Lecture 8
Thur. 9.15.2016

Quadratic form geometry and development of mechanics of Lagrange and Hamilton

(Ch. 12 of Unit 1 and Ch. 4-5 of Unit 7)

Review of partial differential calculus
Chain rule and order $\partial^2\Psi/\partial x\partial y = \partial^2\Psi/\partial y\partial x$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE
Introducing 0^{th} Lagrange and 0^{th} Hamilton differential equations of mechanics
Introducing 1^{st} Lagrange and 1^{st} Hamilton differential equations of mechanics

Introducing the Poincare´ and Legendre contact transformations
Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)
Example from thermodynamics
Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

An elementary contact transformation from sophomore physics
Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"
Intuitive-geometric development of " " " and " " "

Link ⇒ CoulIt - Simulation of the *Volcanoes of Io*

Link ⇒ RelaWavity - Physical Terms *H(p) & L(u)*

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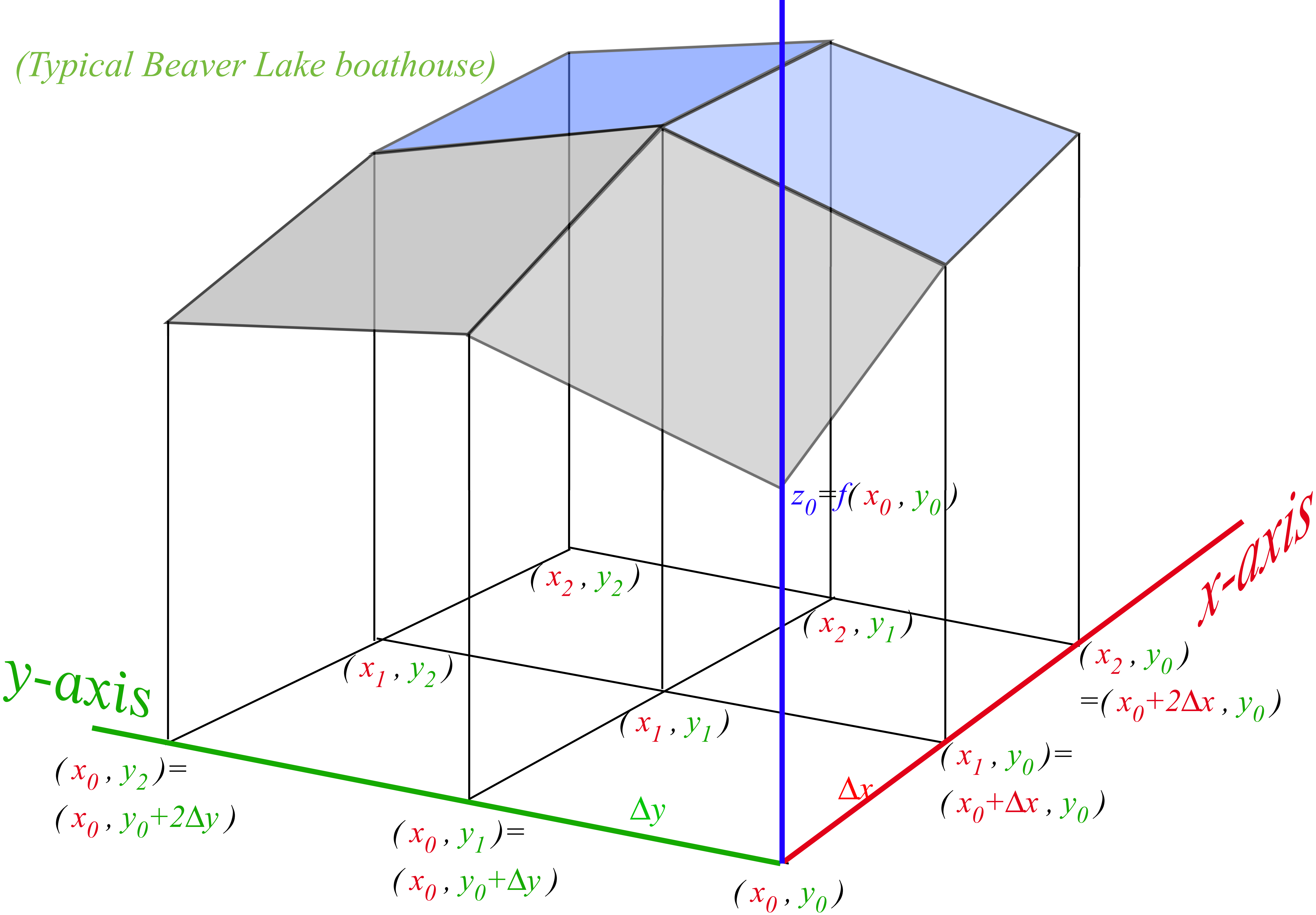
Intuitive-geometric development of " " " and " " "

Begin with a function z=f(z)
of 2-dimensions (x,y) and plotted
in 3-D (Then approximate by cells and tiles.)



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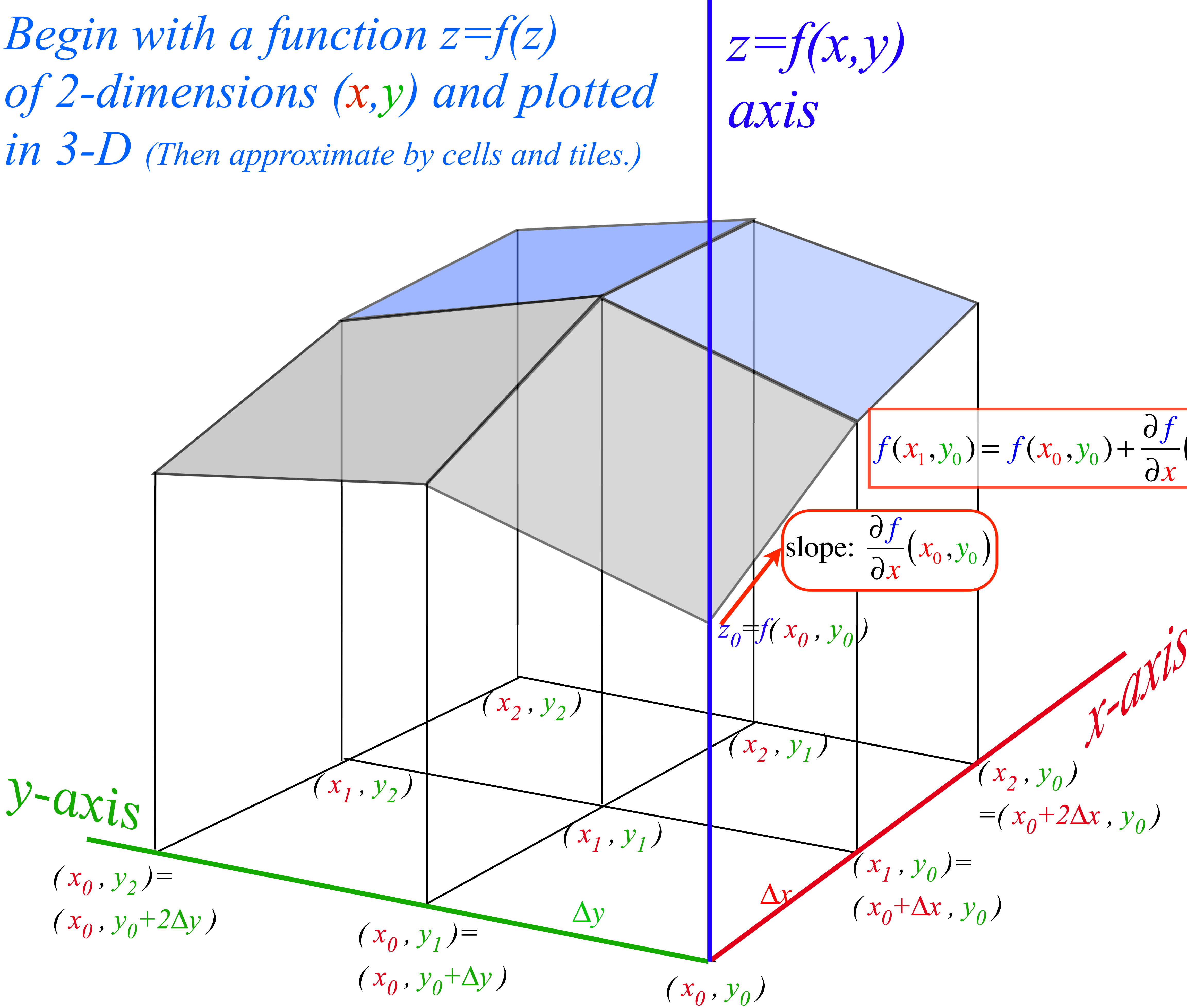

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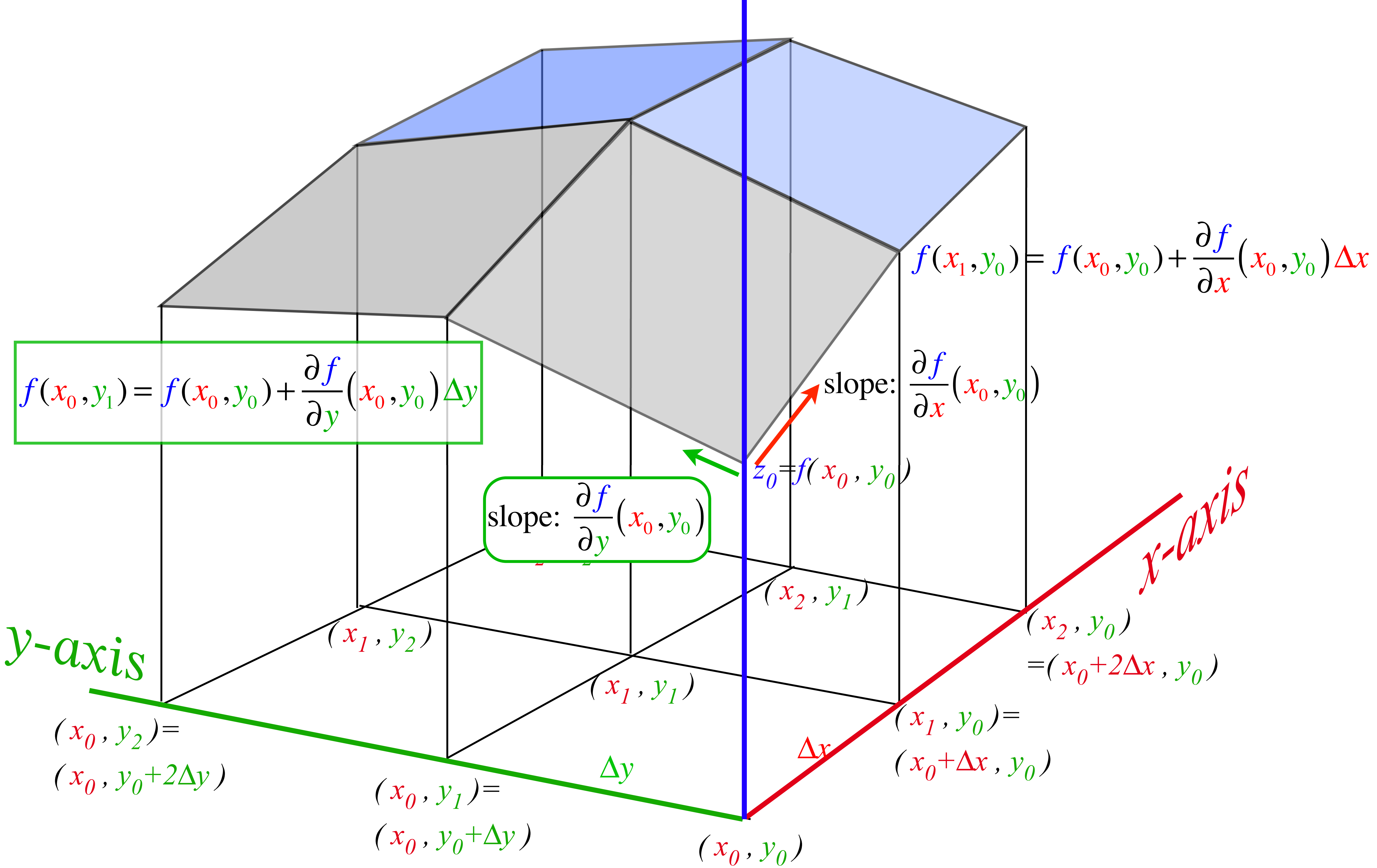

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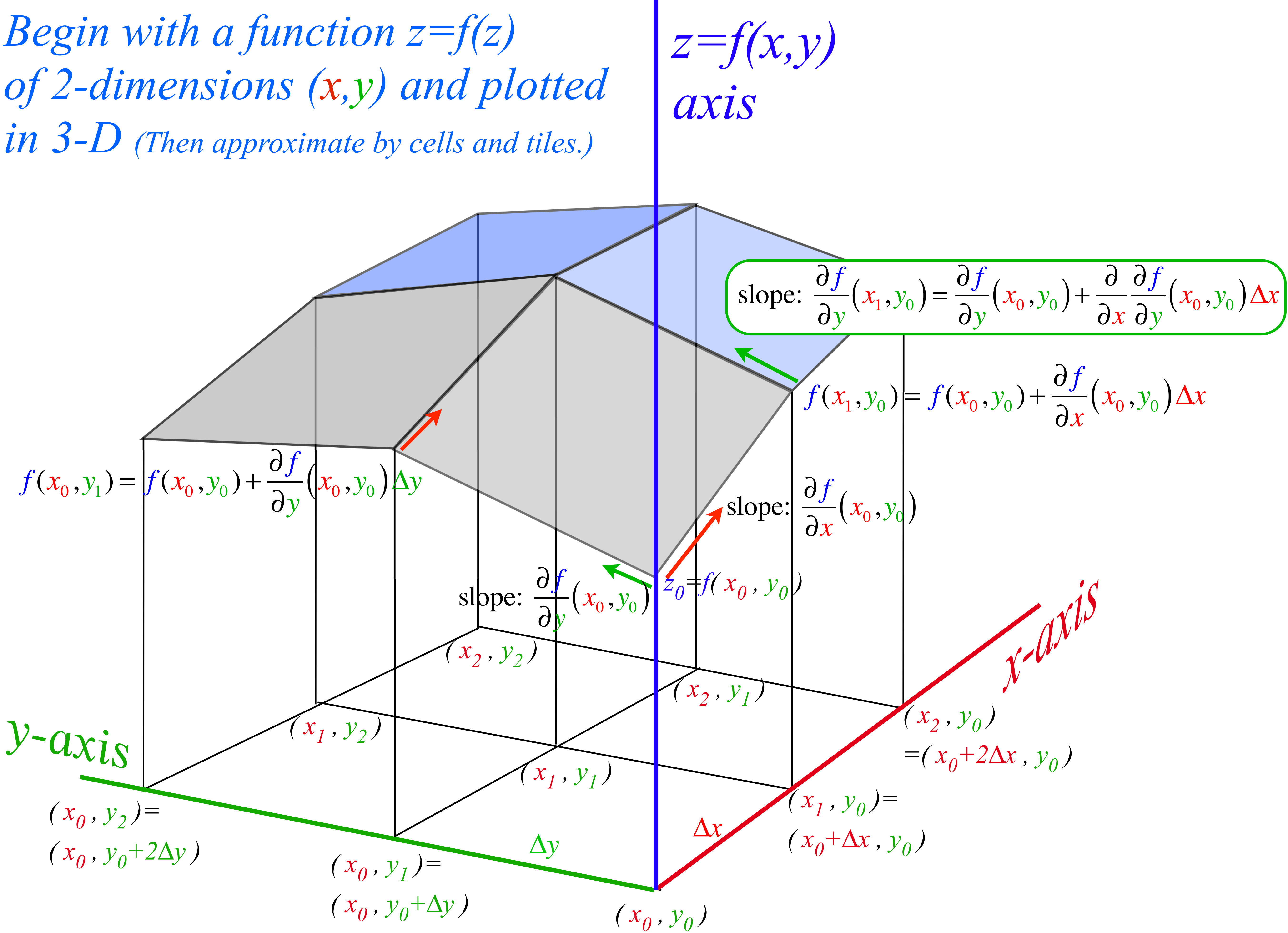

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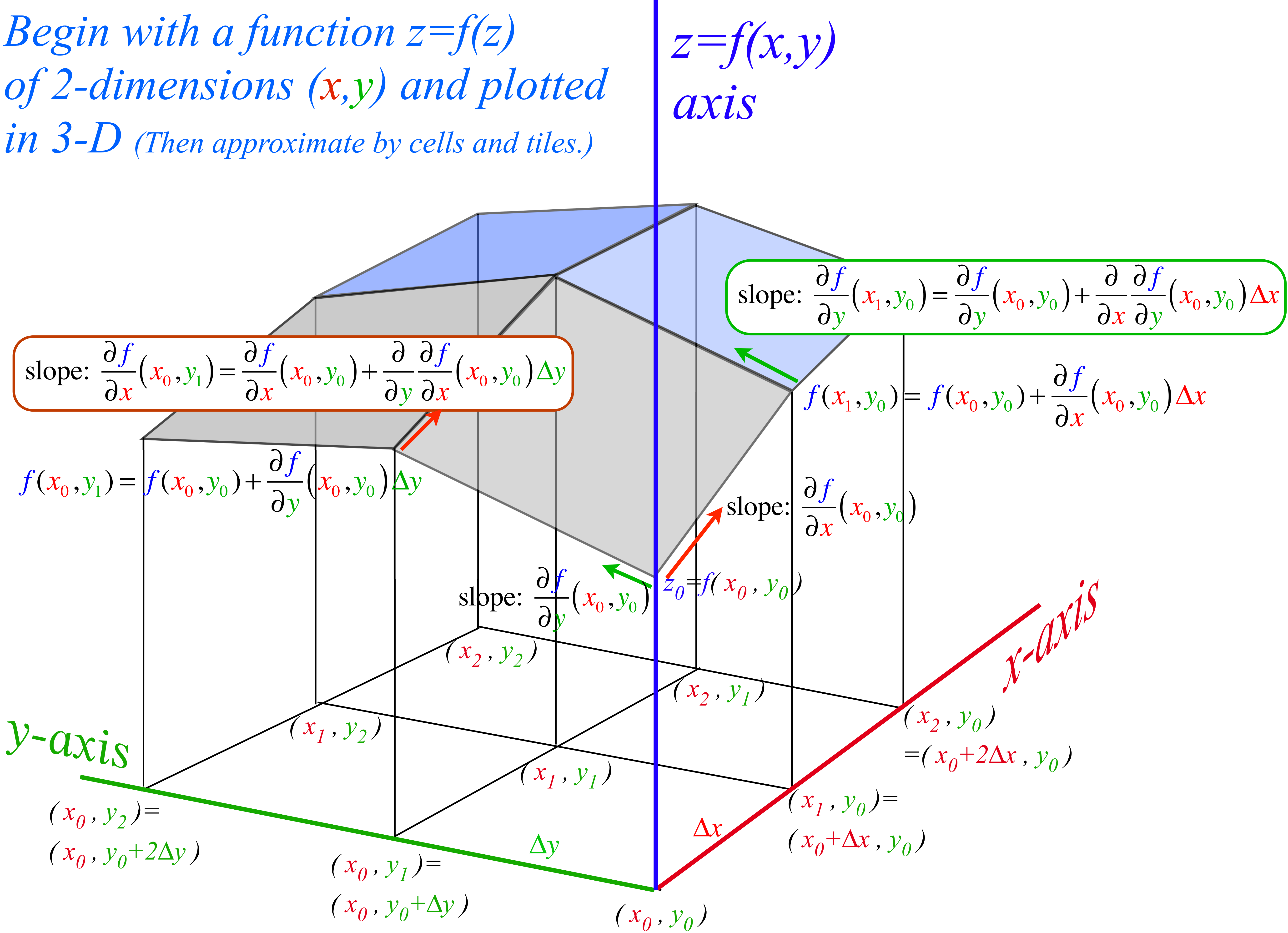

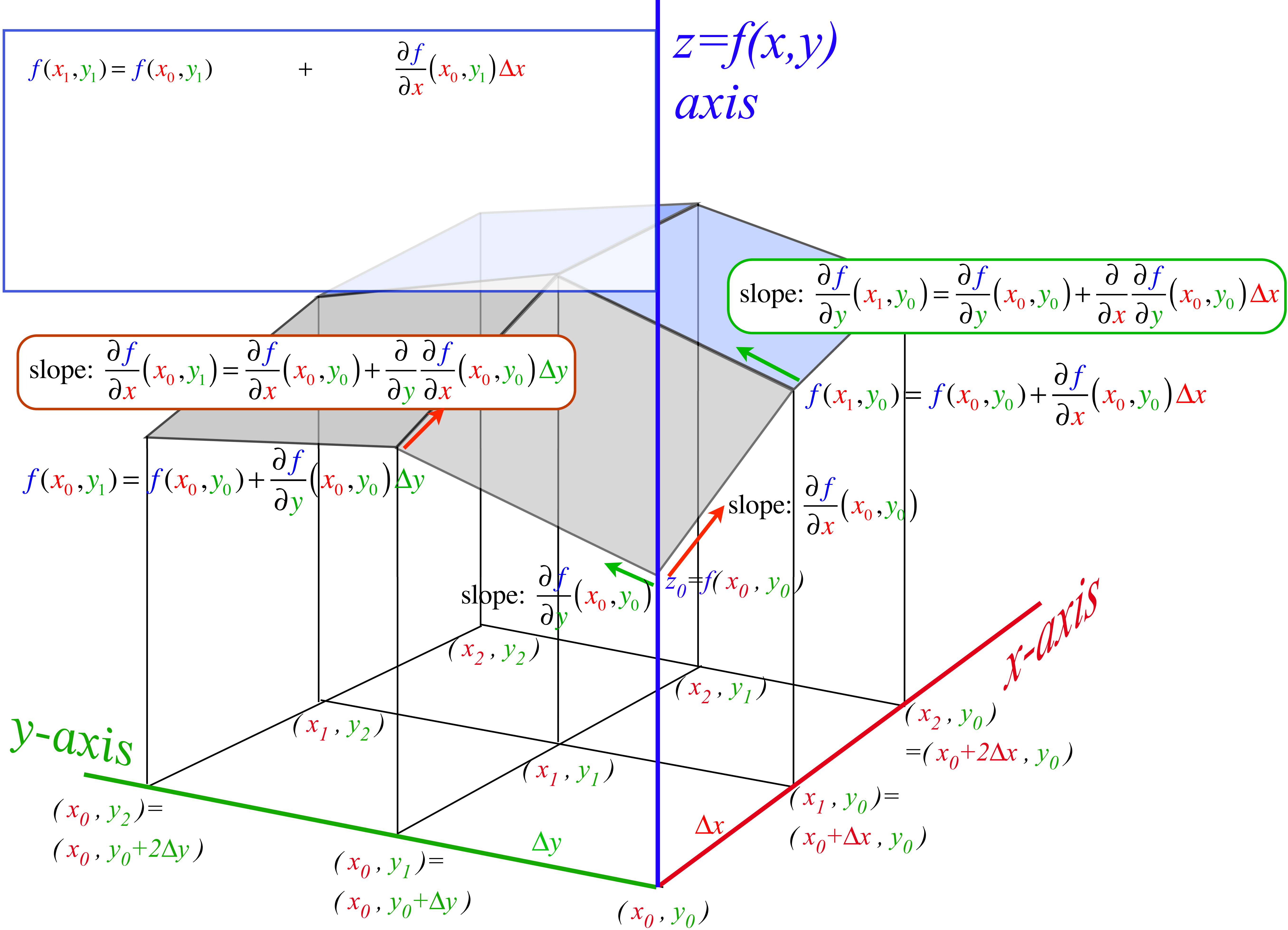

$f(x_1,y_1) = f(x_0,y_1) \quad + \quad \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$
z=f(x,y) axis
slope: $\frac{\partial f}{\partial y}(x_1,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$
slope: $\frac{\partial f}{\partial x}(x_0,y_1) = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$
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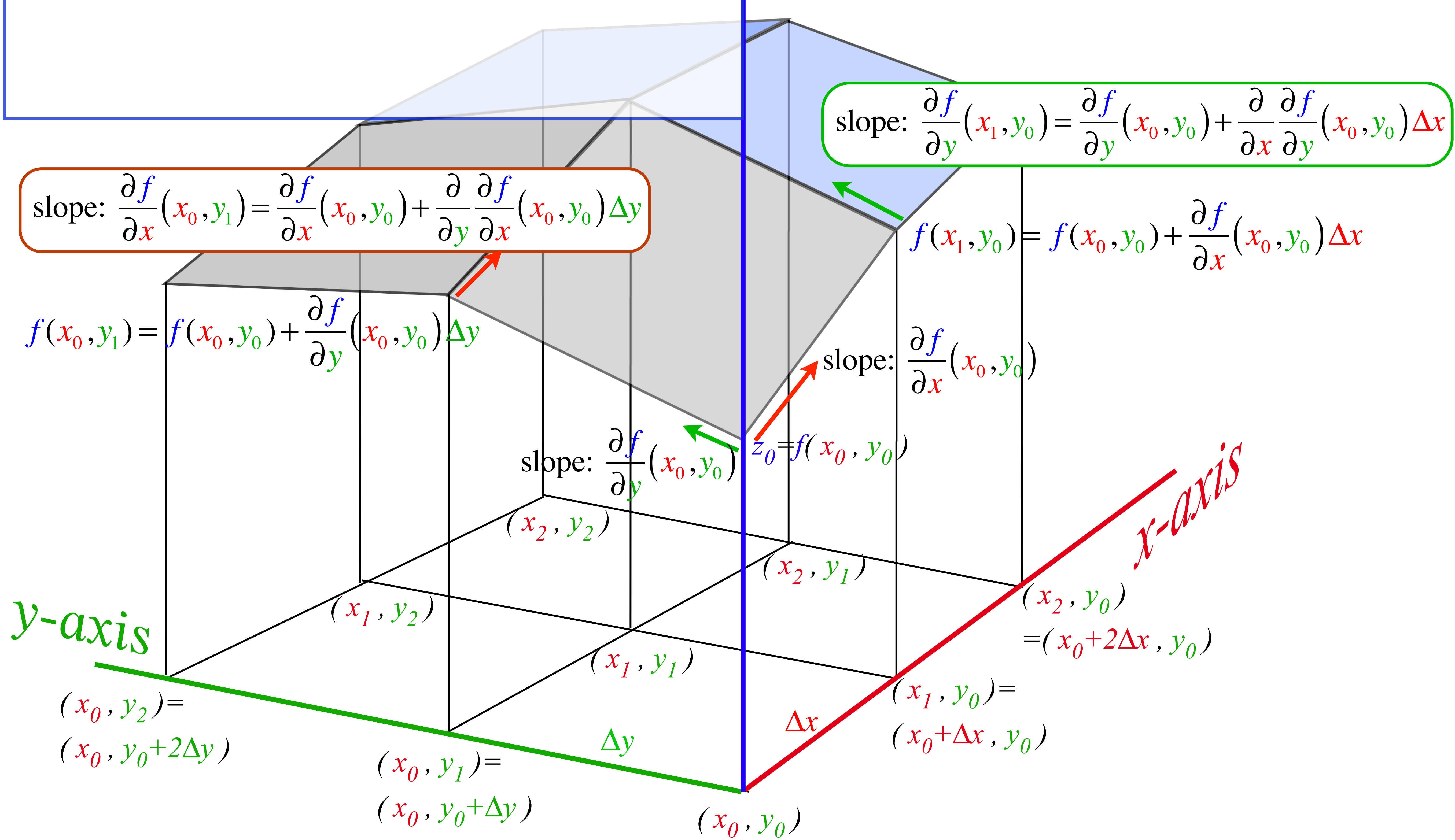
slope: $\frac{\partial f}{\partial x}(x_0,y_0)$
slope: $\frac{\partial f}{\partial y}(x_0,y_0)$
$z_0=f(x_0,y_0)$
x-axis
y-axis
(x_2,y_2)
(x_2,y_1)
(x_1,y_2)
$(x_2,y_0) =(x_0+2\Delta x,y_0)$
(x_1,y_1)
$(x_0,y_2)= (x_0,y_0+2\Delta y)$
$(x_1,y_0)= (x_0+\Delta x,y_0)$
Δx
Δy
$(x_0,y_1)= (x_0,y_0+\Delta y)$
(x_0,y_0)

