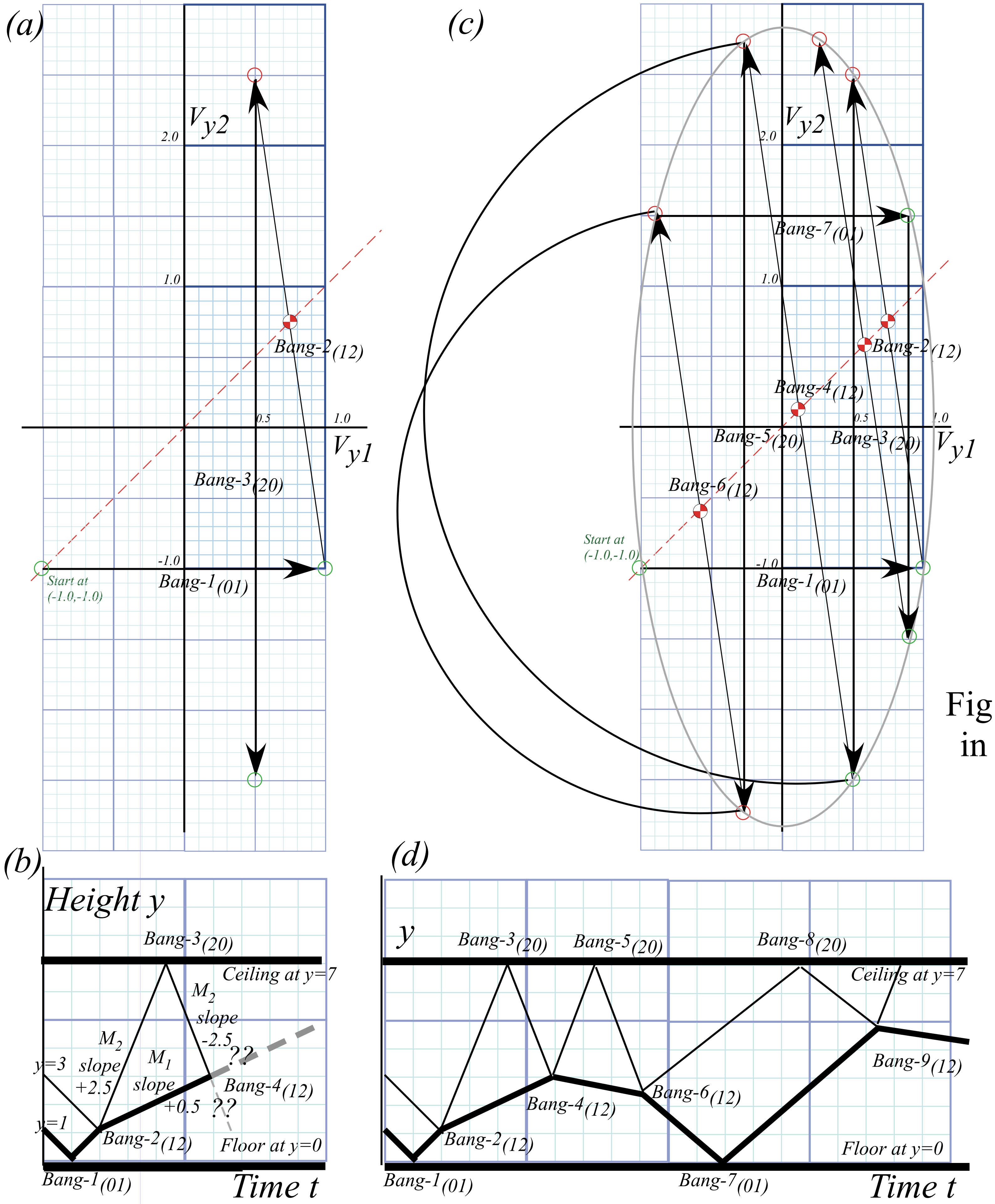


Fig. 4.7a-d
in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2}M_1V_1^2 + \frac{1}{2}M_2V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE/M_1} + \frac{V_2^2}{2KE/M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1}$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/M_2}$$

Geometric "Integration" (Converting Velocity data to Spacetime)

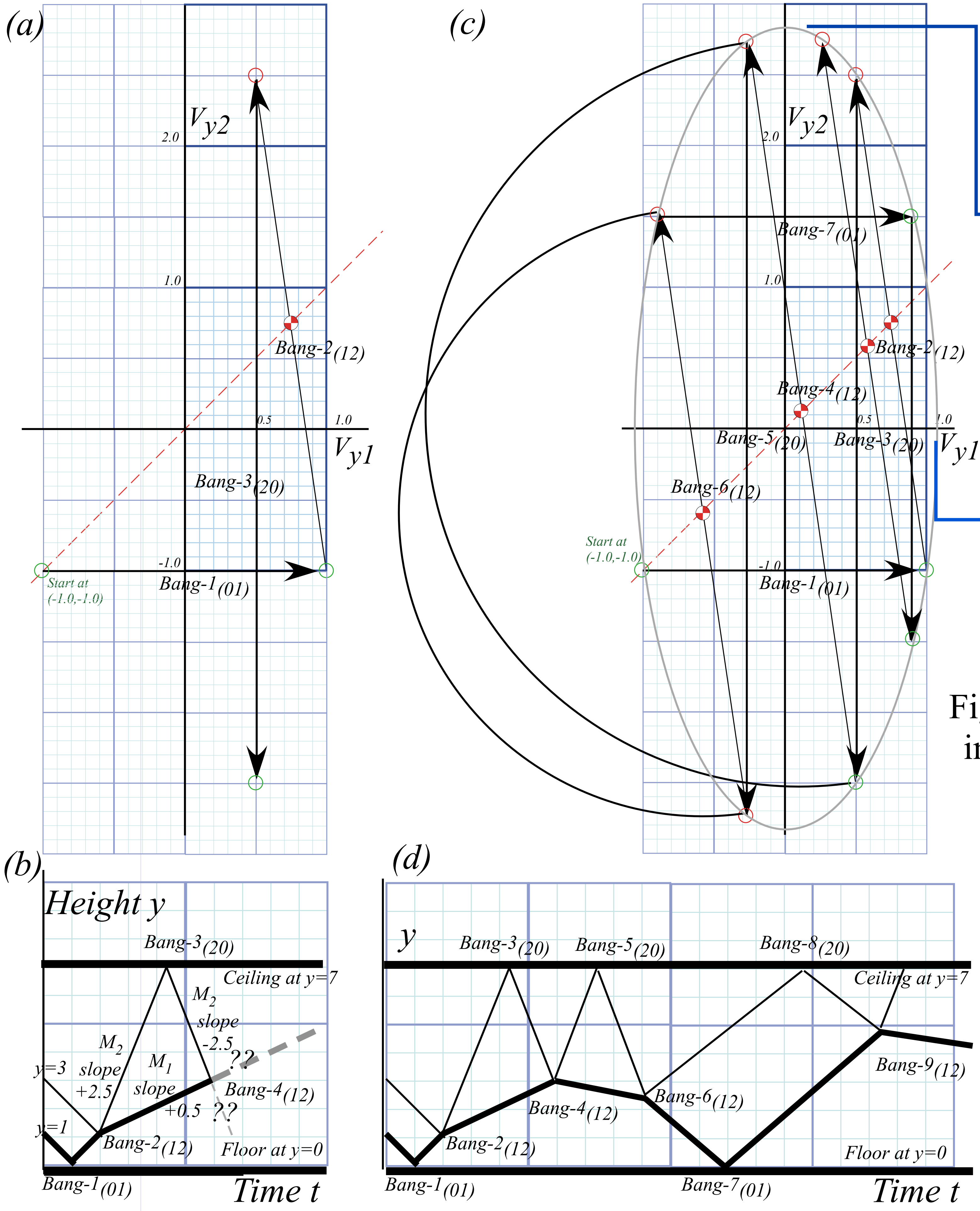


Fig. 4.7a-d
in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2}M_1V_1^2 + \frac{1}{2}M_2V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE/M_1} + \frac{V_2^2}{2KE/M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1} = \sqrt{2KE/7} = \sqrt{8/7} = 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/M_2} = \sqrt{2KE/1} = \sqrt{8/1} = 2.83$$

Geometric "Integration" (Converting Velocity data to Spacetime)

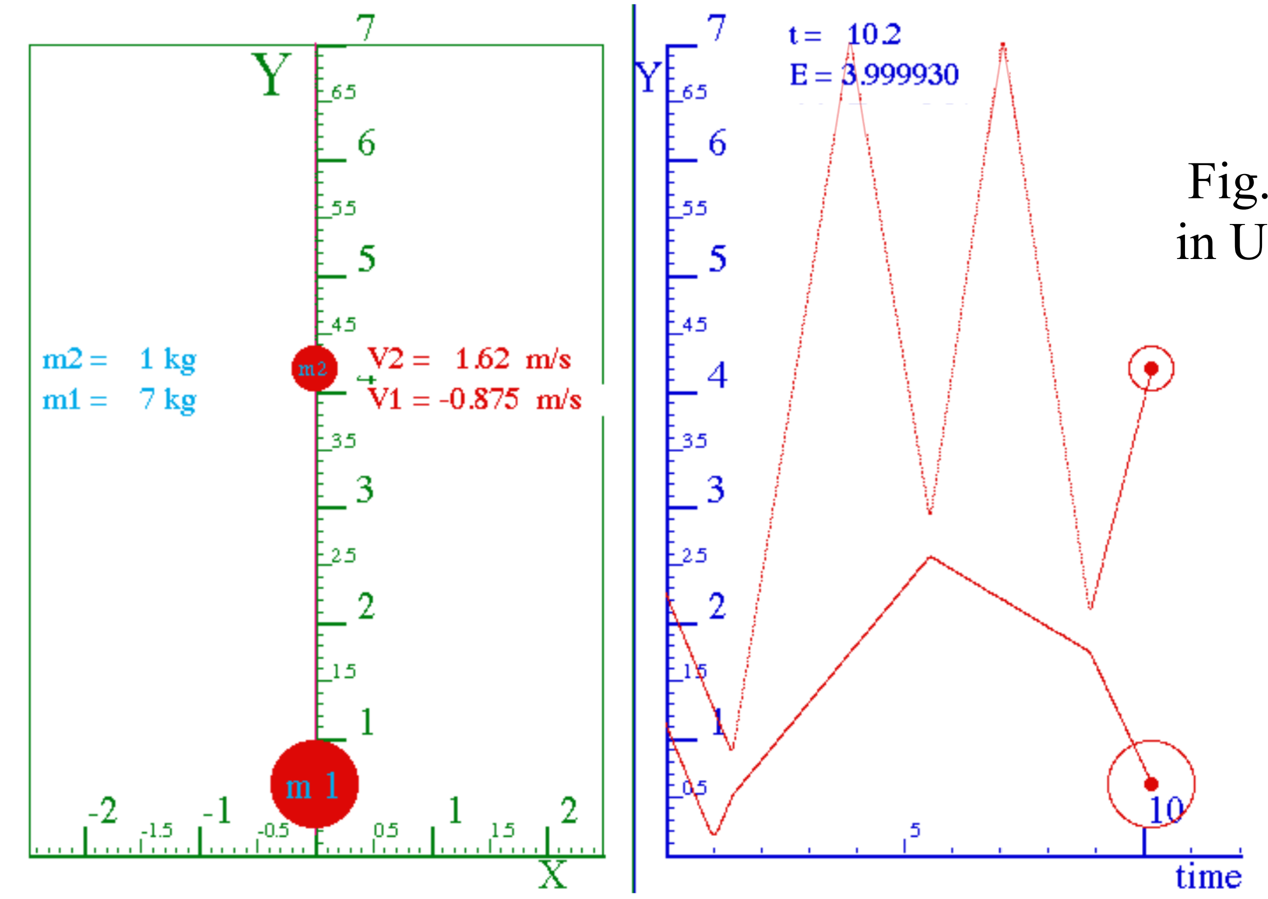


Fig. 4.8
in Unit 1

BounceIt Superball Collision Web Simulator:
M$_1$=70, M$_2$=10 with Newtonian time plot

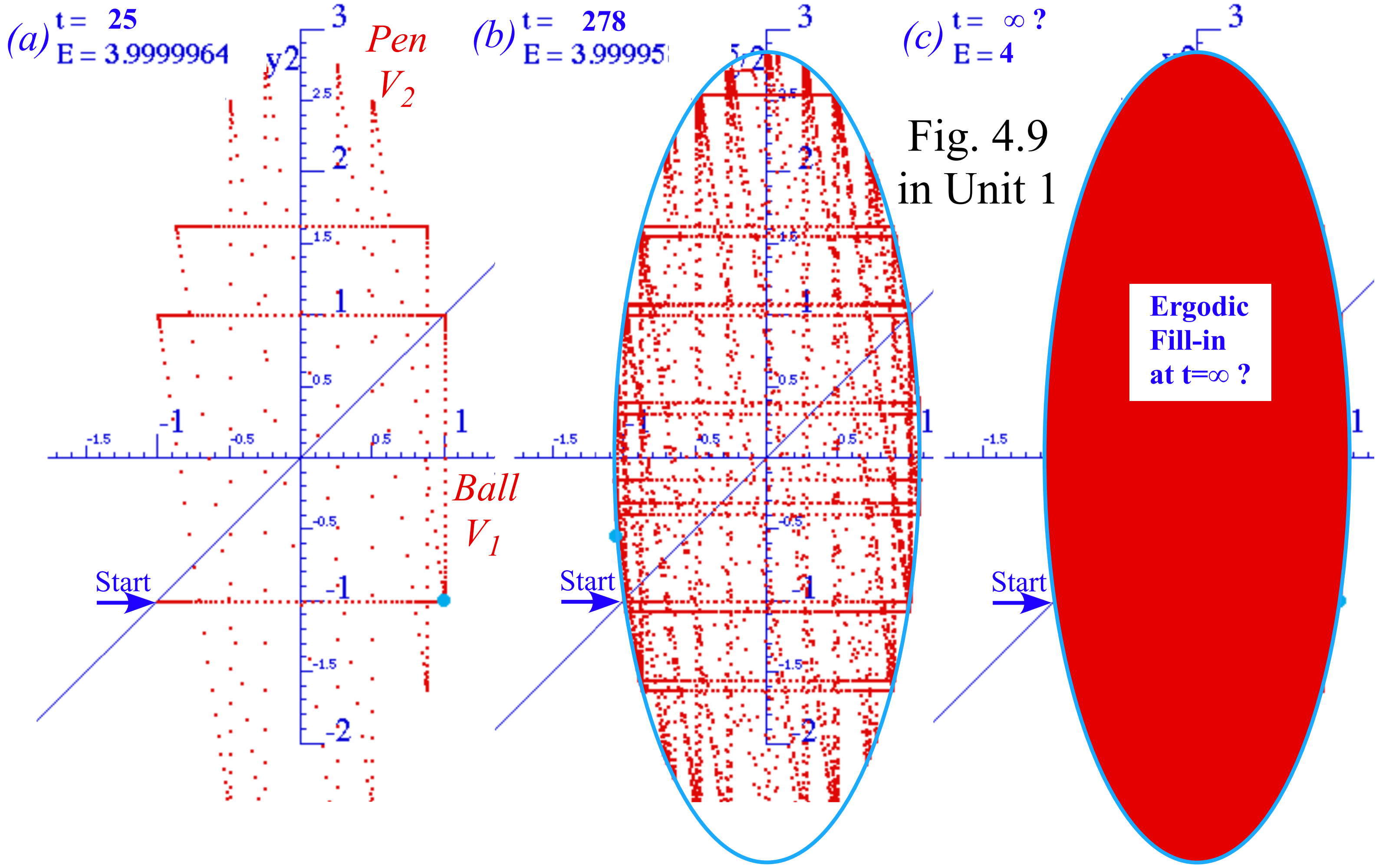


Fig. 4.9
in Unit 1

BounceIt Superball Collision Web Simulator:
M$_1$=70, M$_2$=10 with V$_2$ vs V$_1$ plot

Geometry of X2 launcher bouncing in box (gravity-free)

- *Independent Bounce Model (IBM)*
- *Geometric optimization and range-of-motion calculation(t)*
- *Integration of (V_1,V_2) data to space-time plots $(y_1(t),t)$ and $(y_2(t),t)$ plots*
- *Integration of (V_1,V_2) data to space-space plots (y_1, y_2)* *Examples $(M_1=7, M_2=1)$ and $(M_1=49, M_2=1)$*

Geometric "Integration" (Converting Velocity data to Space-space trajectory)

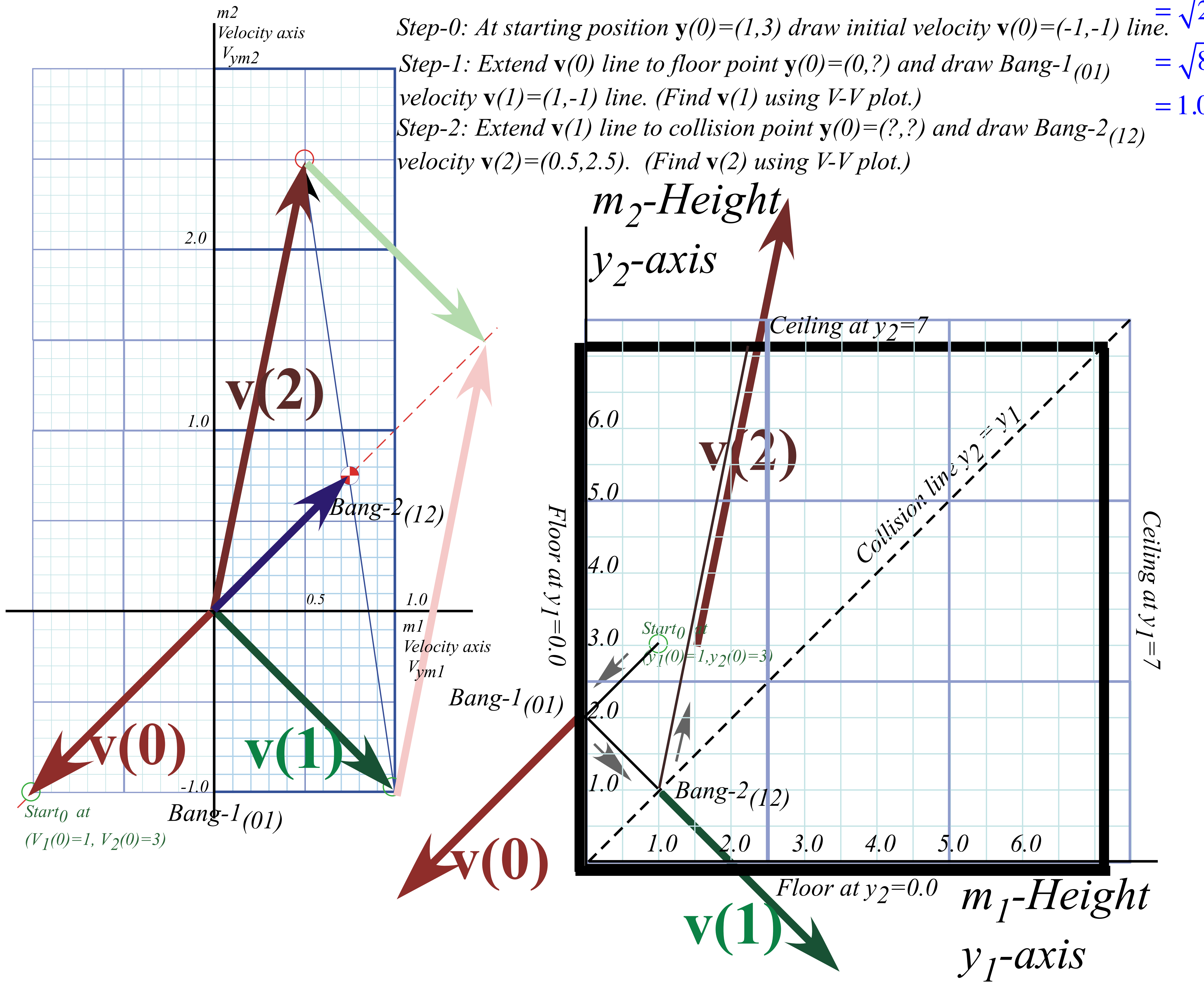
Fig. 4.11
in Unit 1

Ellipse radius 1	*Ellipse radius 2*
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/M_2}$
$= \sqrt{2KE/7}$	$= \sqrt{2KE/1}$
$= \sqrt{8/7}$	$= \sqrt{8/1}$
$= 1.07$	$= 2.83$

Step-0: At starting position **y**(0)=(1,3) *draw initial velocity* **v**(0)=(-1,-1) *line.*

Step-1: Extend **v**(0) *line to floor point* **y**(0)=(0,?) *and draw Bang-1*$_{(01)}$ *velocity* **v**(1)=(1,-1) *line. (Find* **v**(1) *using V-V plot.)*

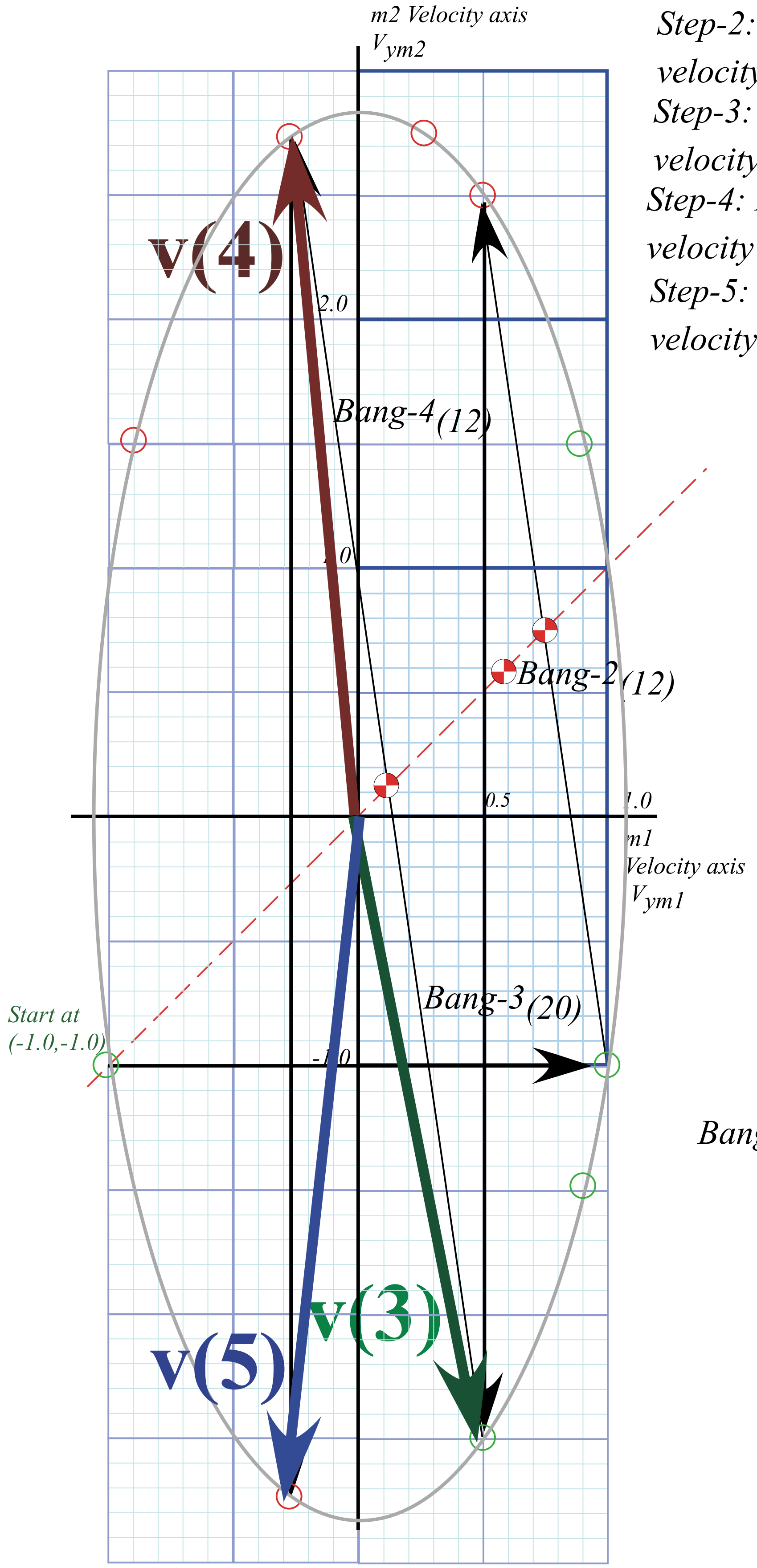
Step-2: Extend **v**(1) *line to collision point* **y**(0)=(?,?) *and draw Bang-2*$_{(12)}$ *velocity* **v**(2)=(0.5,2.5). *(Find* **v**(2) *using V-V plot.)*



Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

Ellipse radius 1	*Ellipse radius 2*
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/M_2}$
$= \sqrt{2KE/7}$	$= \sqrt{2KE/1}$
$= \sqrt{8/7}$	$= \sqrt{8/1}$
$= 1.07$	$= 2.83$

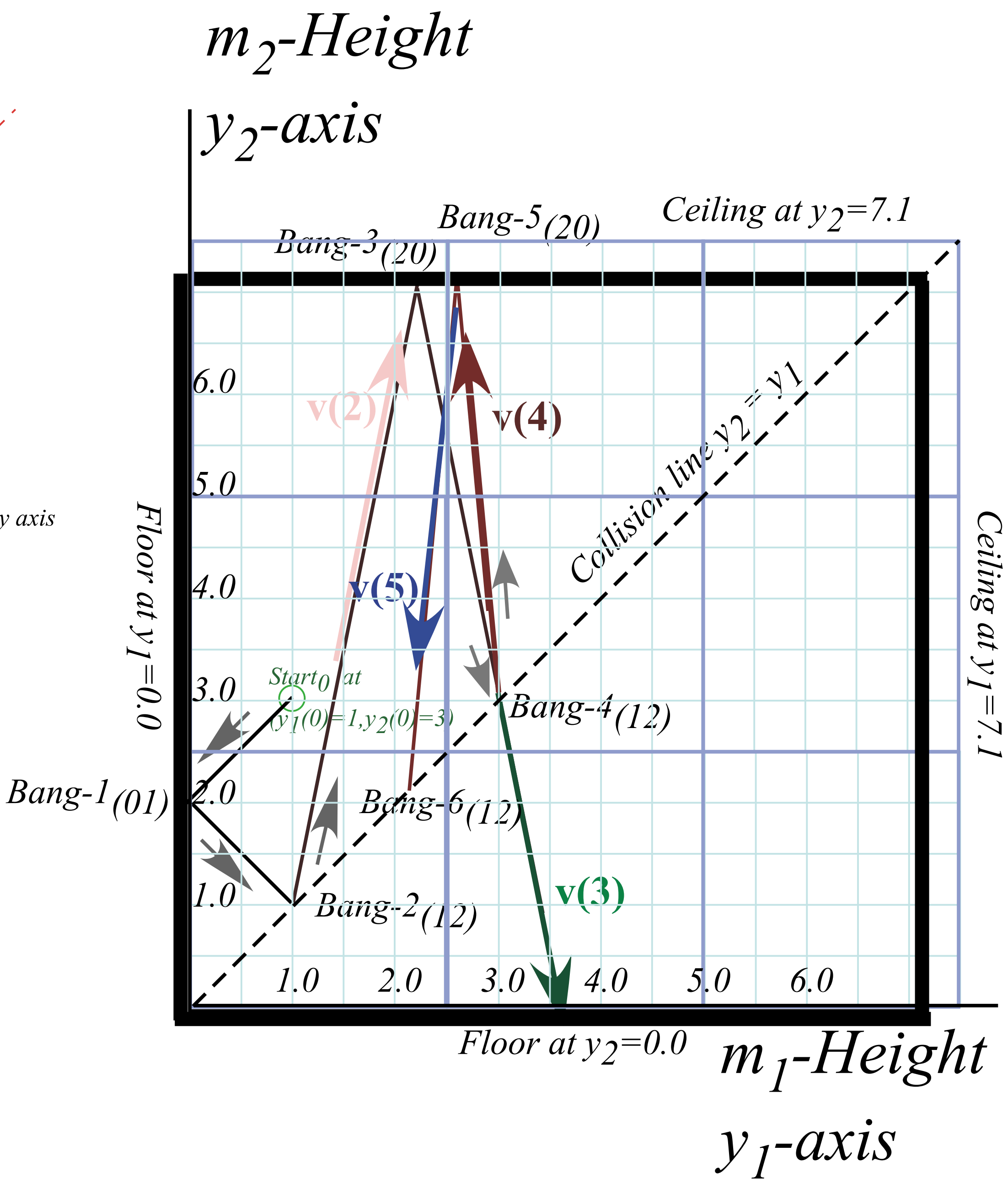


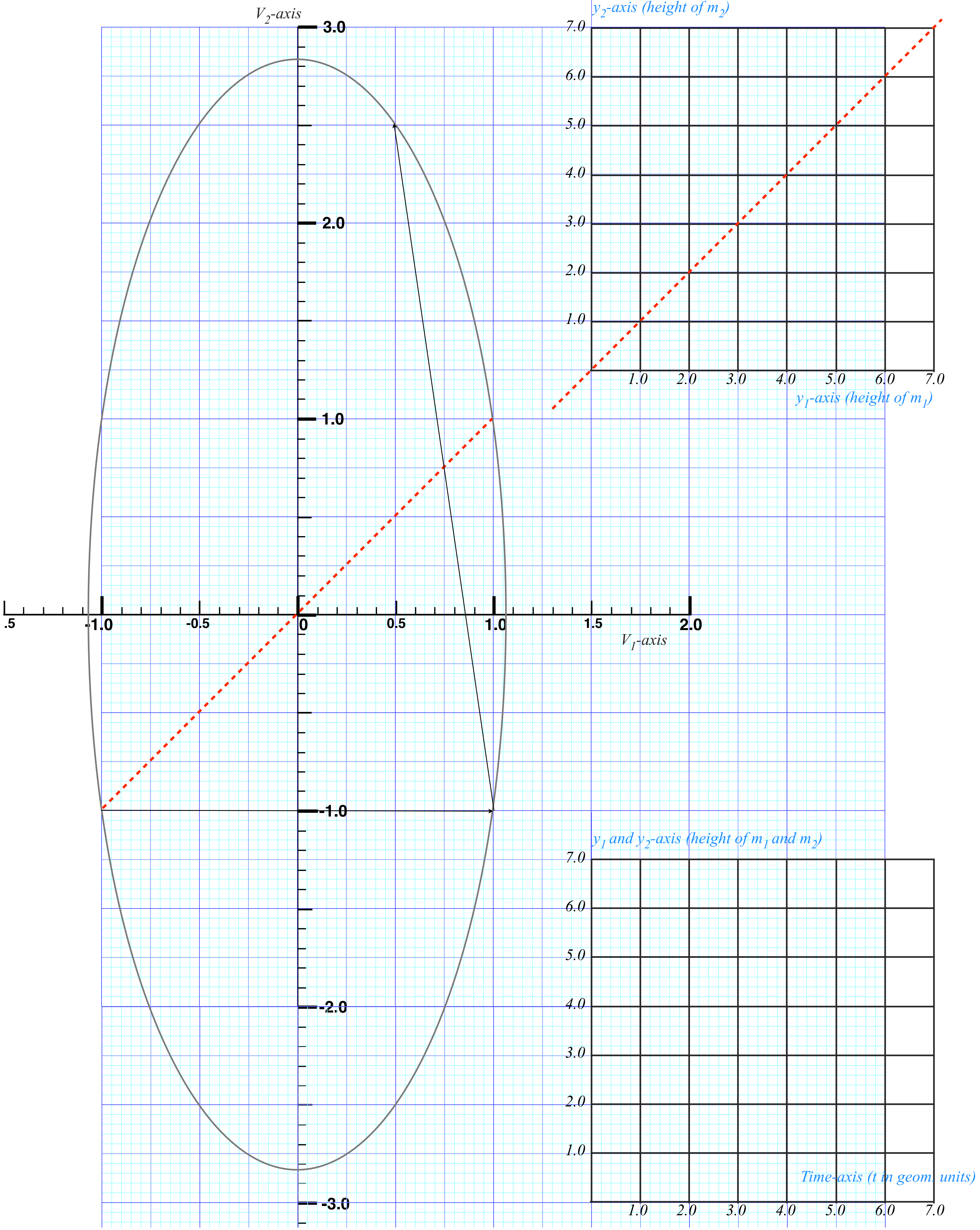
Step-2: Extend **v**(2) *line to ceiling point* **y**(3)=(?,7.1) *and draw Bang-3*$_{(20)}$ *velocity* **v**(3)=(1,-1) *line. (Find* **v**(3) *using V-V plot.)*

Step-3: Extend **v**(3) *line to collision point* **y**(4)=(?,?) *and draw Bang-4*$_{(12)}$ *velocity* **v**(4)=(0.5,2.5). *(Find* **v**(4) *using V-V plot.)*

Step-4: Extend **v**(4) *line to ceiling point* **y**(4)=(?,7.1) *and draw Bang-5*$_{(20)}$ *velocity* **v**(5)=(1,-1) *line. (Find* **v**(5) *using V-V plot.)*

Step-5: Extend **v**(5) *line to collision point* **y**(6)=(?,?) *and draw Bang-6*$_{(12)}$ *velocity* **v**(6)=(0.5,2.5). *(Find* **v**(6) *using V-V plot.)*





Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1}$$
$$= \sqrt{2KE/7}$$
$$= \sqrt{8/7}$$
$$= 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/M_2}$$
$$= \sqrt{2KE/1}$$
$$= \sqrt{8/1}$$
$$= 2.83$$

Example with masses: m_1=49 and m_2=1

Geometric "Integration" (Converting Velocity data to Space-time trajectory)

Example with masses: $m_1=49$ and $m_2=1$

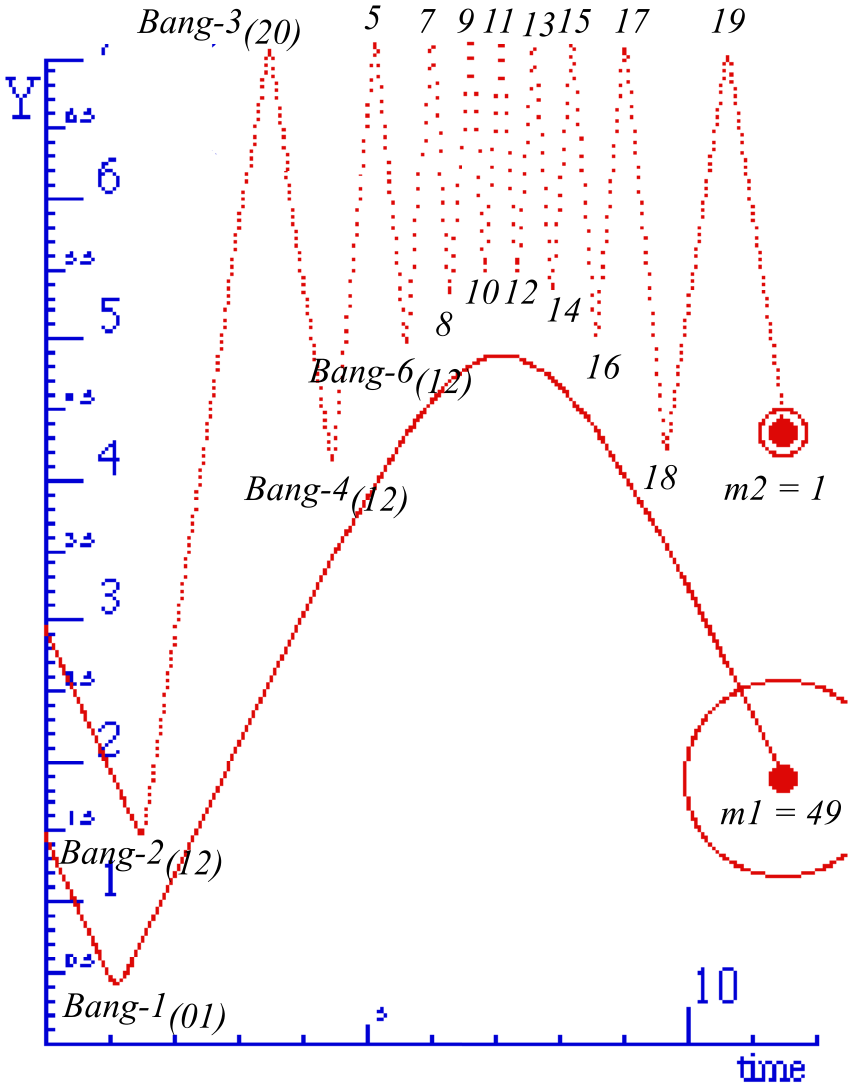
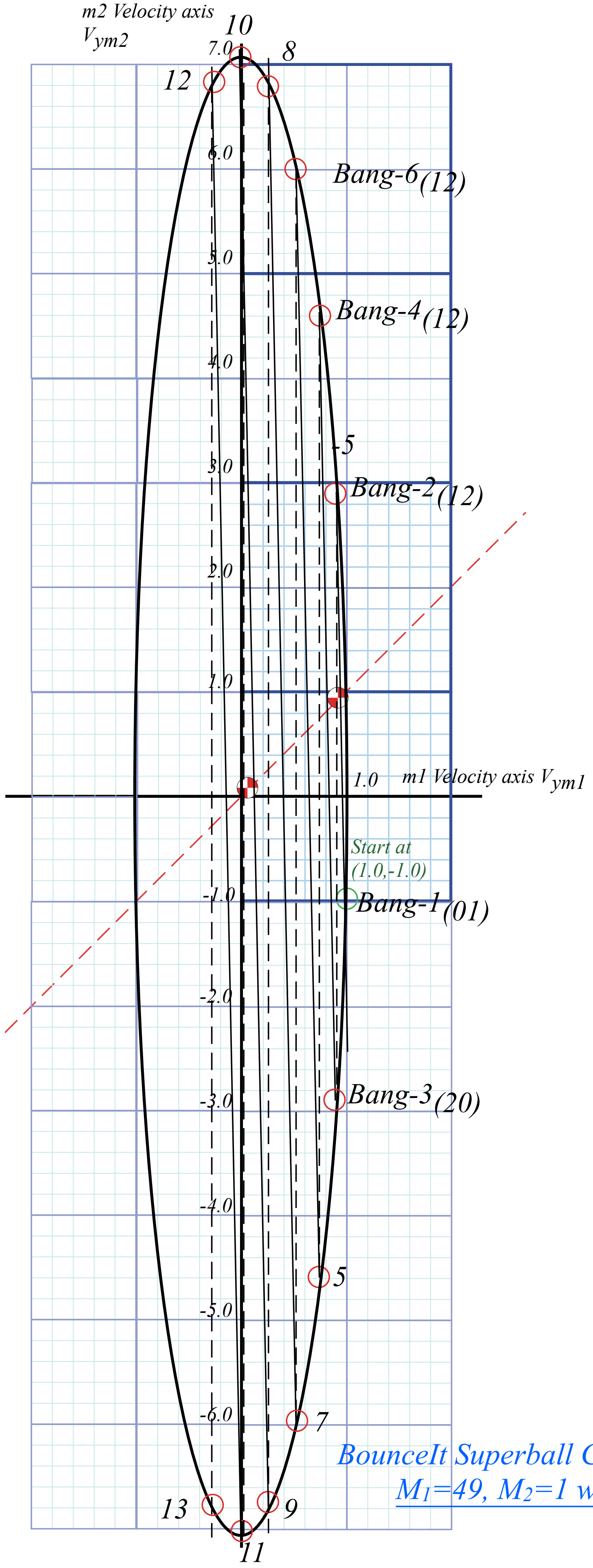


Fig. 5.1
in Unit 1

BounceIt Superball Collision Web Simulator:
$M_1=49$, $M_2=1$ with Newtonian time plot

BounceIt Superball Collision Web Simulator:
$M_1=49$, $M_2=1$ with V_2 vs V_1 plot

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: m_1=49 and m_2=1

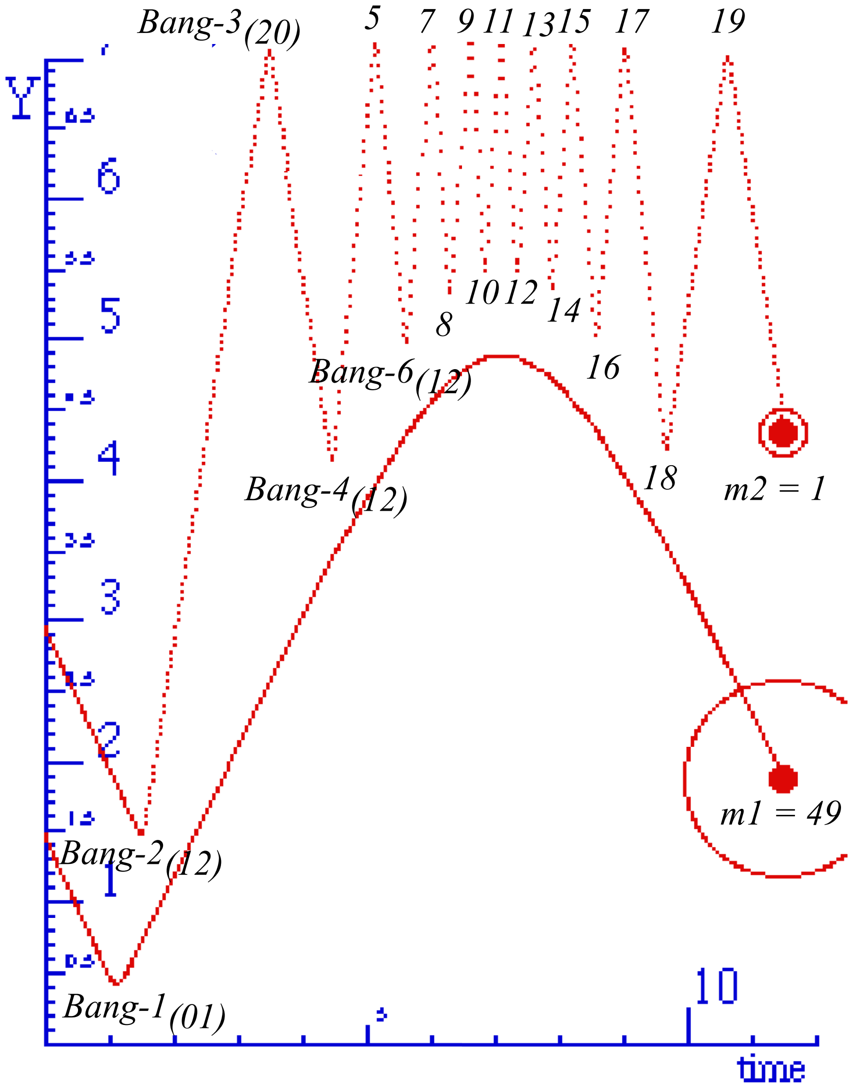


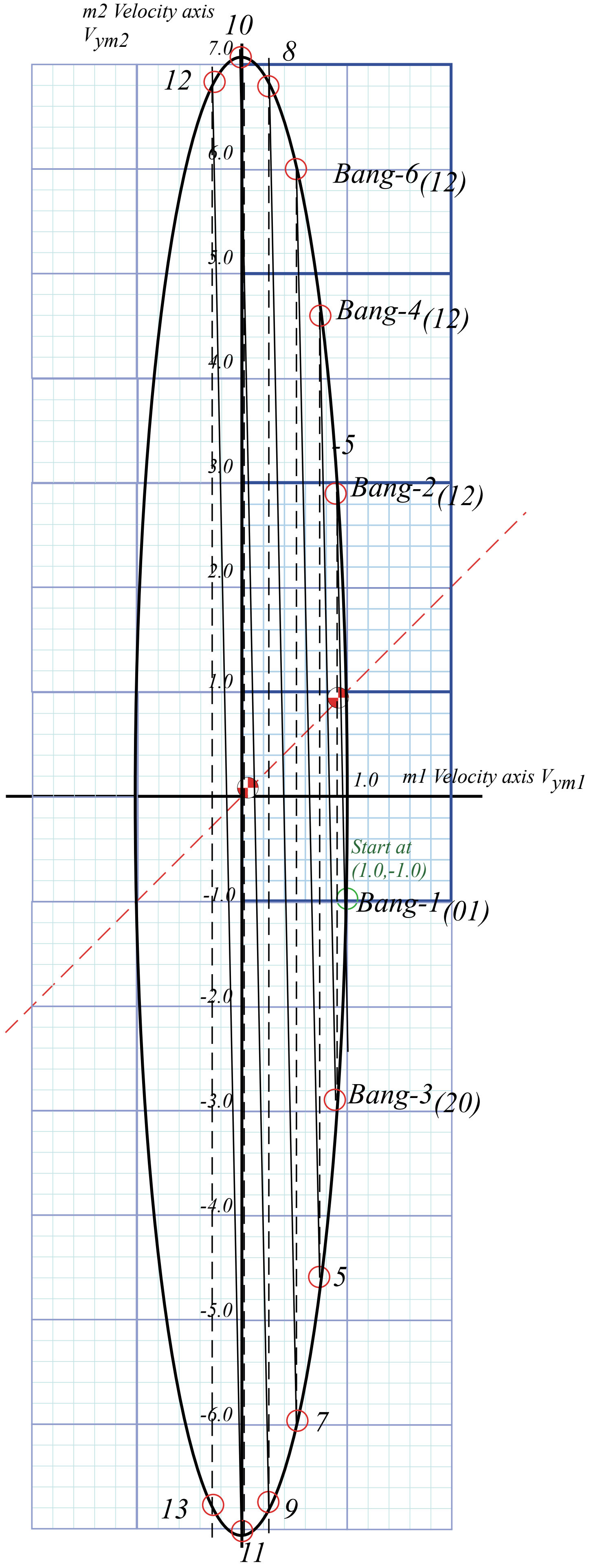
Fig. 5.1
in Unit 1

BounceIt Superball Collision Web Simulator:
M_1=49, M_2=1 with Newtonian time plot

BounceIt Superball Collision Web Simulator:
M_1=49, M_2=1 with V_2 vs V_1 plot

Geometric "Integration" (Converting Velocity data to Space-time trajectory)

Example with masses: m_1=49 and m_2=1



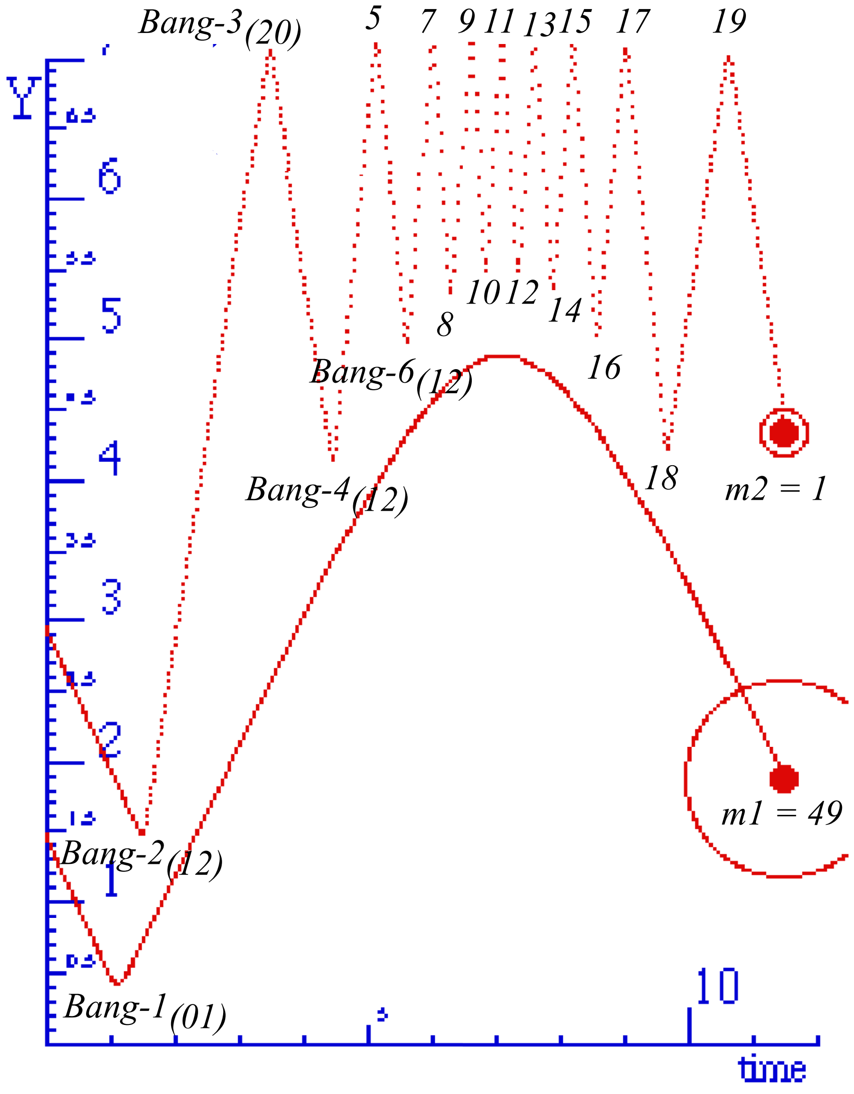


Fig. 5.1
in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1	*Ellipse radius 2*
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/m_2}$
$= \sqrt{2KE/49}$	$= \sqrt{2KE/1}$
$= \sqrt{50/49}$	$= \sqrt{50/1}$
$= 1.01$	$= 7.07$

Multiple collisions calculated by matrix operator products

→ *Matrix or tensor algebra of 1-D 2-body collisions*

"Mass-bang" matrix **M**, *"Floor-bang" matrix* **F**, *"Ceiling-bang" matrix* **C**.

Geometry and algebra of "ellipse-Rotation" group product: **R**= **C**•**M**

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $\mathbf{V}^{COM}=\frac{\mathbf{V}^{FIN}+\mathbf{V}^{IN}}{2}=\frac{m_1v_1+m_2v_2}{m_1+m_2}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}V^{COM}\\V^{COM}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}v_1^{FIN}\\v_2^{FIN}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}v_1^{IN}\\v_2^{IN}\end{pmatrix}$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $$\mathbf{V}^{COM}=\frac{\mathbf{V}^{FIN}+\mathbf{V}^{IN}}{2}=\frac{m_1v_1+m_2v_2}{m_1+m_2}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}V^{COM}\\V^{COM}\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}v_1^{FIN}\\v_2^{FIN}\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}v_1^{IN}\\v_2^{IN}\end{pmatrix}$$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

$$\begin{pmatrix}v_1^{FIN}\\v_2^{FIN}\end{pmatrix}=\begin{pmatrix}2V^{COM}-v_1^{IN}\\2V^{COM}-v_2^{IN}\end{pmatrix}=$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2\dfrac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2\dfrac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN}+\mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1+m_2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}V^{COM}\\V^{COM}\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}v_1^{FIN}\\v_2^{FIN}\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}v_1^{IN}\\v_2^{IN}\end{pmatrix}$$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix}v_1^{FIN}\\v_2^{FIN}\end{pmatrix} = \begin{pmatrix}2V^{COM}-v_1^{IN}\\2V^{COM}-v_2^{IN}\end{pmatrix} = \begin{pmatrix}2\dfrac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_1^{IN}\\[2ex]2\dfrac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_2^{IN}\end{pmatrix} = \frac{\begin{pmatrix}m_1v_1^{IN}-m_2v_1^{IN}+2m_2v_2^{IN}\\2m_1v_1^{IN}+m_2v_2^{IN}-m_1v_2^{IN}\end{pmatrix}}{m_1+m_2} = \frac{\begin{pmatrix}m_1-m_2 & 2m_2\\2m_1 & m_2-m_1\end{pmatrix}\begin{pmatrix}v_1^{IN}\\v_2^{IN}\end{pmatrix}}{m_1+m_2}$$

$$= \begin{pmatrix}\dfrac{2m_1v_1^{IN}+2m_2v_2^{IN}-m_1v_1^{IN}-m_2v_1^{IN}}{m_1+m_2}\\[2ex]\dfrac{2m_1v_1^{IN}+2m_2v_2^{IN}-m_1v_2^{IN}-m_2v_2^{IN}}{m_1+m_2}\end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM}=\frac{V^{FIN}+V^{IN}}{2}=\frac{m_1v_1+m_2v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} 2V^{COM}-v_1^{IN} \\ 2V^{COM}-v_2^{IN} \end{pmatrix}=\begin{pmatrix} 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_1^{IN} \\ 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_2^{IN} \end{pmatrix}=\frac{\begin{pmatrix} m_1v_1^{IN}-m_2v_1^{IN}+2m_2v_2^{IN} \\ 2m_1v_1^{IN}+m_2v_2^{IN}-m_1v_2^{IN} \end{pmatrix}}{m_1+m_2}=\frac{\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Quiz question about linear solution

Linear formula $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$ gives just *one* solution to *quadratic* collision equations.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} \frac{m_1-m_2}{m_1+m_2} & \frac{2m_2}{m_1+m_2} \\ \frac{2m_1}{m_1+m_2} & \frac{m_2-m_1}{m_1+m_2} \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM}=\frac{V^{FIN}+V^{IN}}{2}=\frac{m_1v_1+m_2v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} 2V^{COM}-v_1^{IN} \\ 2V^{COM}-v_2^{IN} \end{pmatrix}=\begin{pmatrix} 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_1^{IN} \\ 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_2^{IN} \end{pmatrix}=\frac{\begin{pmatrix} m_1v_1^{IN}-m_2v_1^{IN}+2m_2v_2^{IN} \\ 2m_1v_1^{IN}+m_2v_2^{IN}-m_1v_2^{IN} \end{pmatrix}}{m_1+m_2}=\frac{\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Quiz question about linear solution

Linear formula $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$ gives just *one* solution to *quadratic* collision equations.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} \frac{m_1-m_2}{m_1+m_2} & \frac{2m_2}{m_1+m_2} \\ \frac{2m_1}{m_1+m_2} & \frac{m_2-m_1}{m_1+m_2} \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Q: What is the *second* solution and to what simple process would it correspond?

Example with friction

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

→ *"Mass-bang" matrix* **M**, *"Floor-bang" matrix* **F**, *"Ceiling-bang" matrix* **C**.

Geometry and algebra of "ellipse-Rotation" group product: **R**= **C**•**M**

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN}+V^{IN}}{2} = \frac{m_1v_1+m_2v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2} - v_1^{IN} \\ 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1v_1^{IN} - m_2v_1^{IN} + 2m_2v_2^{IN} \\ 2m_1v_1^{IN} + m_2v_2^{IN} - m_1v_2^{IN} \end{pmatrix}}{m_1+m_2} = \frac{\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM}=\frac{V^{FIN}+V^{IN}}{2}=\frac{m_1 v_1+m_2 v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} 2V^{COM}-v_1^{IN} \\ 2V^{COM}-v_2^{IN} \end{pmatrix}=\begin{pmatrix} 2\frac{m_1 v_1^{IN}+m_2 v_2^{IN}}{m_1+m_2}-v_1^{IN} \\ 2\frac{m_1 v_1^{IN}+m_2 v_2^{IN}}{m_1+m_2}-v_2^{IN} \end{pmatrix}=\frac{\begin{pmatrix} m_1 v_1^{IN}-m_2 v_1^{IN}+2m_2 v_2^{IN} \\ 2m_1 v_1^{IN}+m_2 v_2^{IN}-m_1 v_2^{IN} \end{pmatrix}}{m_1+m_2}=\frac{\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM}=\frac{V^{FIN}+V^{IN}}{2}=\frac{m_1v_1+m_2v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN}=\mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix}=\begin{pmatrix} 2V^{COM}-v_1^{IN} \\ 2V^{COM}-v_2^{IN} \end{pmatrix}=\begin{pmatrix} 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_1^{IN} \\ 2\frac{m_1v_1^{IN}+m_2v_2^{IN}}{m_1+m_2}-v_2^{IN} \end{pmatrix}=\frac{\begin{pmatrix} m_1v_1^{IN}-m_2v_1^{IN}+2m_2v_2^{IN} \\ 2m_1v_1^{IN}+m_2v_2^{IN}-m_1v_2^{IN} \end{pmatrix}}{m_1+m_2}=\frac{\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M}=\begin{pmatrix} \frac{m_1-m_2}{m_1+m_2} & \frac{2m_2}{m_1+m_2} \\ \frac{2m_1}{m_1+m_2} & \frac{m_2-m_1}{m_1+m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \dfrac{V^{FIN}+V^{IN}}{2} = \dfrac{m_1 v_1 + m_2 v_2}{m_1+m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M}\bullet\mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2\dfrac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1+m_2} - v_1^{IN} \\ 2\dfrac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1+m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1+m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1+m_2}$$

Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \dfrac{m_1 - m_2}{m_1+m_2} & \dfrac{2m_2}{m_1+m_2} \\ \dfrac{2m_1}{m_1+m_2} & \dfrac{m_2 - m_1}{m_1+m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ *and* $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

"Mass-bang" matrix **M**, *"Floor-bang" matrix* **F**, *"Ceiling-bang" matrix* **C**.

→ *Geometry and algebra of "ellipse-Rotation" group product*: **R**= **C•M**

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives $\mathbf{v}^{FIN}$ in terms of $\mathbf{v}^{IN}$...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \bullet \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2\frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2\frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang **F** of m_1:

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang **M** of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang **C** of m_2:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: m_1=49 and m_2=1

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Define *"ellipse-Rotation"* **R** as *group product*: **R**= **C•M**= $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$

$$\left|FIN^9\right\rangle = \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{F} \left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL }(0))}$$

$$\left|FIN^9\right\rangle = \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{F} \left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$\left|FIN^9\right\rangle = \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{F}\left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}$$

"ellipse-Rotation" group product: $\mathbf{R} = \mathbf{C} \bullet \mathbf{M}$

$$\left|FIN^9\right\rangle = \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{C} \bullet \mathbf{M} \bullet \mathbf{F} \left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{\text{(INITIAL (0))}}$$

$$\left|FIN^9\right\rangle = \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{R} \bullet \mathbf{F}\left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}$$

$$= \begin{pmatrix} v_1 = 0.2925 \\ v_2 = -6.768 \end{pmatrix}_{\text{(after Bang-9)}}$$

"ellipse-Rotation" group product: **R**= **C**•**M**

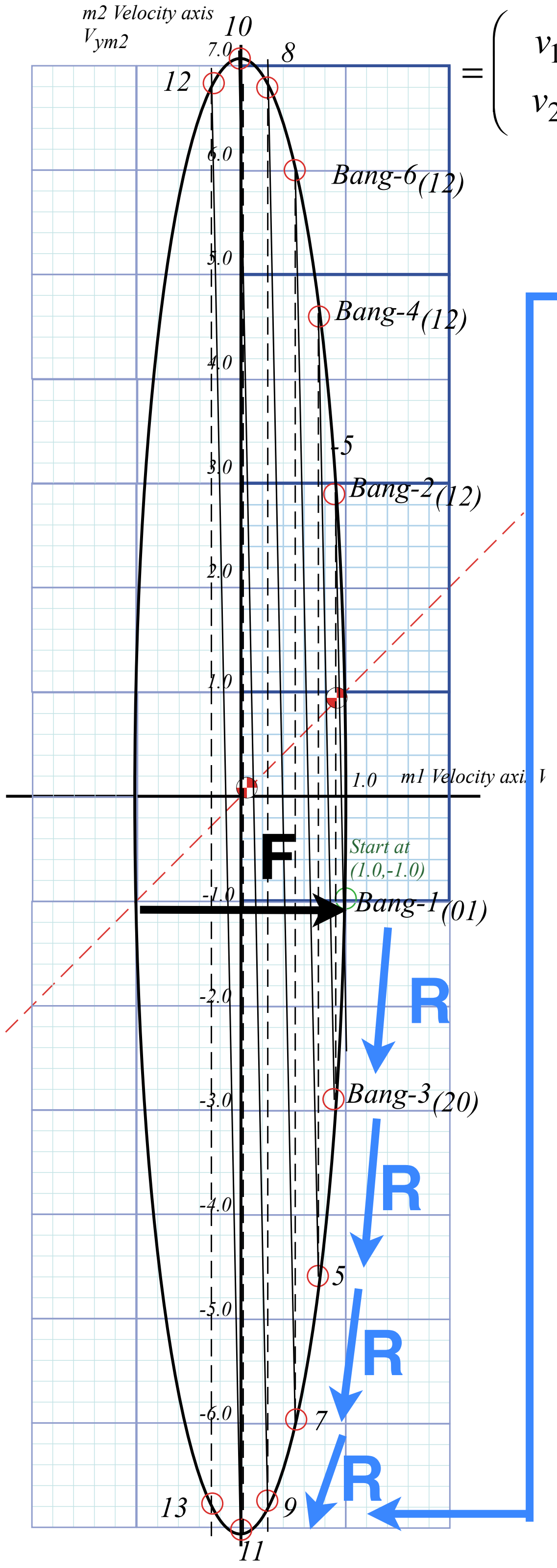
$$|FIN^9\rangle = \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{F}\;|IN^0\rangle$$

$$\begin{pmatrix}v_1^{FIN-9}\\ v_2^{FIN-9}\end{pmatrix} = \underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}\cdot\begin{pmatrix}-1&0\\0&+1\end{pmatrix}\begin{pmatrix}v_1^{IN}=-1\\ v_2^{IN}=-1\end{pmatrix}_{\text{(INITIAL (0))}}$$

$$|FIN^9\rangle = \mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{F}\,|IN^0\rangle$$

$$\begin{pmatrix}v_1^{FIN-9}\\ v_2^{FIN-9}\end{pmatrix} = \begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}v_1=1\\ v_2=-1\end{pmatrix}_{\text{(after Bang-1)}}$$

$$= \begin{pmatrix}v_1=0.2925\\ v_2=-6.768\end{pmatrix}_{\text{(after Bang-9)}}$$



"ellipse-Rotation" group product: **R**= **C**•**M**

Collisions for
mass ratio
m_1:m_2= 49:1

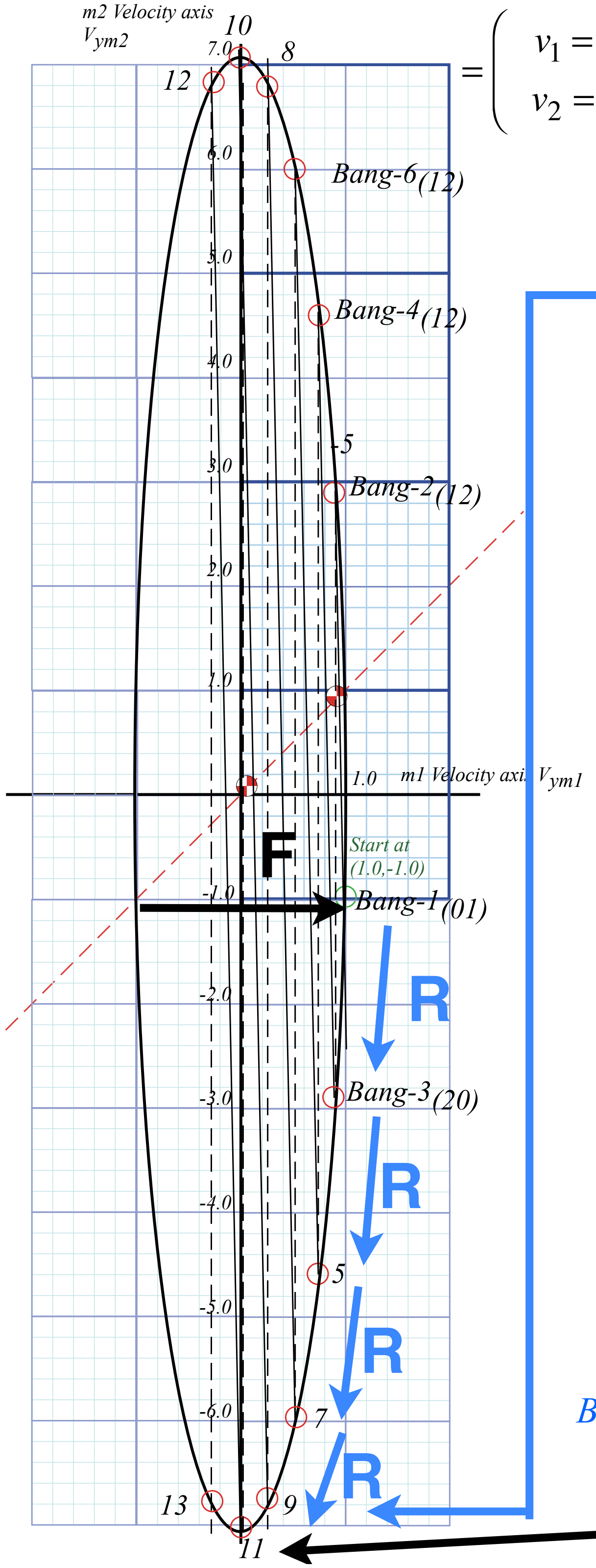
Fig. 5.1a
(revised)

$$\left|FIN^9\right\rangle = \mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{C}\cdot\mathbf{M}\cdot\mathbf{F}\left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9}\\ v_2^{FIN-9}\end{pmatrix} = \underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}_{\mathbf{R}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}_{\mathbf{R}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}_{\mathbf{R}}\cdot\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\1.96&-0.96\end{pmatrix}}_{\mathbf{R}}\cdot\begin{pmatrix}-1&0\\0&+1\end{pmatrix}\begin{pmatrix} v_1^{IN}=-1\\ v_2^{IN}=-1\end{pmatrix}_{\text{(INITIAL (0))}}$$

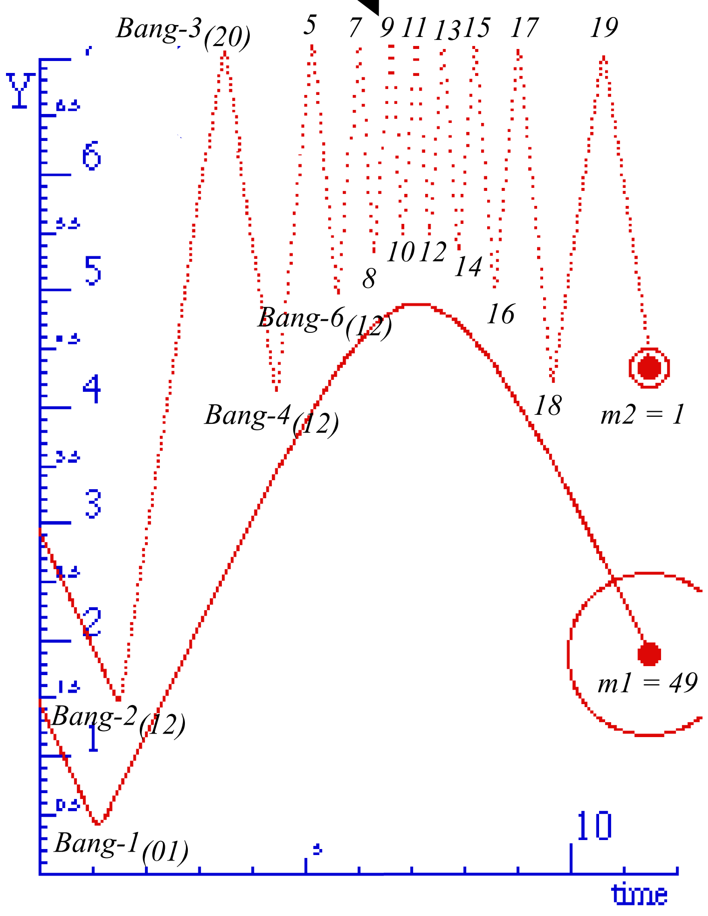
$$\left|FIN^9\right\rangle = \mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{R}\cdot\mathbf{F}\left|IN^0\right\rangle$$

$$\begin{pmatrix} v_1^{FIN-9}\\ v_2^{FIN-9}\end{pmatrix} = \begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix} v_1=1\\ v_2=-1\end{pmatrix}_{\text{(after Bang-1)}}$$



$$= \begin{pmatrix} v_1=0.2925\\ v_2=-6.768\end{pmatrix}_{\text{(after Bang-9)}}$$

"ellipse-Rotation" group product: **R**= **C•M**



$$\begin{pmatrix} v_1^{FIN-11}\\ v_2^{FIN-11}\end{pmatrix} = \begin{pmatrix}0.96&0.04\\-1.96&0.96\end{pmatrix}\cdot\begin{pmatrix} v_1^{FIN-9}\\ v_2^{FIN-9}\end{pmatrix}$$

$$= \begin{pmatrix} v_1=0.0100\\ v_2=-7.071\end{pmatrix}_{\text{(after Bang-11)}}$$

