Lecture 9
Tue. 9.18.2012

Geometry of Dual Quadratic Forms: Lagrange vs Hamilton
(Ch. 11 and Ch. 12 of Unit 1)

Introduction to dual matrix operator geometry

- *Review of dual IHO elliptic orbits (Lecture 7-8)*
  - *Construction by Phasor-pair projection*
  - *Construction by Kepler anomaly projection*
- *Operator geometric sequences and eigenvectors*
- *Rescaled description of matrix operator geometry*
  - *Vector calculus of tensor operation*

Introduction to Lagrangian-Hamiltonian duality

- *Review of partial differential relations*
  - *Chain rule and order symmetry*
- *Duality relations of Lagrangian and Hamiltonian ellipse*
- *Introducing the 1st (partial* $\frac{\partial ?}{\partial ?}$ *) differential equations of mechanics*

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)

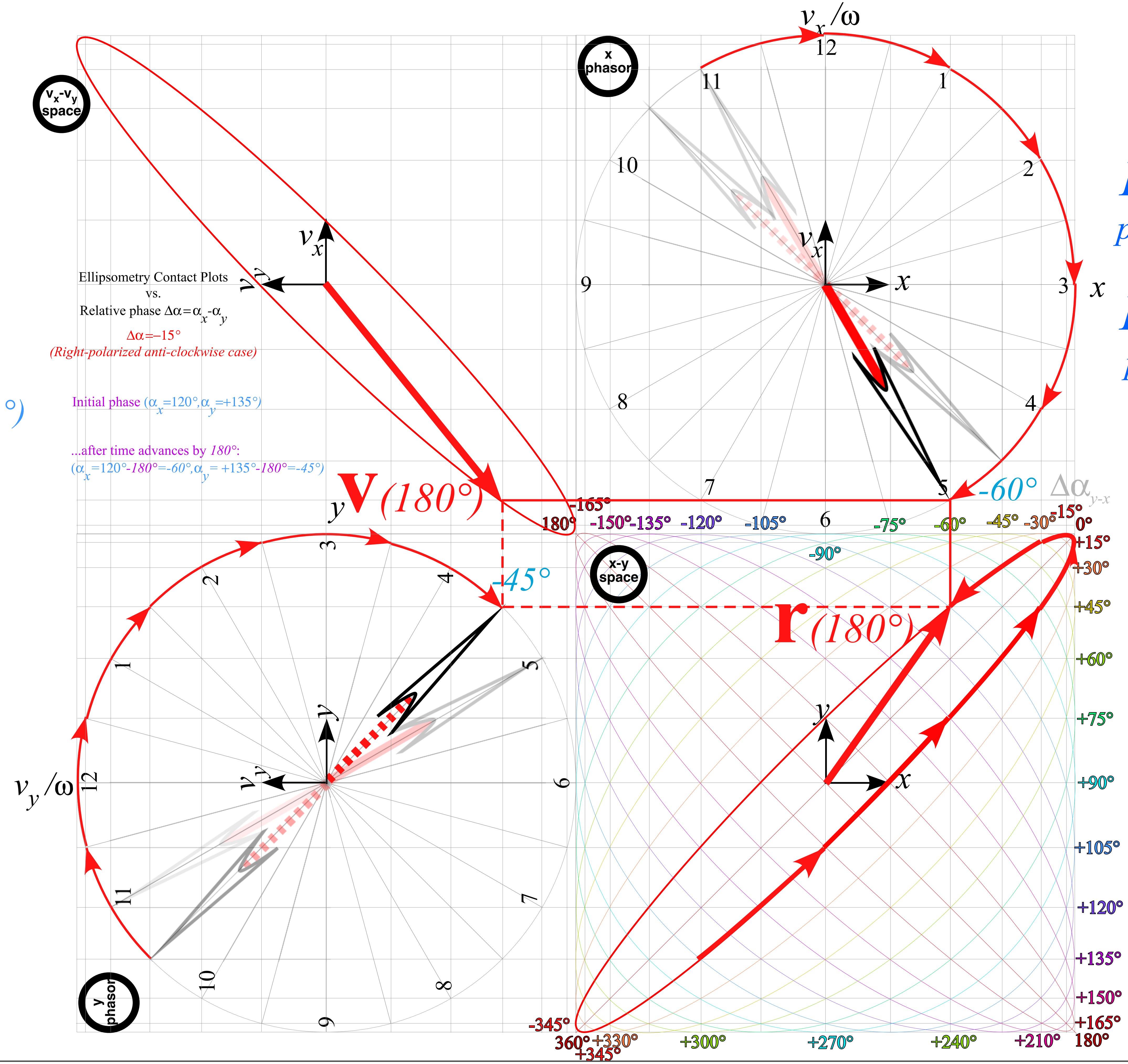
→ *Construction by Phasor-pair projection*

Construction by Kepler anomaly projection

Operator geometric sequences and eigenvectors

Rescaled description of matrix operator geometry

Vector calculus of tensor operation


Ellipsometry Contact Plots
vs.
Relative phase $\Delta\alpha=\alpha_x-\alpha_y$

$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)

Initial phase ($\alpha_x=120°$, $\alpha_y=+135°$)

...after time advances by *180°*:
($\alpha_x=120°$-*180°*=-60°, $\alpha_y=+135°$-*180°*=-45°)

See

Lecture 7

pages 37 to 49

and

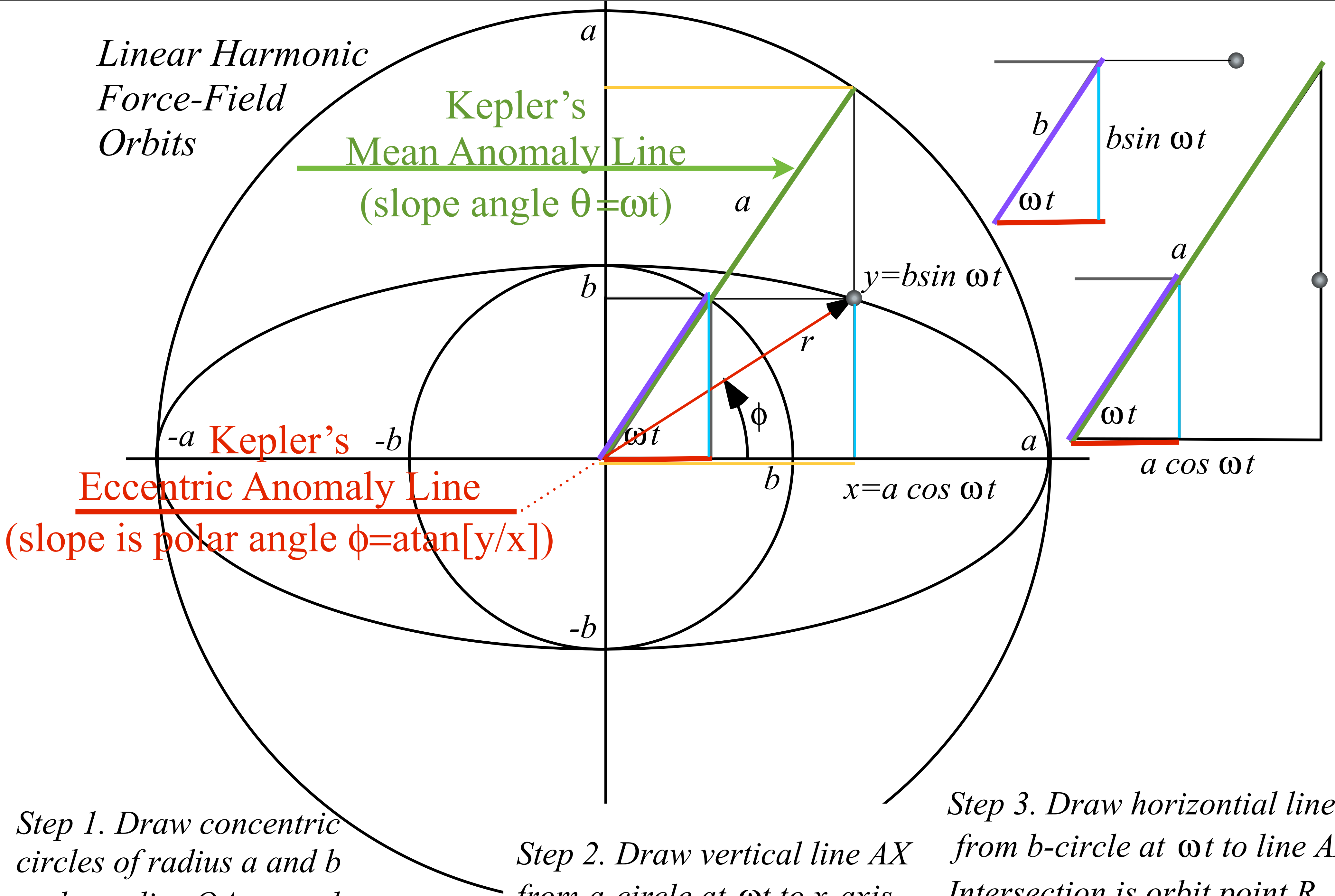
Lecture 8

pages 7 to 15

Introduction to dual matrix operator geometry

- *Review of dual IHO elliptic orbits (Lecture 7-8)*
    - *Construction by Phasor-pair projection*
    - → *Construction by Kepler anomaly projection*
- *Operator geometric sequences and eigenvectors*
- *Rescaled description of matrix operator geometry*
    - *Vector calculus of tensor operation*

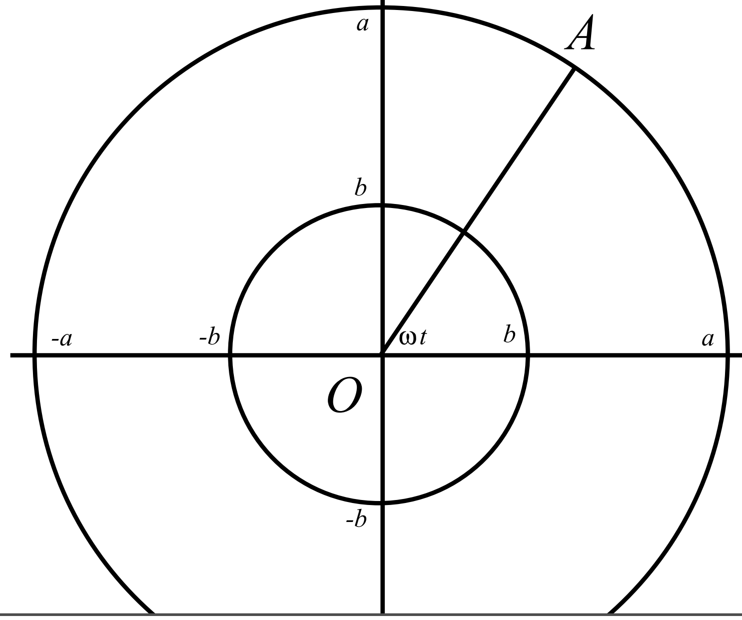
Linear Harmonic Force-Field Orbits



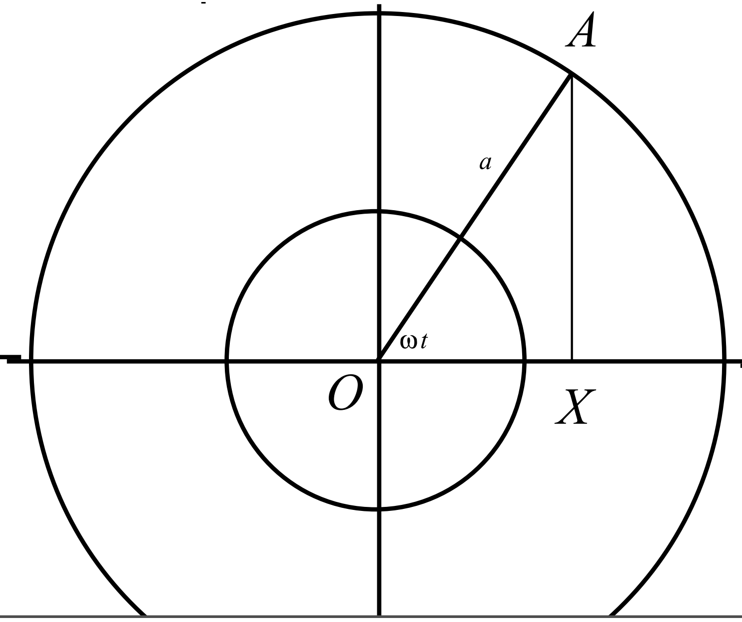
Unit 1
Fig. 11.1
(top 2/3's)

See
Lecture 8
pages 17 to 25

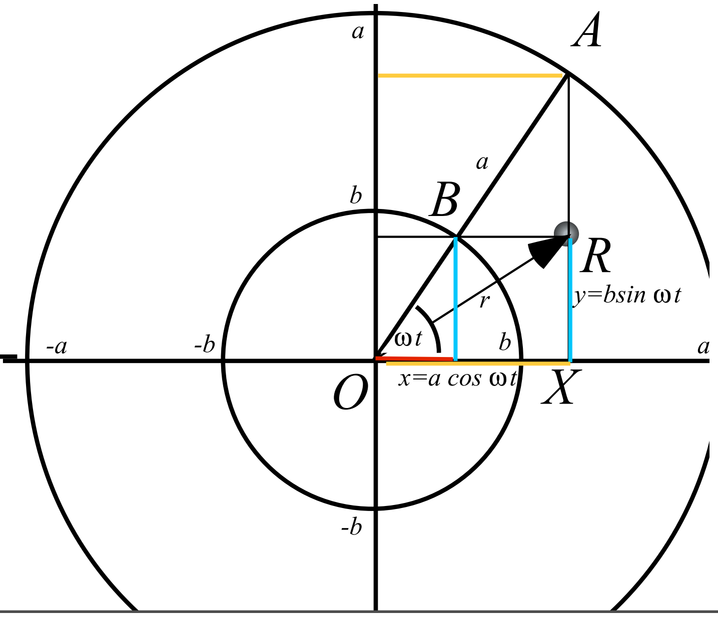
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt



Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.



Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} \qquad\qquad = 1$$

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = 1 = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} *generates an ellipse by* **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} \qquad\qquad = 1$$

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix} = 1 = \begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\overbrace{\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}\qquad\qquad =1}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=1=\overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}}\bullet\overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} *generates an ellipse by* **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

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Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\begin{pmatrix} x & y \end{pmatrix}\bullet\overbrace{\begin{pmatrix}\frac{1}{a^2} & 0\\ 0 & \frac{1}{b^2}\end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}}\bullet\begin{pmatrix}x\\ y\end{pmatrix}\overset{=1}{=1=}\overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}}\bullet\overbrace{\begin{pmatrix}\frac{x}{a^2}\\ \frac{y}{b^2}\end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

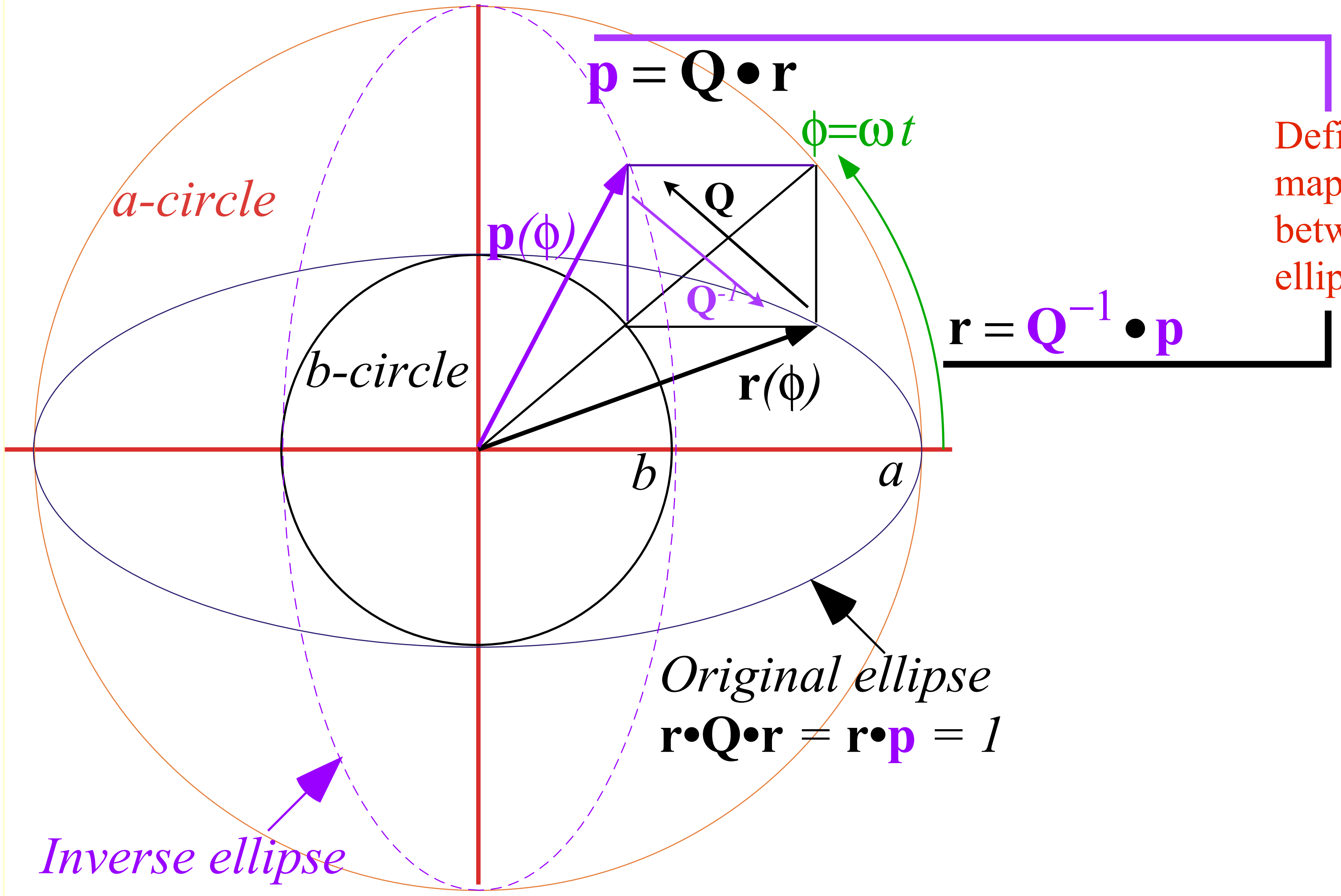
Defined mapping between ellipses

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\overbrace{\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}\overset{=1}{=1=}\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}}\bullet\overbrace{\begin{pmatrix}a^2 p_x\\ b^2 p_y\end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}}=a^2p_x^2+b^2p_y^2$$

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



$\mathbf{p} = \mathbf{Q} \bullet \mathbf{r}$

$\mathbf{r} = \mathbf{Q}^{-1} \bullet \mathbf{p}$

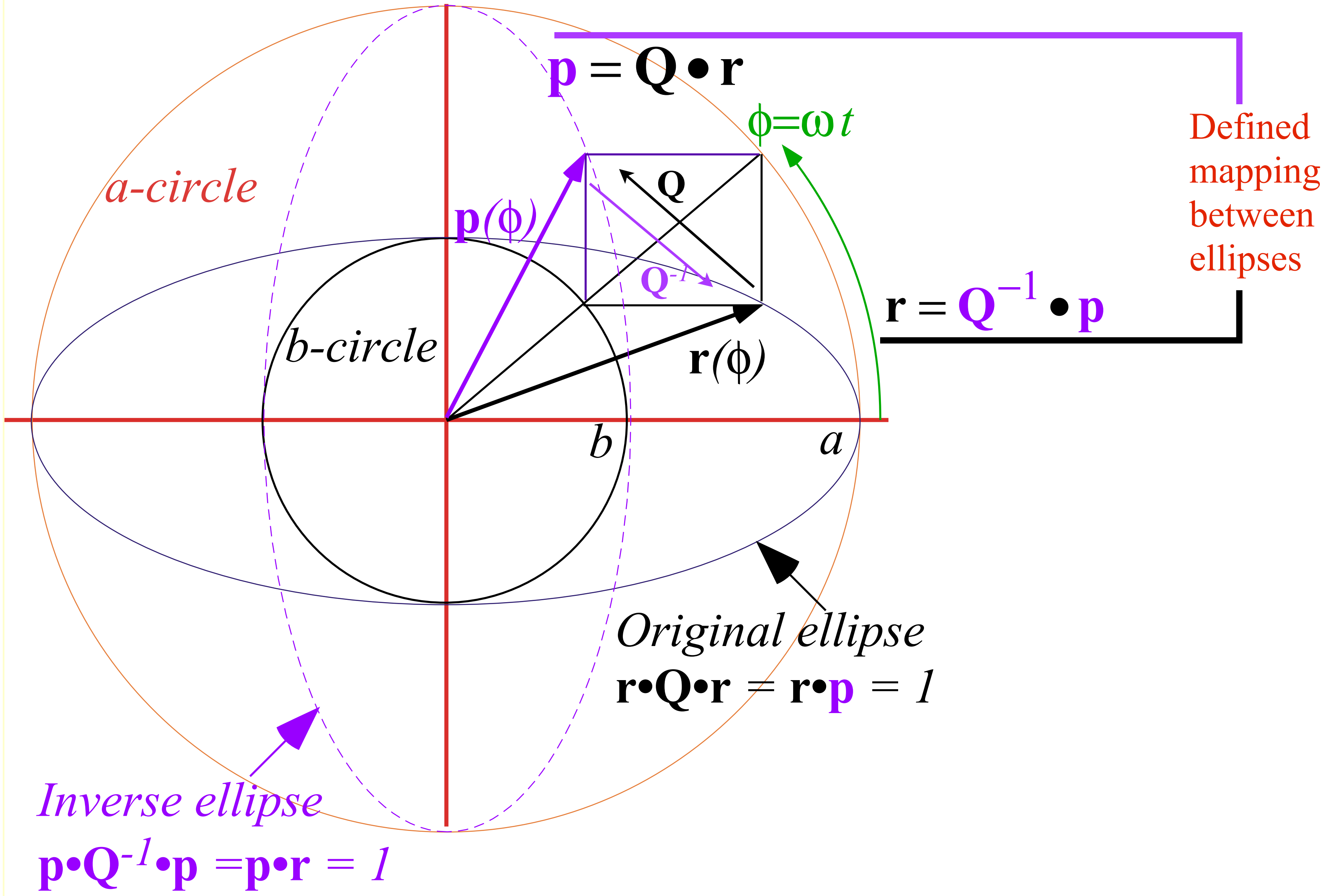
Defined
mapping
between
ellipses

Original ellipse
$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = \mathbf{r}\bullet\mathbf{p} = 1$

Inverse ellipse
$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = \mathbf{p}\bullet\mathbf{r} = 1$

(a) Quadratic form ellipse and Inverse quadratic form ellipse

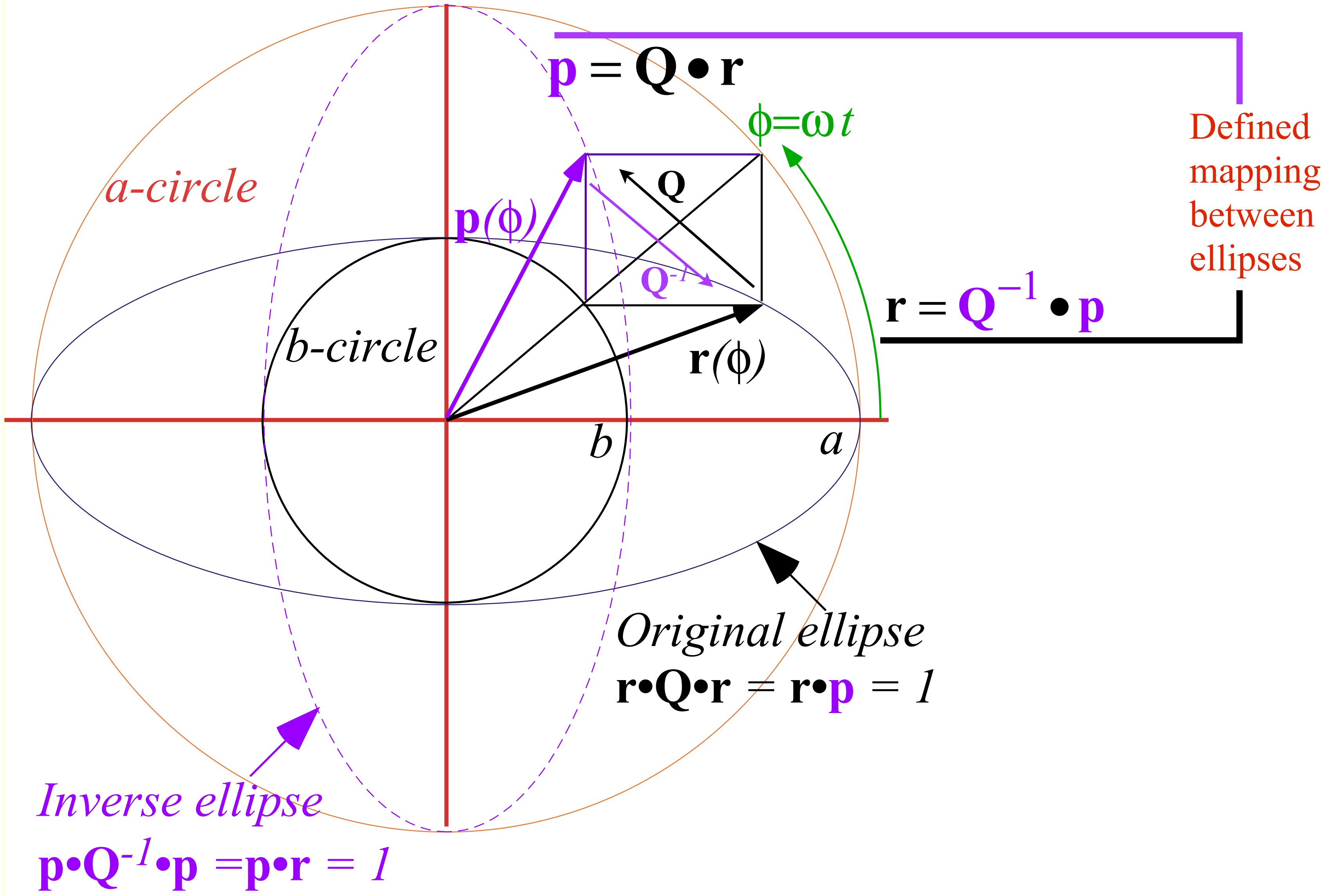
based on
Unit 1
Fig. 11.6



Quadratic form $\mathbf{r}\cdot\mathbf{Q}\cdot\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\cdot\mathbf{Q}^{-1}\cdot\mathbf{p} = 1 = \mathbf{p}\cdot\mathbf{r}$

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

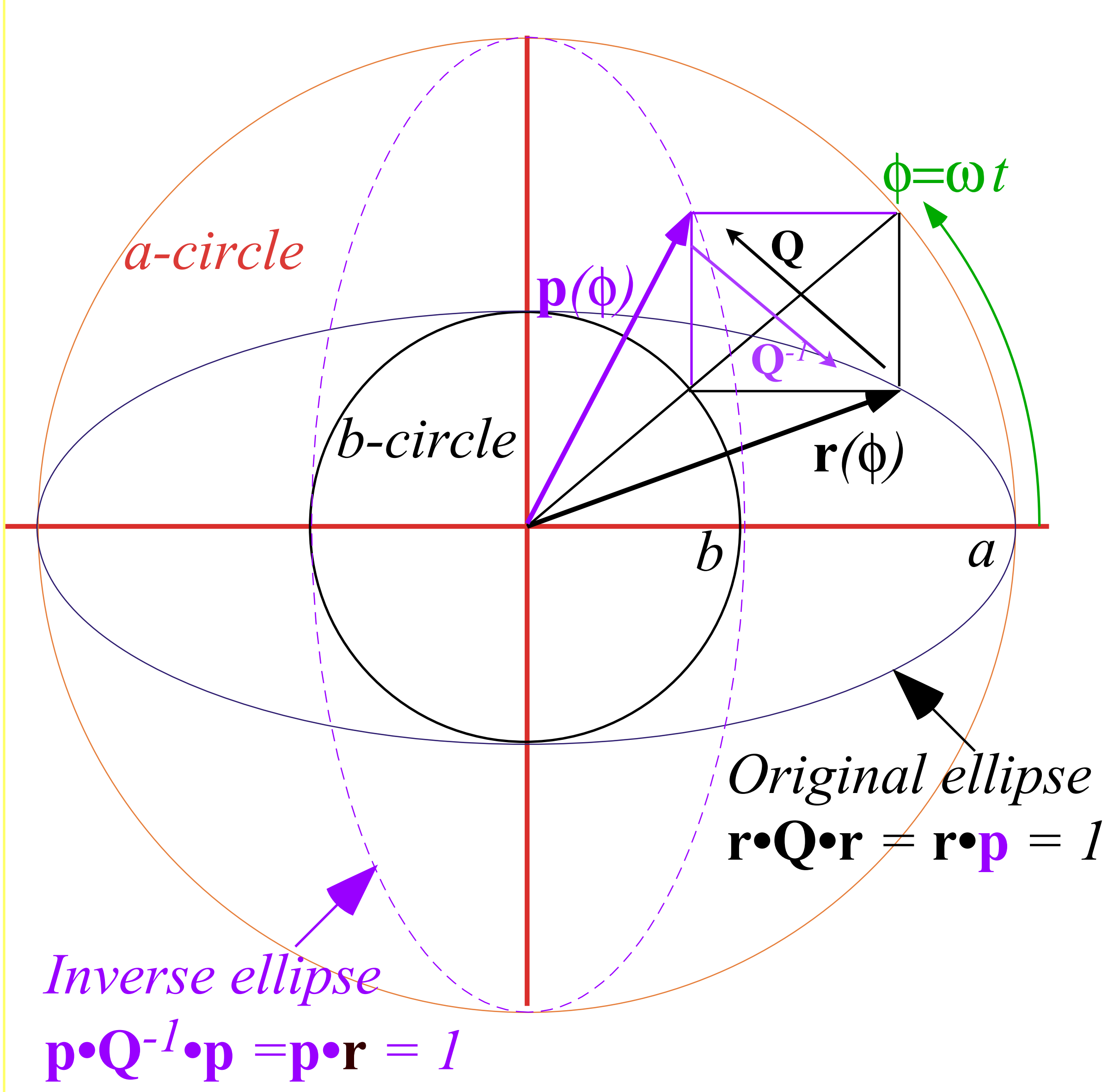
based on
Unit 1
Fig. 11.6



Quadratic form **r•Q•r** =*1 has mutual duality relations with inverse form* **p•Q**$^{-1}$**•p** =*1*= **p•r**