$$f(x_1,y_1) = f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

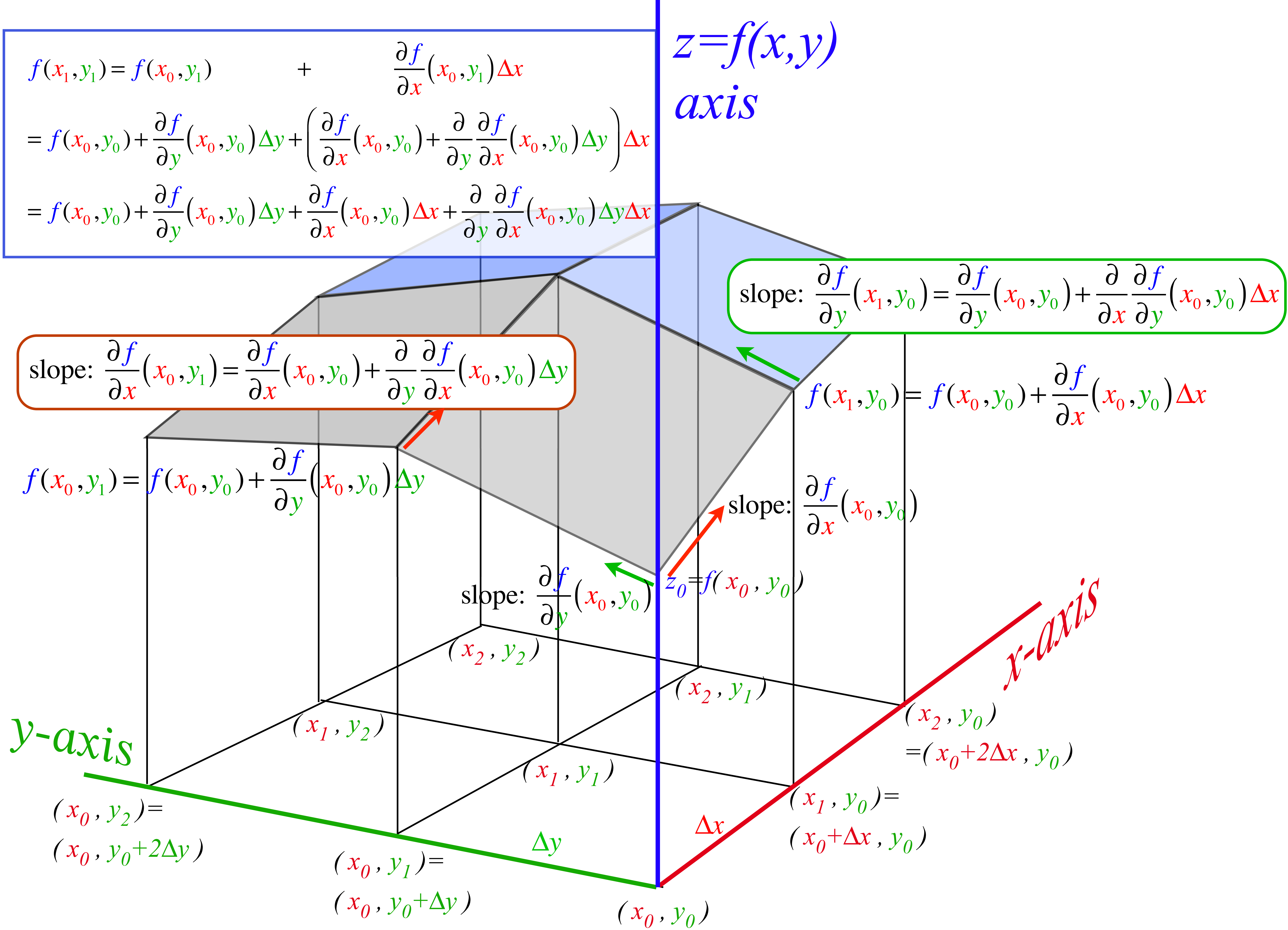
$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \left(\frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$



$$f(x_1,y_1)=f(x_0,y_1)\qquad+\qquad\frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$



$$f(x_1,y_1)=f(x_0,y_1)+\frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

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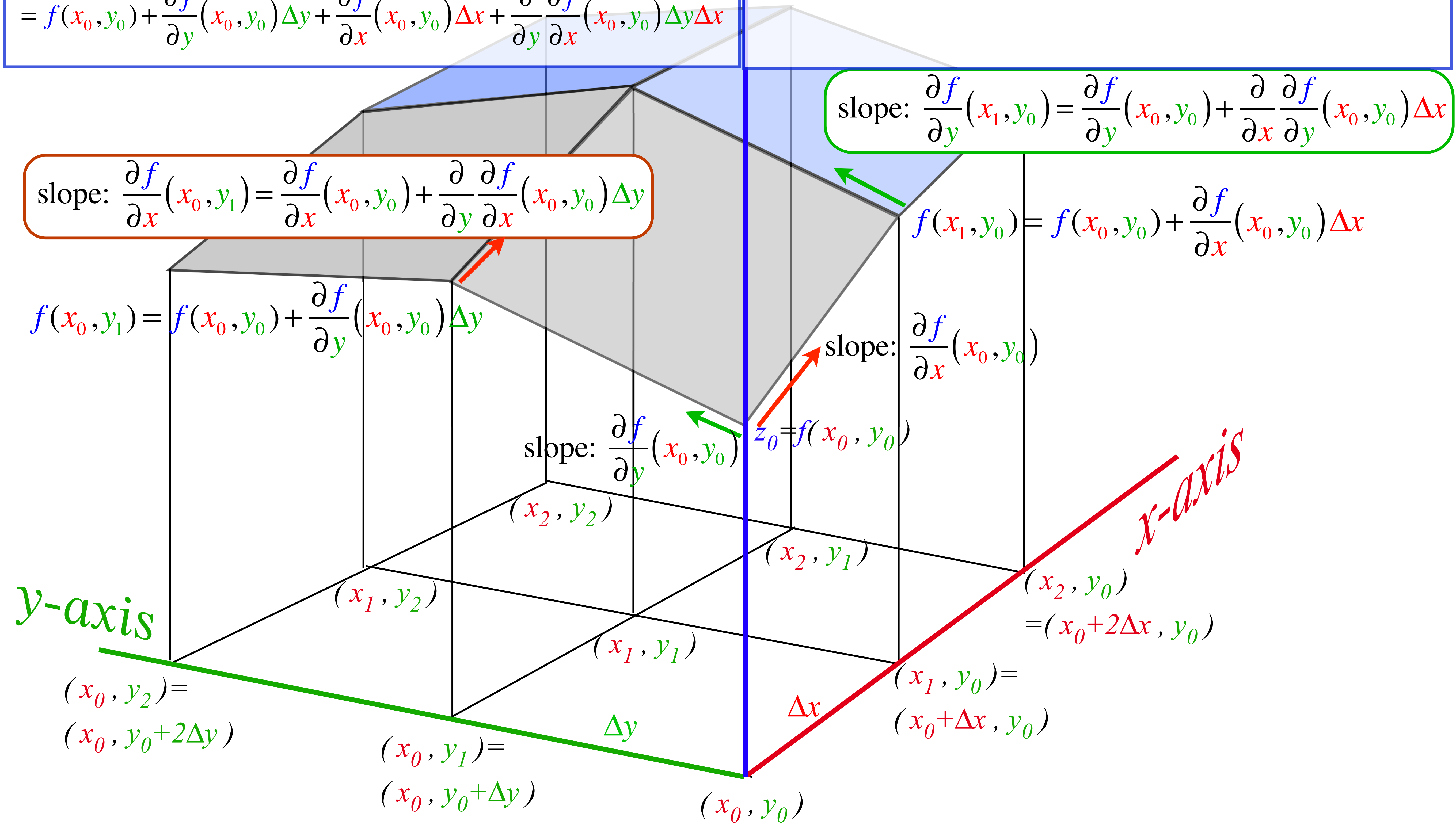

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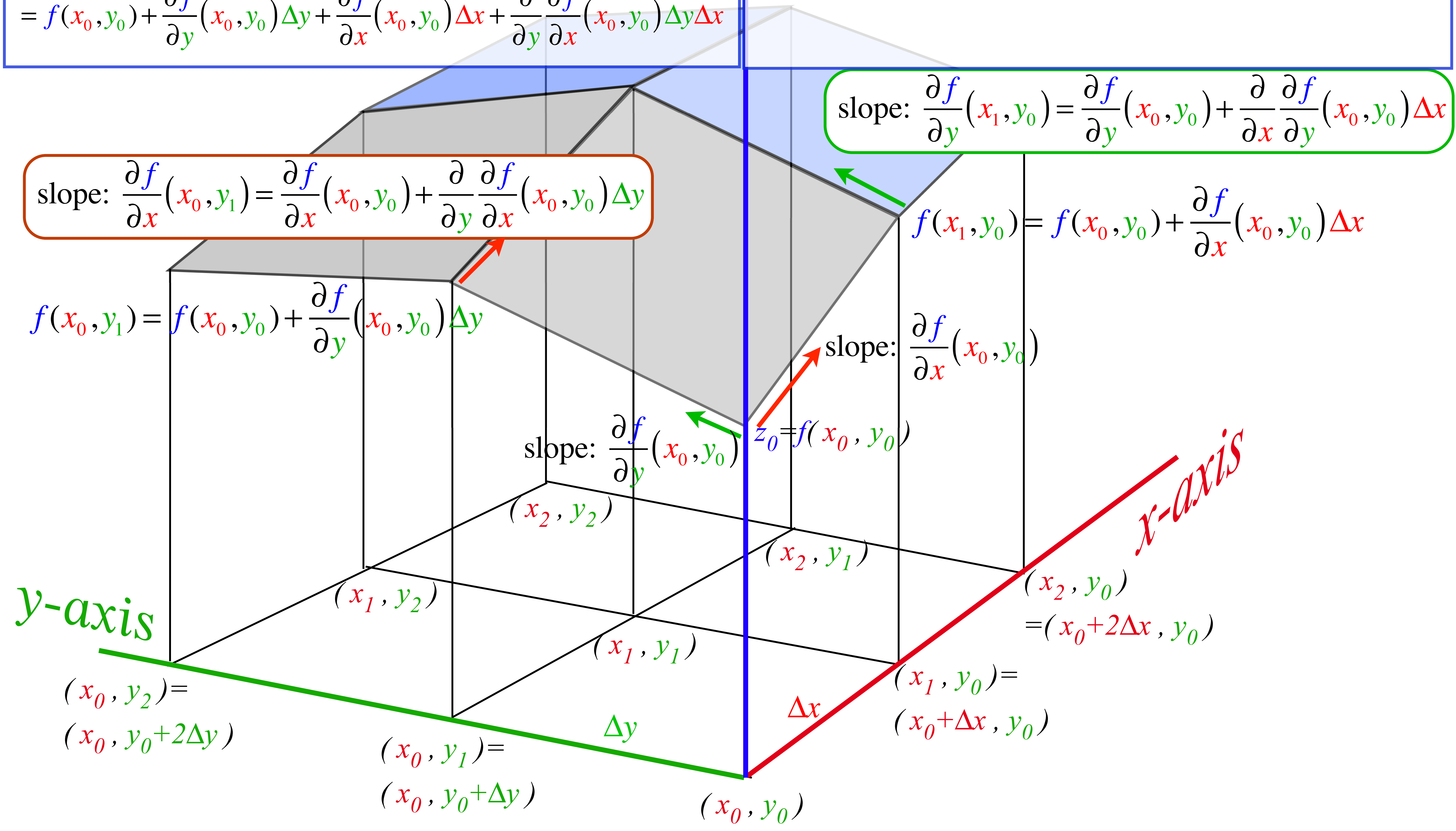

$$
\begin{aligned}
f(x_1,y_1) &= f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x \\
&= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x \\
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\end{aligned}
$$

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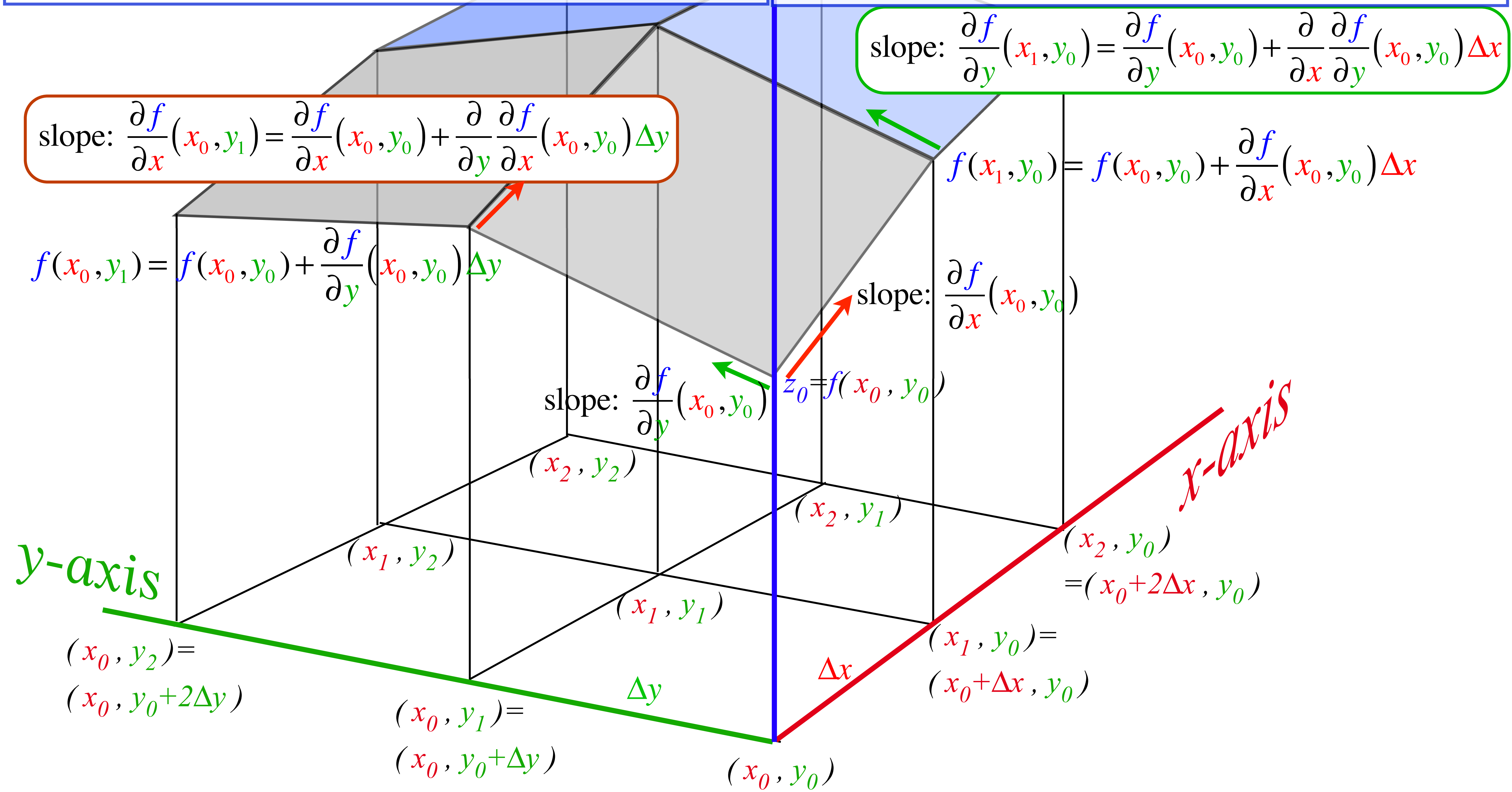
\end{aligned}
$$



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What the geometry indicates....(Two important results)

$$f(x_1,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

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If $f(x,y)$ is contiuous around (x_0,y_0) and (x_1,y_1) then $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ equals $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$

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1. Chain rules

$$\left[f(x_1,y_1)-f(x_0,y_0)\right]=df=\frac{\partial f}{\partial x}(x_0,y_0)dx+\frac{\partial f}{\partial y}(x_0,y_0)dy\ldots_{(\textit{keep } 1^{st}-\textit{order terms only!})}$$

$$\frac{df}{dt}=\frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt}+\frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f}=\frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial y}\dot{y}\qquad(\textit{shorthand notation})$$

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1. Chain rules

$$\left[f(x_1,y_1) - f(x_0,y_0)\right] = df = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy \ldots_{(keep\ 1^{st}-order\ terms\ only!)}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} \quad (shorthand\ notation) \quad = \partial_x f\,\dot{x} + \partial_y f\,\dot{y}$$

2. Symmetry of partial deriv. ordering

(pay attention to the 2^{nd}−order terms, too!)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or:} \quad \partial_y \partial_x f = \partial_x \partial_y f$$

(shorthand notation)

What the geometry indicates....(Two important results)

$$f(x_1,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

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$(\textit{shorthand notation})$

$$Let:\vec{\nabla}=\begin{pmatrix}\partial_x & \partial_y\end{pmatrix}\quad so:\vec{\nabla}f\bullet\mathbf{dr}=\begin{pmatrix}\partial_x f & \partial_y f\end{pmatrix}\bullet\begin{pmatrix}dx\\dy\end{pmatrix}=\partial_x f\,dx+\partial_y f\,dy=df$$

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Three ways to express energy: *Consider kinetic energy (KE) first*

*1. **Lagrangian** is explicit function of **velocity**:* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$L(v_k \ldots) = \tfrac{1}{2}(m_1 v_1^2 + m_2 v_2^2 + \ldots) = L(\mathbf{v}\ldots) = \tfrac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \ldots = \tfrac{1}{2}\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ldots$$

*2. **"Estrangian"** is explicit function of **R**-rescaled velocity:*

(or l'Estrangian

*or: **"speedinum"** V* $\mathbf{V} = \mathbf{R}\bullet\mathbf{v}$ or: $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$E(V_k \ldots) = \tfrac{1}{2}(V_1^2 + V_2^2 + \ldots) = E(\mathbf{V}\ldots) = \tfrac{1}{2}\mathbf{V}\bullet\mathbf{1}\bullet\mathbf{V} + \ldots = \tfrac{1}{2}\begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \ldots$$

*3. **Hamiltonian** is explicit function of **M=R**^{2}-rescaled velocity:*

*or: **momentum p*** $\mathbf{p} = \mathbf{M}\bullet\mathbf{v}$ or: $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 v_1 \\ m_2 v_2 \end{pmatrix}$

$$H(p_k \ldots) = \tfrac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \ldots\right) = H(\mathbf{p}\ldots) = \tfrac{1}{2}\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p} + \ldots = \tfrac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \ldots$$

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Introducing the (partial $\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian*** *have* *<u>no</u>* *explicit dependence on* ***momentum*** $\mathbf{p}=\mathbf{M}\bullet\mathbf{v}$

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian*** *have* *<u>no</u>* *explicit dependence on* ***velocity*** $\mathbf{v}=\mathbf{M}^{-1}\bullet\mathbf{p}$

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian*** *have* *<u>no</u>* *explicit dependence on* ***speedinum*** $\mathbf{V}=\mathbf{M}^{1/2}\bullet\mathbf{v}$

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

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$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections†

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2} = \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2} = \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1st equation(s)

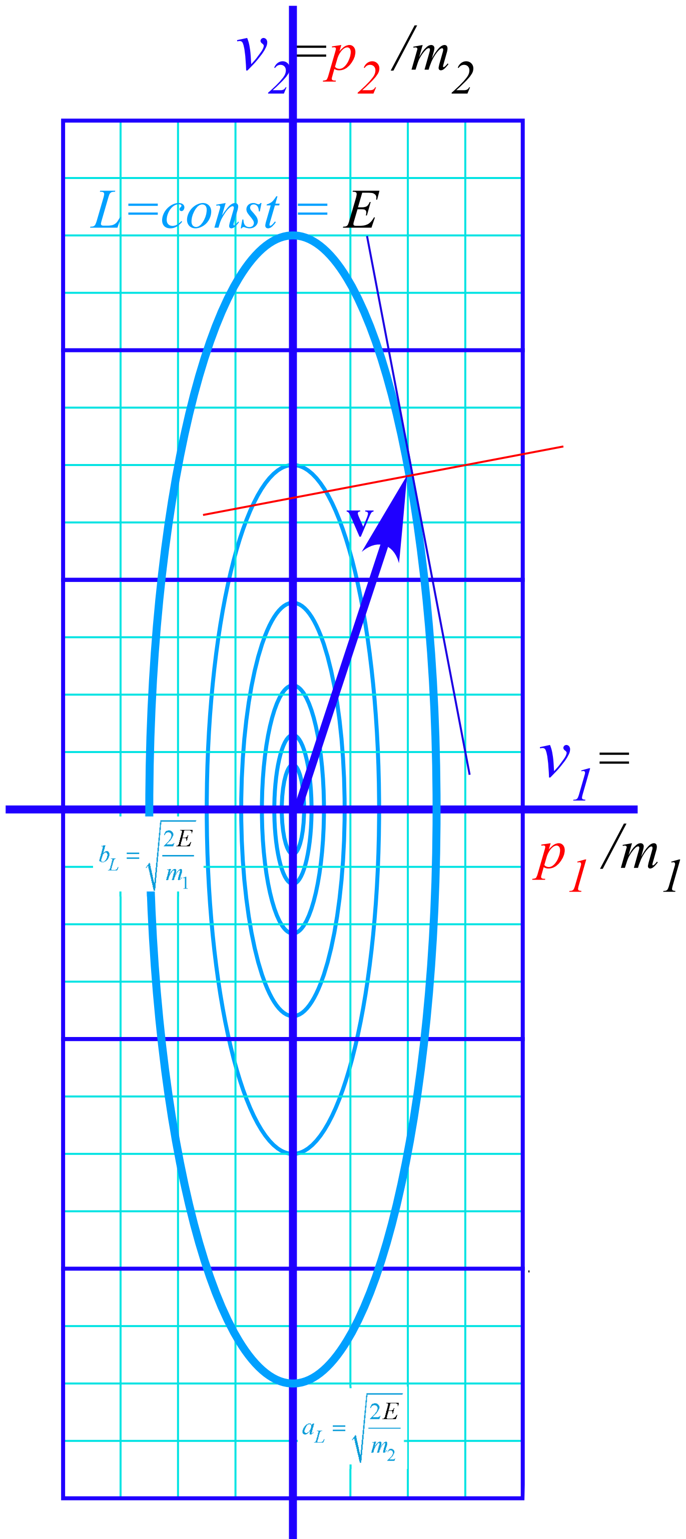
$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

Estrangian is neglected for now.
(It is related to dual ellipse geometry
in Lecture 7 p. 71-79 and 80-85)

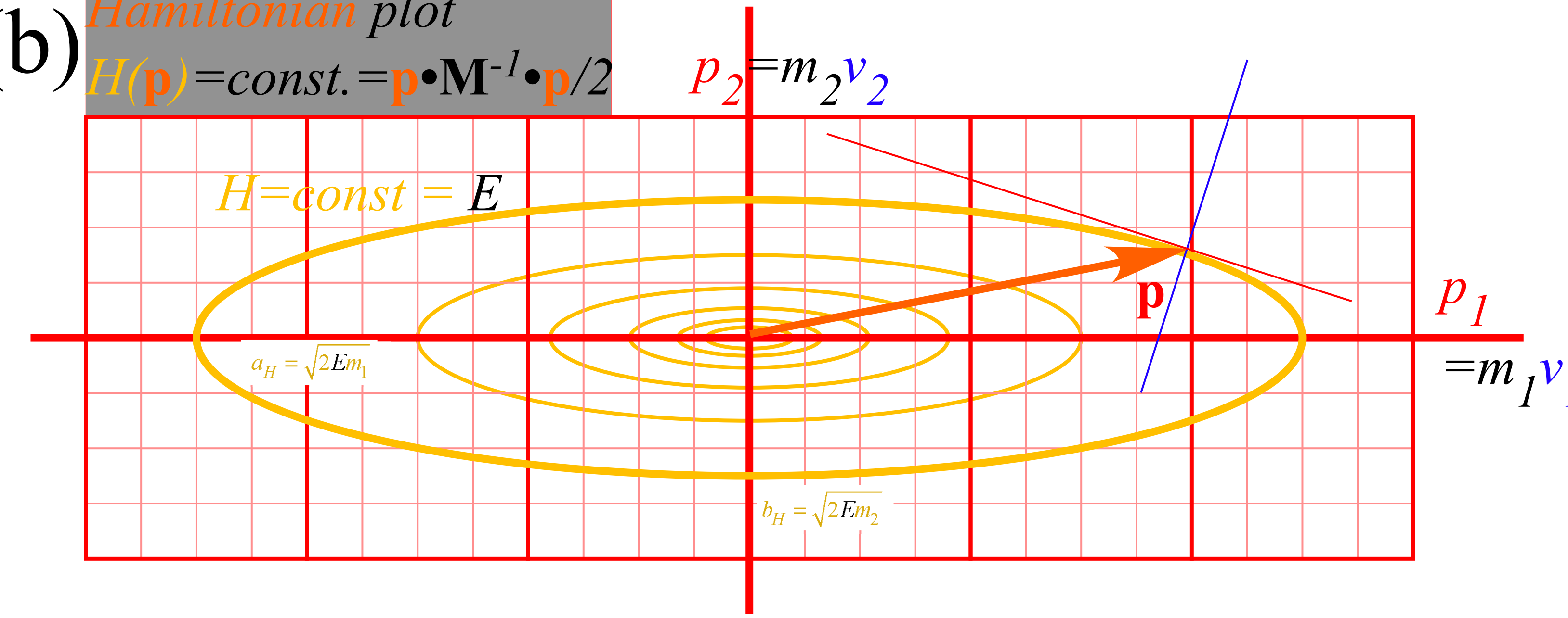
†non-dependency due to
stationary-value effects
as shown on p. 28-31

Unit 1
Fig. 12.2

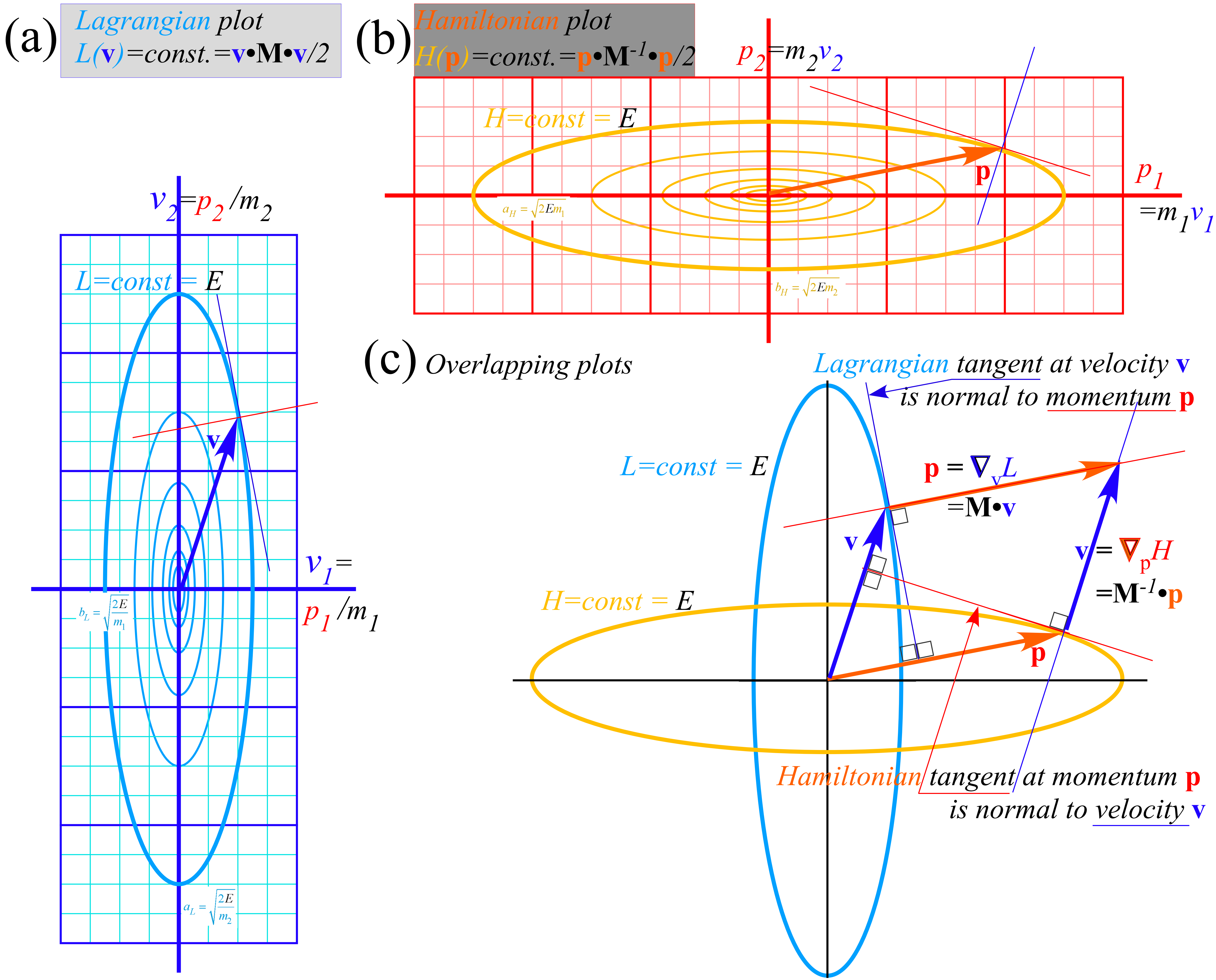
(a) *Lagrangian plot*
$L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$