Collisions for
mass ratio
m_1:m_2= 49:1

BounceIt Superball Collision Web Simulator:
M_1=49, M_2=1 with Newtonian time plot

<<Under Construction>>
***Matrix** Collision Web Simulator*
M_1=49, M_2=1 V_2 vs V_1 plot

BounceIt Superball Collision Web Simulator:
M_1=49, M_2=1 with V_2 vs V_1 plot

Fig. 5.1a-b
(revised)

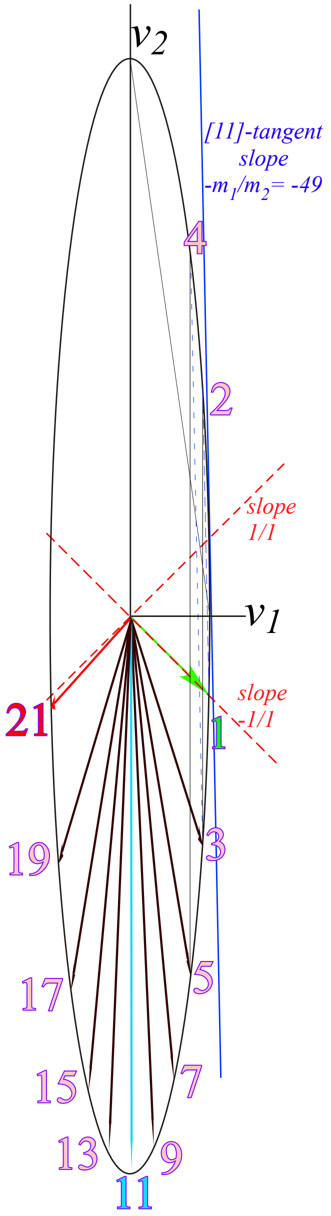
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

- *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mathsf{V}_1^2 + \frac{1}{2} \mathsf{V}_2^2$



Collisions for mass ratio $m_1{:}m_2 = 49{:}1$

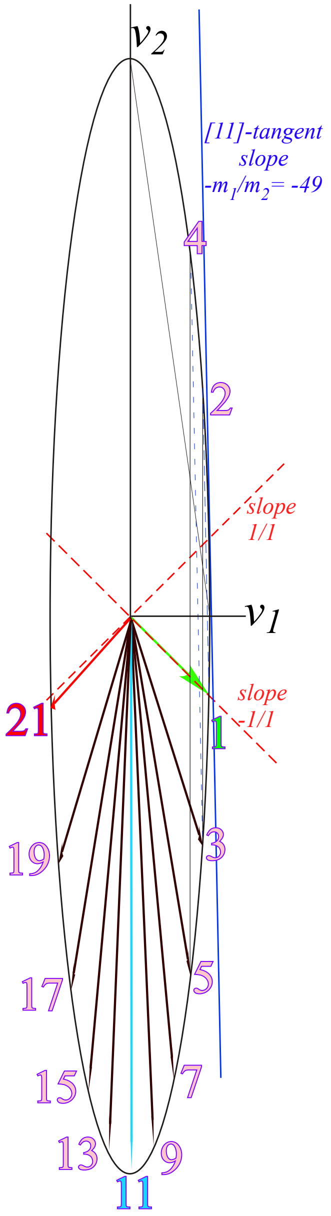
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} \mathsf{V}_1^2 + \tfrac{1}{2} \mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} / \sqrt{m_1} \\ \mathsf{V}_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 / \sqrt{m_1} \\ \mathsf{V}_2 / \sqrt{m_2} \end{pmatrix}$$



Collisions for
mass ratio
m_1:m_2= 49:1

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mathsf{V}_1^2 + \frac{1}{2} \mathsf{V}_2^2$

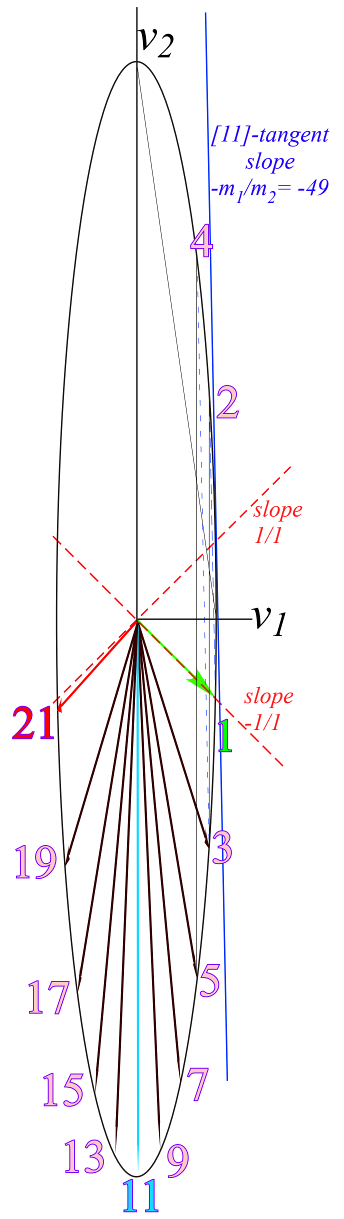
$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} \mathsf{V}_1^{FIN1} / \sqrt{m_1} \\ \mathsf{V}_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 / \sqrt{m_1} \\ \mathsf{V}_2 / \sqrt{m_2} \end{pmatrix}$$

or: $$\begin{pmatrix} \mathsf{V}_1^{FIN1} \\ \mathsf{V}_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{M} \bullet \vec{\mathsf{V}}$$,

or: $$\begin{pmatrix} \mathsf{V}_1^{FIN2} \\ \mathsf{V}_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{C} \bullet \mathsf{M} \bullet \vec{\mathsf{V}}$$



Collisions for mass ratio $m_1{:}m_2 = 49{:}1$

Ellipse rescaling geometry and reflection symmetry analysis

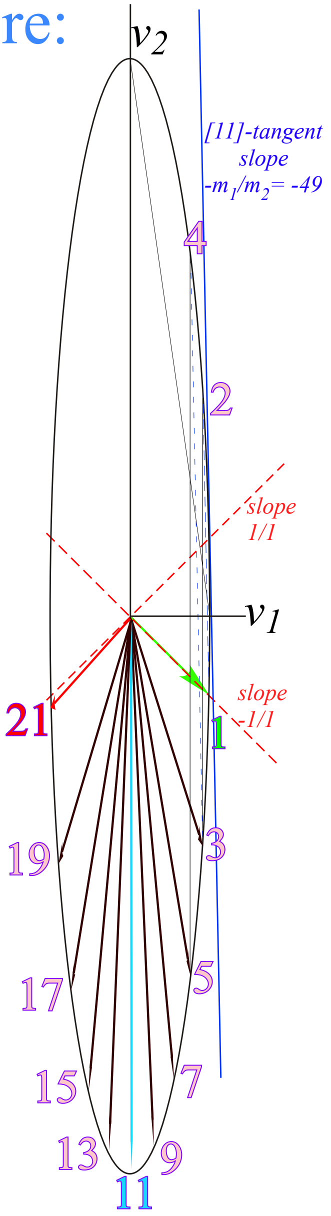
Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}\mathsf{V}_1^2 + \frac{1}{2}\mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} \mathsf{V}_1^{FIN_1}/\sqrt{m_1} \\ \mathsf{V}_2^{FIN_1}/\sqrt{m_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1/\sqrt{m_1} \\ \mathsf{V}_2/\sqrt{m_2} \end{pmatrix}$$

$$\text{or:} \begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{M}\bullet\vec{\mathsf{V}} \ , \quad \text{or:} \begin{pmatrix} \mathsf{V}_1^{FIN_2} \\ \mathsf{V}_2^{FIN_2} \end{pmatrix} = \frac{1}{M}\begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix}\begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{C}\bullet\mathsf{M}\bullet\vec{\mathsf{V}}$$

Then collisions become ***reflections*** $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become ***rotations*** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$



Collisions for mass ratio m_1:m_2= 49:1

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} V_1^2 + \tfrac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} V_1^{FIN1}/\sqrt{m_1} \\ V_2^{FIN1}/\sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1/\sqrt{m_1} \\ V_2/\sqrt{m_2} \end{pmatrix}$$

or:

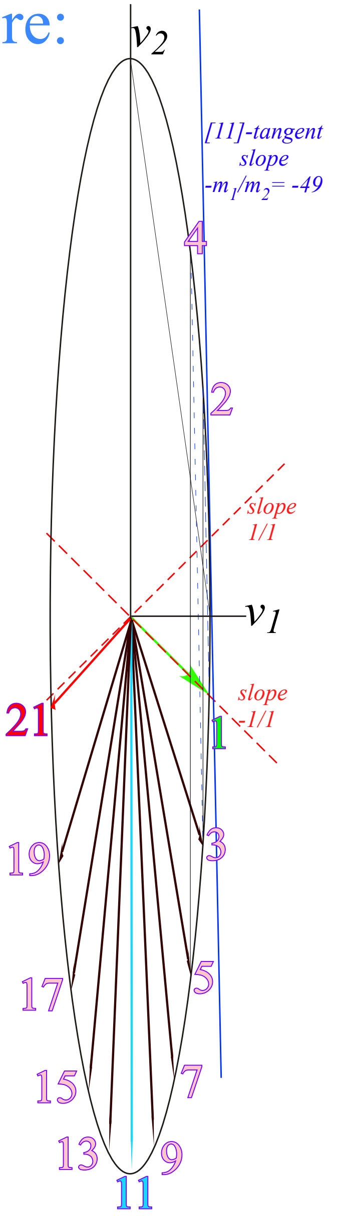
$$\begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \bullet \vec{V} ,$$

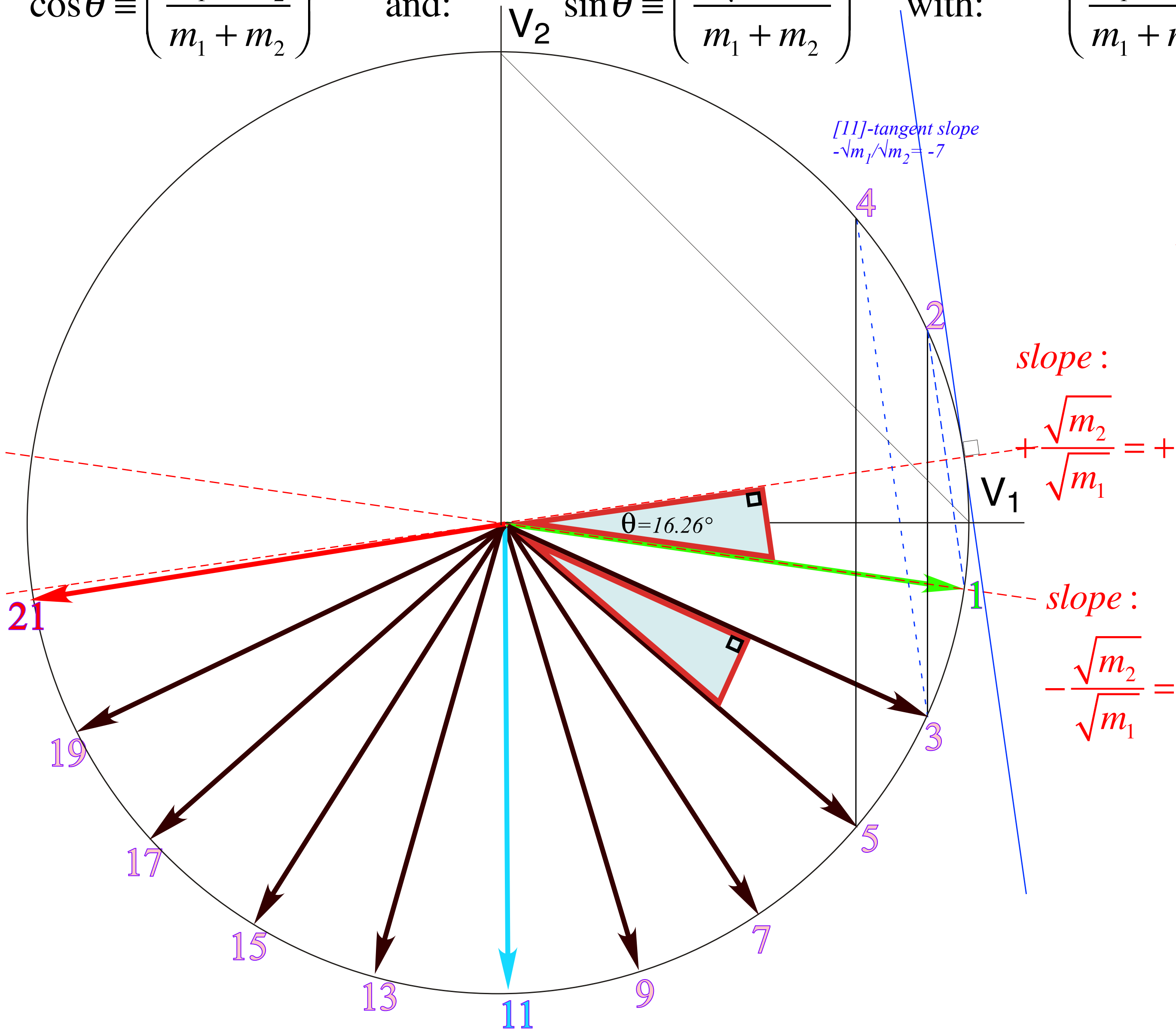
or:

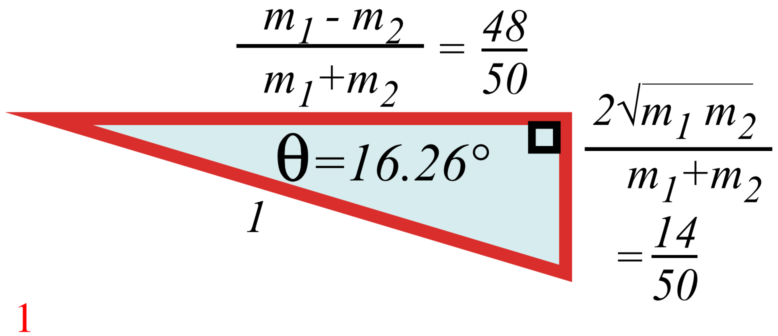
$$\begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \bullet \mathbf{M} \bullet \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$







Collisions for mass ratio $m_1 : m_2 = 49:1$

Fig. 5.2a-c

(revised)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $\mathsf{V}_1 = v_1 \cdot \sqrt{m_1}$, $\mathsf{V}_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mathsf{V}_1^2 + \frac{1}{2} \mathsf{V}_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

becomes:

$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} / \sqrt{m_1} \\ \mathsf{V}_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 / \sqrt{m_1} \\ \mathsf{V}_2 / \sqrt{m_2} \end{pmatrix}$$

or:

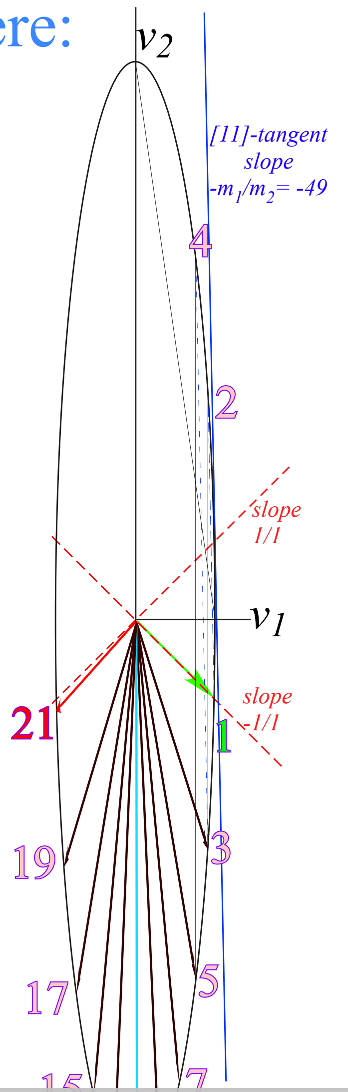
$$\begin{pmatrix} \mathsf{V}_1^{FIN_1} \\ \mathsf{V}_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathbf{M} \bullet \vec{\mathsf{V}} ,$$

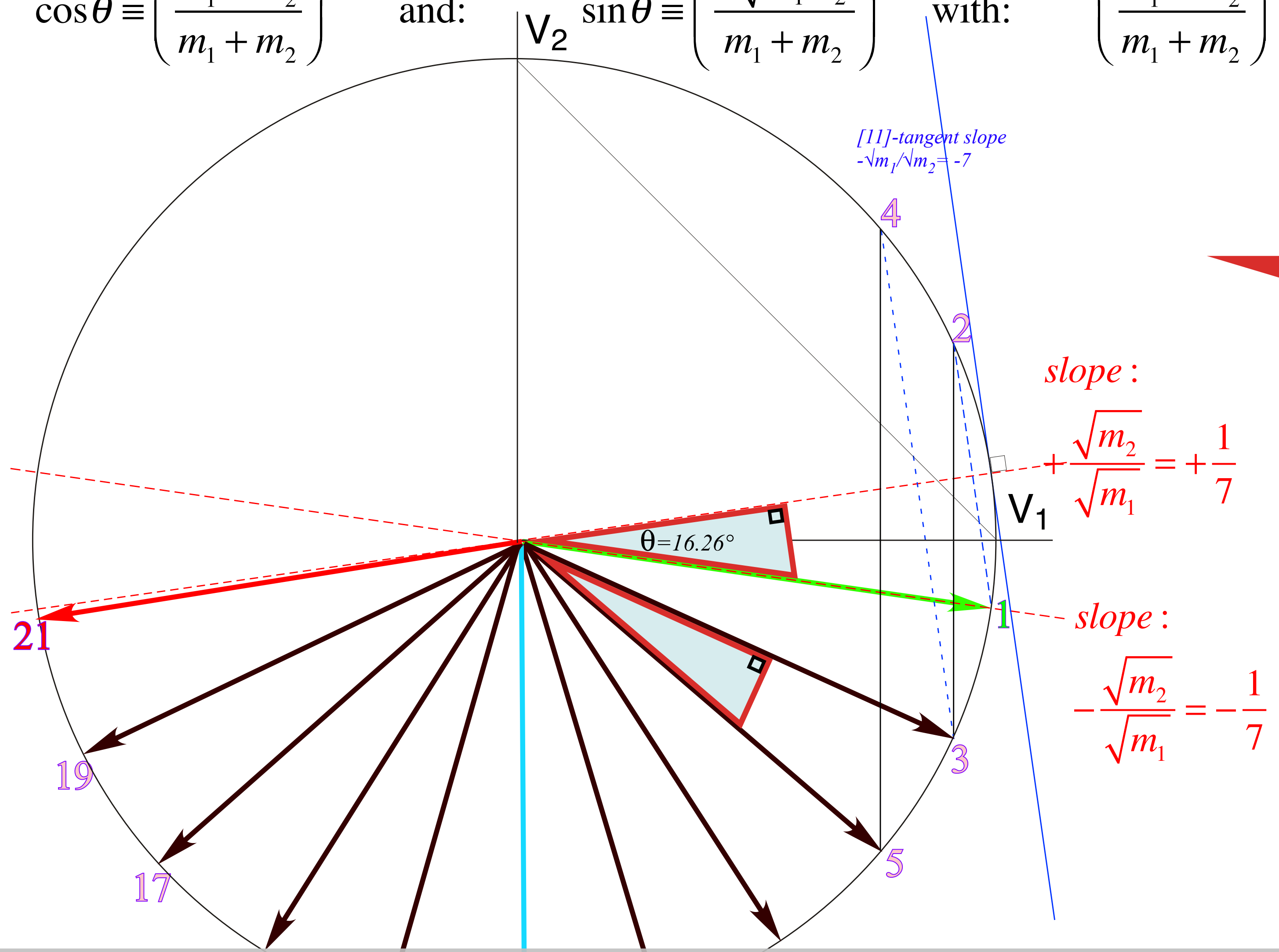
or:

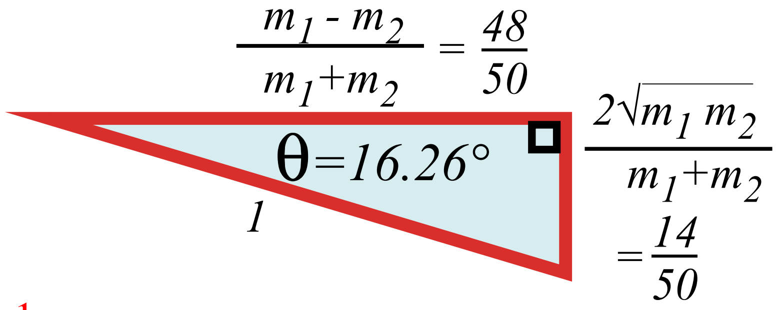
$$\begin{pmatrix} \mathsf{V}_1^{FIN_2} \\ \mathsf{V}_2^{FIN_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} \mathsf{V}_1 \\ \mathsf{V}_2 \end{pmatrix} = \mathsf{C} \bullet \mathbf{M} \bullet \vec{\mathsf{V}}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$







Collisions for mass ratio $m_1 : m_2 = 49:1$

Fig. 5.2a-c

(revised)

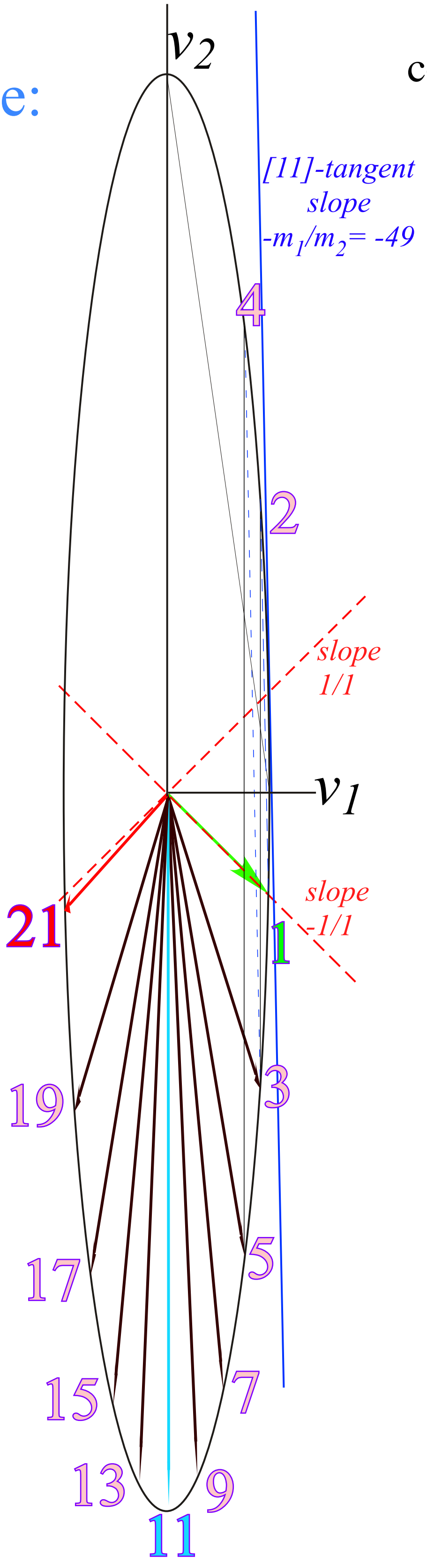
Note: If $m_1 \cdot m_2$ is perfect-square, then θ-triangle is rational ($3^2+4^2=5^2$, etc.)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations*

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$



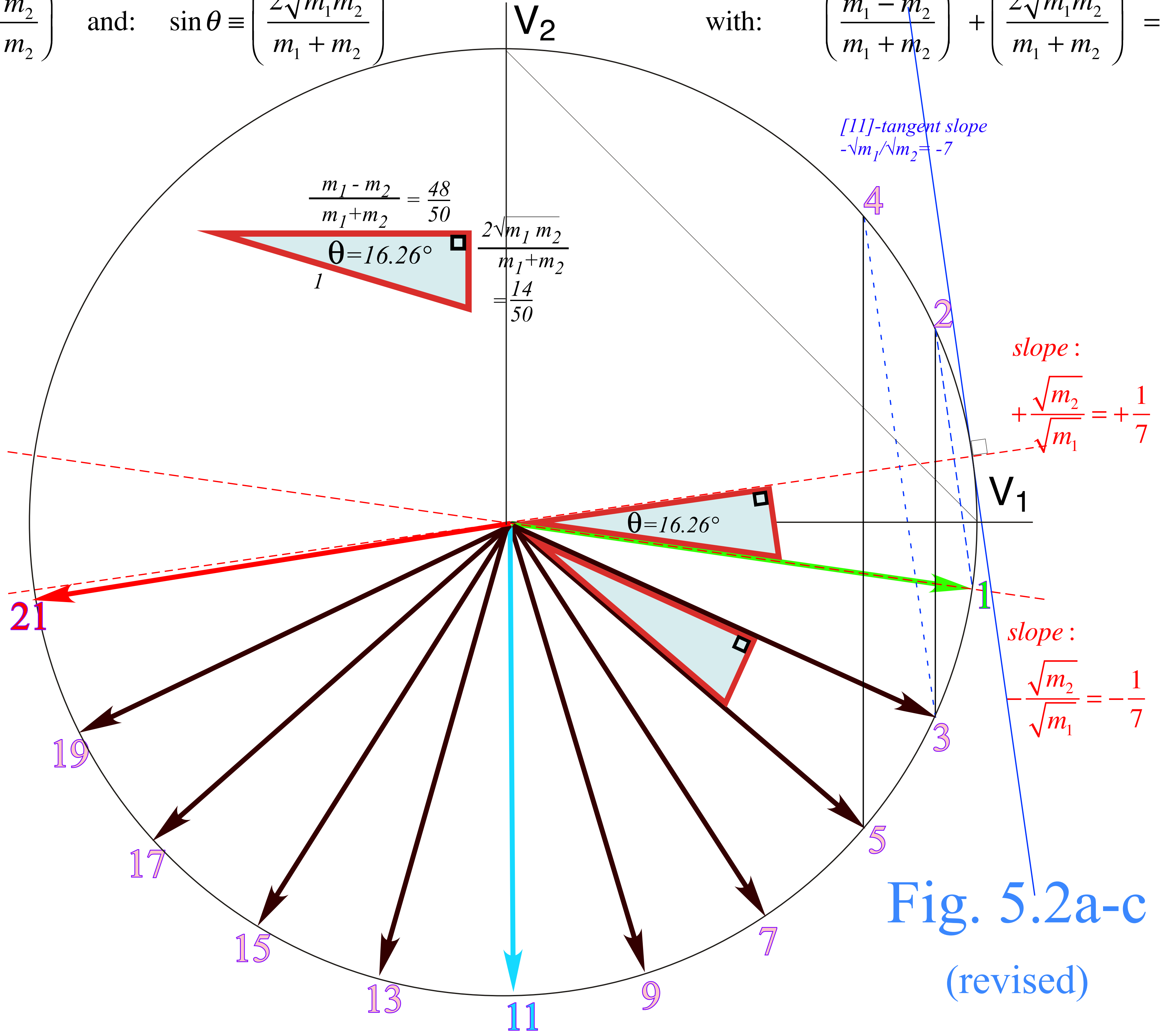


Fig. 5.2a-c

(revised)

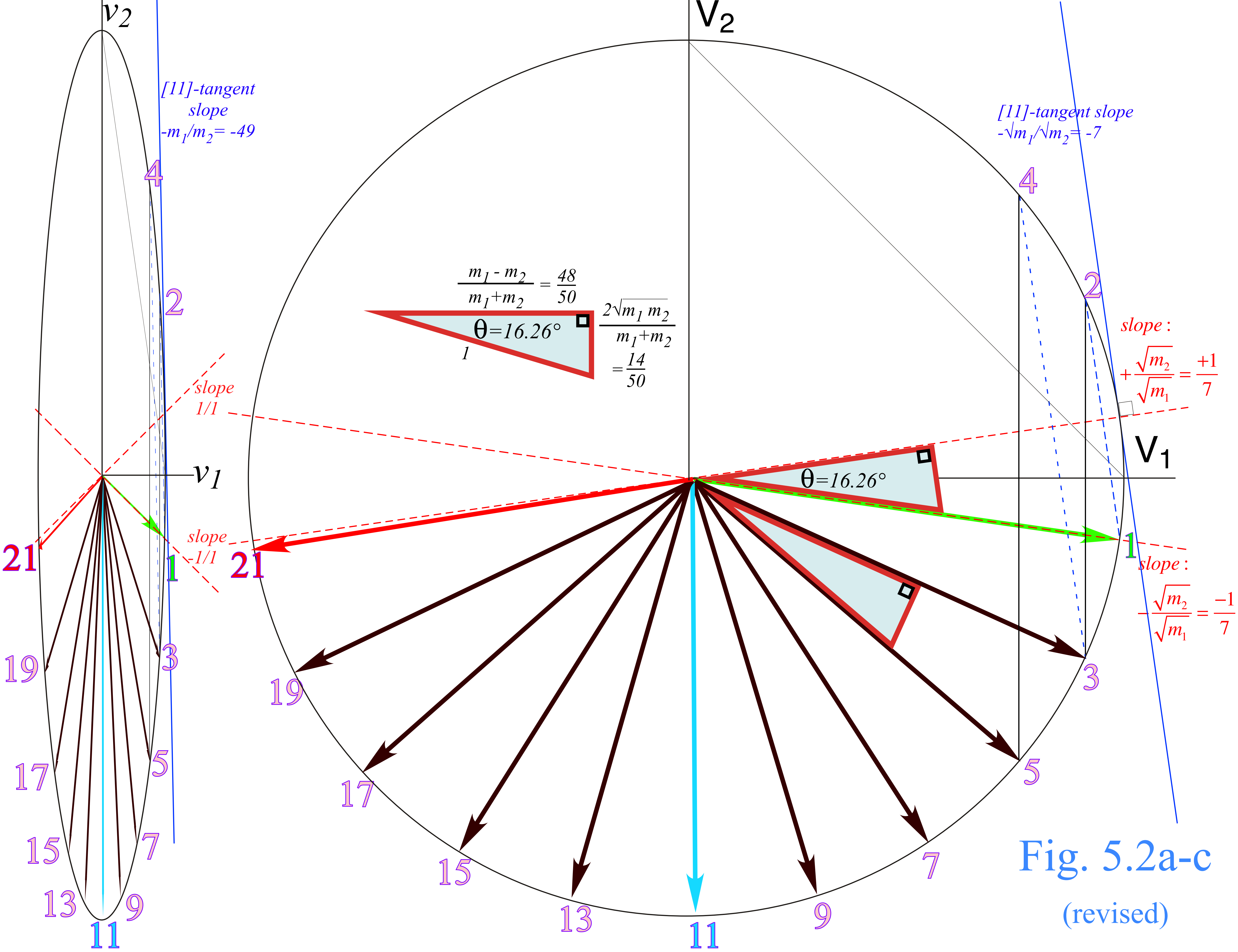


Fig. 5.2a-c

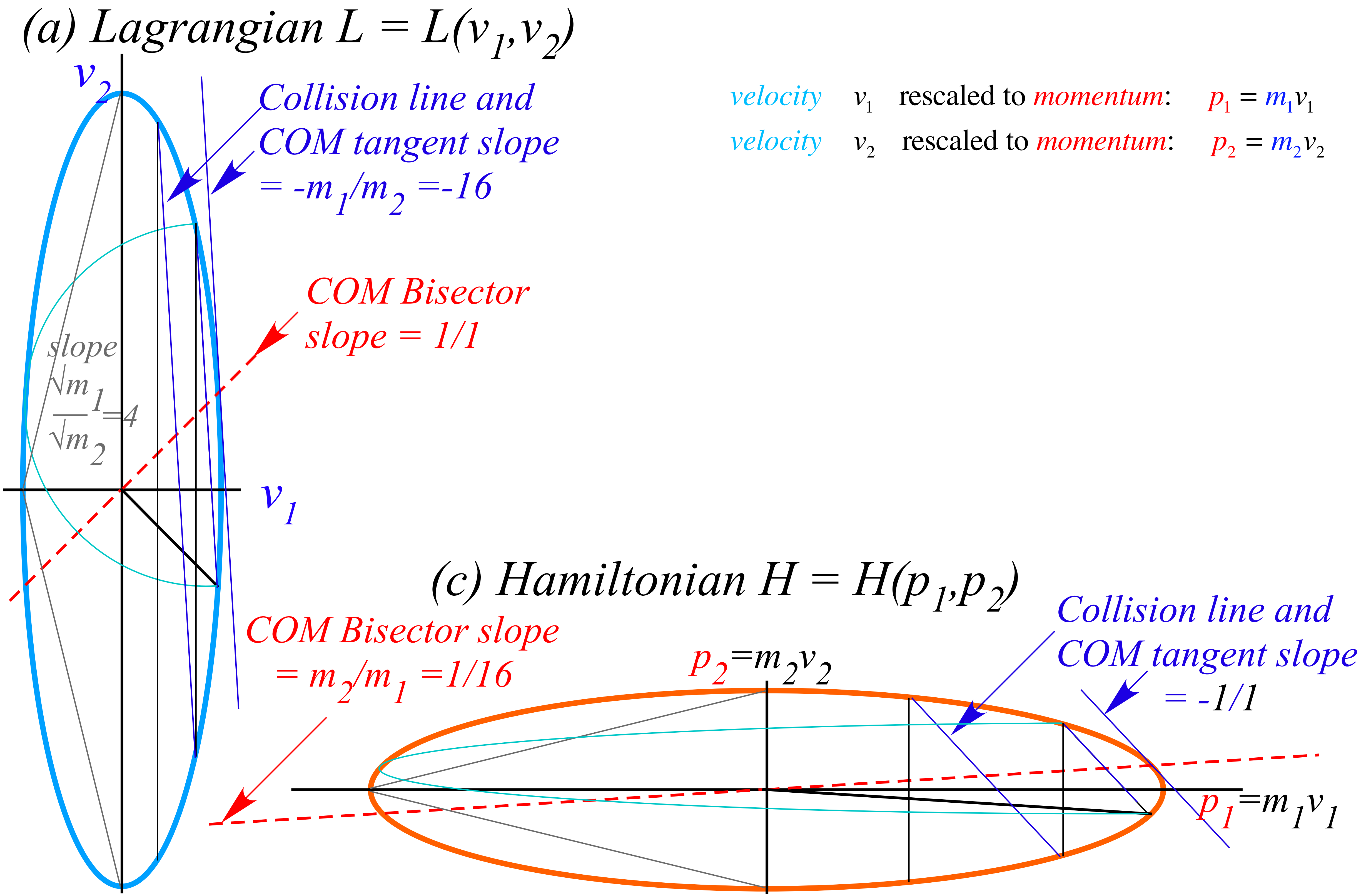
(revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
  - → *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*
- *Reflections in the clothing store: "It's all done with mirrors!"*
- *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*
  - *Group multiplication and product table*
  - *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*
    - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$*

What ellipse rescaling leads to...

How this relates to Lagrangian, and Hamiltonian mechanics later on



(a) Lagrangian $L = L(v_1,v_2)$

(c) Hamiltonian $H = H(p_1,p_2)$

velocity v_1 rescaled to *momentum*: $p_1 = m_1v_1$

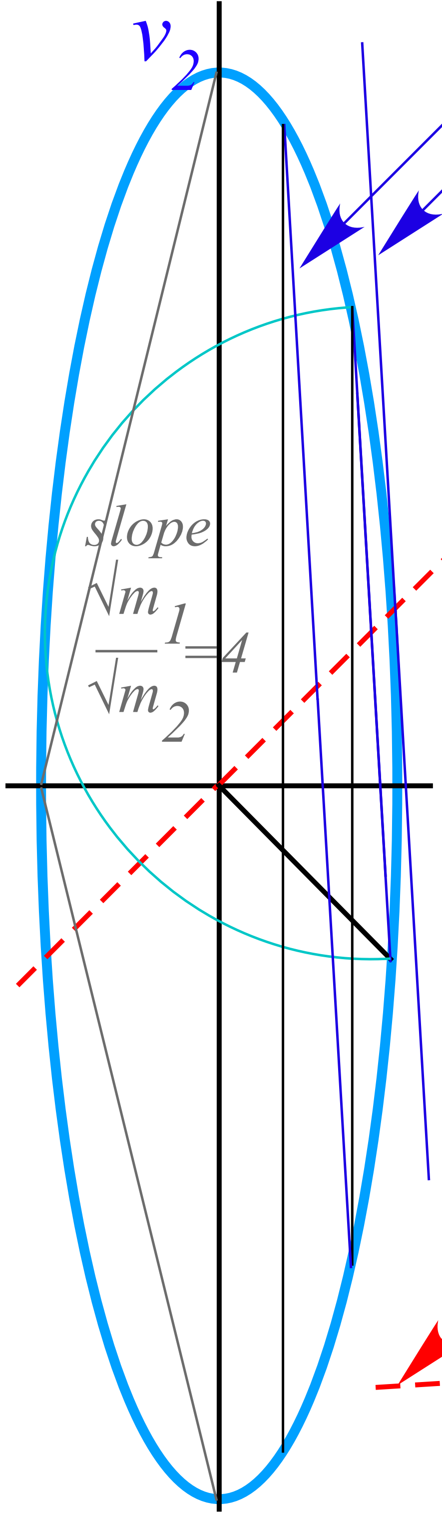
velocity v_2 rescaled to *momentum*: $p_2 = m_2v_2$

What ellipse rescaling leads to...

Fig. 12.1 (Unit 1)

How this relates to Lagrangian, and Hamiltonian mechanics later on

(a) Lagrangian $L = L(v_1,v_2)$



Collision line and COM tangent slope $= -m_1/m_2 = -16$

COM Bisector slope = 1/1

velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$

velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

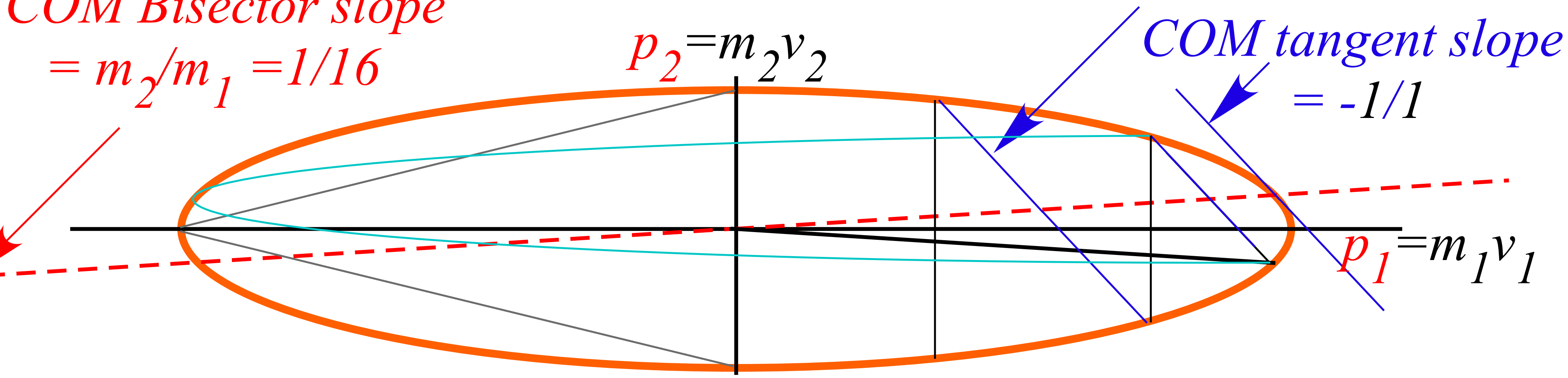
Lagrangian $L(v_1,v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

rescaled to

Hamiltonian $H(p_1,p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

(c) Hamiltonian $H = H(p_1,p_2)$

COM Bisector slope $= m_2/m_1 = 1/16$

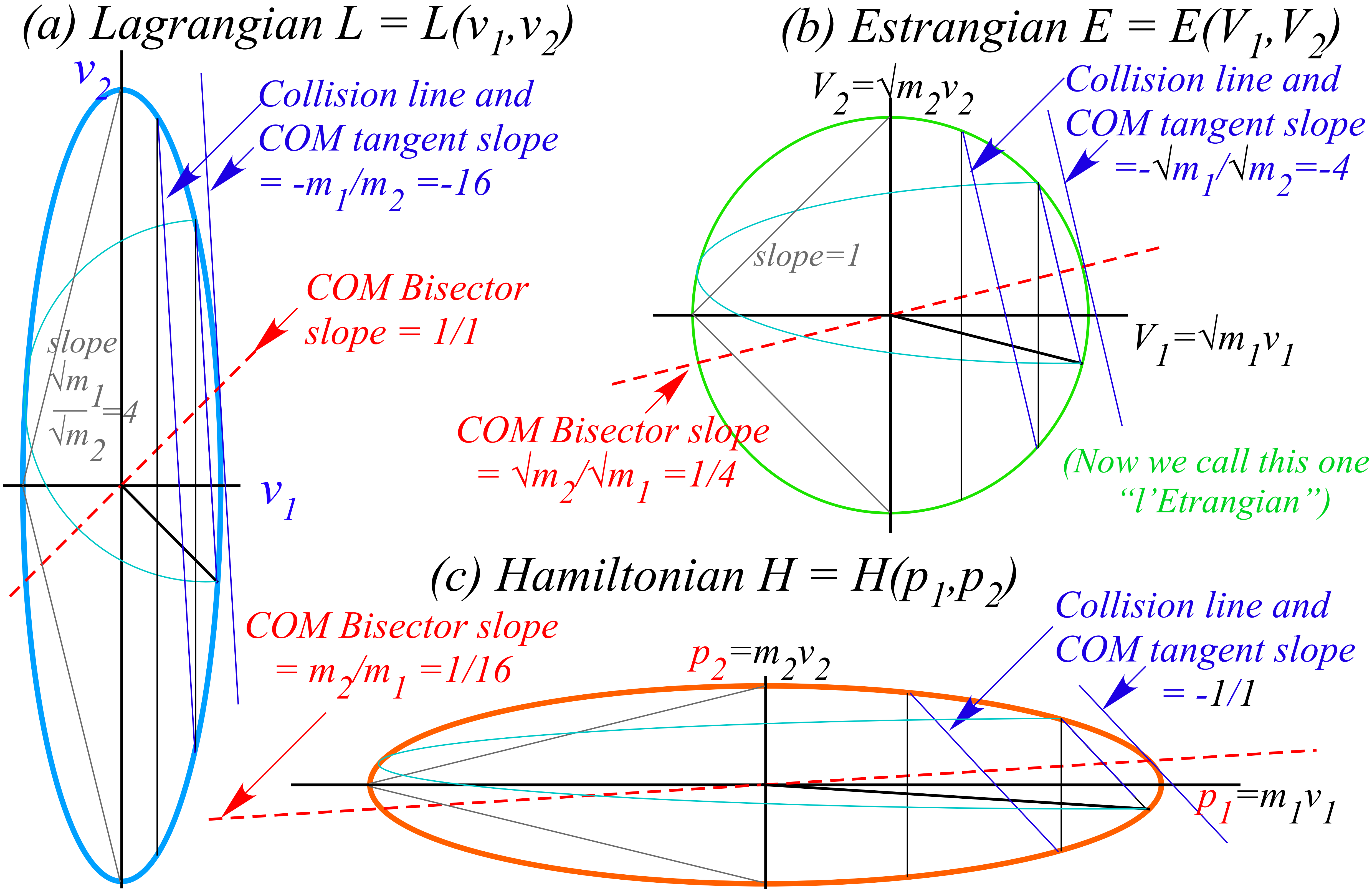


Collision line and COM tangent slope $= -1/1$

What ellipse rescaling leads to...

Fig. 12.1
(Unit 1)

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on



(a) Lagrangian $L = L(v_1,v_2)$

Collision line and
COM tangent slope
$= -m_1/m_2 = -16$

COM Bisector
slope = 1/1

COM Bisector slope
$= m_2/m_1 = 1/16$

(b) Estrangian $E = E(V_1,V_2)$

Collision line and
COM tangent slope
$= -\sqrt{m_1}/\sqrt{m_2} = -4$

COM Bisector slope
$= \sqrt{m_2}/\sqrt{m_1} = 1/4$

(Now we call this one
"l'Etrangian")

(c) Hamiltonian $H = H(p_1,p_2)$

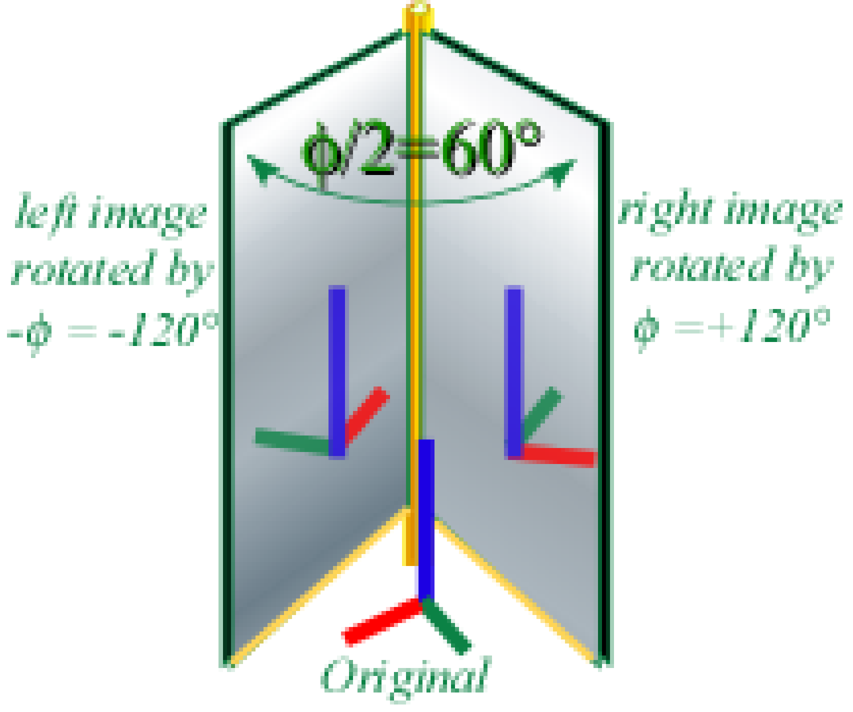
Collision line and
COM tangent slope
$= -1/1$

Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
  - *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*
- *Reflections in the clothing store: "It's all done with mirrors!"*
- *Introducing hexagonal symmetry D_6~C_{6v} (Resulting for $m_1/m_2=3$)*
  - *Group multiplication and product table*
  - *Classical collision paths with D_6~C_{6v} (Resulting from $m_1/m_2=3$)*
    - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$*

Reflections in clothing store mirrors

(a) $\phi=\pm 120°$ rotations



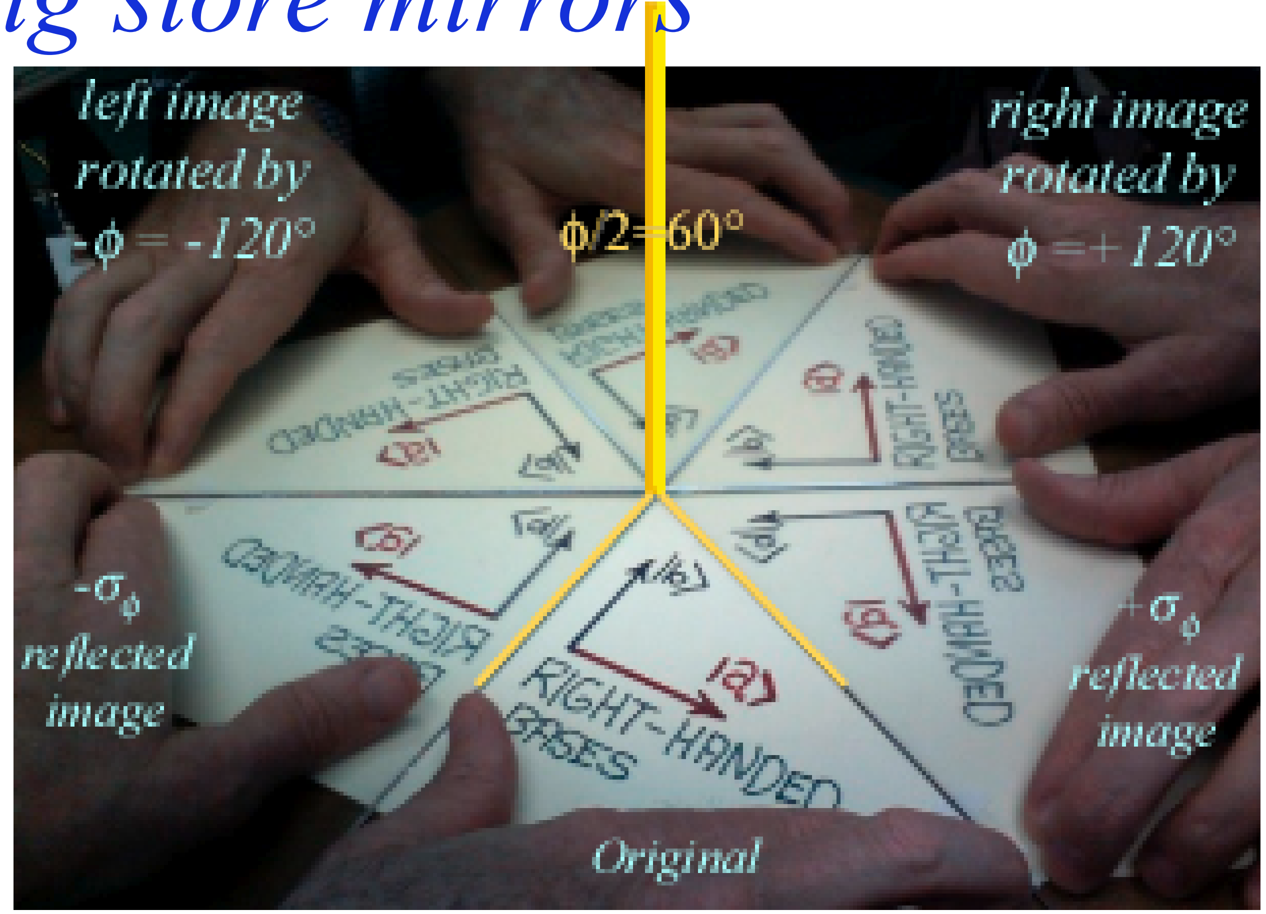
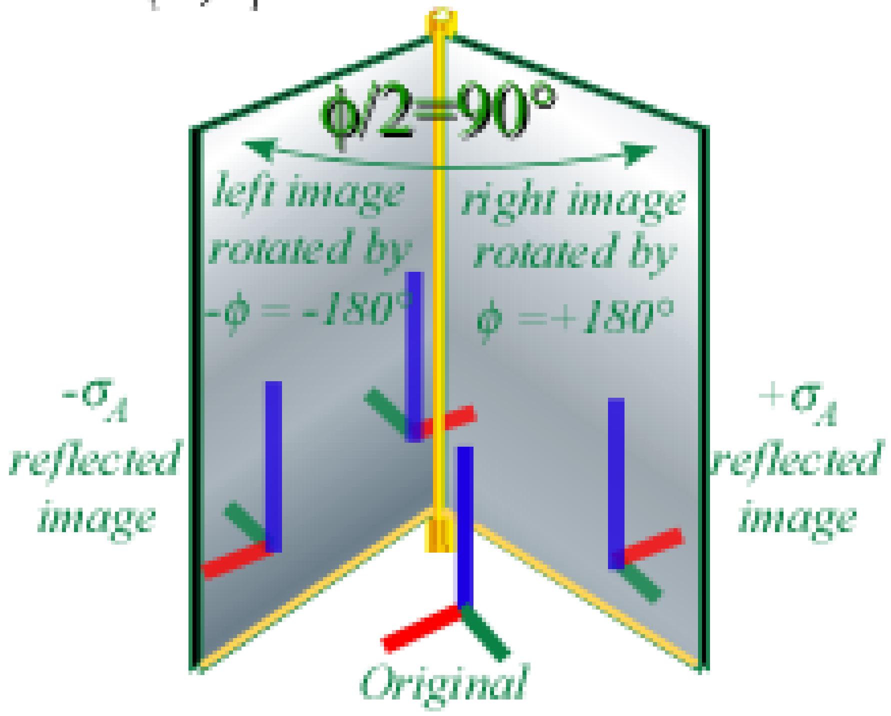
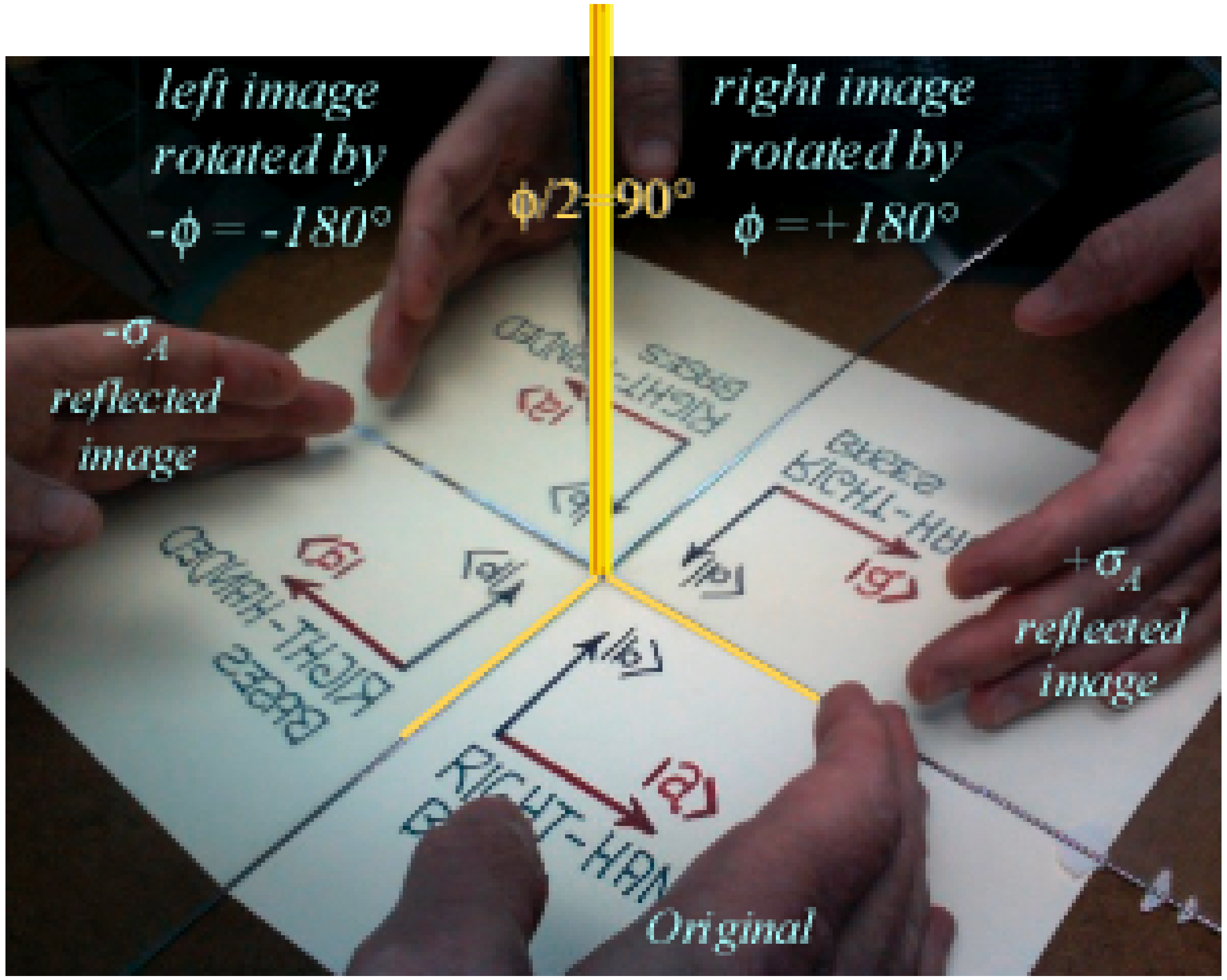


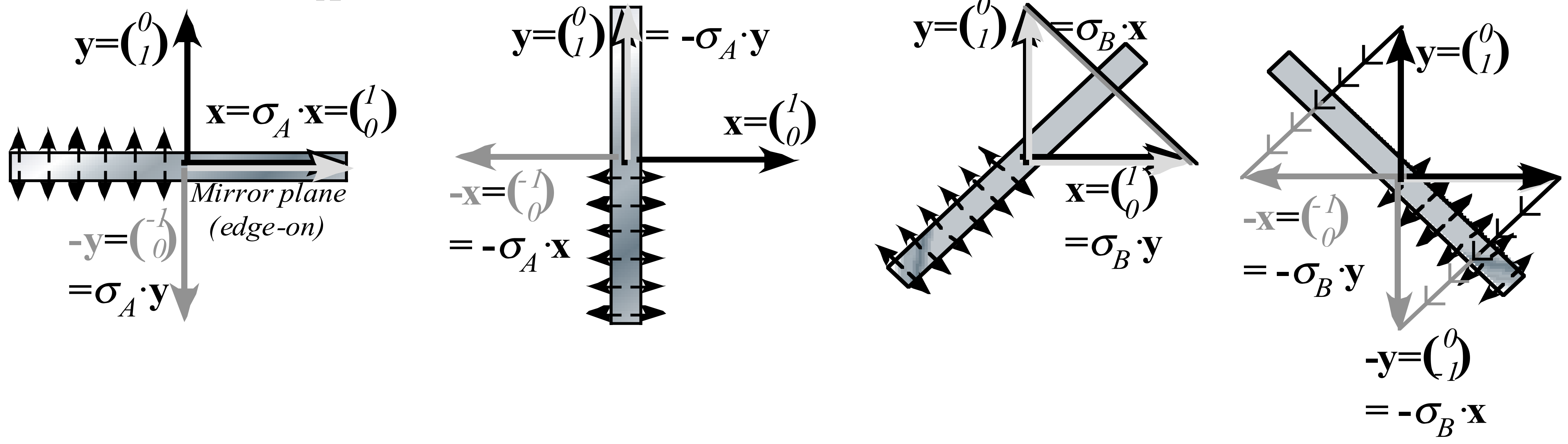
Fig. 5.4a-b

(b) $\phi=\pm 180°$ rotations





Symmetry: It's all done with mirrors!



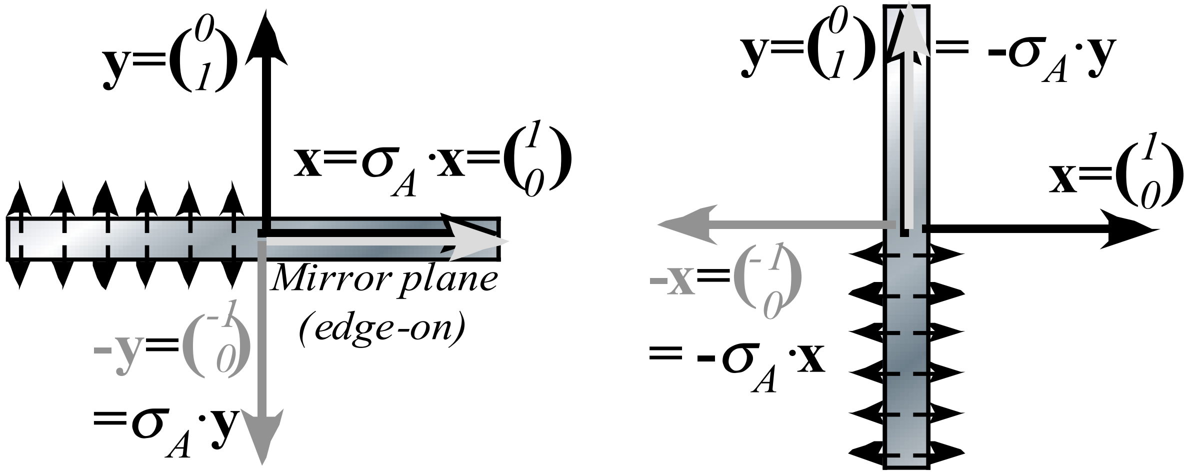
(a)Reflections $\sigma_A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $-\sigma_A=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

(b)Reflections $\sigma_B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $-\sigma_B=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

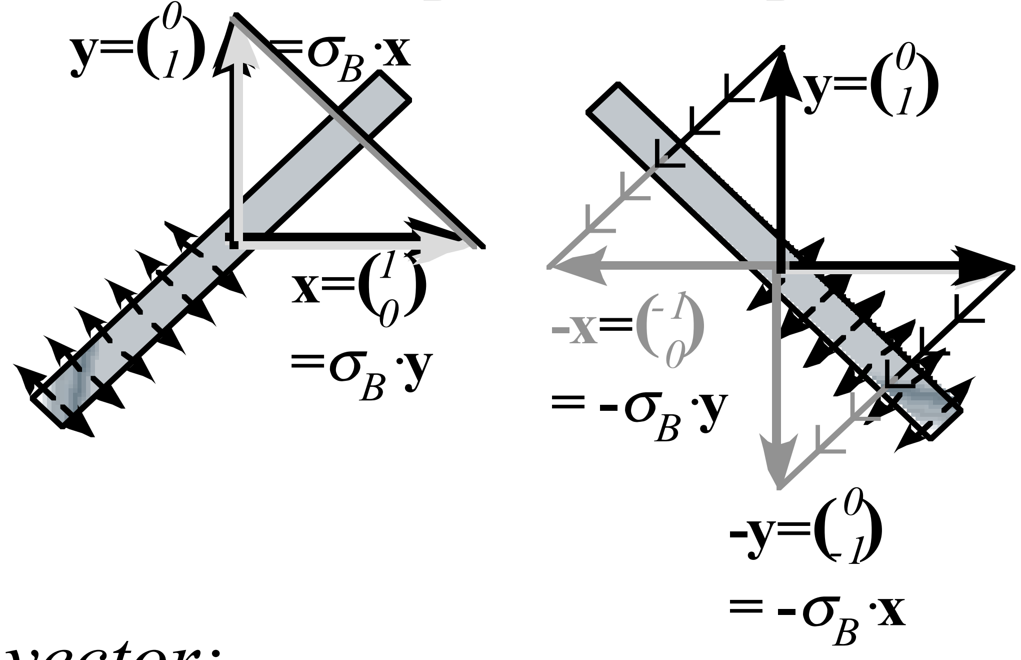
Fig.
5.3a-e

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

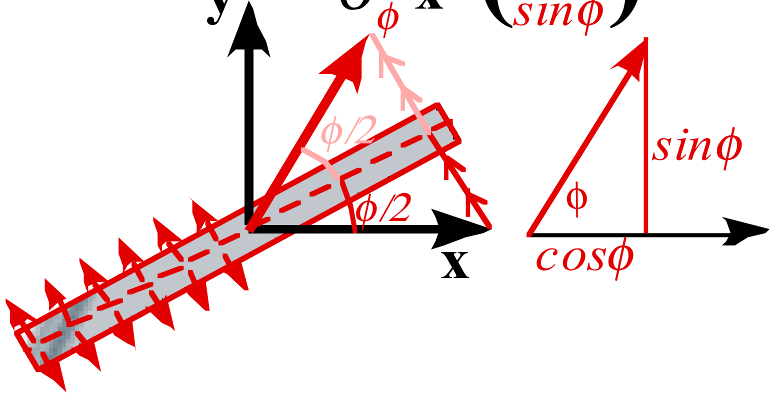


(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

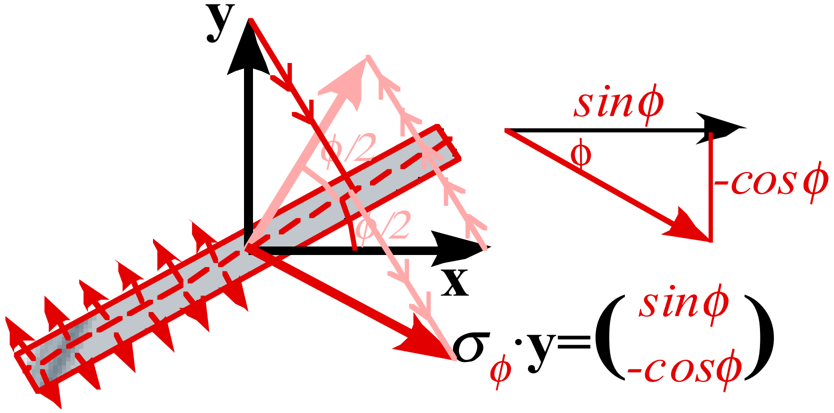


(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of **x**-vector: $\sigma_\phi \cdot \mathbf{x} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$



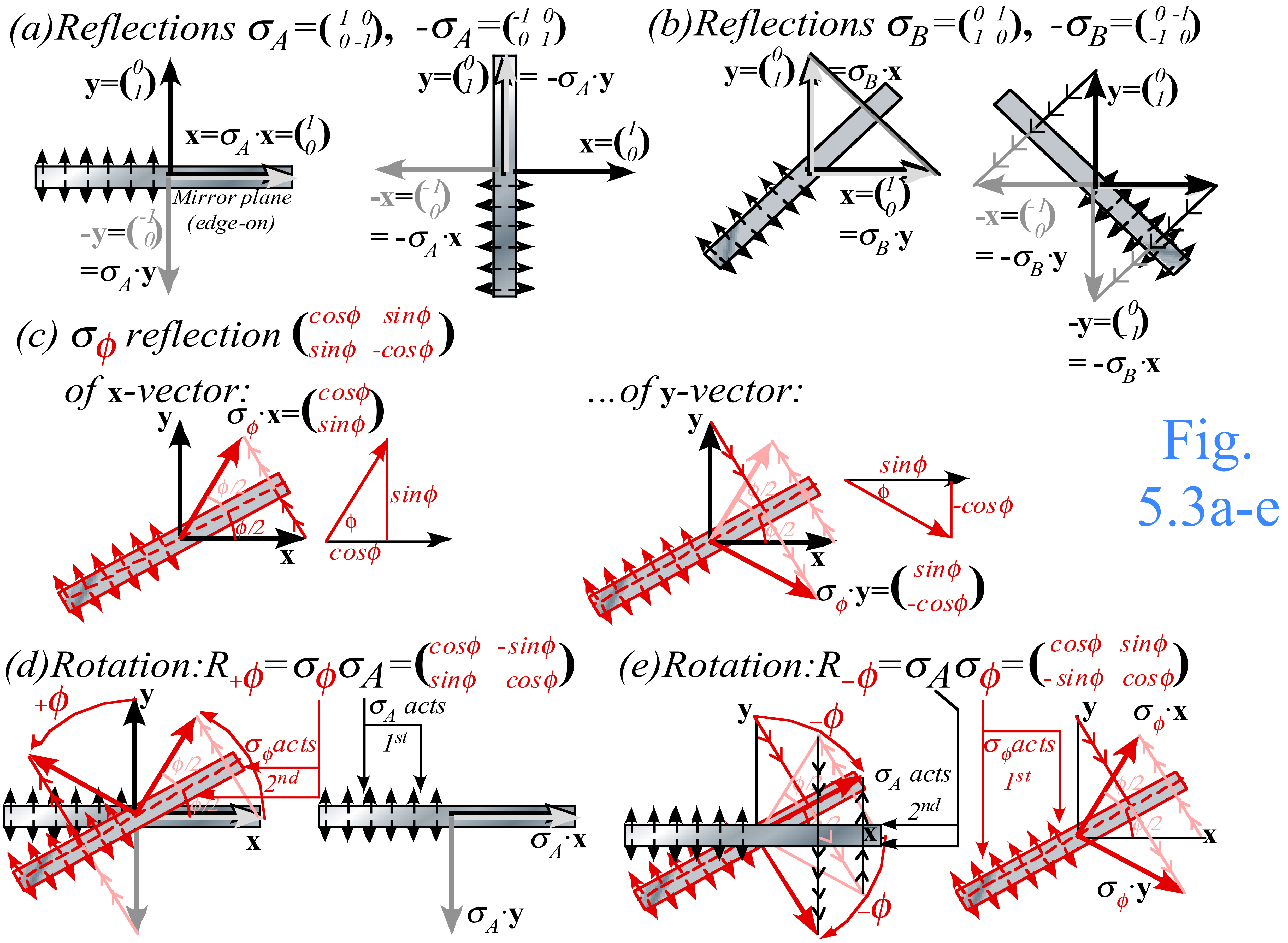
...of **y**-vector:



$\sigma_\phi \cdot \mathbf{y} = \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}$

Fig. 5.3a-e

Symmetry: It's all done with mirrors!