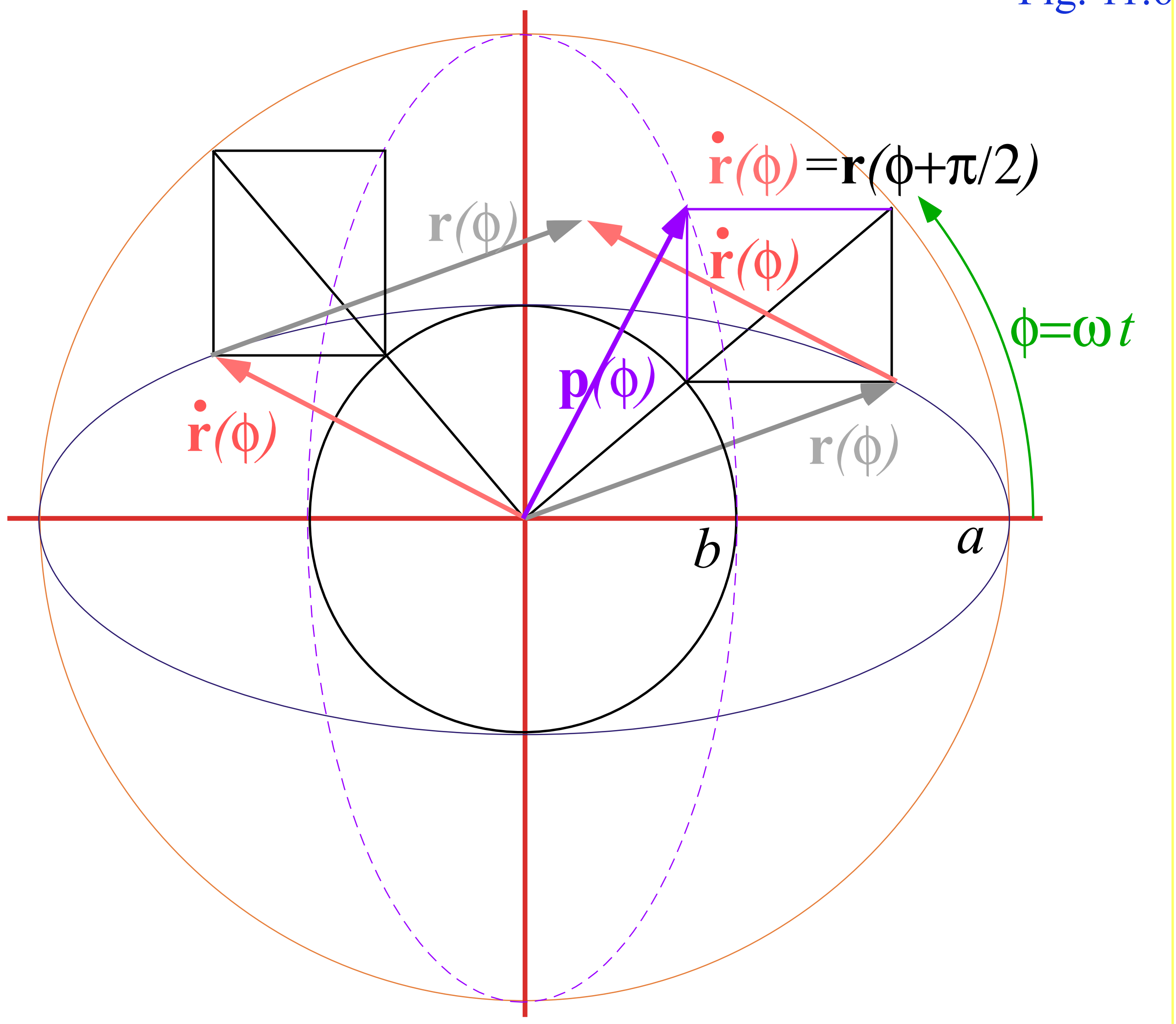
$$\mathbf{p} = \mathbf{Q} \bullet \mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where:} \quad \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi \quad = b\sin\omega t \end{matrix} \quad \text{so:} \quad \boxed{\mathbf{p} \bullet \mathbf{r} = 1}$$

(a) Quadratic form ellipse and Inverse quadratic form ellipse



(b) Ellipse tangents

based on
Unit 1
Fig. 11.6



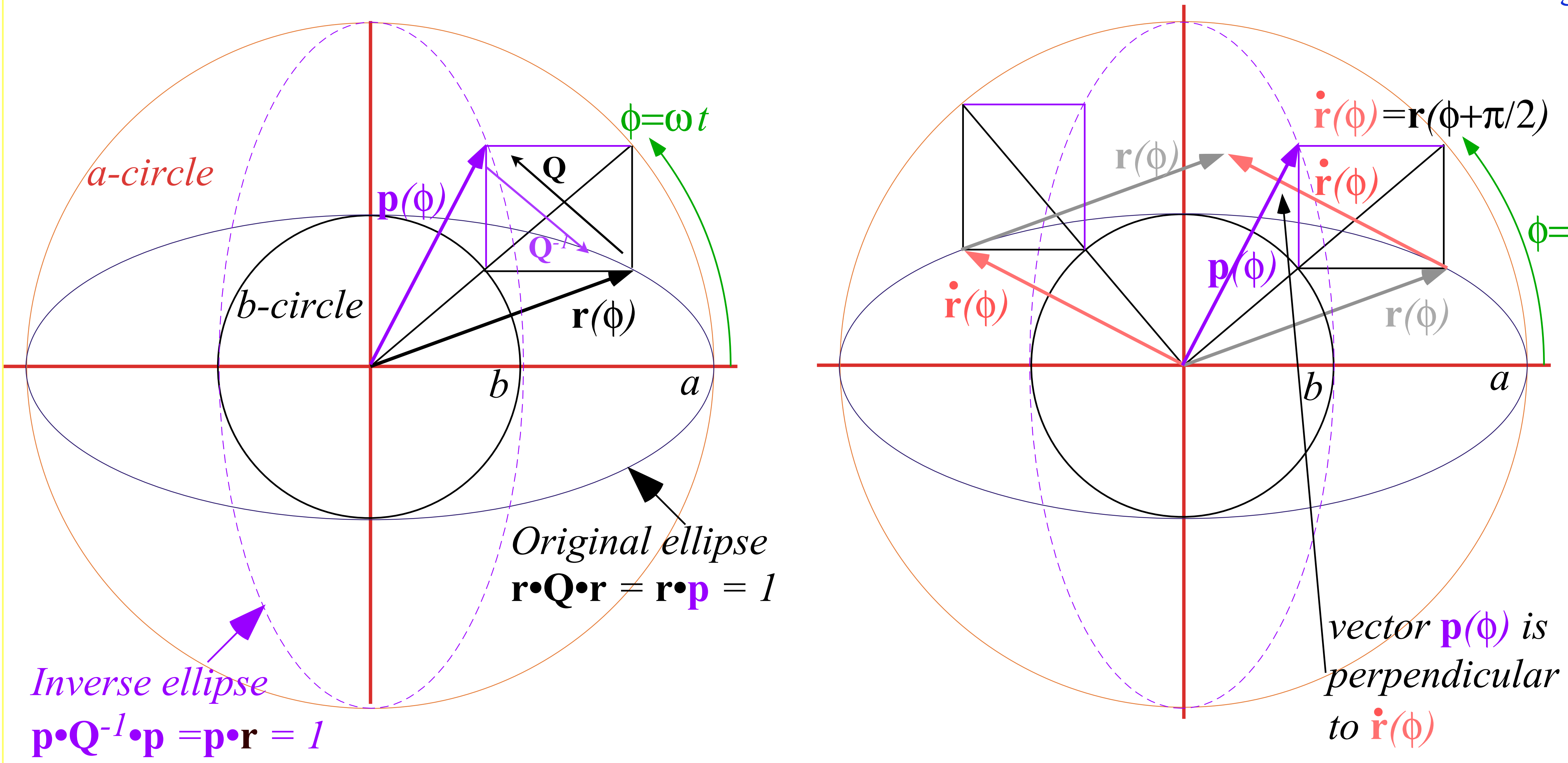
Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q**$^{-1}$**•p** *=1=* **p•r**

$$\mathbf{p}=\mathbf{Q}\cdot\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad\text{where: }\begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{matrix}\quad\text{so: }\boxed{\mathbf{p}\cdot\mathbf{r}=1}$$

(a) Quadratic form ellipse and
Inverse quadratic form ellipse

(b) Ellipse tangents

based on
Unit 1
Fig. 11.6



Original ellipse
$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = \mathbf{r}\bullet\mathbf{p} = 1$

Inverse ellipse
$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = \mathbf{p}\bullet\mathbf{r} = 1$

vector $\mathbf{p}(\phi)$ *is perpendicular to* $\dot{\mathbf{r}}(\phi)$

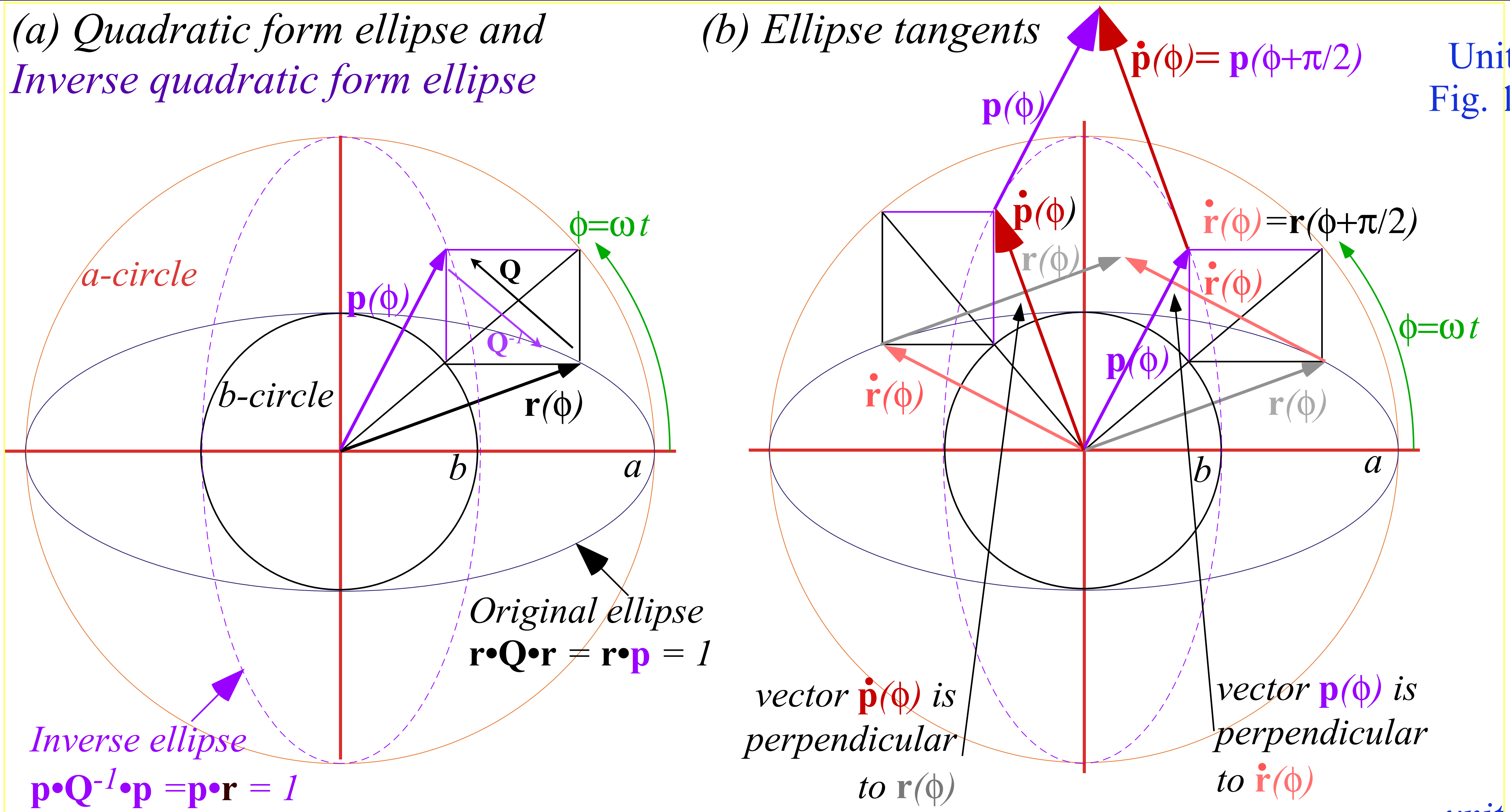
Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where:} \quad \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi \;\; = b\sin\omega t \end{matrix} \quad \text{so:} \quad \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

$\mathbf{p}$ *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p} = 0} = \begin{pmatrix} \dot{r}_x & \dot{r}_y \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -a\sin\phi & b\cos\phi \end{pmatrix} \bullet \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where:} \quad \begin{matrix} \dot{r}_x = -a\sin\phi \\ \dot{r}_y = b\cos\phi \end{matrix} \quad \text{and:} \quad \begin{matrix} p_x = (1/a)\cos\phi \\ p_y = (1/b)\sin\phi \end{matrix}$$

Unit 1
Fig. 11.6



Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1*

unit mutual projection

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $x=r_x=a\cos\phi=a\cos\omega t$

$y=r_y=b\sin\phi\ \ =b\sin\omega t$

so: $\boxed{\mathbf{p}\bullet\mathbf{r}=1}$

p *is perpendicular to velocity* **v**=**ṙ** *, a mutual orthogonality. So is* **r** *perpendicular to* **ṗ**: $\boxed{\dot{\mathbf{p}}\bullet\mathbf{r}=0}$