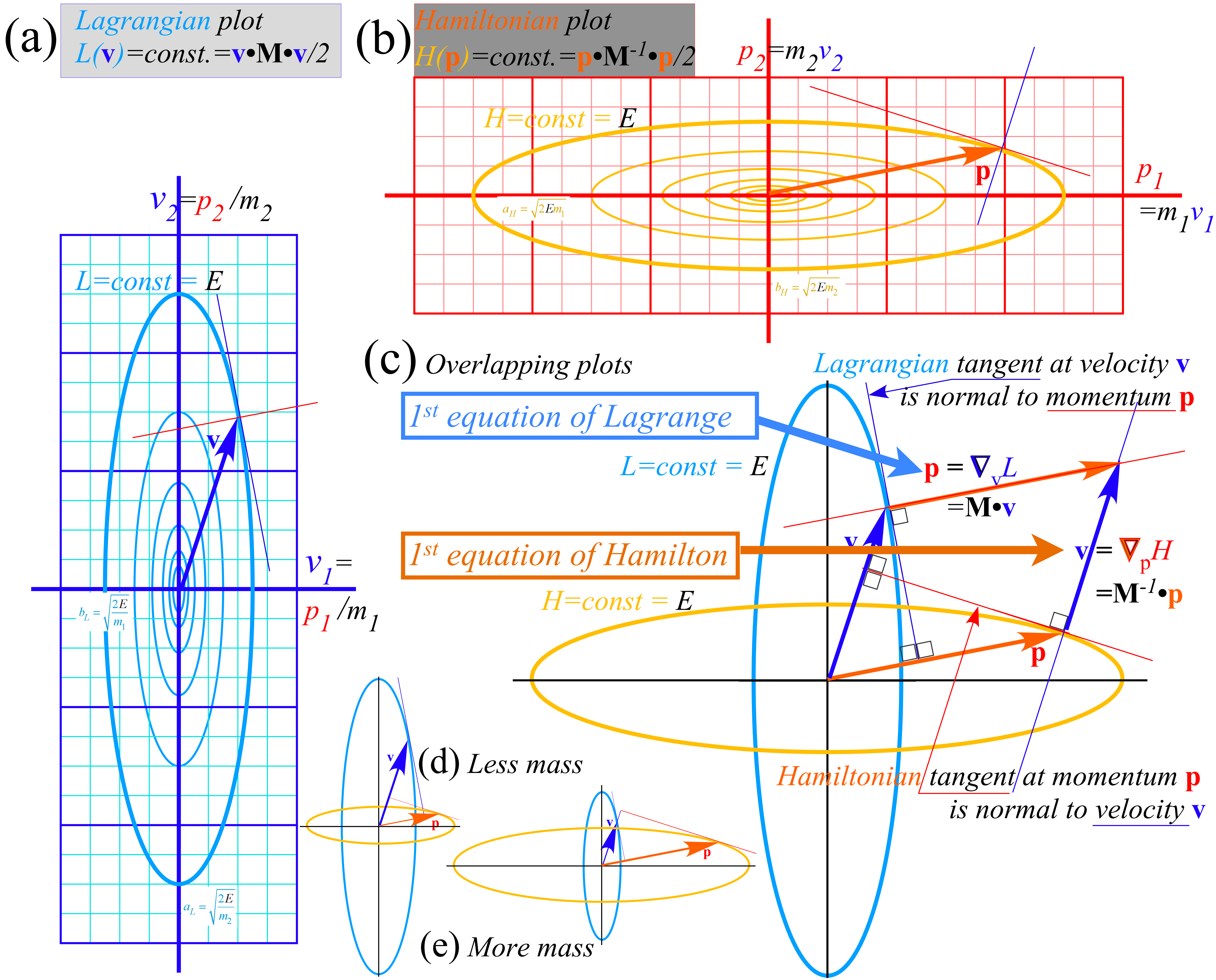
(b) *Hamiltonian plot*
$H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$



Unit 1
Fig. 12.2



Unit 1
Fig. 12.2



Review of partial differential calculus
Chain rule and order $\partial^2\Psi/\partial x\partial y = \partial^2\Psi/\partial y\partial x$ symmetry

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Introducing the Poincare′ and Legendre contact transformations

Given matrix relation: $\mathbf{p}=\mathbf{M}\bullet\mathbf{v}$ *or its inverse:* $\mathbf{v}=\mathbf{M}^{-1}\bullet\mathbf{p}$ *you might be tempted to rewrite*

Q-forms $L(\mathbf{v}..)=(1/2)\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *or* $H(\mathbf{p}..)=(1/2)\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}$ *to be* $H=(1/2)\mathbf{p}\bullet\mathbf{v}$ *or equivalently* $L=(1/2)\mathbf{v}\bullet\mathbf{p}$.

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Numerically-CORRECT, but Differentially-WRONG!

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Numerically-CORRECT, but Differentially-WRONG! *(In classical physics* $\mathbf{p}\cdot\mathbf{v}$ *and* $\mathbf{v}\cdot\mathbf{p}$ *are identical)*

Instead try: $H(\mathbf{p}..)=\mathbf{p}\cdot\mathbf{v}-(1/2)\mathbf{v}\cdot\mathbf{p}=\mathbf{p}\cdot\mathbf{v}-L(\mathbf{v}..)$ *or else:* $L(\mathbf{v}..)=\mathbf{p}\cdot\mathbf{v}-H(\mathbf{p}..)$

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That is ... the ***Legendre contact transformation***

$$L(\mathbf{v})=\mathbf{p}\bullet\mathbf{v}-H(\mathbf{p}) \qquad \text{or:} \qquad H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$$

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Now explicit dependency (non)-relations give the right derivatives

$$\frac{\partial L(\mathbf{v})}{\partial \mathbf{p}}=\frac{\partial}{\partial \mathbf{p}}\mathbf{p}\bullet\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad \frac{\partial H(\mathbf{p})}{\partial \mathbf{v}}=\frac{\partial}{\partial \mathbf{v}}\mathbf{p}\bullet\mathbf{v}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

$$0=\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad 0=\mathbf{p}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

Introducing the Poincare′ and Legendre contact transformations

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$$0=\mathbf{v}-\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad 0=\mathbf{p}-\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

That is *Hamilton's 1st equation(s)* *and* *Lagrange's 1st equation(s)*

$$\mathbf{v}=\frac{\partial H(\mathbf{p})}{\partial \mathbf{p}} \qquad \mathbf{p}=\frac{\partial L(\mathbf{v})}{\partial \mathbf{v}}$$

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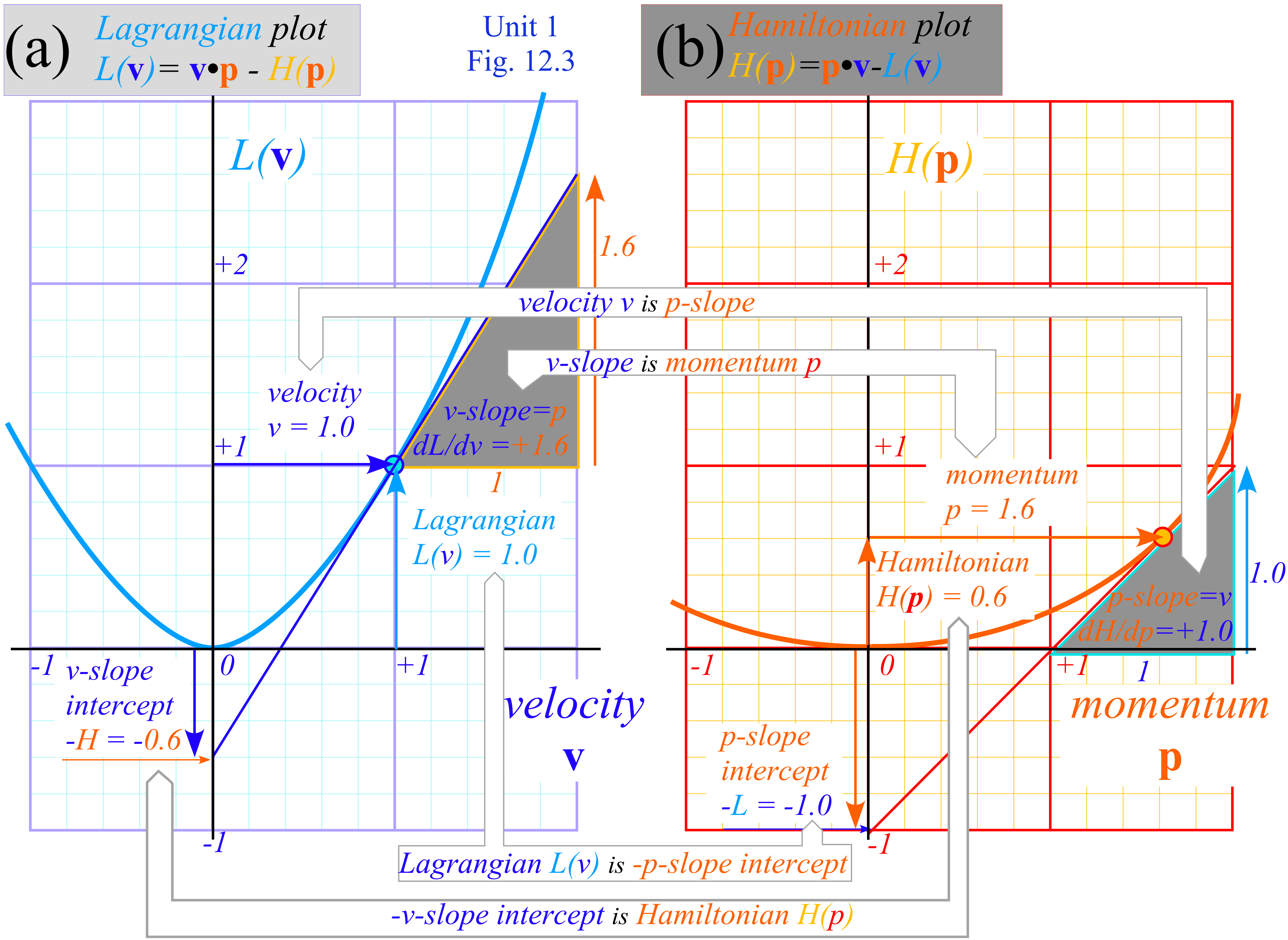
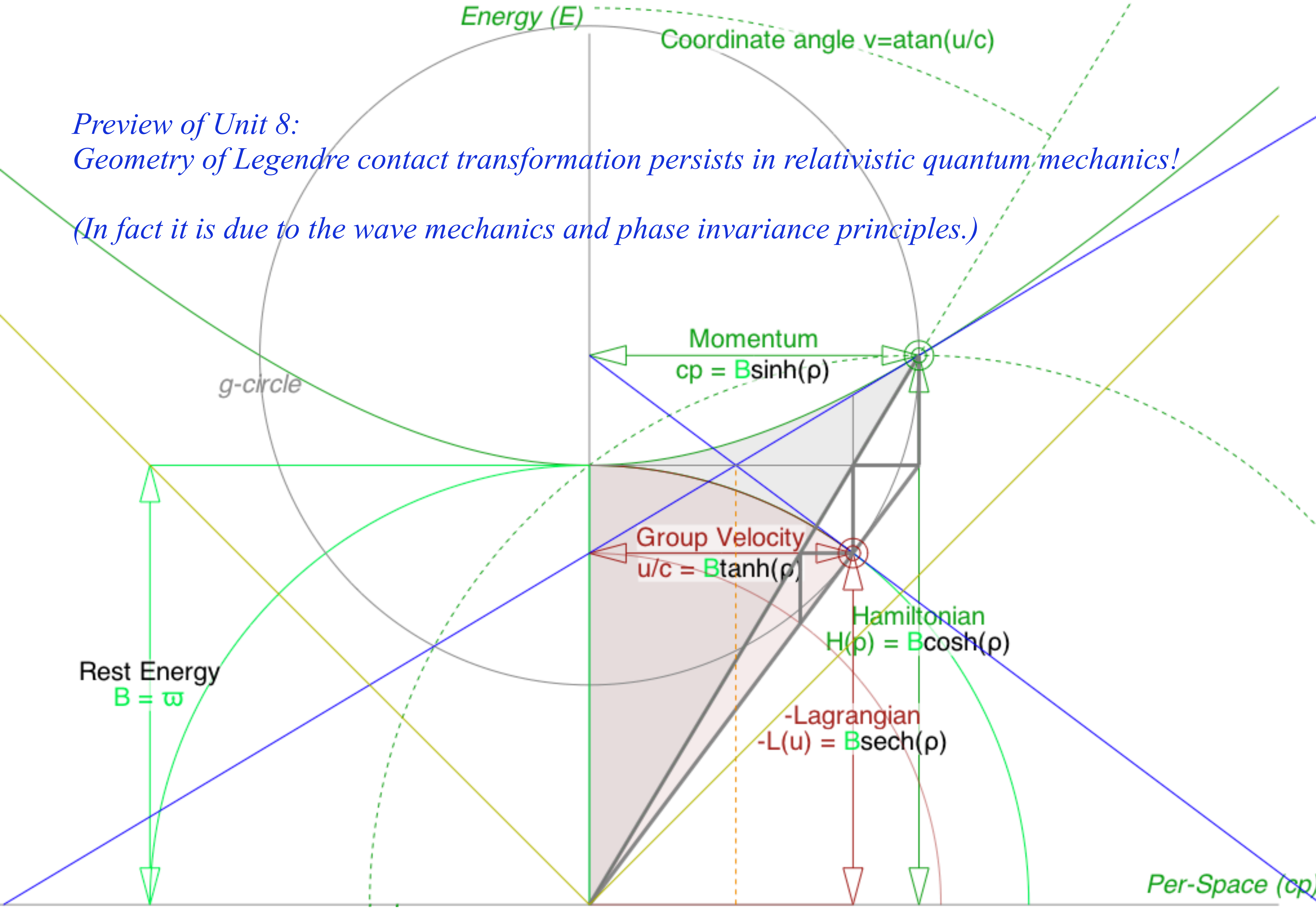


(a) Lagrangian plot
L(v)= v•p - H(p)
Unit 1
Fig. 12.3
(b) Hamiltonian plot
H(p)=p•v-L(v)
L(v)
H(p)
+2
+1
-1
0
+1
1.6
velocity v is p-slope
v-slope is momentum p
velocity
v = 1.0
v-slope=p
dL/dv =+1.6
1
Lagrangian
L(v) = 1.0
v-slope
intercept
-H = -0.6
velocity
v
momentum
p = 1.6
Hamiltonian
H(p) = 0.6
p-slope=v
dH/dp=+1.0
1.0
1
momentum
p
p-slope
intercept
-L = -1.0
Lagrangian L(v) is -p-slope intercept
-v-slope intercept is Hamiltonian H(p)

Preview of Unit 8:
Geometry of Legendre contact transformation persists in relativistic quantum mechanics!

(In fact it is due to the wave mechanics and phase invariance principles.)



[Link ⇒ RelaWavity - Physical Terms H(p) & L(u)]

Preview of Unit 8:
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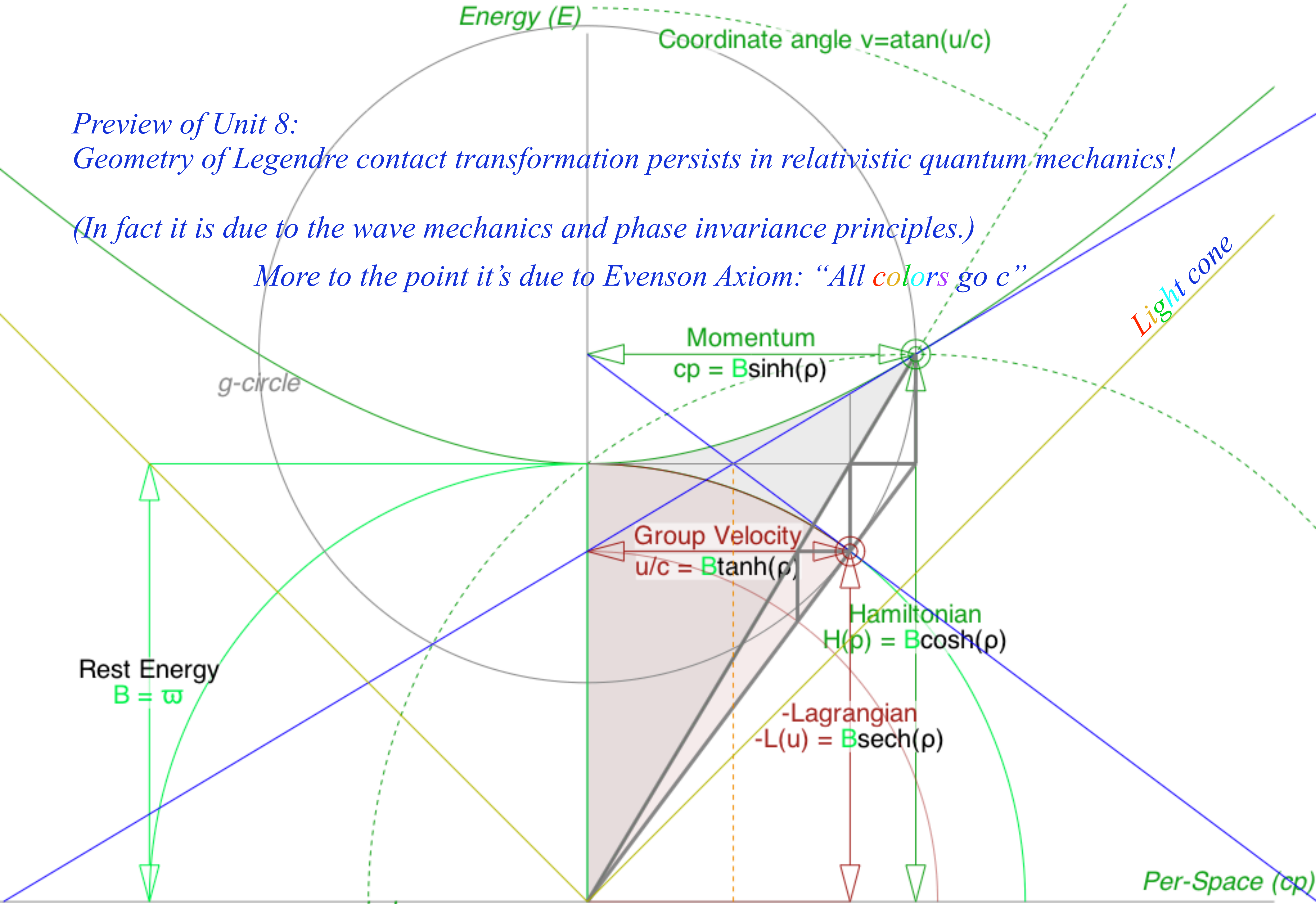
(In fact it is due to the wave mechanics and phase invariance principles.)

More to the point it's due to Evenson Axiom: "All colors go c"



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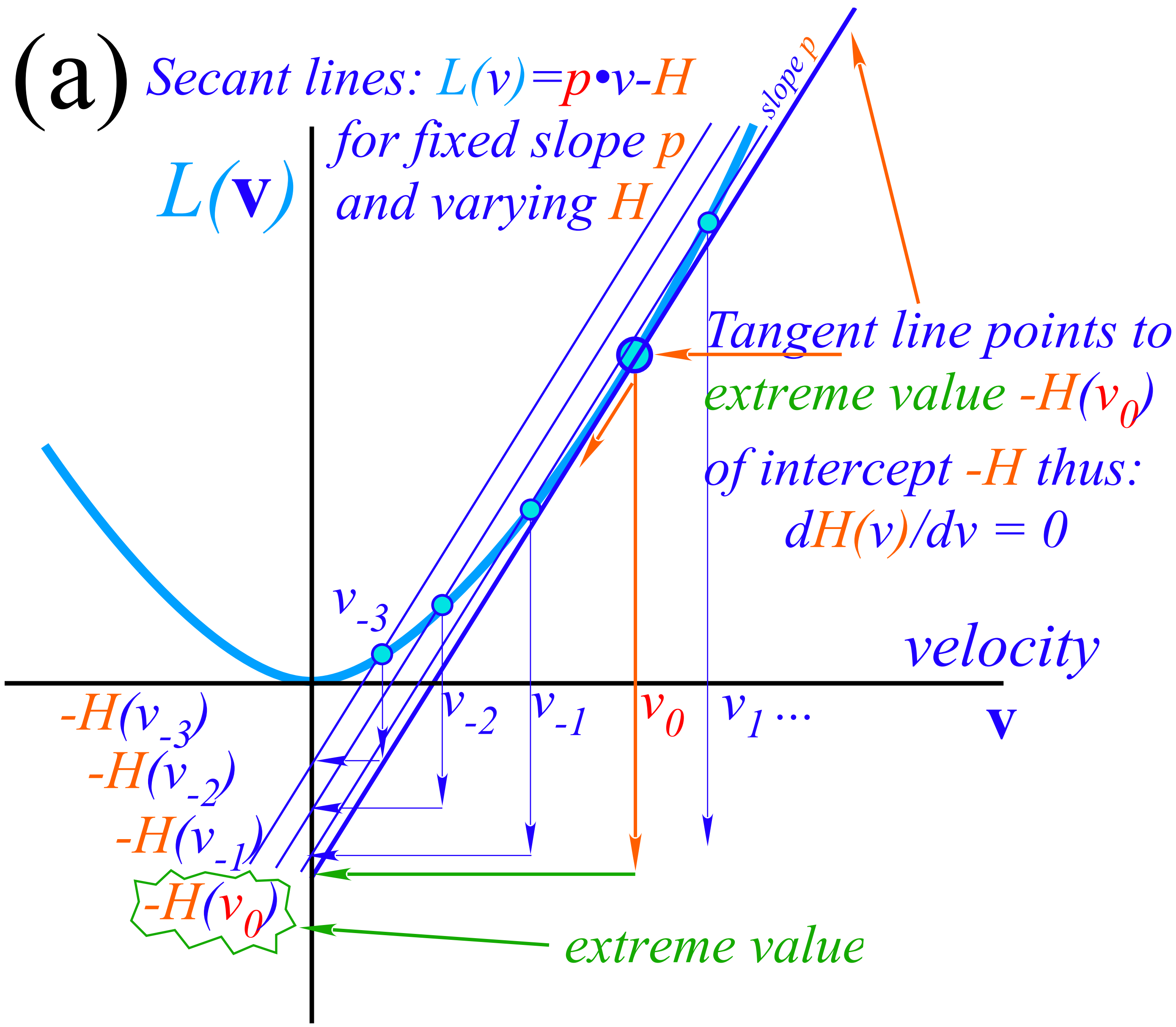
How Legendre contact transformations work...(to make $\frac{\partial H}{\partial v}=0$ or $\frac{\partial L}{\partial p}=0$)

Secant lines $L(\mathbf{v}) = p \bullet v - H$ of fixed slope $p=\frac{\partial L}{\partial v}$
and decreasing intercept $-H(v_{-2}) > -H(v_{-1}) > ...$
for increasing velocity $v_{-2} > v_{-1} > ... > v_0$
lead to unique tangent to $L(\mathbf{v})$-curve at the
tangent contact point $v=v_0$ that has max $H(p\ v_0)$
Thus $\frac{\partial H}{\partial v}=0$

Unit 1
Fig. 12.4



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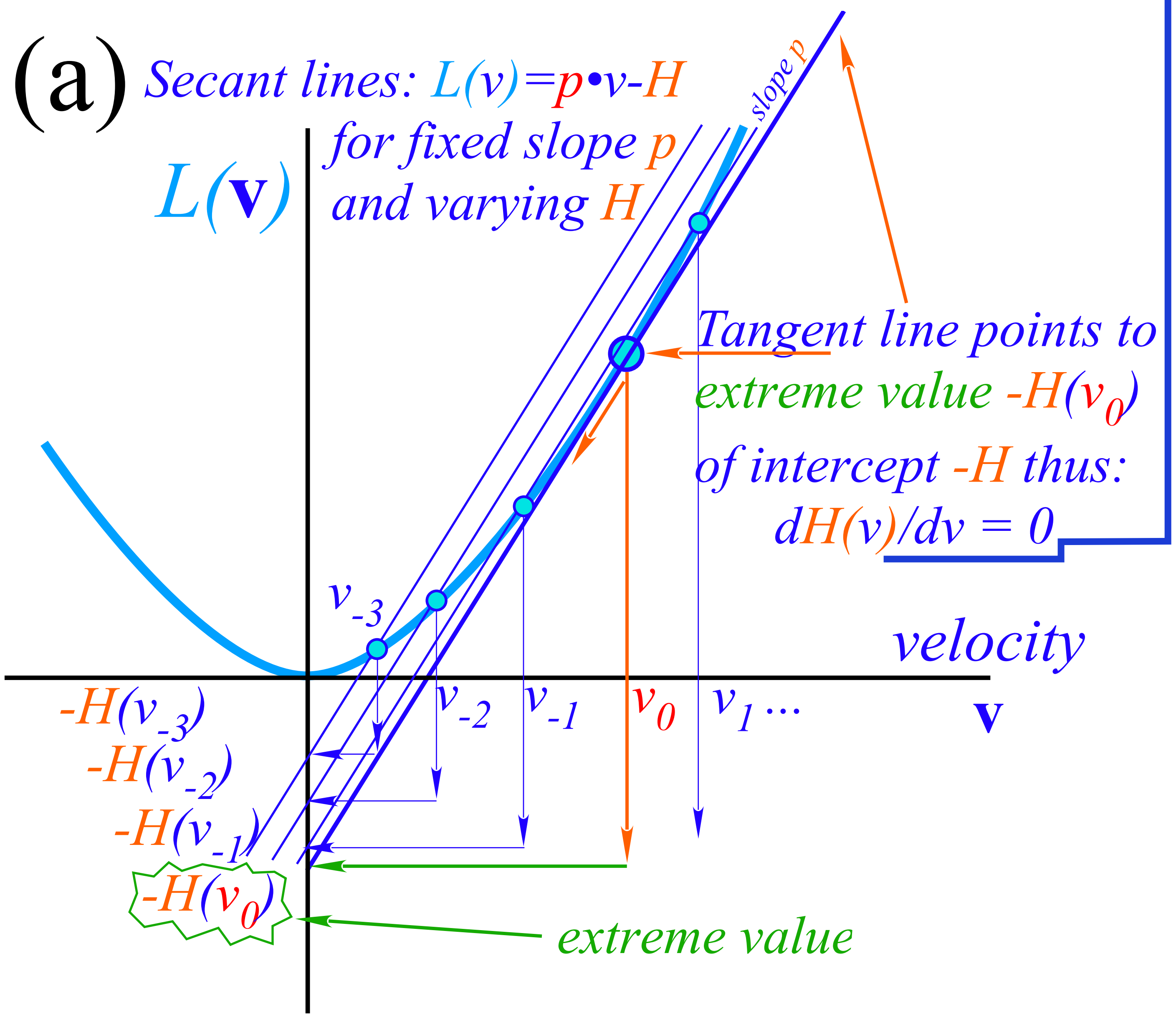
Unit 1
Fig. 12.4

(a)