(a) Reflections $\sigma_A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $-\sigma_A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$

(b) Reflections $\sigma_B=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $-\sigma_B=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$

(c) σ_ϕ reflection $\begin{pmatrix}\cos\phi&\sin\phi\\\sin\phi&-\cos\phi\end{pmatrix}$

of **x**-vector: ...of **y**-vector:

(d) Rotation: $R_{+\phi}=\sigma_\phi\sigma_A=\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}$

(e) Rotation: $R_{-\phi}=\sigma_A\sigma_\phi=\begin{pmatrix}\cos\phi&\sin\phi\\-\sin\phi&\cos\phi\end{pmatrix}$

Fig. 5.3a-e

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

Product of odd number of reflections is a reflection

... even number of reflections is a rotation (or unit-op **1***)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily
Waves patterns are non-rigid and reflect easily

Why reflections underlie all symmetry analyses

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Classical objects are semi-rigid and rotate easily
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∴ *...wave reflections underlie modern physics*

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

→ *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*

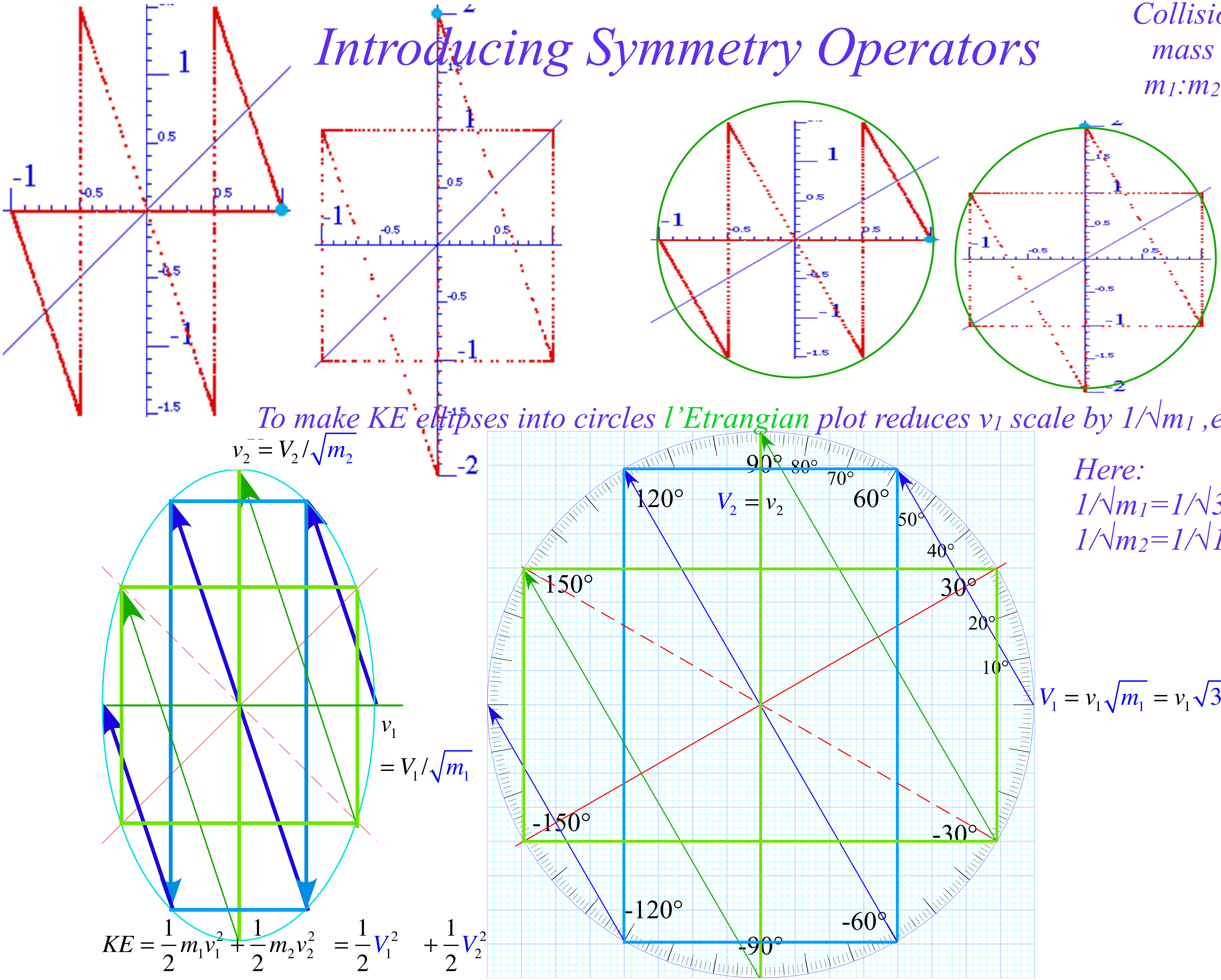
Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

Introducing Symmetry Operators

Collisions for mass ratio $m_1{:}m_2 = 3{:}1$



To make KE ellipses into circles l'Etrangian plot reduces v_1 scale by $1/\sqrt{m_1}$,etc.

$$v_2 = V_2/\sqrt{m_2}$$

$$v_1 = V_1/\sqrt{m_1}$$

$$V_2 = v_2$$

$$V_1 = v_1\sqrt{m_1} = v_1\sqrt{3}$$

$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2$$

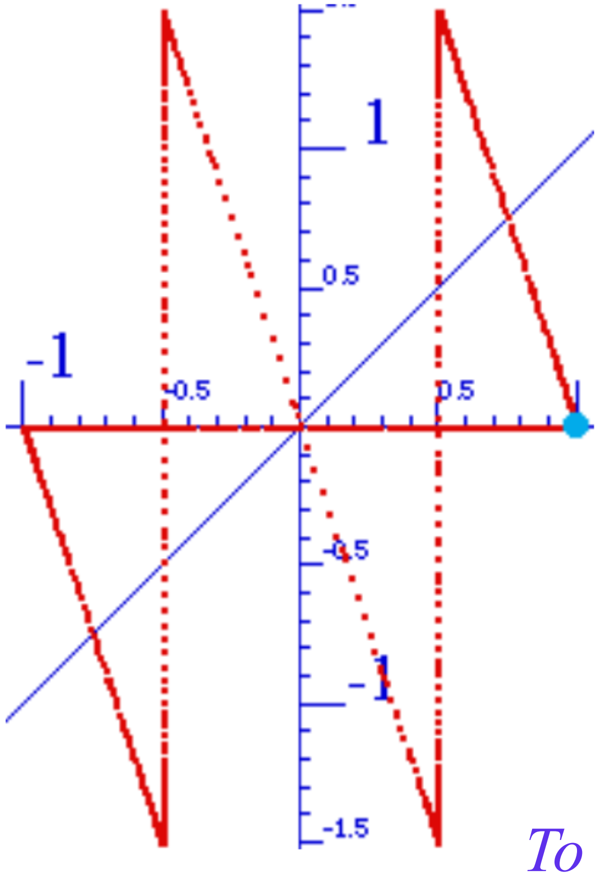
Here:

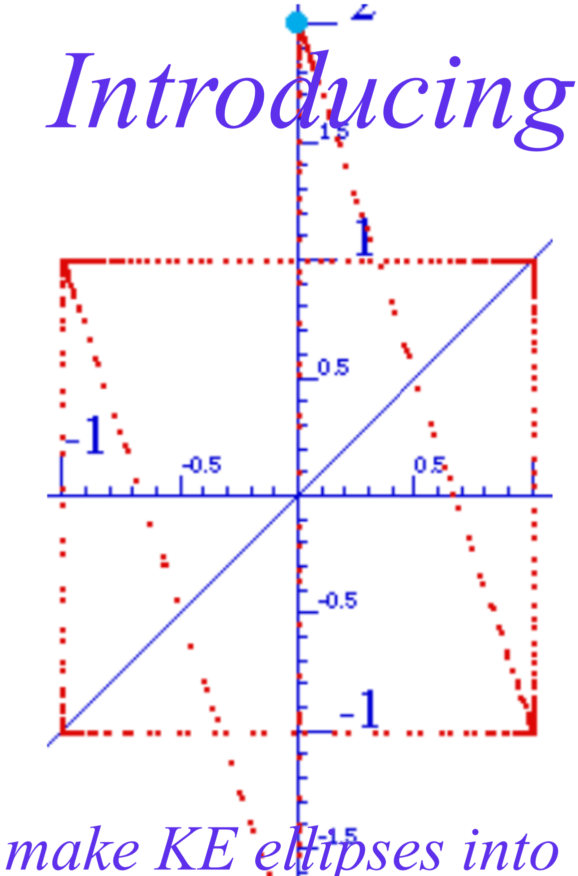
$1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$

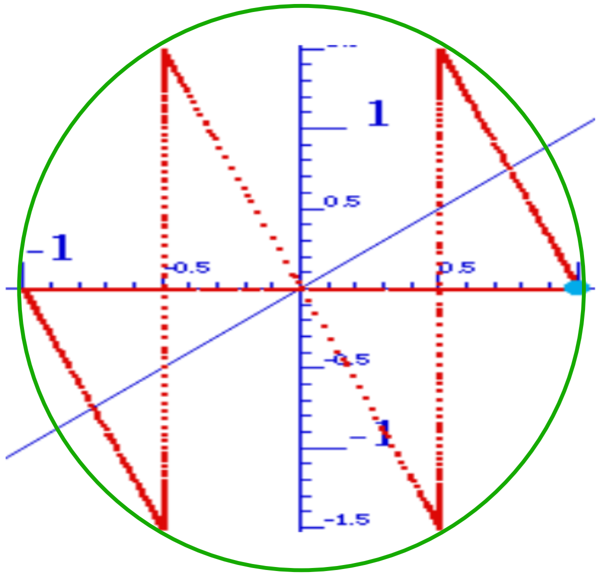
$1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

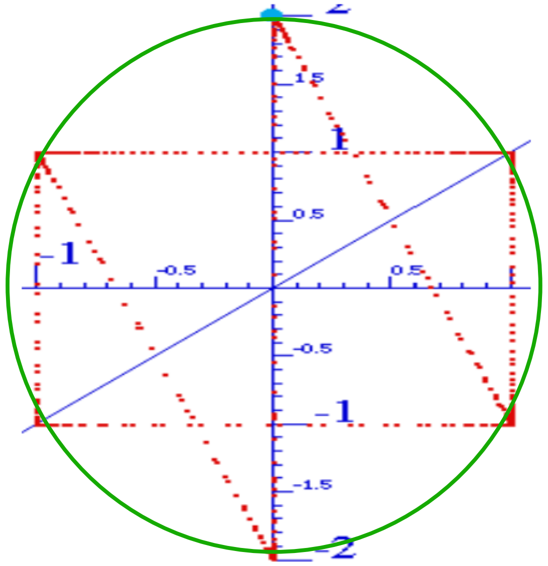
Introducing Symmetry Operators

Collisions for mass ratio
m_1:m_2= 3:1









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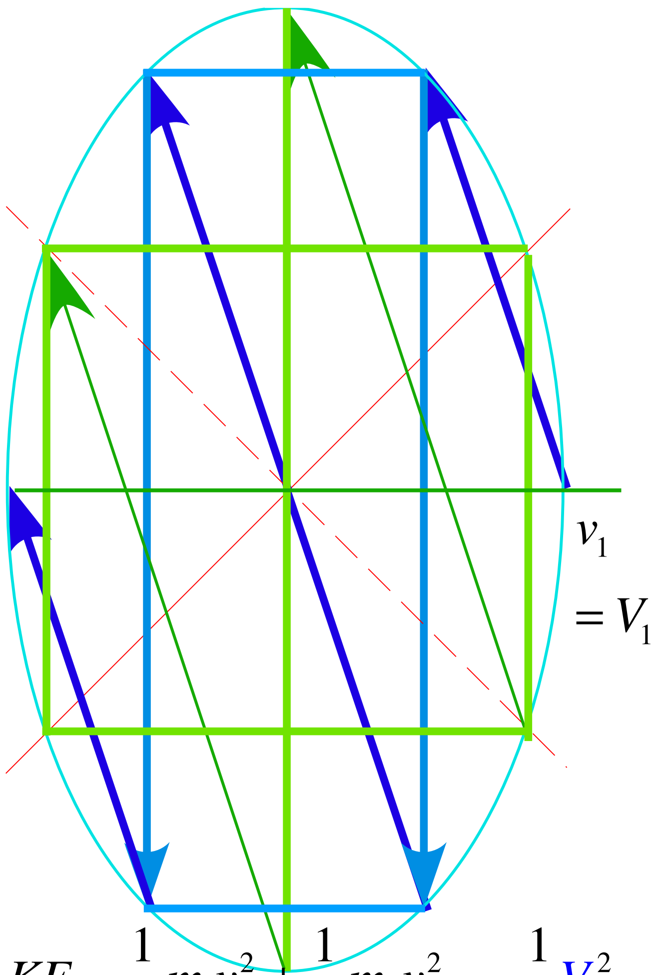
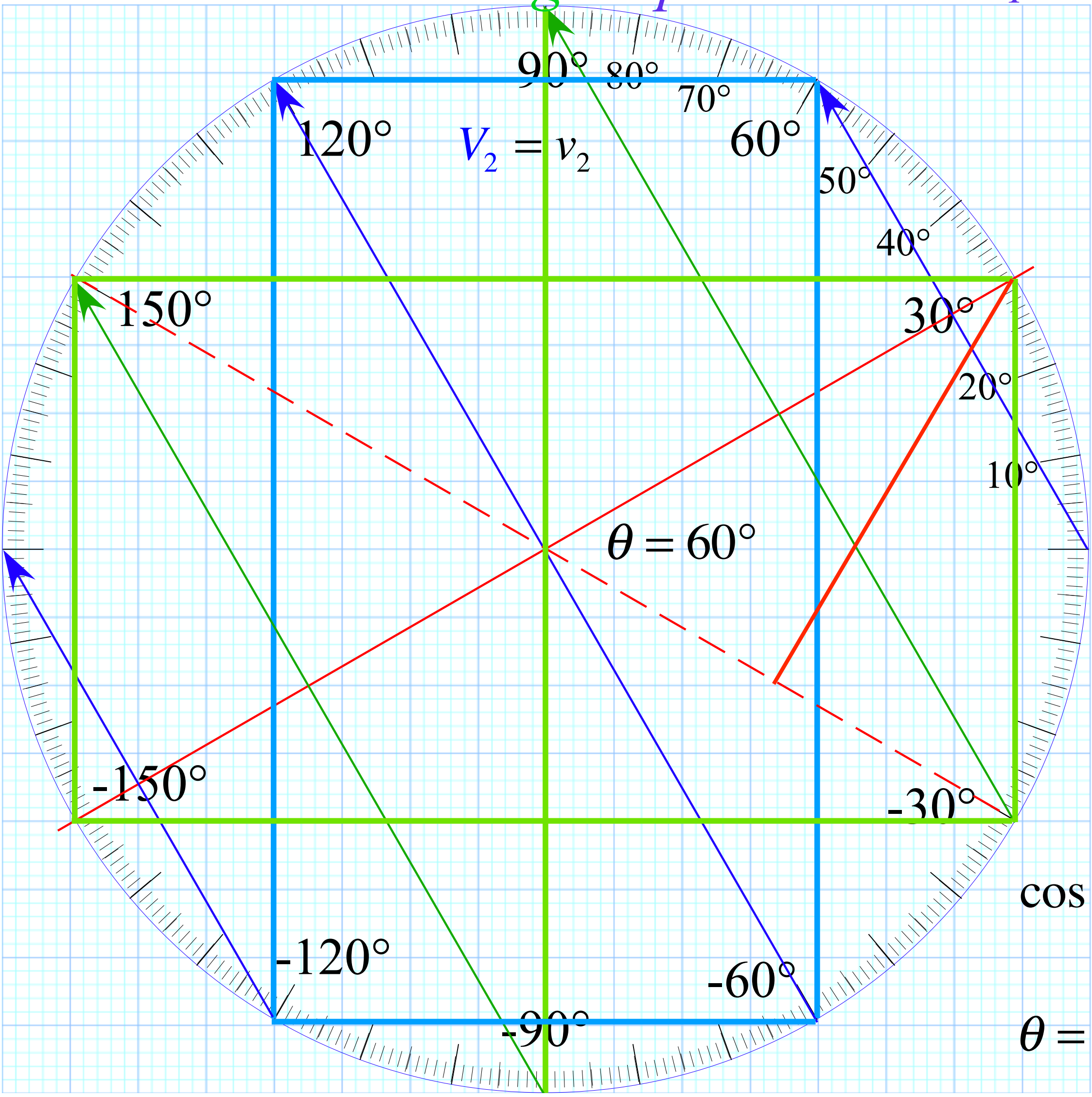
$v_2 = V_2/\sqrt{m_2}$



v_1

$= V_1/\sqrt{m_1}$

$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad = \frac{1}{2}V_1^2 \quad + \frac{1}{2}V_2^2$$



$V_2 = v_2$

$\theta = 60°$

$V_1 = v_1\sqrt{m_1} = v_1\sqrt{3}$

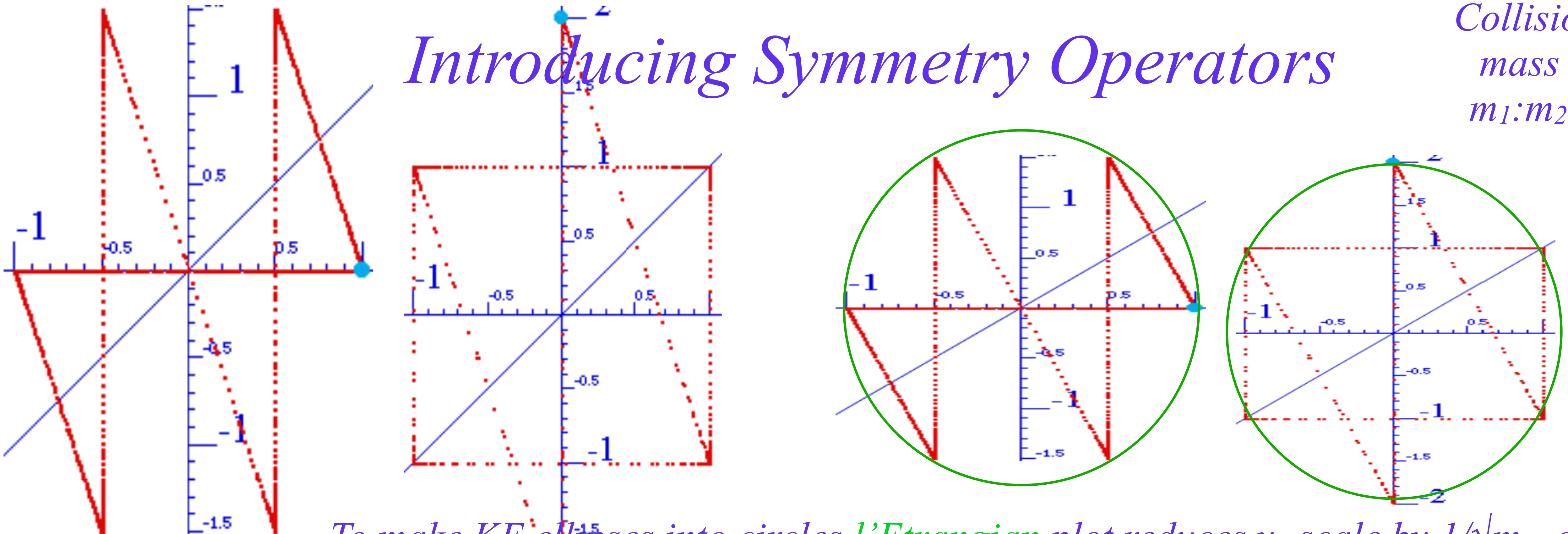
Here:
$1/\sqrt{m_1}=1/\sqrt{3}=0.577$
$1/\sqrt{m_2}=1/\sqrt{1}=1.0$

$$\cos\theta = \frac{m_1-m_2}{m_1+m_2} = \frac{\frac{m_1}{m_2}-1}{\frac{m_1}{m_2}+1} = \frac{2}{4} = \frac{1}{2}$$

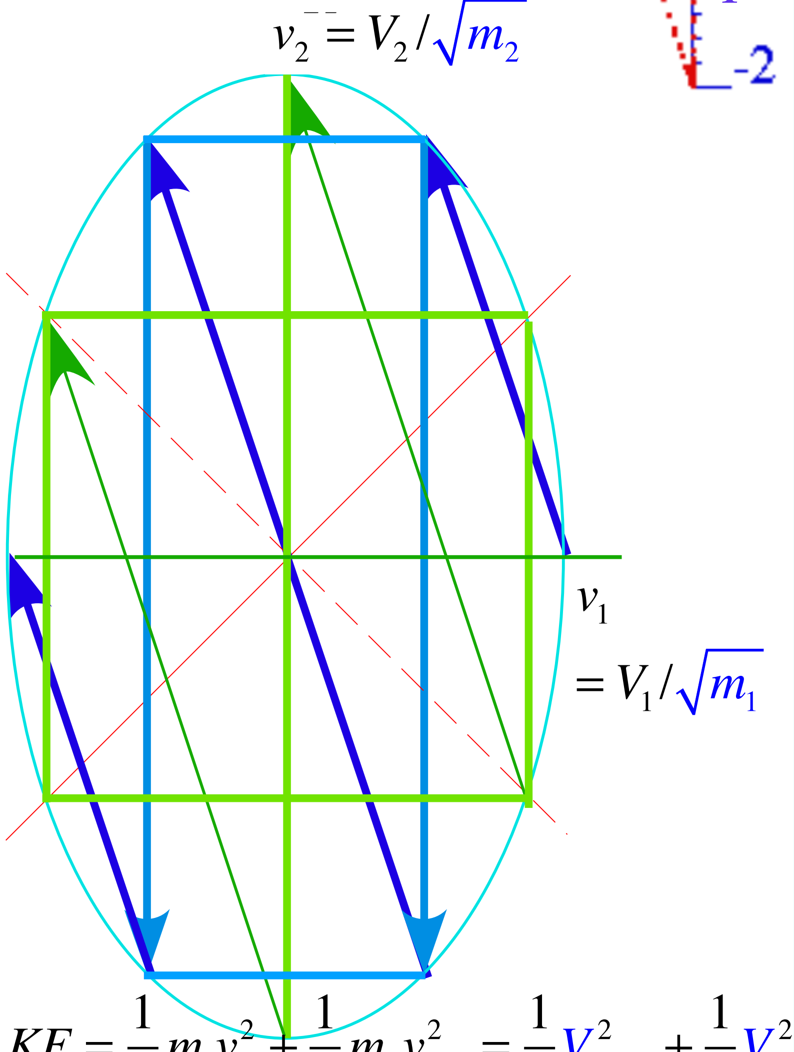
$\theta = 60°$

Introducing Symmetry Operators

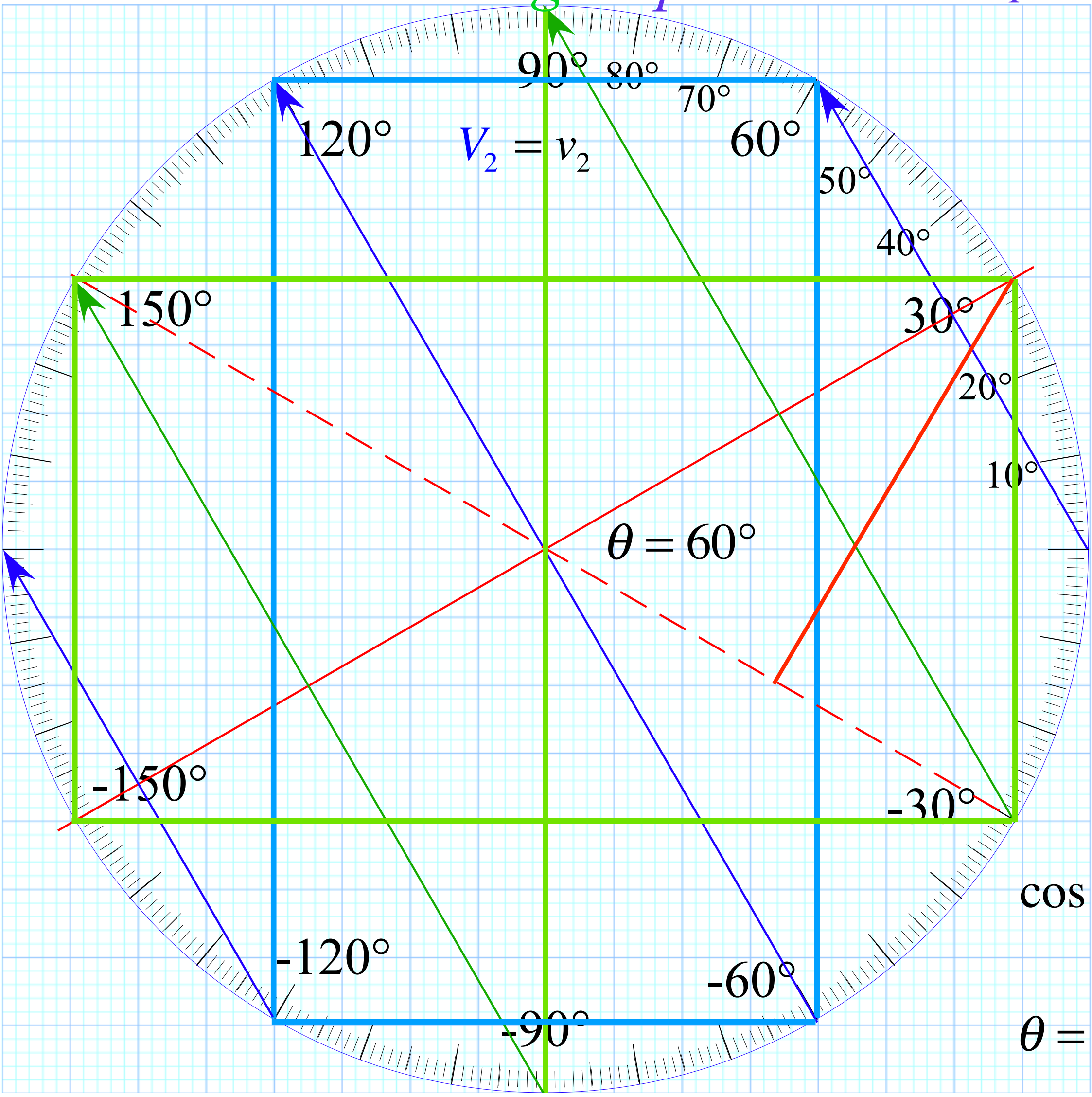
Collisions for mass ratio m_1:m_2= 3:1



To make KE ellipses into circles l'Etrangian plot reduces v_1 scale by 1/√m_1 ,etc.



$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad = \frac{1}{2}V_1^2 \quad + \frac{1}{2}V_2^2$$



$V_1 = v_1\sqrt{m_1} = v_1\sqrt{3}$

Here:
1/√m_1=1/√3=0.577
1/√m_2=1/√1=1.0

$$\frac{m_1}{m_2} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{3/2}{1/2} = 3$$

$$\cos\theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} = \frac{2}{4} = \frac{1}{2}$$

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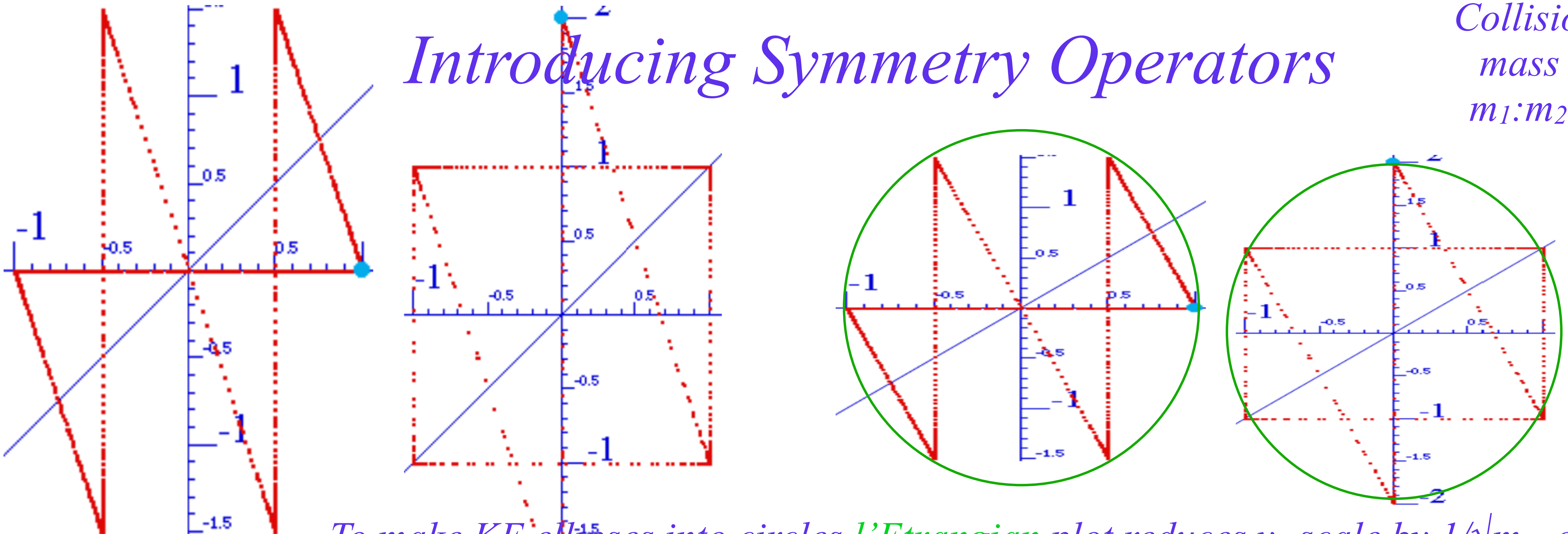
Introducing Symmetry Operators

Collisions for mass ratio m_1:m_2= 3:1



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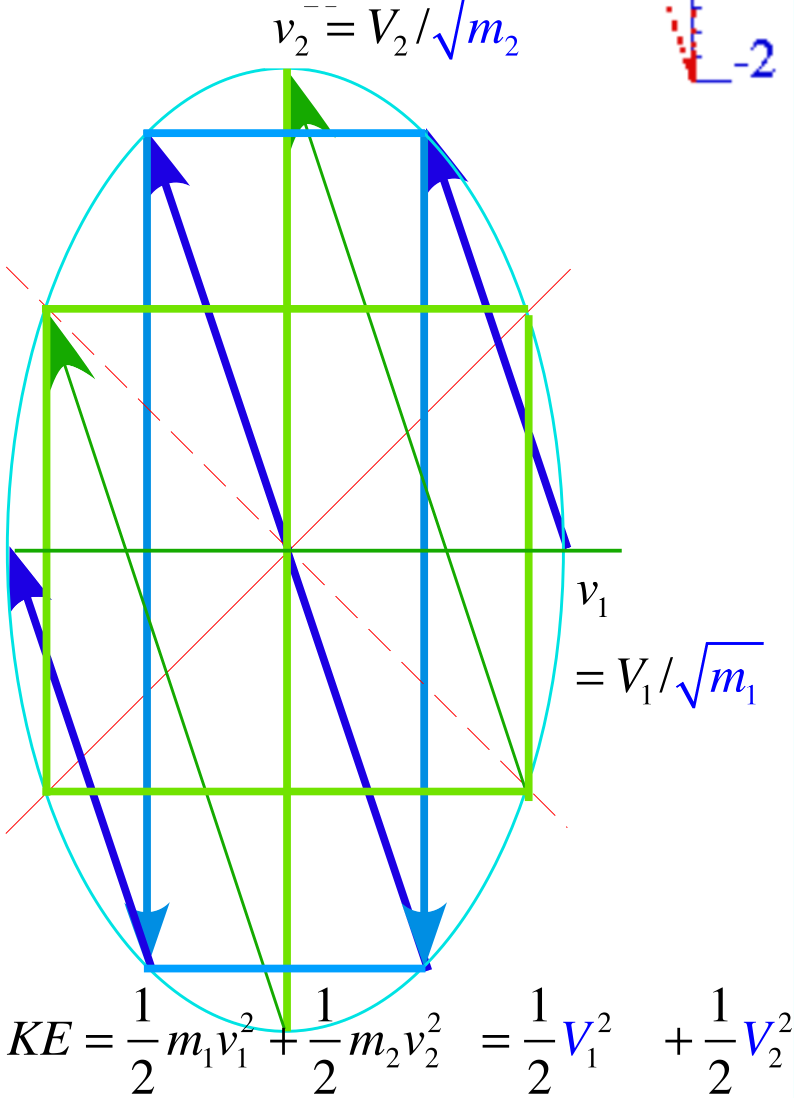
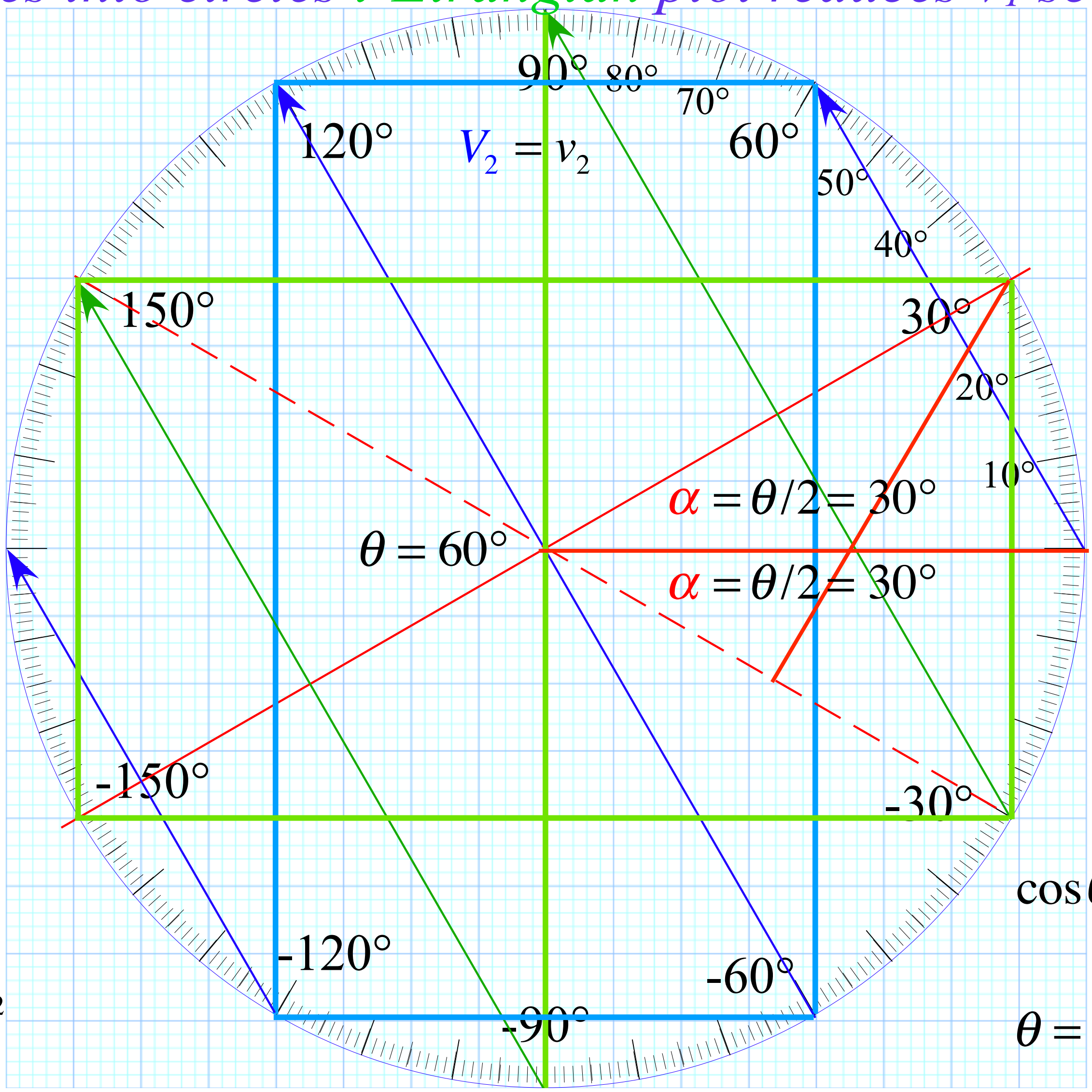


$$v_1 = V_1/\sqrt{m_1}$$



Here:
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$$V_1 = v_1\sqrt{m_1} = v_1\sqrt{3}$$

$$\frac{m_1}{m_2} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{\sin^2\alpha}{\cos^2\alpha} = \tan^2\alpha$$

$$\alpha = \theta/2$$

$$\cos\theta = \frac{m_1-m_2}{m_1+m_2} = \frac{\frac{m_1}{m_2}-1}{\frac{m_1}{m_2}+1} = \frac{2}{4} = \frac{1}{2}$$

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$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2$$

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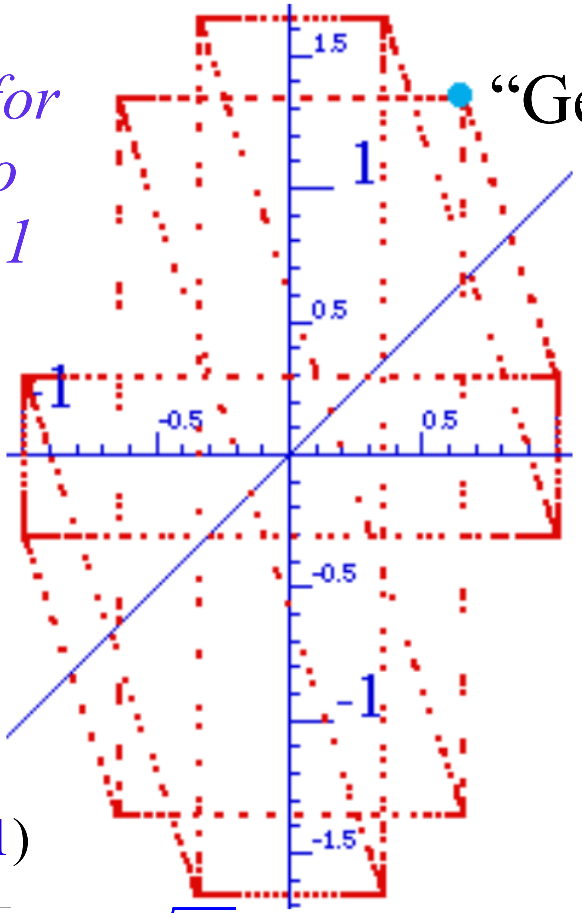
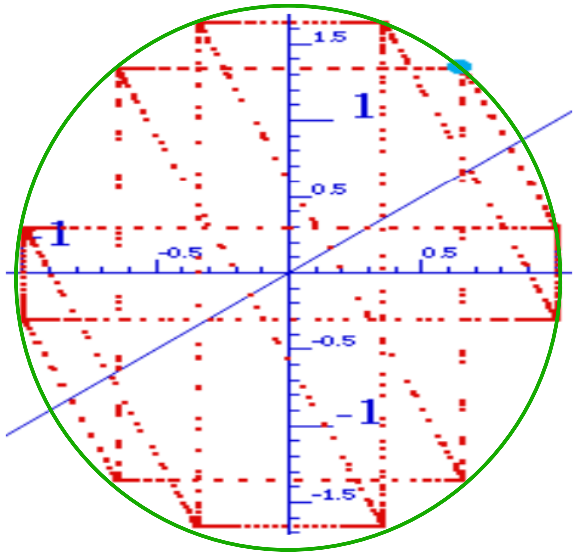
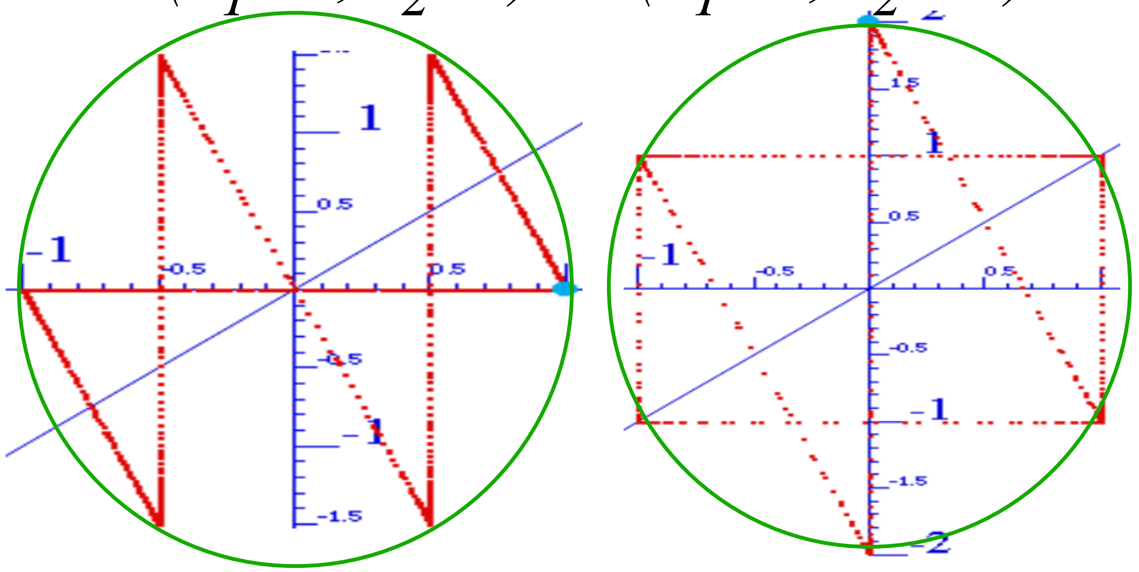


"Generic" initial velocity
(v_1=1.0, v_2=0.1)

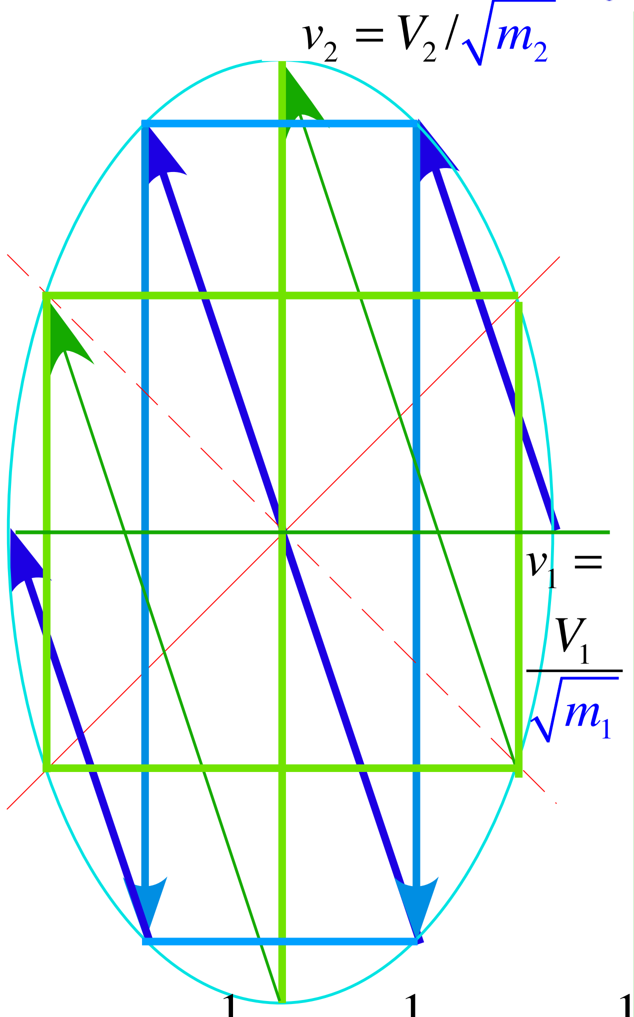


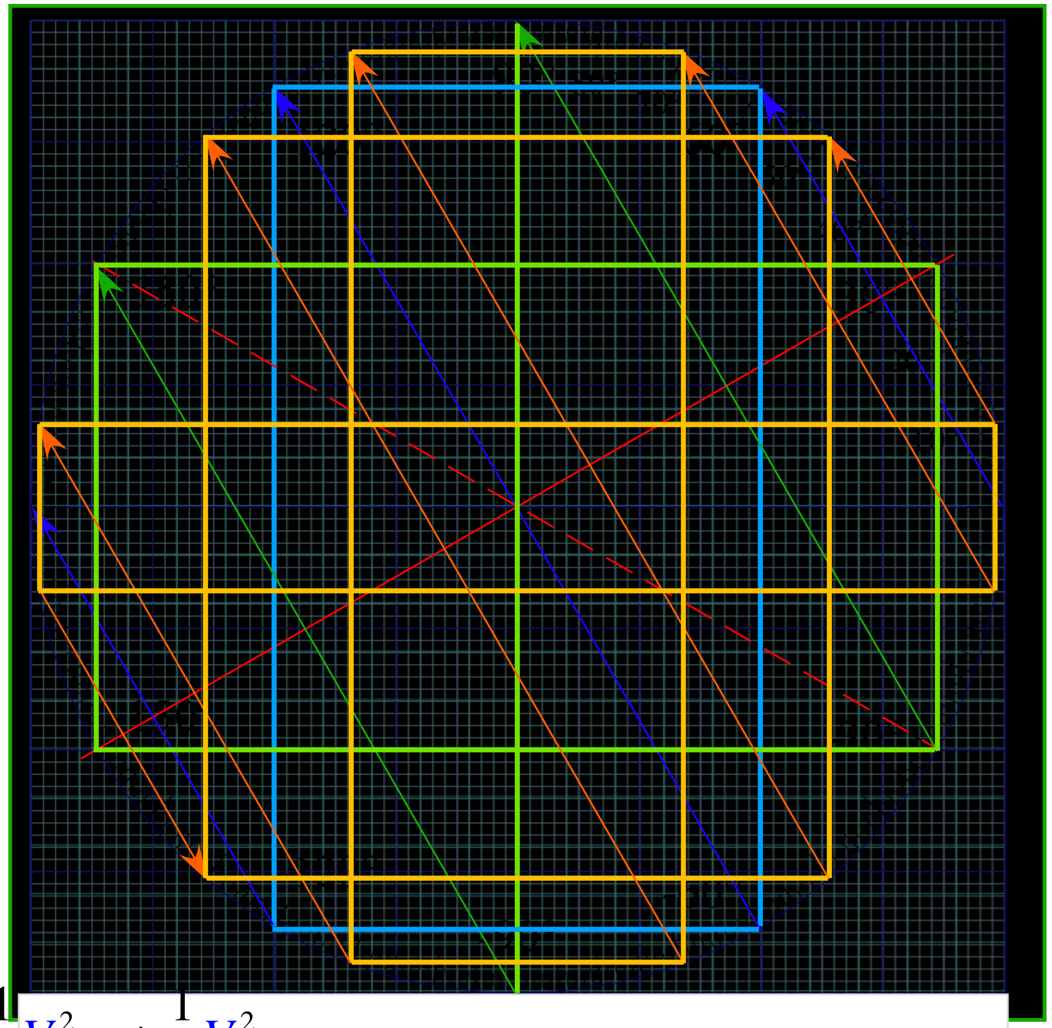
"Symmetric" initial velocity
(v_1=1, v_2=0) or (v_1=1, v_2=-1)

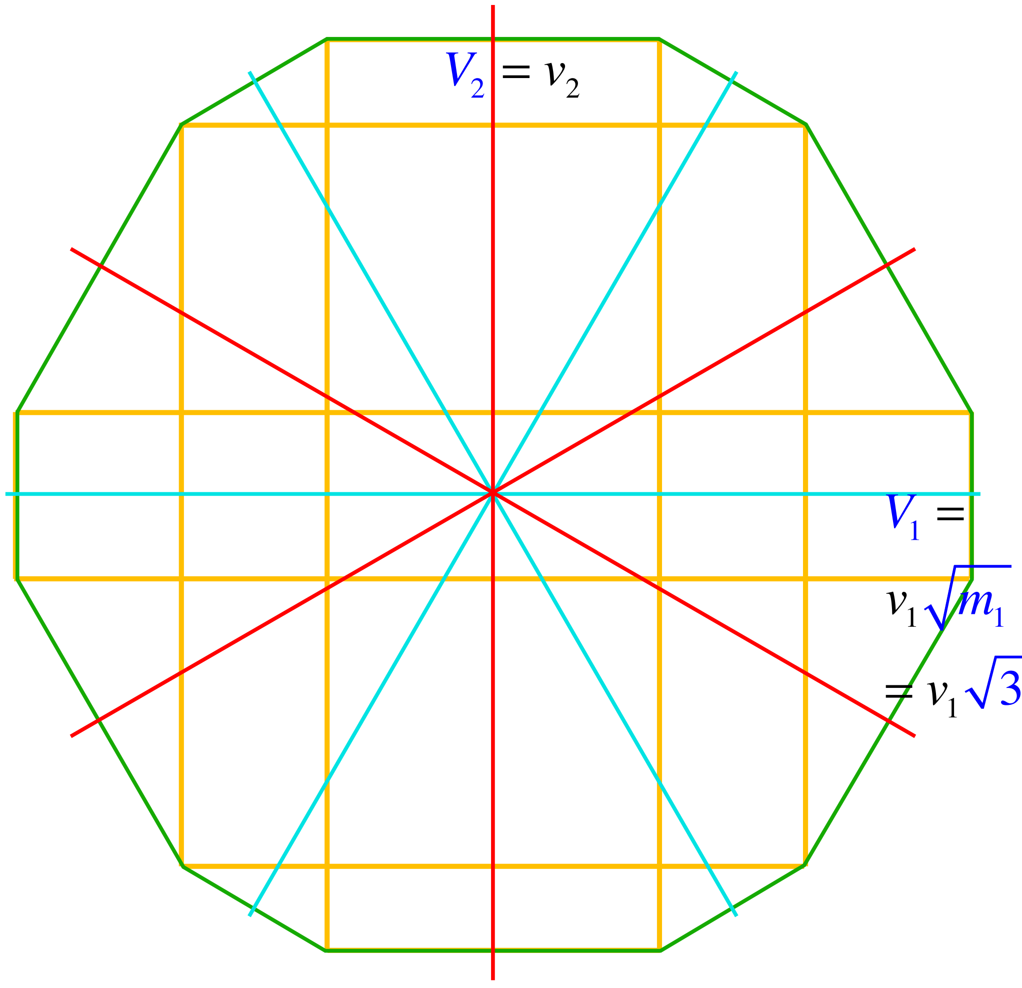


m_1/m_2=(3)/(1)

reduce v_1 scale by $1/\sqrt{m_1} = 1/\sqrt{3}=0.577$







$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}V_1^2 + \frac{1}{2}V_2^2$$

Ellipse rescaling-geometry and reflection-symmetry analysis

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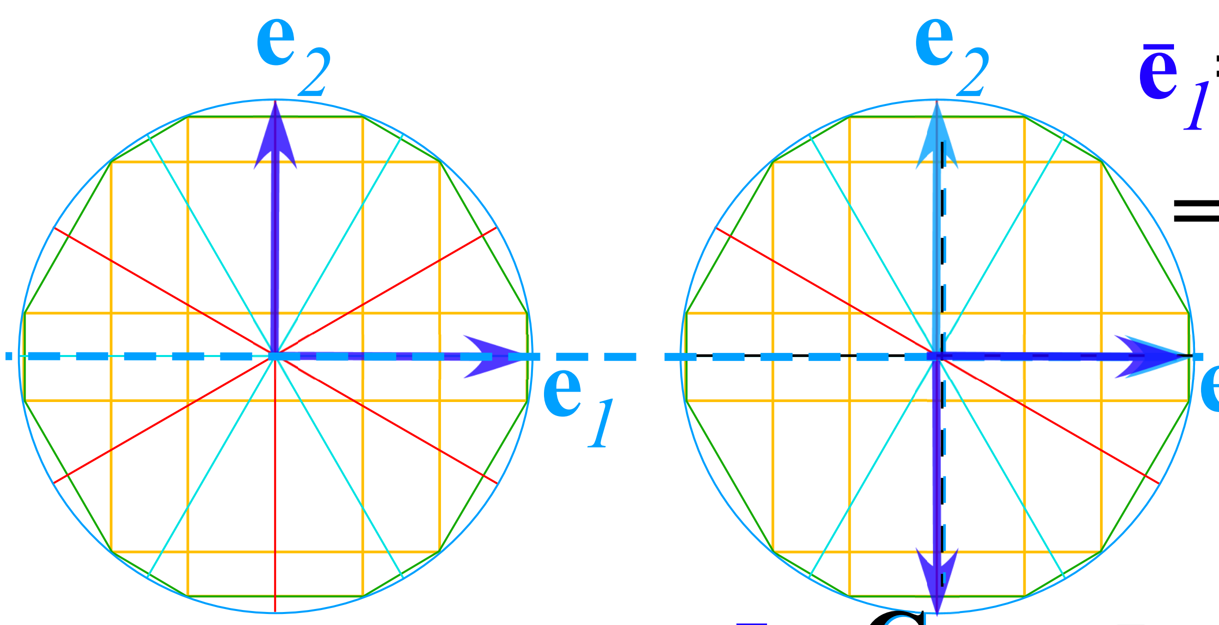
Effects of Ceiling Bang Matrix $\mathbf{C}=\boldsymbol{\sigma}_z=\begin{pmatrix} \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_2 \end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

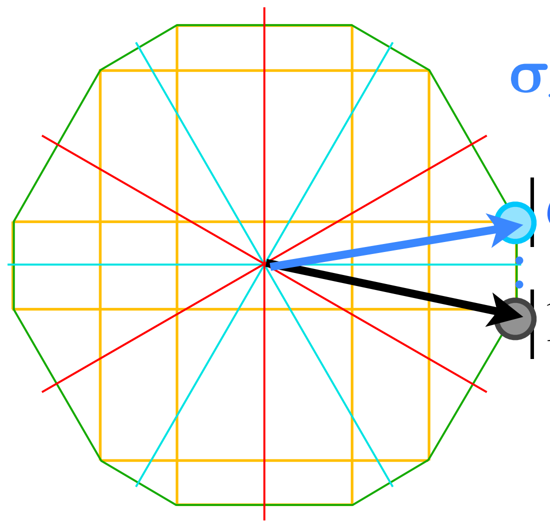


$\bar{\mathbf{e}}_1=\mathbf{C}\bullet\mathbf{e}_1$

$=\boldsymbol{\sigma}_z\bullet\mathbf{e}_1=\mathbf{e}_1$

$\mathbf{e}_1=\bar{\mathbf{e}}_1$

$\bar{\mathbf{e}}_2=\mathbf{C}\bullet\mathbf{e}_2=\boldsymbol{\sigma}_z\bullet\mathbf{e}_2=-\mathbf{e}_2$



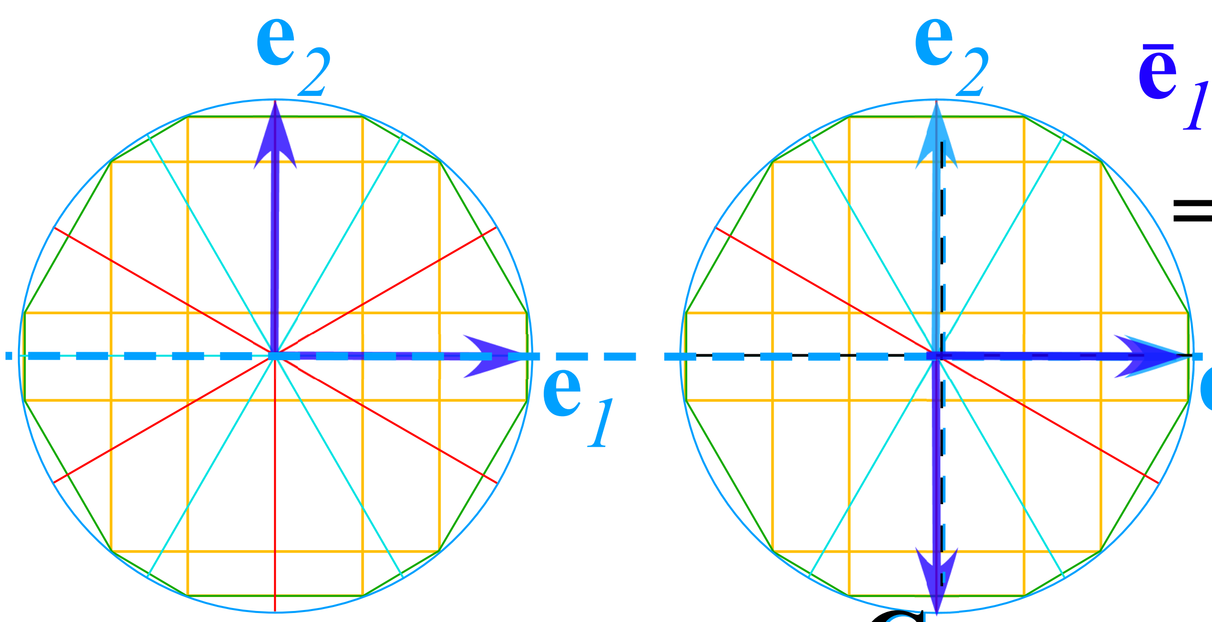
$\boldsymbol{\sigma}_x$-reflected state

$|C\rangle=\mathbf{C}|1\rangle$

$|1\rangle=\mathbf{1}|1\rangle$

Generic Initial state

Effects of Ceiling Bang Matrix $\mathbf{C}=\sigma_z=\begin{pmatrix} \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{C}\bullet\mathbf{e}_2\\ \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{C}\bullet\mathbf{e}_2\end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2\\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

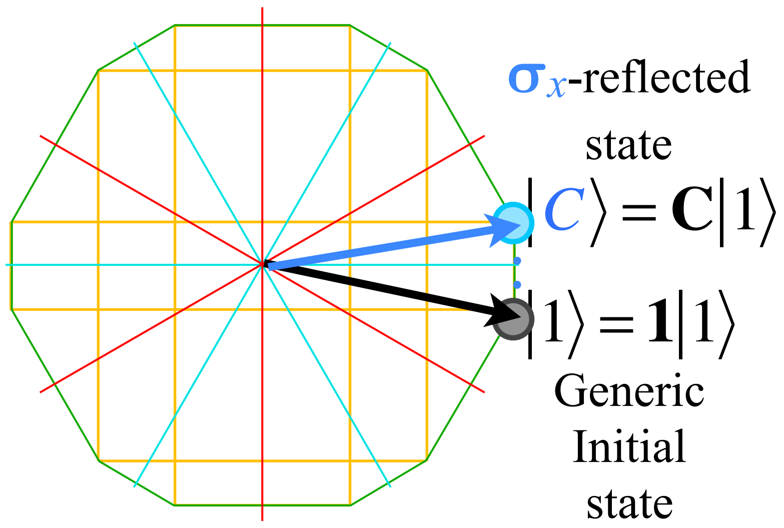


$\mathbf{e}_1=\bar{\mathbf{e}}_1$

$\bar{\mathbf{e}}_1=\mathbf{C}\bullet\mathbf{e}_1$

$=\sigma_z\bullet\mathbf{e}_1=\mathbf{e}_1$

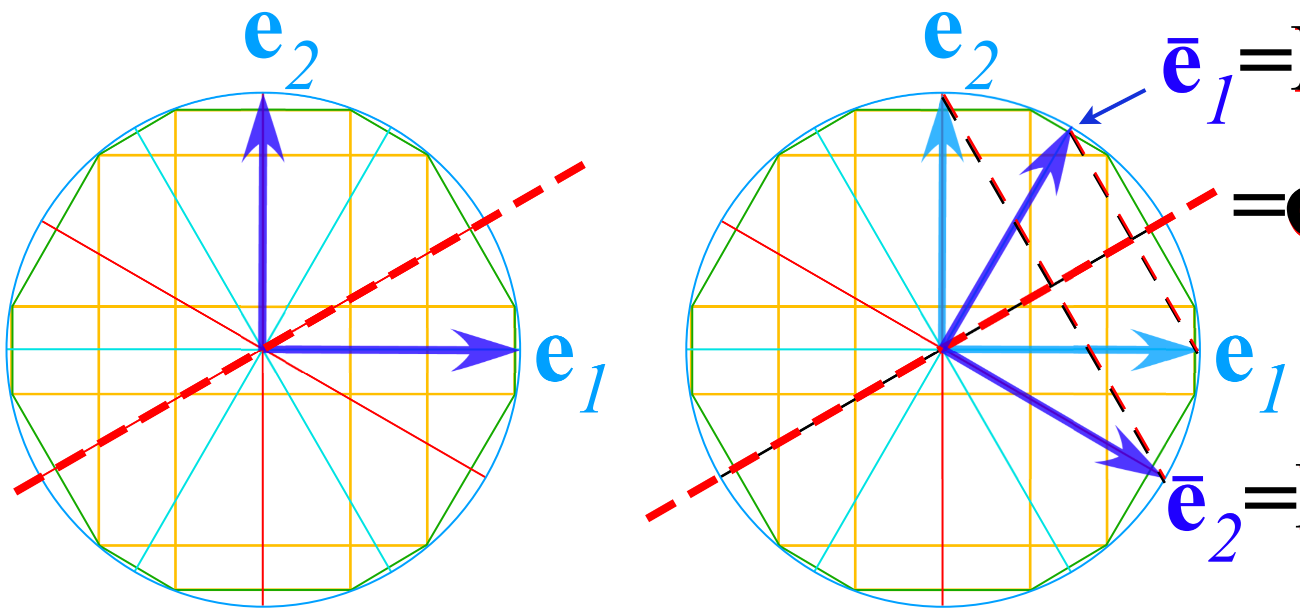
$\bar{\mathbf{e}}_2=\mathbf{C}\bullet\mathbf{e}_2=\sigma_z\bullet\mathbf{e}_2=-\mathbf{e}_2$



Known as *matrix elements* or *components*

Known as *relative direction cosines*

Effects of Mass Bang Matrix $\mathbf{M}=\sigma_{30^\circ}=\begin{pmatrix} \mathbf{e}_1\bullet\mathbf{M}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{M}\bullet\mathbf{e}_2\\ \mathbf{e}_2\bullet\mathbf{M}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{M}\bullet\mathbf{e}_2\end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2\\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2\end{pmatrix}=\begin{pmatrix}\cos 60^\circ & \sin 60^\circ\\ \sin 60^\circ & -\cos 60^\circ\end{pmatrix}$

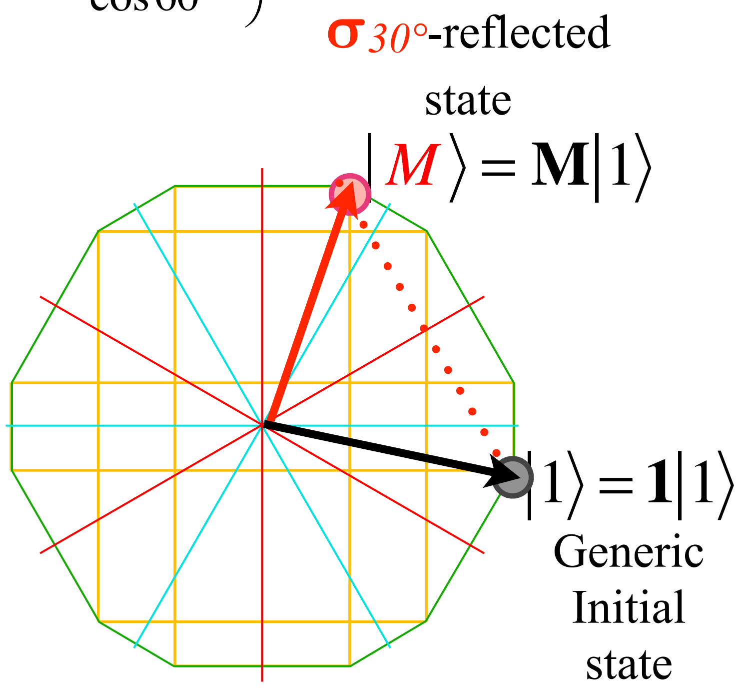


$\bar{\mathbf{e}}_1=\mathbf{M}\bullet\mathbf{e}_1$

$=\sigma_{30^\circ}\bullet\mathbf{e}_1$

$\bar{\mathbf{e}}_2=\mathbf{M}\bullet\mathbf{e}_2$

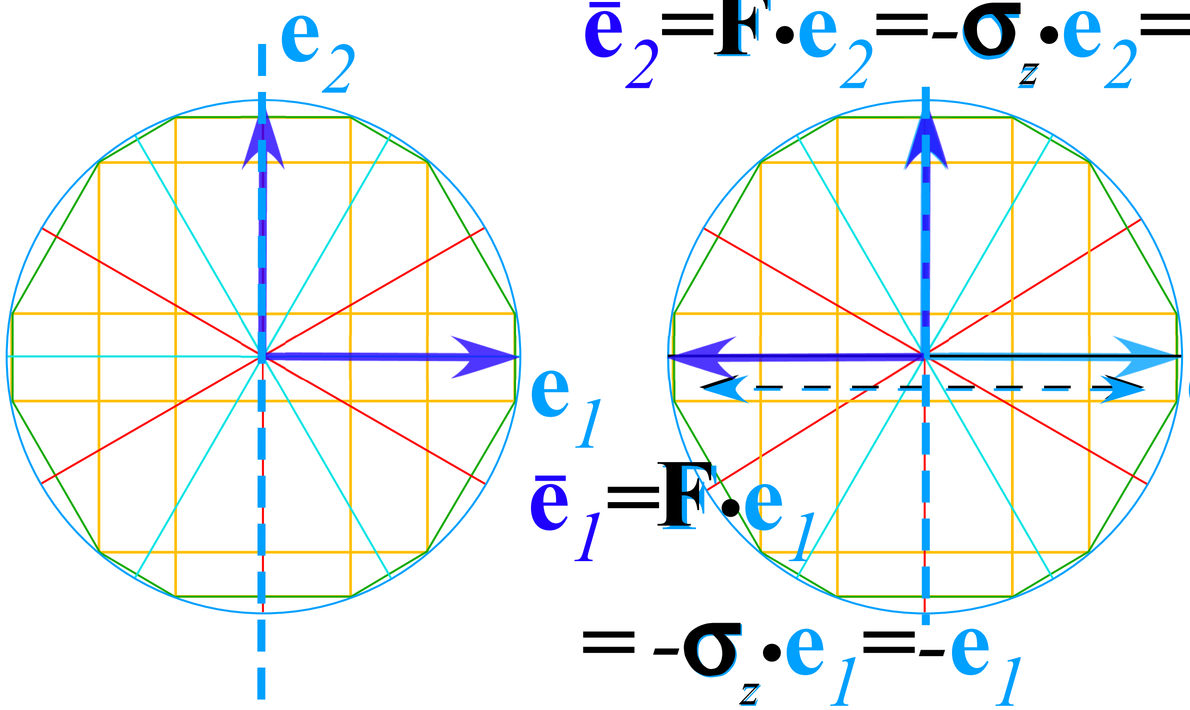
$=\sigma_{30^\circ}\bullet\mathbf{e}_2$



Effects of Floor Bang Matrix

$$\mathbf{F}=-\boldsymbol{\sigma}_z=\begin{pmatrix} \mathbf{e}_1\cdot\mathbf{F}\cdot\mathbf{e}_1 & \mathbf{e}_1\cdot\mathbf{F}\cdot\mathbf{e}_2 \\ \mathbf{e}_2\cdot\mathbf{F}\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{F}\cdot\mathbf{e}_2 \end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\cdot\bar{\mathbf{e}}_1 & \mathbf{e}_1\cdot\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\cdot\bar{\mathbf{e}}_1 & \mathbf{e}_2\cdot\bar{\mathbf{e}}_2 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{\mathbf{e}}_2=\mathbf{F}\cdot\mathbf{e}_2=-\boldsymbol{\sigma}_z\cdot\mathbf{e}_2=+\mathbf{e}_2$$