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot r_x & \dot r_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}$$

where: $\dot r_x=-a\sin\phi$

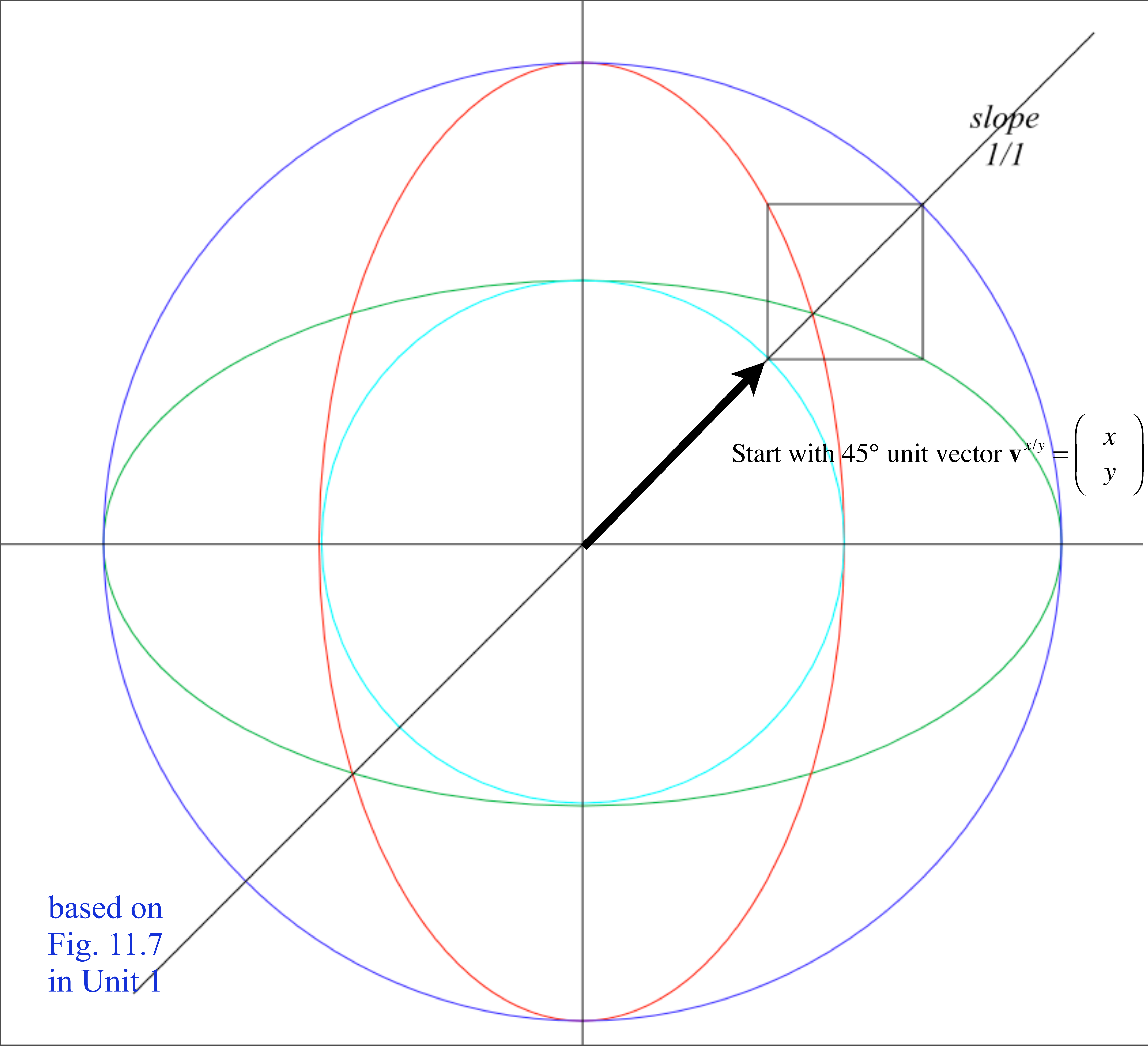
$\dot r_y=b\cos\phi$

and: $p_x=(1/a)\cos\phi$

$p_y=(1/b)\sin\phi$

Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)
    Construction by Phasor-pair projection
    Construction by Kepler anomaly projection
→ *Operator geometric sequences and eigenvectors*
Rescaled description of matrix operator geometry
    Vector calculus of tensor operation



Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

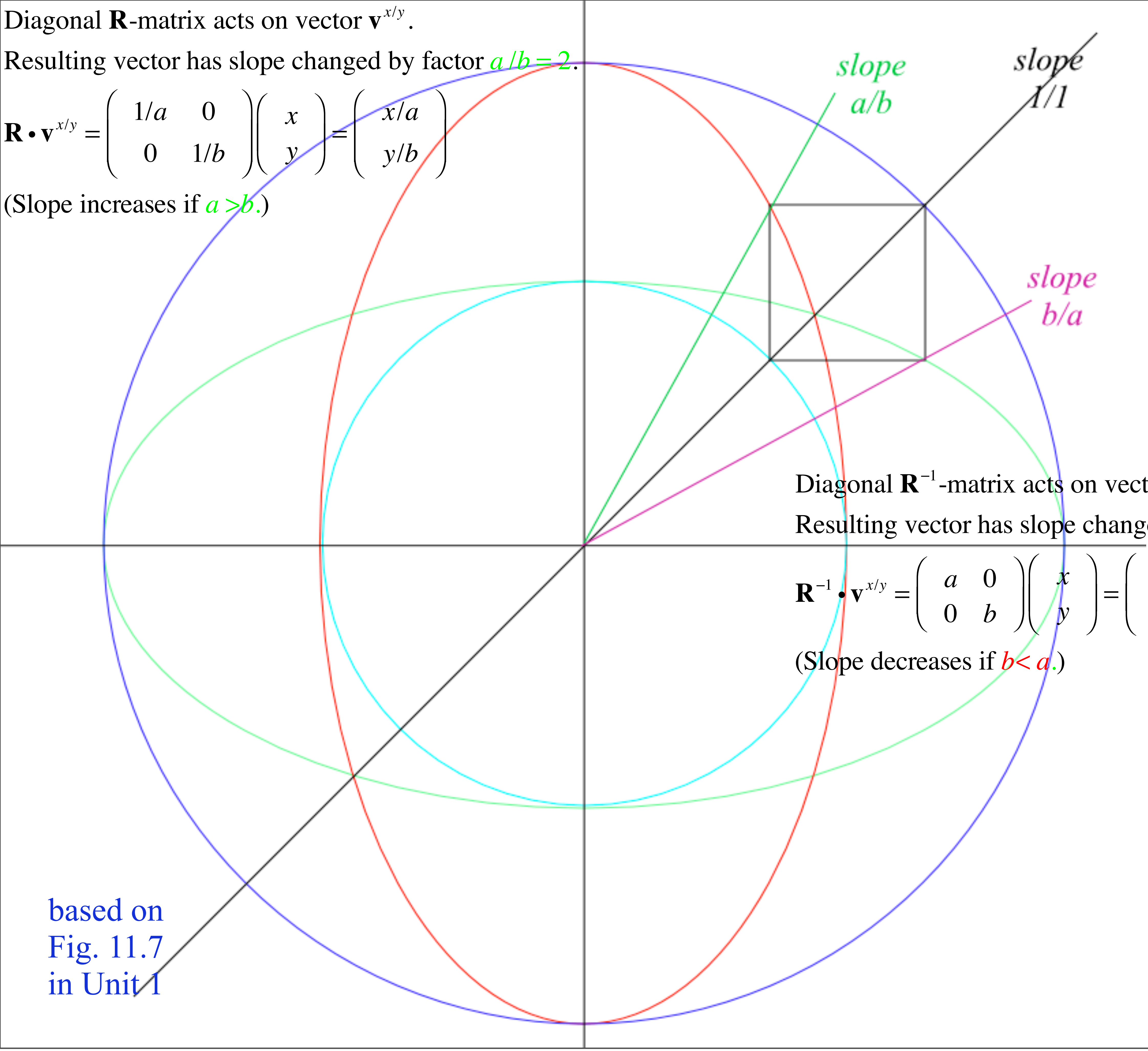
based on
Fig. 11.7
in Unit 1

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(Slope increases if $a > b$.)



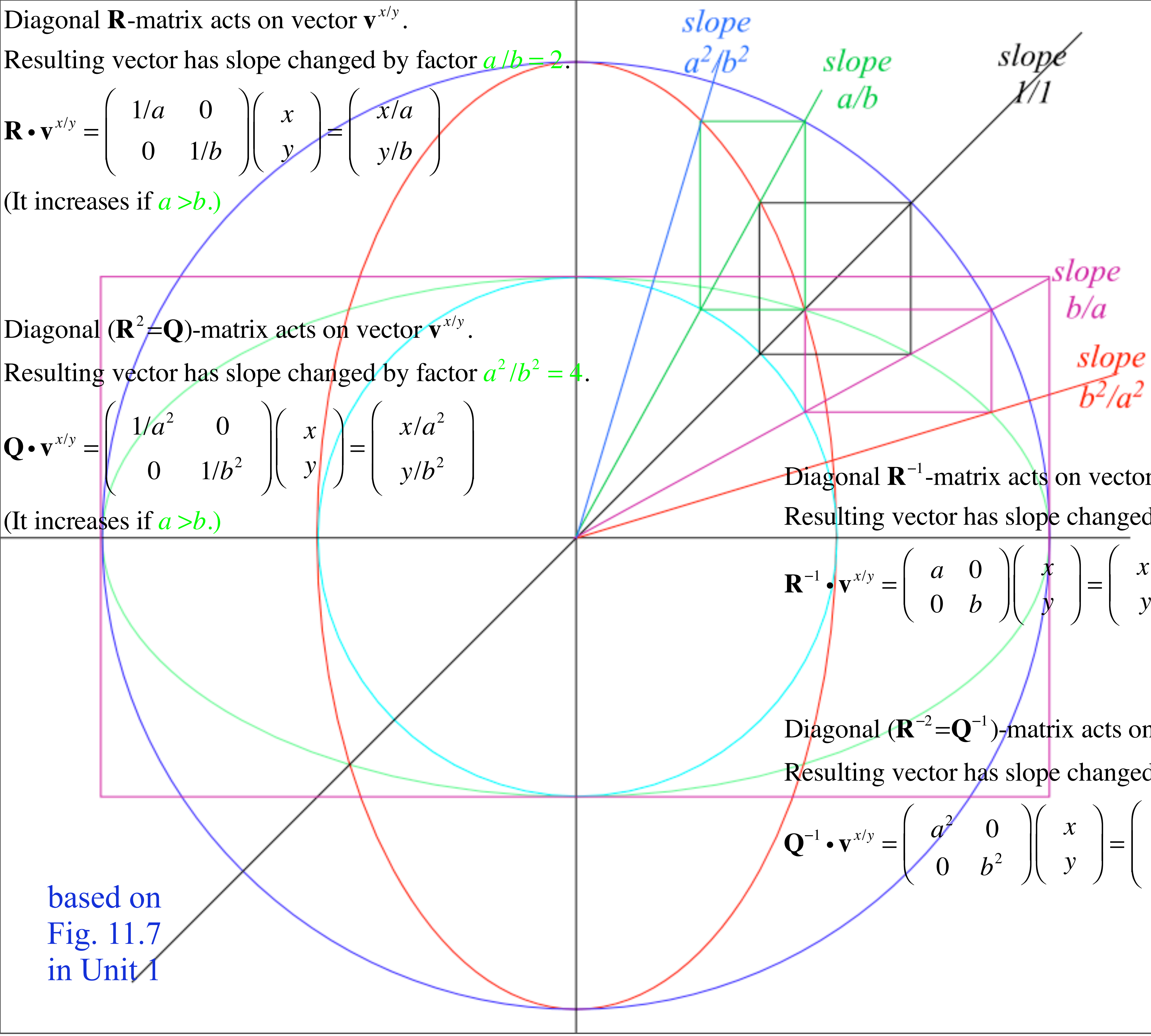
Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cdot a \\ y\cdot b \end{pmatrix}$$

(Slope decreases if $b < a$.)

based on
Fig. 11.7
in Unit 1



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b/a=1/2$.

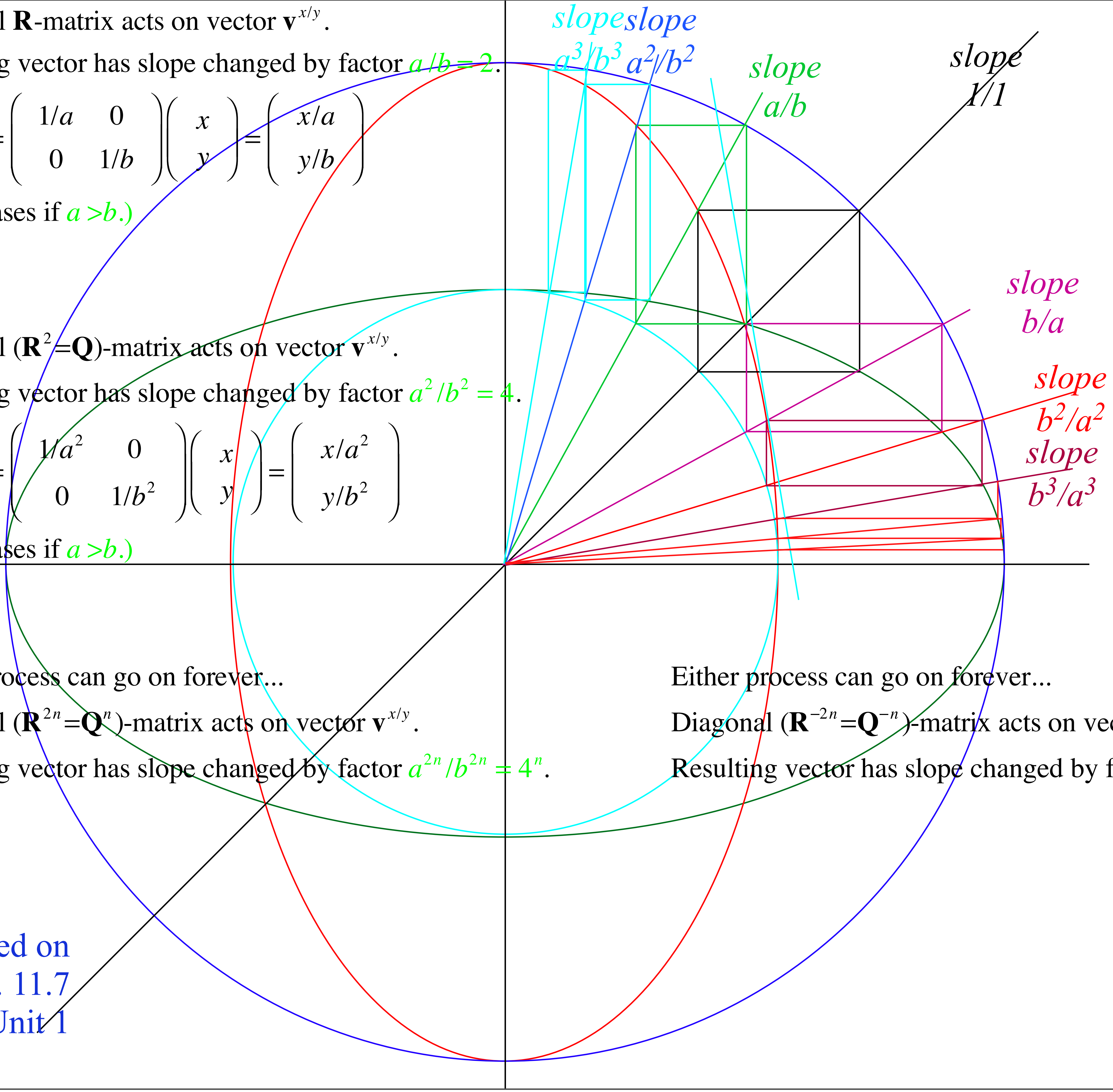
$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a\\ y\cdot b\end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2}$=$\mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^2/a^2=1/4$.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a^2\\ y\cdot b^2\end{pmatrix}$$

based on
Fig. 11.7
in Unit 1



Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2$=$\mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