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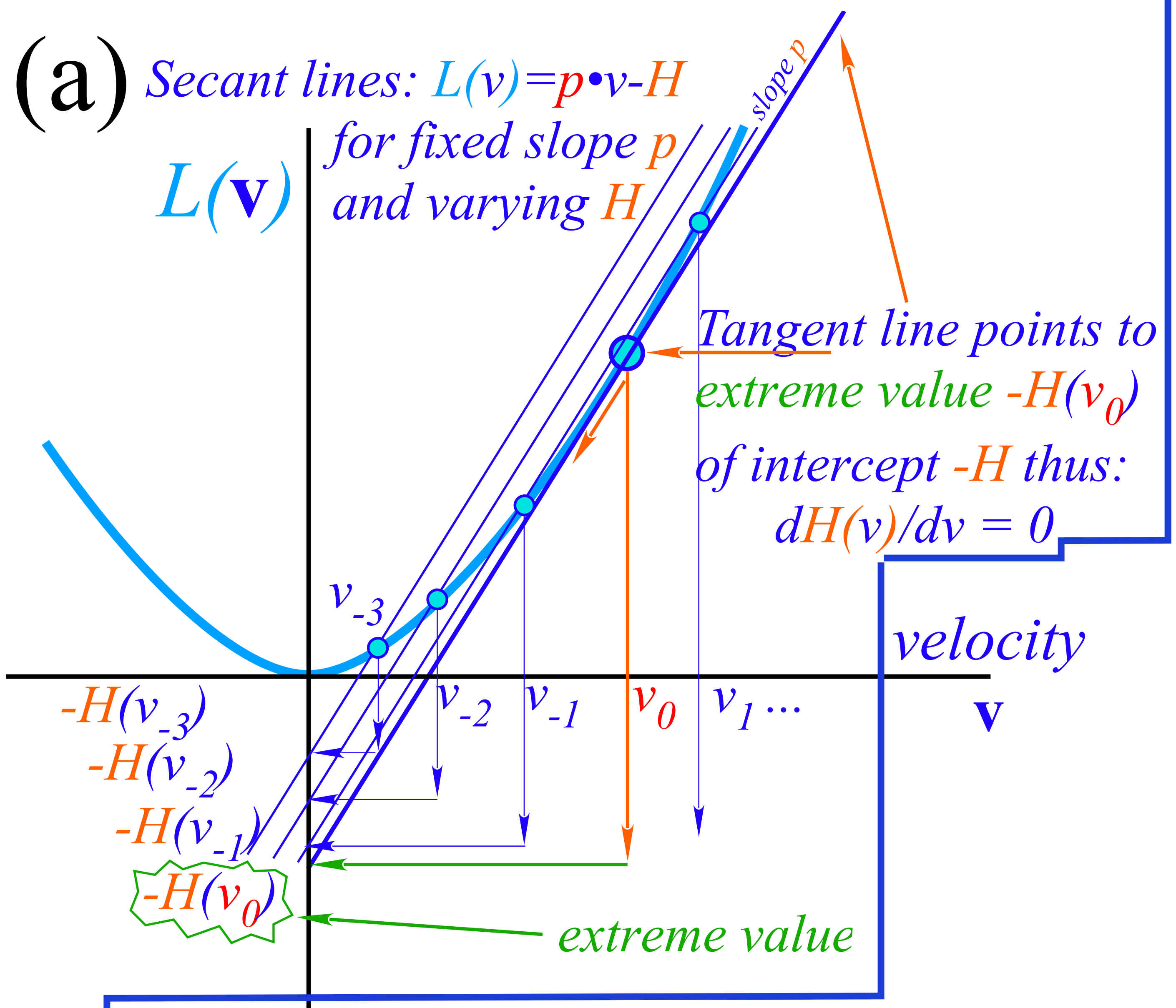
Unit 1
Fig. 12.4



$\frac{\partial H}{\partial v}=0$ at each point $v=\frac{\partial H}{\partial p}$ of $L(v)$ with slope $p=\frac{\partial L}{\partial v}$

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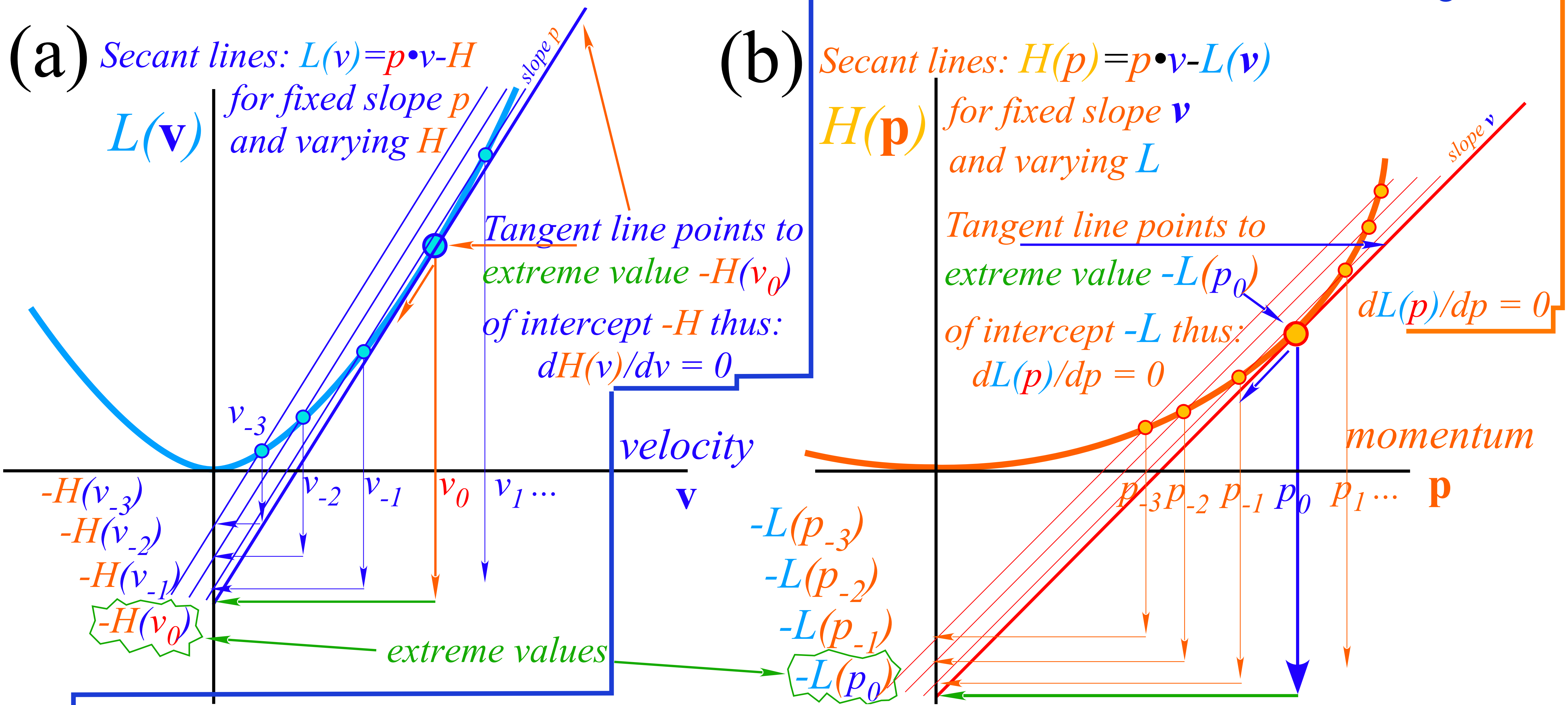
(Similarly...)

Unit 1
Fig. 12.4



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Unit 1
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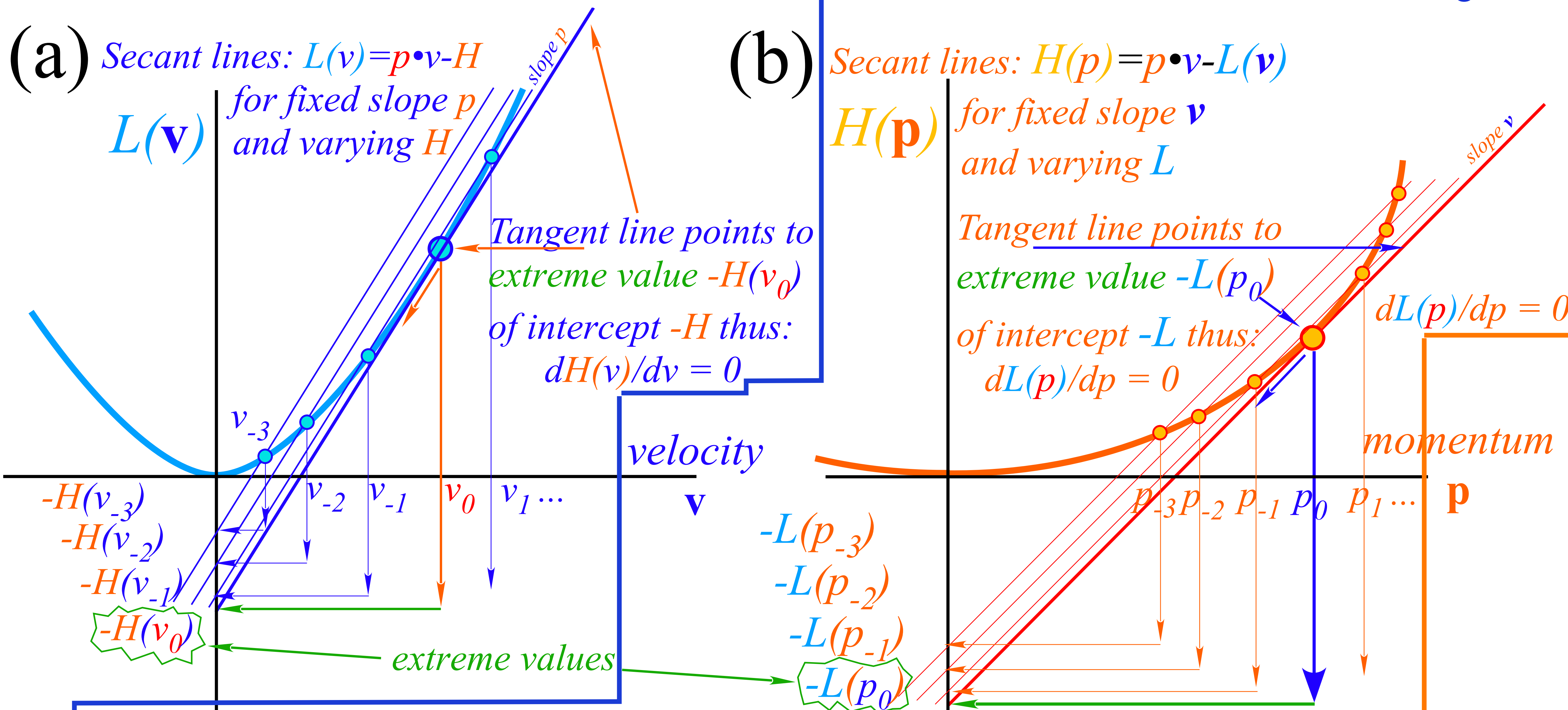

$\frac{\partial H}{\partial v}=0$ at each point $v=\frac{\partial H}{\partial p}$ of $L(v)$ with slope $p=\frac{\partial L}{\partial v}$

$\frac{\partial L}{\partial p}=0$ at each point $p=\frac{\partial L}{\partial v}$ of $H(p)$ with slope $v=\frac{\partial H}{\partial p}$

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Example of Legendre contact transformation in thermodynamics

Internal energy $U(S,V)$ is defined as a function of entropy S and volume V.

A new function *enthalpy* $H(S,P)$ depends on entropy and *pressure* P.

It is a Legendre transform $H(S,P)=P\cdot V+U$ of energy $U(S,V)$ to new variable $P=-\left(\frac{\partial U}{\partial V}\right)_S$.

Example of Legendre contact transformation in thermodynamics

Lagrangian $L(r,v)$ position r velocity v

Internal energy $U(S,V)$ is defined as a function of entropy S and volume V.

Hamiltonian $H(r,p)$ position r *momentum* p

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$H(r,p) = p \cdot v - L$ Lagrangian $L(r,v)$ $p = (\frac{\partial L}{\partial v})_r$

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Except for ±signs, it's our Hamiltonian $H(p)=p \cdot v-L(v)$ going from Lagrangian $L(v)$

to use new variable momentum $p = (\frac{\partial L}{\partial v})_x$.

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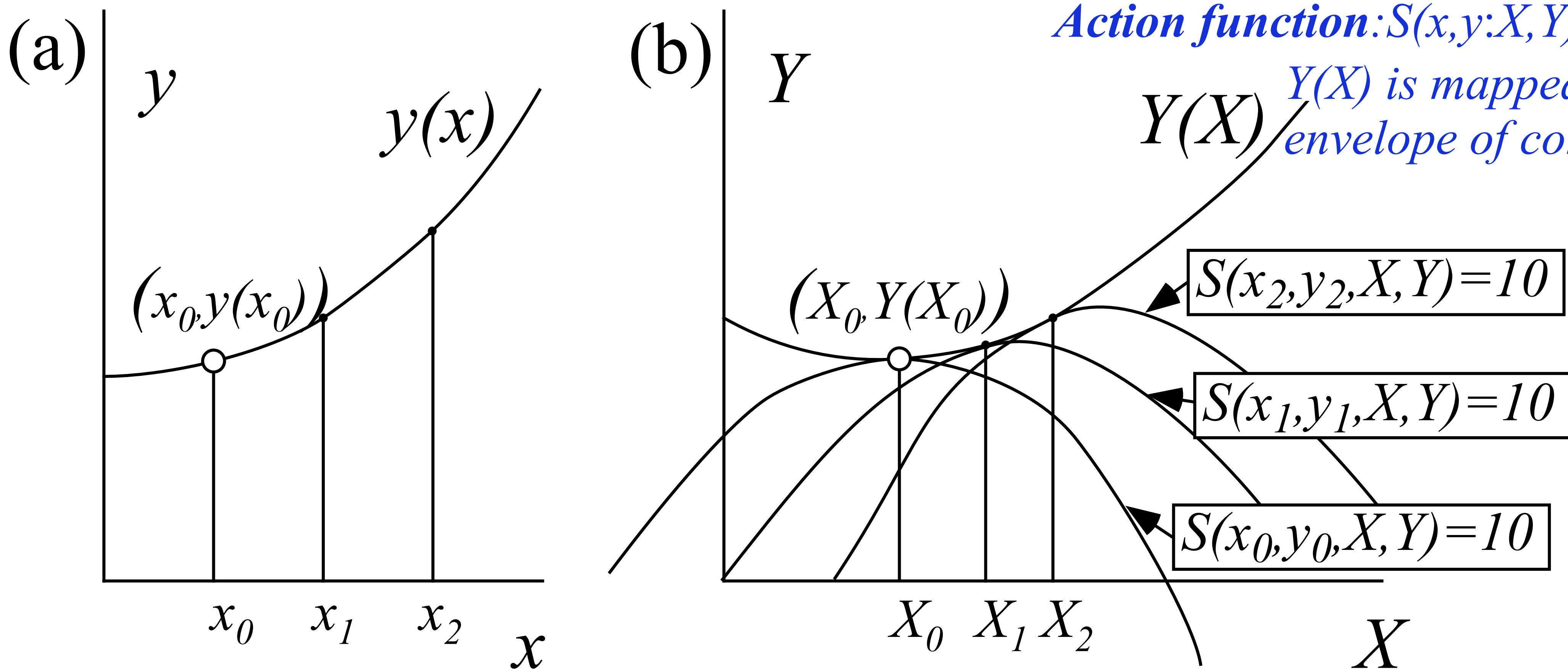
Active-Contact-Transformation Generator or ***Action function****: $S(x,y:X,Y)$=const. does mapping.*

$Y(X)$ is mapped from $y(x)$ as an envelope of contacting S=const. curves.



Unit 1
Fig. 12.7

Legendre transform: special case of General Contact Transformation

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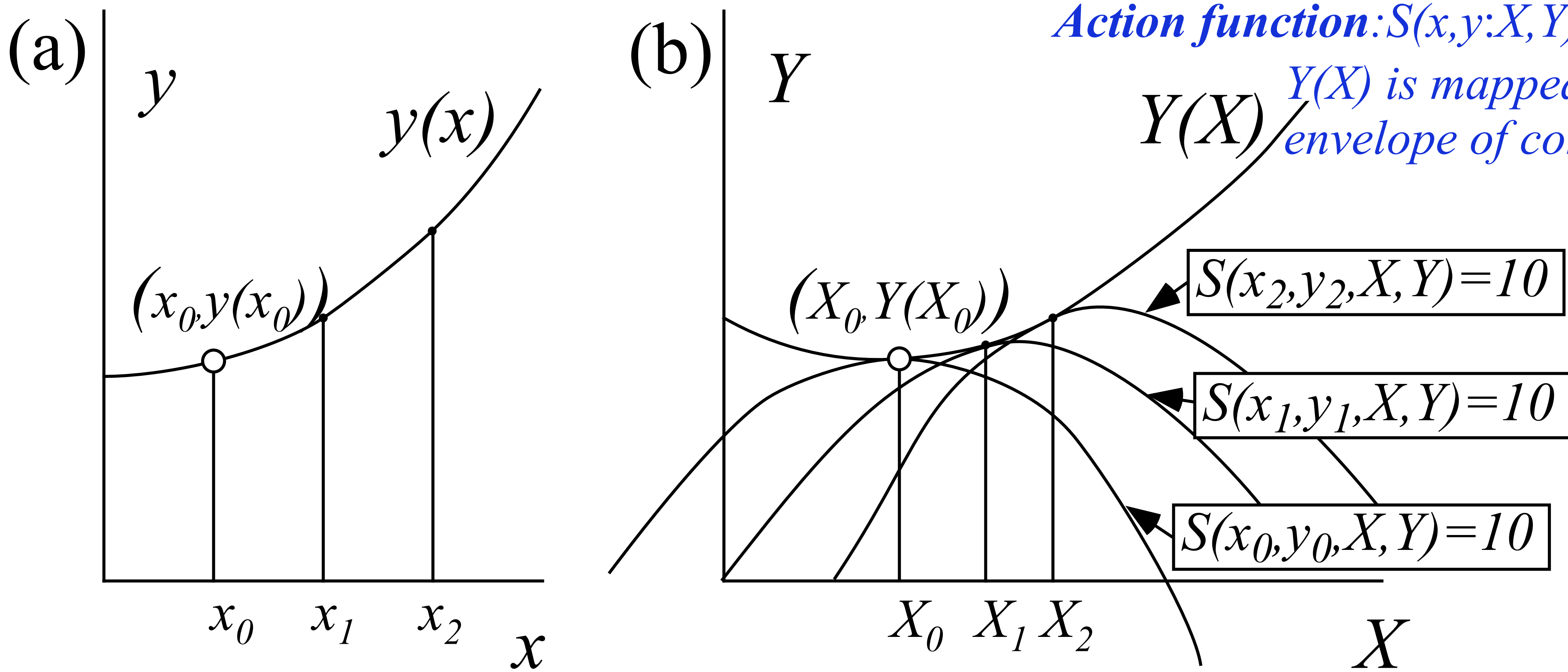
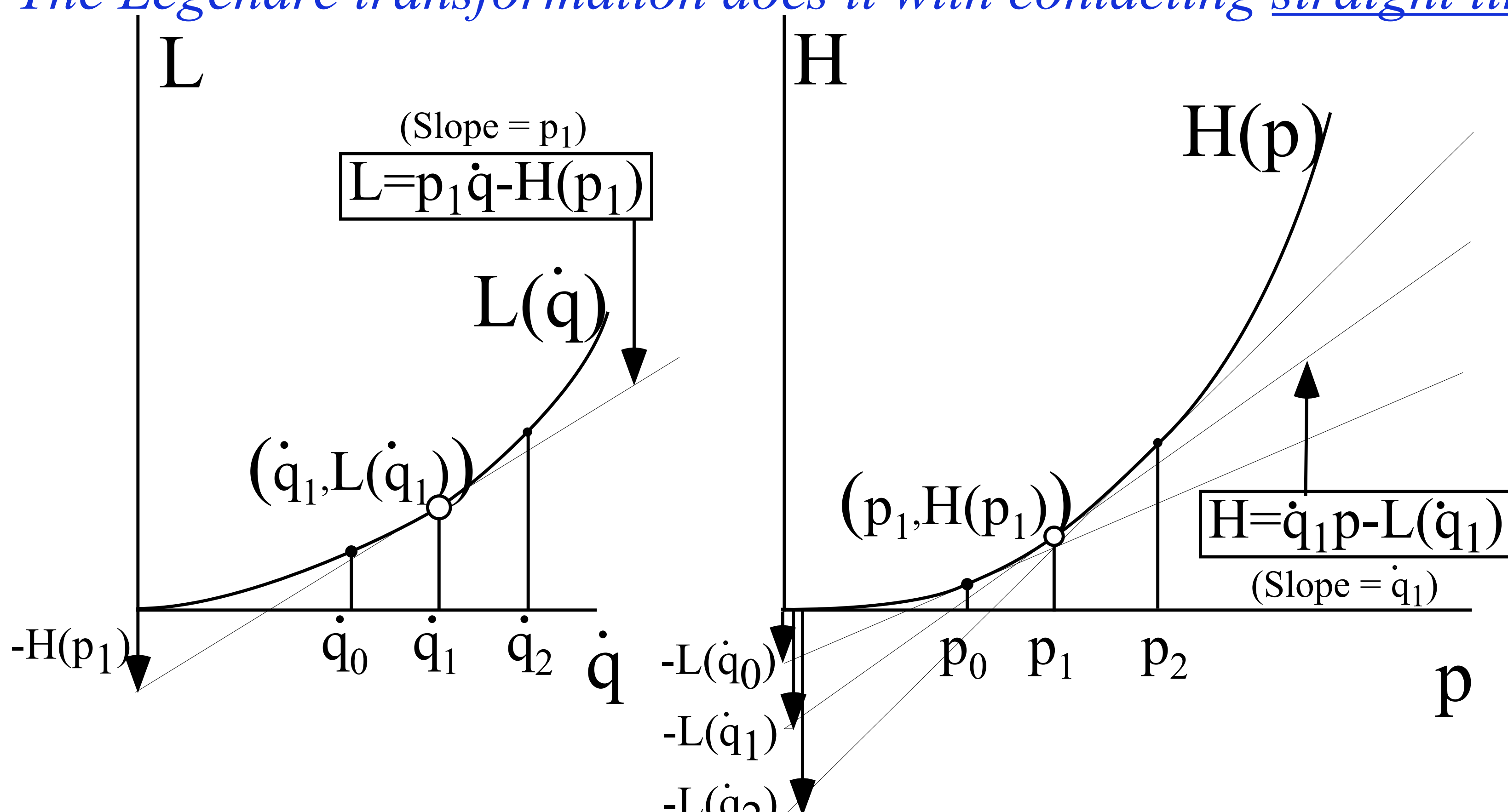


Unit 1
Fig. 12.7

The Legendre transformation does it with contacting *<u>straight line</u>* *tangents.*



Unit 1
Fig. 12.9

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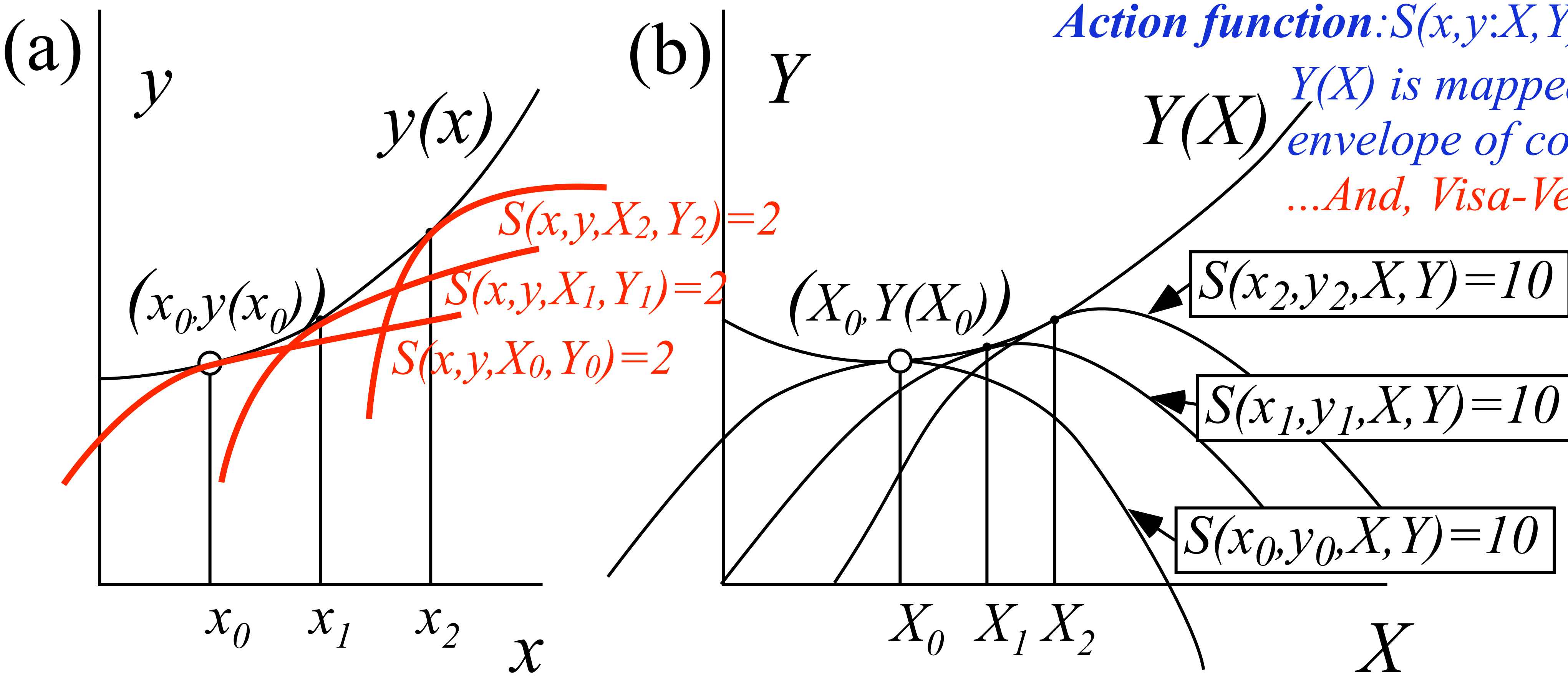
...And, Visa-Versa !...