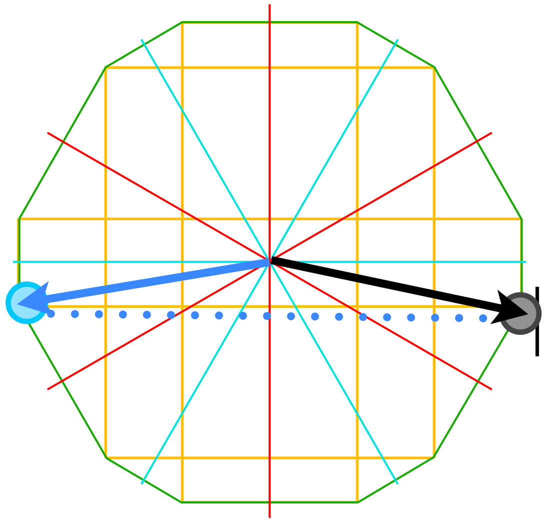


$$\bar{\mathbf{e}}_1=\mathbf{F}\cdot\mathbf{e}_1$$

$$=-\boldsymbol{\sigma}_z\cdot\mathbf{e}_1=-\mathbf{e}_1$$

$-\boldsymbol{\sigma}_z$ -reflected state

$|F\rangle=\mathbf{F}|1\rangle$

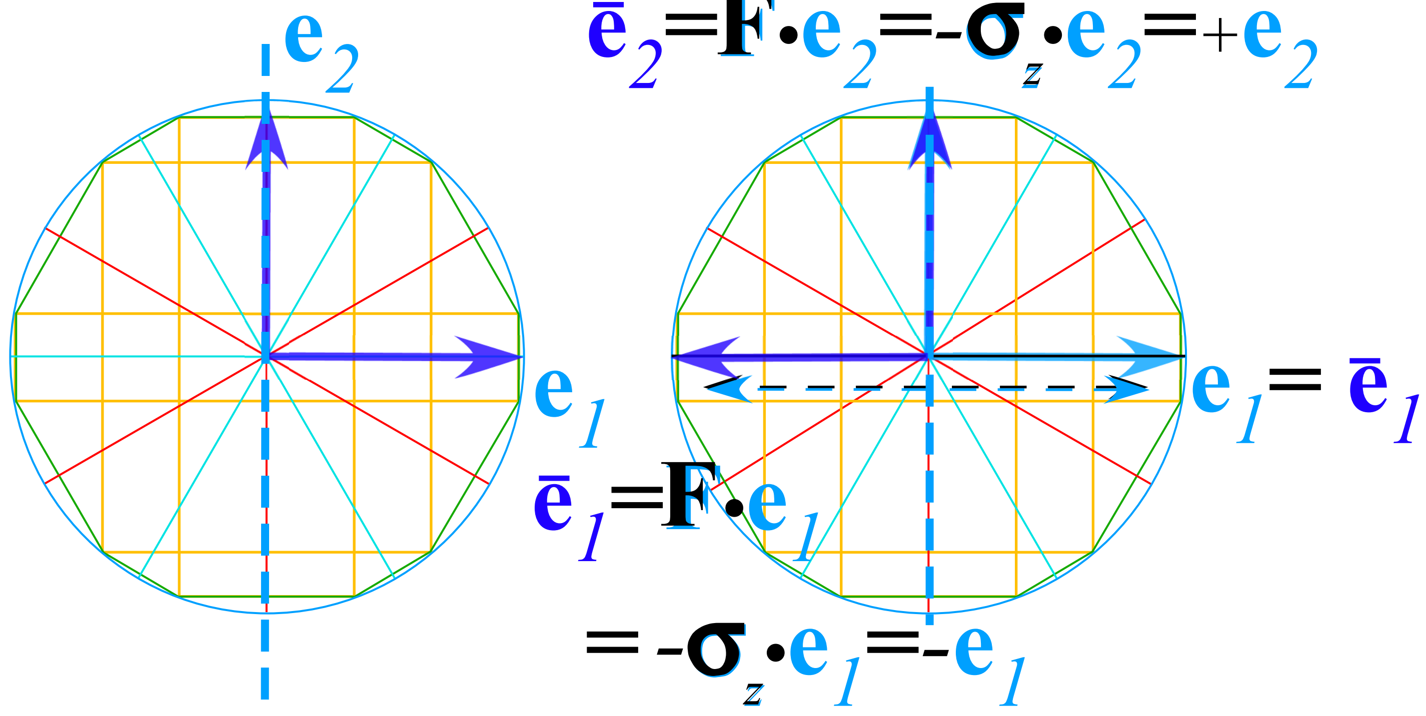


$|1\rangle=\mathbf{1}|1\rangle$

Generic Initial state

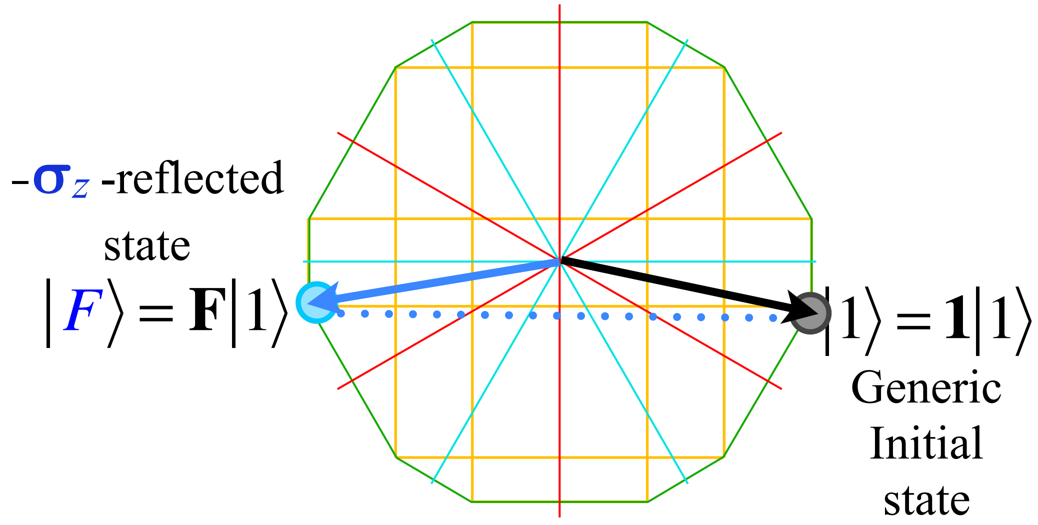
Effects of Floor Bang Matrix $\mathbf{F}=-\boldsymbol{\sigma}_z=\begin{pmatrix} \mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_1\bullet\mathbf{F}\bullet\mathbf{e}_2 \\ \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_1 & \mathbf{e}_2\bullet\mathbf{F}\bullet\mathbf{e}_2 \end{pmatrix}=\begin{pmatrix} \mathbf{e}_1\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_1\bullet\bar{\mathbf{e}}_2 \\ \mathbf{e}_2\bullet\bar{\mathbf{e}}_1 & \mathbf{e}_2\bullet\bar{\mathbf{e}}_2 \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\bar{\mathbf{e}}_2=\mathbf{F}\bullet\mathbf{e}_2=-\boldsymbol{\sigma}_z\bullet\mathbf{e}_2=+\mathbf{e}_2$$

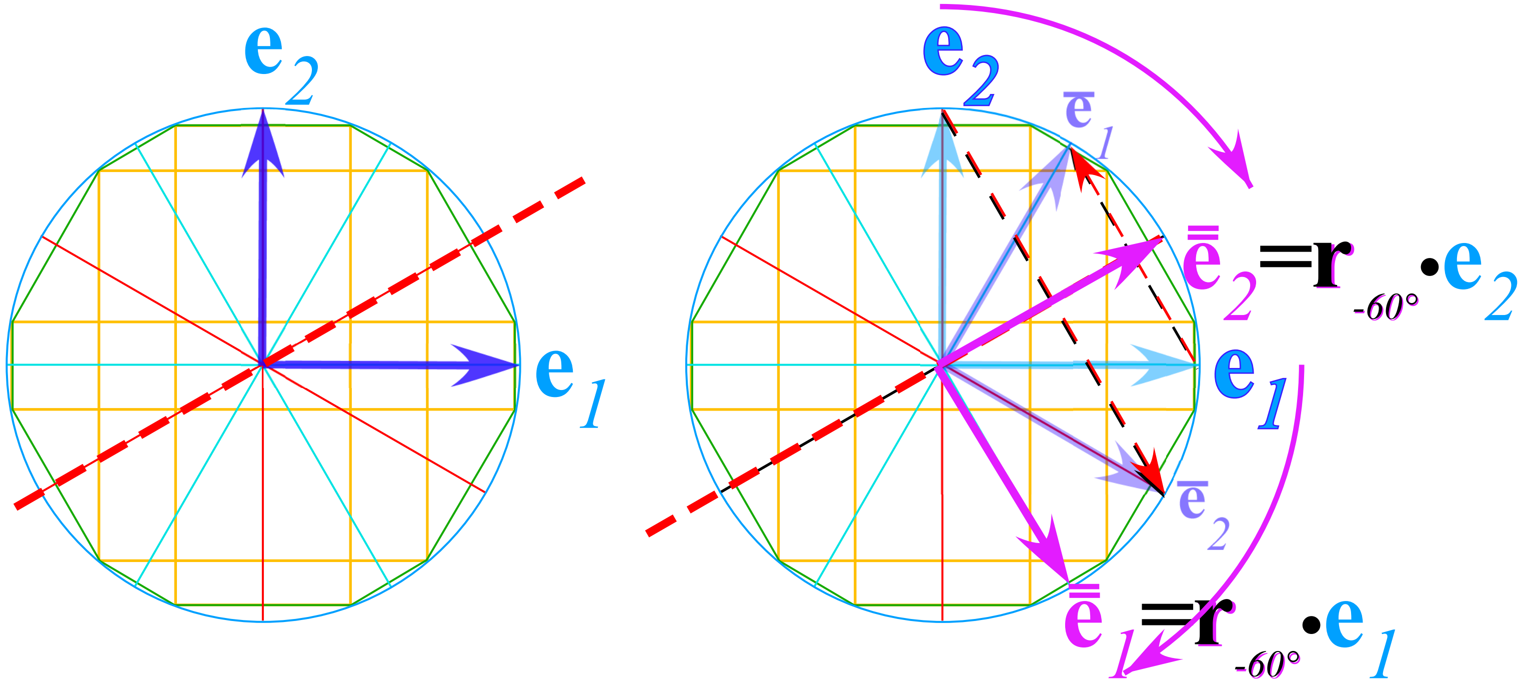


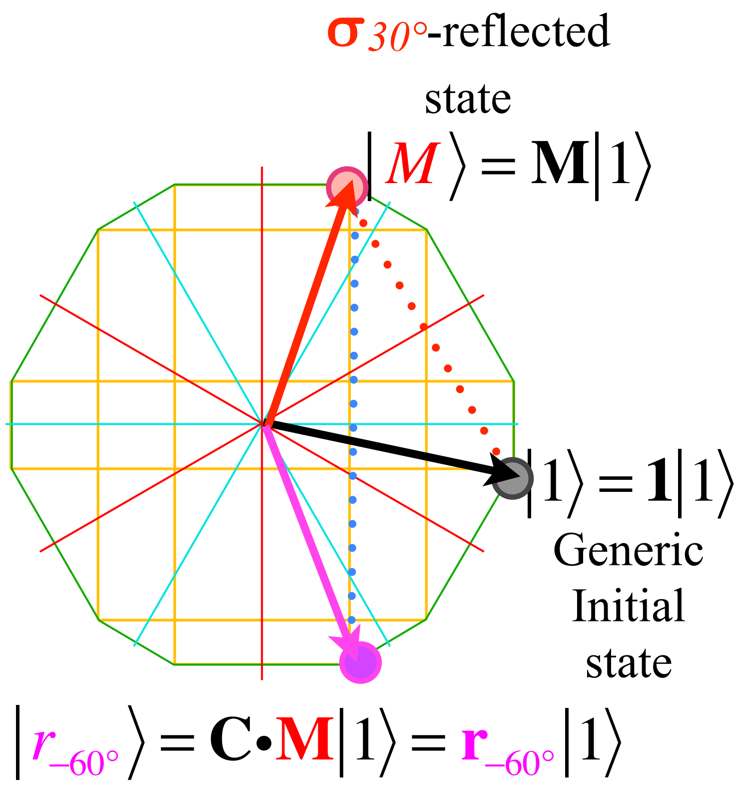
$$\bar{\mathbf{e}}_1=\mathbf{F}\bullet\mathbf{e}_1$$

$$=-\boldsymbol{\sigma}_z\bullet\mathbf{e}_1=-\mathbf{e}_1$$



Effects of Ceiling **C** after Bang **M**: $\mathbf{r}_{-60°}=\mathbf{C}\bullet\mathbf{M}=\boldsymbol{\sigma}_z\bullet\boldsymbol{\sigma}_{30°}$





$$|r_{-60°}\rangle=\mathbf{C}\bullet\mathbf{M}|1\rangle=\mathbf{r}_{-60°}|1\rangle$$

$\boldsymbol{\sigma}_{30°}\,\boldsymbol{\sigma}_{30°}$-reflected state is a $\mathbf{r}_{-60°}$-rotated state

Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
  - *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*
- *Reflections in the clothing store: "It's all done with mirrors!"*
- *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*
  - → *Group multiplication and product table* ←
  - *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*
    - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$*

D_6	1	r_{120}	$\bar{r}_{120}$	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	$\bar{r}_{60}$	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
1	1											
$\bar{r}_{120}$		1										
r_{120}			1									
σ_{60}				1								
$\bar{\sigma}_{60}$					1							
σ_z						1					$\bar{r}_{60}$	
I							1					
r_{60}								1				
$\bar{r}_{60}$									1			
$\bar{\sigma}_{30}$										1		
σ_{30}						r_{60}					1	
$\bar{\sigma}_z$												1

Note: $\bar{r}_{60} = I r_{120} = r_{120} I = r_{-60}$ and: $I = r_{\pm 180}$

$\bar{r}_{120} = I r_{60} = r_{60} I = r_{-120}$ and: $I^2 = 1$

$\sigma_{60} = I \bar{\sigma}_{30} = \bar{\sigma}_{30} I$

$\bar{\sigma}_{60} = I \sigma_{30} = \sigma_{30} I$

$\bar{\sigma}_z = I \sigma_z = \sigma_z I$

Easy to make hexagonal (D_6) symmetry group table:

Example 1: Find $\boldsymbol{\sigma}_{30°} \bullet \boldsymbol{\sigma}_{-60°} =$ ______?

Solution: Find $\boldsymbol{\sigma}_{30°}$*-plane and state-*$|\boldsymbol{\sigma}_{-60°}\rangle$

Operate former on latter to get: $\boldsymbol{\sigma}_{30°}|\boldsymbol{\sigma}_{-60°}\rangle = |\mathbf{I}\rangle$

That gives answer: $\boldsymbol{\sigma}_{30°} \bullet \boldsymbol{\sigma}_{-60°} = \mathbf{I}$.

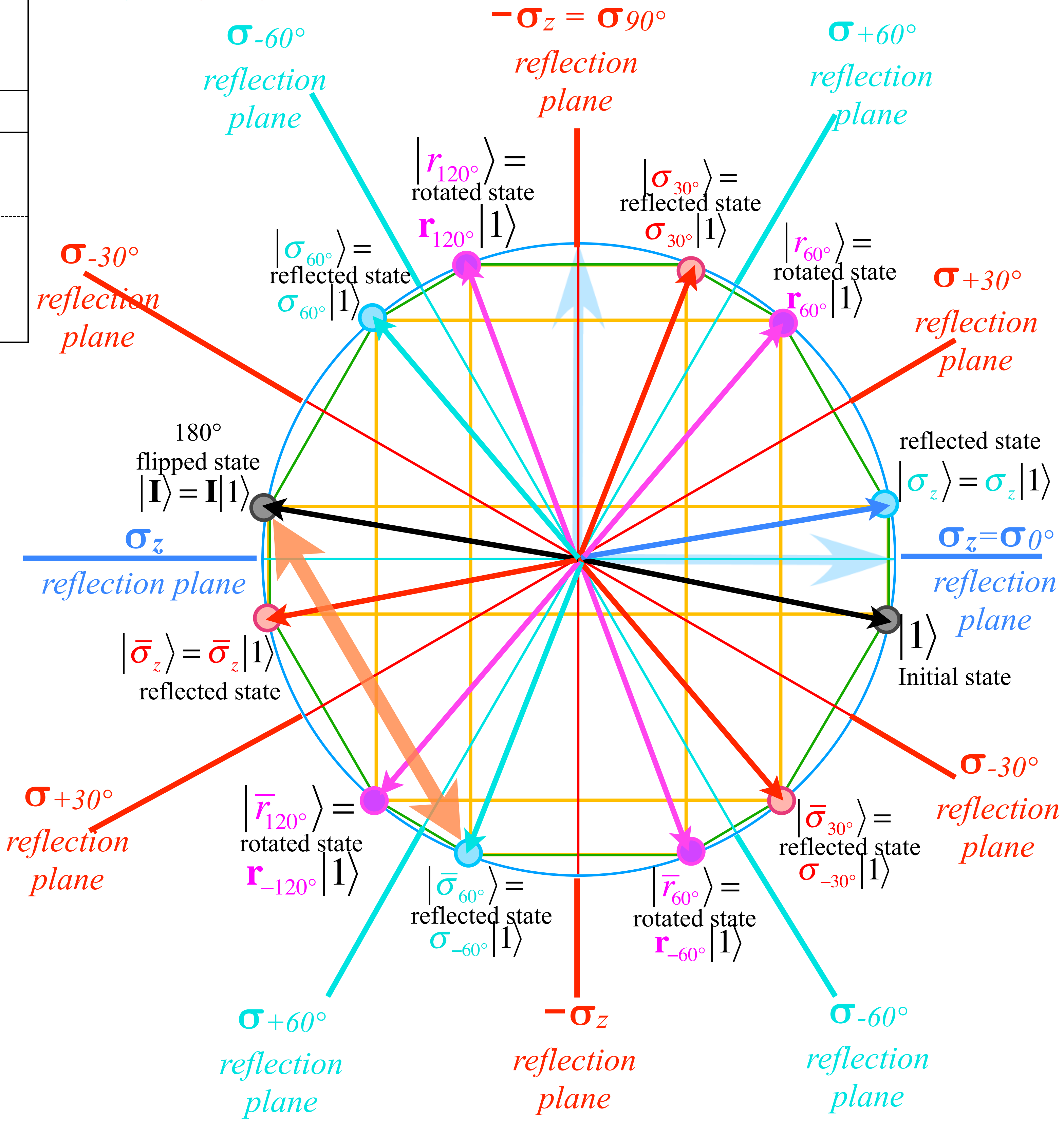
Rest of $\boldsymbol{\sigma}_{30°}$ *row follows:*

11^{th} row	1	r_{120}	$\bar{r}_{120}$	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	$\bar{r}_{60}$	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
σ_{30}	σ_{30}	$\bar{\sigma}_{30}$	$\bar{\sigma}_z$	$\bar{r}_{60}$	I	r_{60}	$\bar{\sigma}_{60}$	σ_{60}	σ_z	r_{120}	1	$\bar{r}_{120}$

Example 2: Find $\mathbf{r}_{60°} \bullet \boldsymbol{\sigma}_{-60°} =$ ______?

Solution: Do $\mathbf{r}_{60°}$*-rotation* $\mathbf{r}_{60°}|\boldsymbol{\sigma}_{-60°}\rangle = |\boldsymbol{\sigma}_{-30°}\rangle$

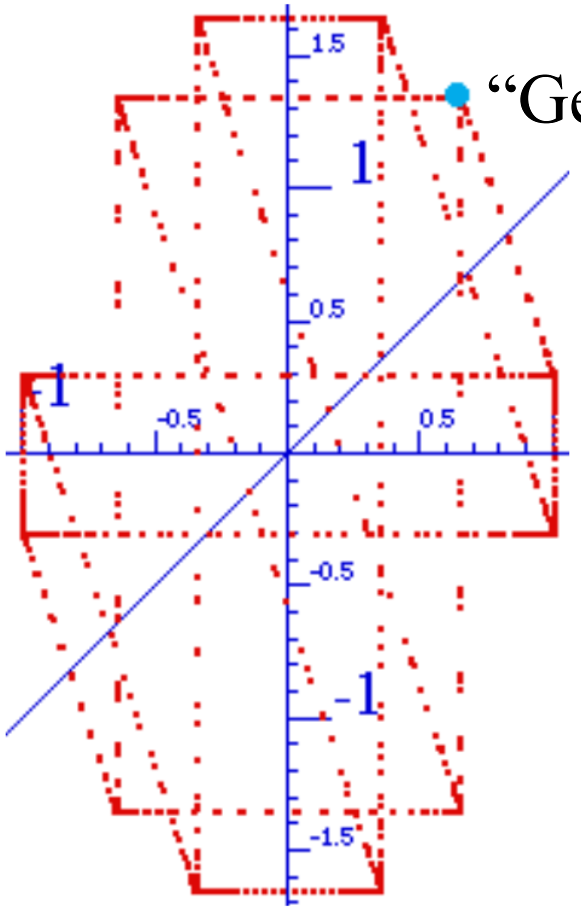
That gives answer: $\mathbf{r}_{60°} \bullet \boldsymbol{\sigma}_{-60°} = \boldsymbol{\sigma}_{-30°}$



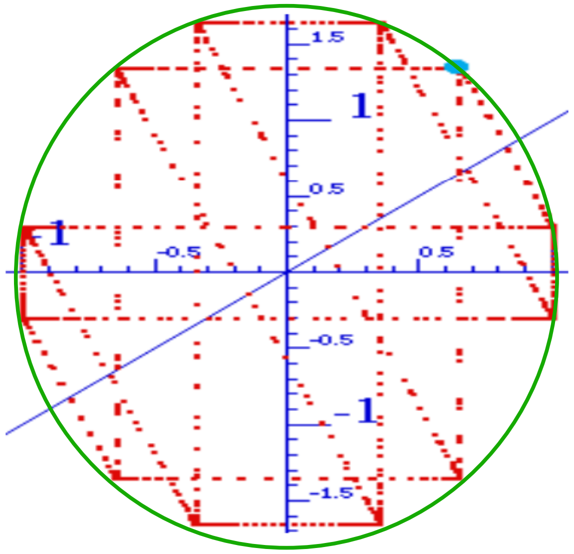
Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
  - *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*
- *Reflections in the clothing store: "It's all done with mirrors!"*
- *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*
  - *Group multiplication and product table*
  - → *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*
    - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$*

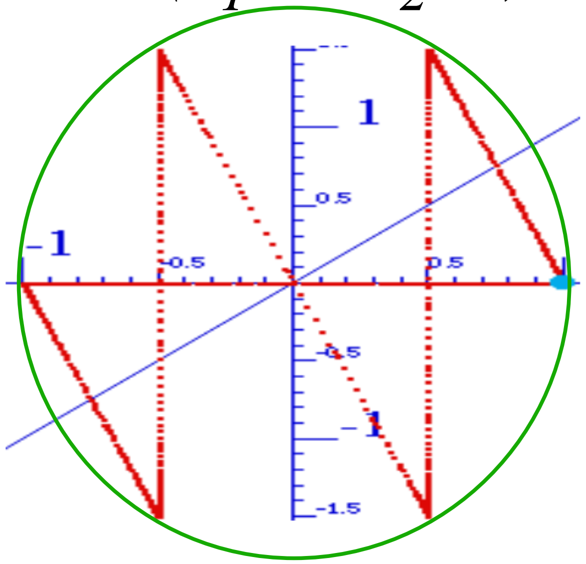
Collisions for mass ratio m_1:m_2= 3:1

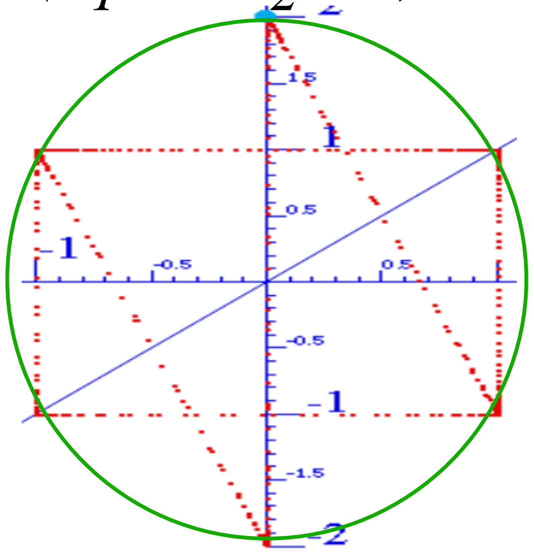


“Generic” initial velocity
(v_1=1.0, v_2=0.1)

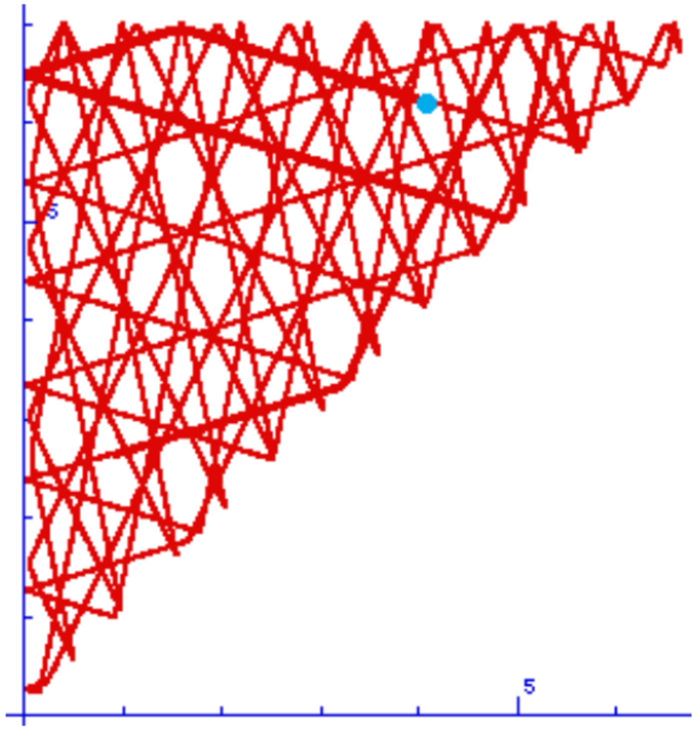


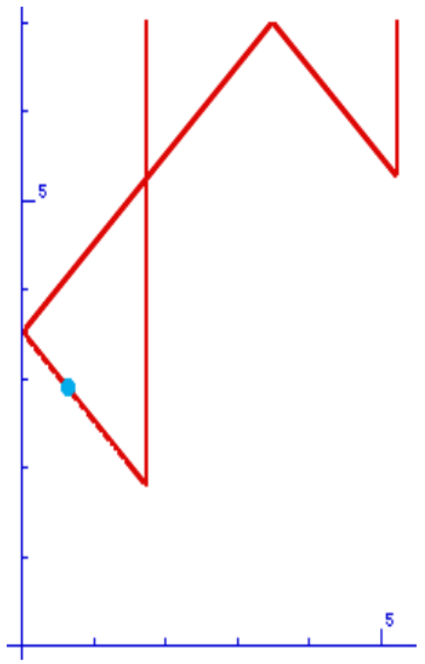
“Symmetric” initial velocity
(v_1=1, v_2=0) or (v_1=1, v_2=-1)

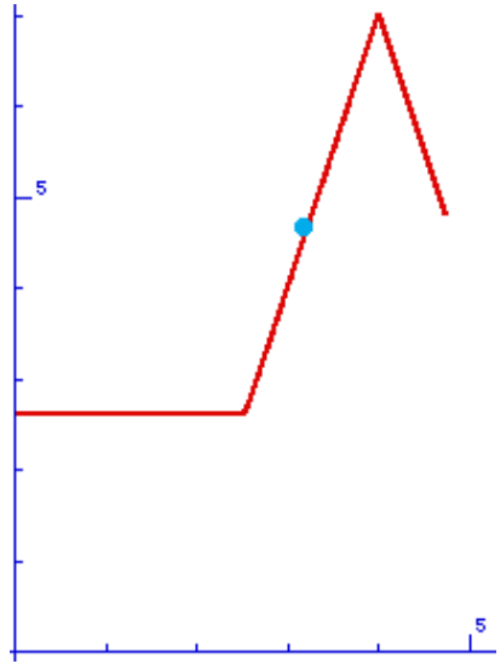




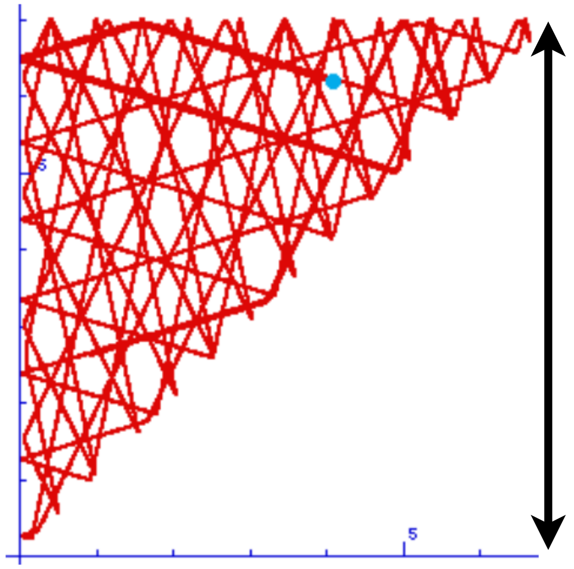
Corresponding space-space (y_1, y_2) paths

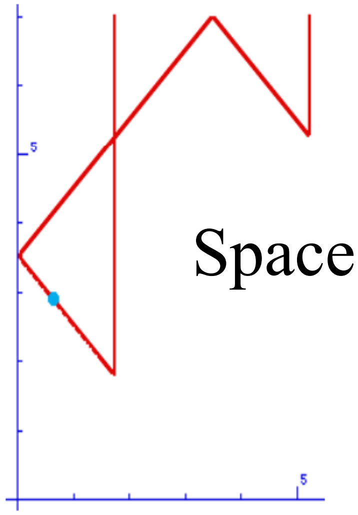




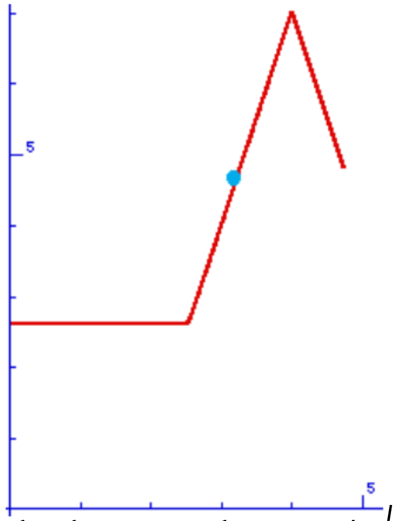


Collisions for mass ratio $m_1{:}m_2 = 3{:}1$



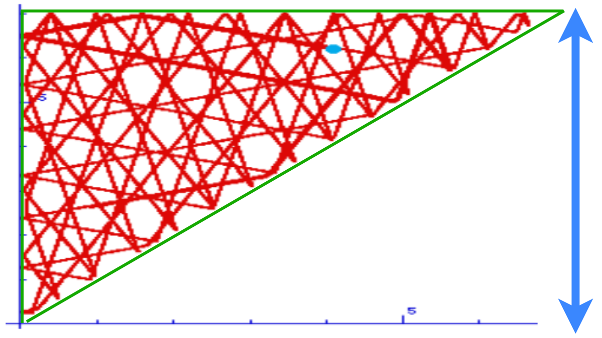


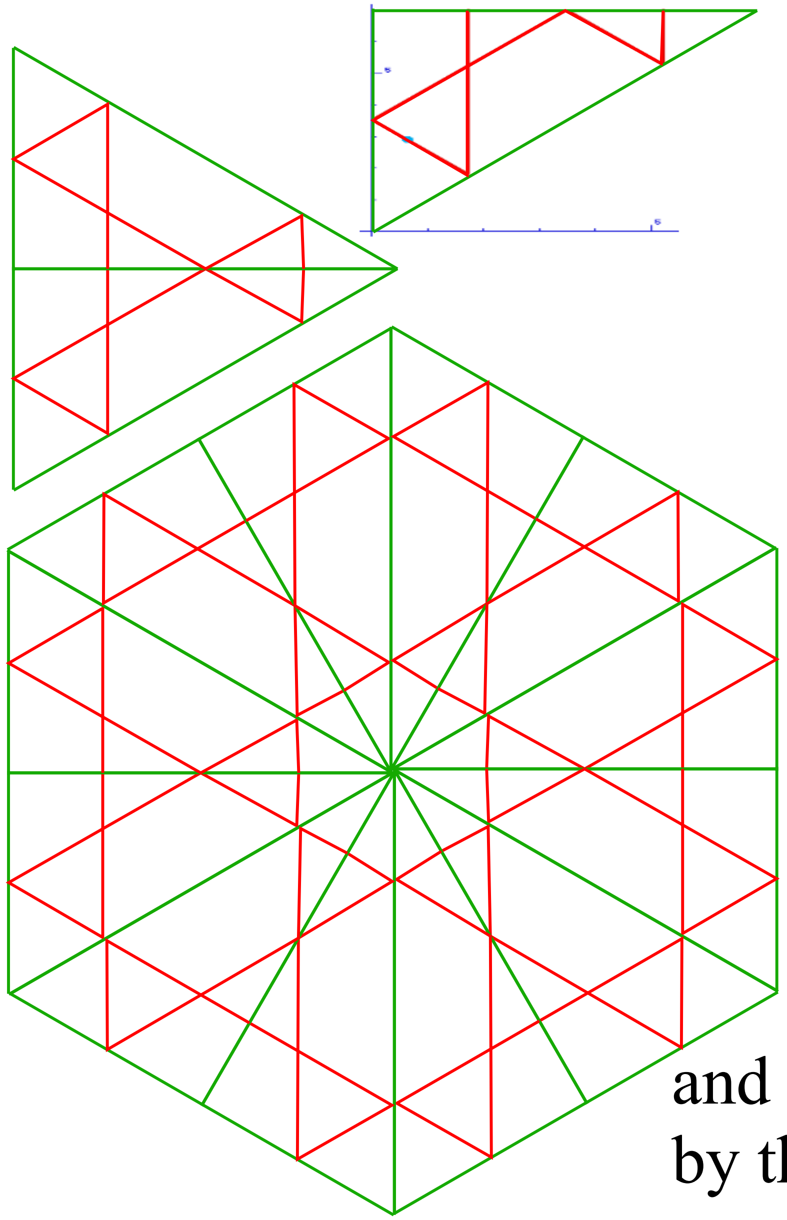
Space-space (y_1, y_2) paths

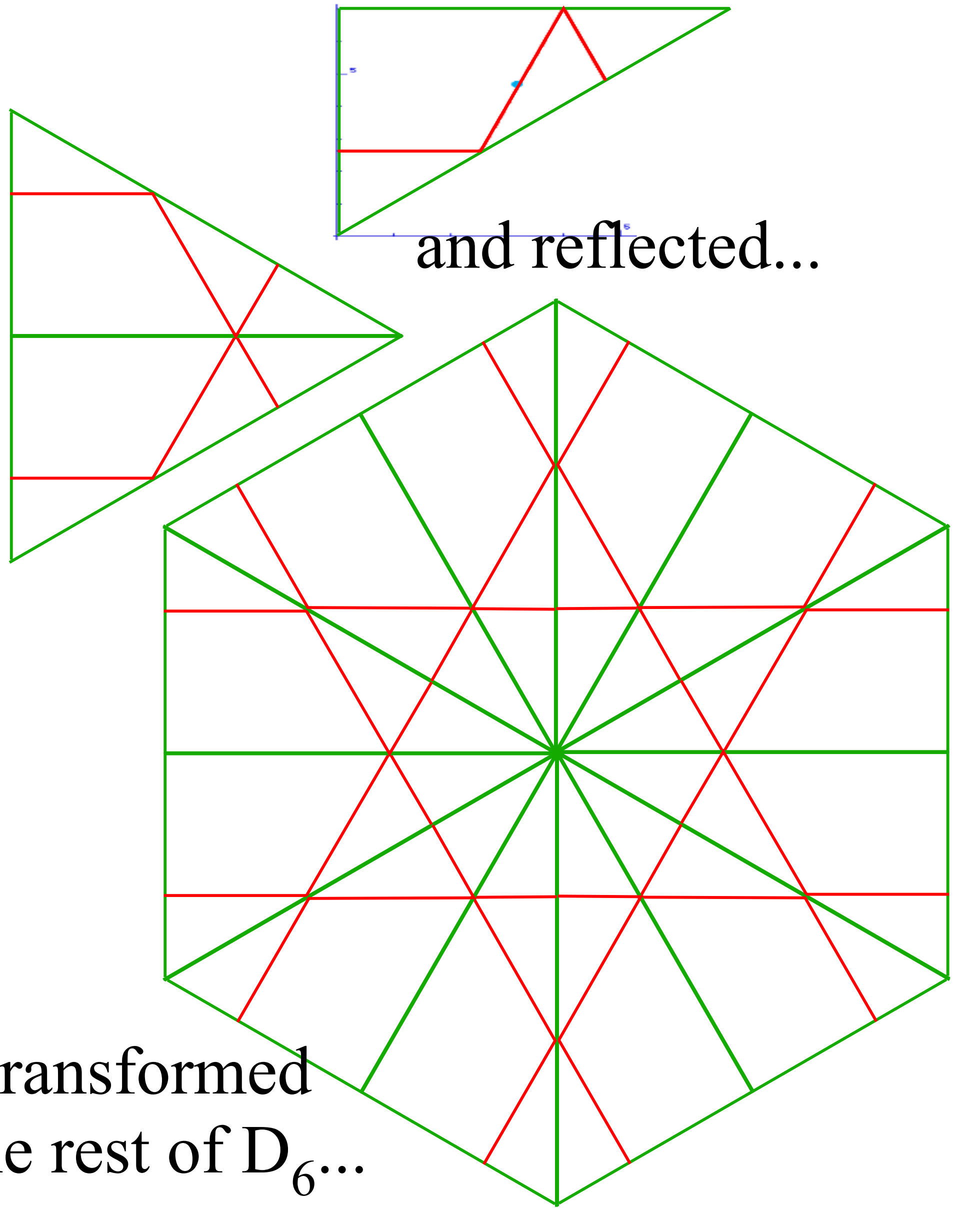


Space-space (y_1, y_2) paths scaled down by $1/\sqrt{3}$...

Scaled y down by $1/\sqrt{3}=0.577$





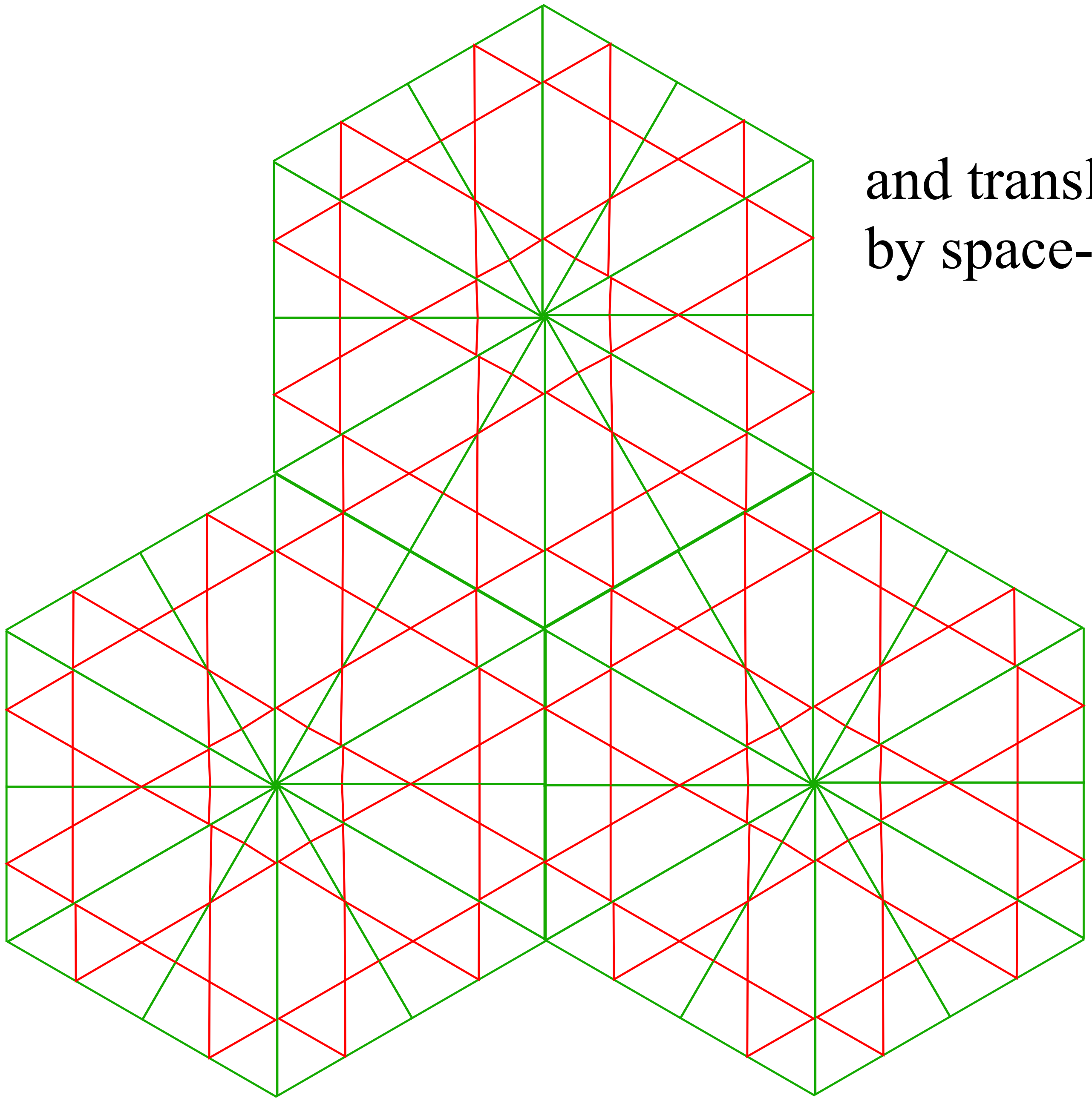


and reflected...

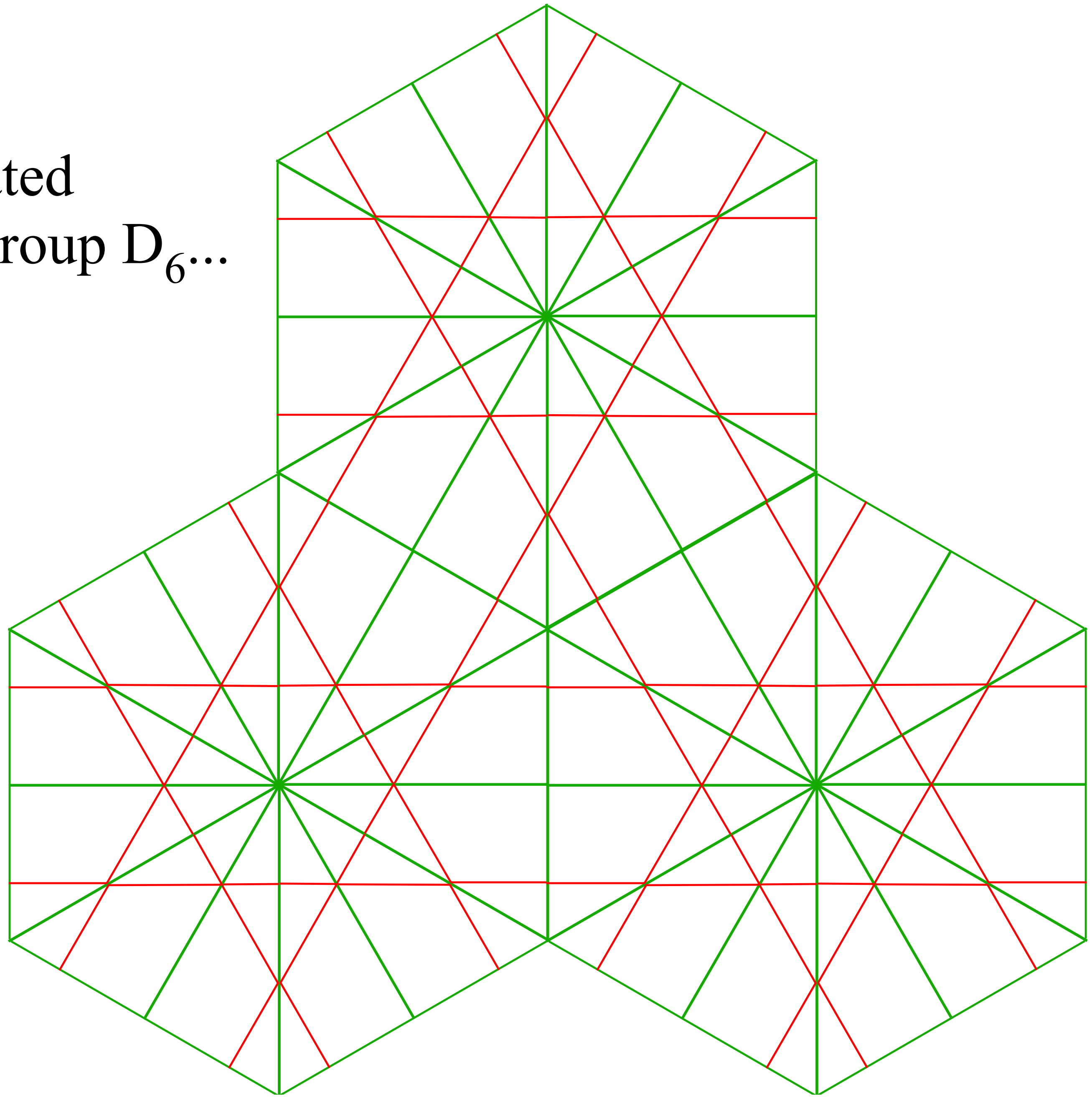
..or could have scaled x up by $\sqrt{3}=1.732$

and transformed by the rest of D_6...

Collisions for mass ratio $m_1:m_2= 3:1$

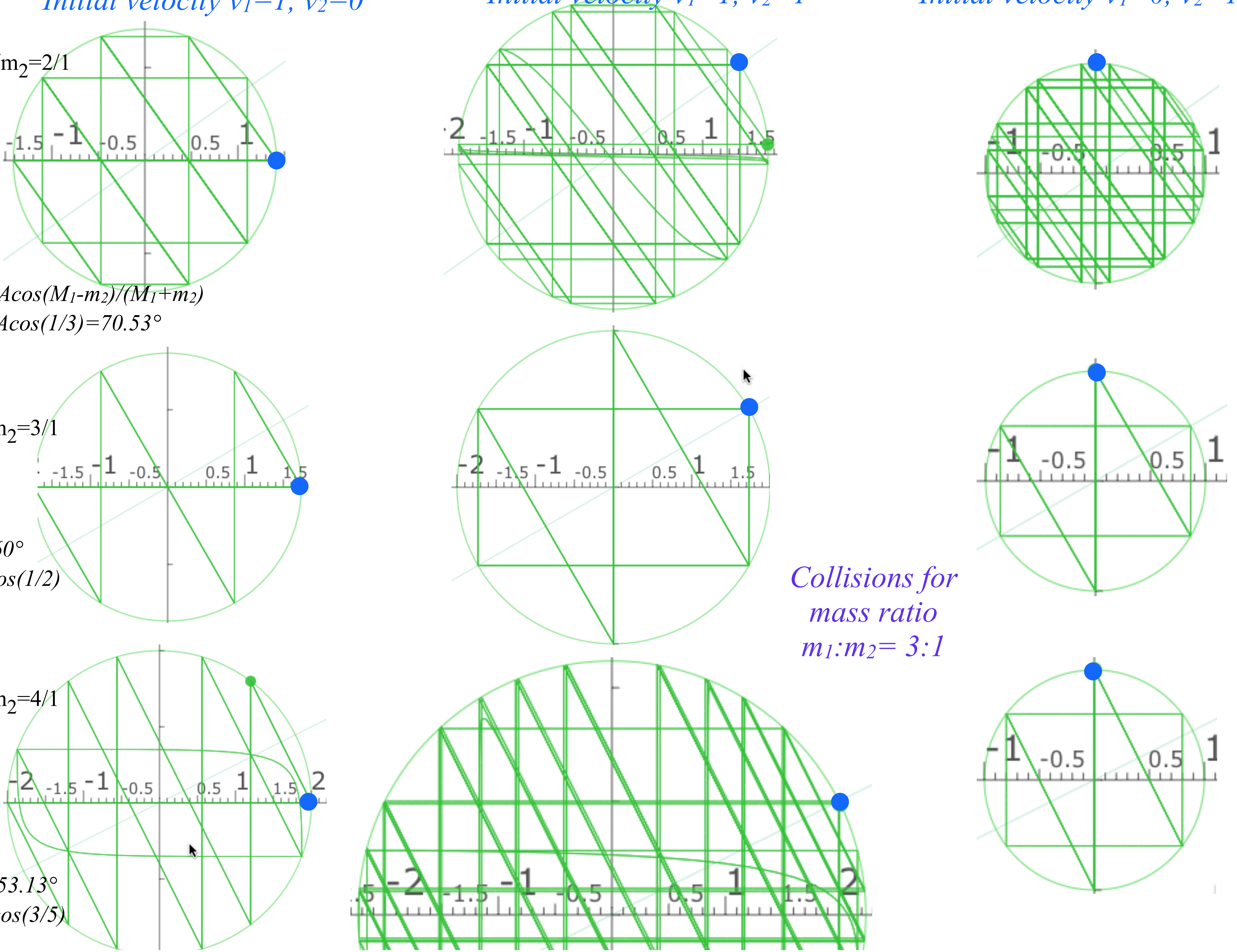


and translated
by space-group D_6...



...they’re just straight lines going forever.

Initial velocity $v_1=1$, $v_2=0$ *Initial velocity $v_1=1$, $v_2=1$* *Initial velocity $v_1=0$, $v_2=1$*



$M_1/m_2=2/1$

$\phi = A\cos(M_1-m_2)/(M_1+m_2)$
$= A\cos(1/3) = 70.53°$

$M_1/m_2=3/1$

$\phi = 60°$
$= A\cos(1/2)$

$M_1/m_2=4/1$

$\phi = 53.13°$
$= A\cos(3/5)$

Collisions for
mass ratio
$m_1 : m_2 = 3:1$

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

  How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

  Group multiplication and product table

  Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

→ *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$* ←

Geometric “Integration” (Converting Velocity data to Spacetime)

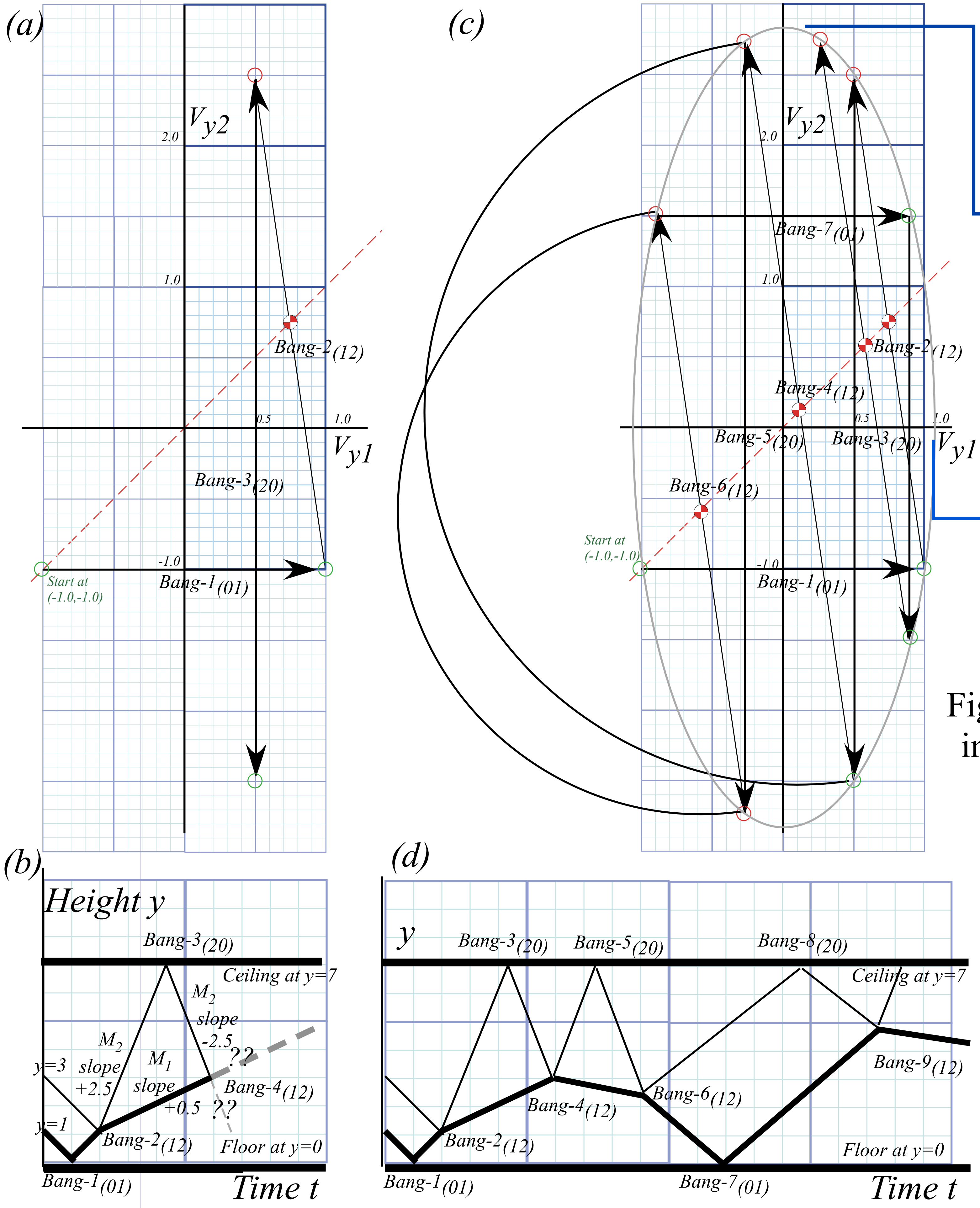


Fig. 4.7a-d
in Unit 1

Kinetic Energy Ellipse

$$KE = \frac{1}{2}M_1V_1^2 + \frac{1}{2}M_2V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE/M_1} + \frac{V_2^2}{2KE/M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE/M_1} = \sqrt{2KE/7} = \sqrt{8/7} = 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE/M_2} = \sqrt{2KE/1} = \sqrt{8/1} = 2.83$$

Collisions for
mass ratio
m₁:m₂= 7:1

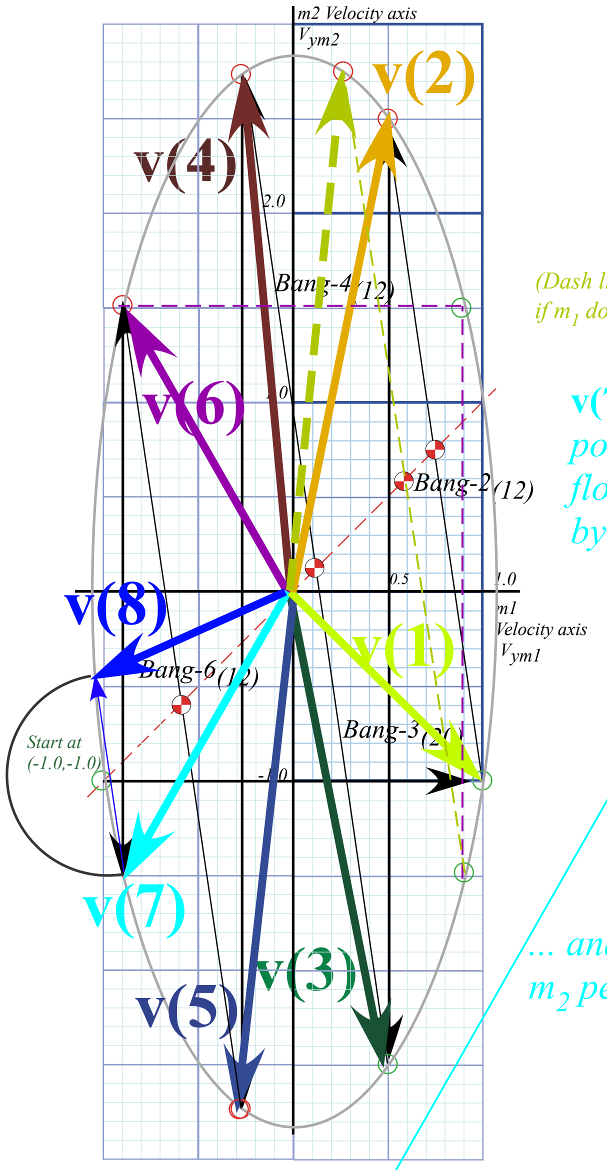
Collisions for mass ratio m_1:m_2= 7:1

Step-2: Extend $\mathbf{v}(2)$ *line to ceiling point* $\mathbf{y}(3)=(?,7.1)$ *and draw Bang-3*$_{(20)}$ *velocity* $\mathbf{v}(3)=(1,-1)$ *line. (Find* $\mathbf{v}(3)$ *using V-V plot.)*

Step-3: Extend $\mathbf{v}(3)$ *line to collision point* $\mathbf{y}(4)=(?,?)$ *and draw Bang-4*$_{(12)}$ *velocity* $\mathbf{v}(4)=(0.5,2.5)$*. (Find* $\mathbf{v}(4)$ *using V-V plot.)*

Step-4: Extend $\mathbf{v}(4)$ *line to ceiling point* $\mathbf{y}(4)=(?,7.1)$ *and draw Bang-5*$_{(20)}$ *velocity* $\mathbf{v}(5)=(1,-1)$ *line. (Find* $\mathbf{v}(5)$ *using V-V plot.)*

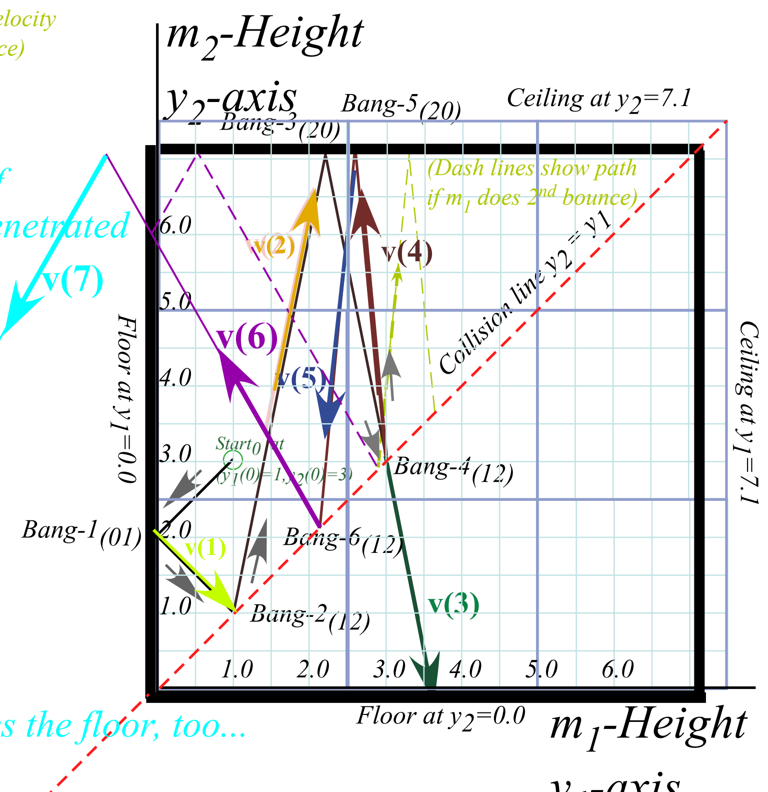
Step-5: Extend $\mathbf{v}(5)$ *line to collision point* $\mathbf{y}(6)=(?,?)$ *and draw Bang-6*$_{(12)}$ *velocity* $\mathbf{v}(6)=(0.5,2.5)$*. (Find* $\mathbf{v}(6)$ *using V-V plot.)*

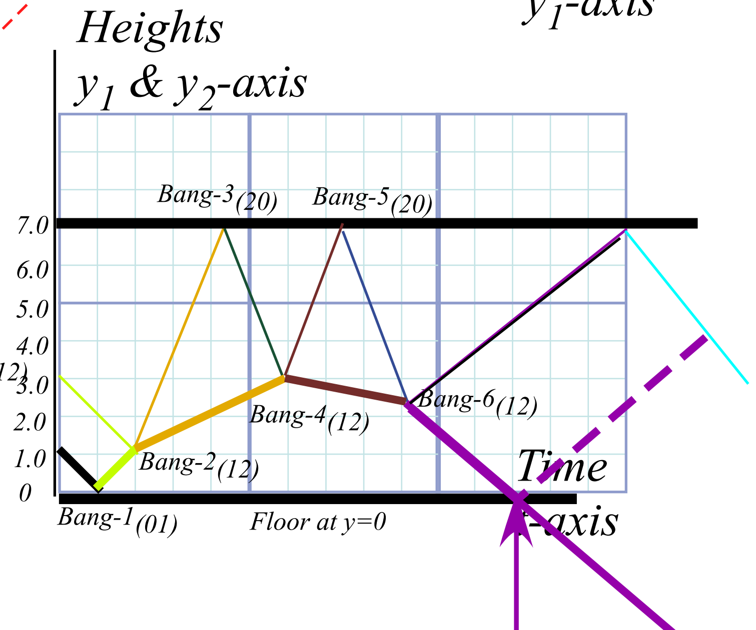


(Dash lines show velocity if m_1 *does* 2^{nd} *bounce)*

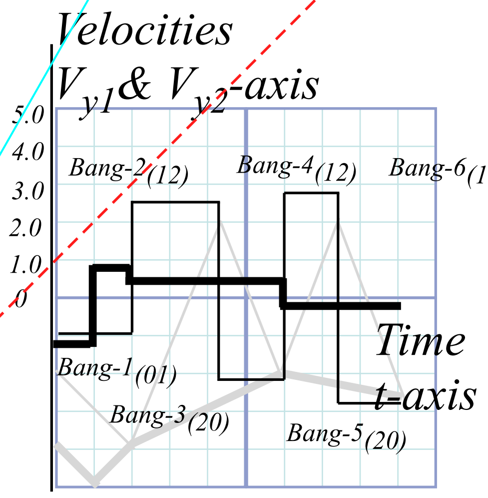
v(7) only possible if floor is penetrated by m_1 *...*

... and later m_2 *penetrates the floor, too...*





floor is penetrated by m_1 *.*



"Gameover collision" occurs way down here!

v(8)

Ellipse rescaling-geometry and reflection-symmetry analysis

- *Rescaling KE ellipse to circle*
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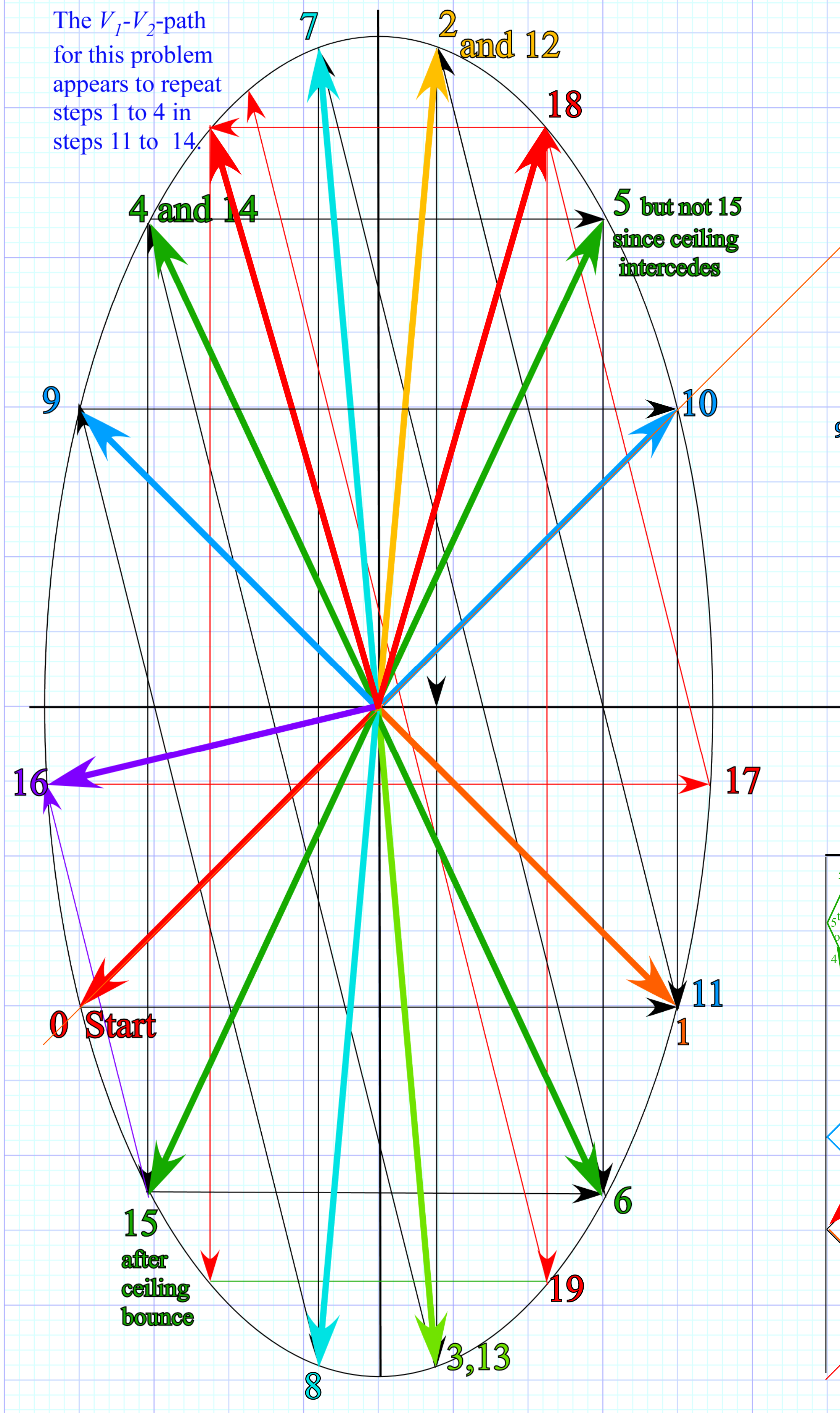
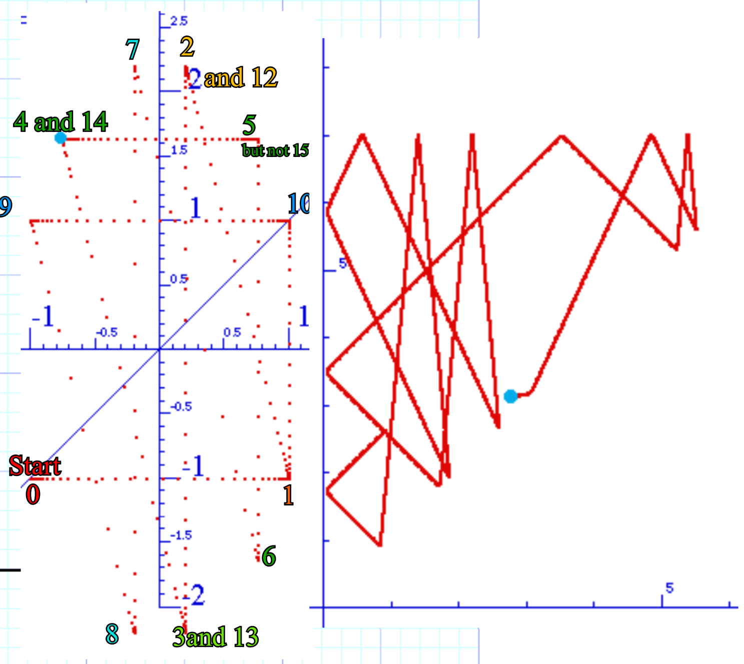
        - *Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$*

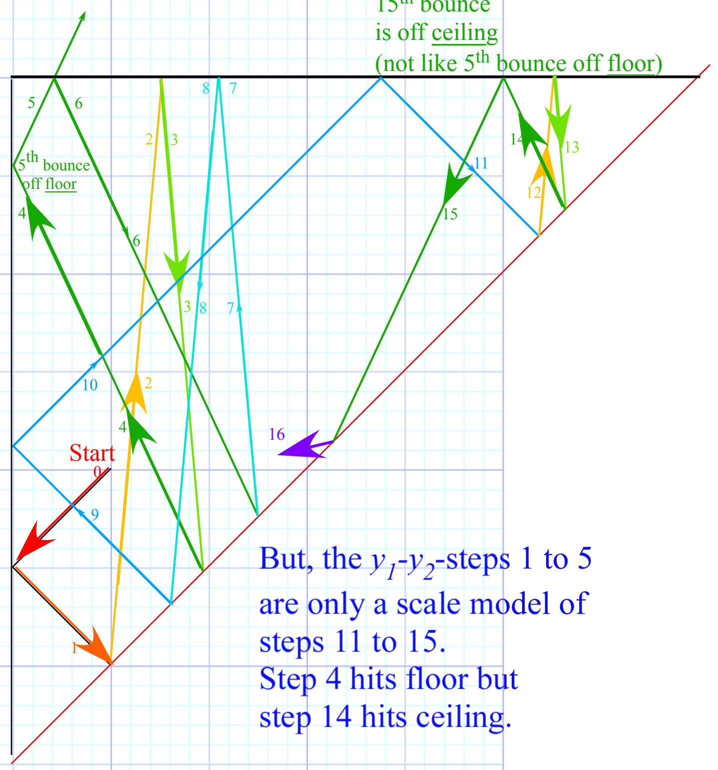
First part of Exercise 1.4.1 has pen-ball initial values $v_1(0)=-1=v_2(0)$

Collisions for mass ratio $m_1:m_2= 4:1$



The V_1-V_2-path for this problem appears to repeat steps 1 to 4 in steps 11 to 14.