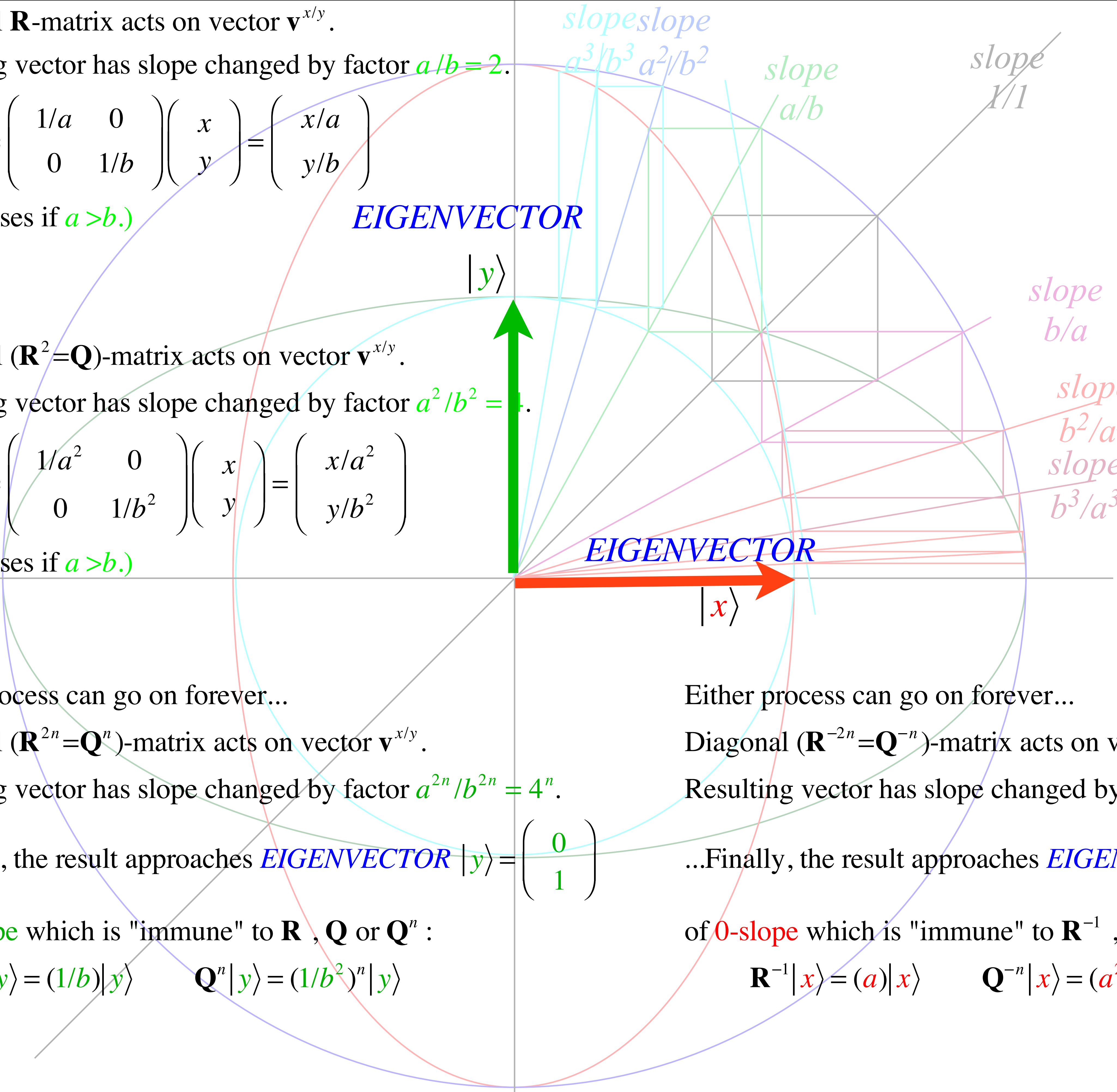
based on
Fig. 11.7
in Unit 1

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

(It increases if $a > b$.)



Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}$=$\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$$\mathbf{R}|y\rangle = (1/b)|y\rangle \qquad \mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}$=$\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

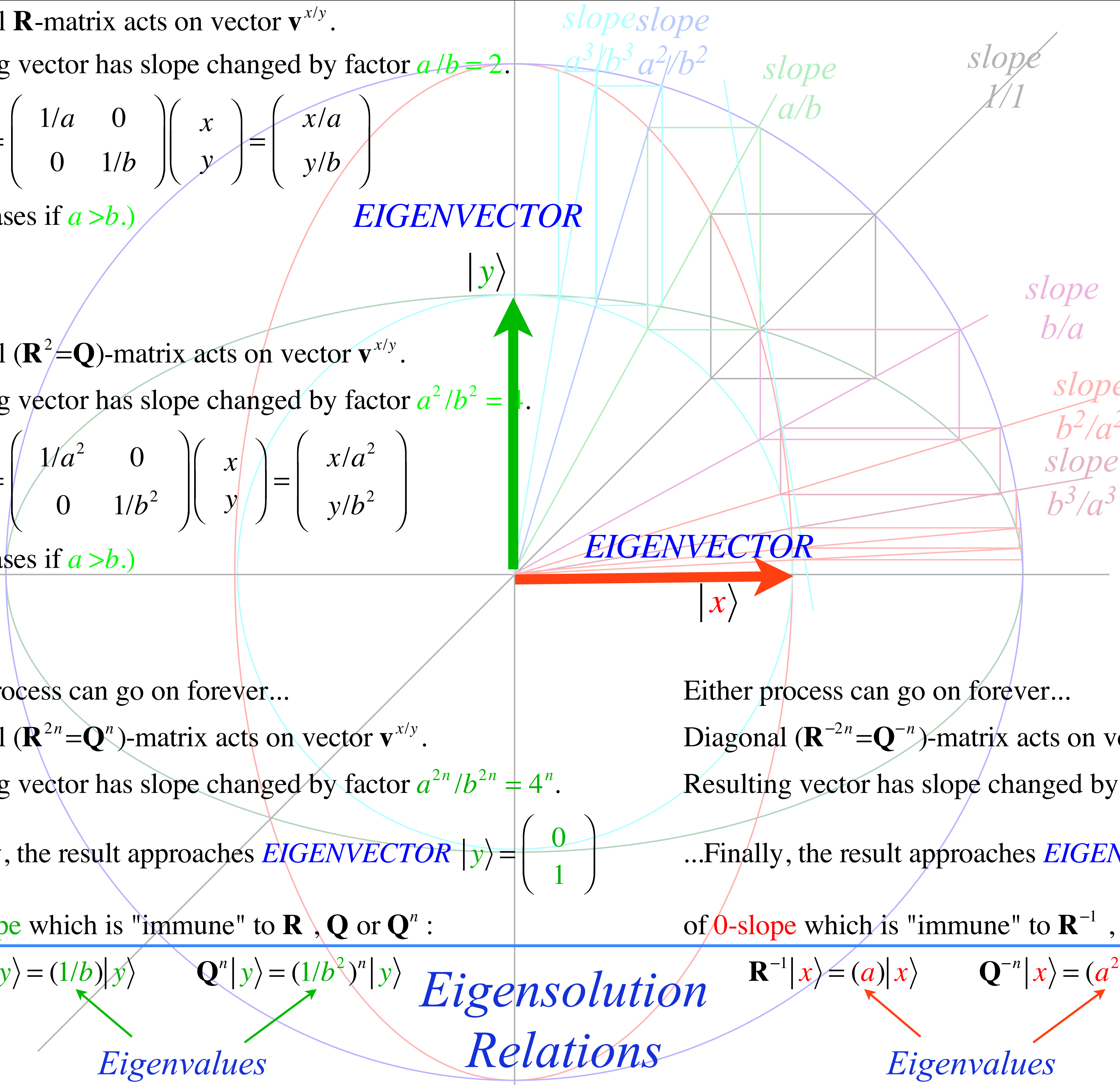
$$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle \qquad \mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)



Diagonal ($\mathbf{R}^2=\mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}=\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n}=4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}=\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n}=4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle=\begin{pmatrix}1\\ 0\end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$\mathbf{R}|y\rangle=(1/b)|y\rangle$ $\mathbf{Q}^n|y\rangle=(1/b^2)^n|y\rangle$

Eigenvalues

Eigensolution Relations

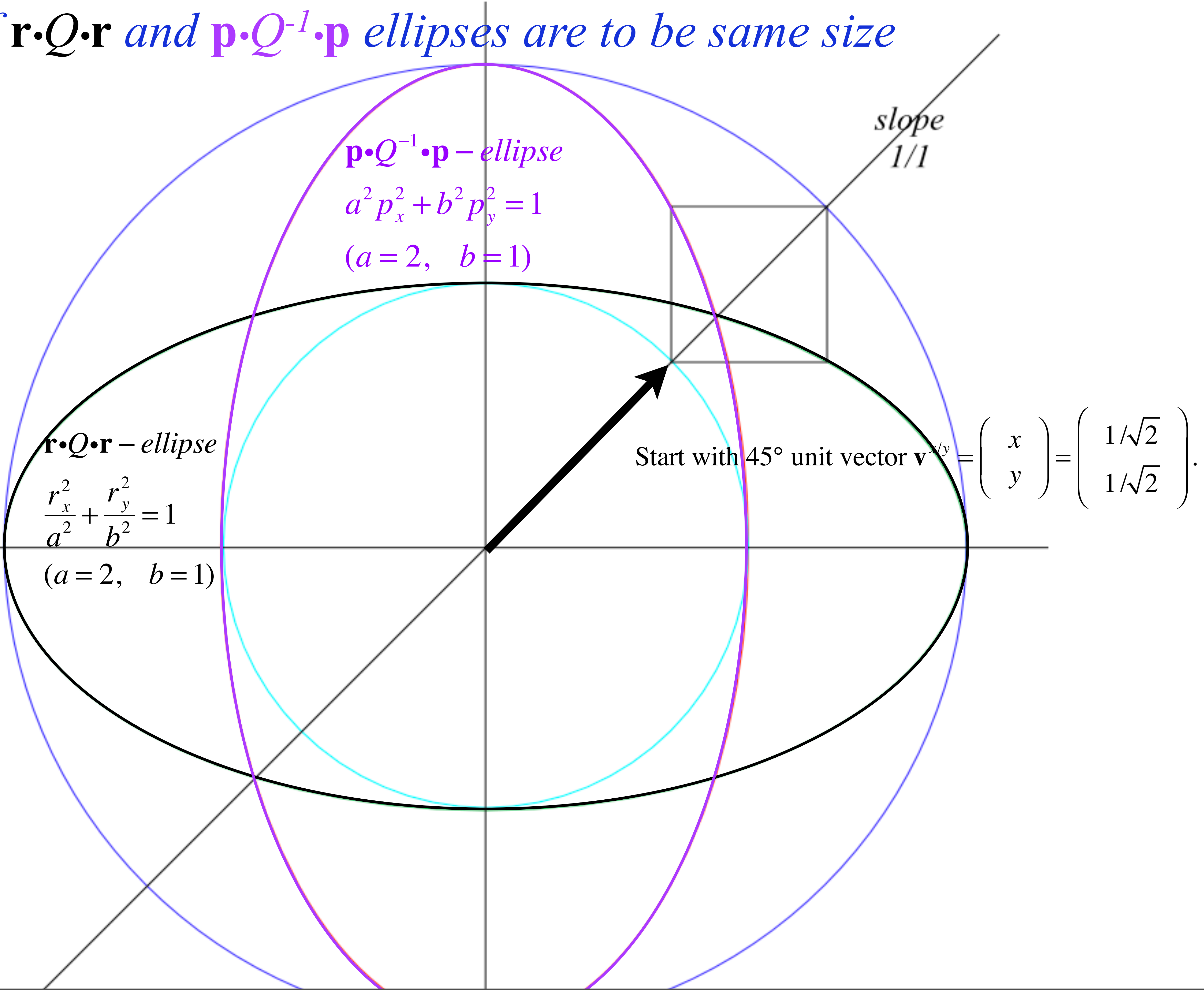
$\mathbf{R}^{-1}|x\rangle=(a)|x\rangle$ $\mathbf{Q}^{-n}|x\rangle=(a^2)^n|x\rangle$

Eigenvalues

Introduction to dual matrix operator geometry

- *Review of dual IHO elliptic orbits (Lecture 7-8)*
  - *Construction by Phasor-pair projection*
  - *Construction by Kepler anomaly projection*
- *Operator geometric sequences and eigenvectors*
- → *Rescaled description of matrix operator geometry*
  - *Vector calculus of tensor operation*

Need to rescale by geometric mean √(a·b) *if* $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses are to be same size*



$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}-ellipse$

$a^2p_x^2+b^2p_y^2=1$

$(a=2,\quad b=1)$

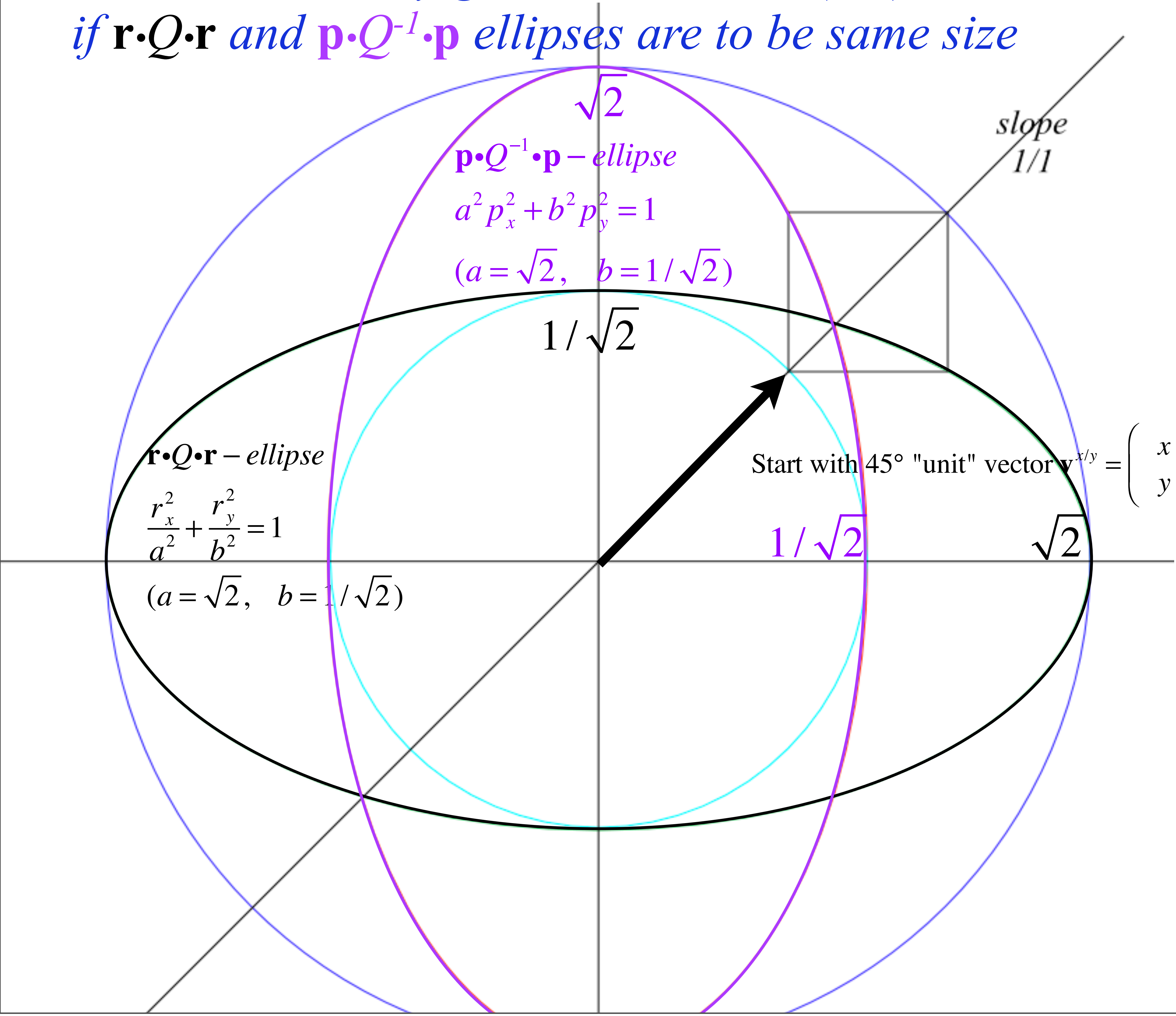
$\mathbf{r}\bullet Q\bullet\mathbf{r}-ellipse$