Unit 1
Fig. 12.7

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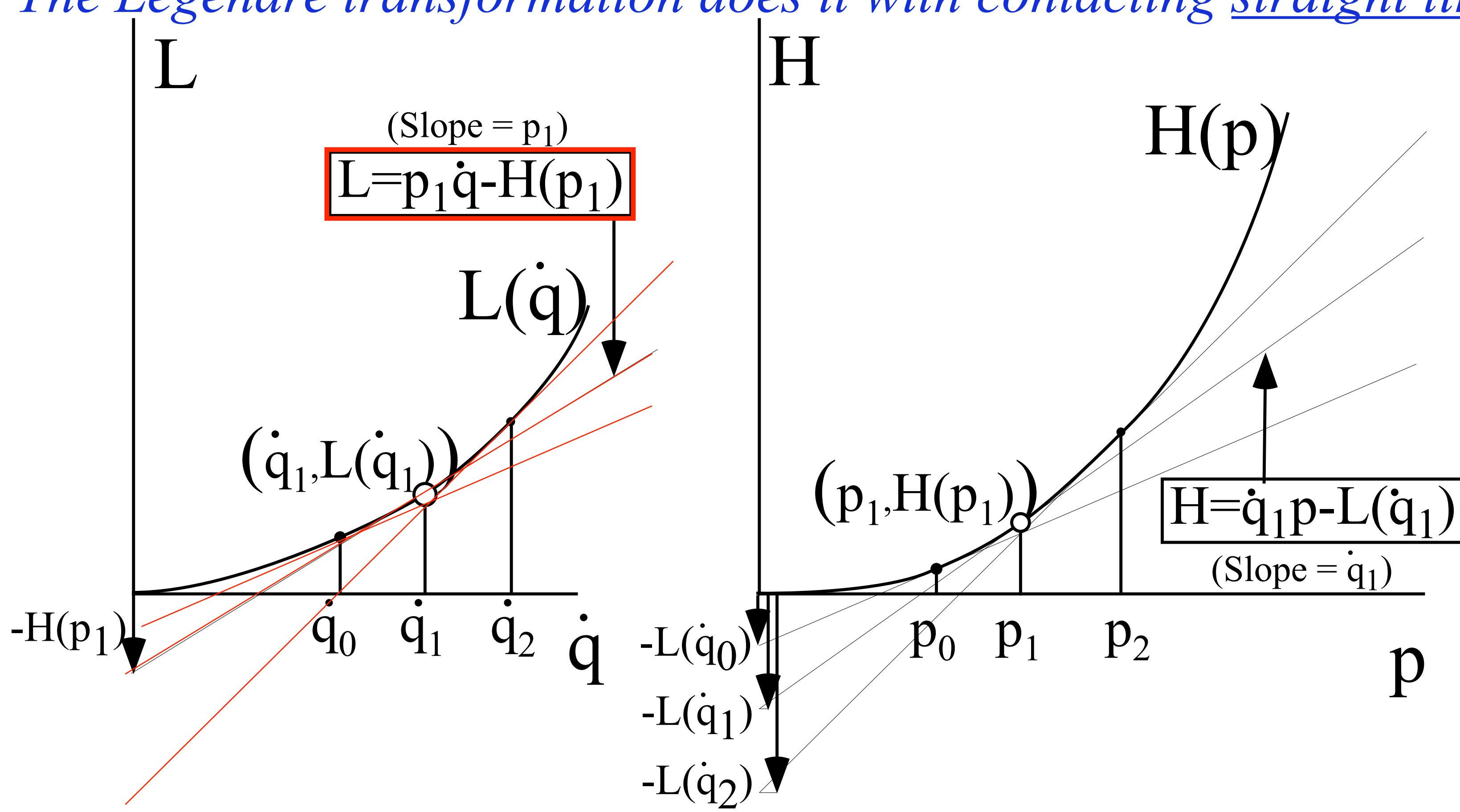

Unit 1
Fig. 12.9

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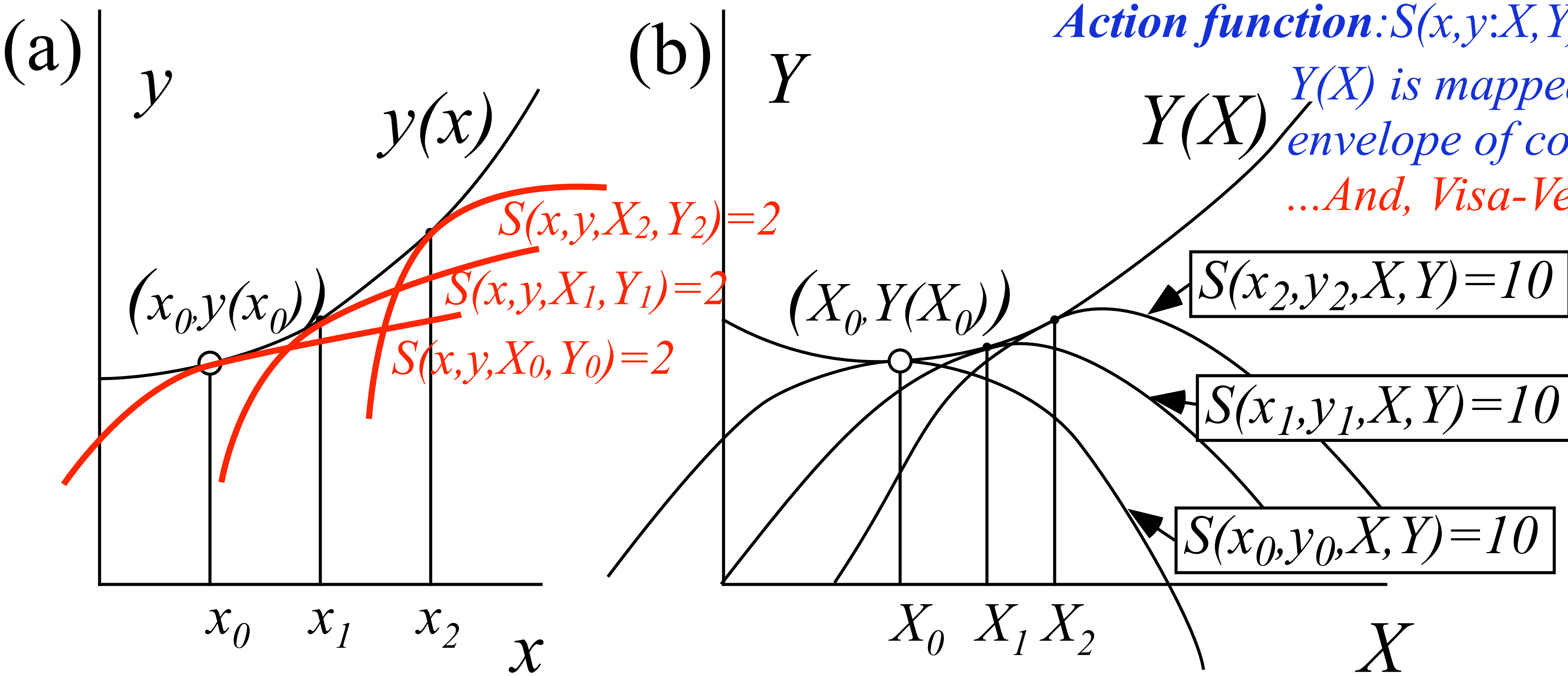

Unit 1
Fig. 12.7

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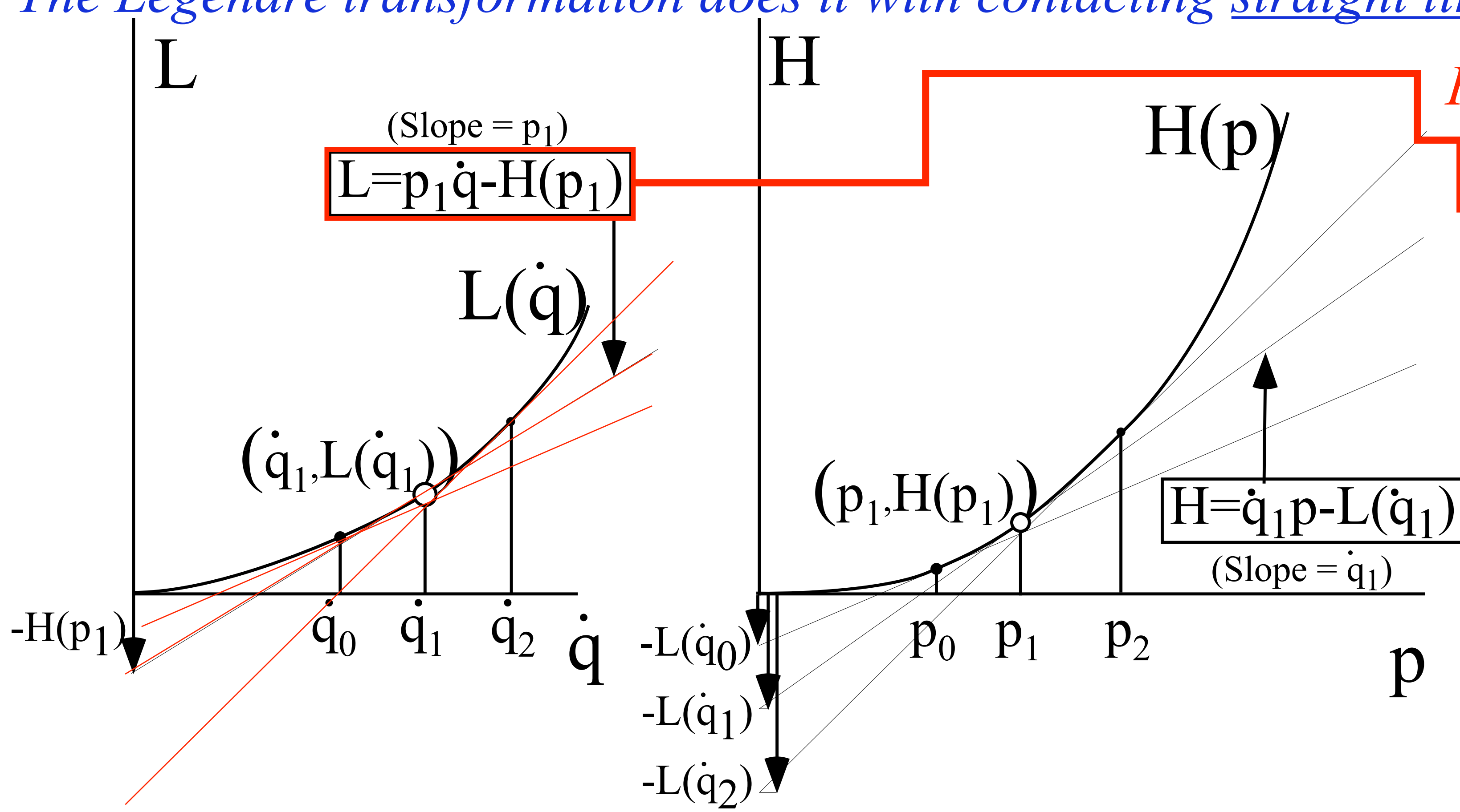


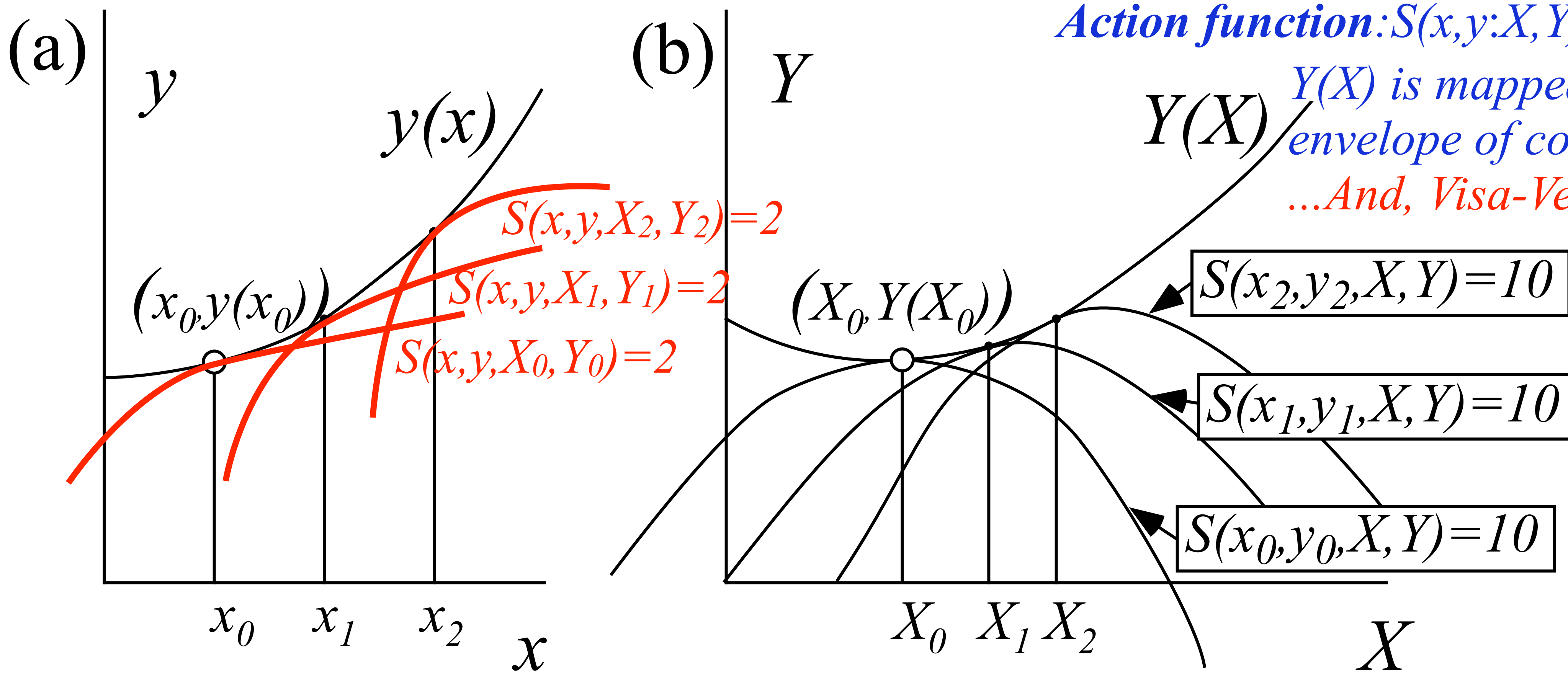
Poincare's differential action

$$dS = L\,dt = p\cdot\dot{q}\,dt - H\cdot dt$$
$$= p\cdot dq - H\cdot dt$$

Unit 1
Fig. 12.9

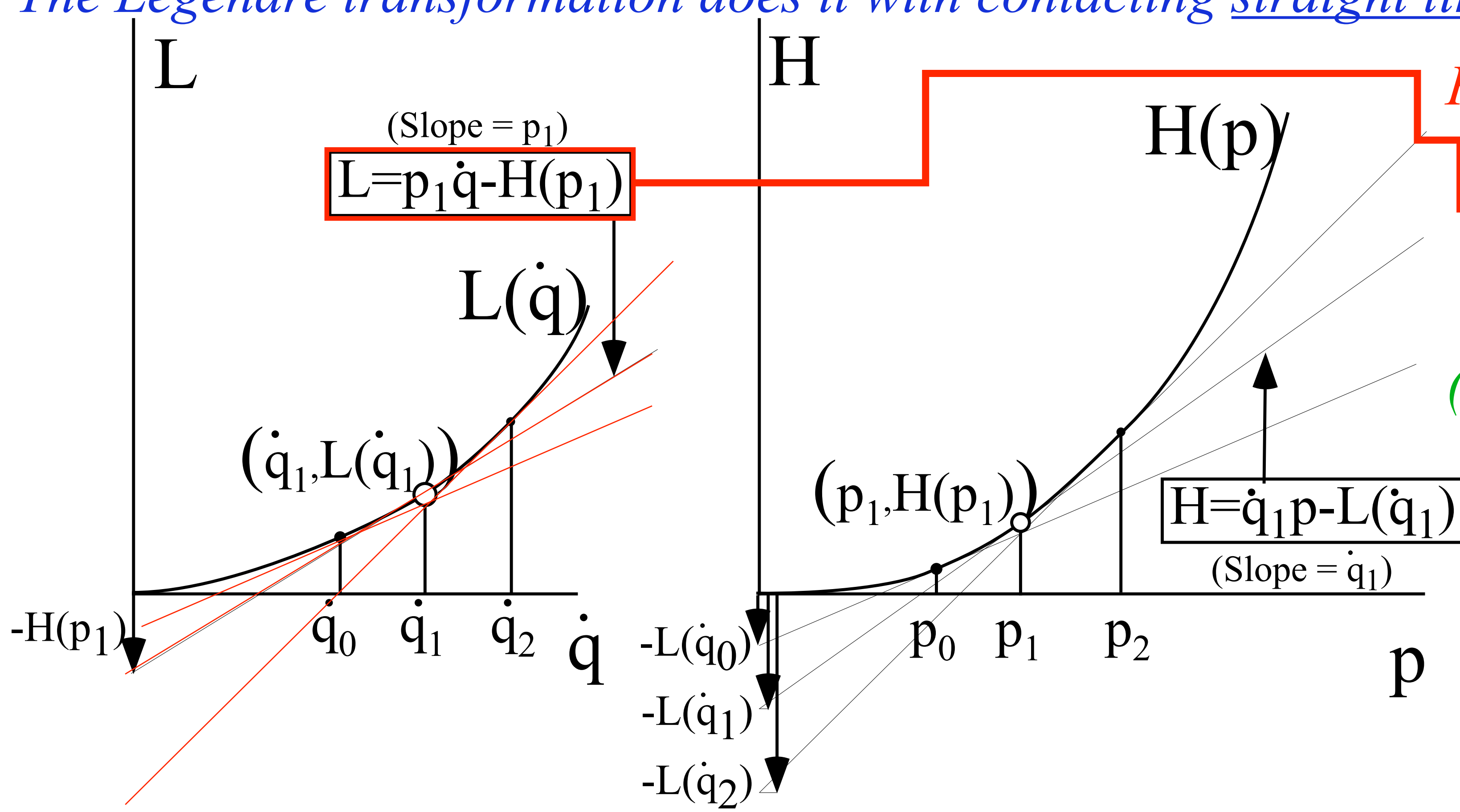
Legendre transform: special case of General Contact Transformation

Active-Contact-Transformation Generator or
***Action function**: $S(x,y{:}X,Y)$=const. does mapping.*
$Y(X)$ is mapped from $y(x)$ as an envelope of contacting S=const. curves.
...And, Visa-Versa !...



Unit 1
Fig. 12.7

The Legendre transformation does it with contacting <u>straight line</u> tangents.



Poincare's differential action

$$dS = L\,dt = p\cdot\dot{q}\,dt - H\cdot dt$$
$$= p\cdot dq - H\cdot dt$$

$$dS = L\,dt = \hbar k\cdot dr - \hbar\omega\cdot dt$$
(Quantum phase differential)

Unit 1
Fig. 12.9

Legendre transform: special case of General Contact Transformation

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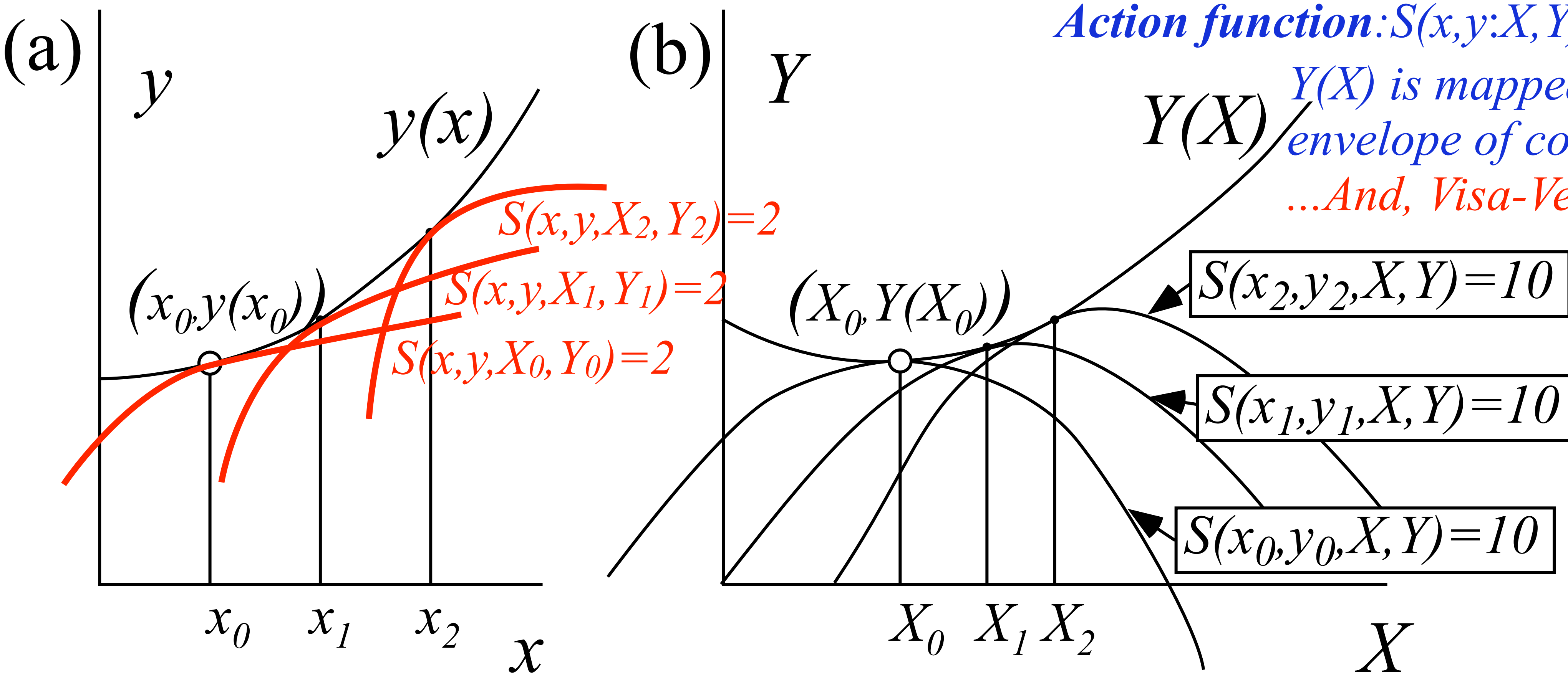
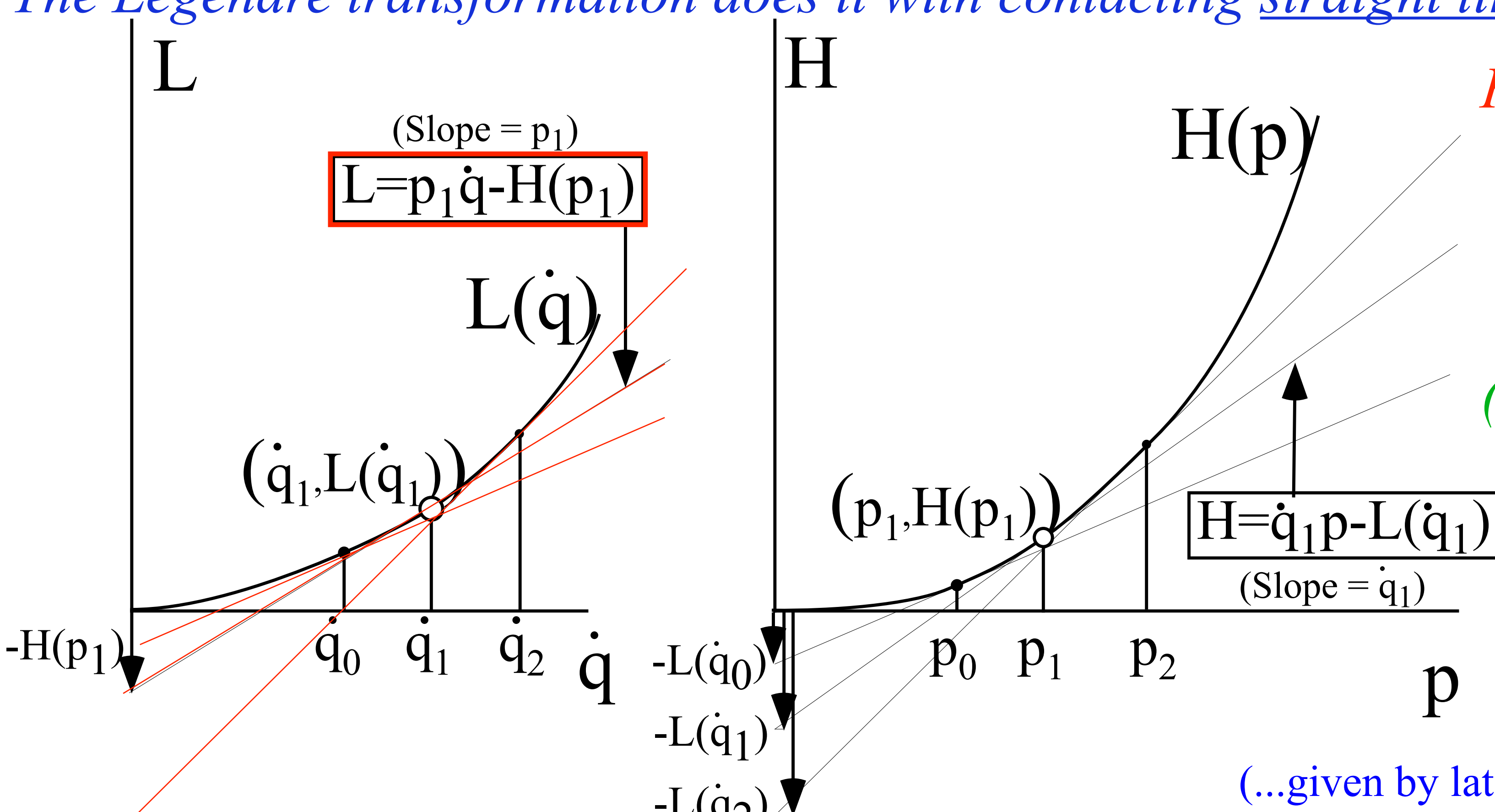


Unit 1
Fig. 12.7

The Legendre transformation does it with contacting straight line tangents.



Poincare's differential action

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$$= p\cdot dq - H\cdot dt$$
$$dS = L\,dt = \hbar k\cdot dr - \hbar\omega\cdot dt$$

(Quantum phase differential)

Unit 1
Fig. 12.9

This extraordinary claim needs extraordinary proof!

(...given by later lectures for Ch. 12 Unit 1 and Unit 8.)

Review of partial differential calculus

Chain rule and order $\partial^2\Psi/\partial x\partial y = \partial^2\Psi/\partial y\partial x$ symmetry

Scaling transformation between Lagrangian and Hamiltonian views of KE

Introducing 0th Lagrange and 0th Hamilton differential equations of mechanics

Introducing 1st Lagrange and 1st Hamilton differential equations of mechanics

Introducing the Poincare´ and Legendre contact transformations

Geometry of Legendre contact transformation (Preview of Unit 8 relativistic quantum mechanics)

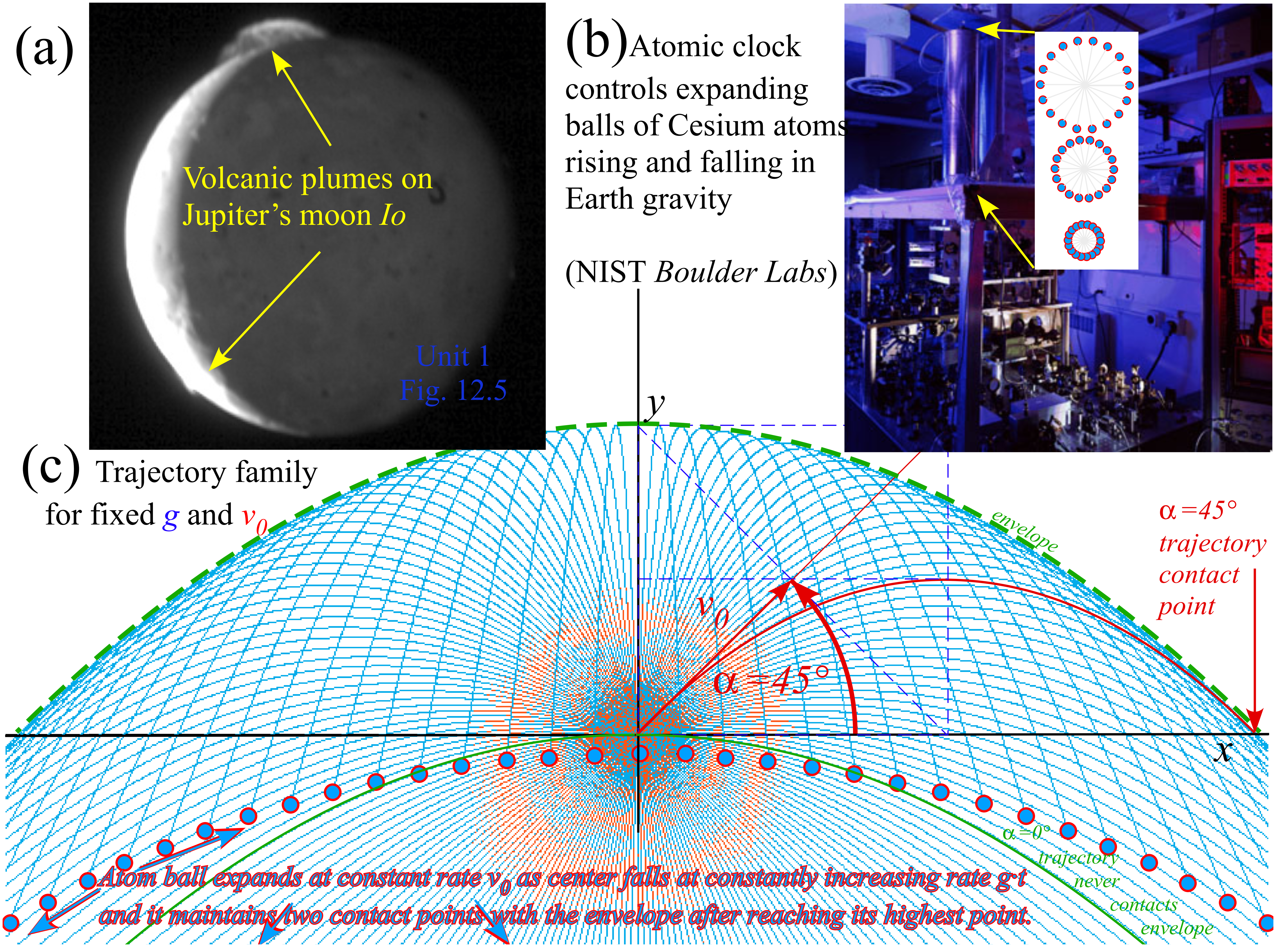
Example from thermodynamics

Legendre transform: special case of General Contact Transformation (lights,camera, ACTION!)