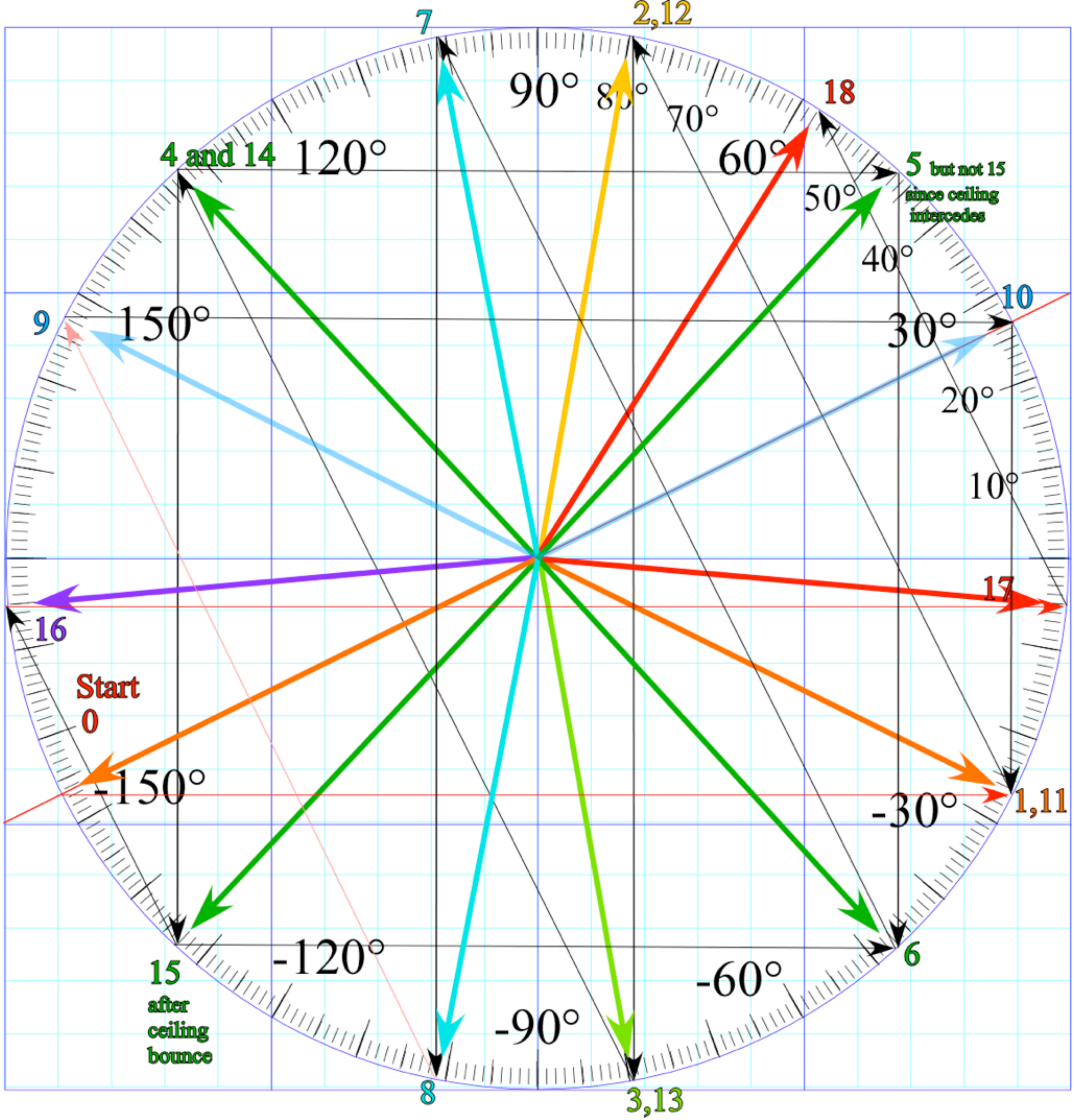




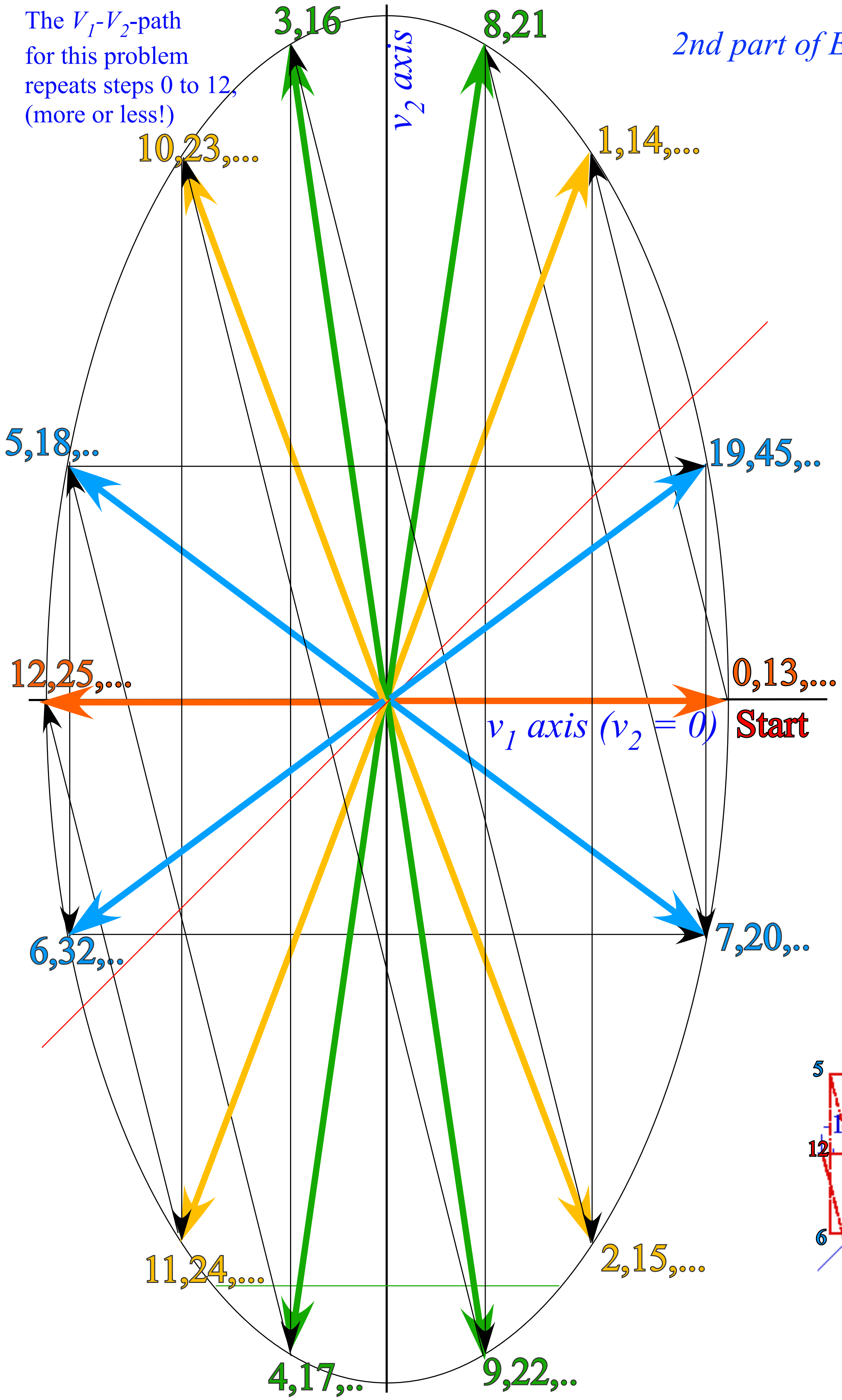
15th bounce is off ceiling (not like 5th bounce off floor)

But, the y_1-y_2-steps 1 to 5 are only a scale model of steps 11 to 15. Step 4 hits floor but step 14 hits ceiling.

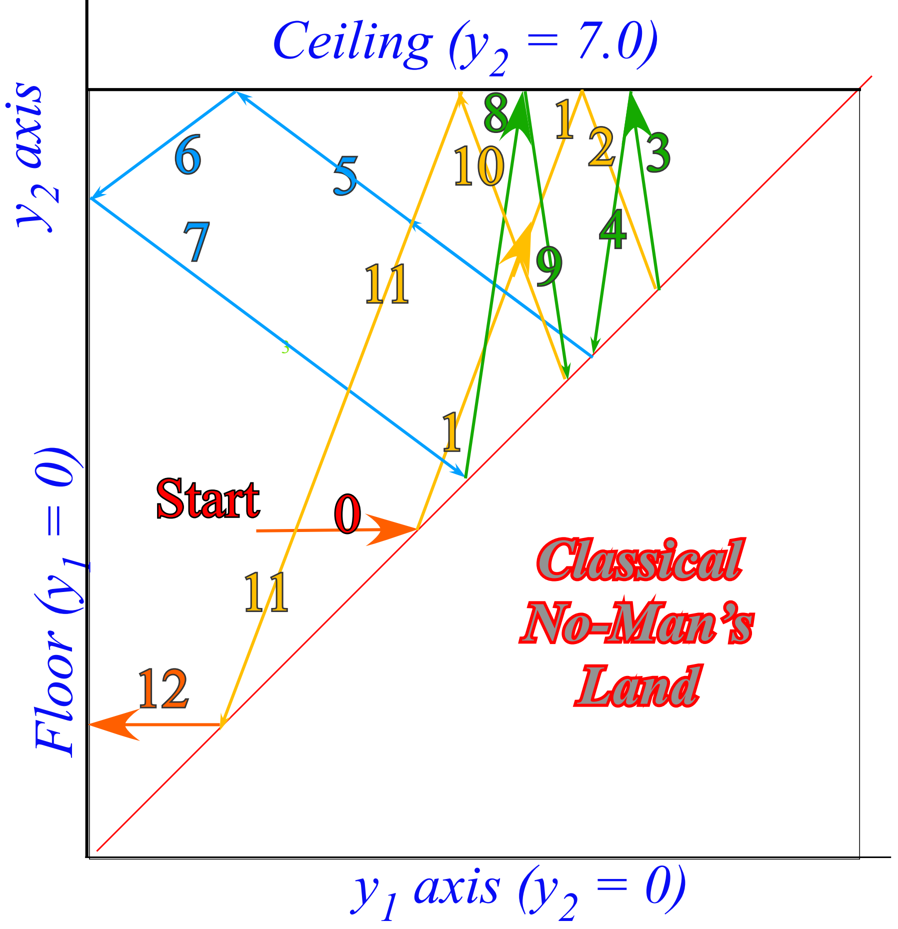
Collisions for mass ratio m_1:m_2= 4:1



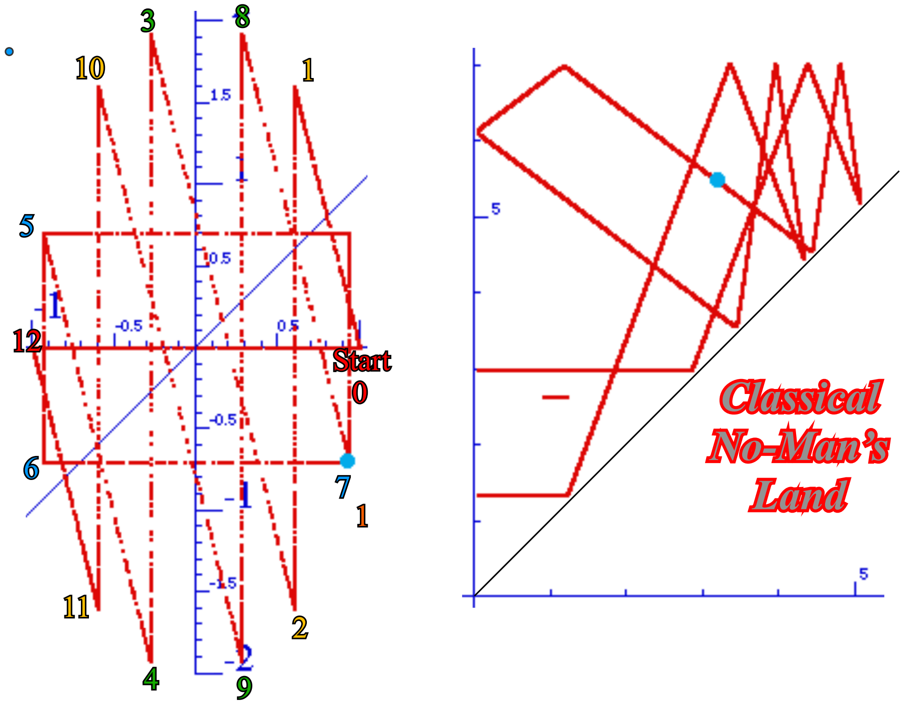
The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)



2nd part of Exercise 1.4.1 has pen-ball initial velocity values $v_1(0)=1$ and $v_2(0)=0$ at: $x_1(0)=1.5$ and $x_2(0)=3.0$

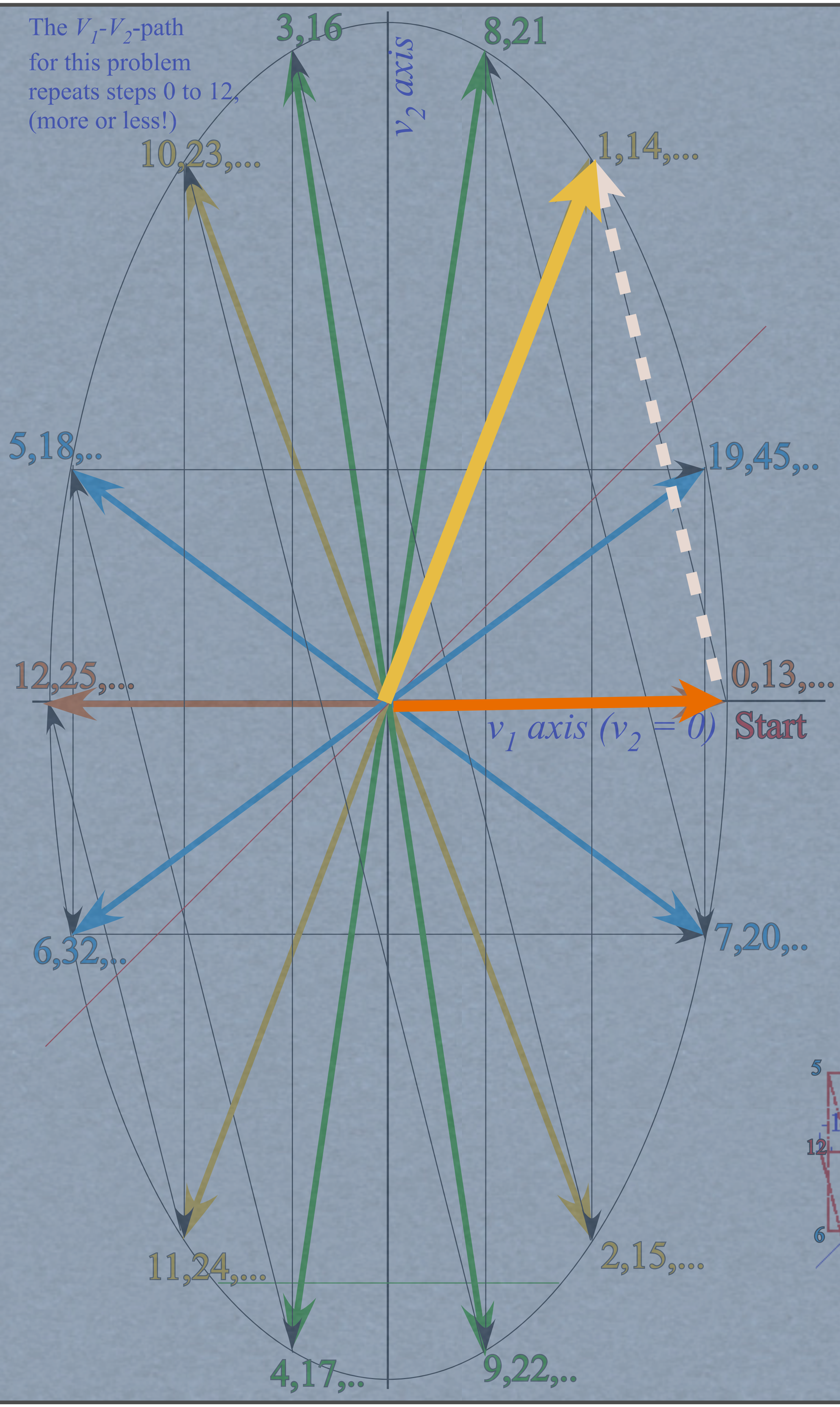


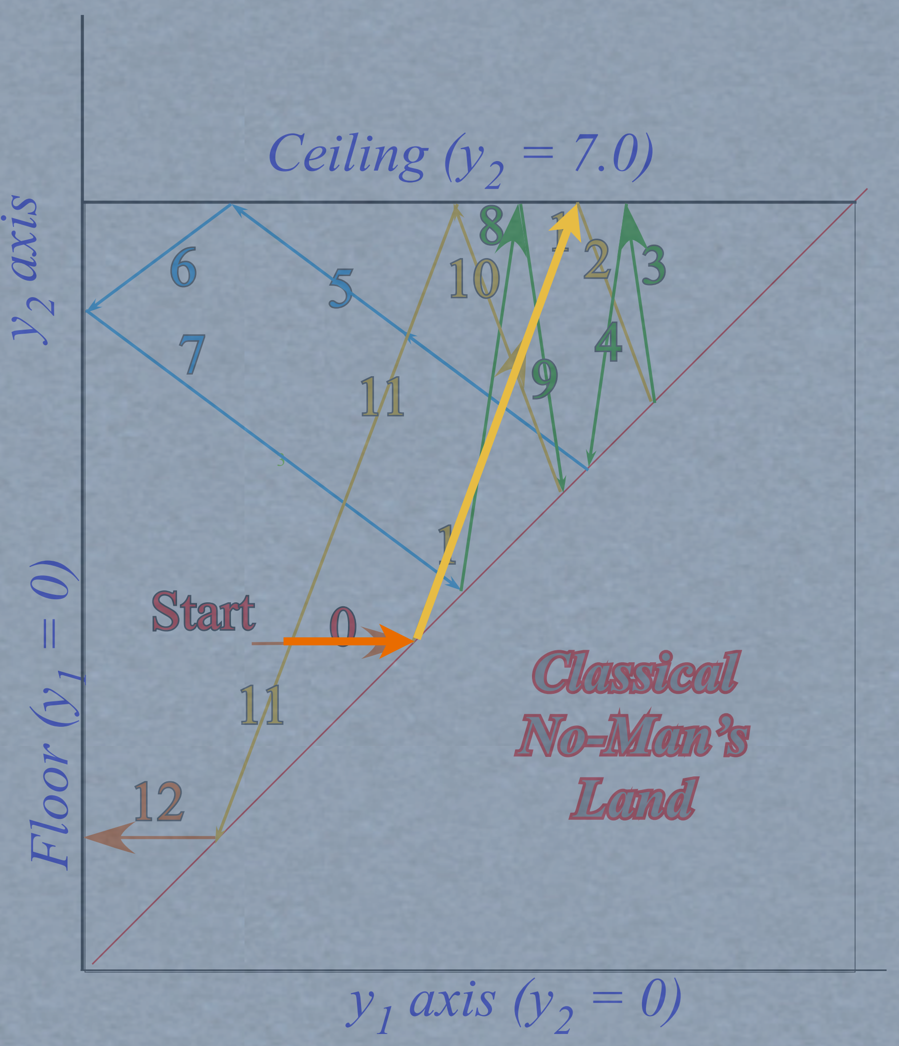
Simulations by *BounceIt*



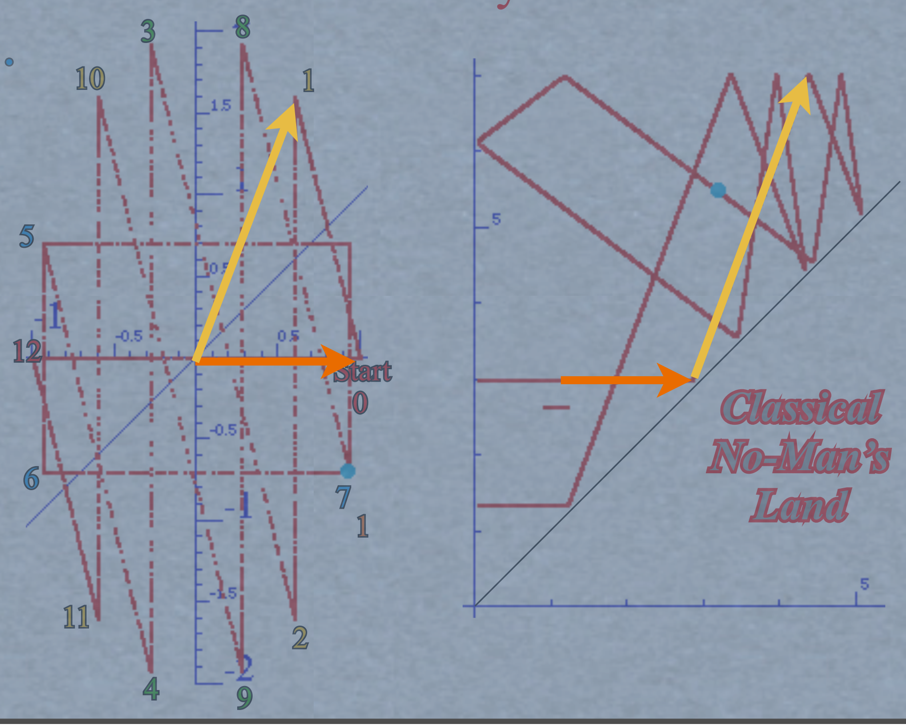
Collisions for mass ratio $m_1:m_2= 4:1$

The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

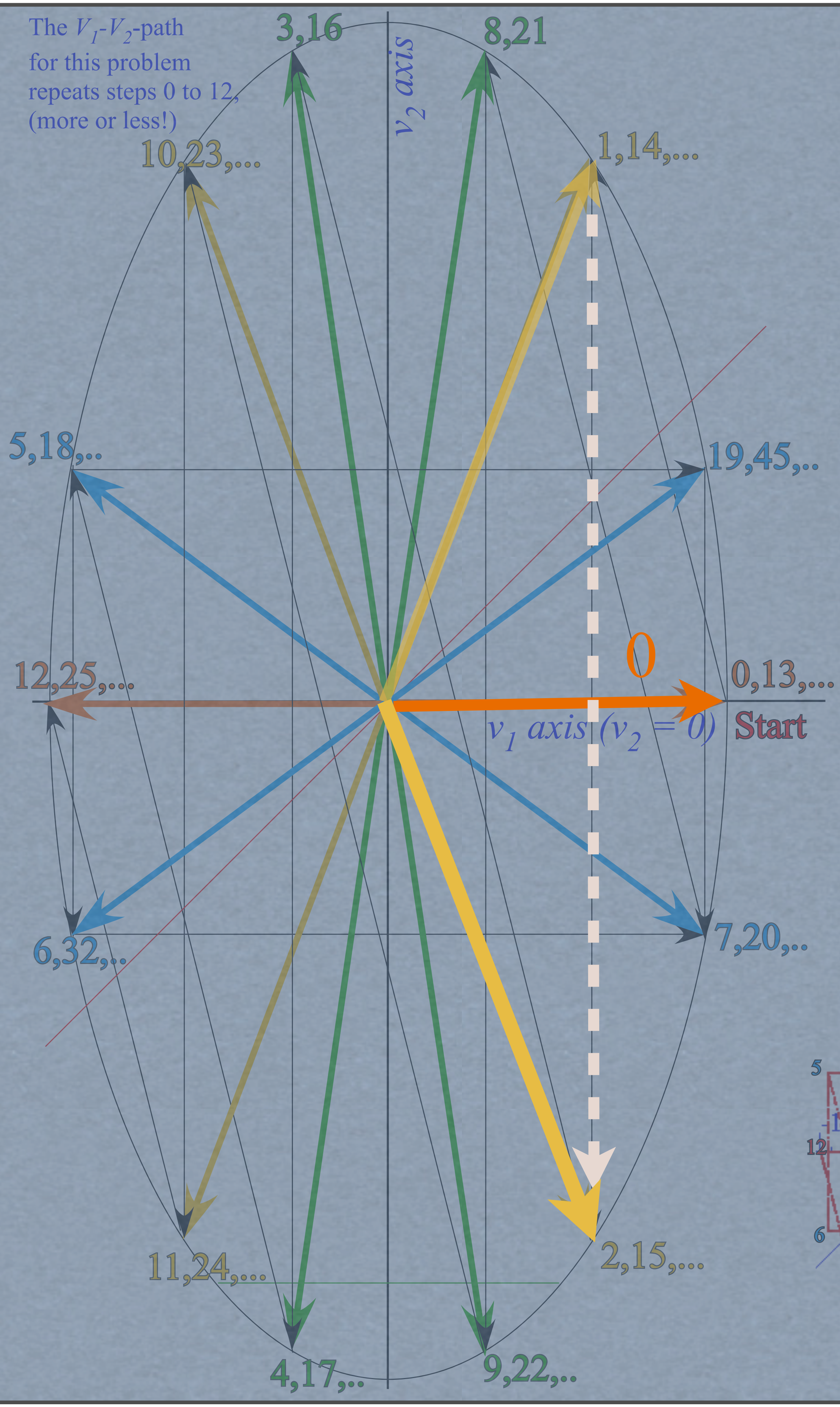


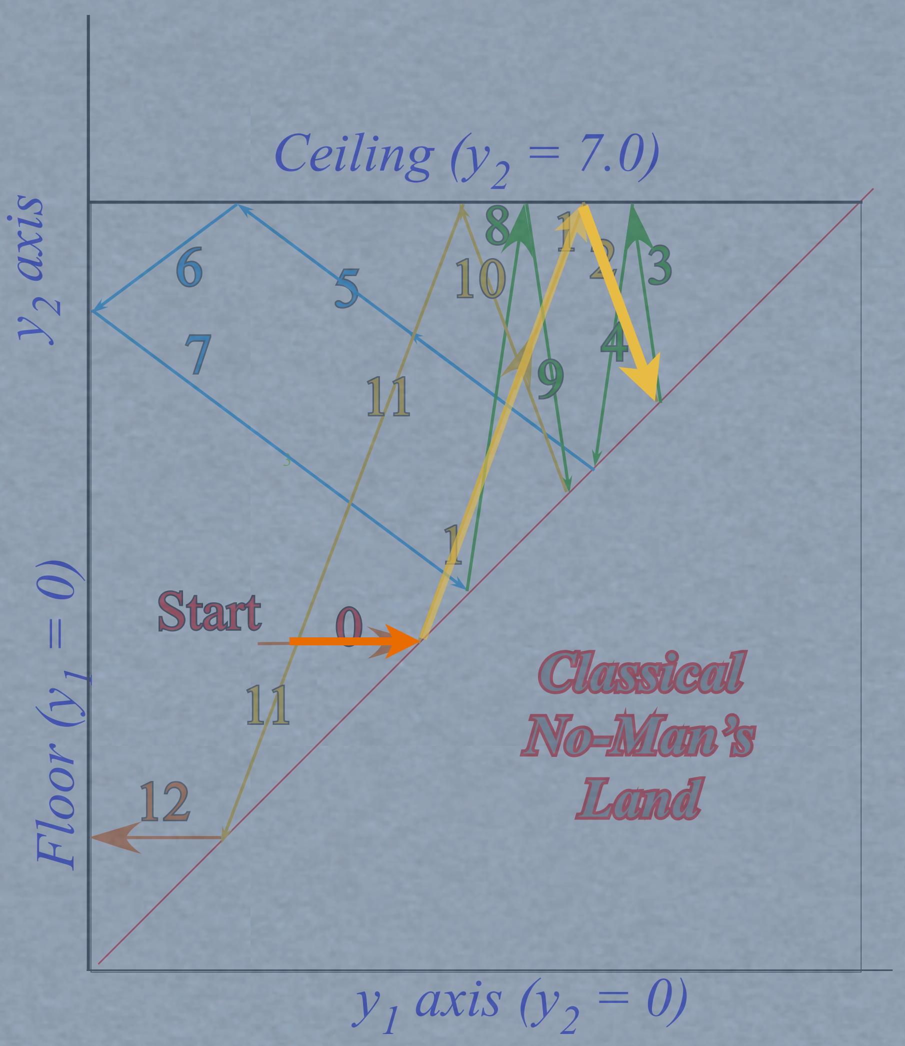


Simulations by *BounceIt*

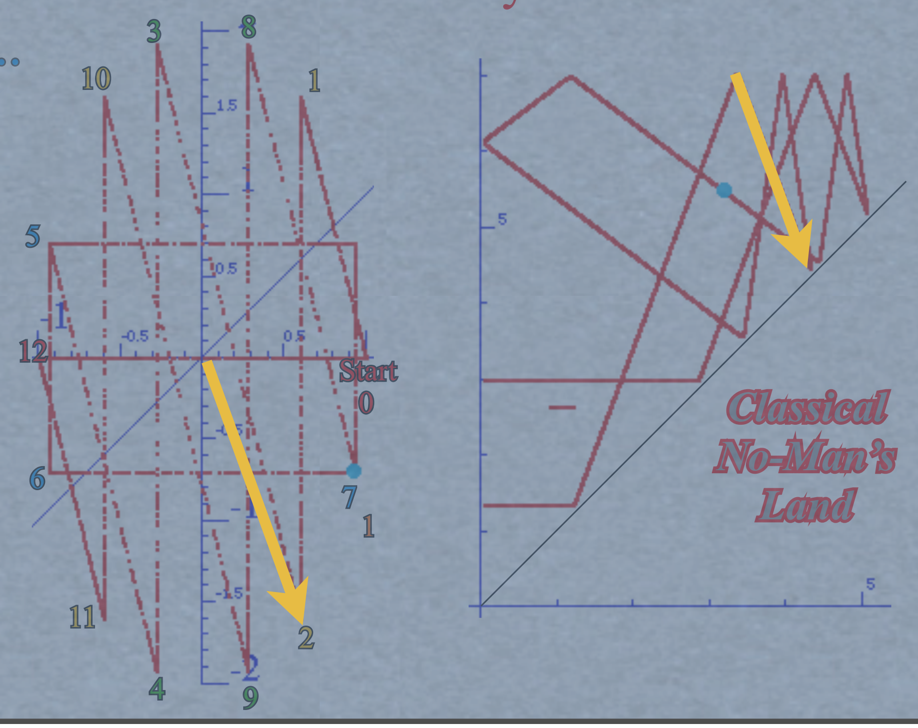


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

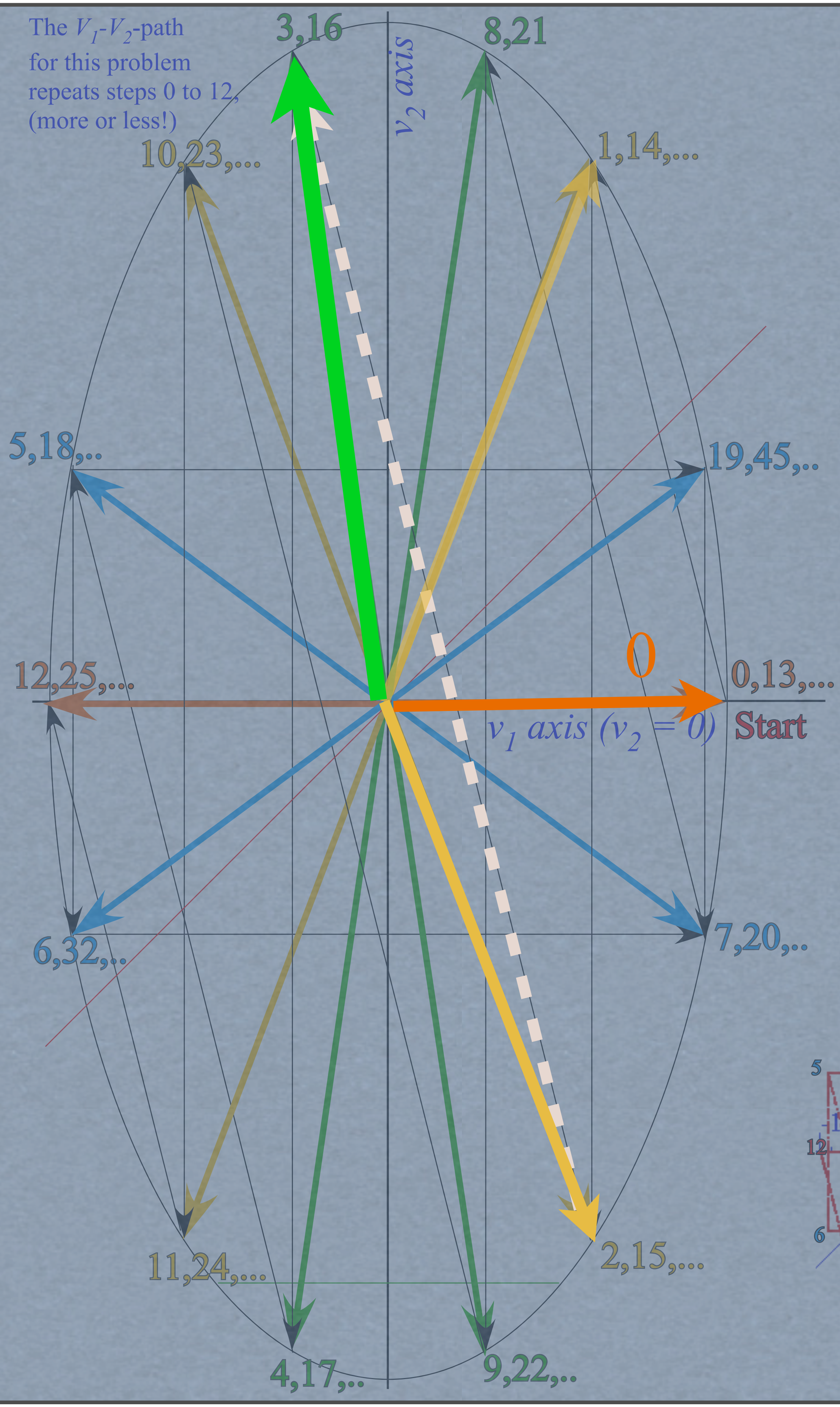


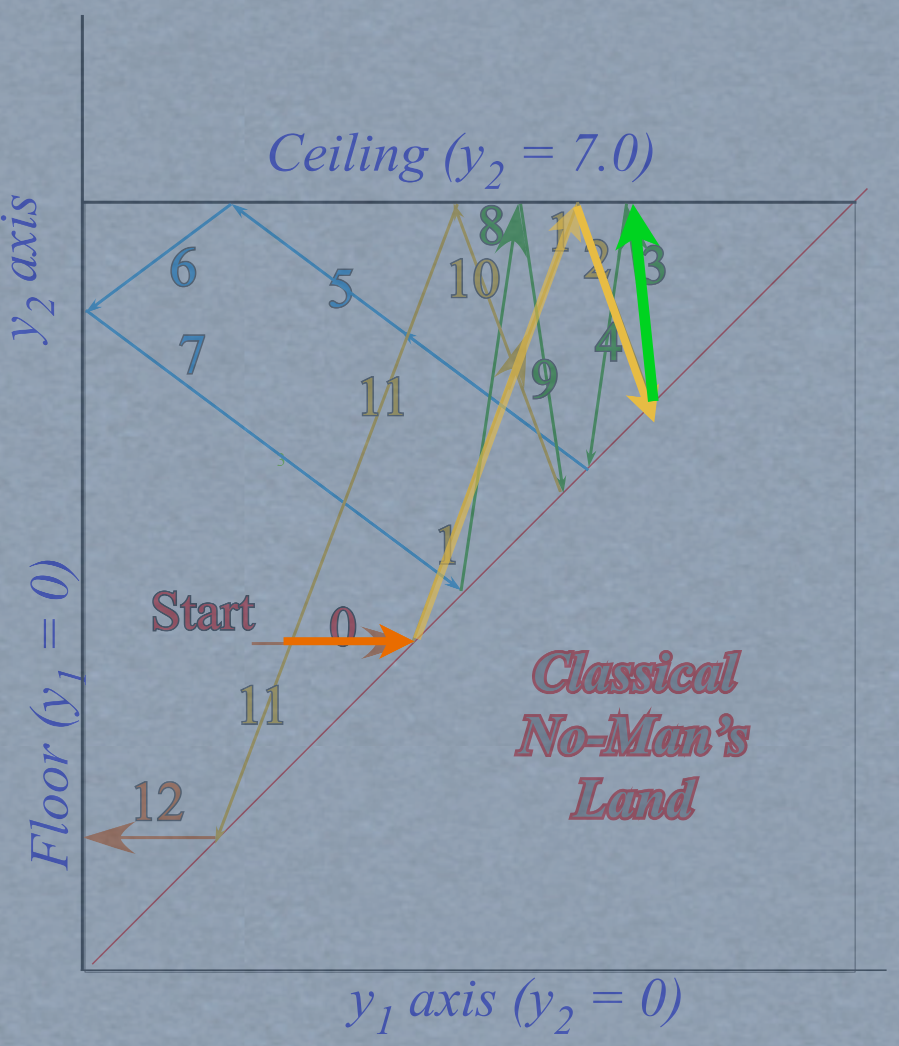


Simulations by *BounceIt*

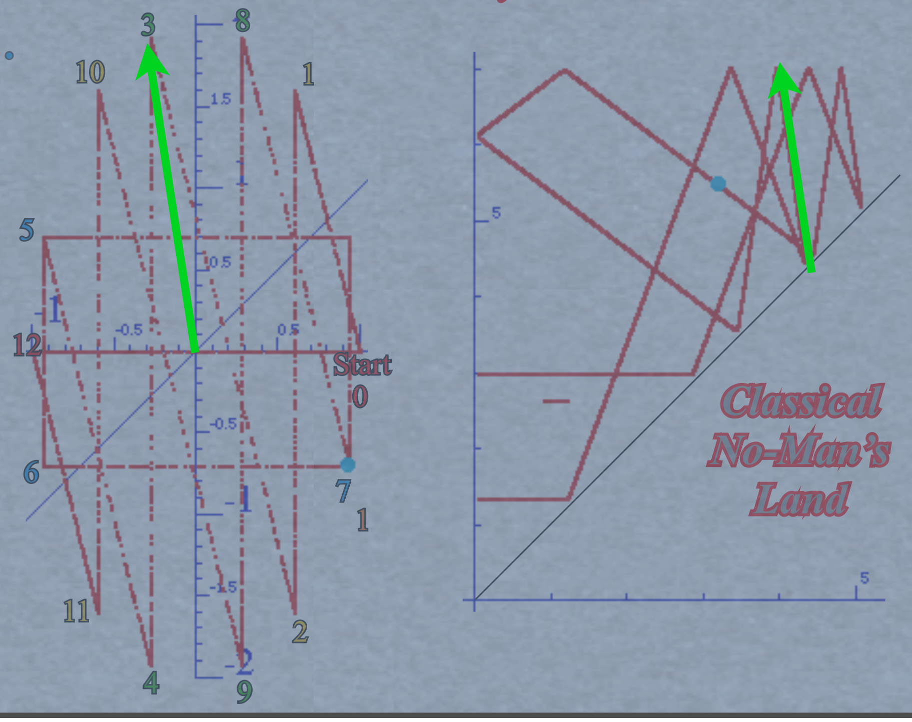


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

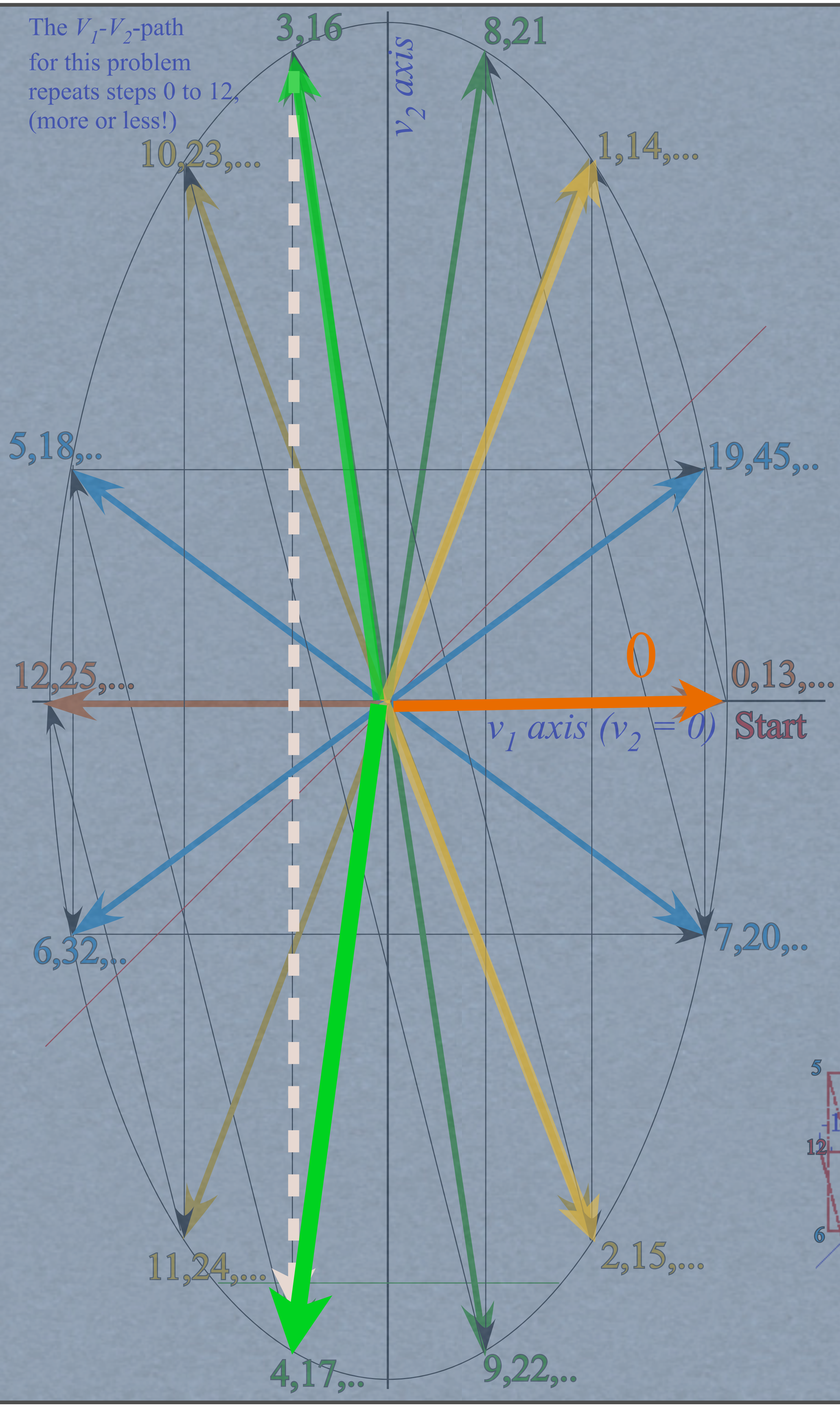


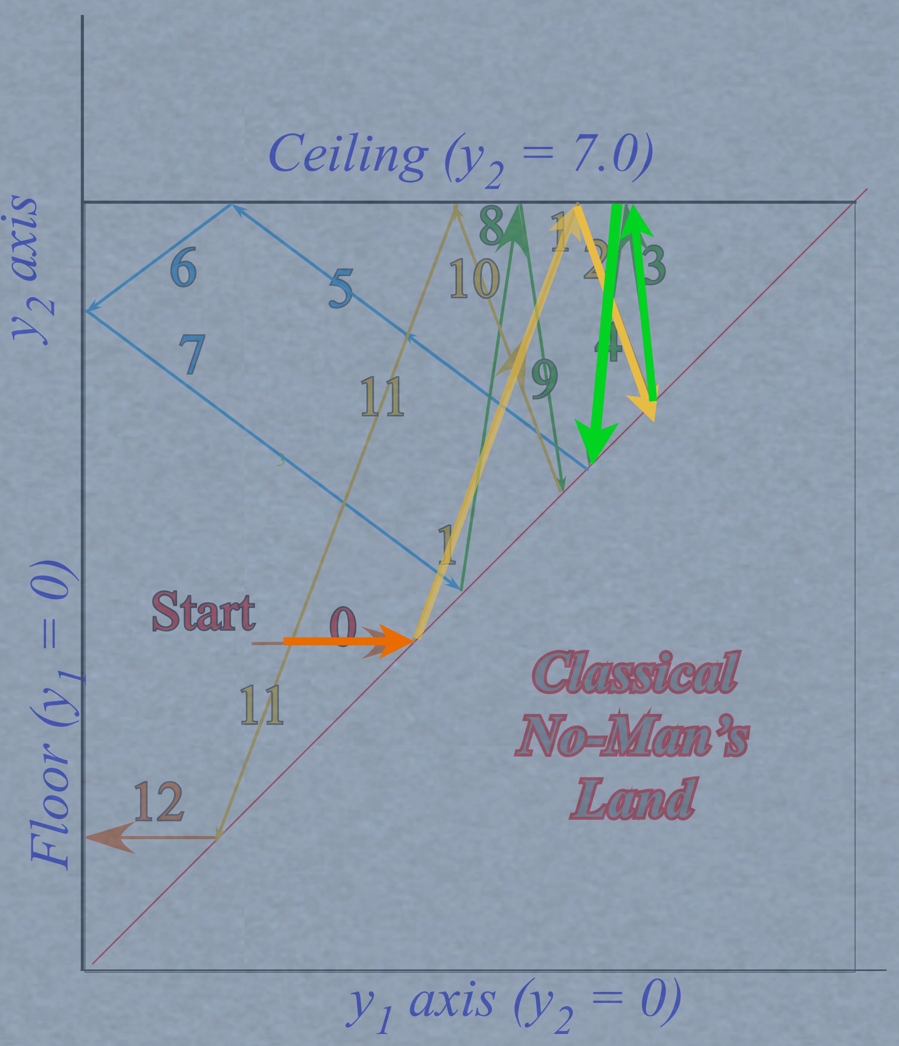


Simulations by *BounceIt*

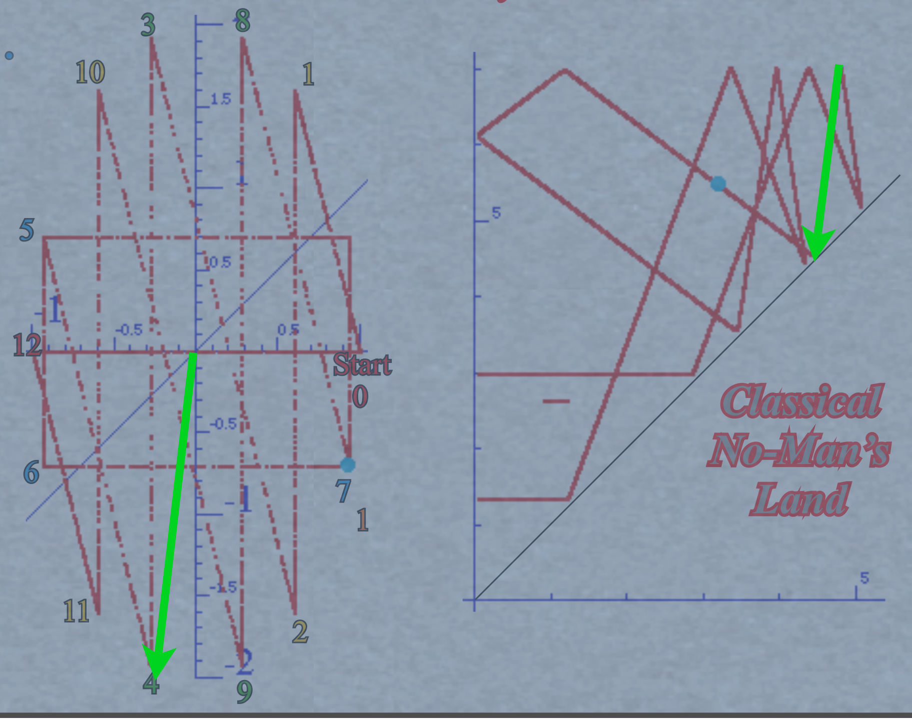


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)

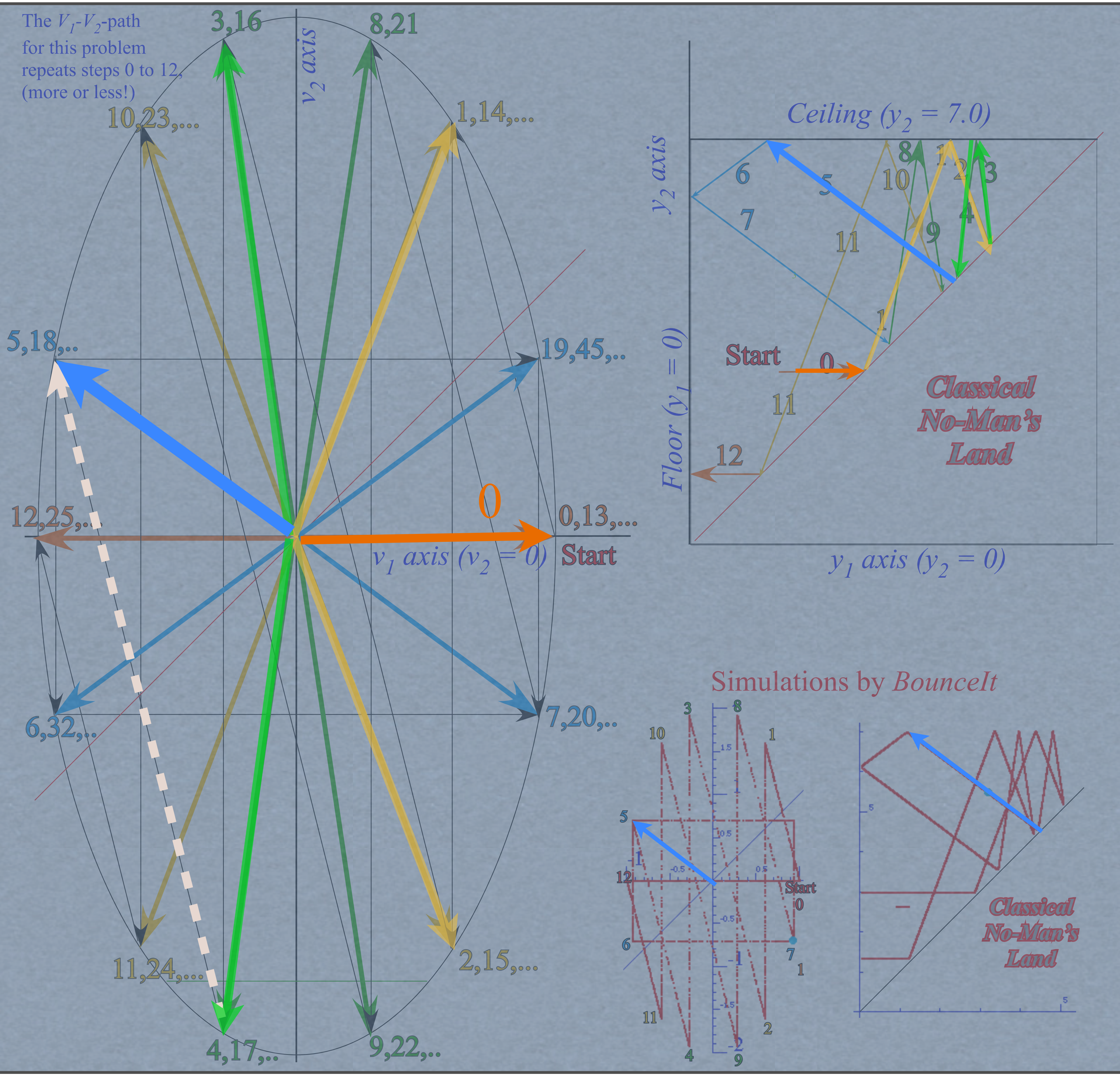




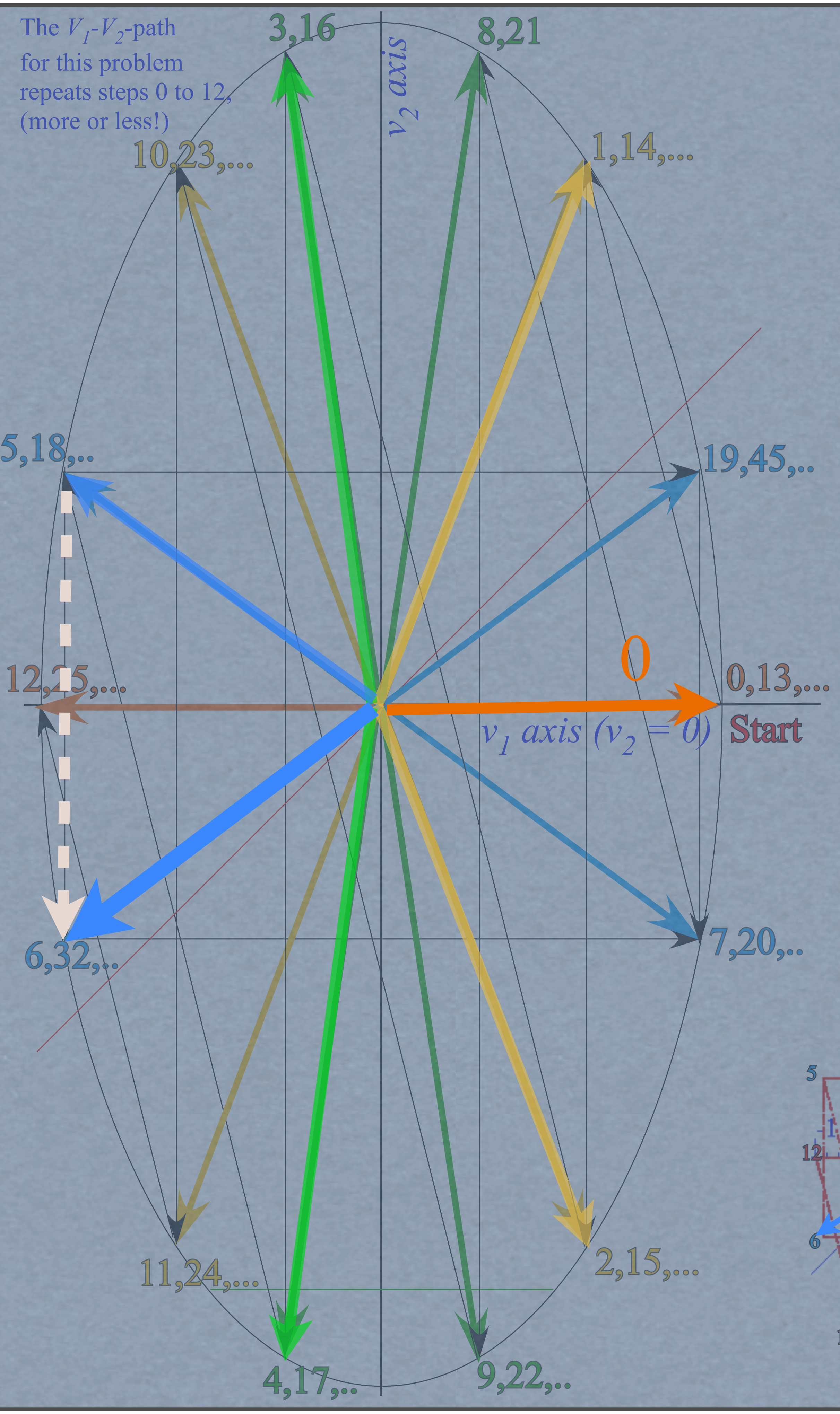
Simulations by *BounceIt*

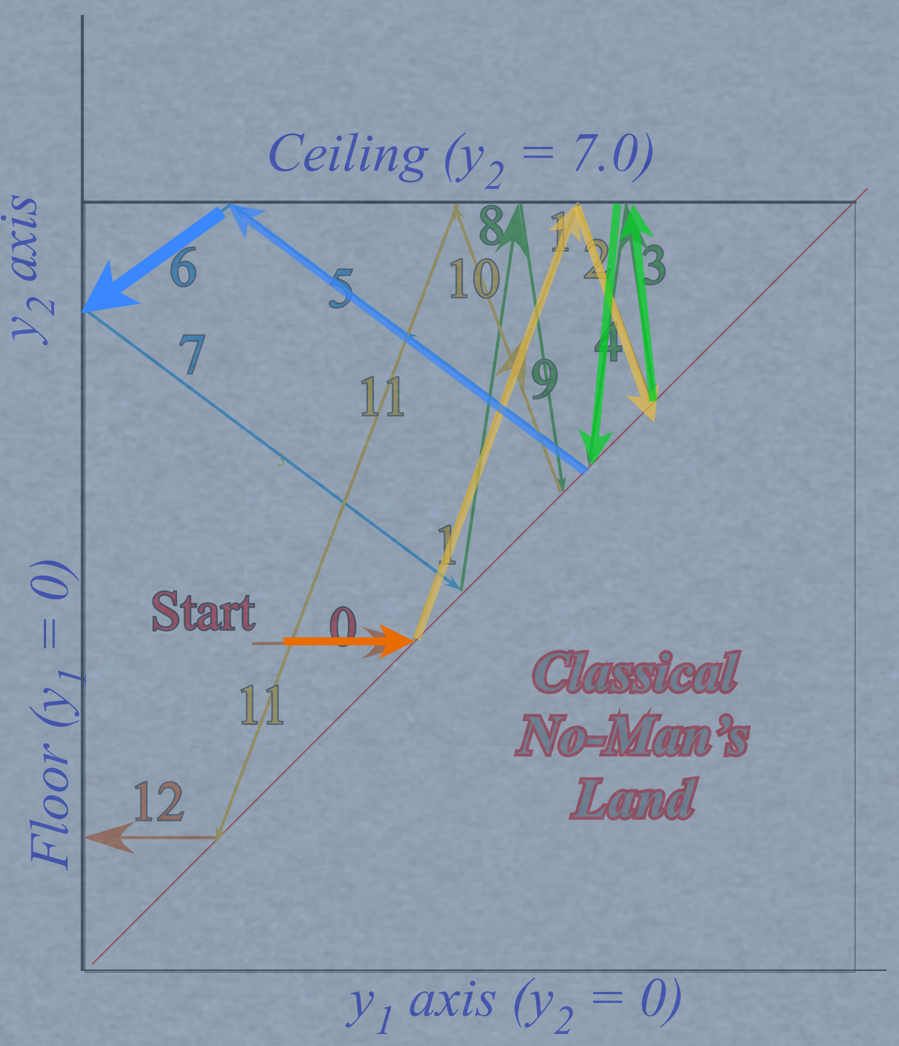


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)



The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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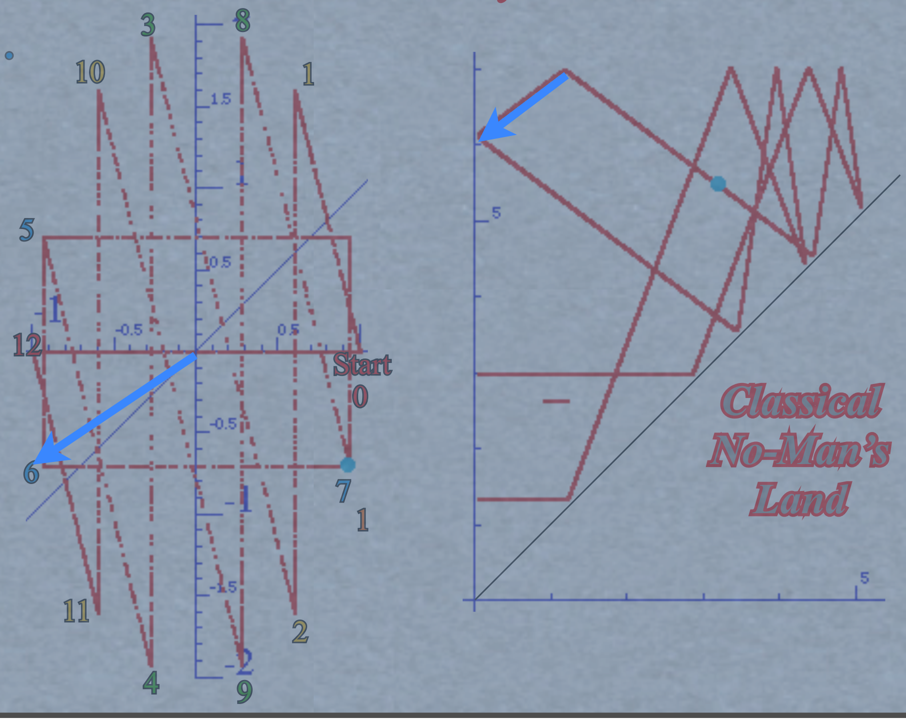
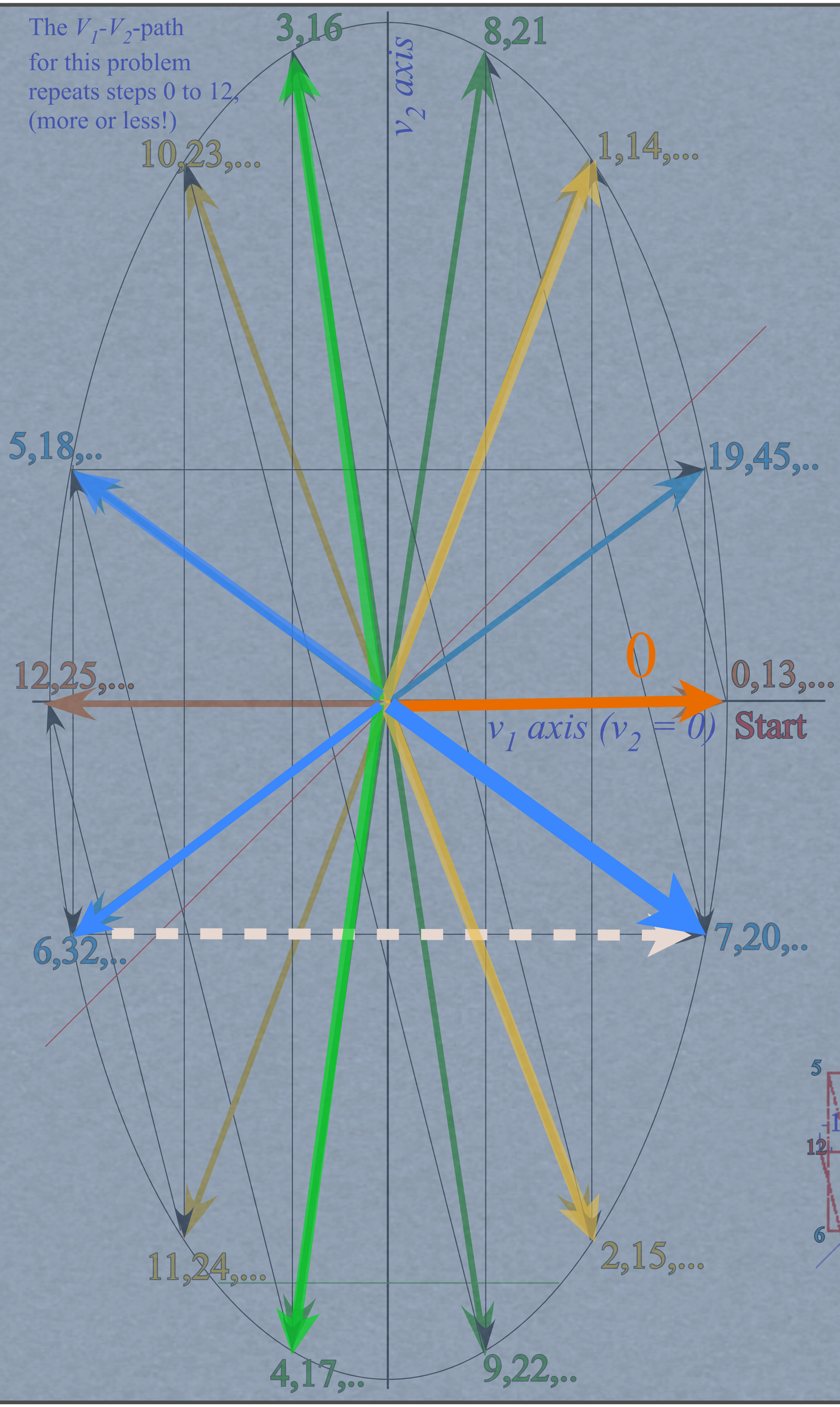
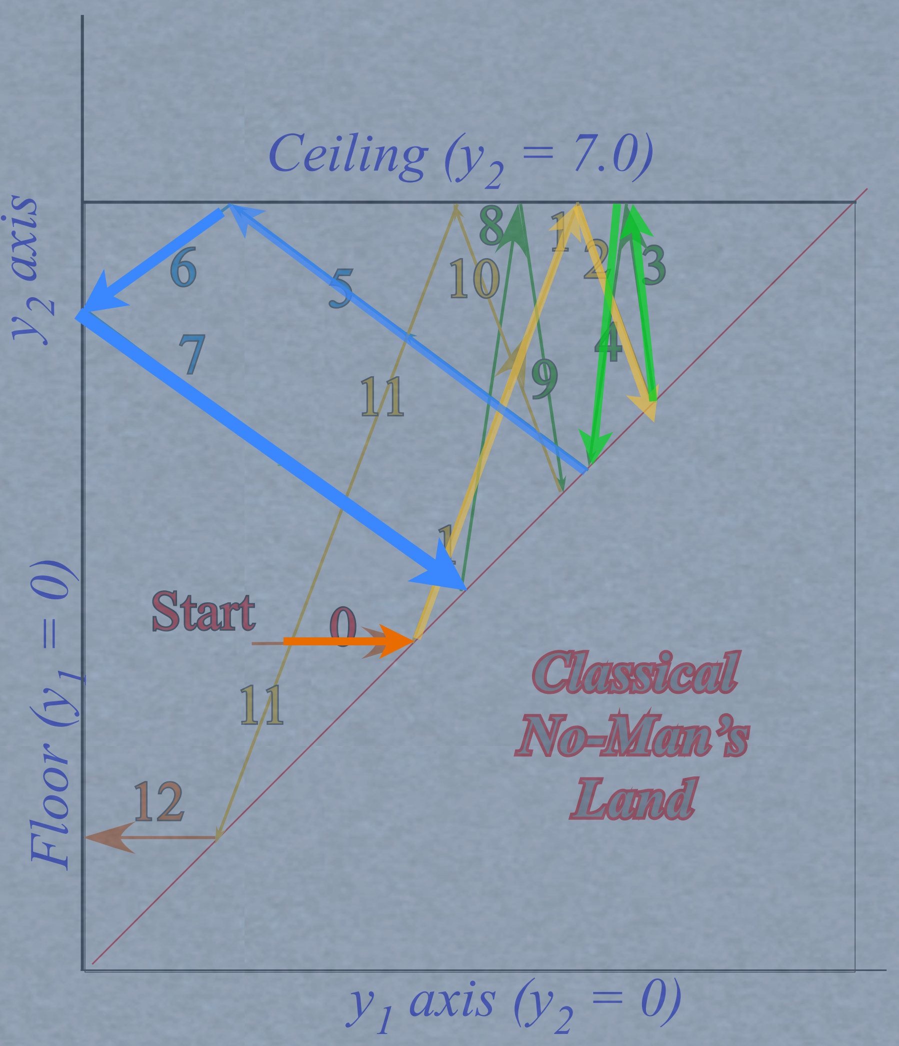
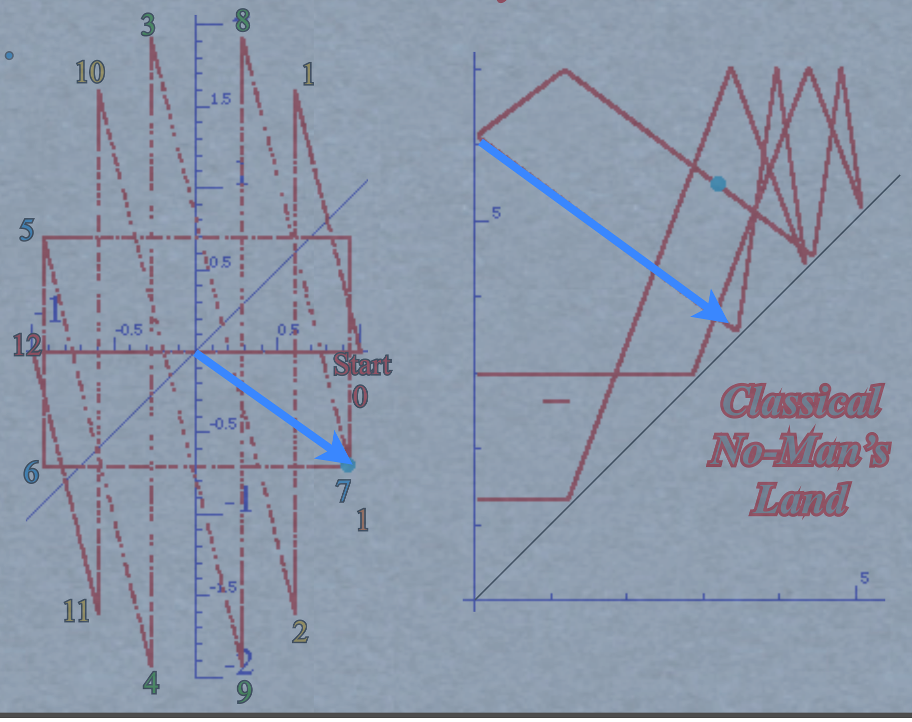