$\frac{r_x^2}{a^2}+\frac{r_y^2}{b^2}=1$

$(a=2,\quad b=1)$

Start with 45° unit vector $\mathbf{v}^{x/y}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$

Need to rescale by geometric mean $\sqrt{(a\cdot b)}$ *(so* $a\cdot b=1$*)* *if* $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses are to be same size*



$\sqrt{2}$

$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}-ellipse$

$a^2 p_x^2 + b^2 p_y^2 = 1$

$(a=\sqrt{2},\quad b=1/\sqrt{2})$

slope 1/1

$1/\sqrt{2}$

$\mathbf{r}\bullet Q\bullet\mathbf{r}-ellipse$

$$\frac{r_x^2}{a^2}+\frac{r_y^2}{b^2}=1$$

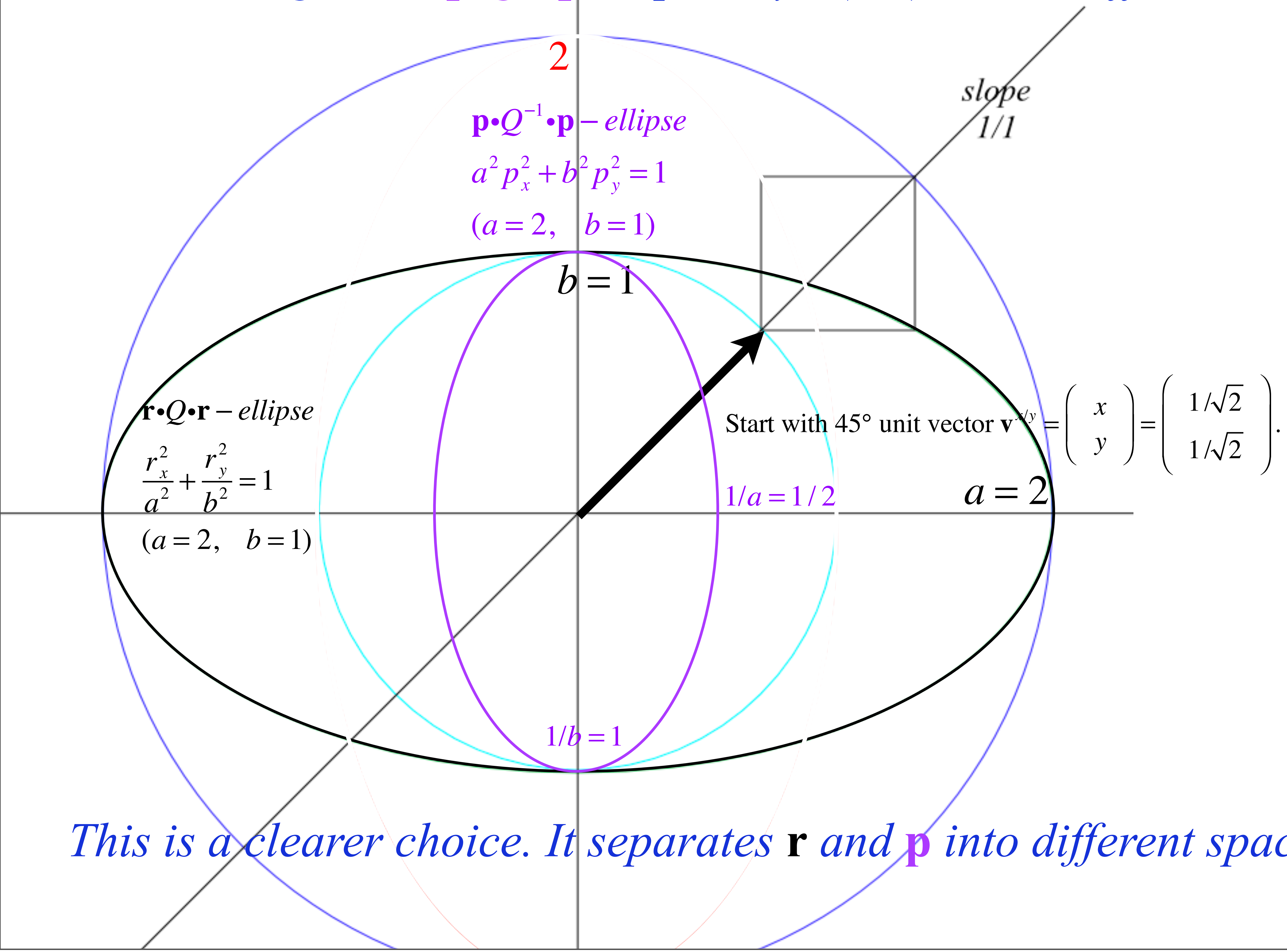
$(a=\sqrt{2},\quad b=1/\sqrt{2})$

Start with 45° "unit" vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

$1/\sqrt{2}$

$\sqrt{2}$

...or rescale $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses by* $\sqrt{(a\cdot b)}=\sqrt{2}$ *to different size*



$\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ – *ellipse*

$$a^2 p_x^2 + b^2 p_y^2 = 1$$

$$(a=2, \quad b=1)$$

$\mathbf{r}\cdot Q\cdot\mathbf{r}$ – *ellipse*

$$\frac{r_x^2}{a^2}+\frac{r_y^2}{b^2}=1$$

$$(a=2, \quad b=1)$$

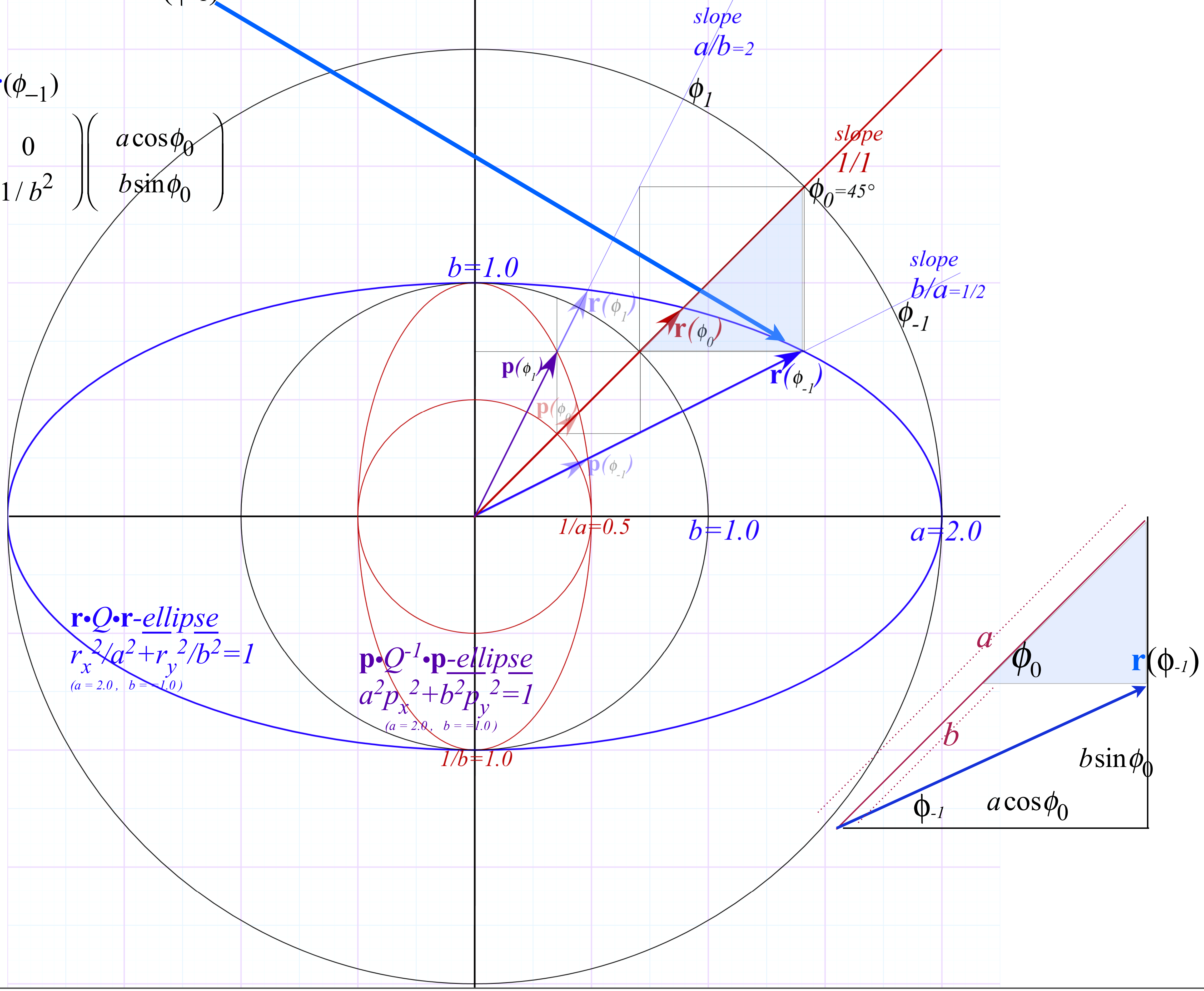
Start with 45° unit vector $\mathbf{v}^{x/y}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

This is a clearer choice. It separates **r** *and* **p** *into different spaces*

Action of matrix Q that generates an **r***-ellipse (***r***•Q•* **r** *=1)*
on a single **r***-vector* **r**(ϕ_{-1})*...*

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



based on
Fig. 11.7
in Unit 1

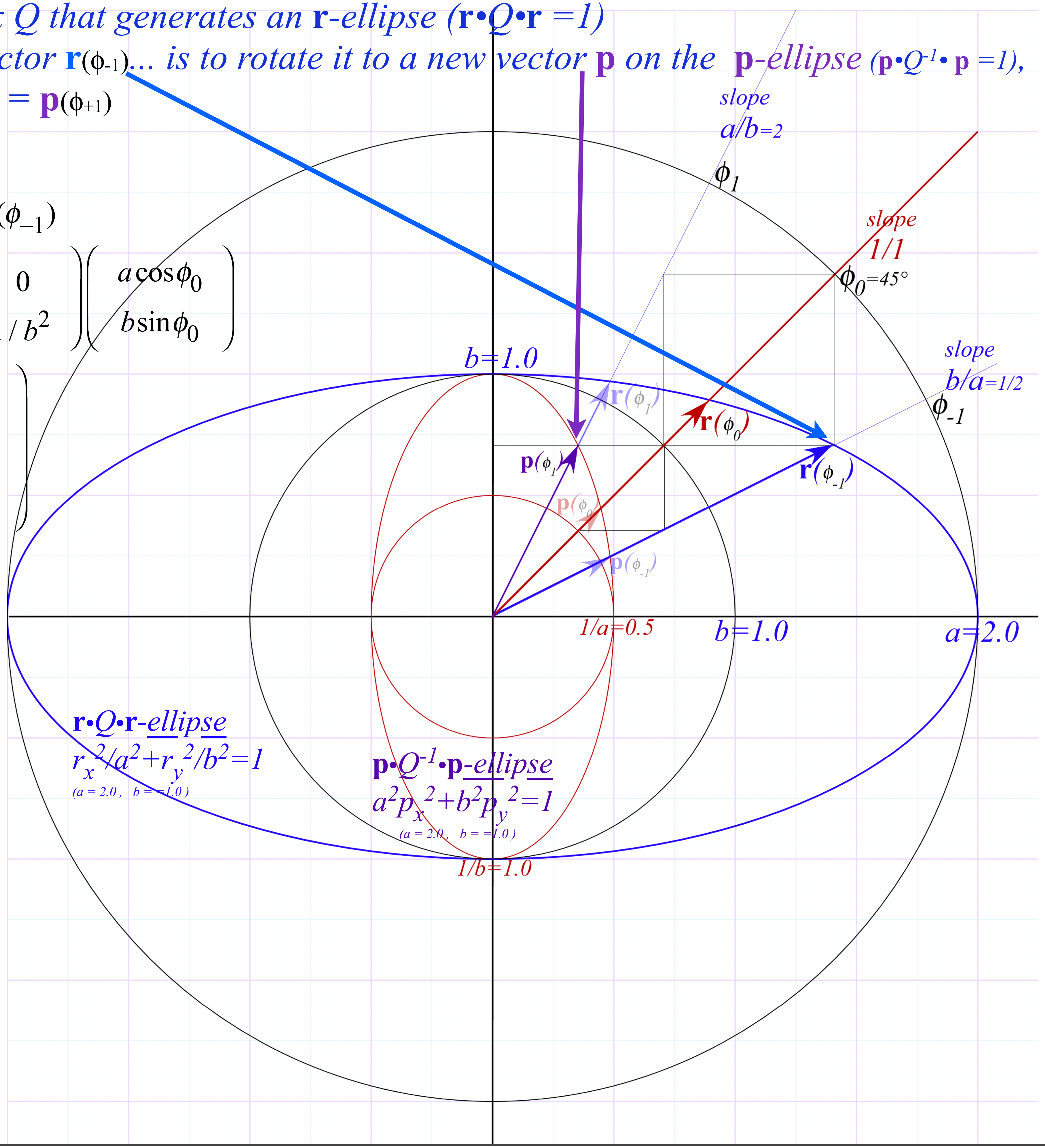
Action of matrix Q that generates an **r***-ellipse (***r***•Q•***r** *=1) on a single* **r***-vector* **r**(ϕ-1)*... is to rotate it to a new vector* **p** *on the* **p***-ellipse (***p***•Q⁻¹•* **p** *=1), that is, Q•***r**(ϕ-1) = **p**(ϕ+1)

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



based on
Fig. 11.7
in Unit 1

Action of matrix Q that generates an **r***-ellipse (*$\mathbf{r}\bullet Q\bullet\mathbf{r}=1$*) on a single* **r***-vector* $\mathbf{r}(\phi_{-1})$*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* ($\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$)*, that is,* $Q\bullet\mathbf{r}(\phi_{-1}) = \mathbf{p}(\phi_{+1})$

$$\mathbf{p}(\phi_1)=\mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}\cos\phi_0\\ \frac{1}{b}\sin\phi_0\end{pmatrix}$$