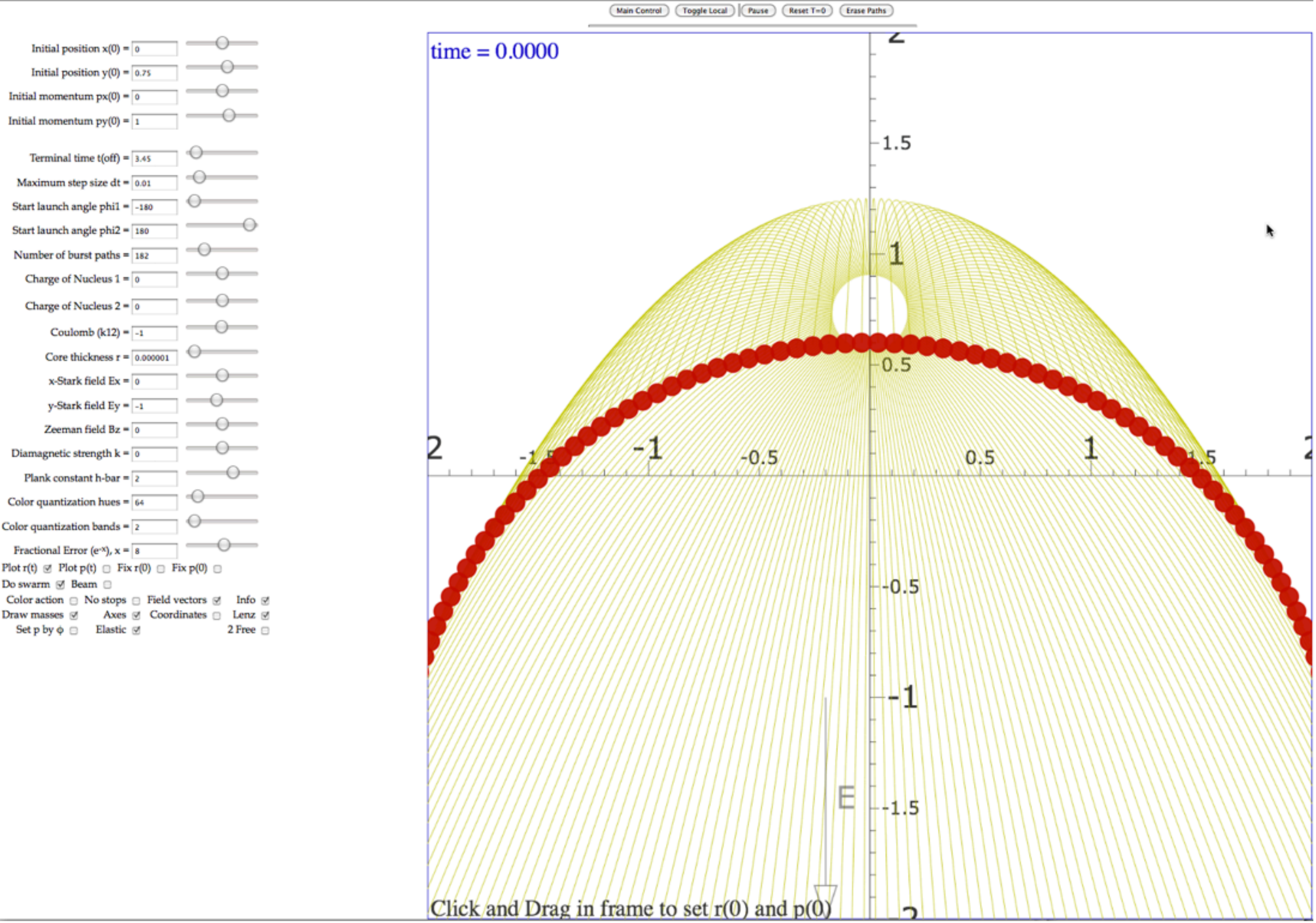
→ *An elementary contact transformation from sophomore physics*

Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"

Intuitive-geometric development of " " " and " " "



(a)
Volcanic plumes on Jupiter's moon *Io*
Unit 1 Fig. 12.5
(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity
(NIST *Boulder Labs*)
(c) Trajectory family for fixed g and v_0
y
x
envelope
v_0
$\alpha=45°$
$\alpha=45°$ trajectory contact point
$\alpha=0°$ trajectory never contacts envelope
Atom ball expands at constant rate v_0 as center falls at constantly increasing rate $g\cdot t$
and it maintains two contact points with the envelope after reaching its highest point.



[Link ⇒ CoulIt - Simulation of the *Volcanoes of Io*]

Review of partial differential calculus

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→ *Algebra-calculus development of "The Volcanoes of Io" and "The Atoms of NIST"*

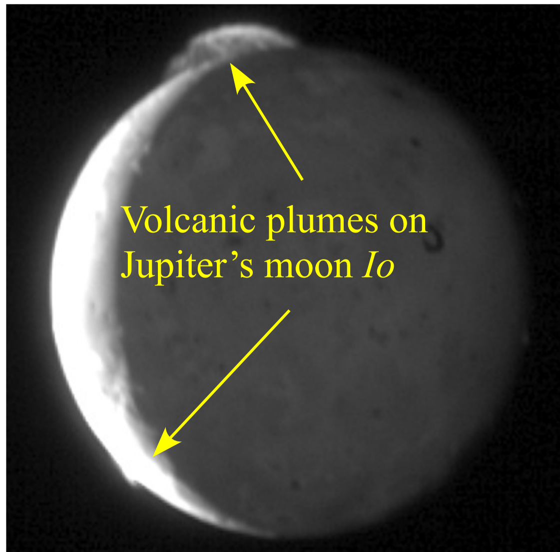
Intuitive-geometric development of " " " and " " "

Constant gravity g assumed here...

Excellent for NIST

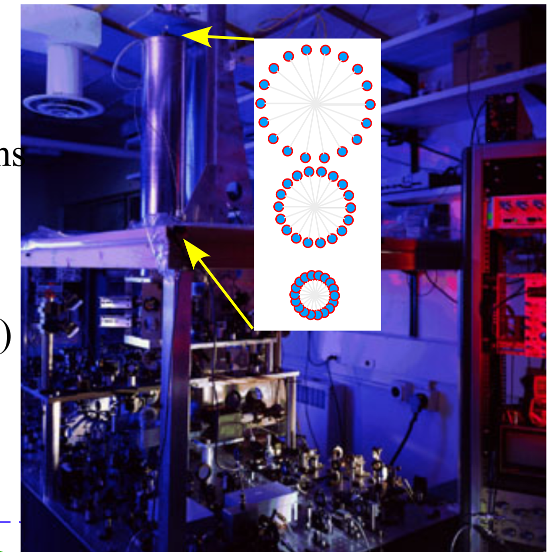
OK for Io (fixed in Unit5)

(a)

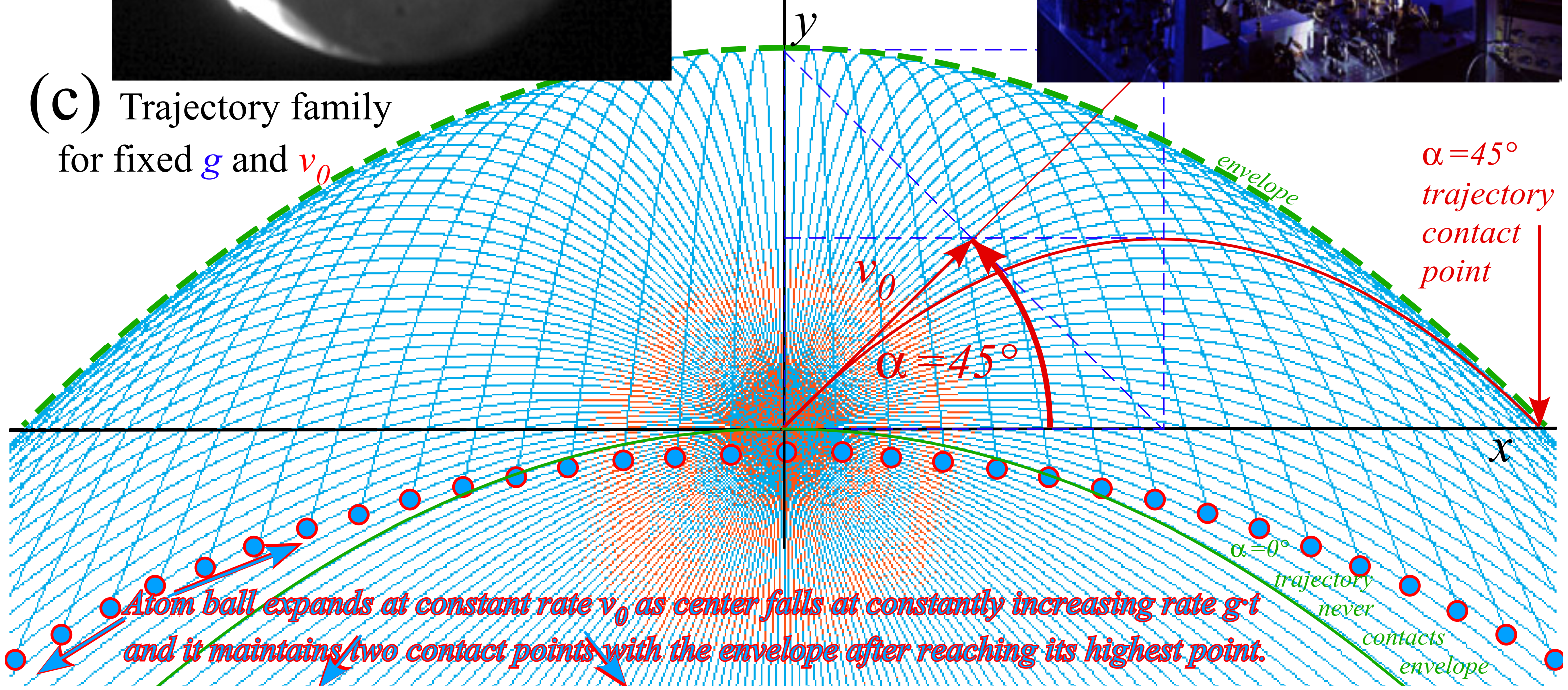


(b) Atomic clock controls expanding balls of Cesium atoms rising and falling in Earth gravity

(NIST *Boulder Labs*)



(c) Trajectory family for fixed g and v_0



Unit 1
Fig. 12.5

UP-1 formulas for trajectories in constant gravity g

$$x(t) = (v_0 \cos\alpha)t \qquad y(t) = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$\dot{x}(0) = v_x(0) = v_0 \cos\alpha \qquad \dot{y}(0) = v_y(0) = v_0 \sin\alpha$$

Substitute time $t = x/(v_0 \cos\alpha)$ into $y(t)$

$$y(x) = \frac{v_0 \sin\alpha}{v_0 \cos\alpha}x - \frac{gx^2}{2v_0^2 \cos^2\alpha}$$

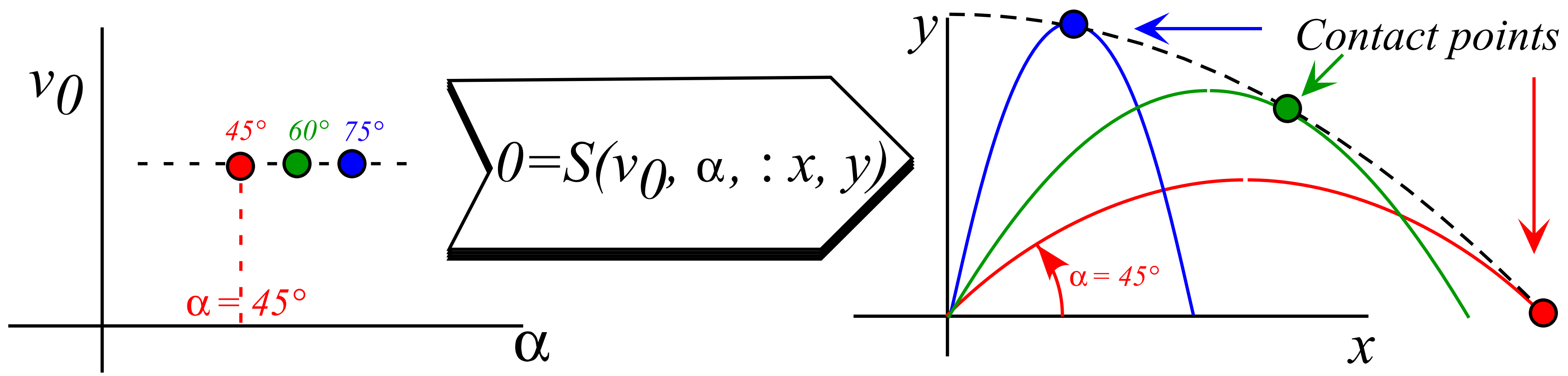
$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha}$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

becomes:

$$S(v_0, \alpha : x, y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Unit 1
Fig. 12.6

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

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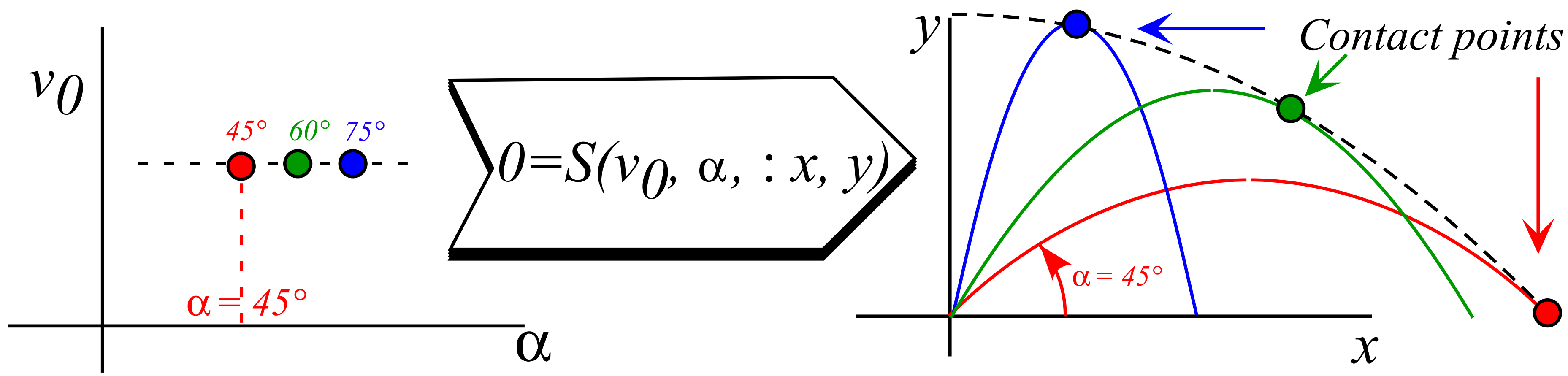
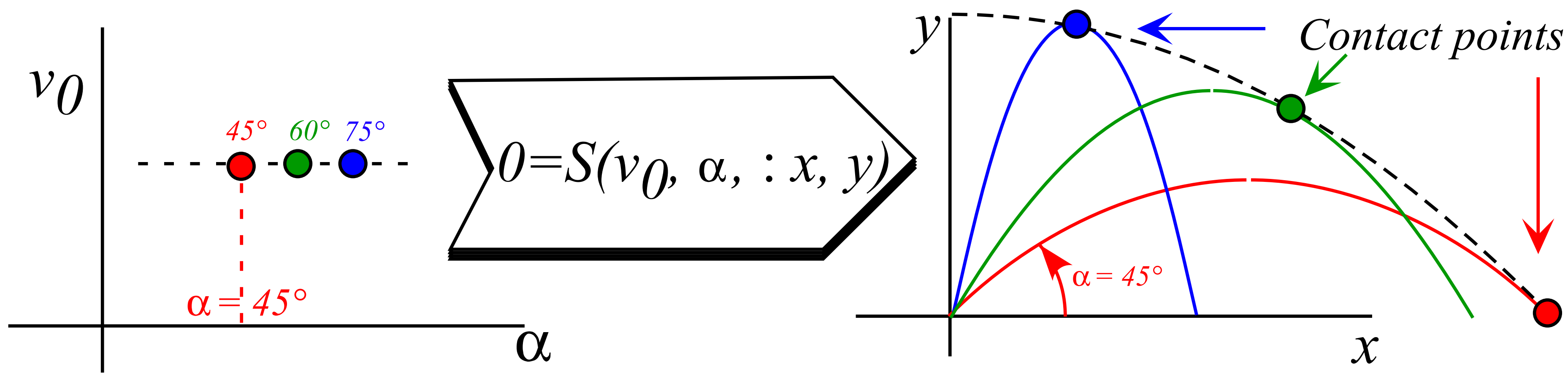


Unit 1
Fig. 12.6

Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $\frac{\partial S(v_0, \alpha : x, y)}{\partial \alpha} = 0$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \qquad \textit{becomes:} \qquad S(v_0,\alpha:x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Unit 1
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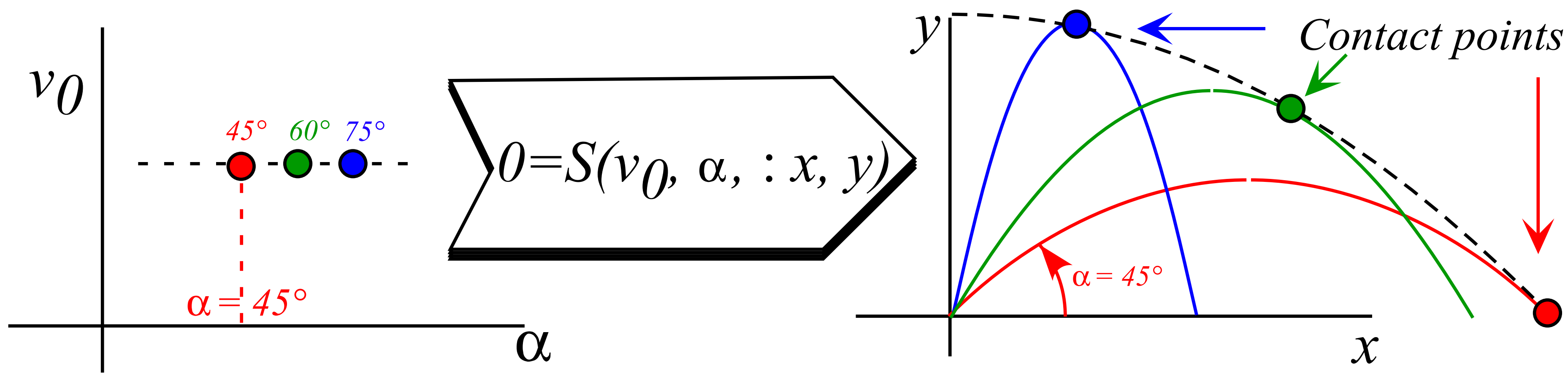
Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $\frac{\partial S(v_0,\alpha:x,y)}{\partial\alpha} = 0$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha}$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \qquad \textit{becomes:} \qquad S(v_0,\alpha : x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



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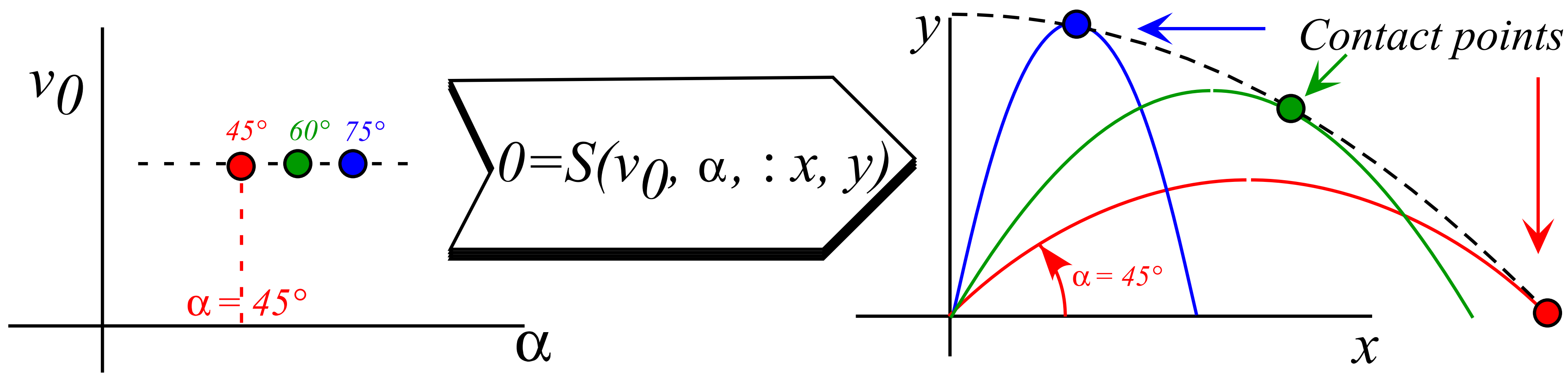
$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha} \qquad \textit{gives:} \qquad \tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}\ .$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$ *becomes:* $$S(v_0,\alpha:x,y) = -y + x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} = 0$$



Unit 1
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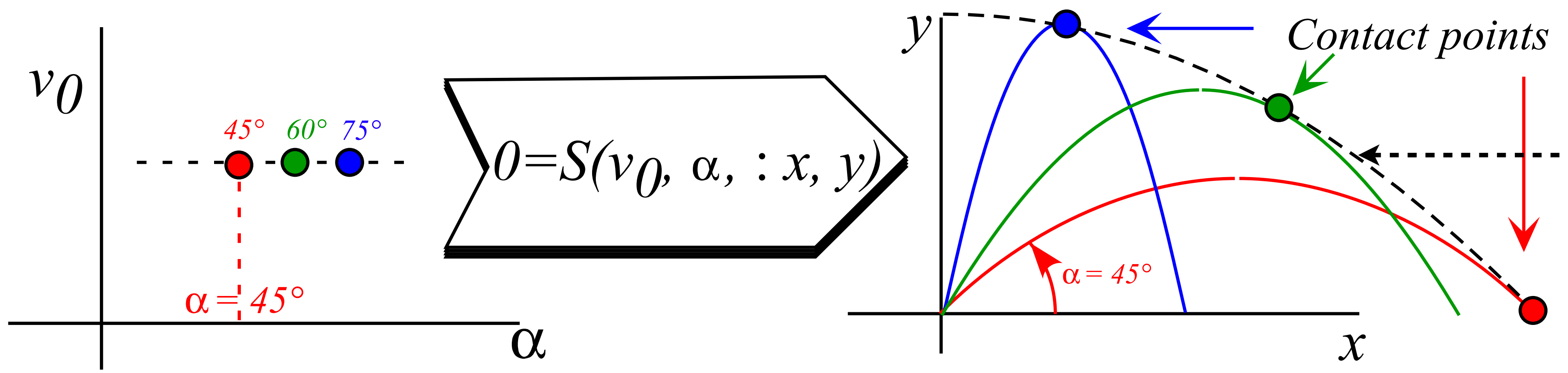
Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $\frac{\partial S(v_0,\alpha:x,y)}{\partial\alpha} = 0$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2v_0^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2v_0^2}\frac{2\sin\alpha}{\cos^3\alpha}$$ $$\tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}.$$

$$y_{env}(x) = x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1+\tan^2\alpha\right) \Rightarrow y_{env}(x) = x\frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2}\left(1+\frac{v_0^4}{g^2x^2}\right)$$

Convert y(x) solution into Active Contact Transformation Generator $S(v_0, \alpha : x, y)$

$$y(x) = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$ *becomes:* $$S(v_0,\alpha : x,y) = -y + x\tan\alpha - \frac{gx^2}{2{v_0}^2\cos^2\alpha} = 0$$



Envelopes of the v_0-trajectory region contain extremal *contact points* with each trajectory where: $$\frac{\partial S(v_0,\alpha : x,y)}{\partial\alpha} = 0$$

$$x\frac{\partial\tan\alpha}{\partial\alpha} - \frac{gx^2}{2{v_0}^2}\frac{\partial\cos^{-2}\alpha}{\partial\alpha} = 0 = \frac{x}{\cos^2\alpha} - \frac{gx^2}{2{v_0}^2}\frac{2\sin\alpha}{\cos^3\alpha}$$

$$\tan\alpha = \frac{v_0^2}{gx} \quad \text{or:} \quad x = \frac{v_0^2}{g\tan\alpha}.$$

$$y_{env}(x) = x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1+\tan^2\alpha\right) \Rightarrow y_{env}(x) = x\frac{v_0^2}{gx} - \frac{gx^2}{2v_0^2}\left(1+\frac{v_0^4}{g^2x^2}\right)$$

$$y_{env}(x) = \frac{v_0^2}{g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2}\frac{v_0^4}{g^2x^2} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

Envelope function

Review of partial differential calculus
Chain rule and order $\partial^2\Psi/\partial x\partial y = \partial^2\Psi/\partial y\partial x$ symmetry

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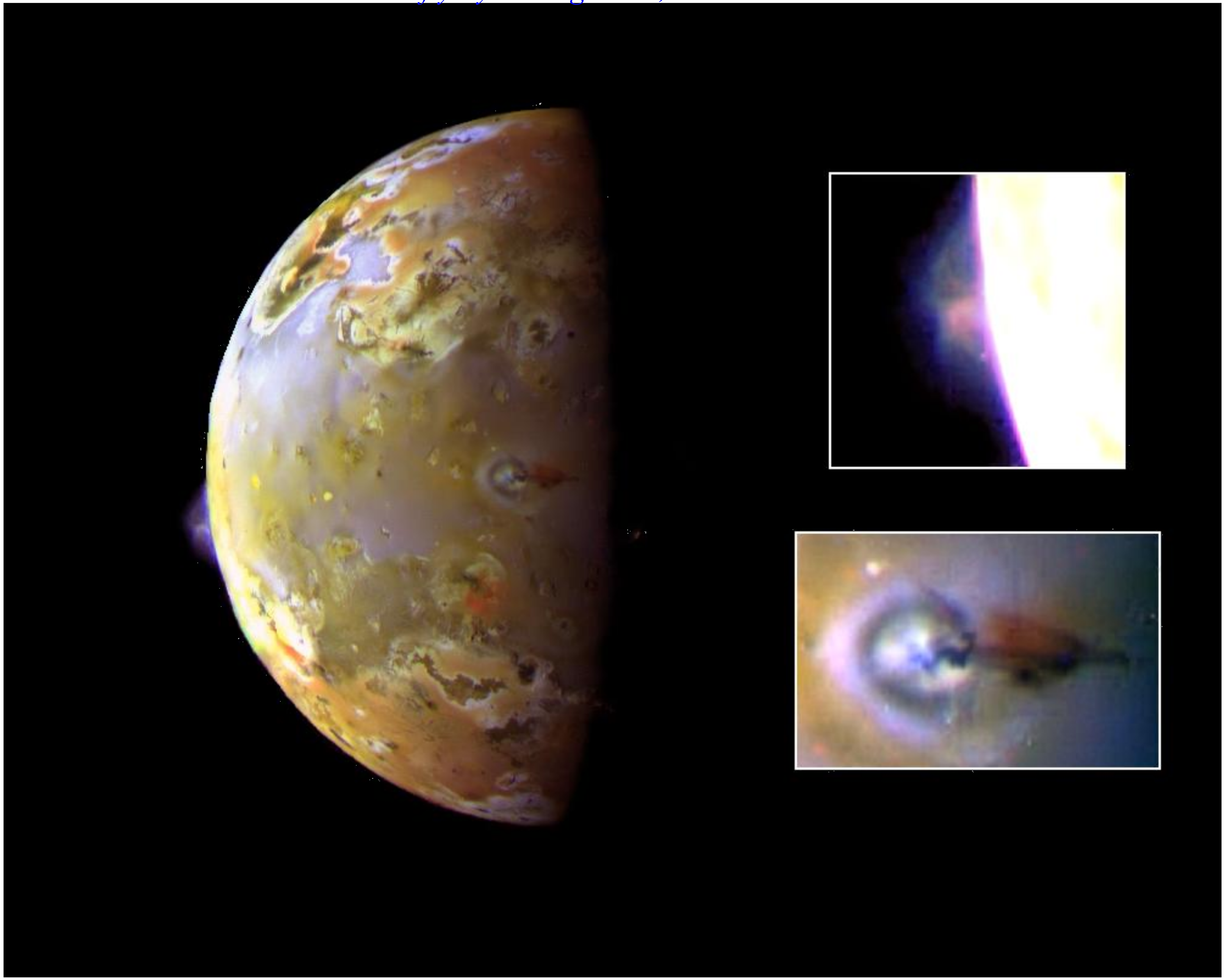
→ *Intuitive-geometric development of " " " and " " "*

The Plumes of Prometheus

NASA-Galileo Project

Io fly-by on August 18, 1997



NASA Astronomy Picture of the Day - *Io: The Prometheus Plume* *(Just Image)*

NASA Galileo - *Io's Alien Volcanoes*

New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*

NASA Galileo - A Hawaiian-Style Volcano on Io

Io's Alien Volcanoes

Inform Inspire Involve
science.nasa.gov

Space Science News home

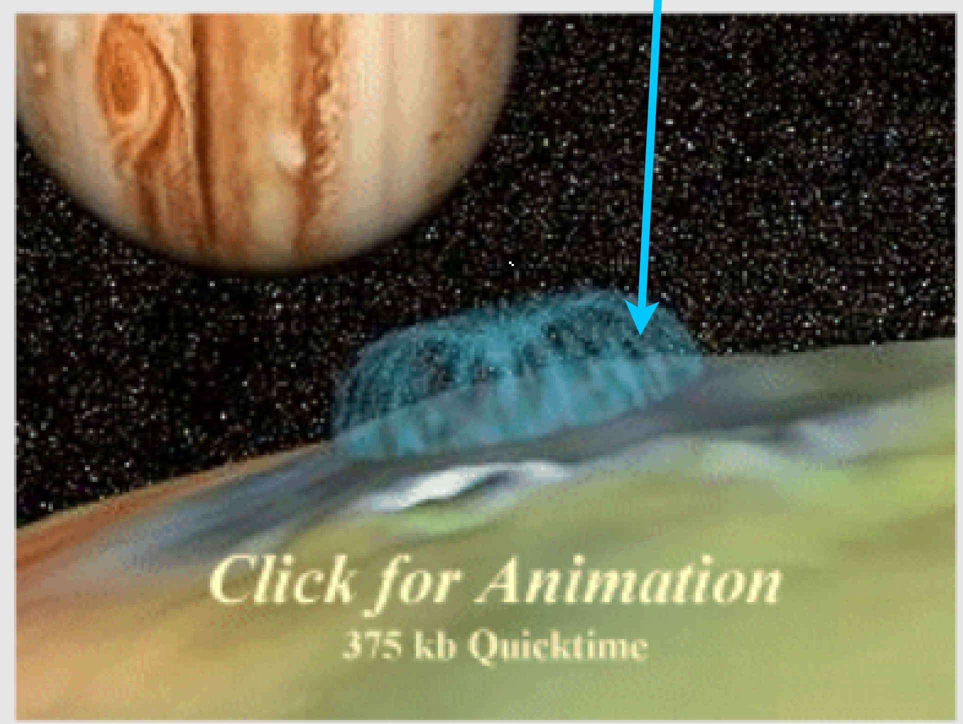
Pretty bad sketch of plumes
(LasVegas model of planetary ejecta?)