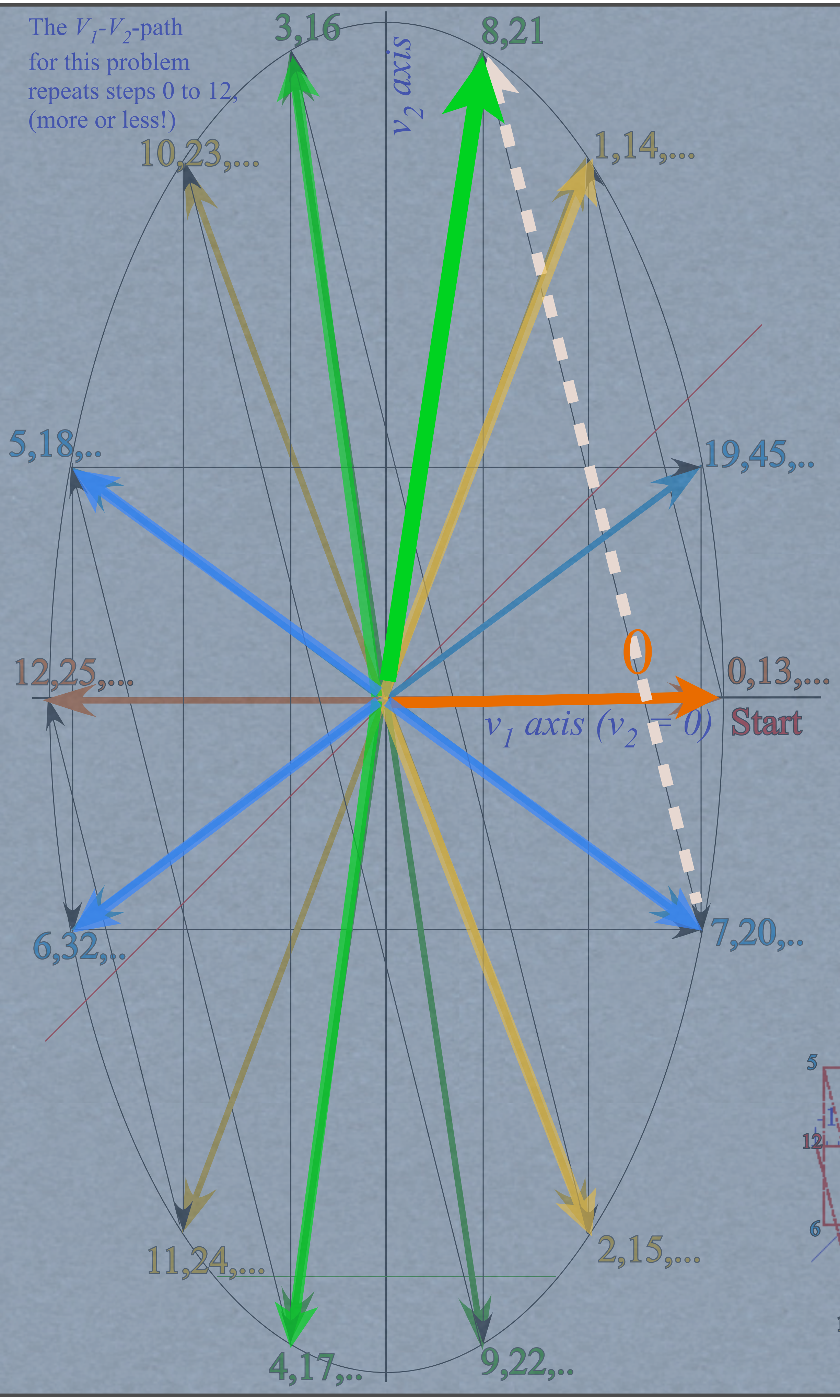
The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

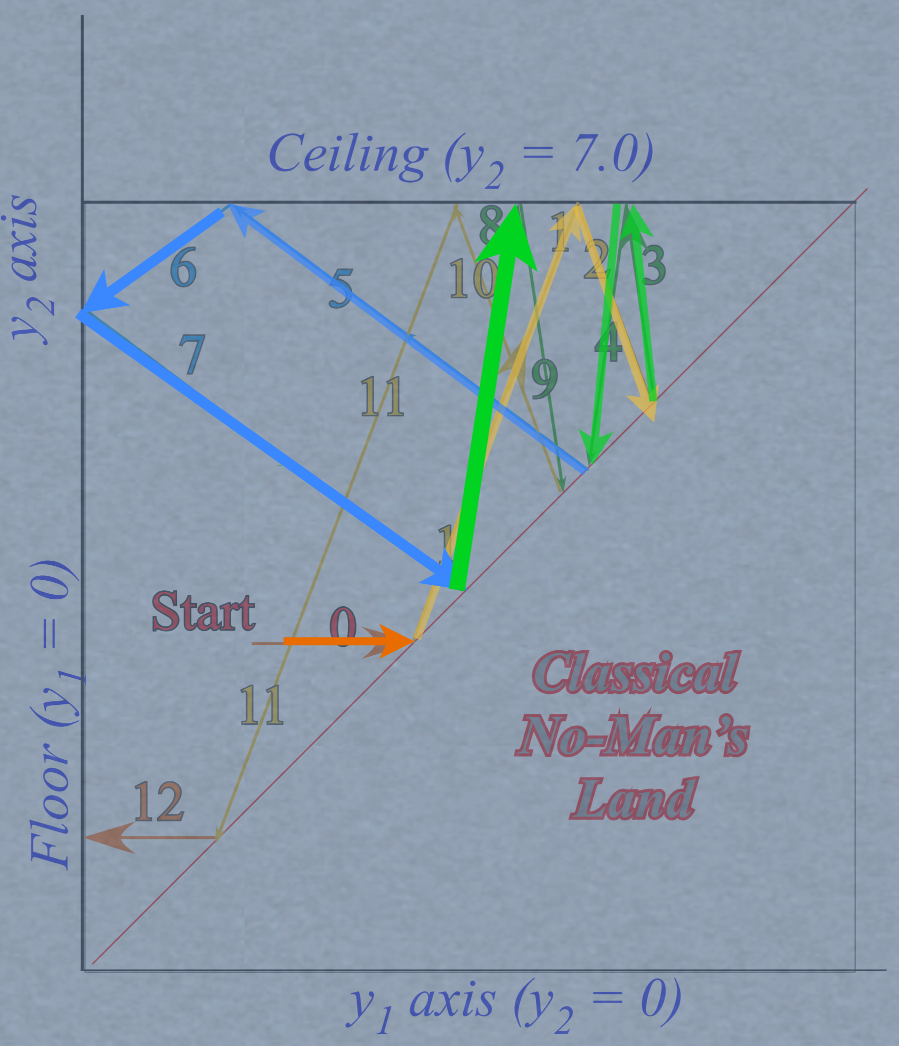


Simulations by *BounceIt*



The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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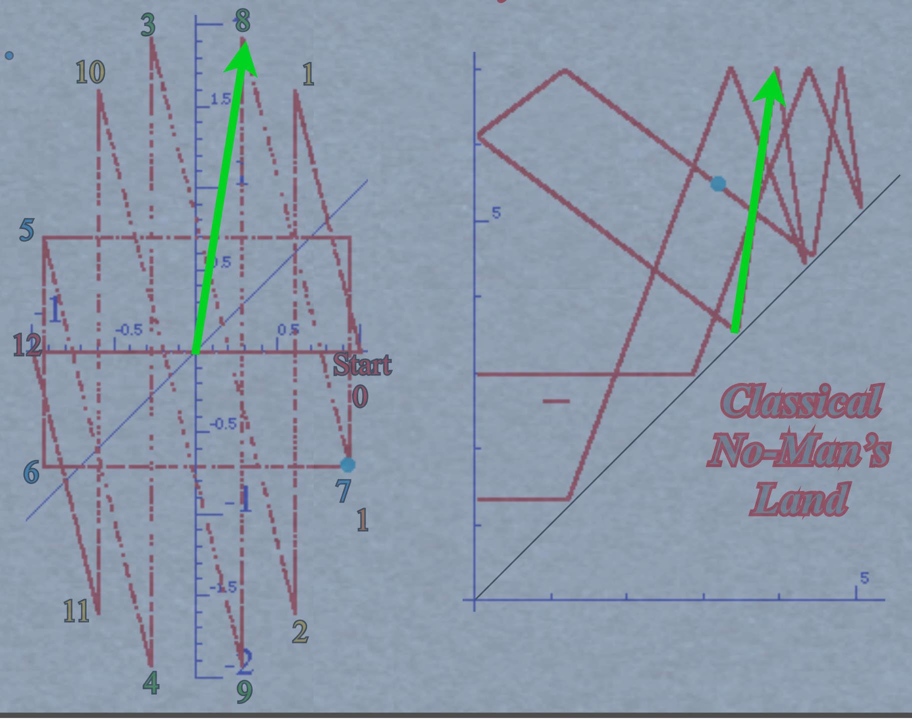
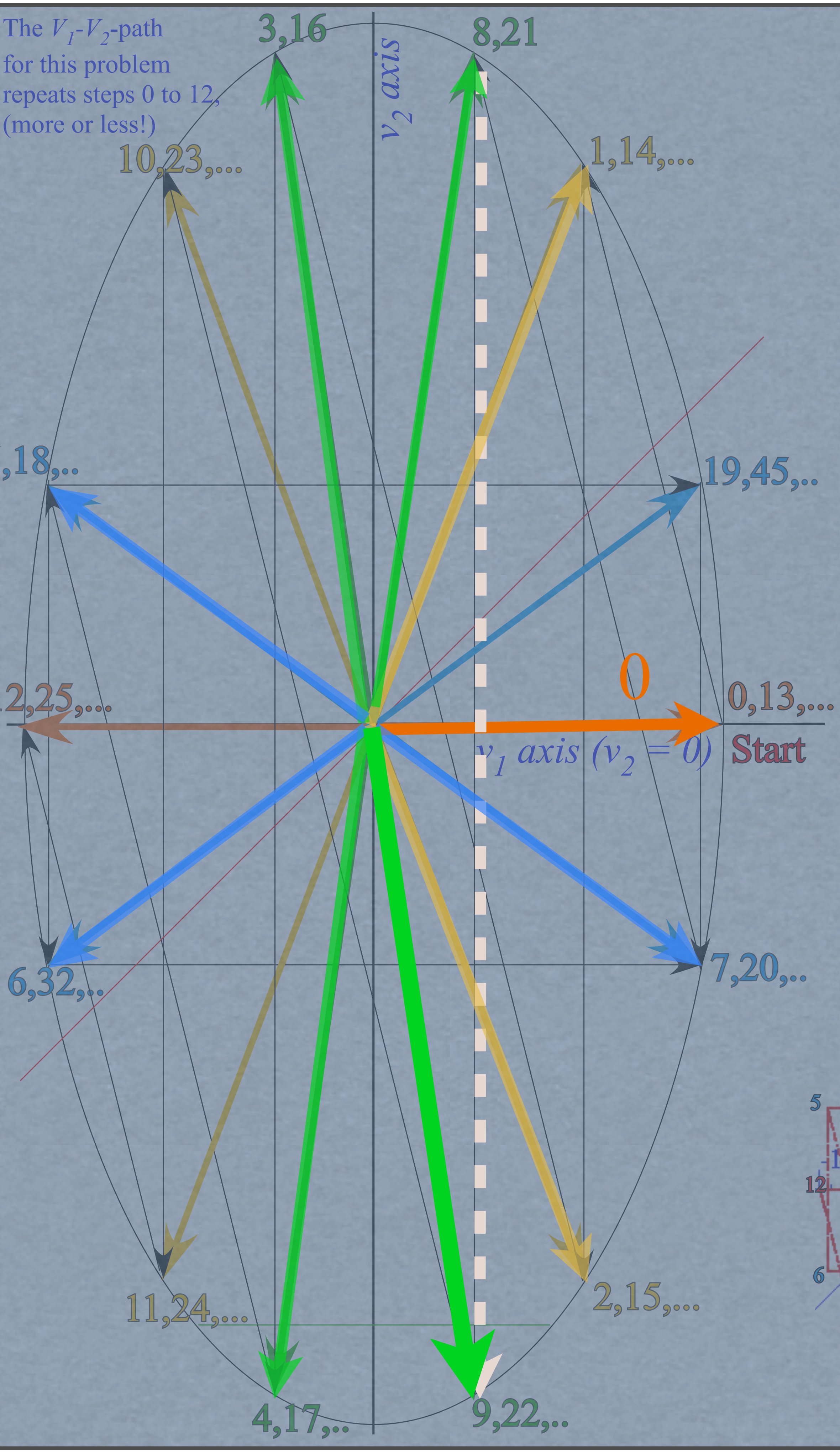
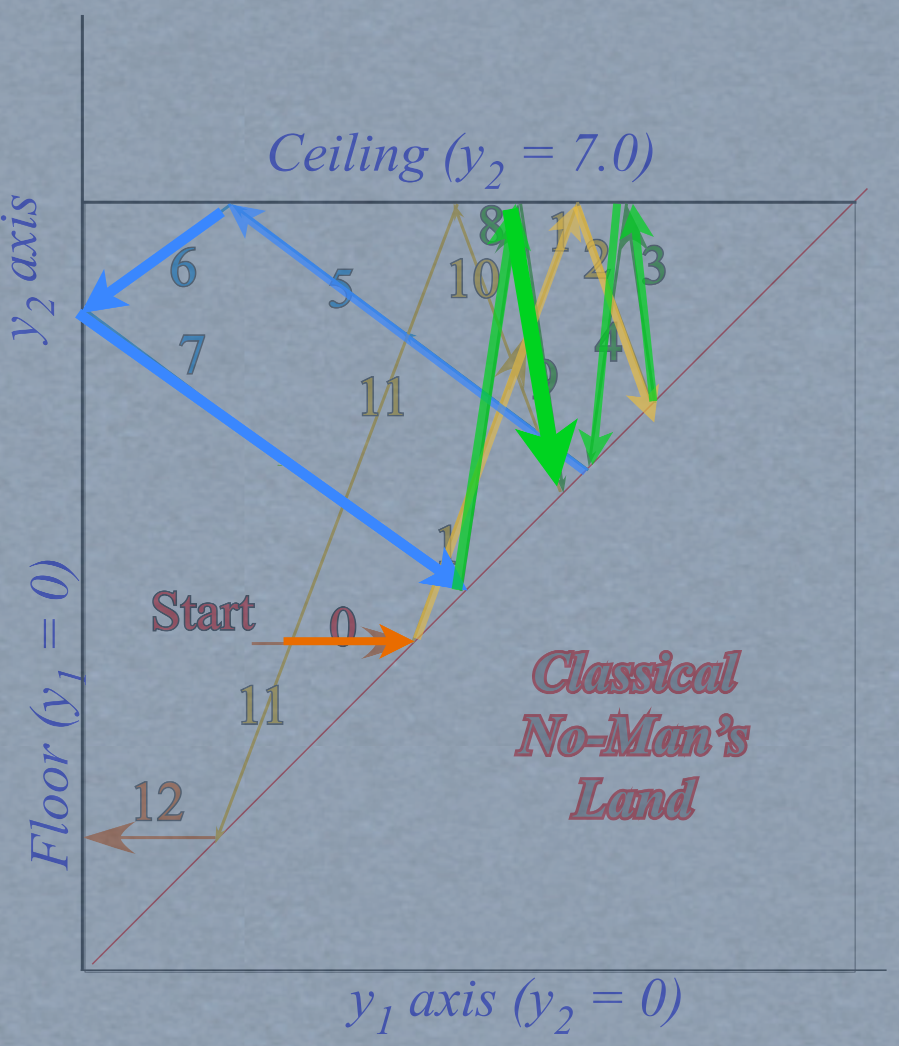
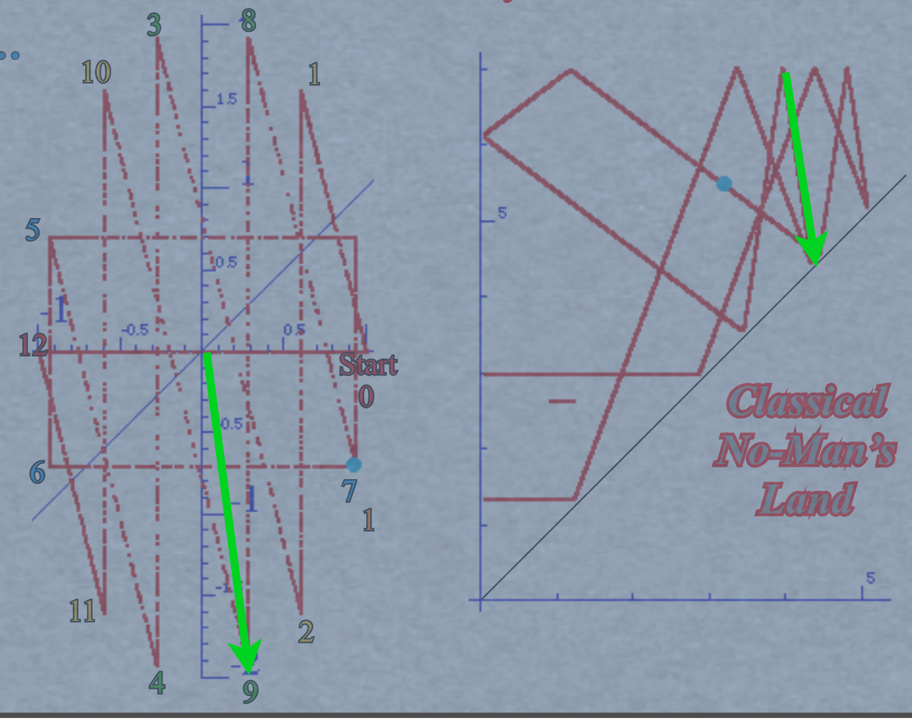


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)

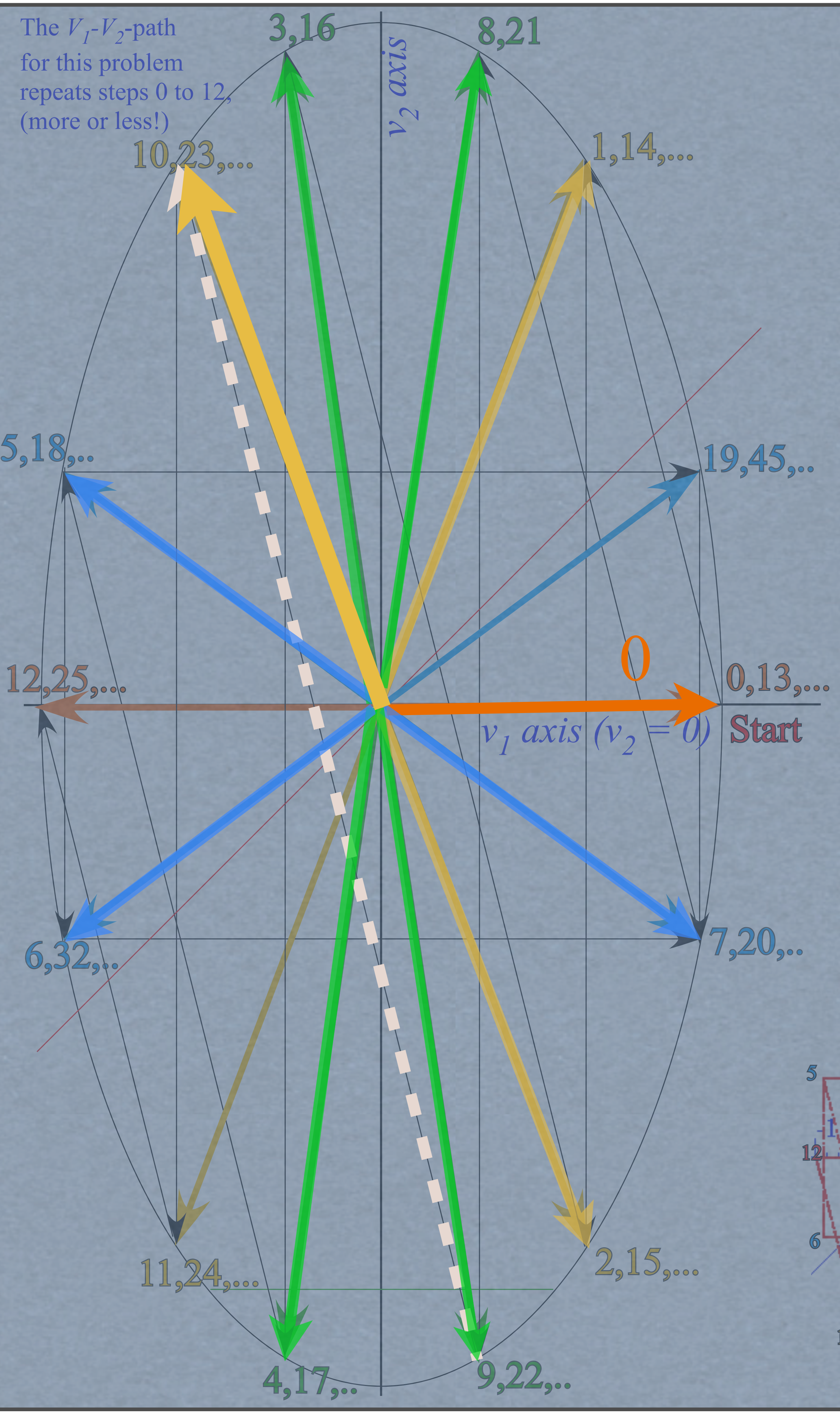


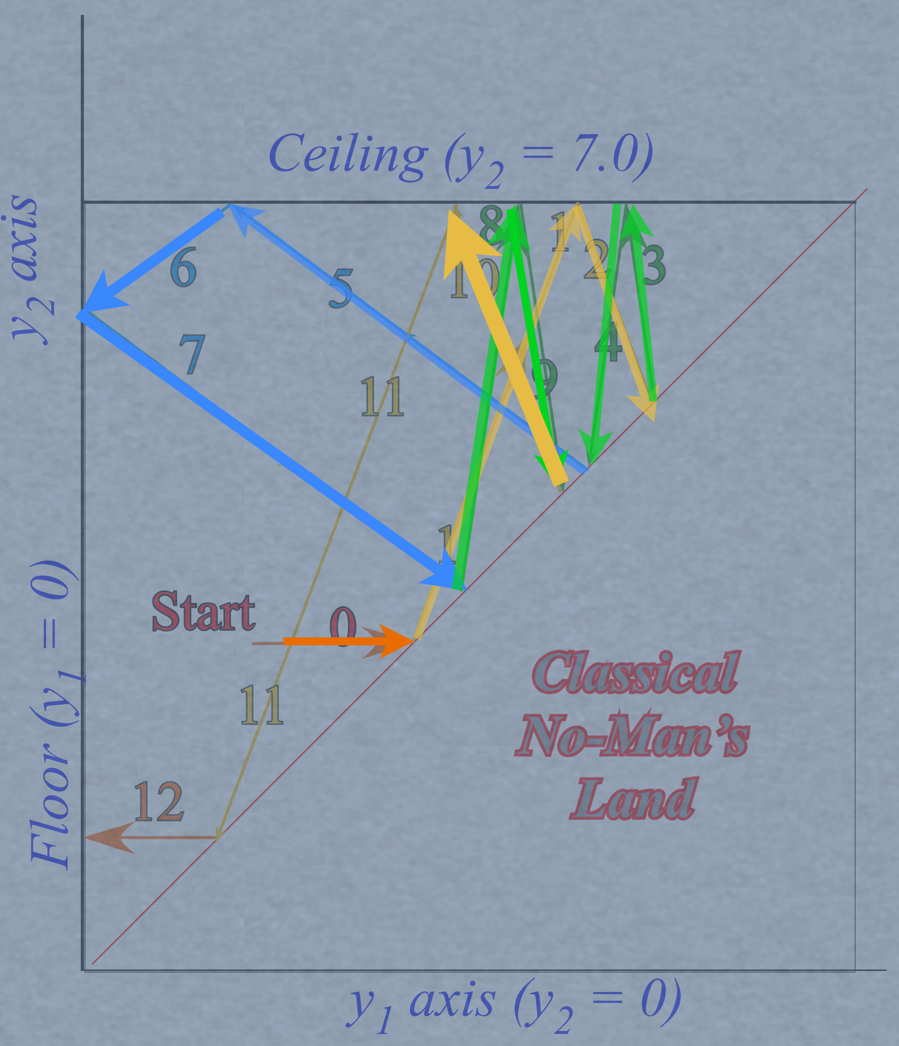


Simulations by *BounceIt*



The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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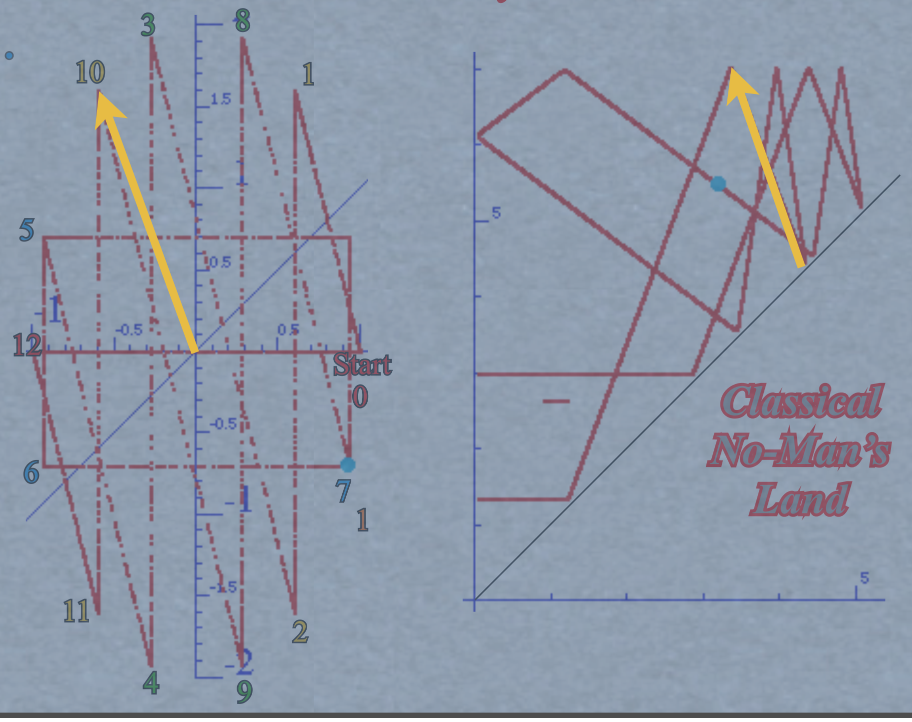
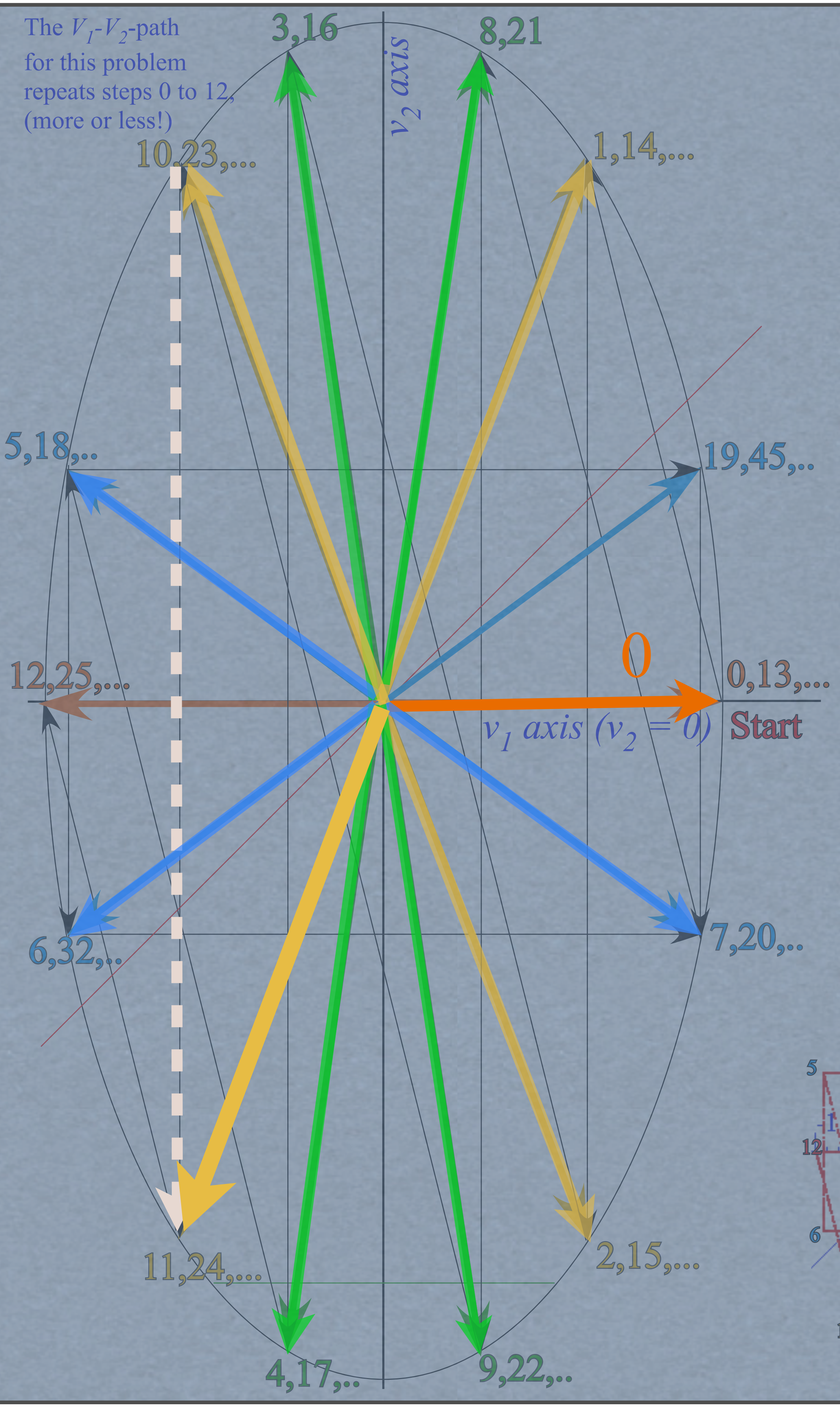
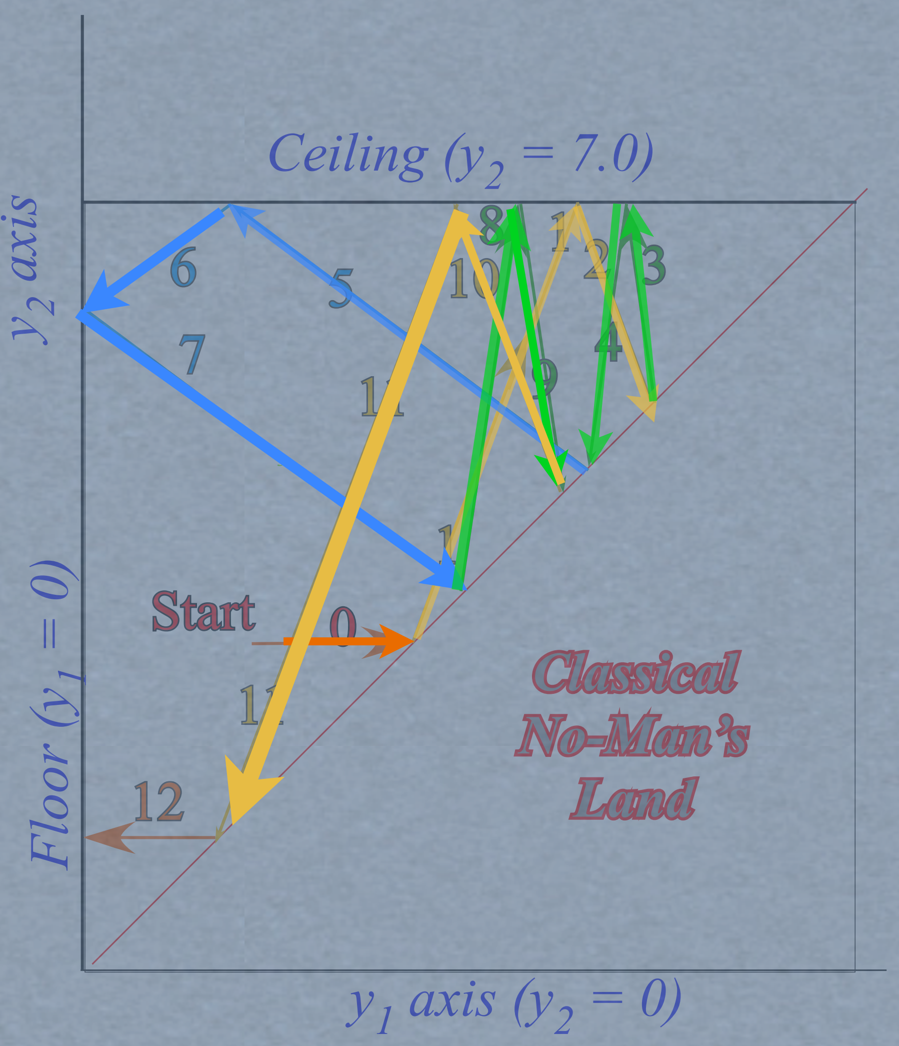


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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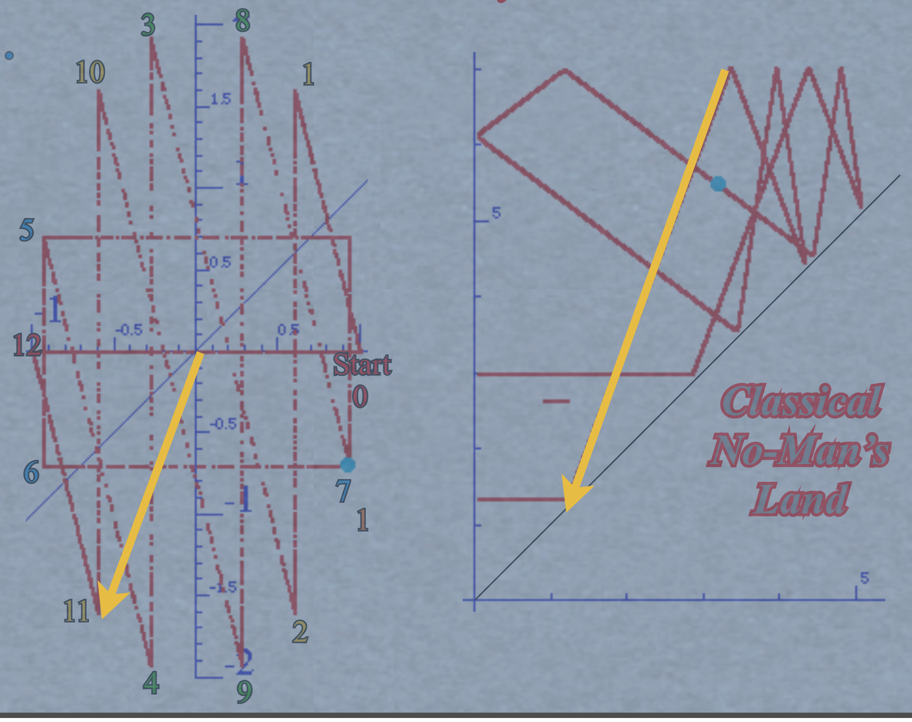
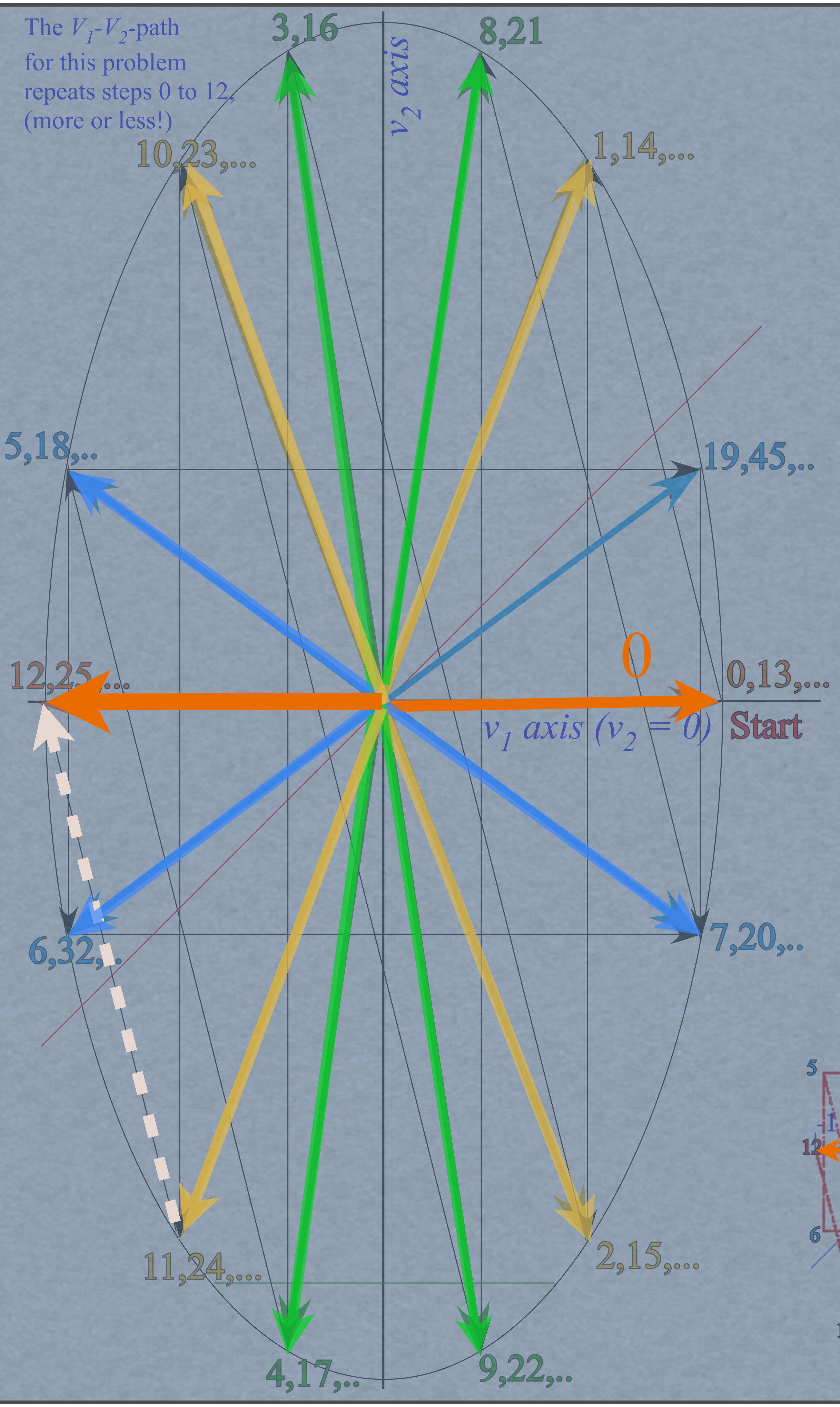
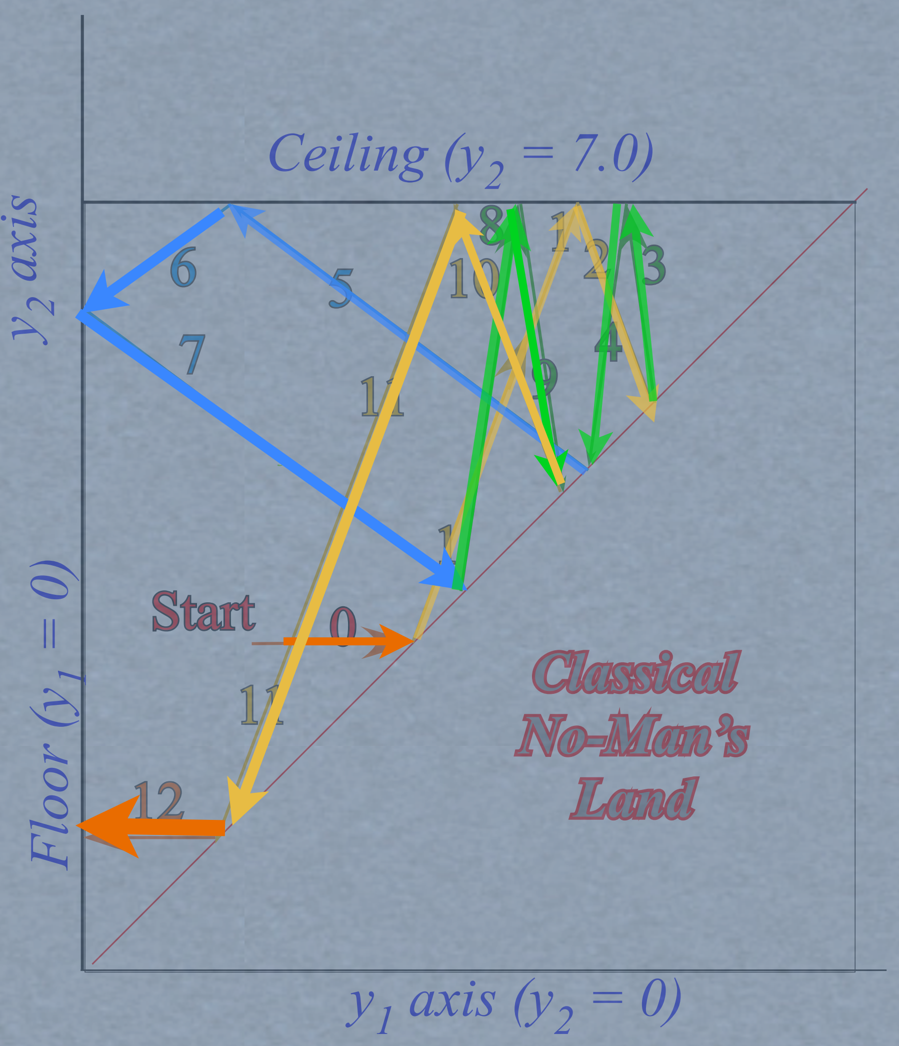


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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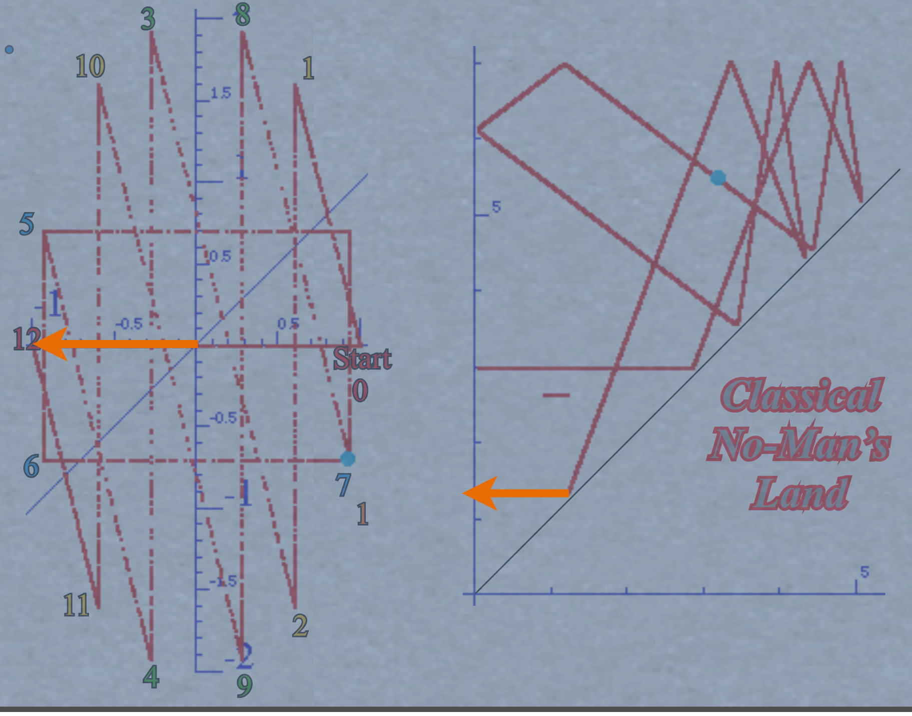
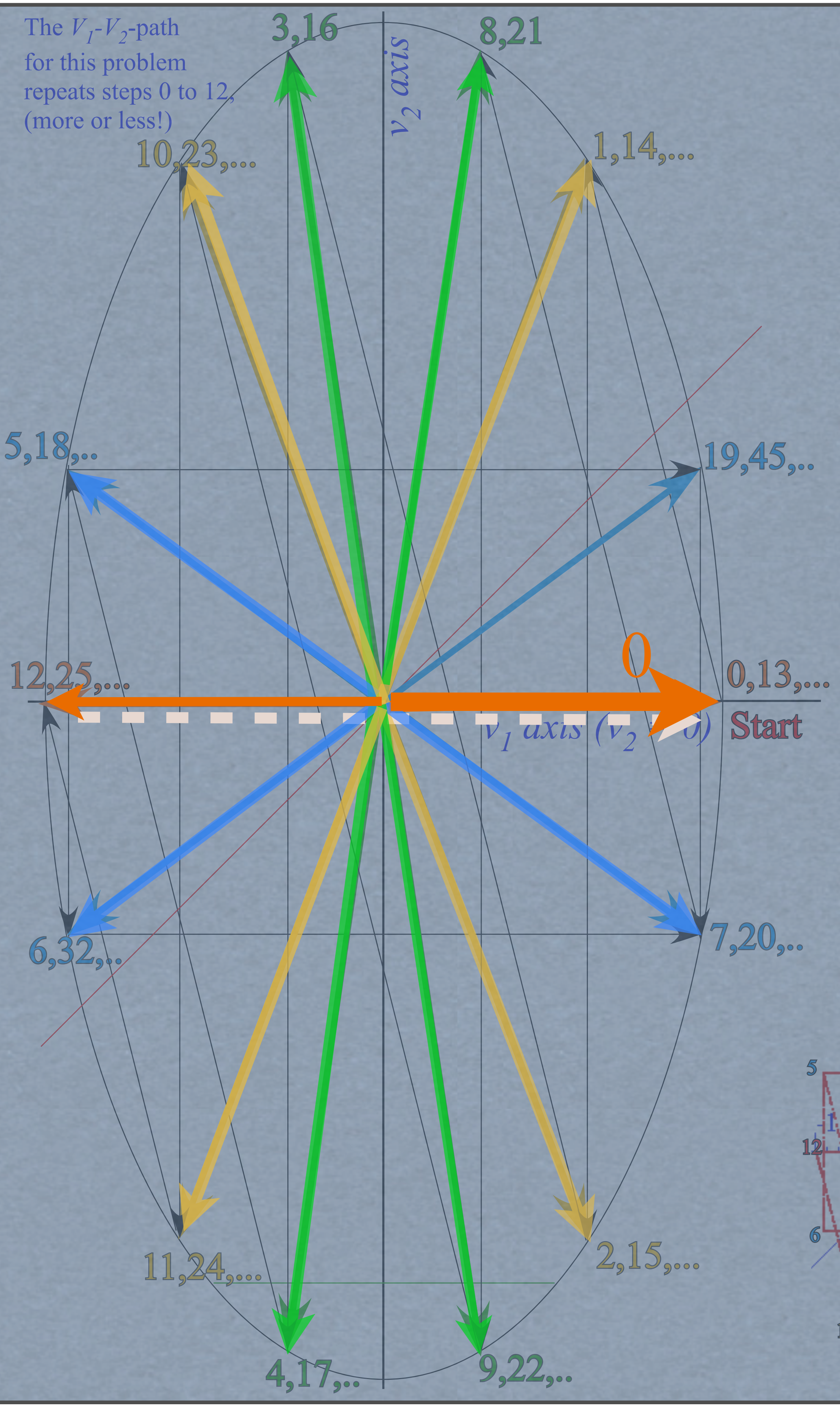
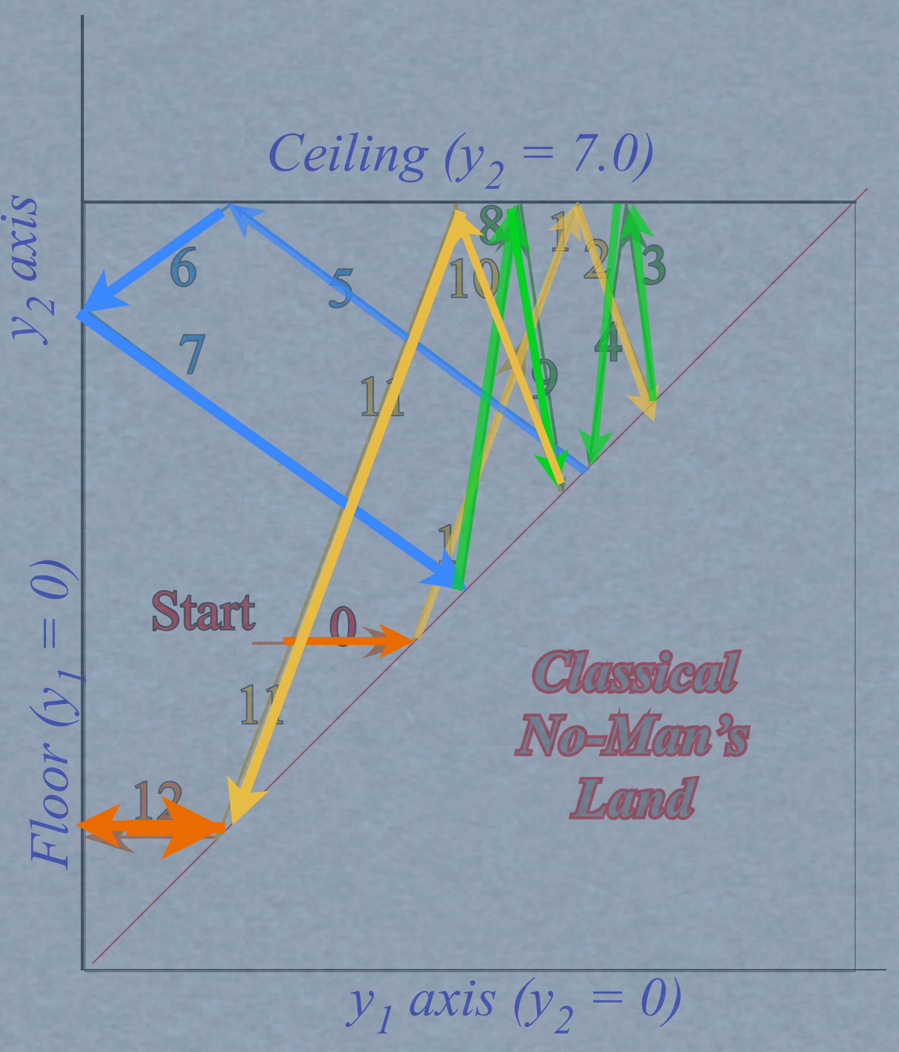
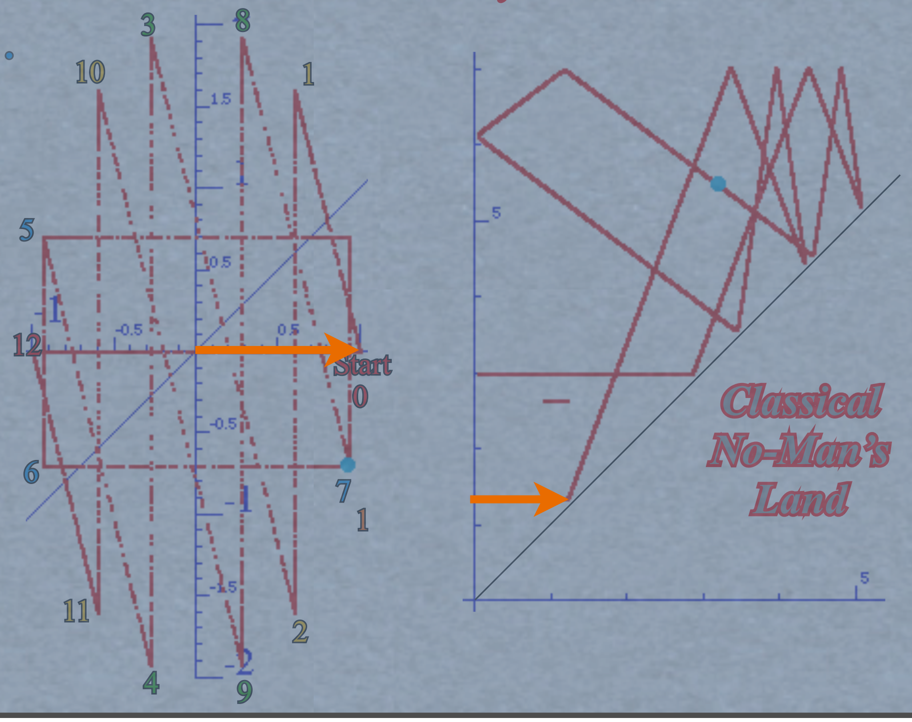


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)

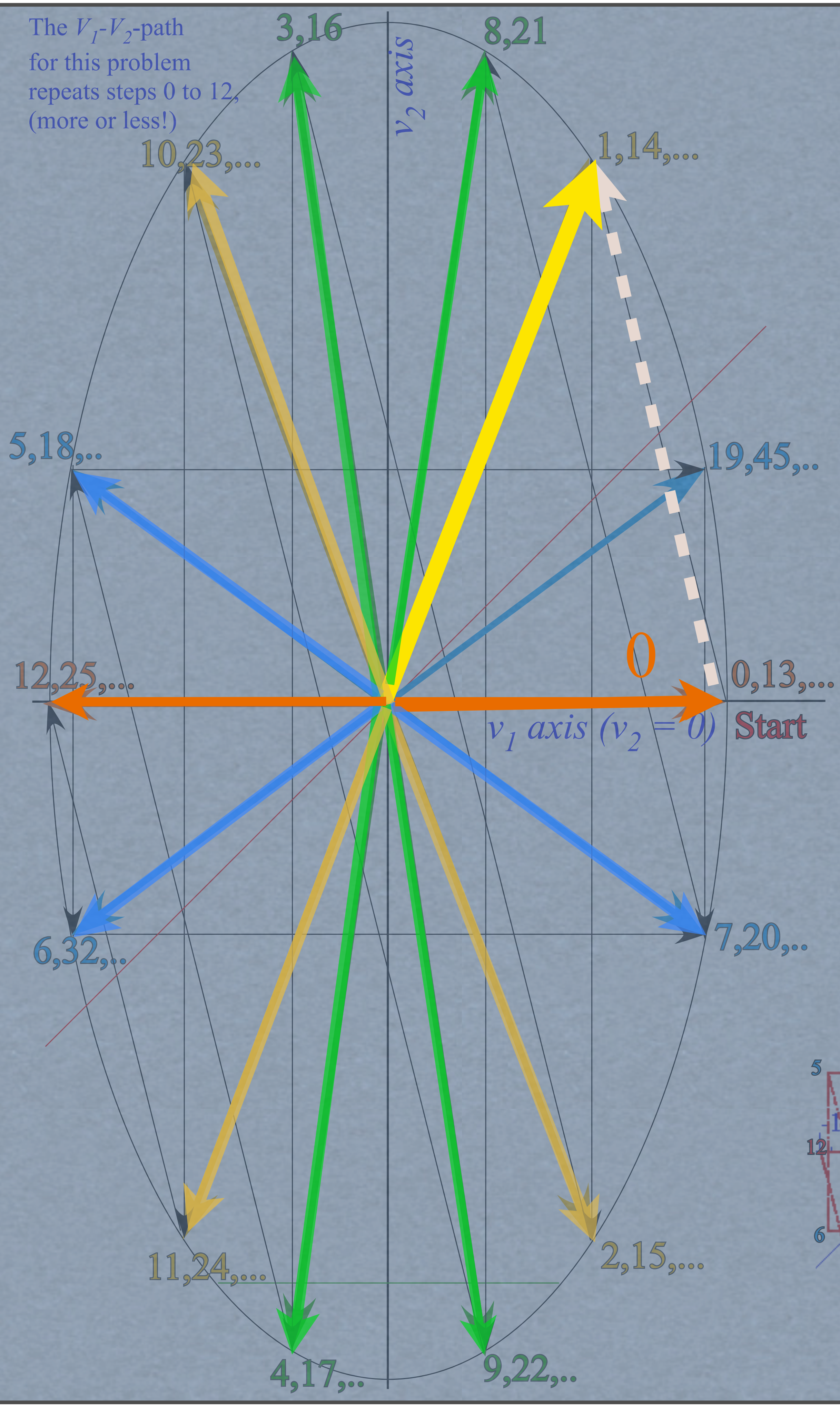


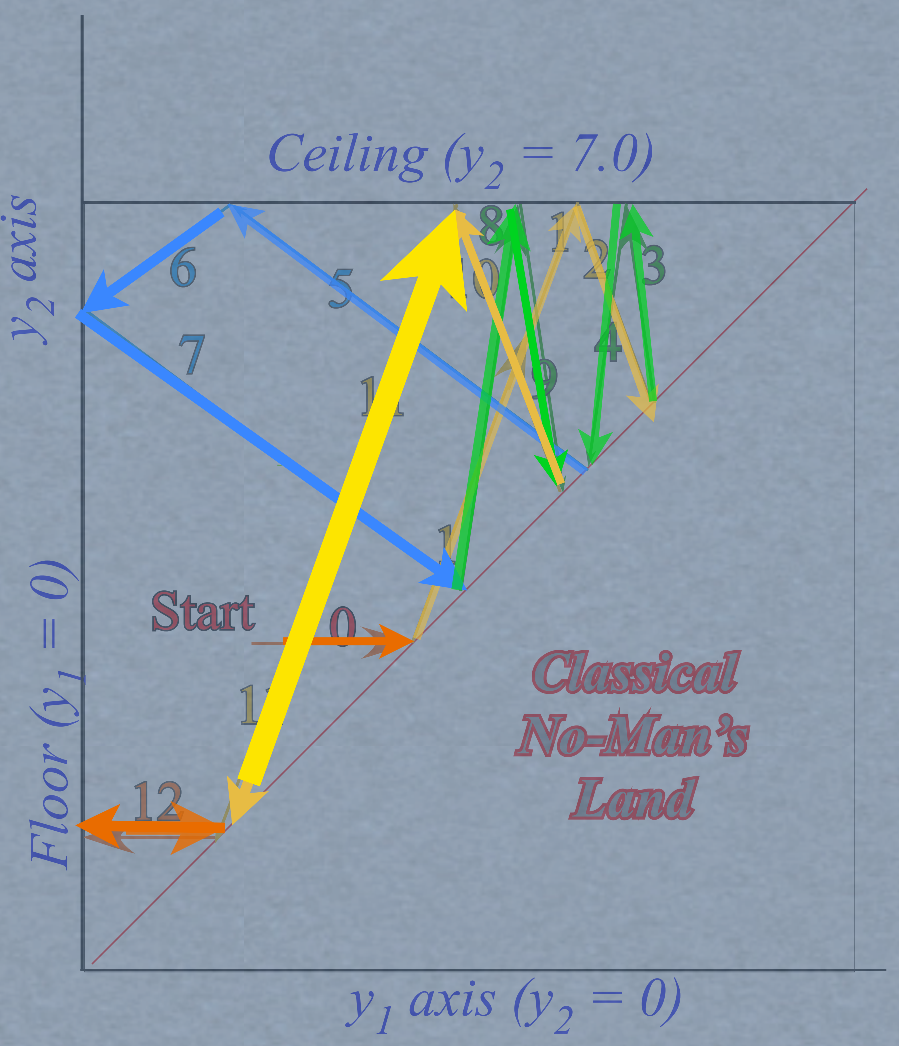


Simulations by *BounceIt*

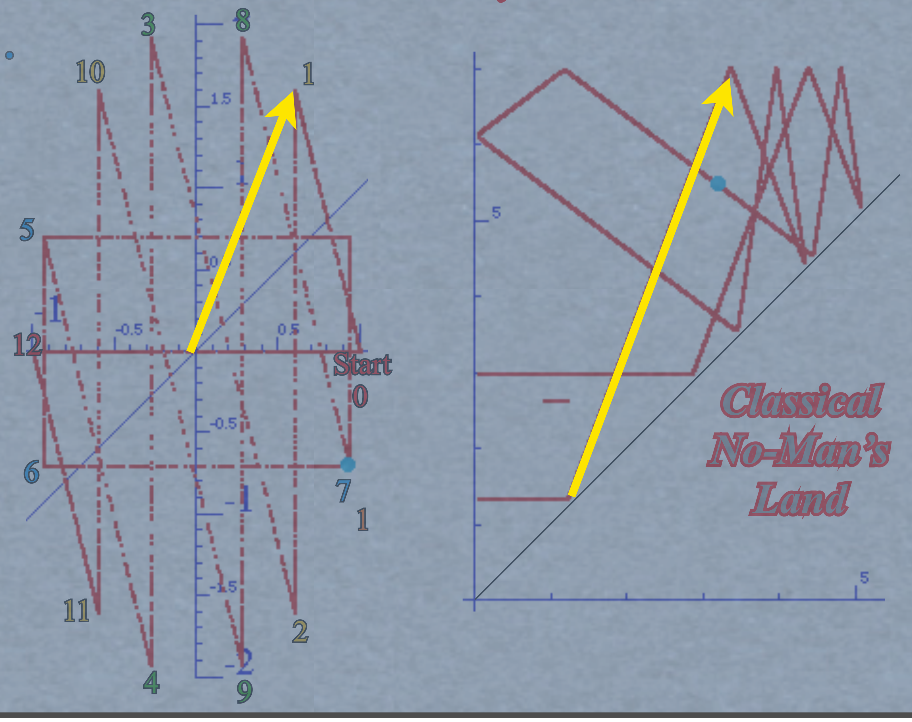


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)

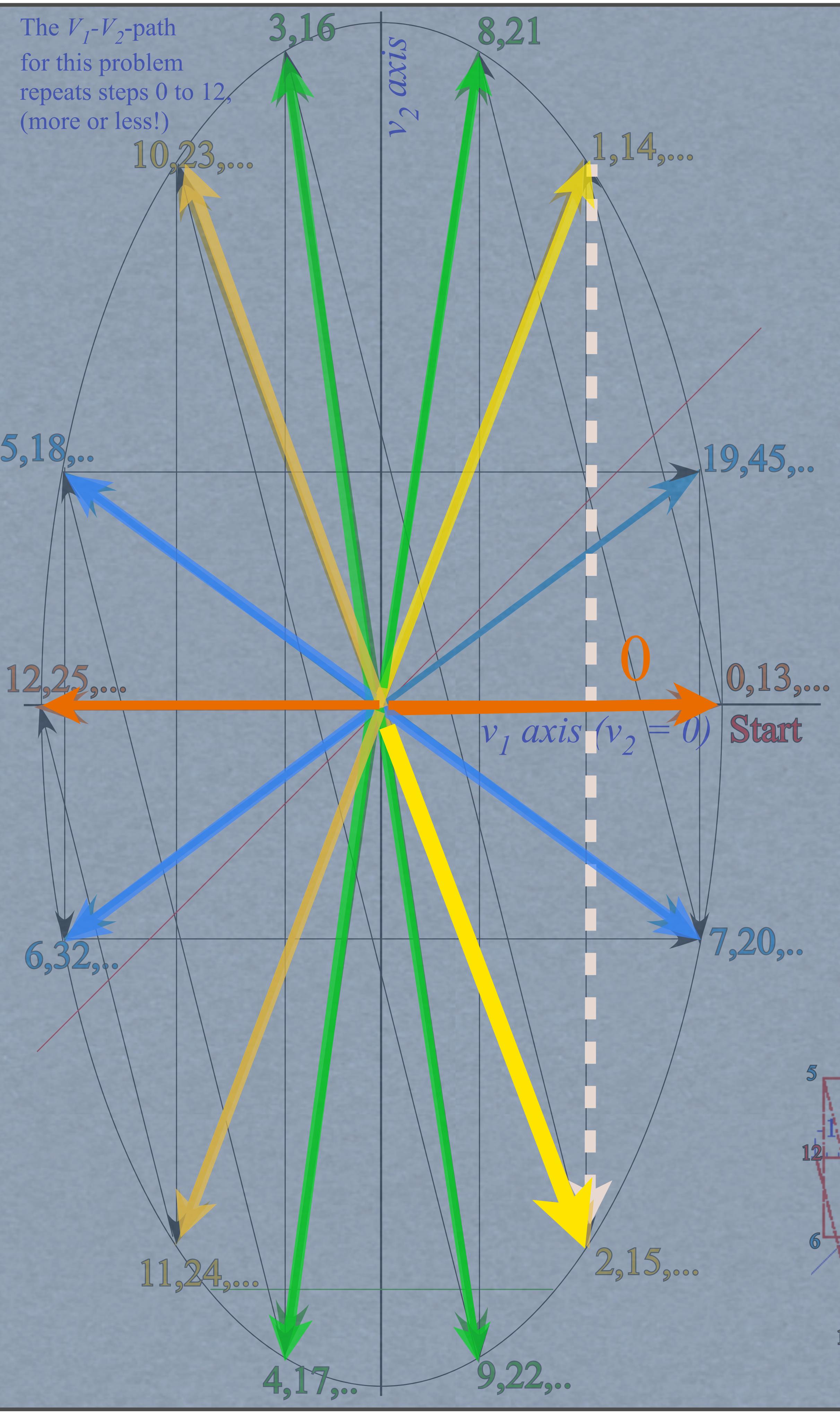


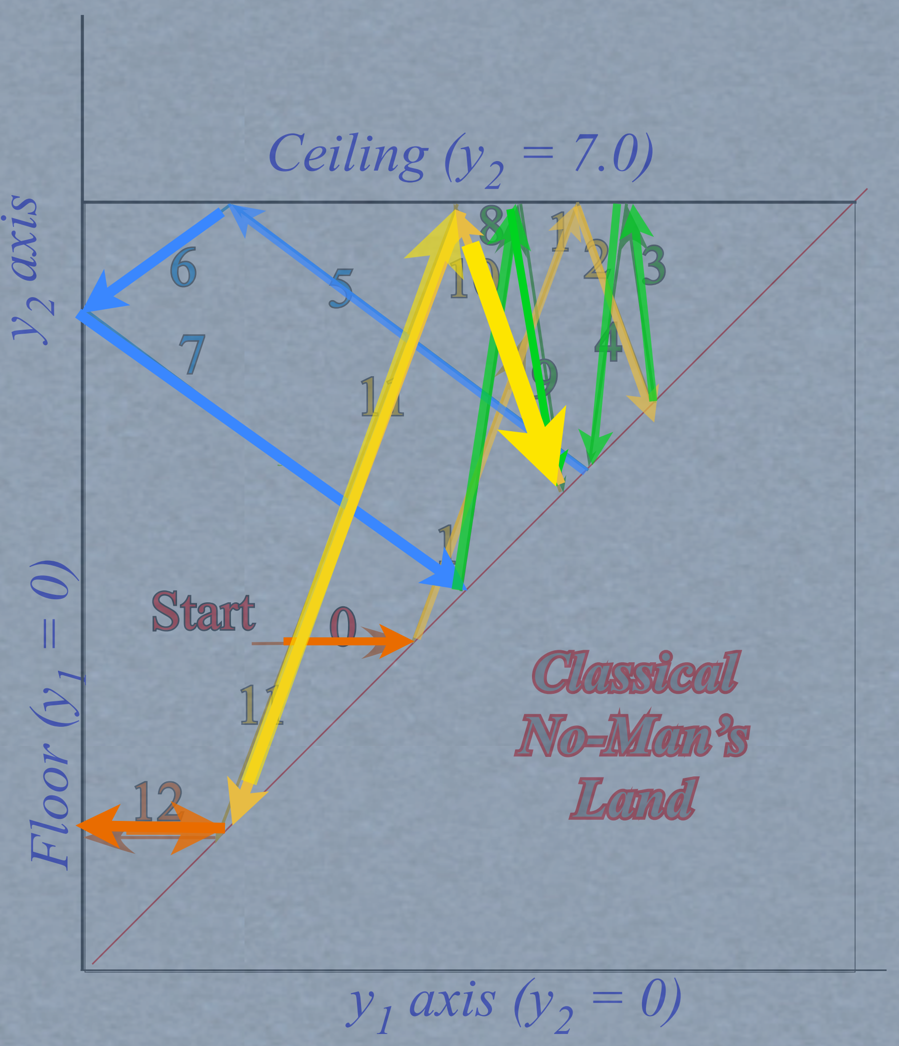


Simulations by *BounceIt*

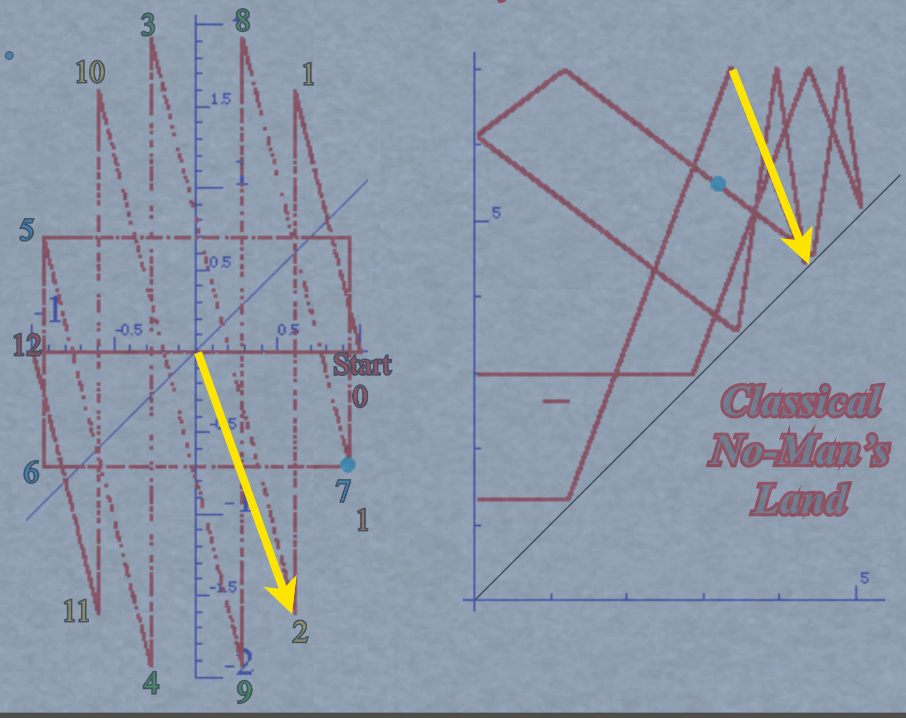


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)