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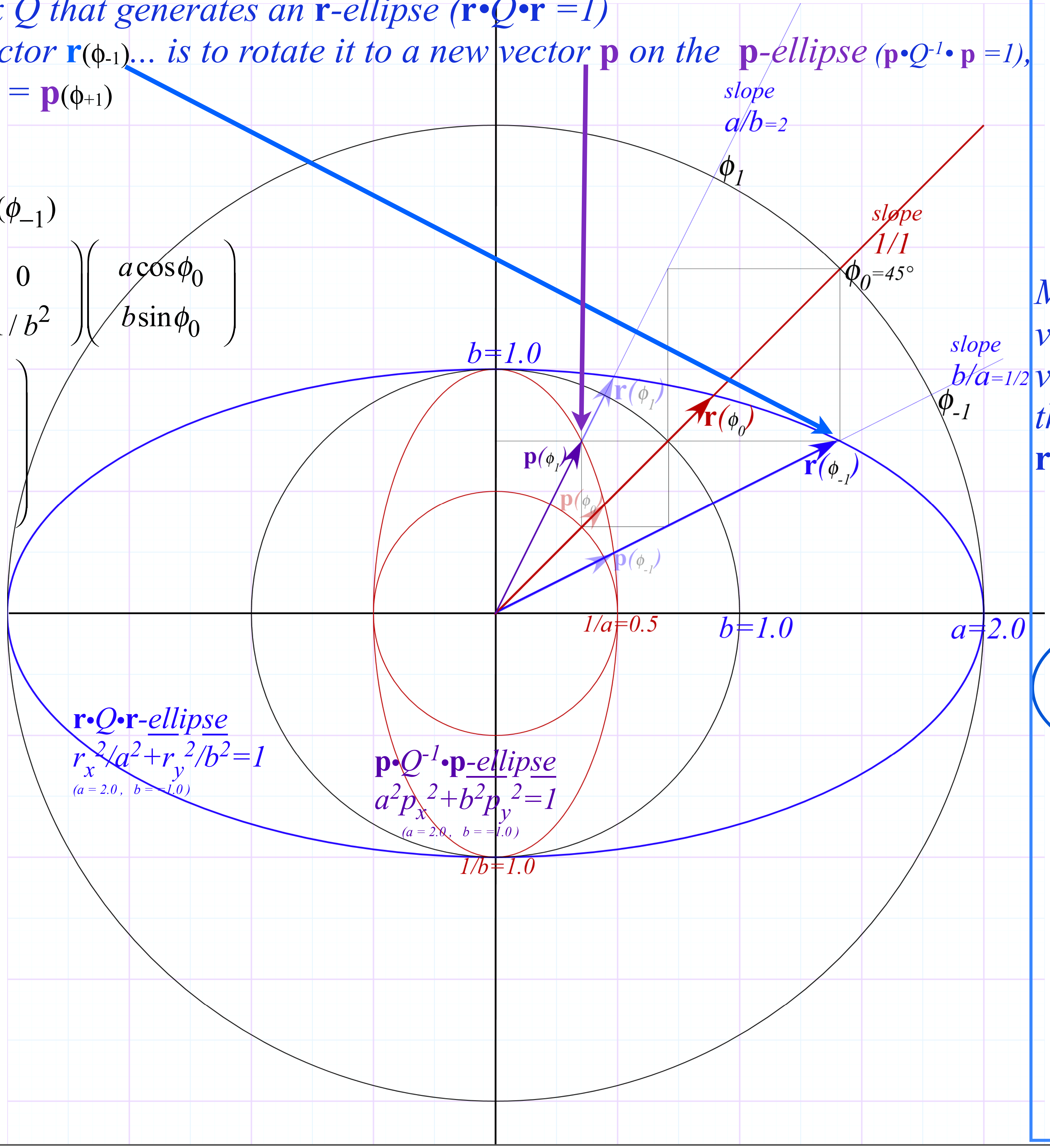
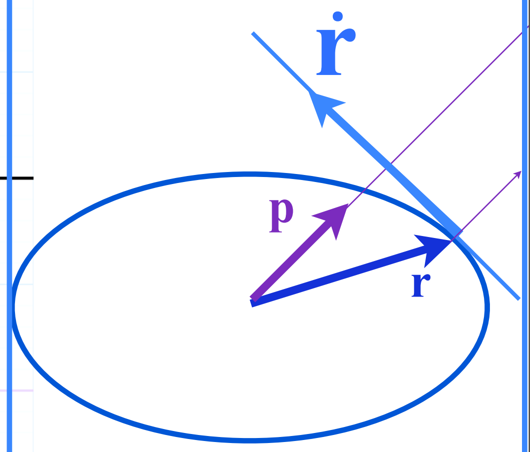


based on
Fig. 11.7
in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* $\dot{\mathbf{r}}$ *to its* $\mathbf{r}\bullet Q\bullet\mathbf{r}$*-ellipse.*



Action of matrix Q that generates an **r***-ellipse (***r***•Q•***r** *=1)*
on a single **r***-vector* **r**(ϕ_{-1})*... is to rotate it to a new vector* **p** *on the* **p***-ellipse (***p***•Q^{-1}•* **p** *=1),*
*that is, Q•***r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

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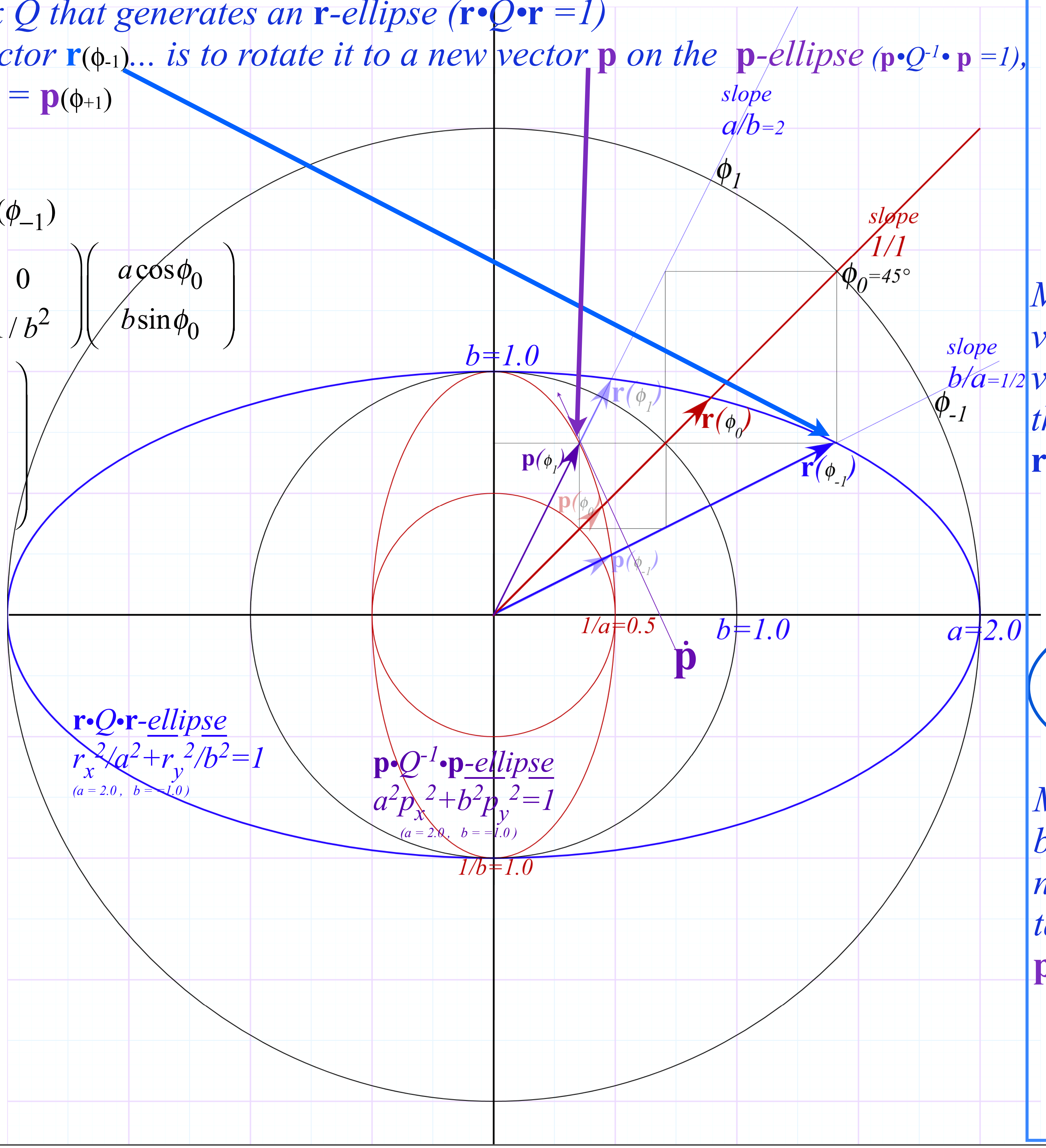
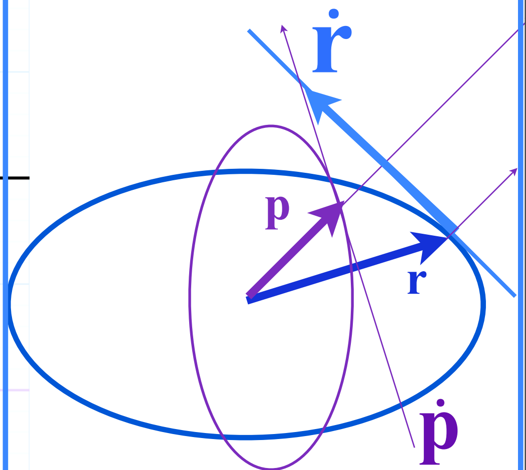


based on
Fig. 11.7
in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r***•Q•***r***-ellipse.*



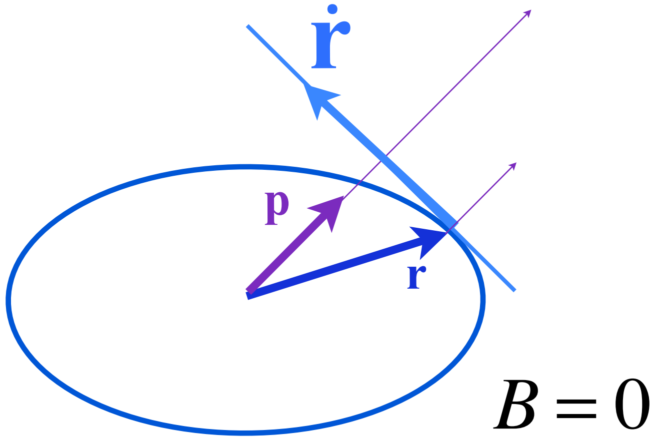
Matrix Q^{-1} maps **p** *back to* **r** *that is normal to the tangent* **ṗ** *to its* **p***• Q^{-1}•* **p***-ellipse.*

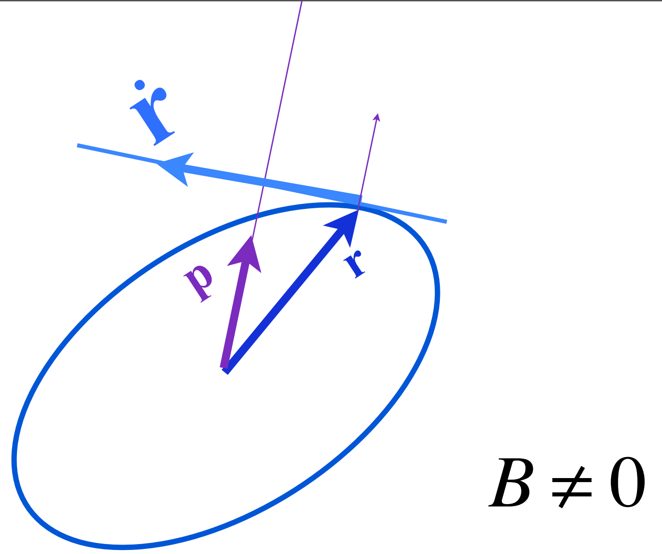
Introduction to dual matrix operator geometry

Review of dual IHO elliptic orbits (Lecture 7-8)
  - *Construction by Phasor-pair projection*
  - *Construction by Kepler anomaly projection*

Operator geometric sequences and eigenvectors

Rescaled description of matrix operator geometry
  - → *Vector calculus of tensor operation*

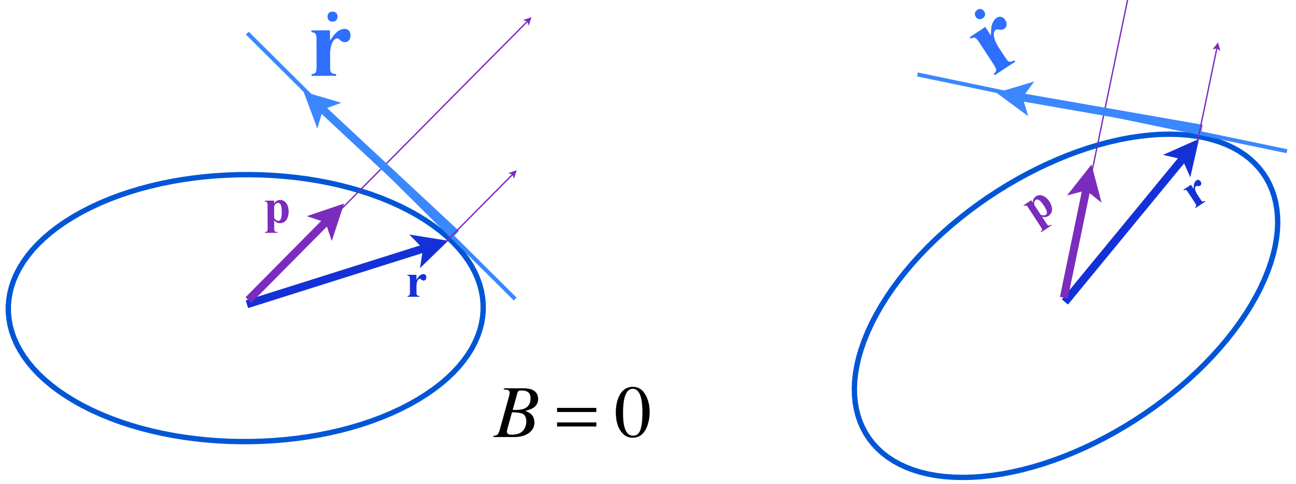




Derive matrix "normal-to-ellipse"geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1=\mathbf{r}\bullet Q\bullet\mathbf{r} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix} = A\cdot x^2+2B\cdot xy+D\cdot y^2=1$