Io's Alien Volcanoes

Do these guys need a geometry lesson?

Need to fly parabola kite geometry...

Scientists are eager for a closer look at the solar system's strangest and most active volcanoes when Galileo flies by Io on October 11.



October 4, 1999: Thirty years ago, before the Voyager probes visited Jupiter, if you had described Io to a literary critic it would have been declared overwrought science fiction. Jupiter's strange moon is literally bursting with volcanoes. Dozens of active vents pepper the landscape which also includes gigantic frosty plains, towering mountains and volcanic rings the size of California. The volcanoes themselves are the hottest spots in the solar system with temperatures exceeding 1800 K (1527 C). The plumes which rise 300 km into space are so large they can be seen from Earth by the Hubble Space Telescope. Confounding common sense, these high-rising ejecta seem to be made up of, not blisteringly hot lava, but frozen sulfur dioxide. And to top it all off, Io bears a striking resemblance to a pepperoni pizza. Simply unbelievable.

Right: Digital Radiance simulation of Pillan Patera just before the Galileo flyby. click for animation → .

NASA Astronomy Picture of the Day - *Io: The Prometheus Plume* *(Just Image)*

NASA Galileo - *Io's Alien Volcanoes*

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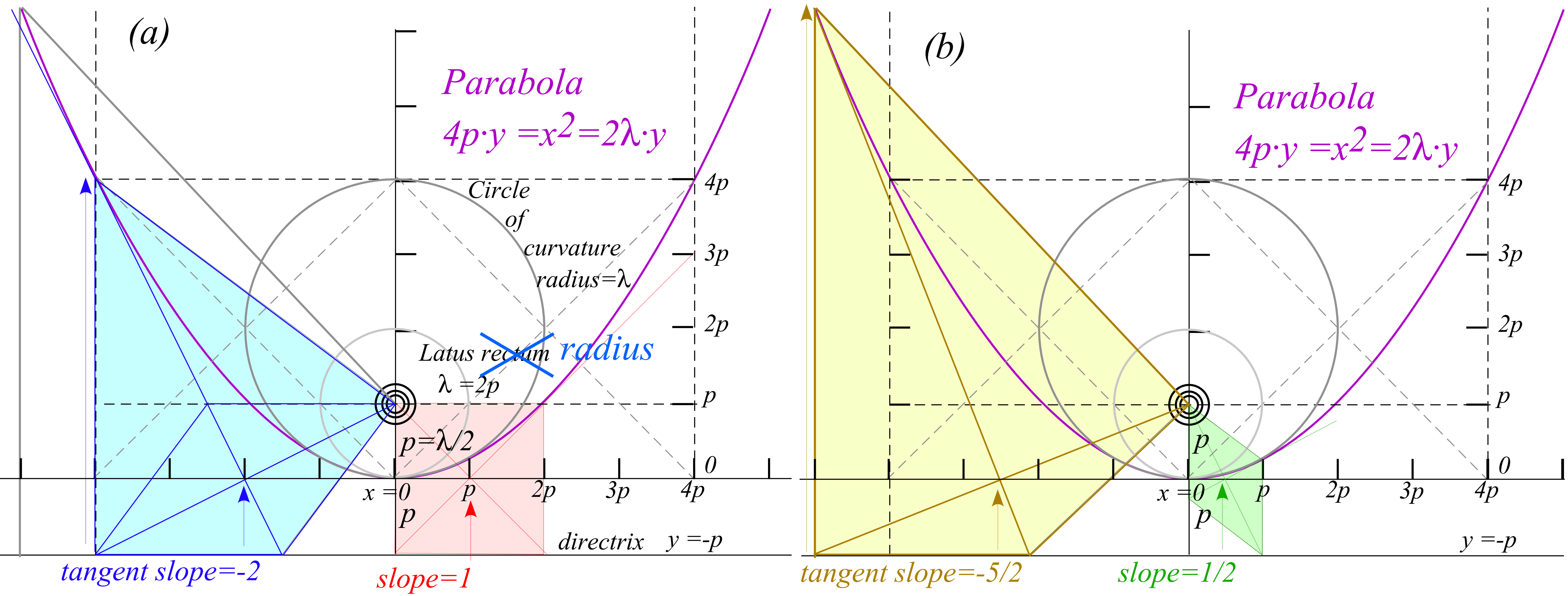
...conventional parabolic geometry...carried to extremes...

Recall Lecture 6 p.26 and p. 48-49 for kite geometry and application



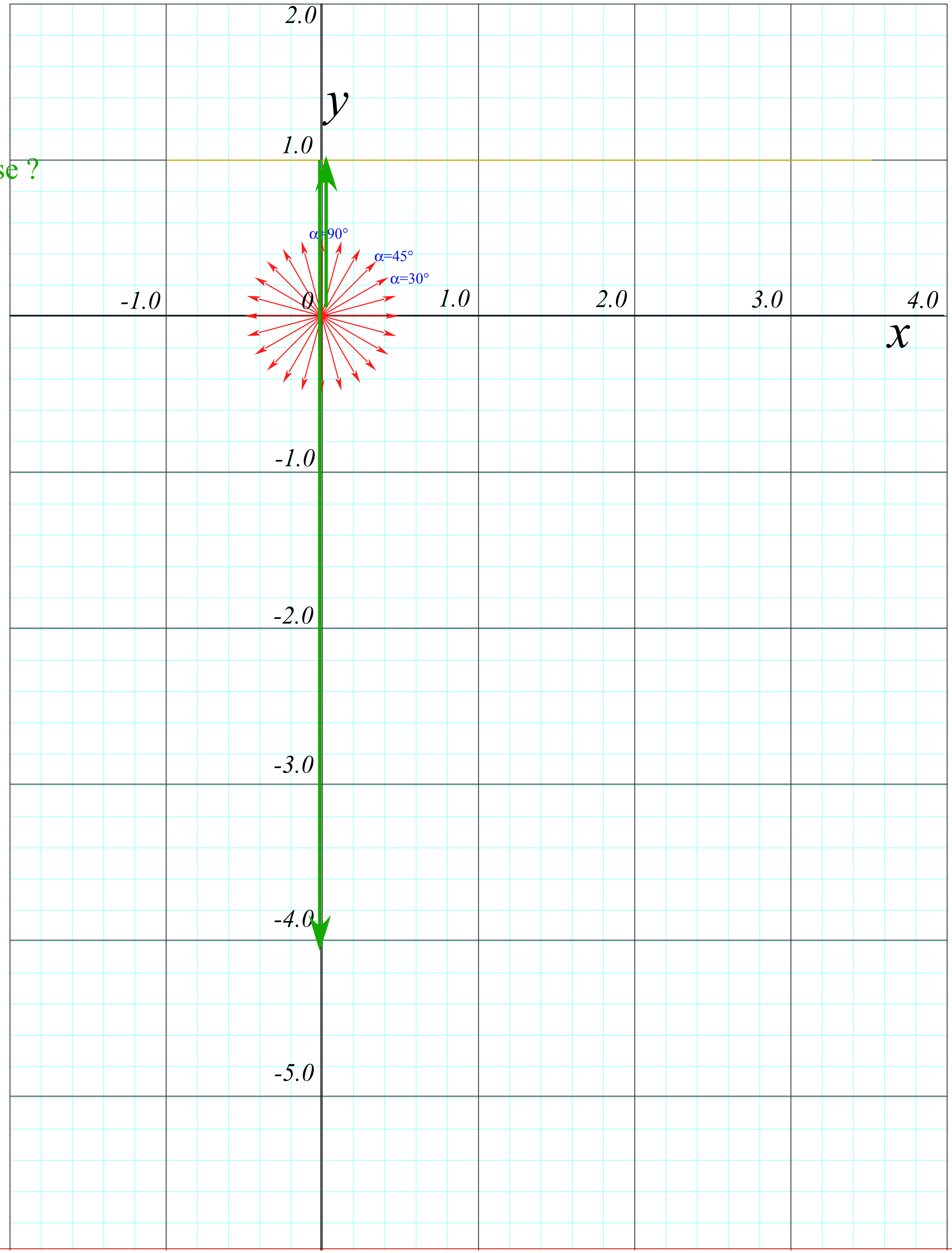
Unit 1
Fig. 9.4

Say α=90° path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

Q2. ...where is the blast wave?

Q3. ...how high can α=45° path path rise ?

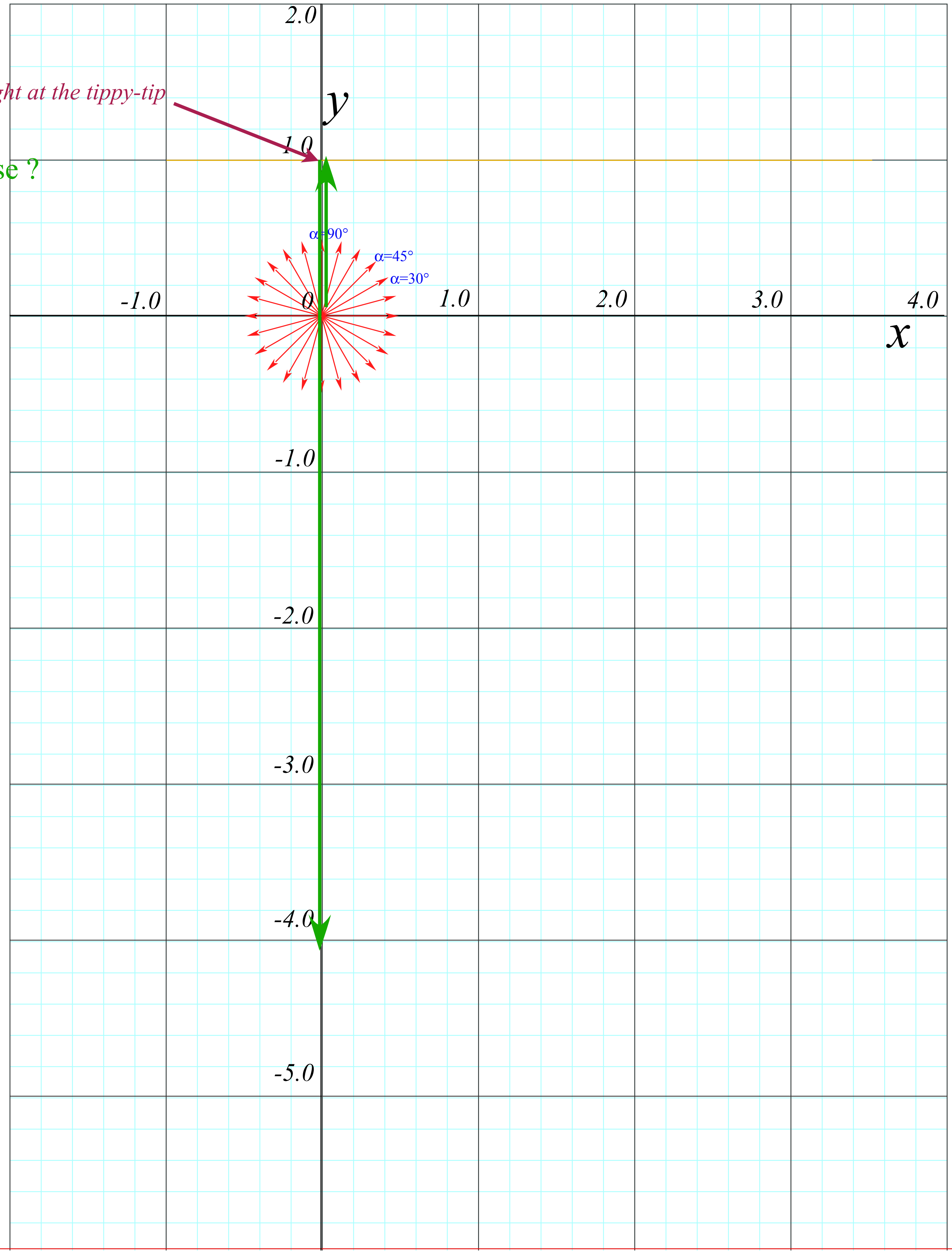


Say $\alpha=90°$ path rises to *1.0* then drops. When at $y=1.0$...

Q1. ...where is its focus? → *Right at the tippy-tip*

Q2. ...where is the blast wave?

Q3. ...how high can $\alpha=45°$ path path rise ?



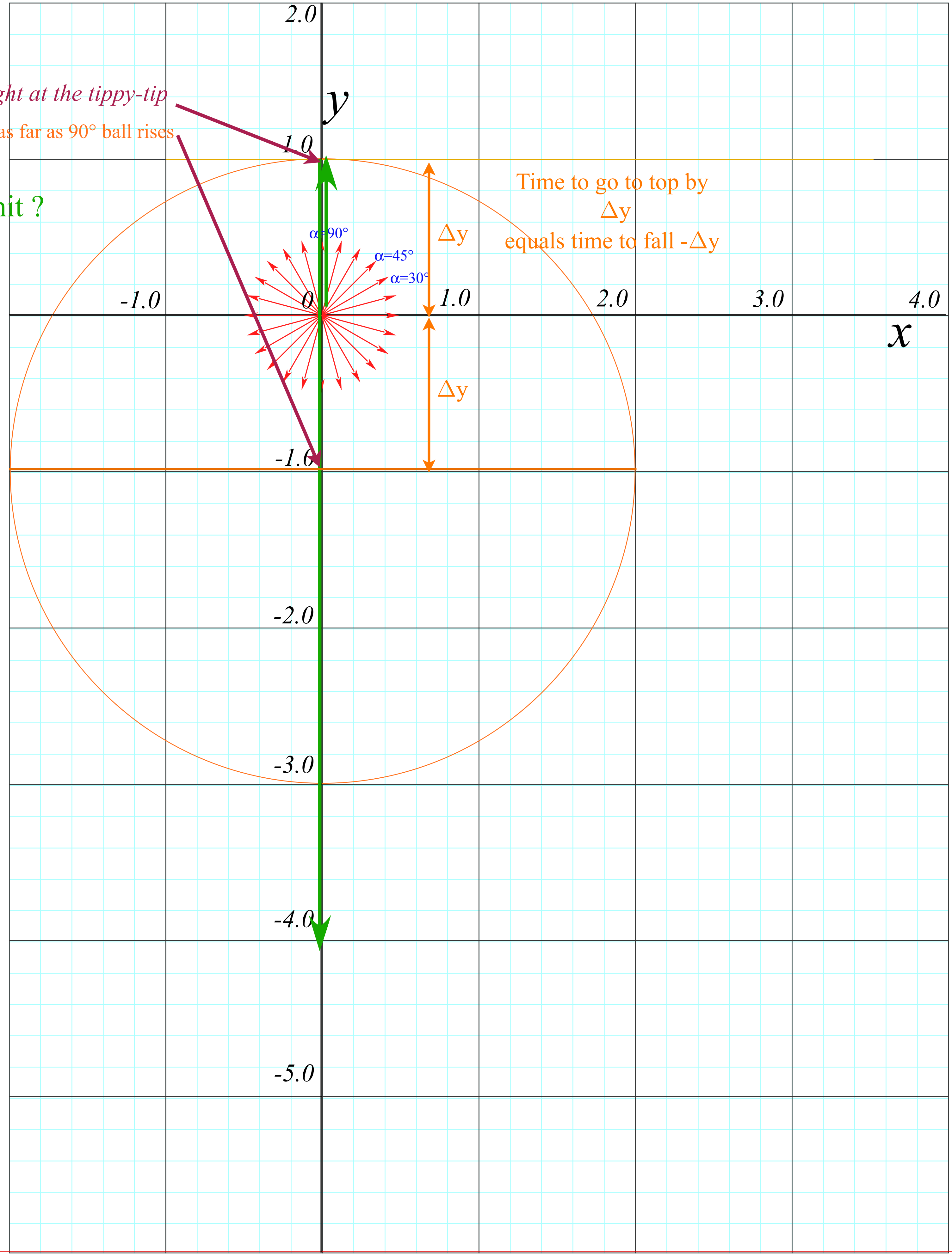
Say $\alpha=90°$ path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

Q2. ...where is the blast wave?

Q3. How high can $\alpha=45°$ path rise ?

Q4. Where on *x-axis* does $\alpha=45°$ path hit ?



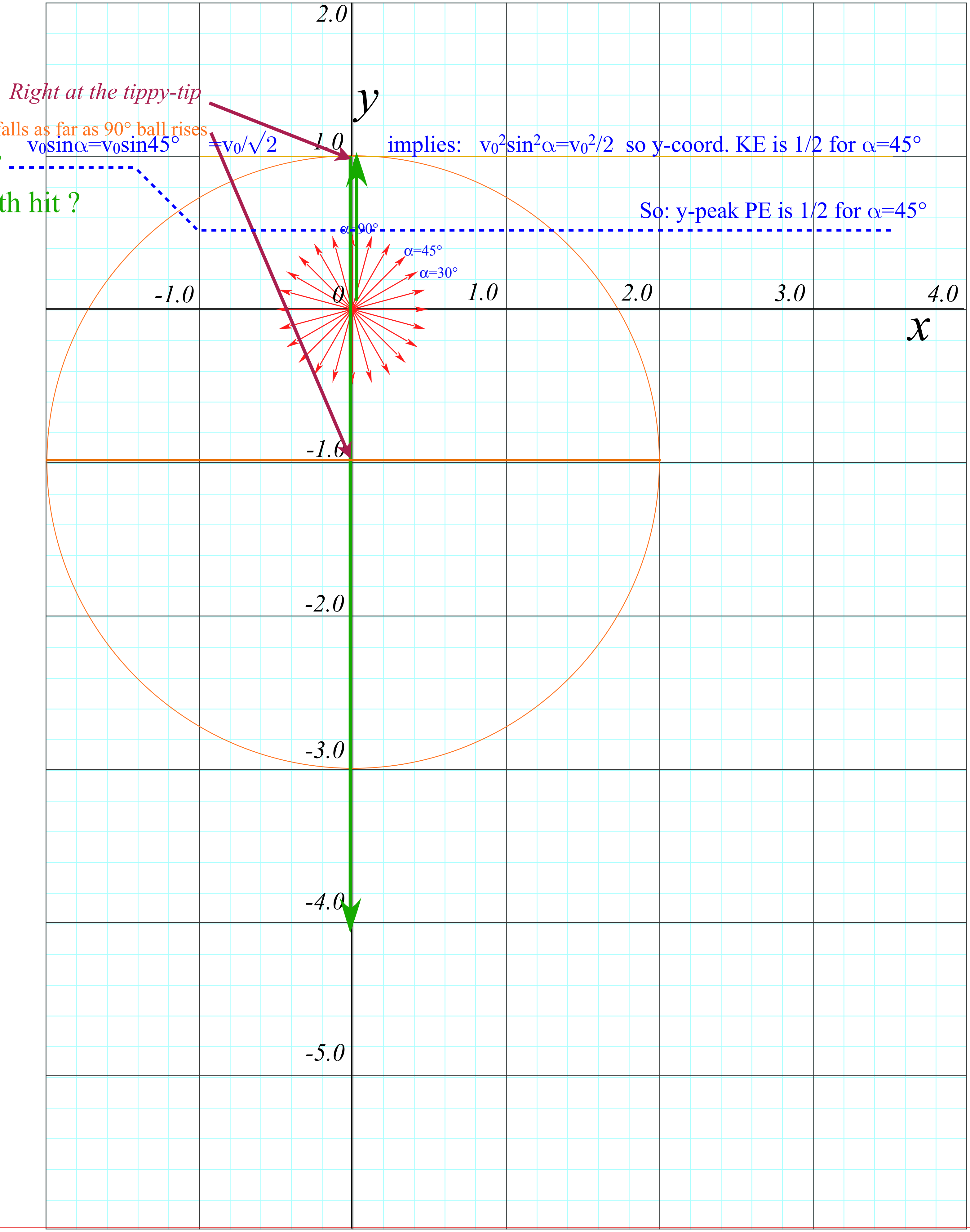
Say α=90° path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as 90° ball rises

Q3. How high can α=45° path rise ?

Q4. Where on *x-axis* does α=45° path hit ?



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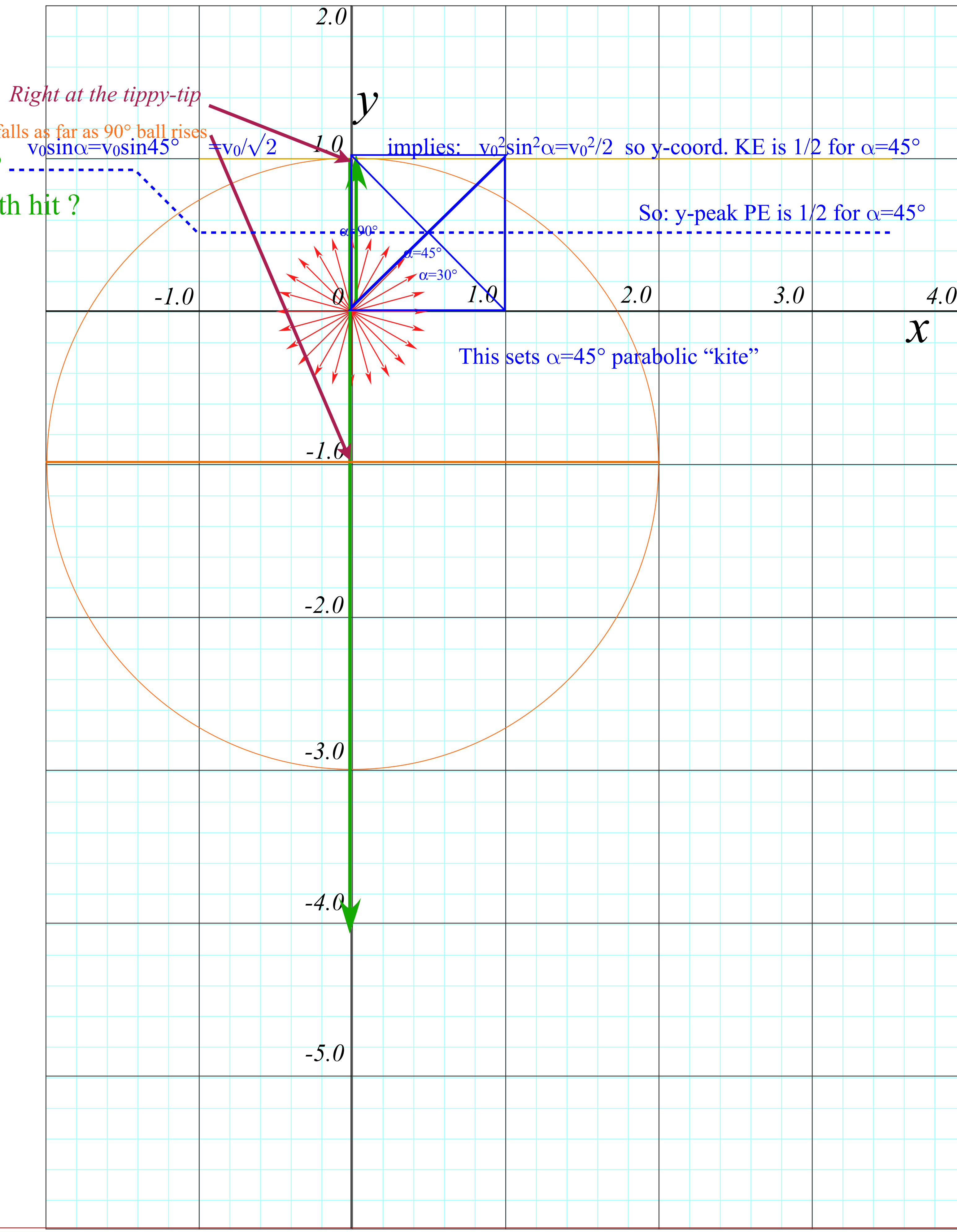
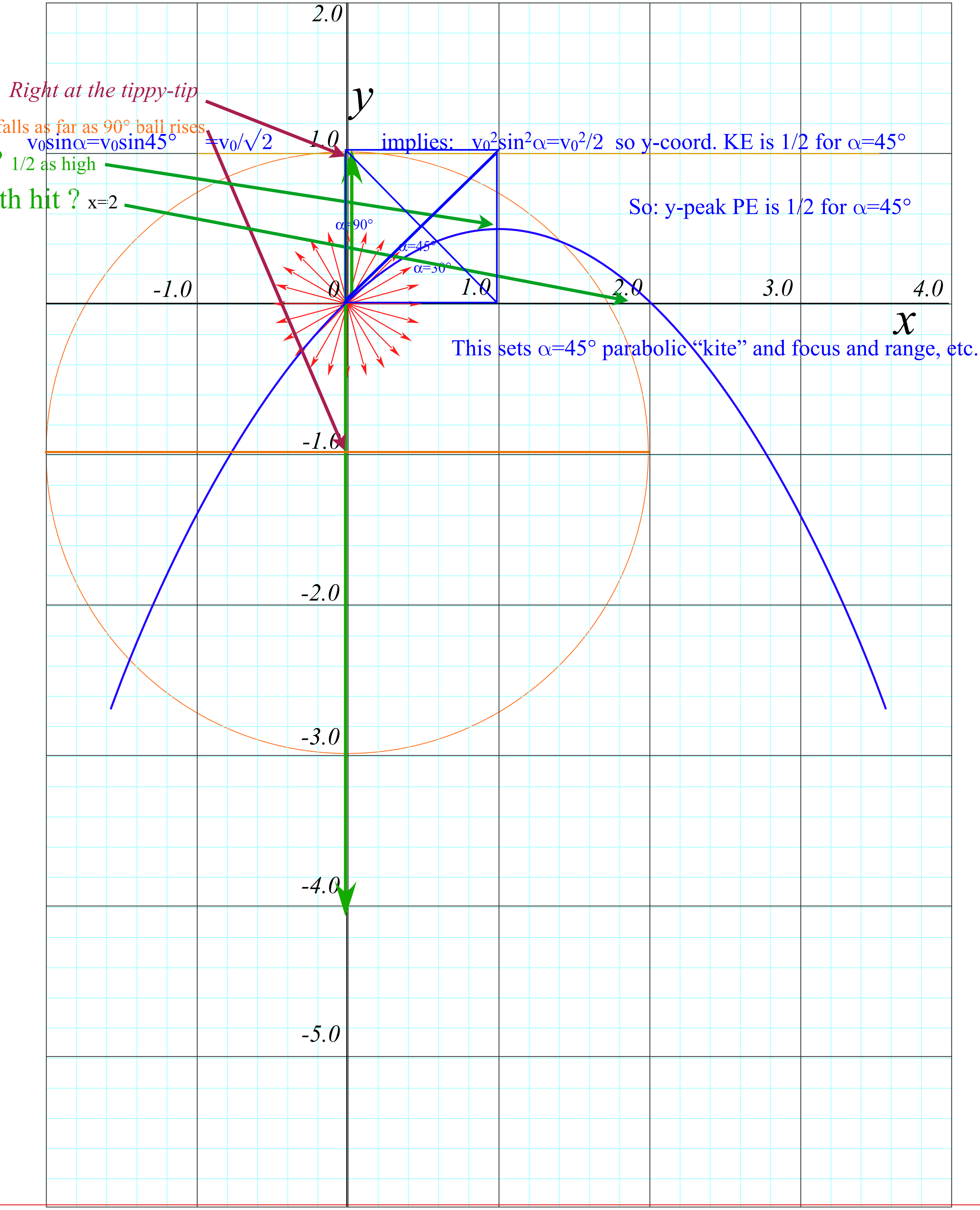


Say α=90° path rises to *1.0*
then drops. When at *y=1.0*...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can α=45° path rise ?
Q4. Where on *x-axis* does α=45° path hit ?
Q5. Where is blast wave then?
Q6 Where is α=45° path focus?
Q7 Guess for all-path envelope?
and its focus? directrix?