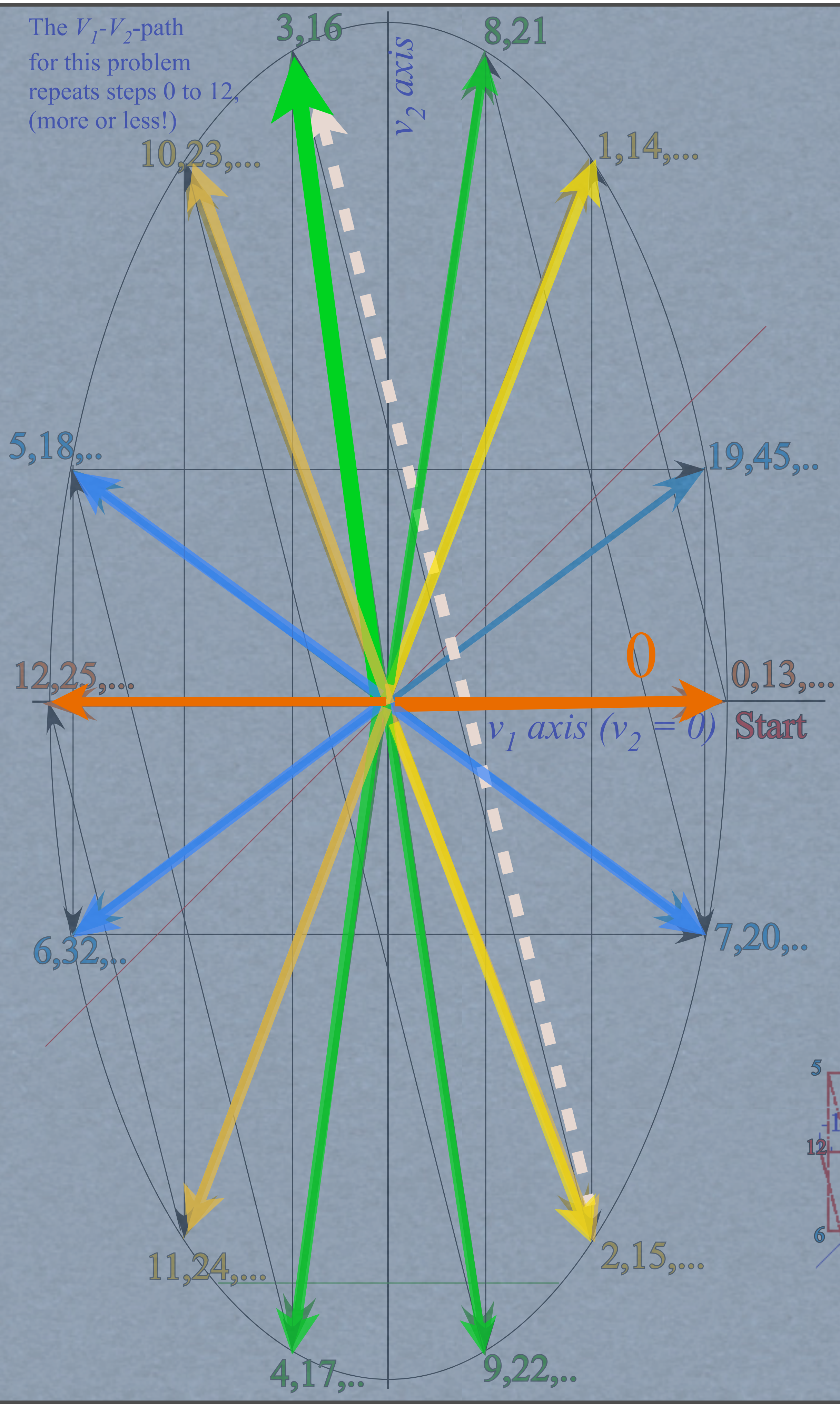


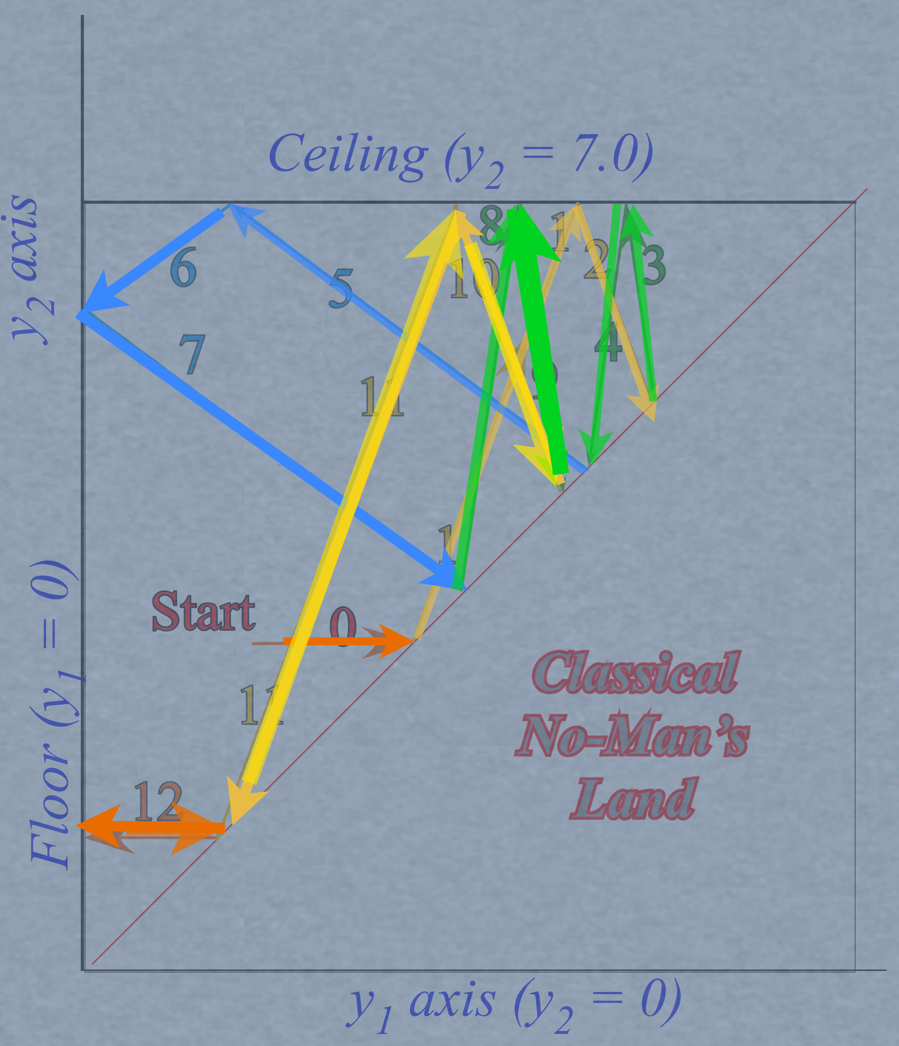


Simulations by *BounceIt*



The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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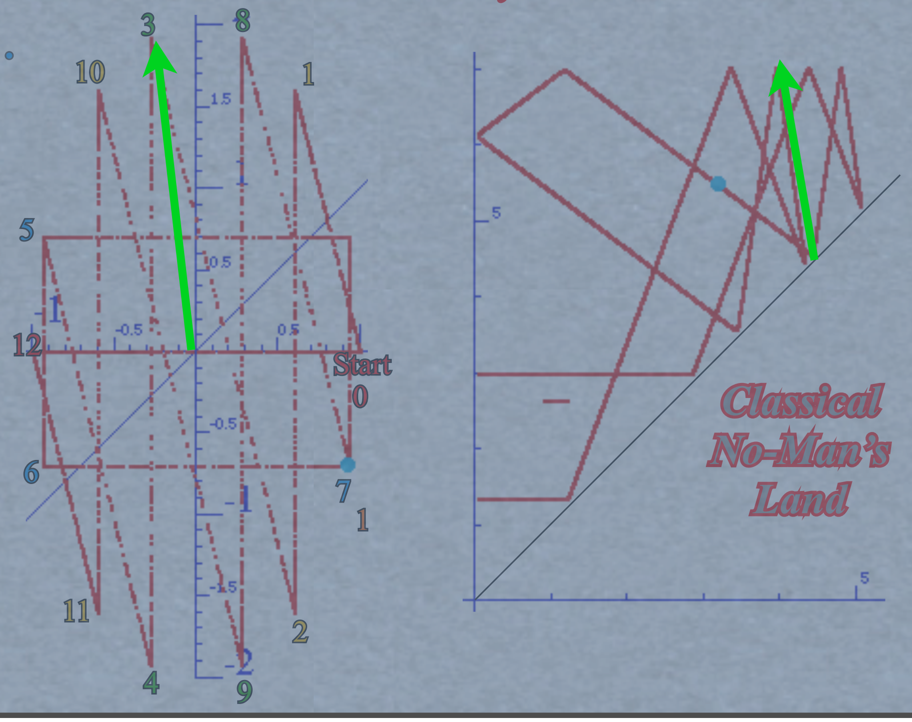
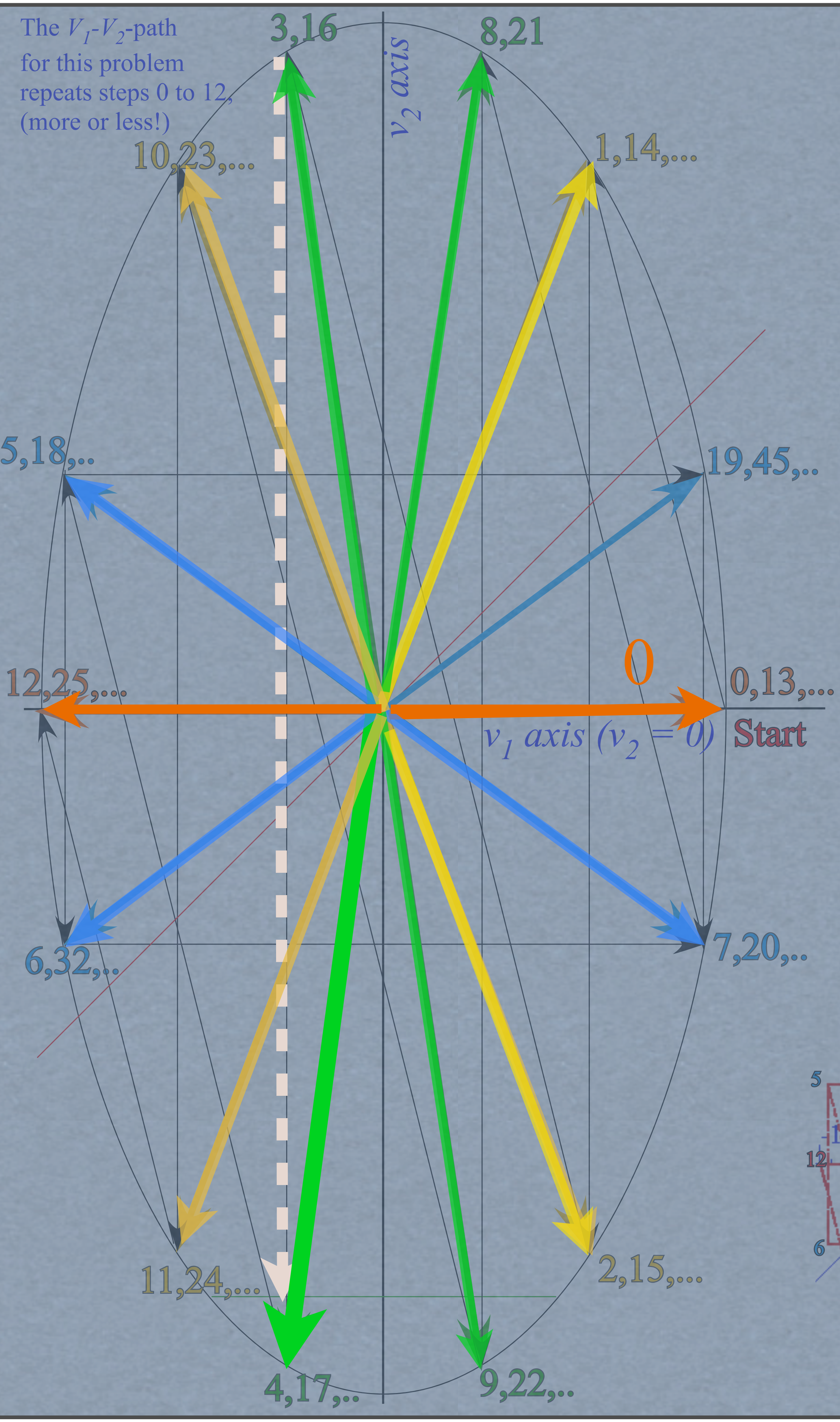
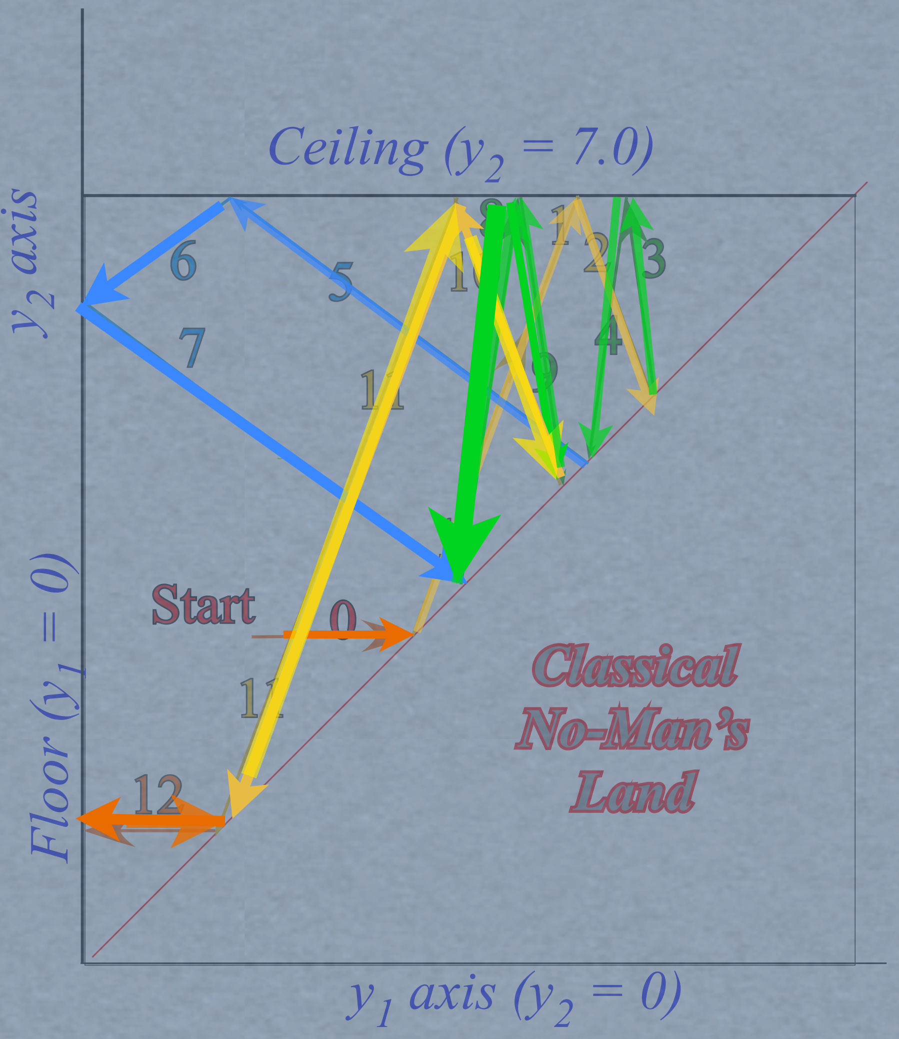


The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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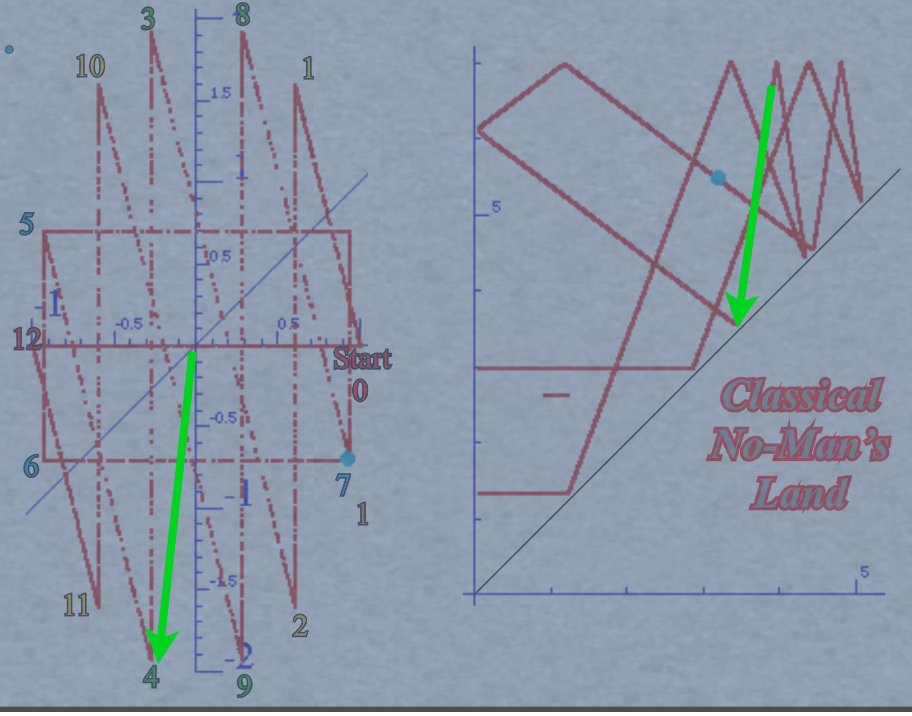
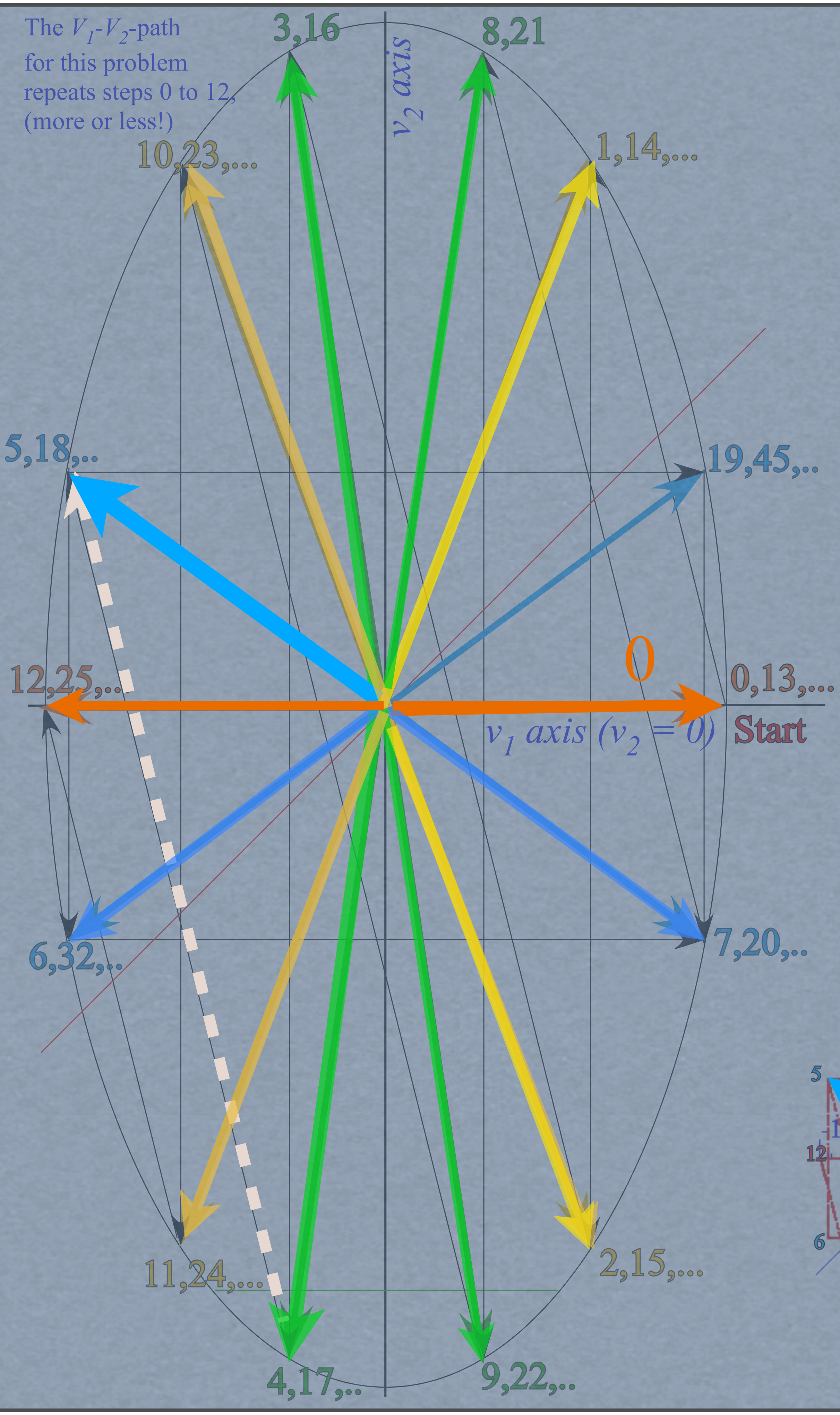
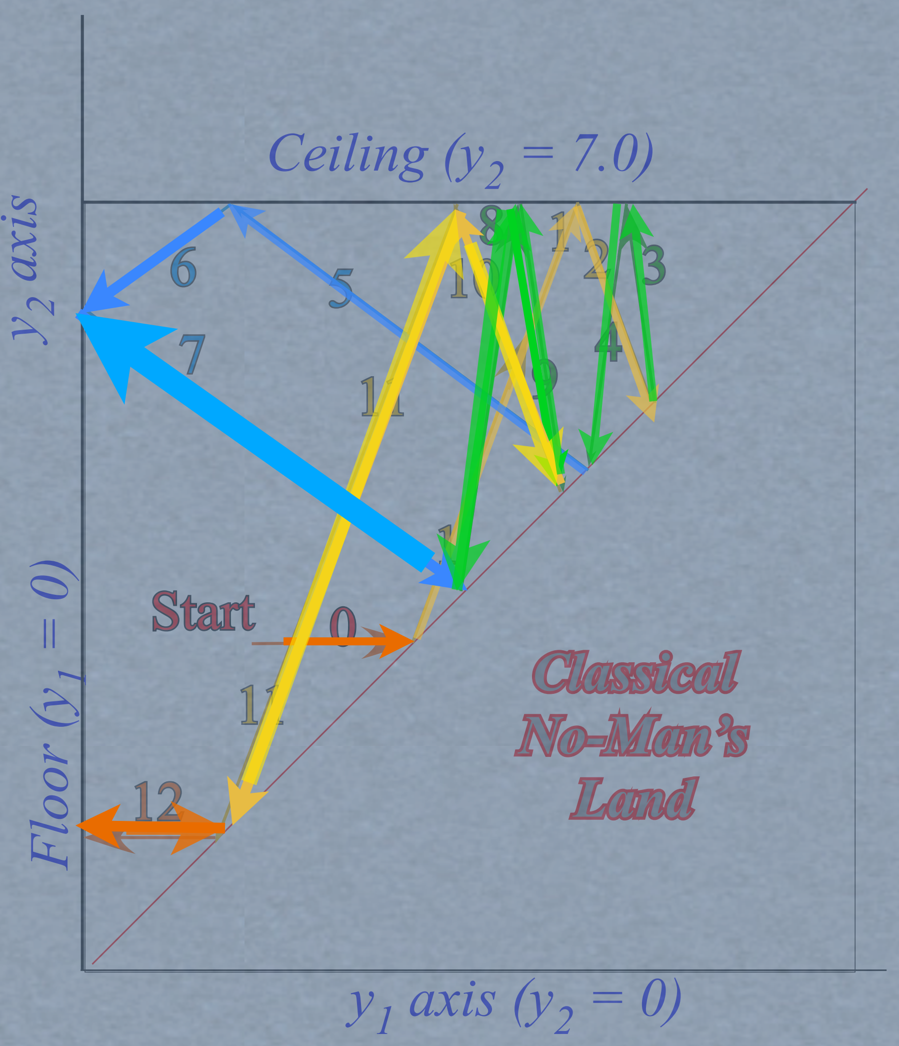


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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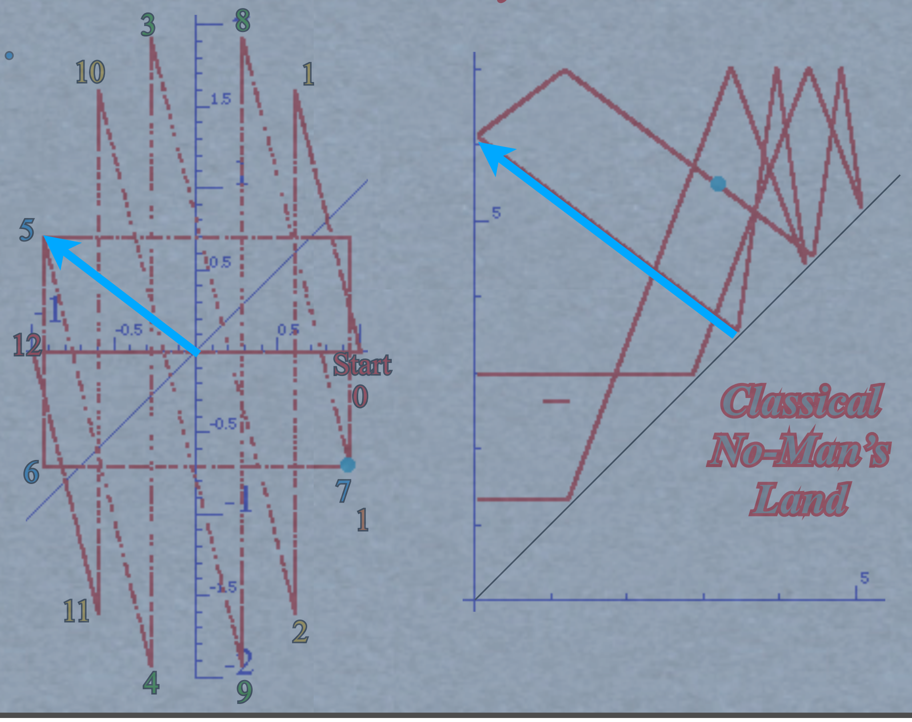
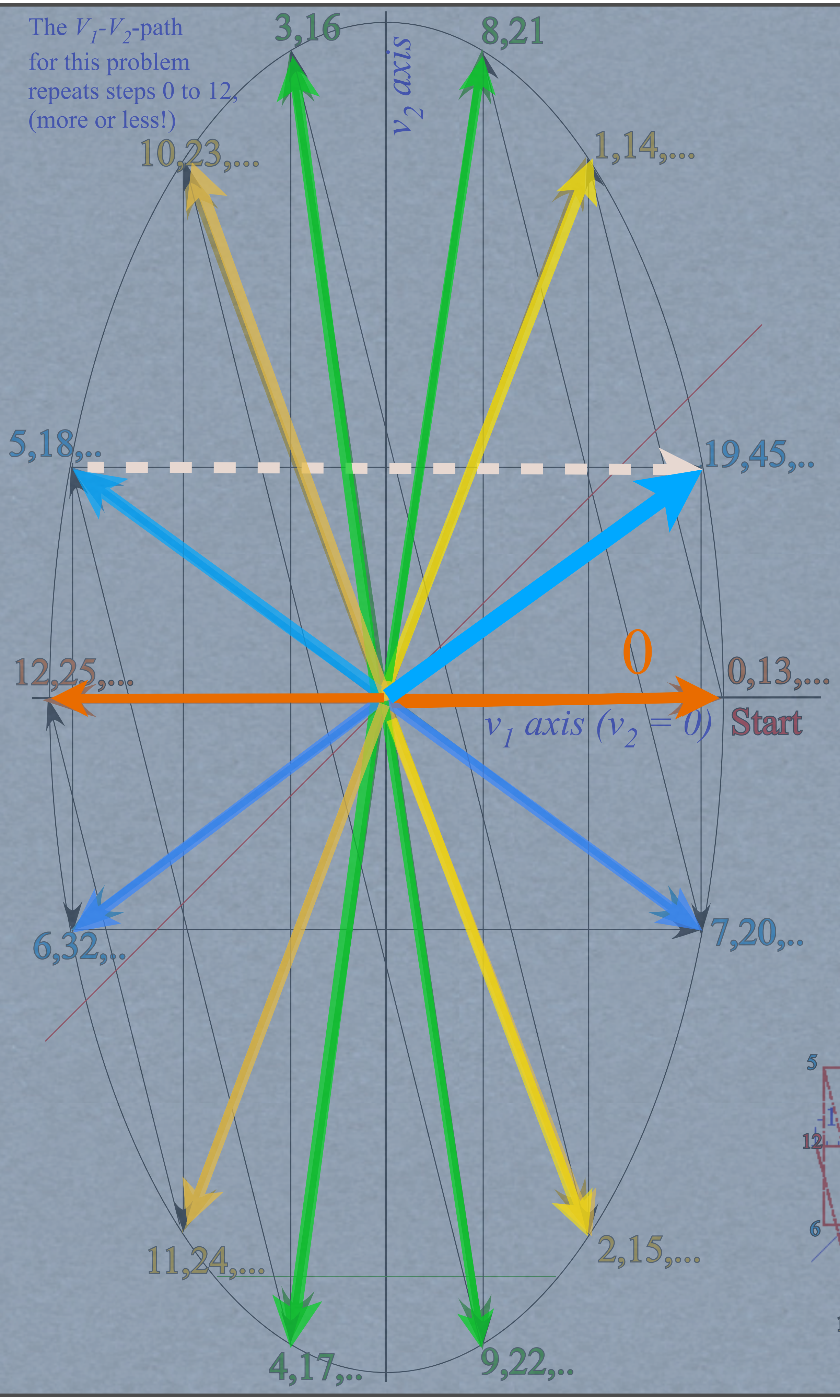
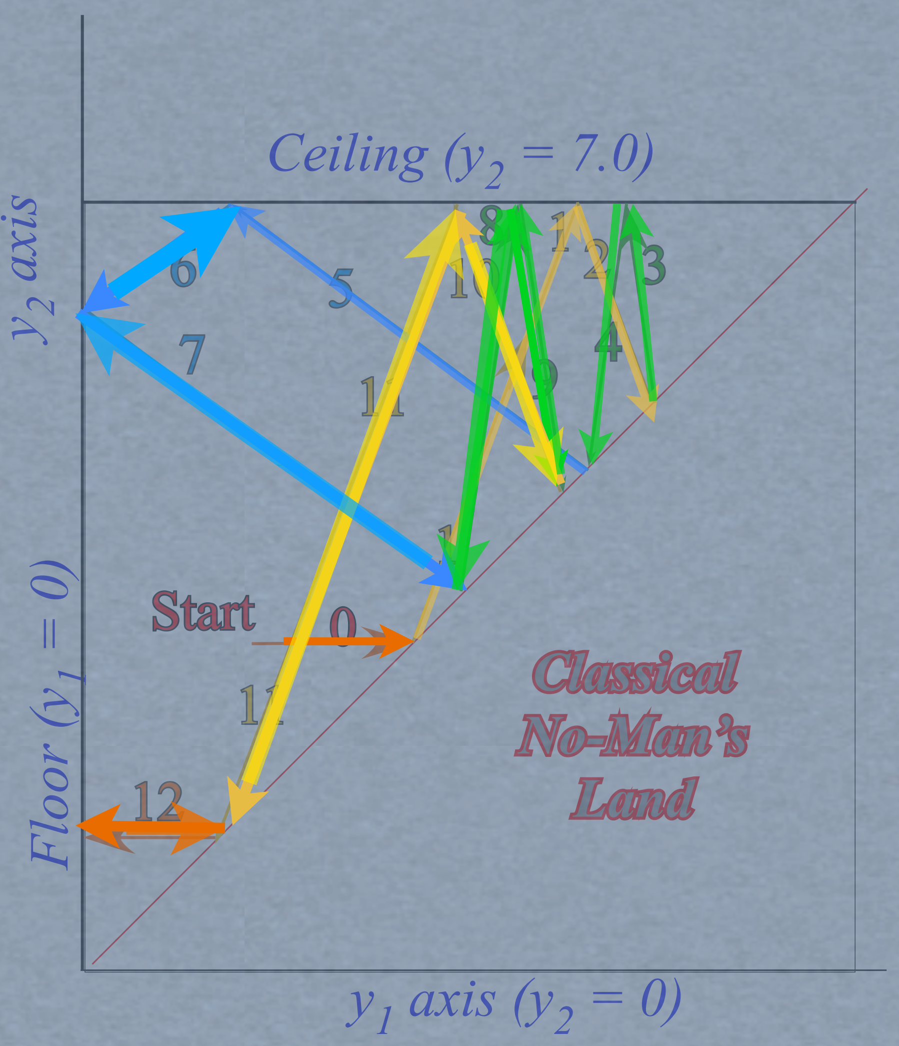
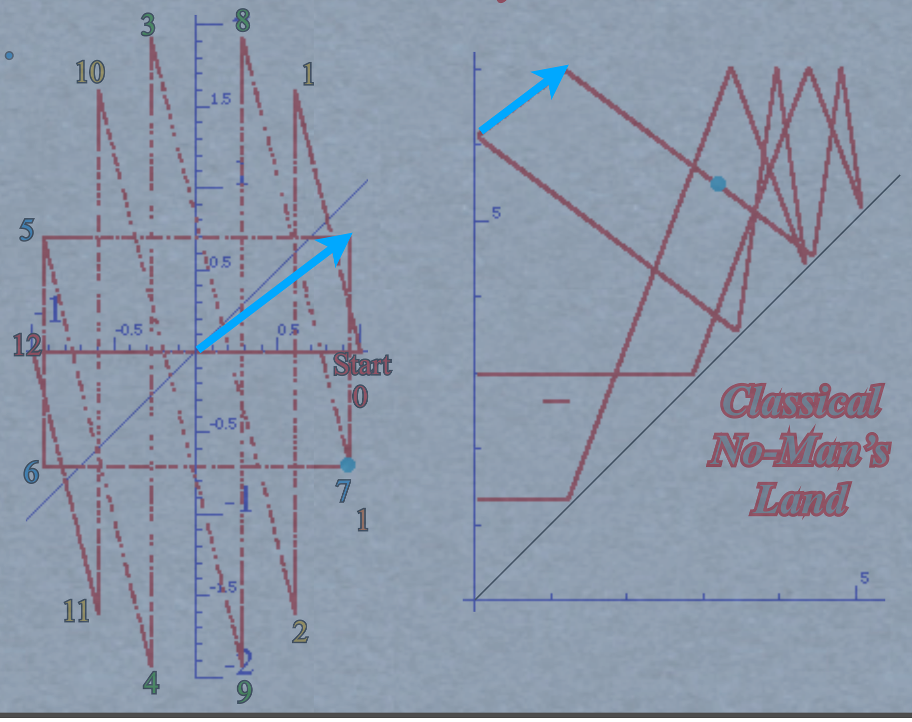


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

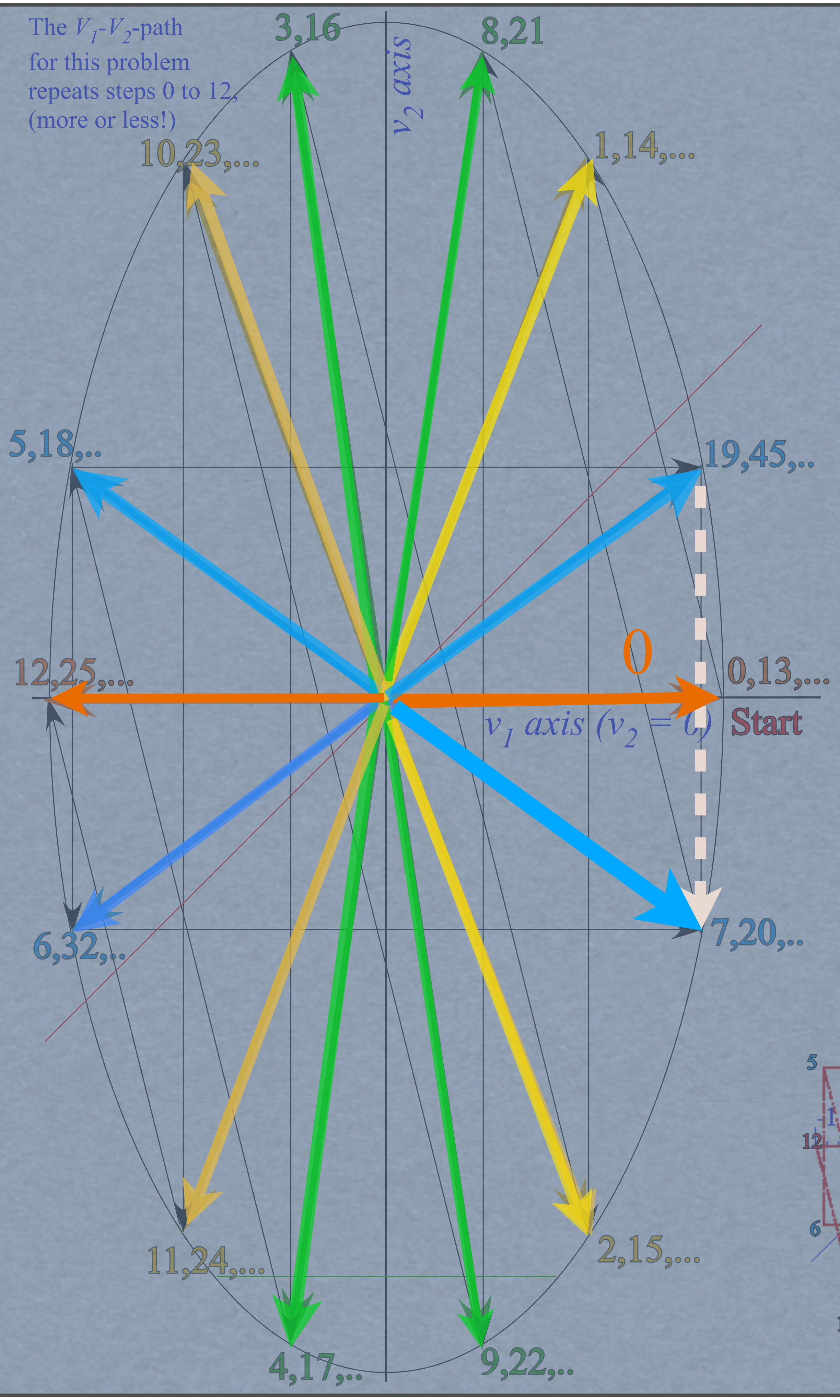


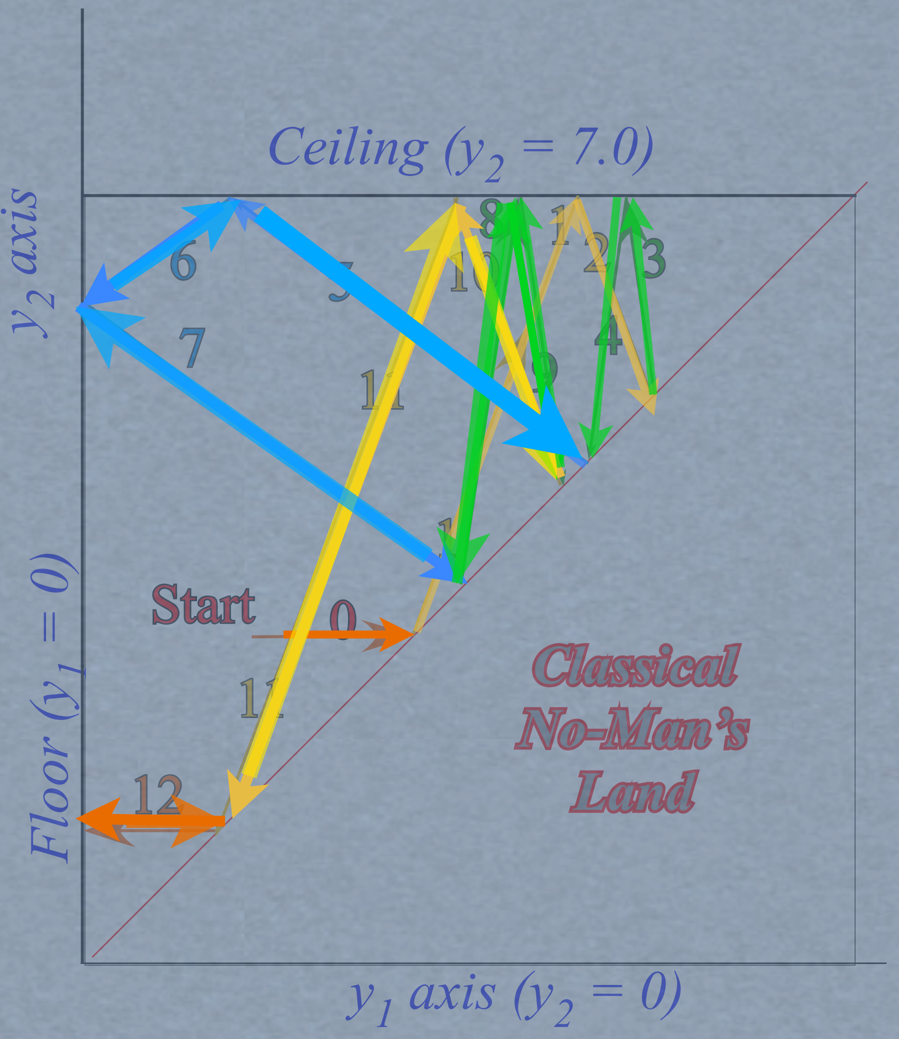


Simulations by *BounceIt*



The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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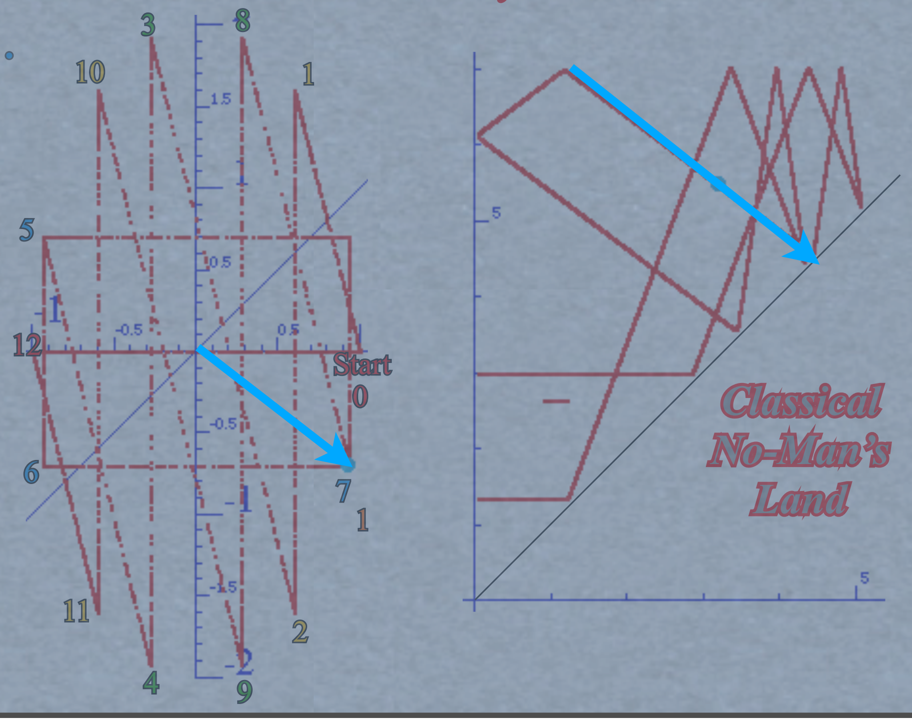
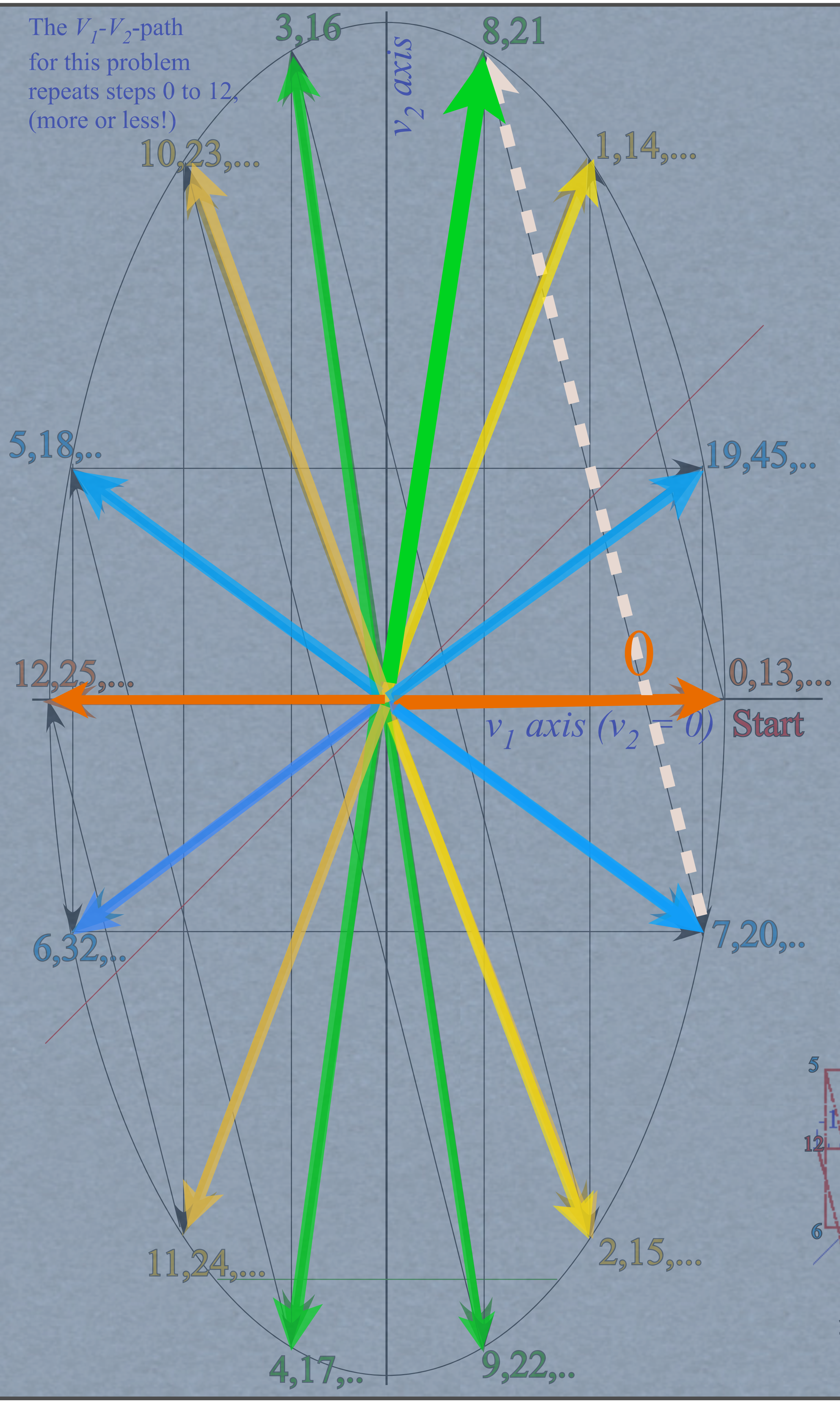
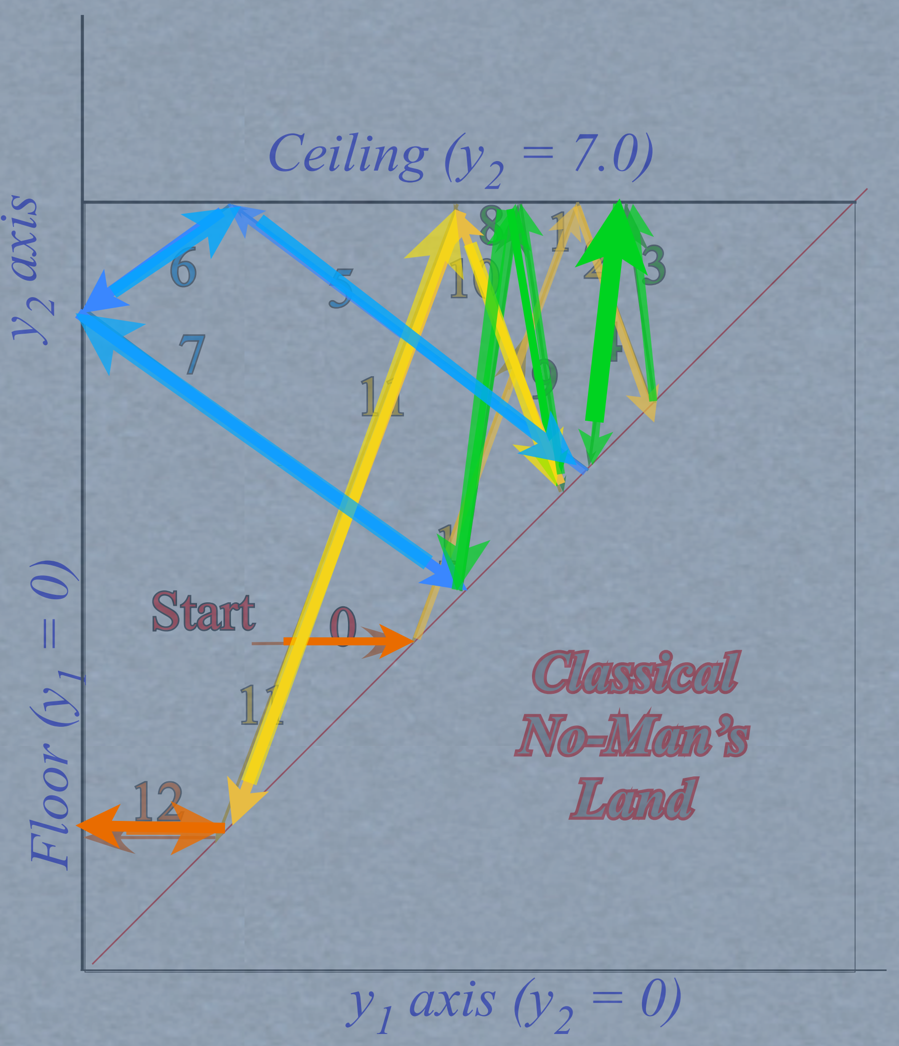
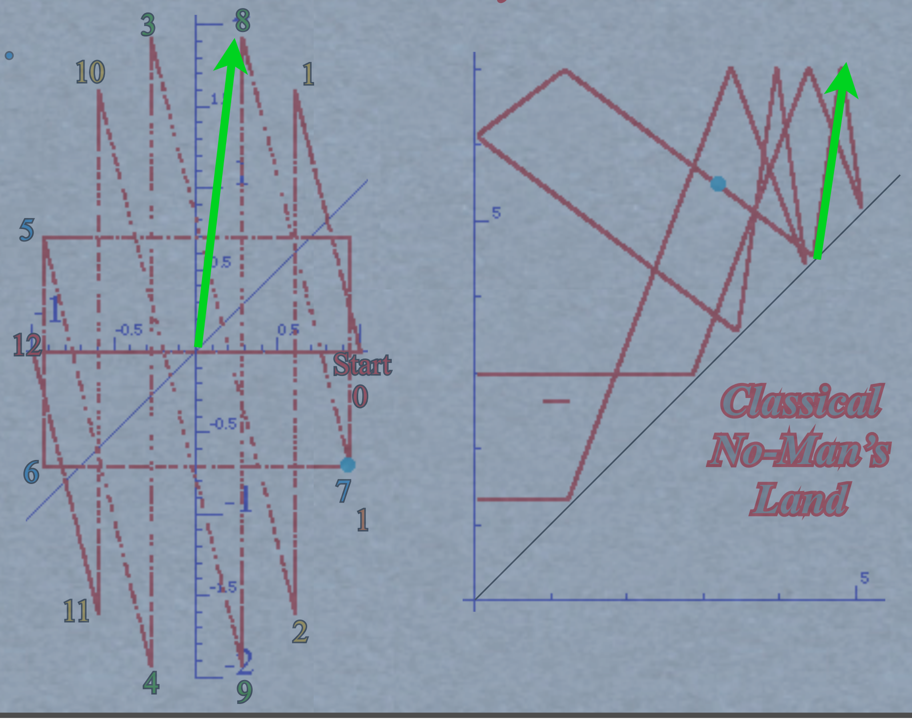




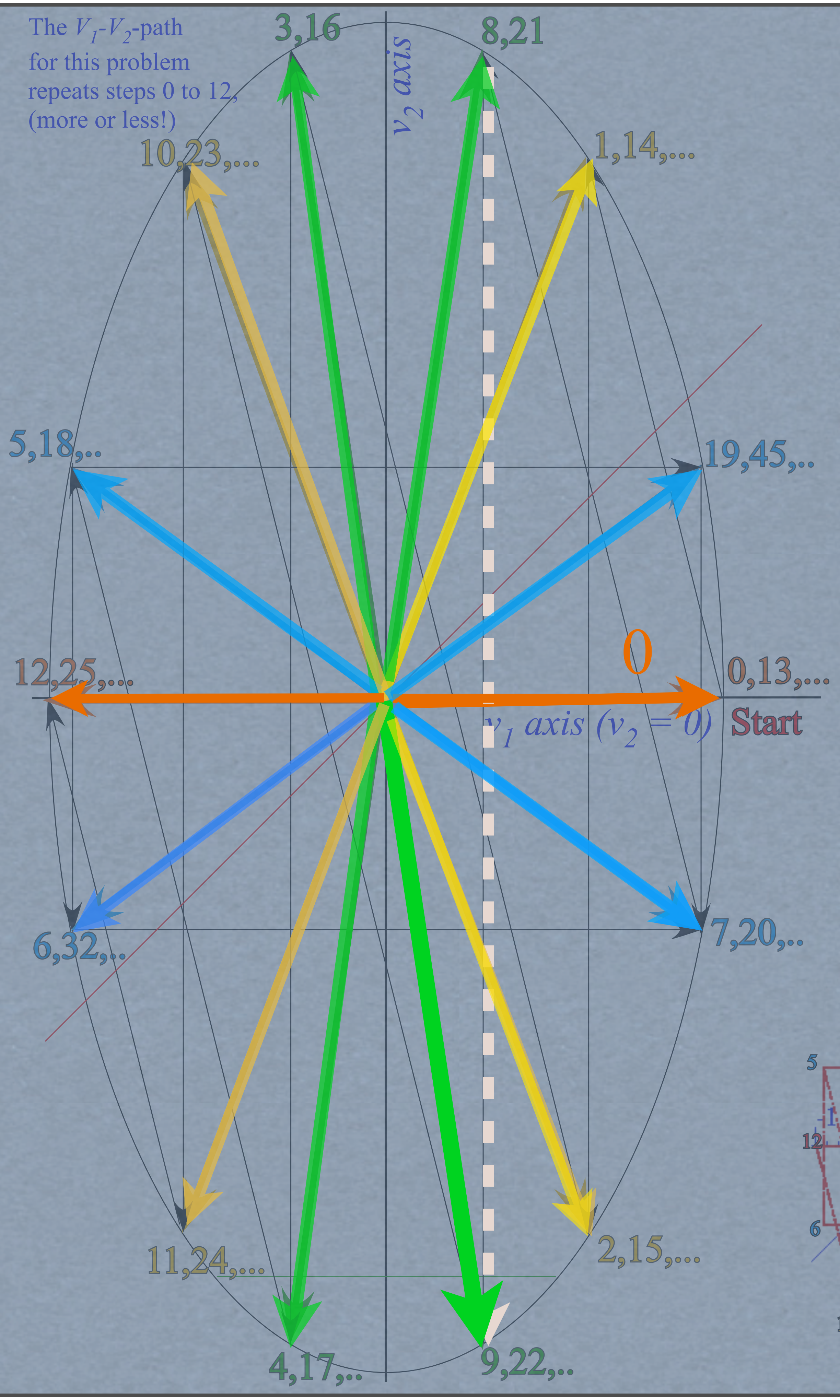
The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

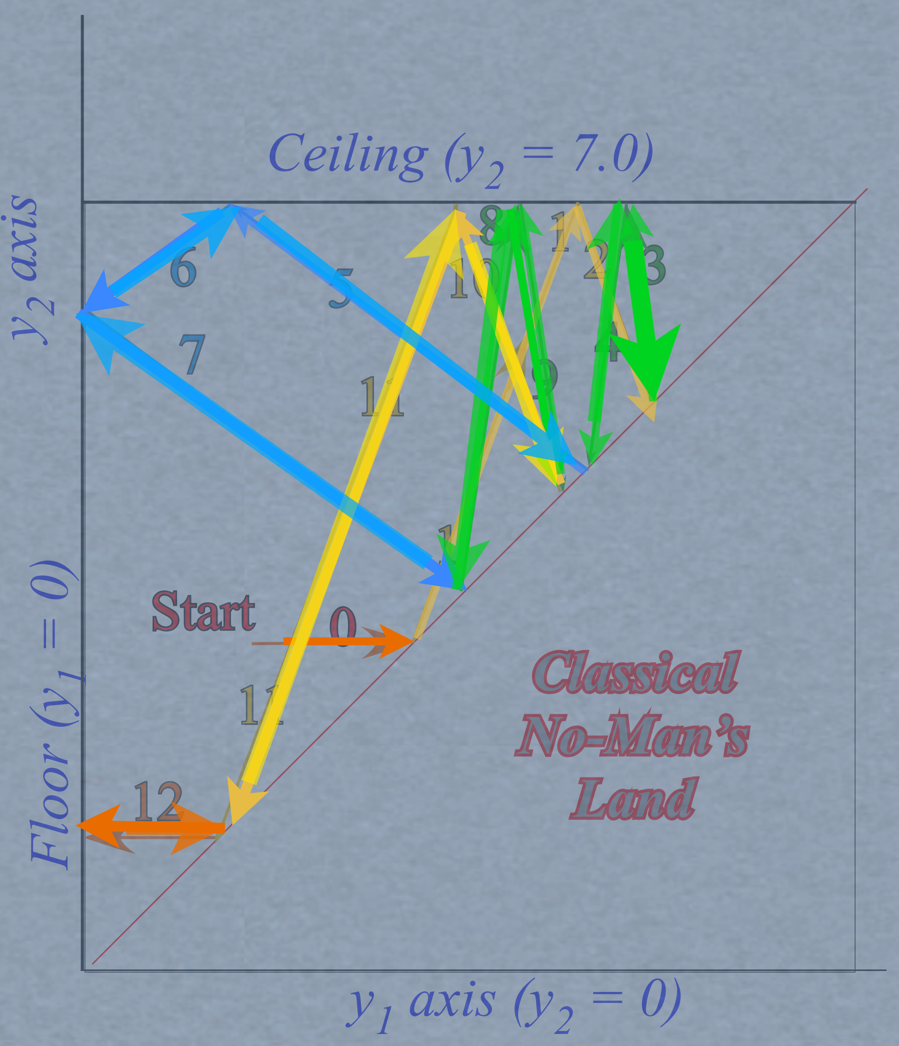


Simulations by *BounceIt*

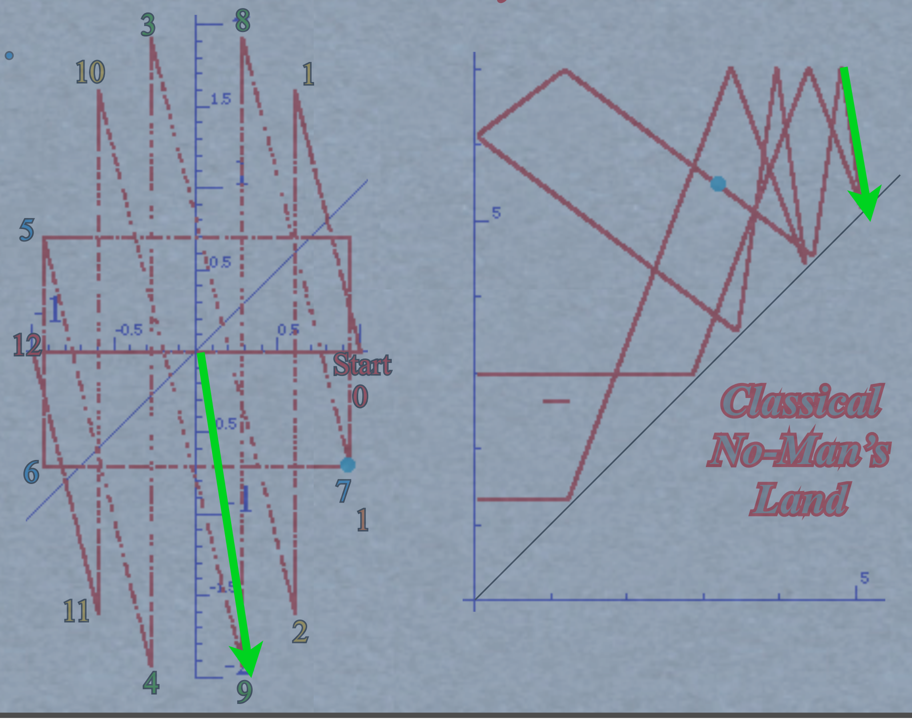


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)

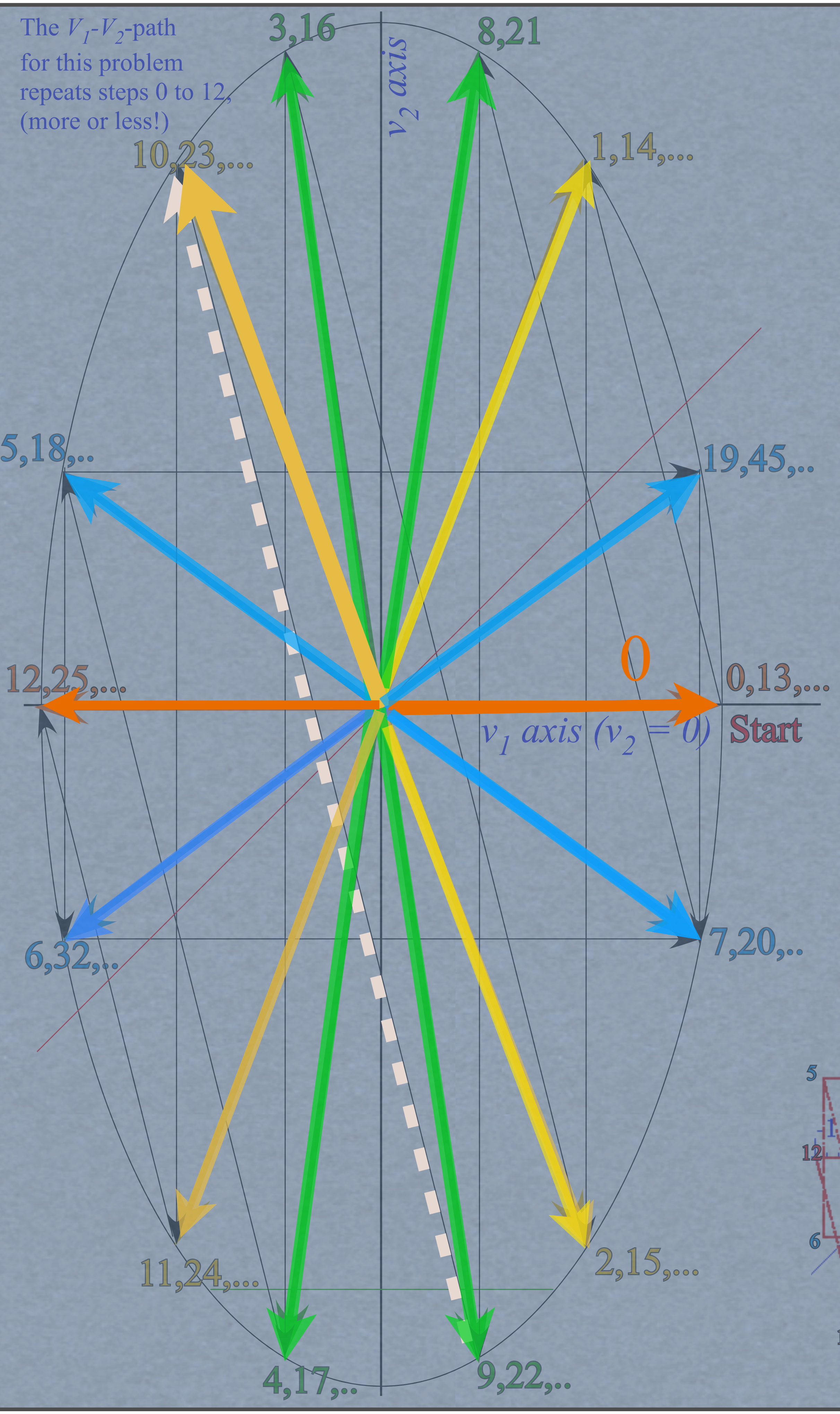


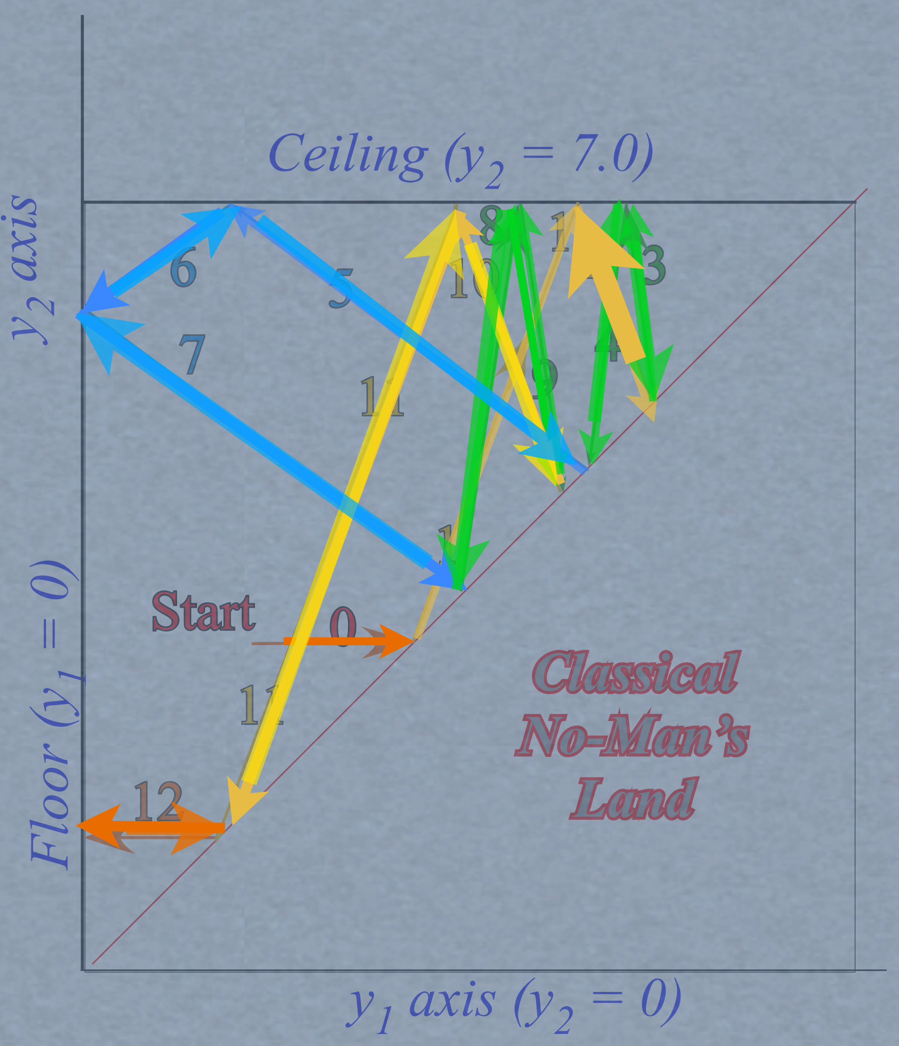


Simulations by *BounceIt*



The V_1-V_2-path
for this problem
repeats steps 0 to 12,
(more or less!)





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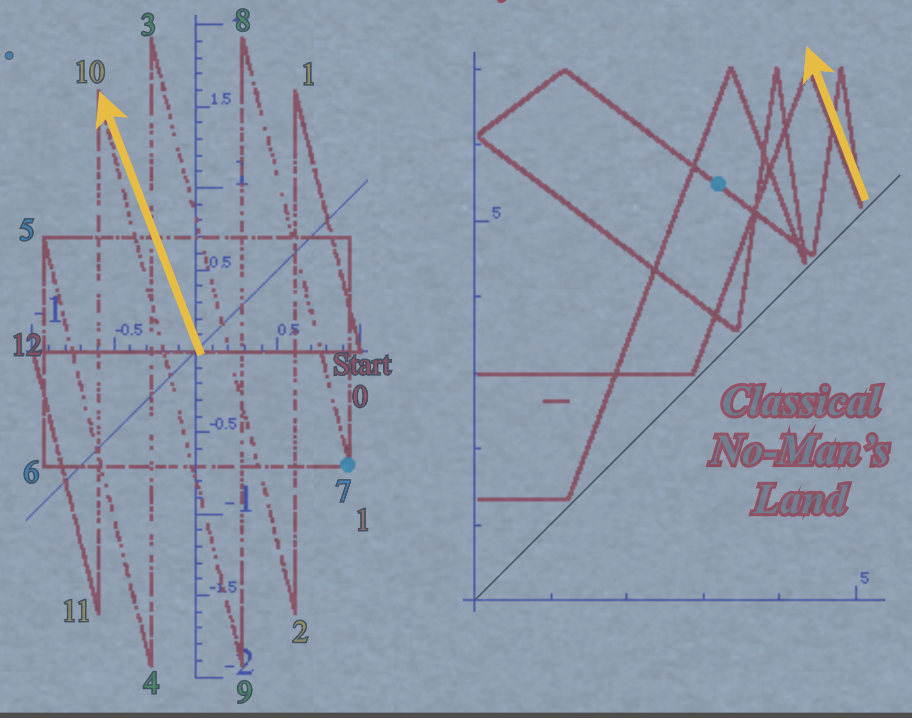
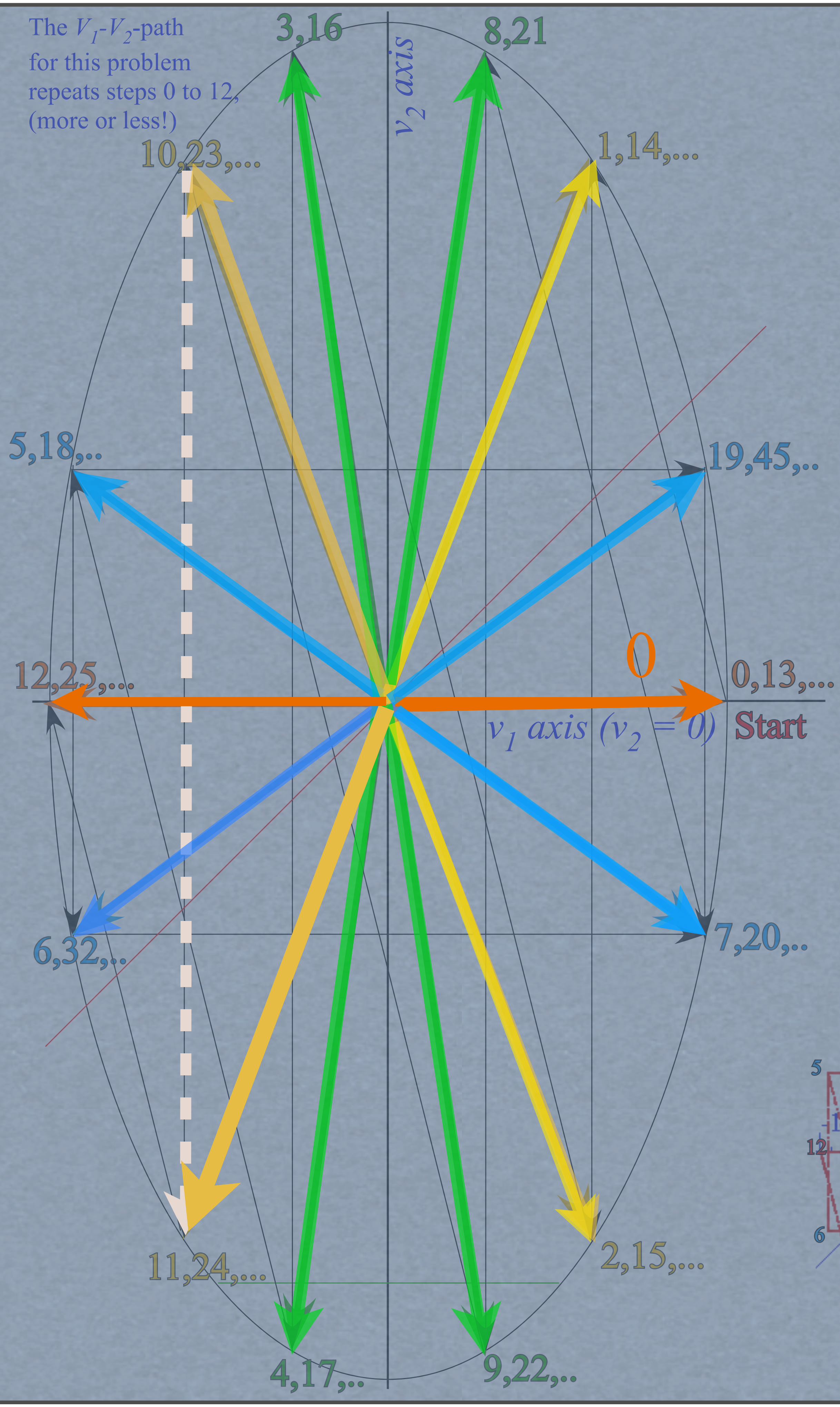
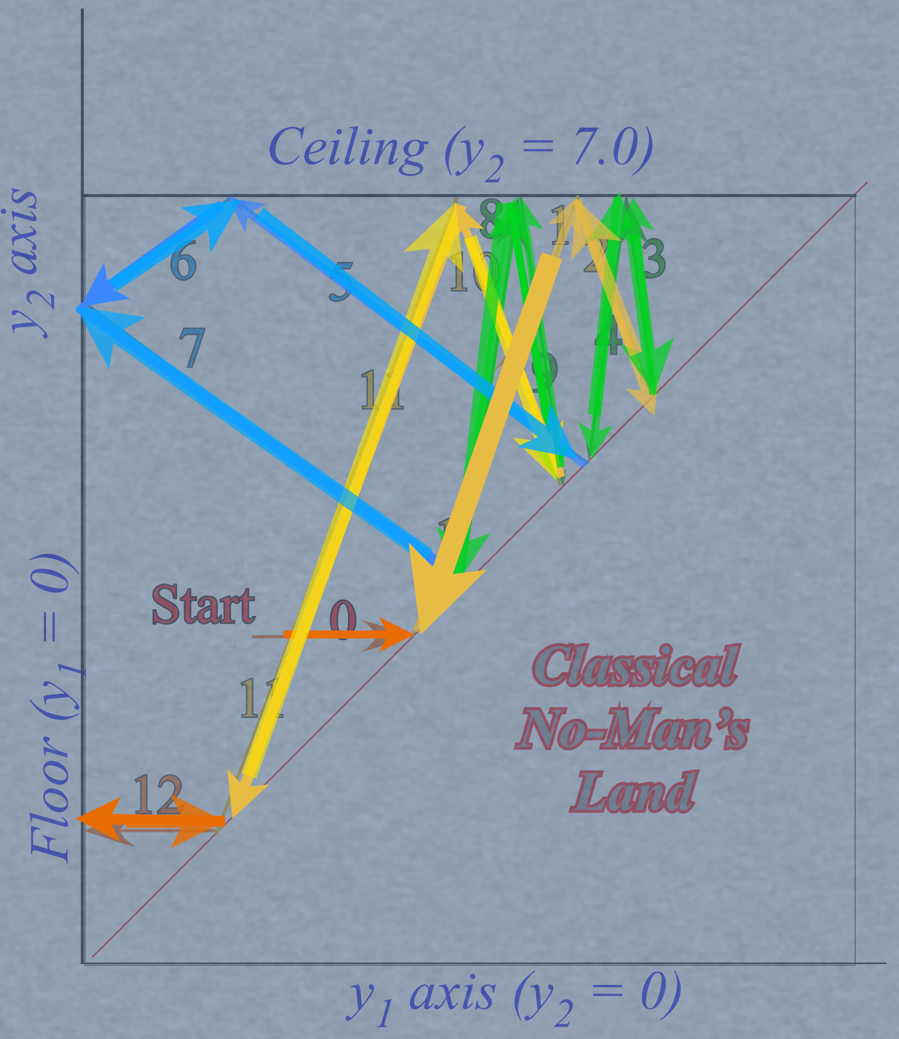


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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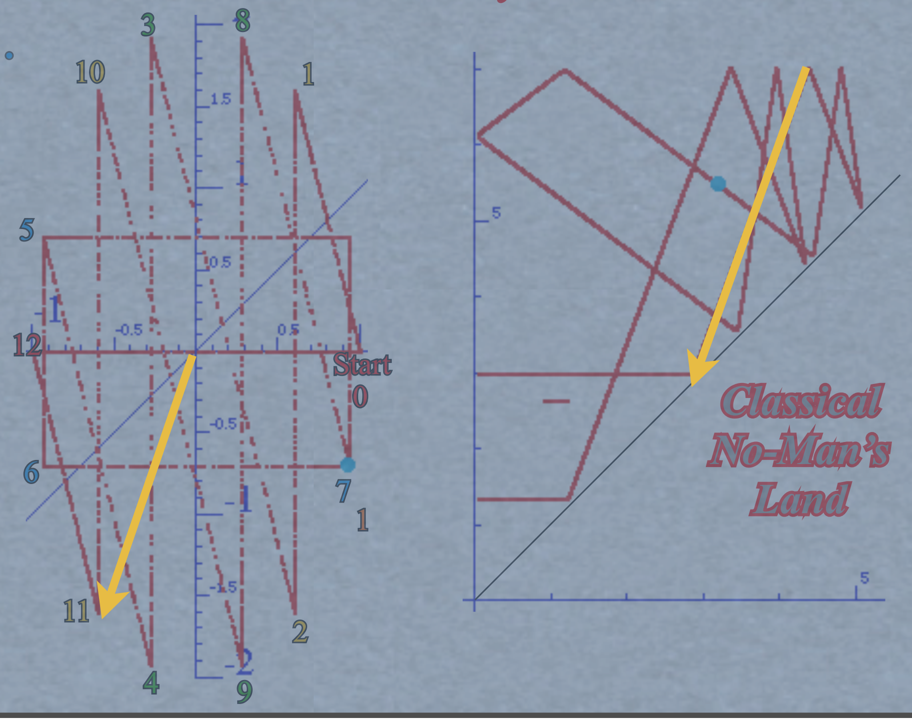
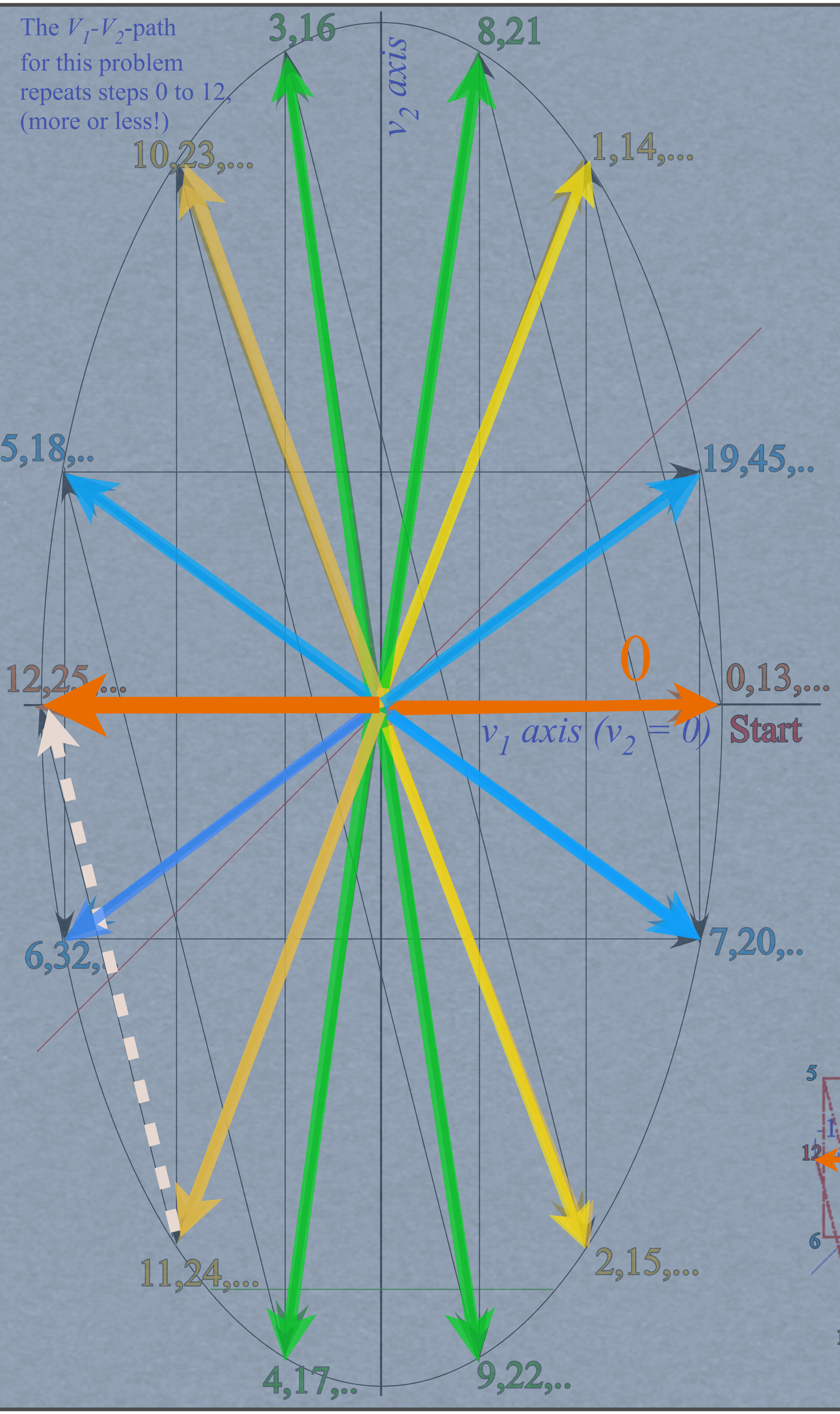
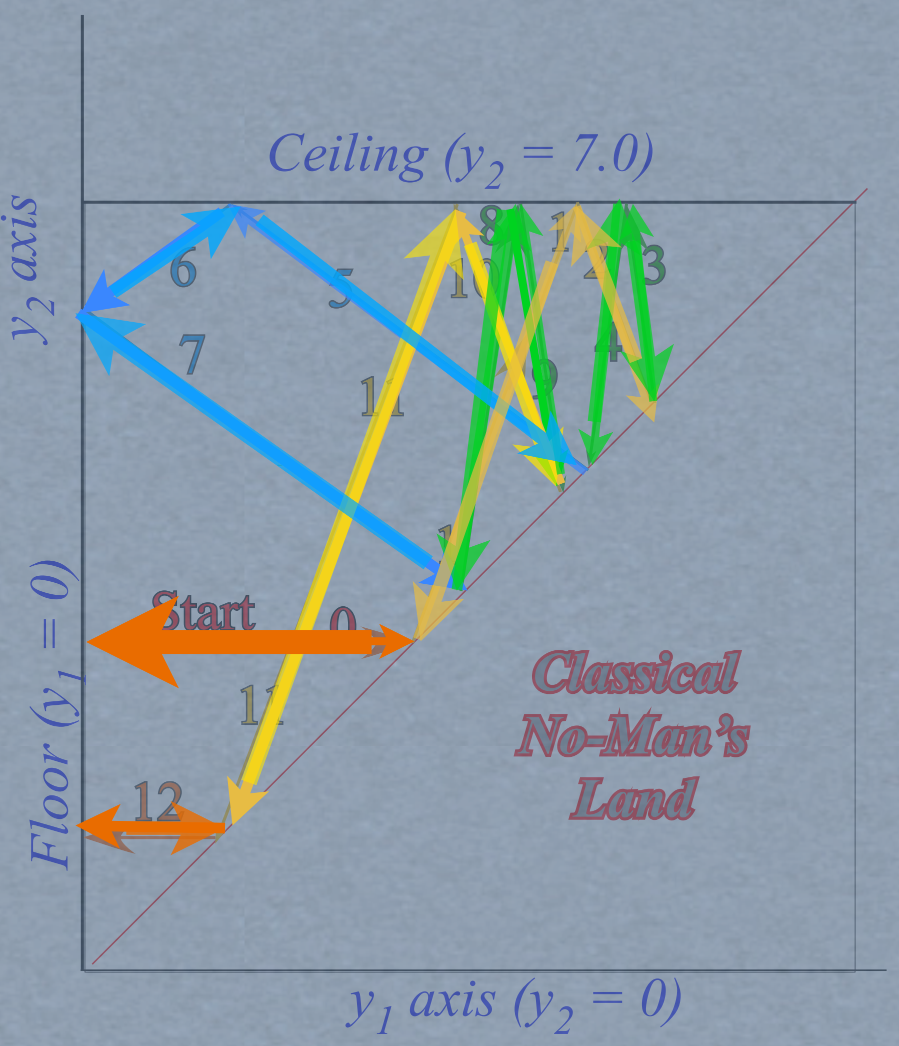


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)





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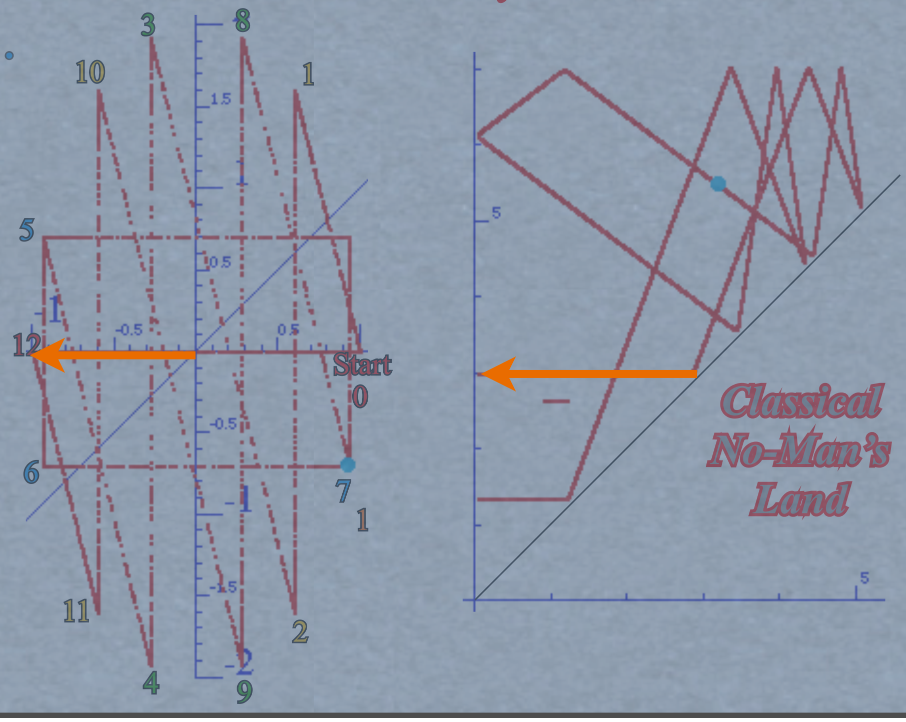
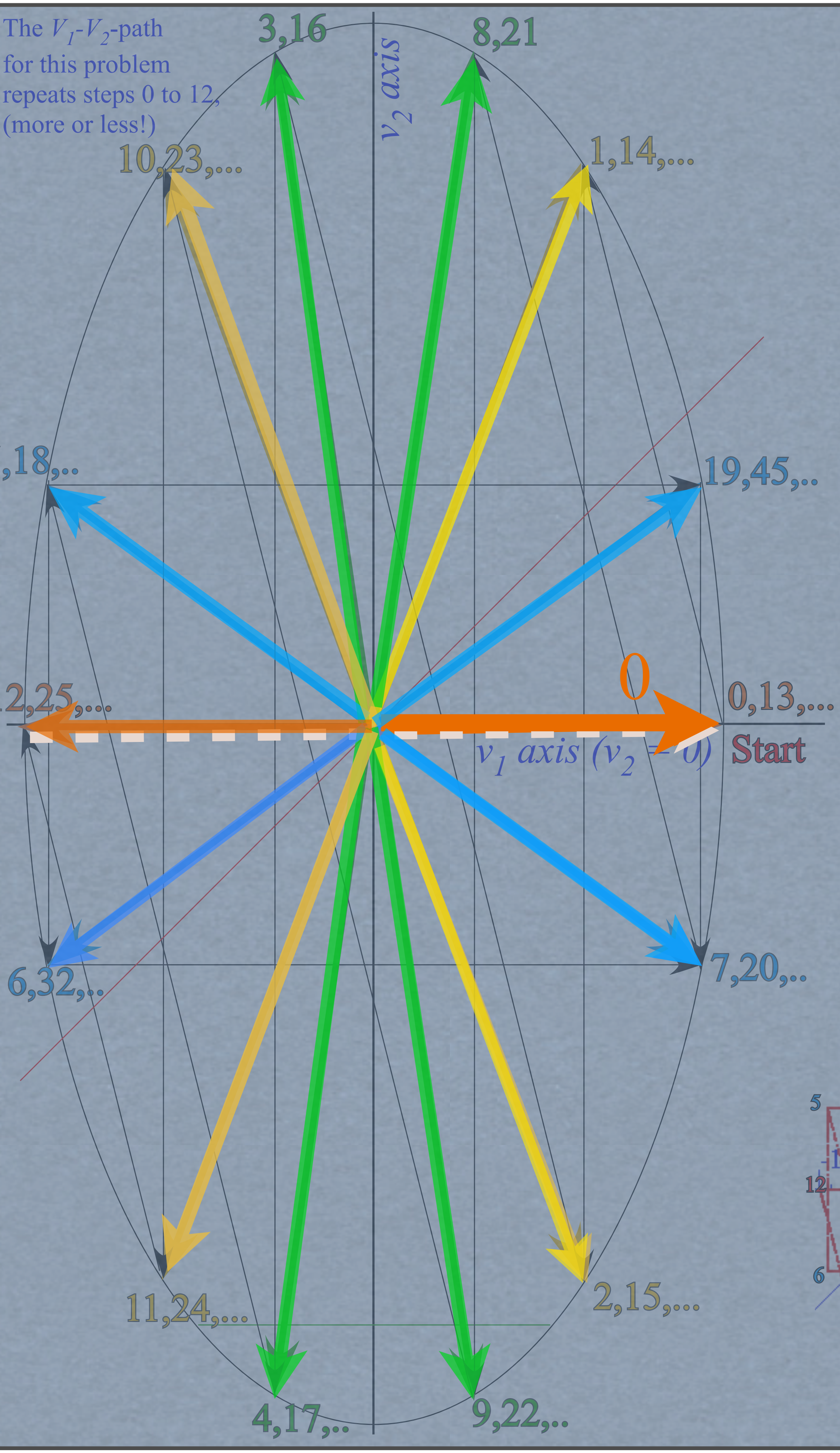
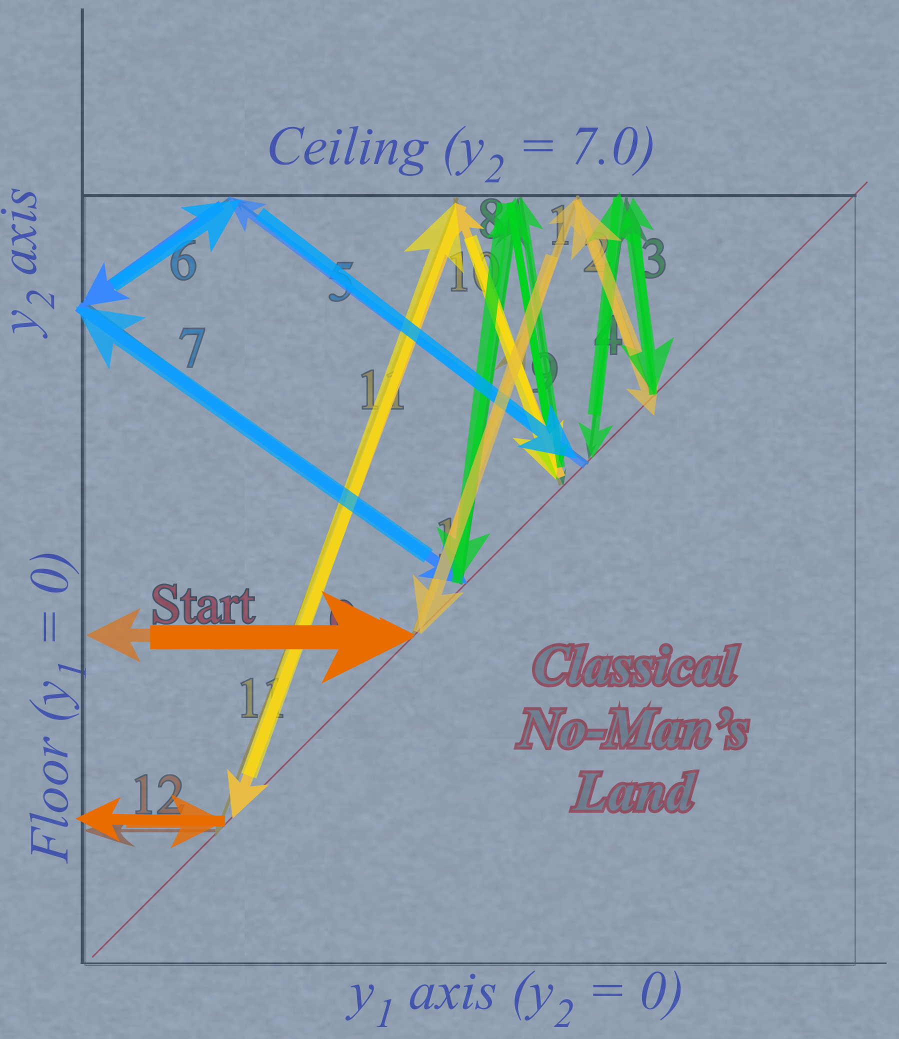
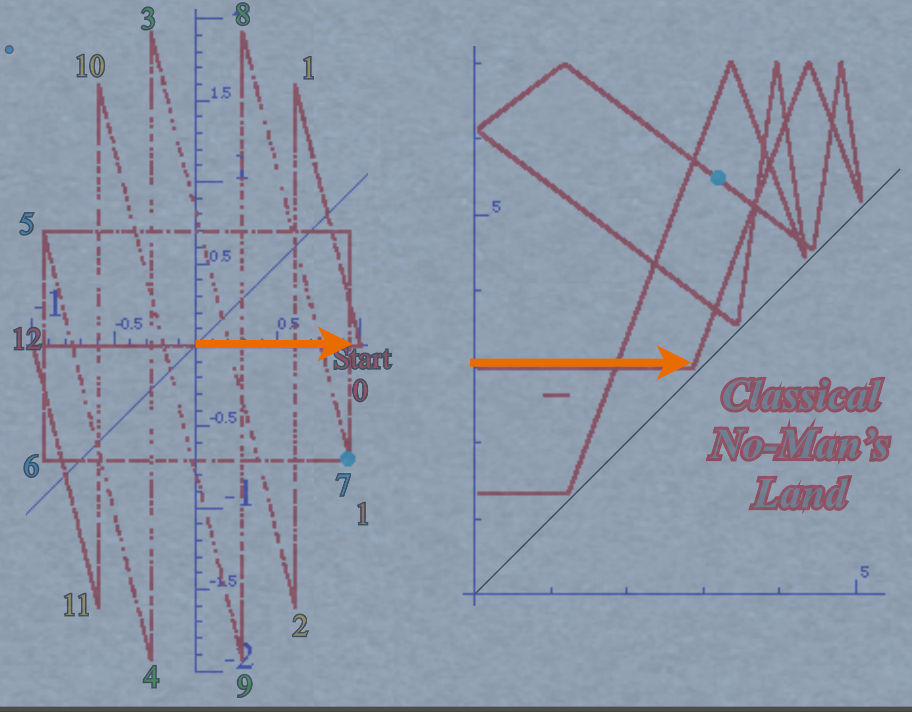


The V_1-V_2-path for this problem repeats steps 0 to 12, (more or less!)



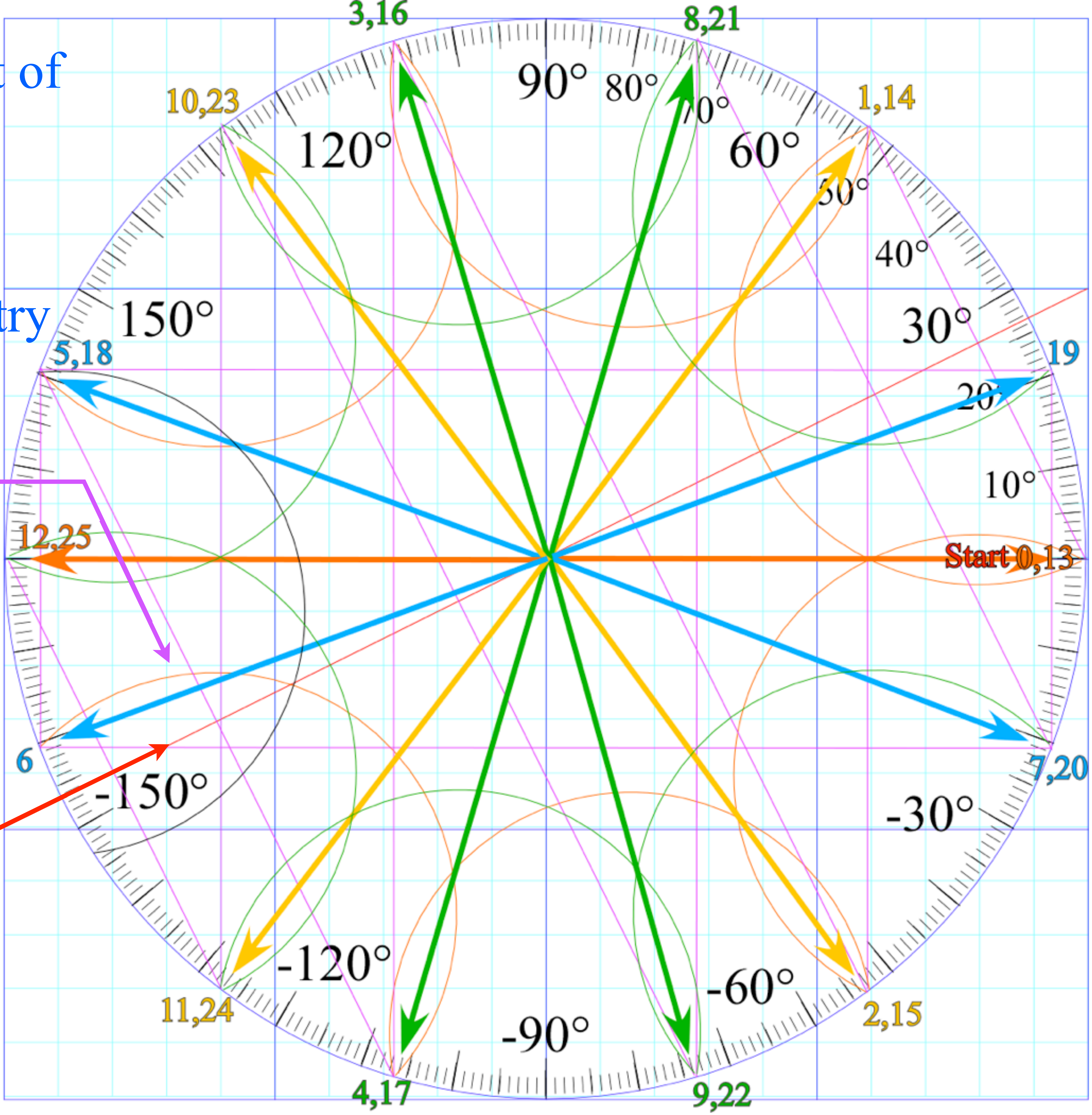


Simulations by *BounceIt*



Estrangian plot of $m_1/m_2=4/1$ collision sequence shows symmetry

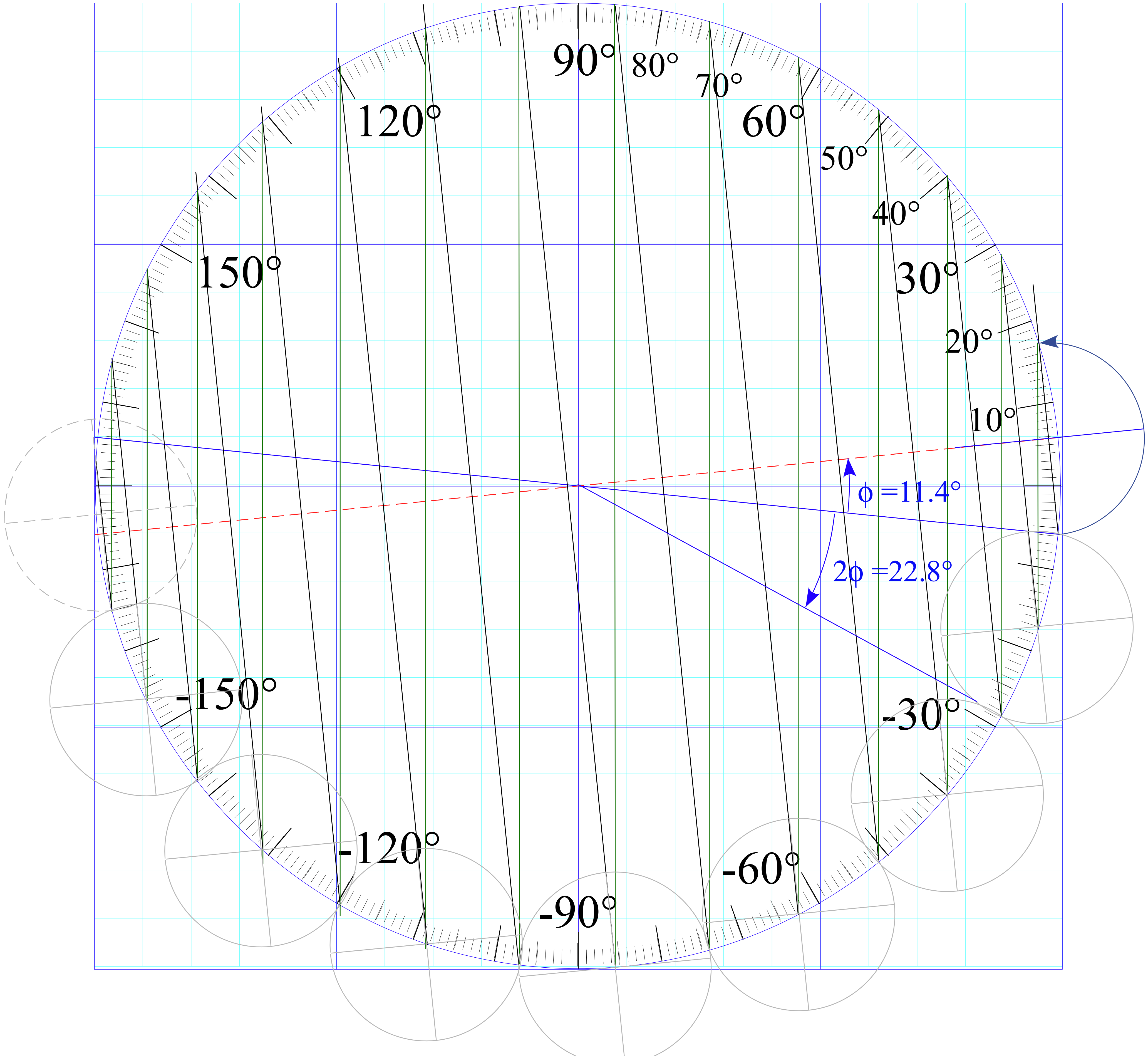
(sort of)



c.o.m. lines (cons. of mom.) have slope $-\sqrt{m_2}/\sqrt{m_1}=-2/1$

COM line has slope $\sqrt{m_2}/\sqrt{m_1}=1/2$

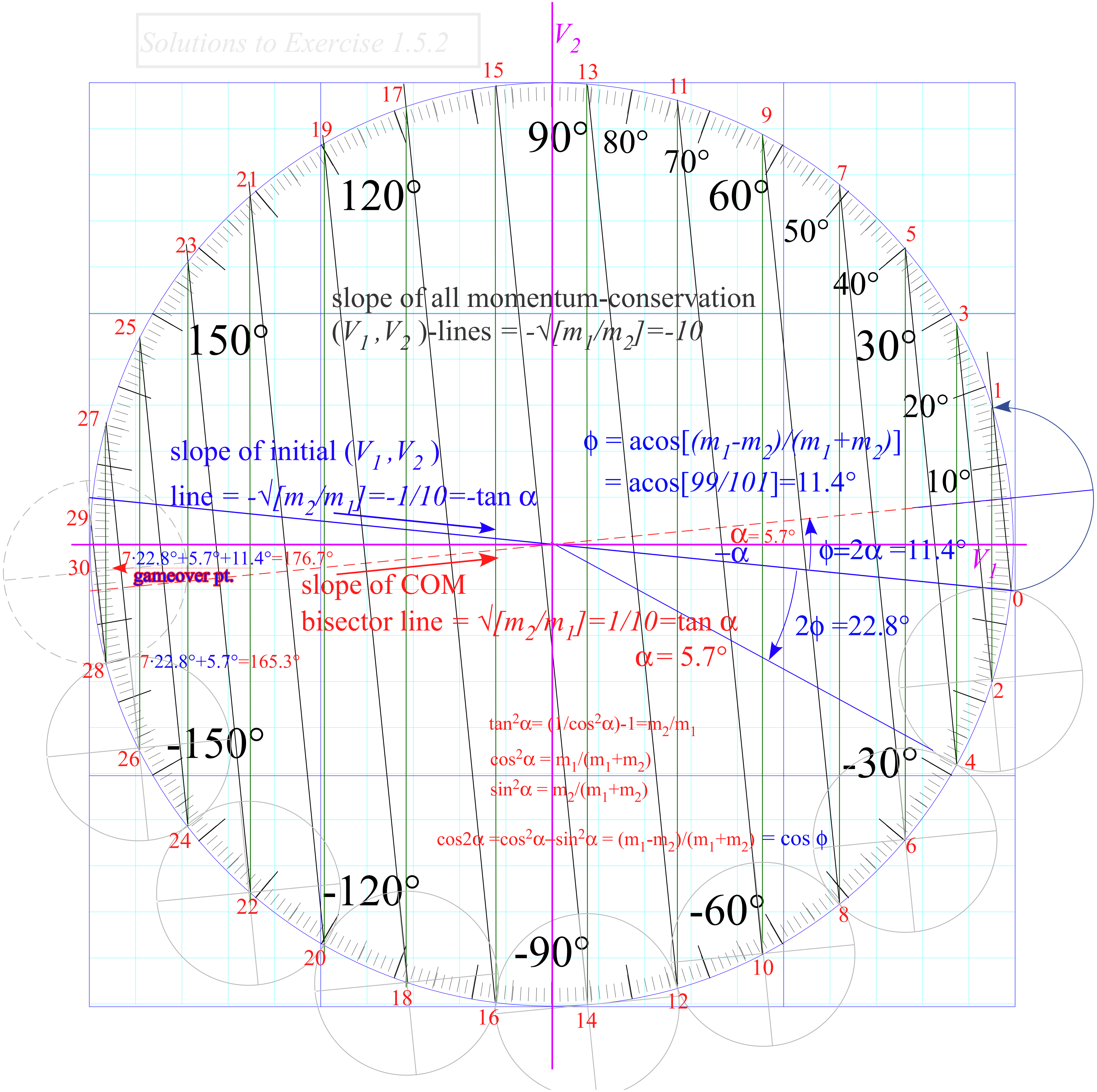
$\phi = \text{acos}[(m_1\text{-}m_2)/(m_1+m_2)] = \text{acos}[99/101]=11.4°$



Collisions for mass ratio m_1:m_2= 100:1

Solutions to Exercise 1.5.2

Collisions for mass ratio $m_1:m_2= 100:1$



slope of all momentum-conservation (V_1, V_2)-lines $= -\sqrt{[m_1/m_2]} = -10$

slope of initial (V_1, V_2) line $= -\sqrt{[m_2/m_1]} = -1/10 = -\tan\alpha$

$\phi = \text{acos}[(m_1-m_2)/(m_1+m_2)]$

$= \text{acos}[99/101] = 11.4°$

slope of COM bisector line $= \sqrt{[m_2/m_1]} = 1/10 = \tan\alpha$

$\alpha = 5.7°$

$\tan^2\alpha = (1/\cos^2\alpha)-1 = m_2/m_1$

$\cos^2\alpha = m_1/(m_1+m_2)$

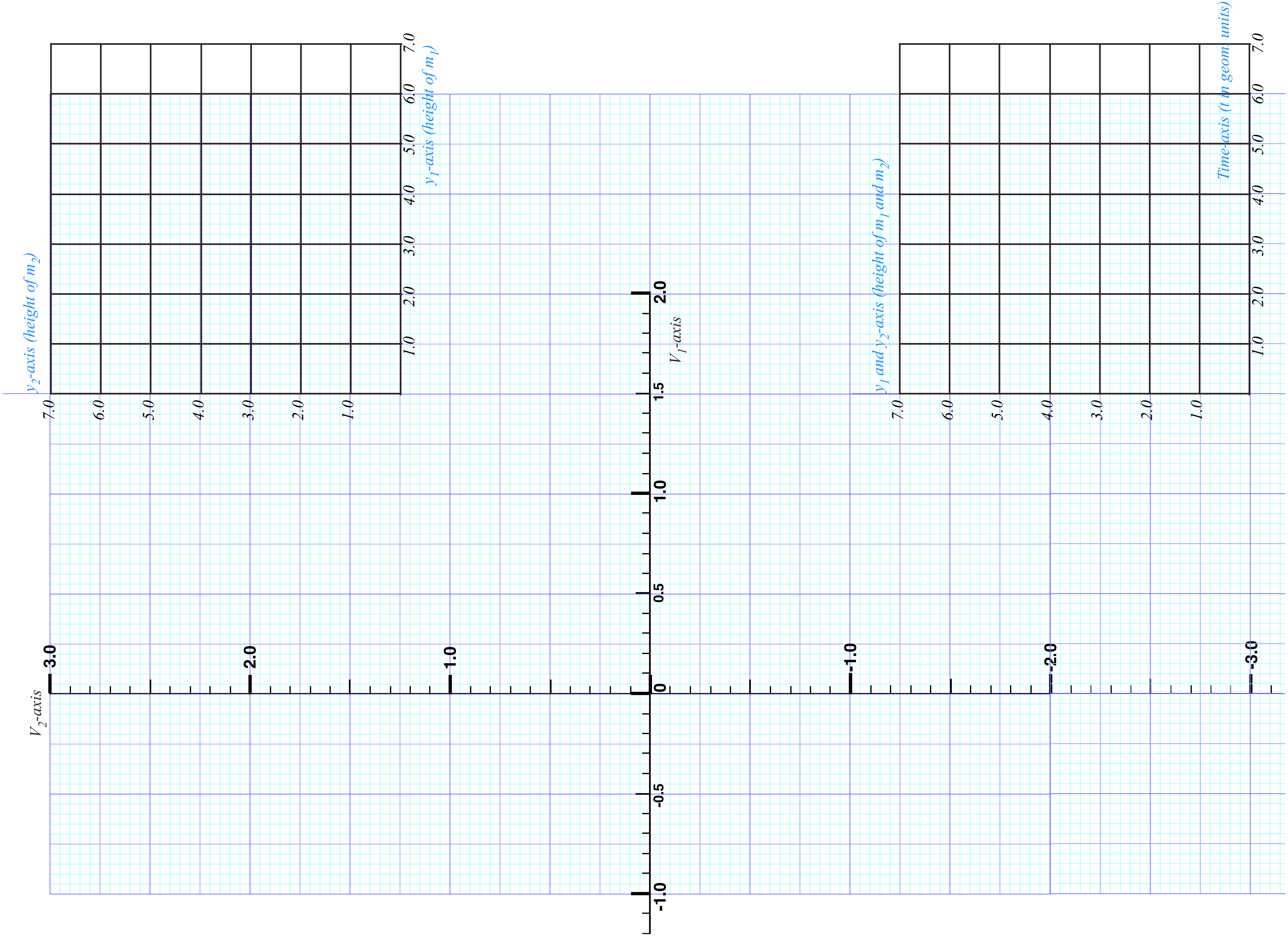
$\sin^2\alpha = m_2/(m_1+m_2)$

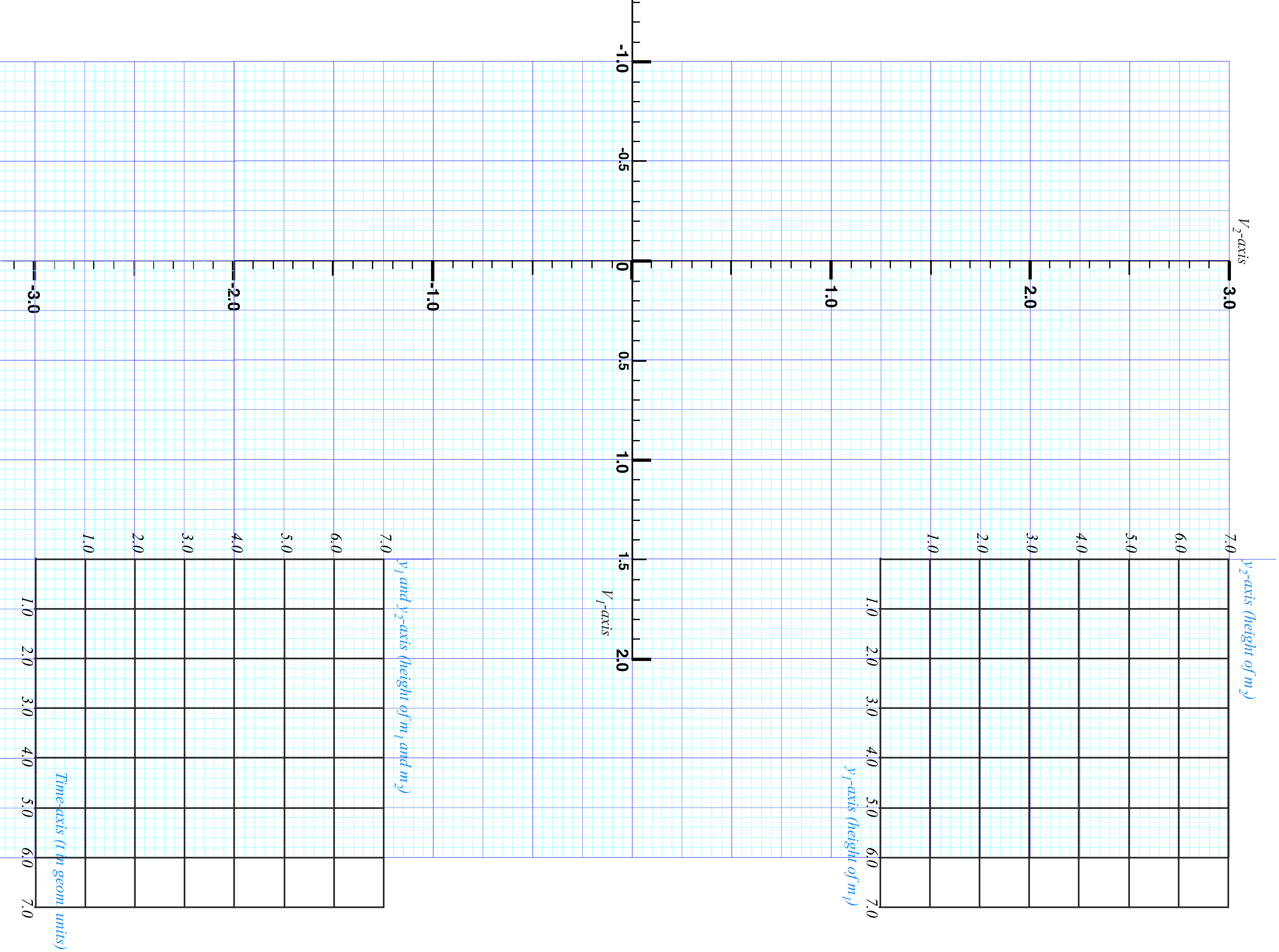
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (m_1-m_2)/(m_1+m_2) = \cos\phi$

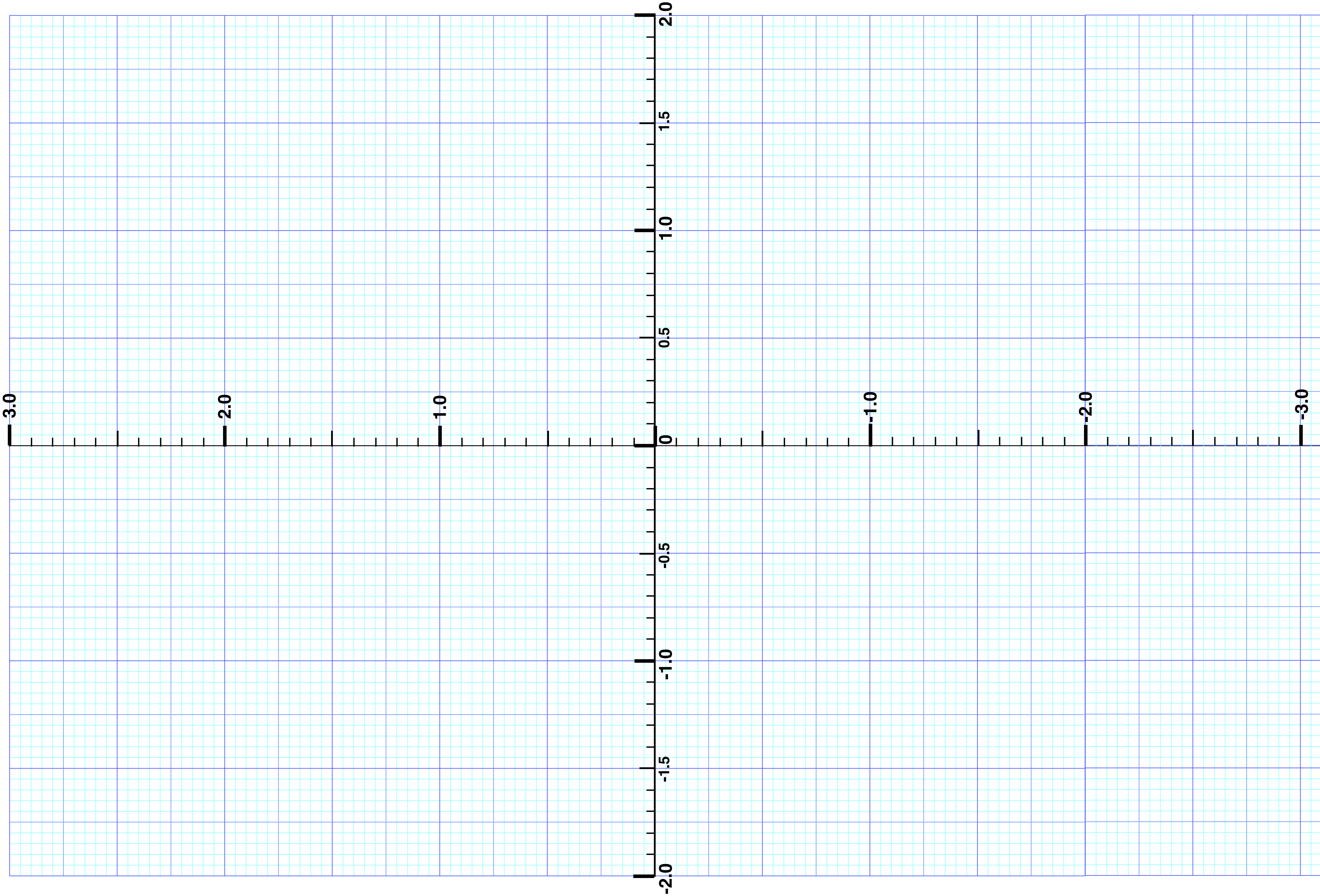
$7\cdot 22.8° + 5.7° + 11.4° = 176.7°$

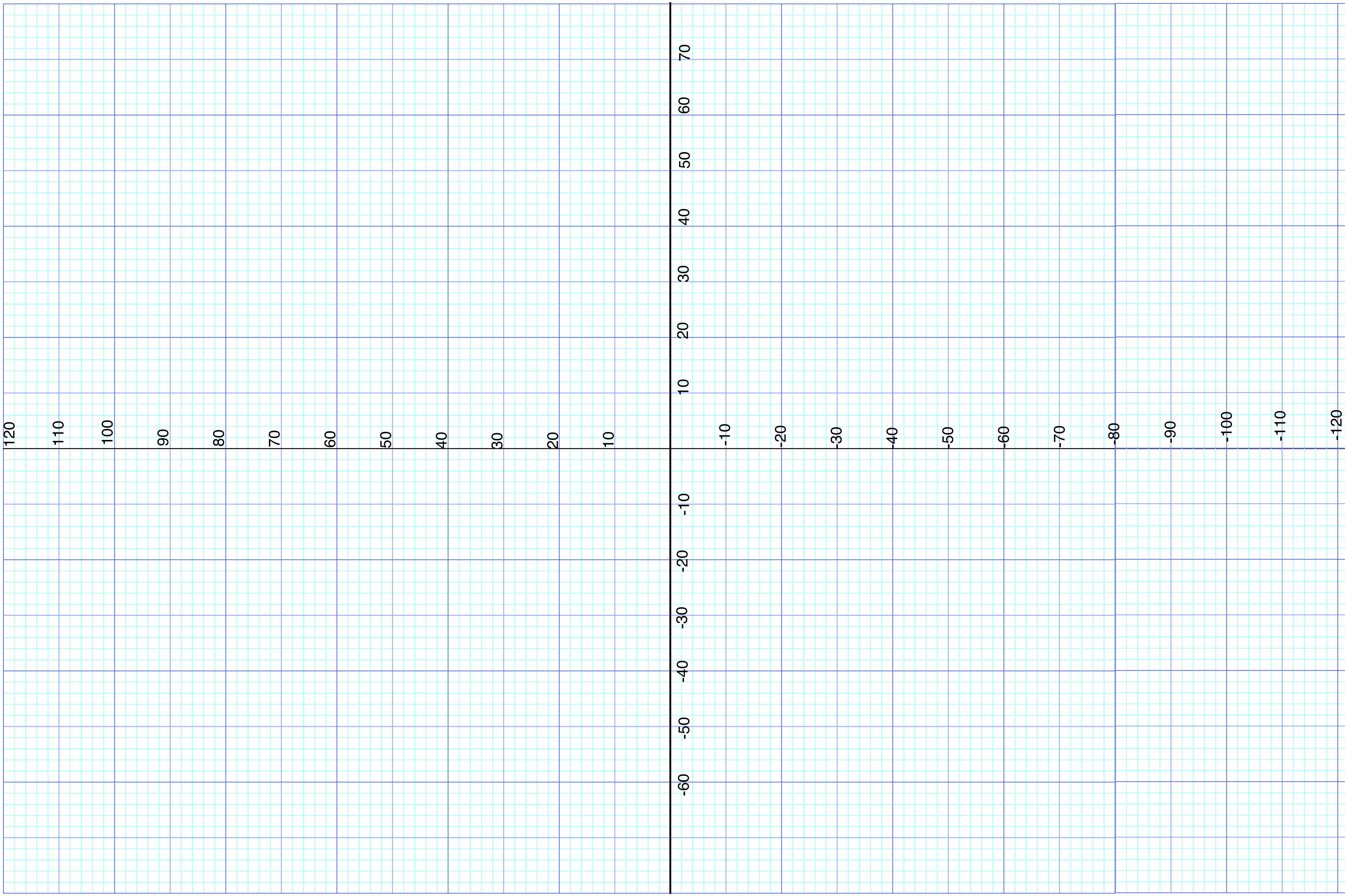
gameover pt.

$7\cdot 22.8° + 5.7° = 165.3°$


y_2-axis (height of m_2)
7.0
6.0
5.0
4.0
3.0
2.0
1.0
y_1-axis (height of m_1)
7.0
6.0
5.0
4.0
3.0
2.0
1.0
V_1-axis
2.0
1.5
1.0
0.5
0
-0.5
-1.0
V_2-axis
3.0
2.0
1.0
-1.0
-2.0
-3.0
y_1 and y_2-axis (height of m_1 and m_2)
7.0
6.0
5.0
4.0
3.0
2.0
1.0
Time-axis (t in geom. units)
7.0
6.0
5.0
4.0
3.0
2.0
1.0


-1.0
-0.5
0
0.5
1.0
1.5
2.0
v_1-axis
v_2-axis
-3.0
-2.0
-1.0
1.0
2.0
3.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
y_1 and y_2-axis (height of m_1 and m_2)
Time-axis (t in geom. units)
y_2-axis (height of m_2)
y_1-axis (height of m_1)


2.0
1.5
1.0
0.5
3.0
2.0
1.0
0
-1.0
-2.0
-3.0
-0.5
-1.0
-1.5
-2.0


120
110
100
90
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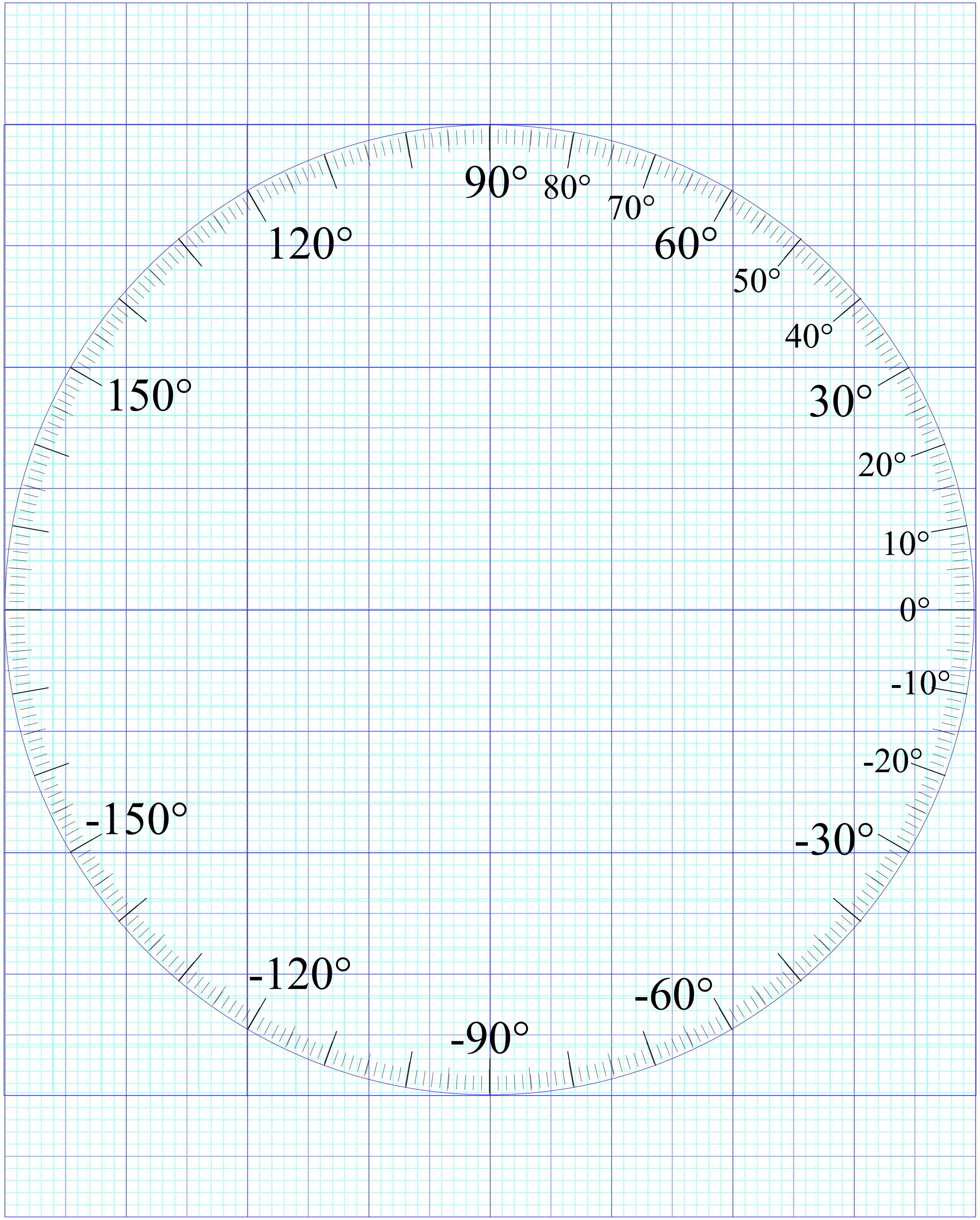
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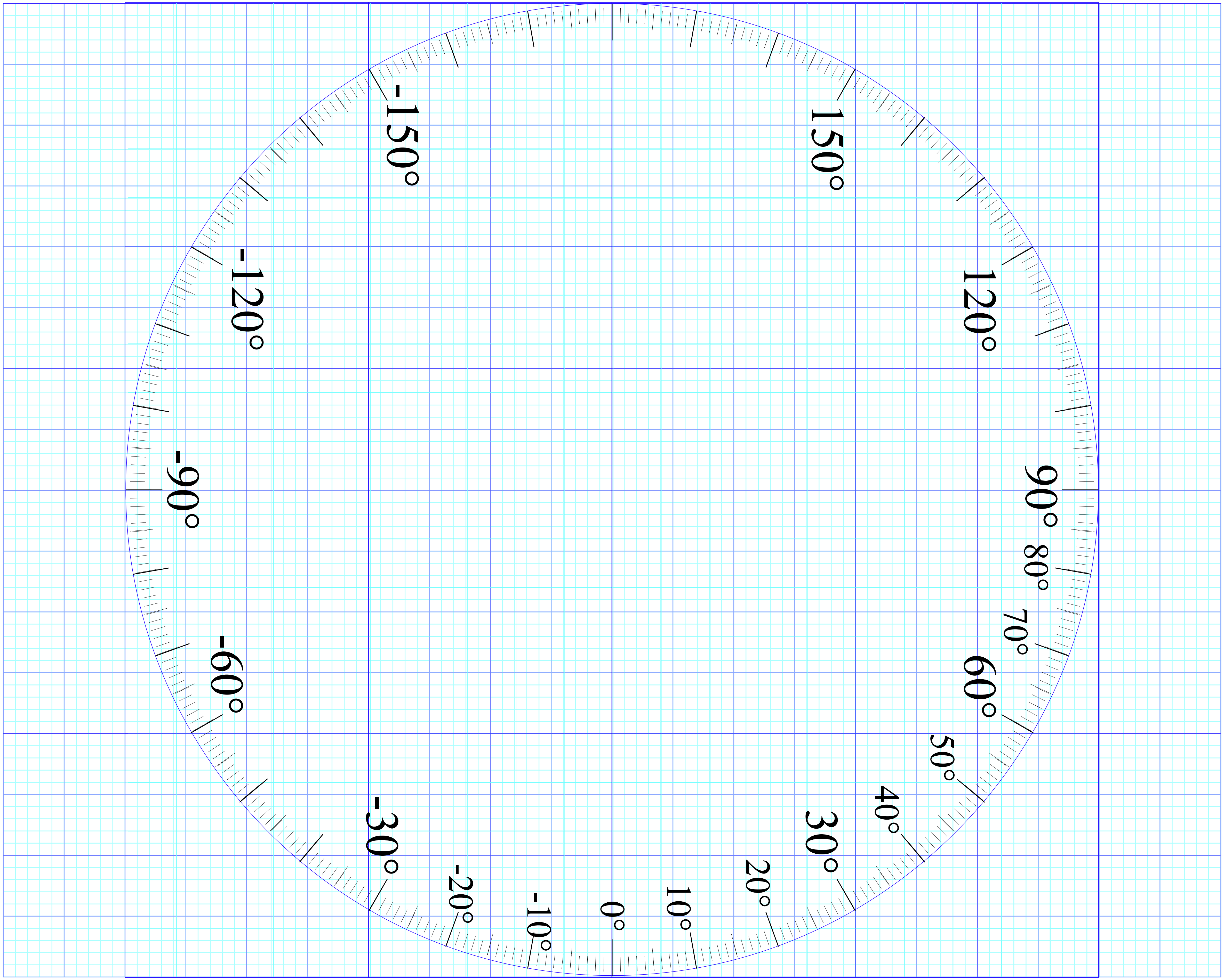
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