Say $\alpha=90°$ path rises to *1.0* then drops. When at $y=1.0$...

Q1. ...where is its focus?

Q2. ...where is the blast wave? center falls as far as 90° ball rises

Q3. How high can $\alpha=45°$ path rise ? 1/2 as high

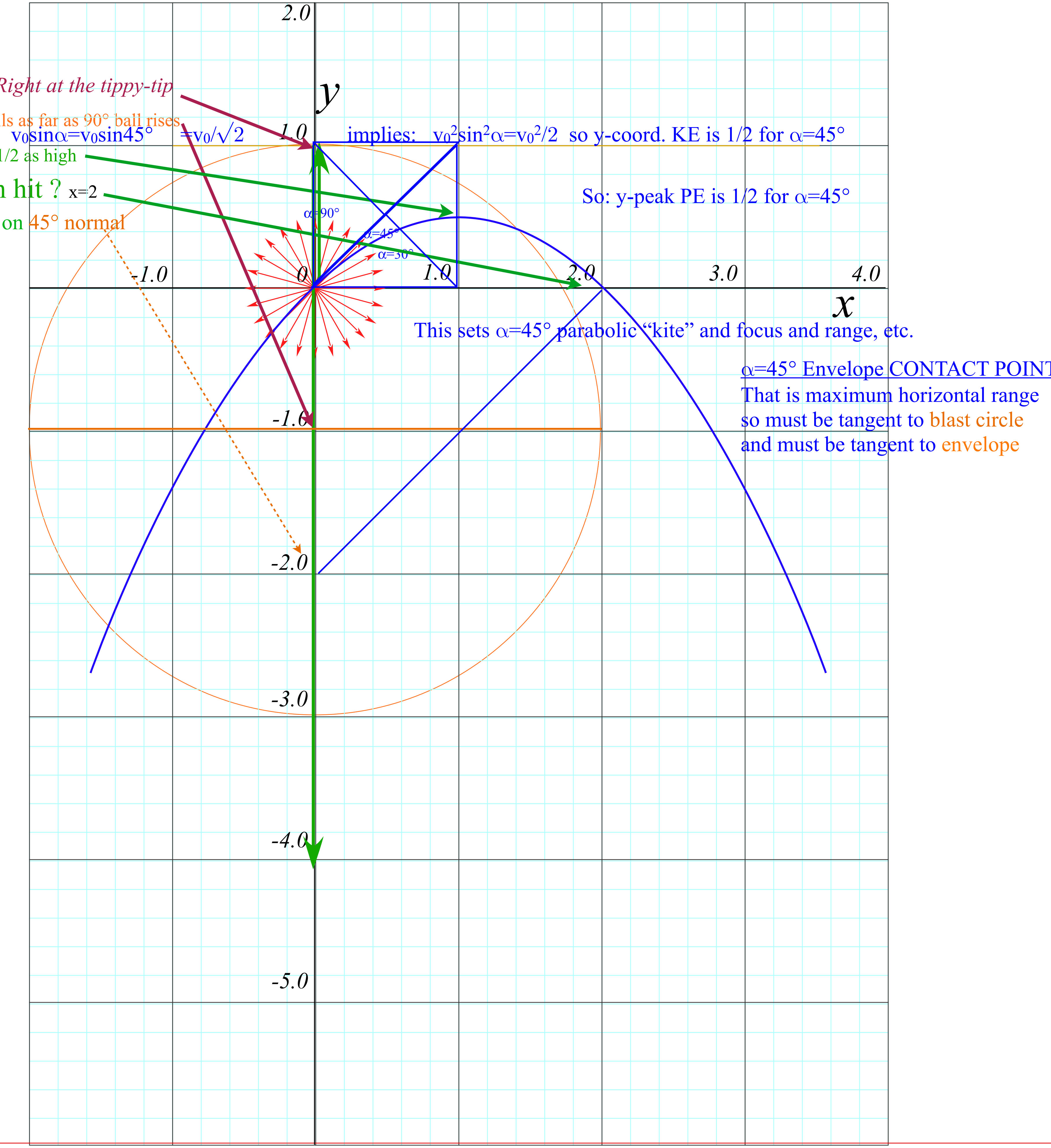
Q4. Where on *x-axis* does $\alpha=45°$ path hit ? x=2

Q5. Where is blast wave then? centered on 45° normal

Q6 Where is $\alpha=45°$ path focus?

Q7 Guess for all-path envelope?

and its focus? directrix?



Say $\alpha=90°$ path rises to *1.0*
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Q4. Where on *x-axis* does $\alpha=45°$ path hit ? x=2

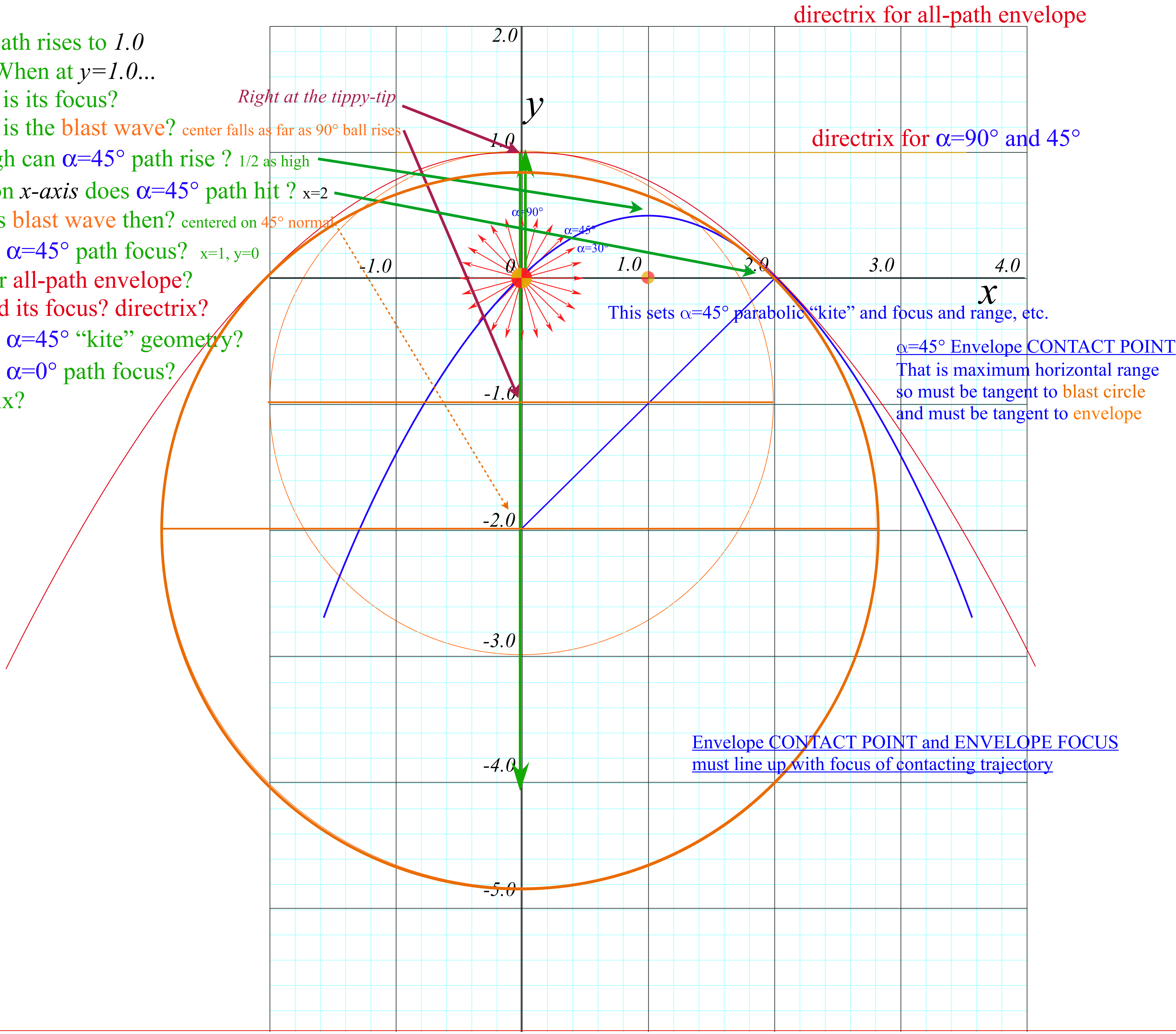
Q5. Where is blast wave then? centered on 45° normal

Q6 Where is $\alpha=45°$ path focus? x=1, y=0

Q7 Guess for all-path envelope?
and its focus? directrix?

Q7 Where is $\alpha=45°$ "kite" geometry?

Q8 Where is $\alpha=0°$ path focus?
directrix?



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Q4. Where on *x-axis* does α=45° path hit ? x=2

Q5. Where is blast wave then? centered on 45° normal

Q6 Where is α=45° path focus? x=1, y=0

Q7 Guess for all-path envelope?

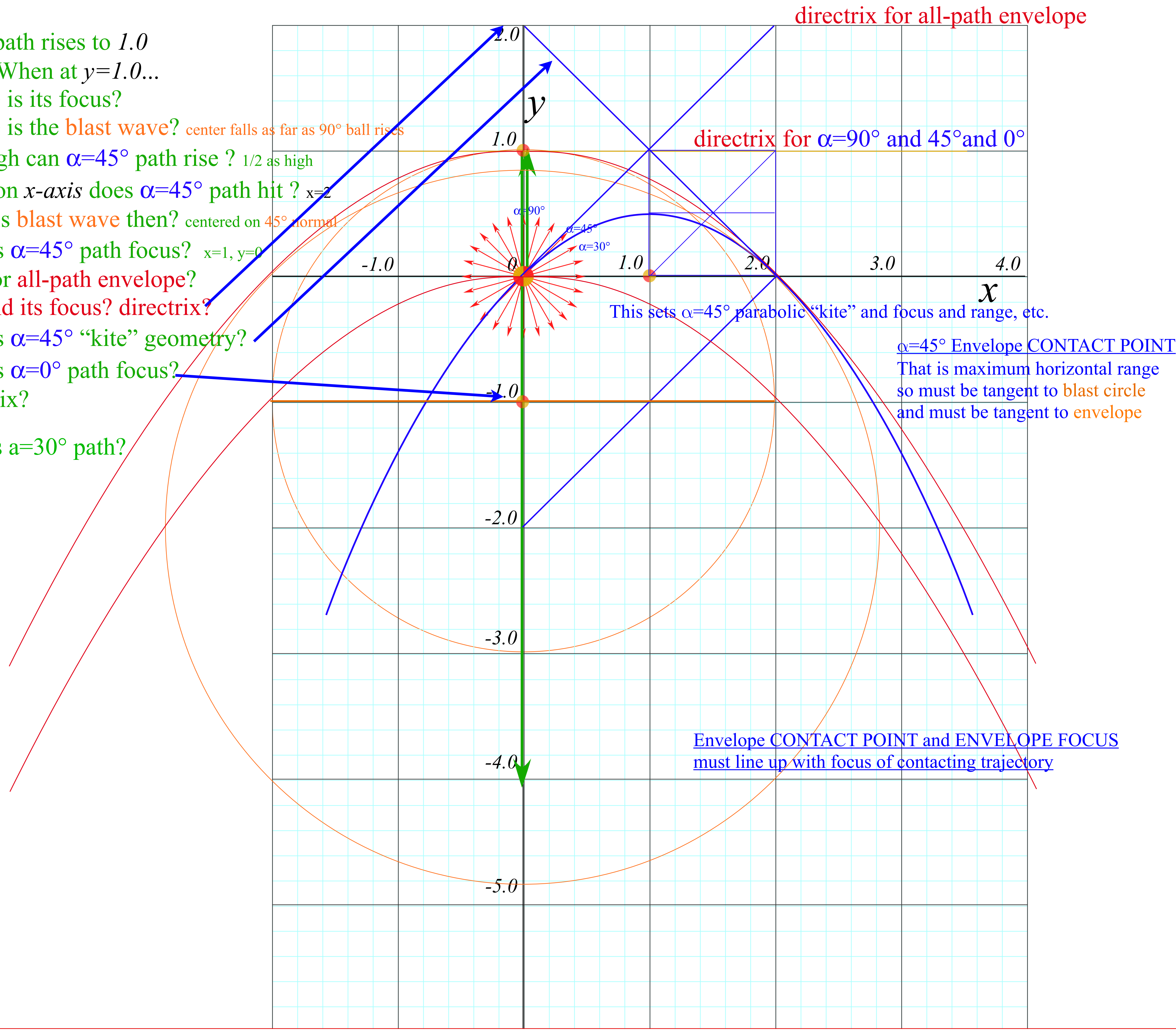
and its focus? directrix?

Q7 Where is α=45° "kite" geometry?

Q8 Where is α=0° path focus?

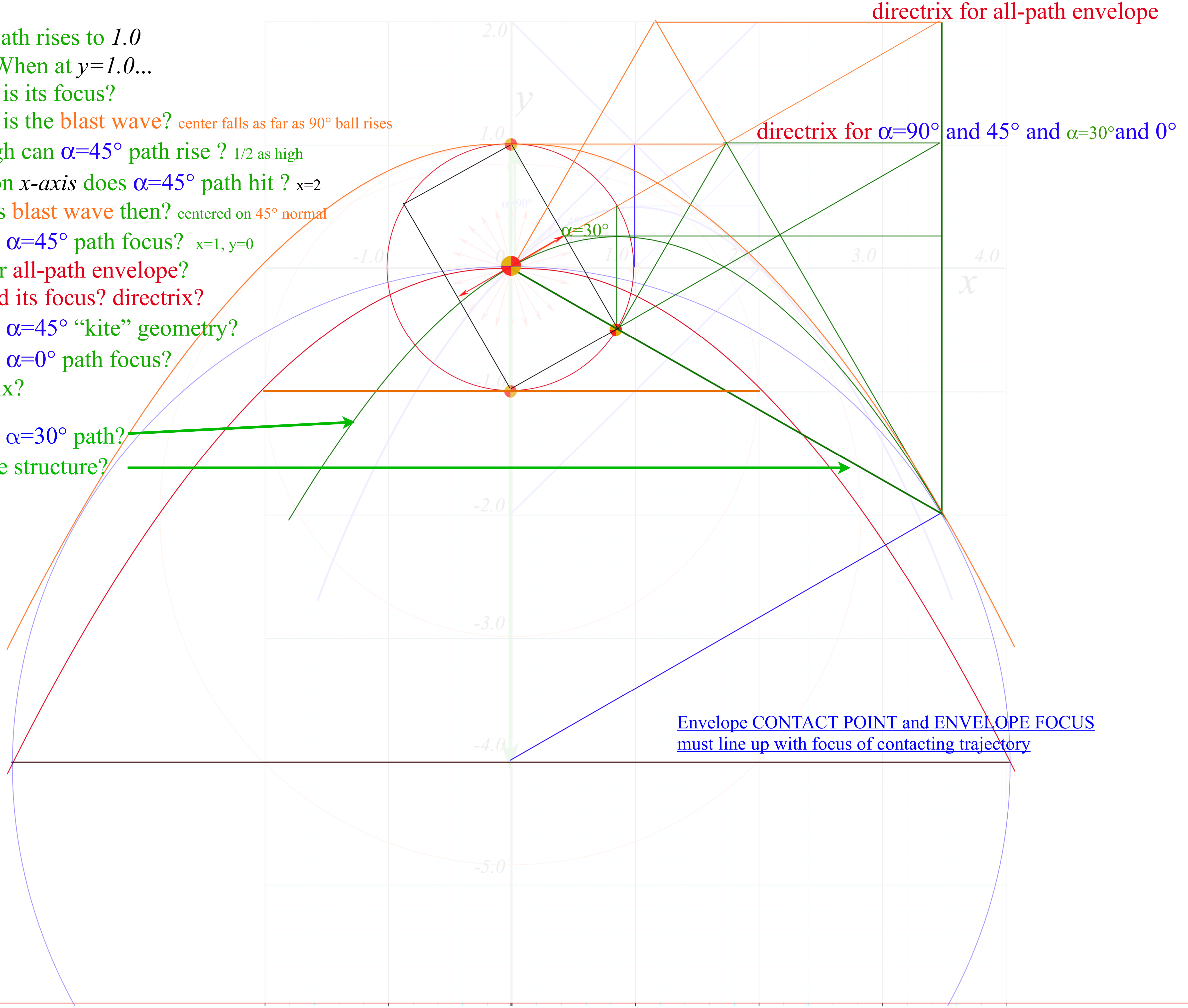
directrix?

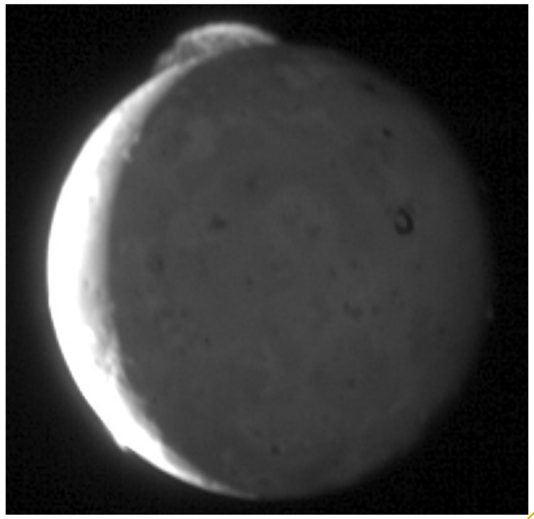
Where is a=30° path?

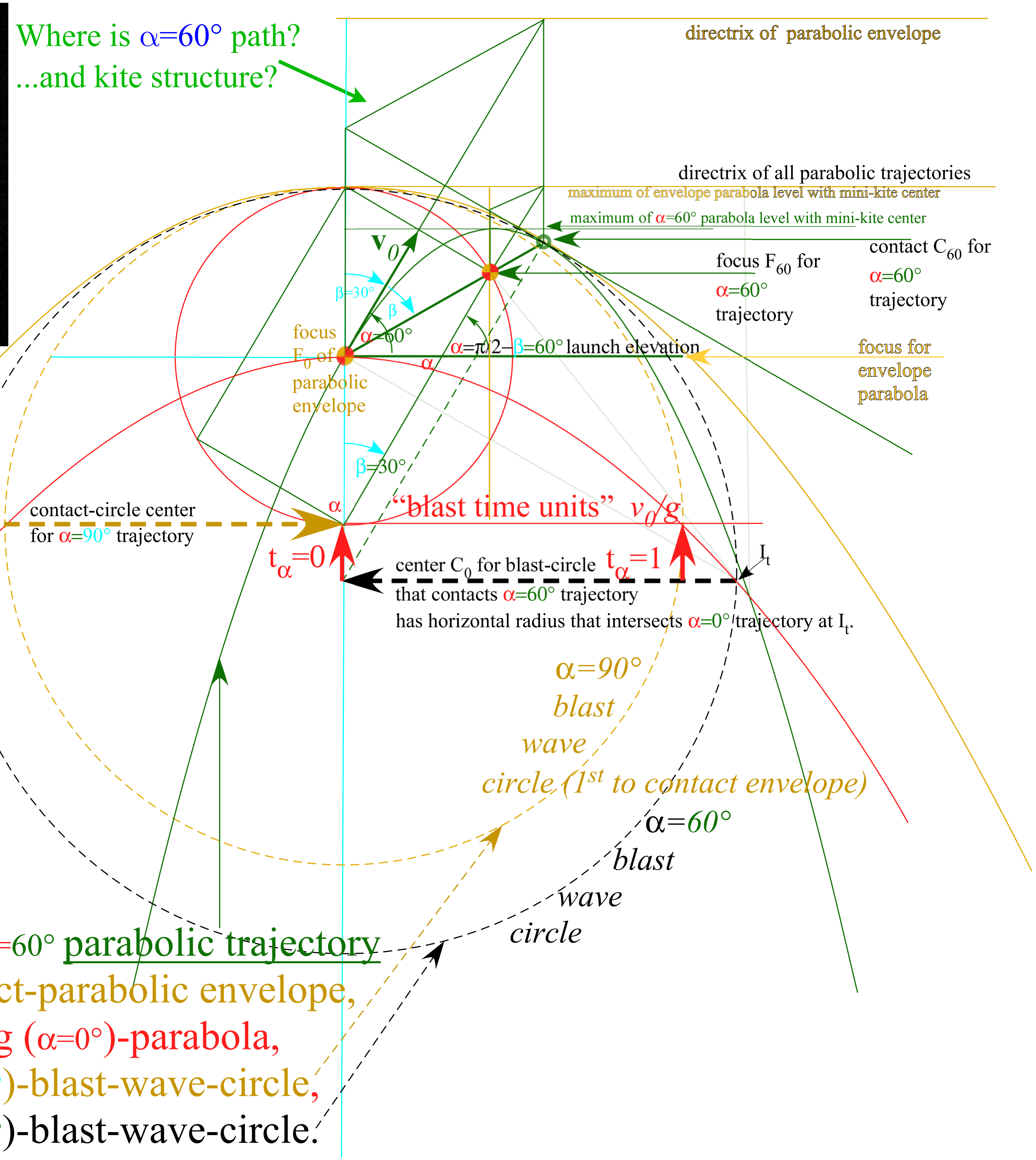


Say α=90° path rises to *1.0*
then drops. When at *y=1.0*...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Guess for all-path envelope?
and its focus? directrix?
Q7 Where is α=45° "kite" geometry?
Q8 Where is α=0° path focus?
directrix?

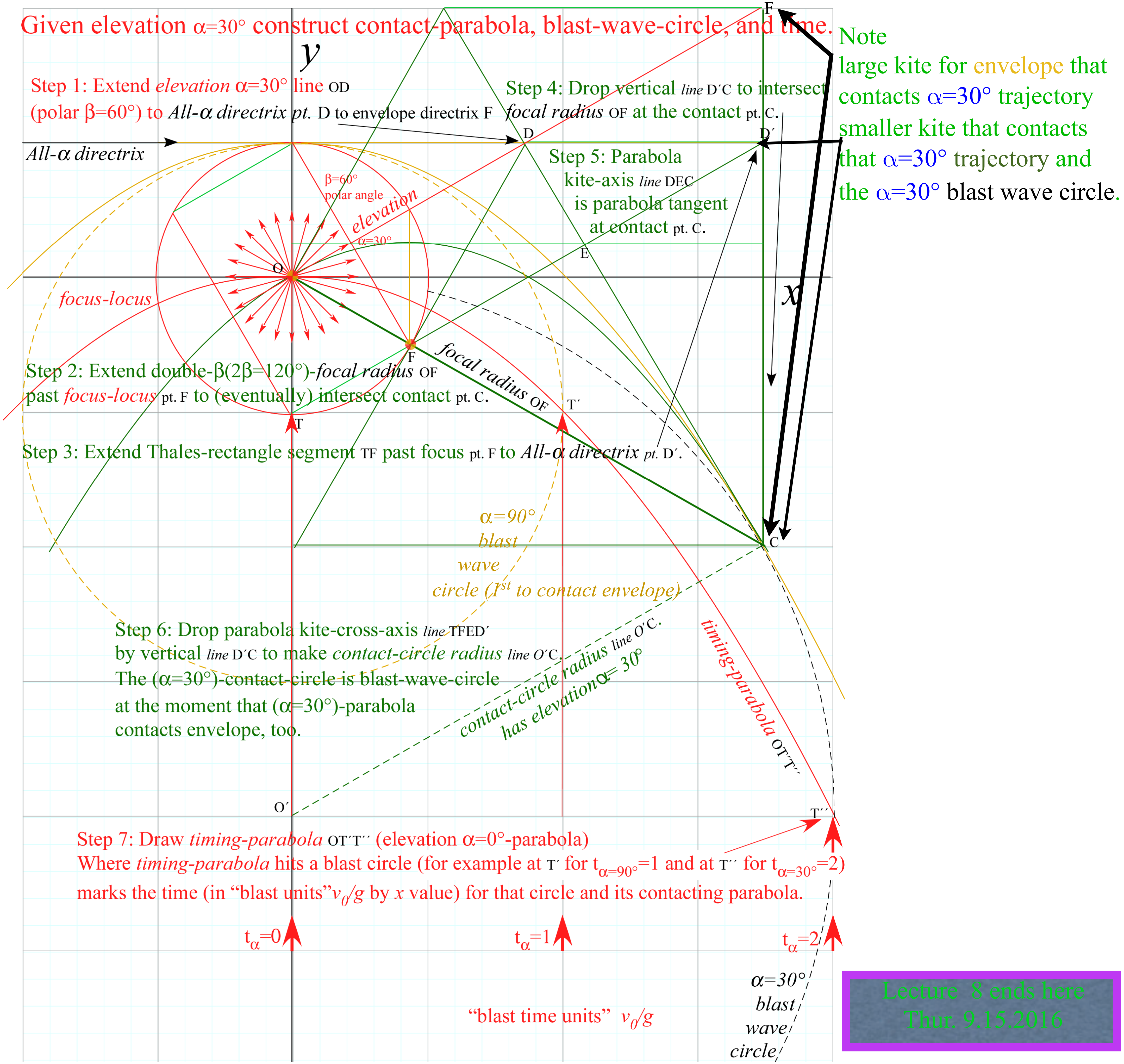
Where is α=30° path?
...and kite structure?







Where is α=60° path?
...and kite structure?
directrix of parabolic envelope
directrix of all parabolic trajectories
maximum of envelope parabola level with mini-kite center
maximum of α=60° parabola level with mini-kite center
contact C60 for α=60° trajectory
focus F60 for α=60° trajectory
v0
β=30°
β
α=60°
focus F0 of parabolic envelope
α=π/2−β=60° launch elevation
focus for envelope parabola
α
β=30°
contact-circle center for α=90° trajectory
"blast time units" v0/g
tα=0
tα=1
It
center C0 for blast-circle that contacts α=60° trajectory has horizontal radius that intersects α=0° trajectory at It.
α=90° blast wave circle (1st to contact envelope)
α=60° blast wave circle
For α=60° parabolic trajectory
contact-parabolic envelope,
timing (α=0°)-parabola,
(α=90°)-blast-wave-circle,
(α=60°)-blast-wave-circle.


Given elevation α=30° construct contact-parabola, blast-wave-circle, and time.
Note
large kite for envelope that
contacts α=30° trajectory
smaller kite that contacts
that α=30° trajectory and
the α=30° blast wave circle.
y
Step 1: Extend elevation α=30° line OD
(polar β=60°) to All-α directrix pt. D to envelope directrix F
Step 4: Drop vertical line D′C to intersect
focal radius OF at the contact pt. C.
All-α directrix
D
D′
F
Step 5: Parabola
kite-axis line DEC
is parabola tangent
at contact pt. C.
β=60°
polar angle
elevation
α=30°
E
O
x
focus-locus
F
focal radius OF
Step 2: Extend double-β(2β=120°)-focal radius OF
past focus-locus pt. F to (eventually) intersect contact pt. C.
T
T′
Step 3: Extend Thales-rectangle segment TF past focus pt. F to All-α directrix pt. D′.
α=90°
blast
wave
circle (1st to contact envelope)
C
Step 6: Drop parabola kite-cross-axis line TFED′
by vertical line D′C to make contact-circle radius line O′C.
The (α=30°)-contact-circle is blast-wave-circle
at the moment that (α=30°)-parabola
contacts envelope, too.
contact-circle radius line O′C.
has elevation α= 30°
timing-parabola OT′T′′
O′
T′′
Step 7: Draw timing-parabola OT′T′′ (elevation α=0°-parabola)
Where timing-parabola hits a blast circle (for example at T′ for tα=90°=1 and at T′′ for tα=30°=2)
marks the time (in “blast units” v0/g by x value) for that circle and its contacting parabola.
tα=0
tα=1
tα=2
α=30°
blast
wave
circle
“blast time units” v0/g
Lecture 8 ends here
Thur. 9.15.2016