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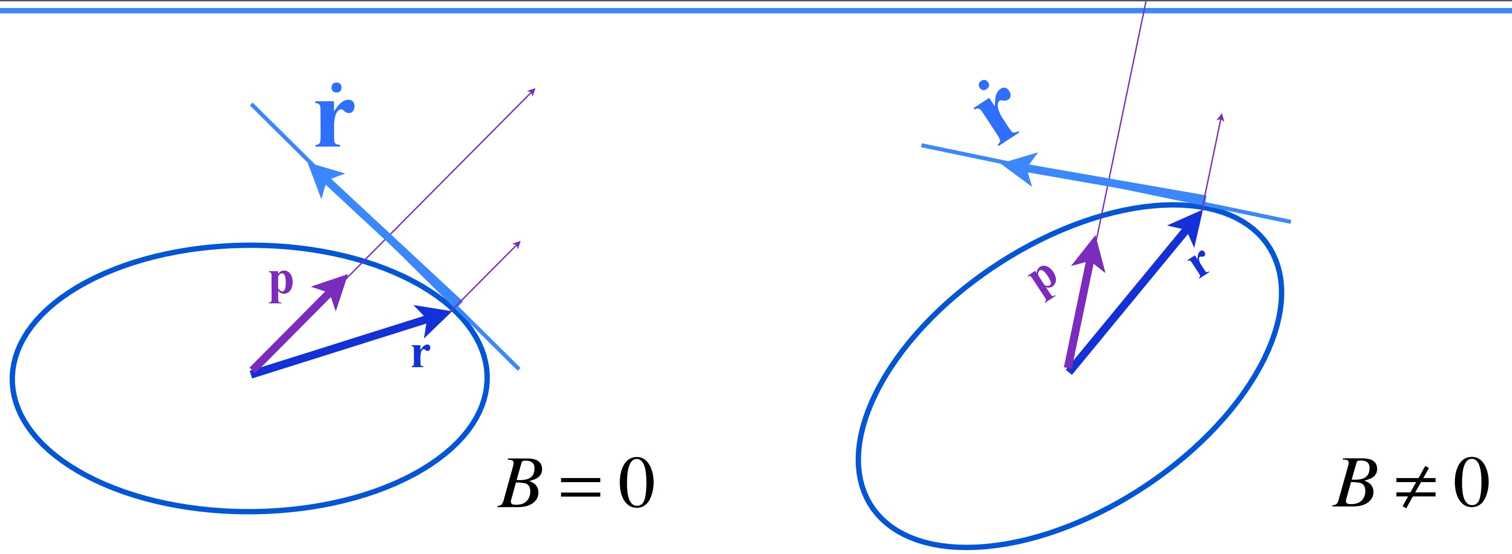
Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r})=\nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2+2B\cdot xy+D\cdot y^2\right)=\begin{pmatrix} 2A\cdot x+2B\cdot y \\ 2B\cdot x+2D\cdot y \end{pmatrix}$$



Derive matrix “normal-to-ellipse”geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

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Compare operation by Q on vector **r** *with* *vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\frac{\partial}{\partial \mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2 + 2B\cdot xy + D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x + 2B\cdot y \\ 2B\cdot x + 2D\cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial \mathbf{r}}\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = \nabla\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = Q\bullet\mathbf{r}$$

Action of “sqrt-”matrix $R=\sqrt{Q}$ *(R generates another ellipse* $\mathbf{r}\bullet R\bullet\mathbf{r} = 1$ *not shown)*

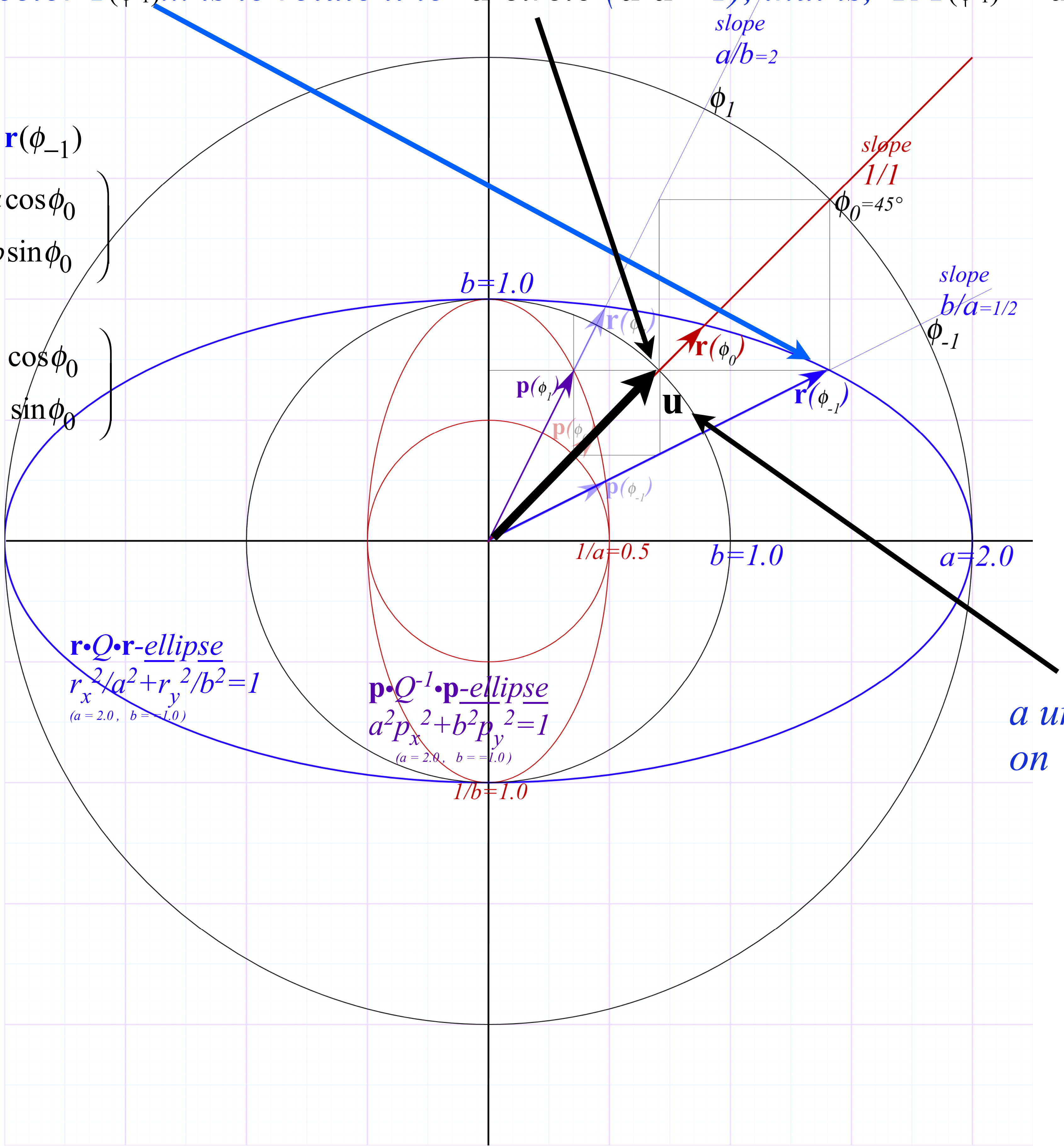
on a single **r**-*vector* $\mathbf{r}(\phi_{-1})$... *is to rotate it to* **u**-*circle* ($\mathbf{u}\bullet\mathbf{u} = 1$), *that is,* $R\bullet\mathbf{r}(\phi_{-1}) = \mathbf{u} = (\text{const.})\,\mathbf{r}(\phi_0)$

$$\mathbf{u} = \sqrt{\mathbf{Q}} \bullet \mathbf{r}(\phi_{-1}) = \mathbf{R} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}a\cos\phi_0 \\ \frac{1}{b}b\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

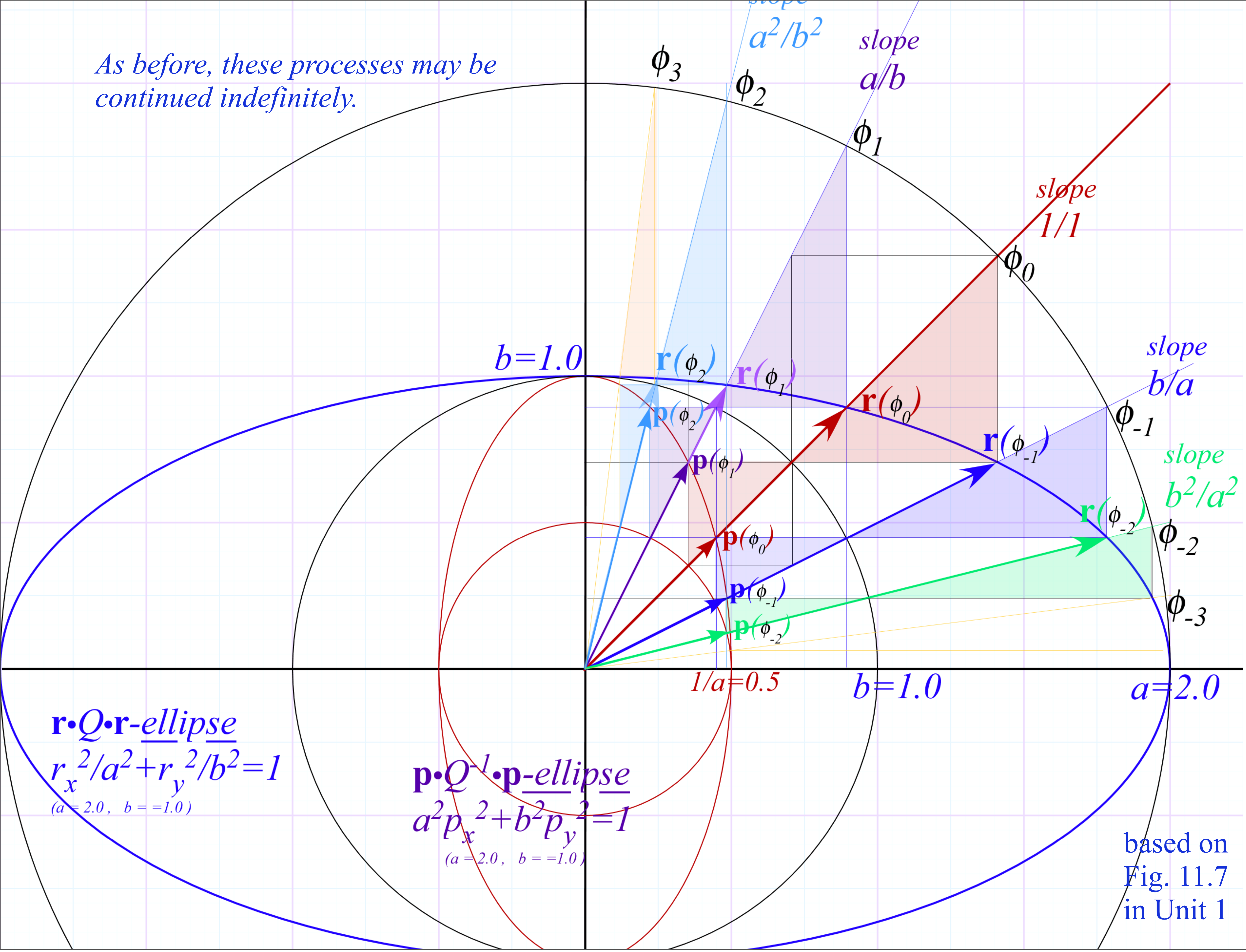
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



a unit vector
on unit-circle

based on
Fig. 11.7
in Unit 1

As before, these processes may be continued indefinitely.



r•*Q*•**r**-*ellipse*

$r_x{}^2/a^2+r_y{}^2/b^2=1$

(a = 2.0 , b = =1.0)

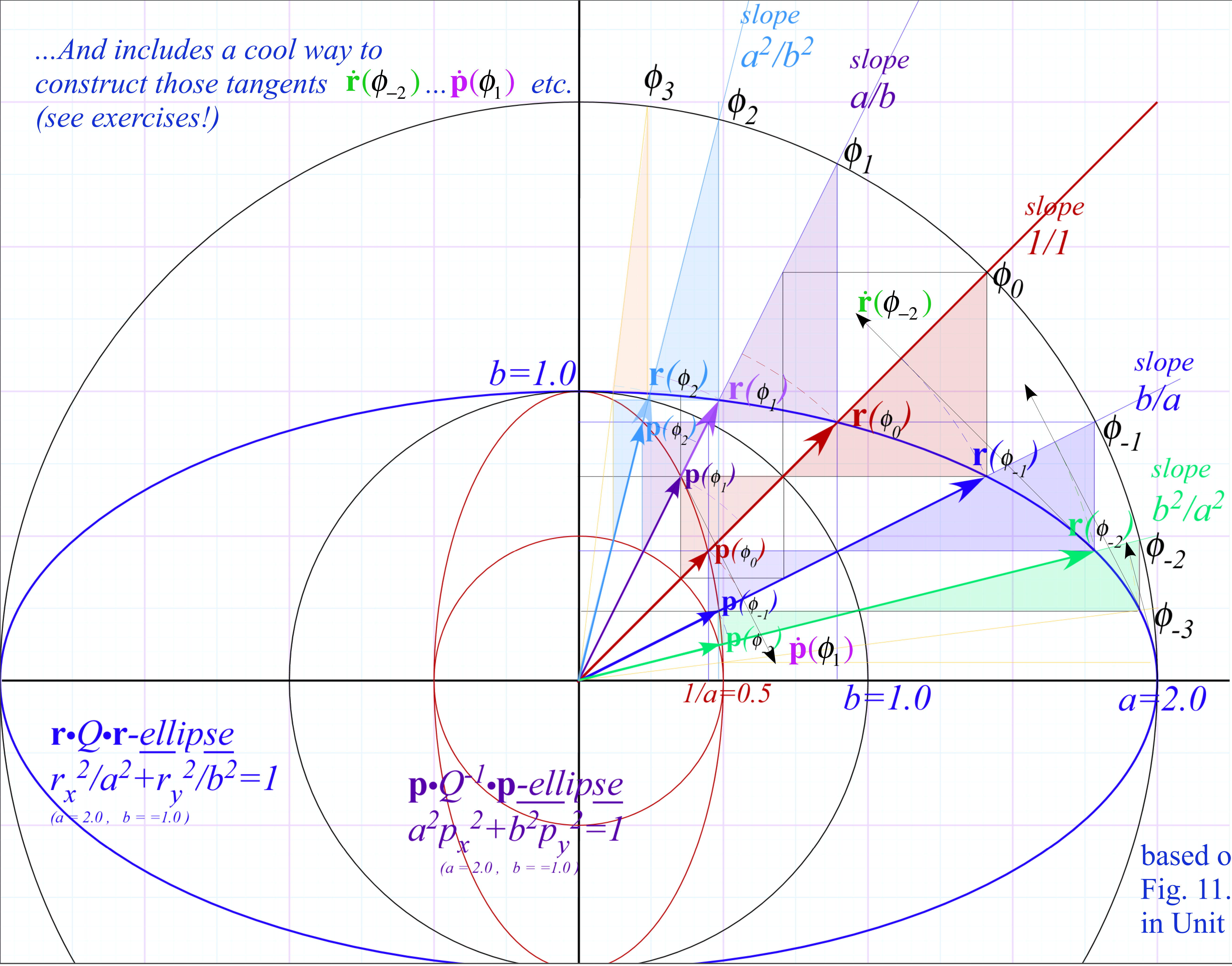
p•*Q*$^{-1}$•**p**-*ellipse*

$a^2p_x{}^2+b^2p_y{}^2=1$

(a = 2.0 , b = =1.0)

based on
Fig. 11.7
in Unit 1

...And includes a cool way to construct those tangents $\dot{\mathbf{r}}(\phi_{-2})\ldots\dot{\mathbf{p}}(\phi_1)$ *etc. (see exercises!)*



based on
Fig. 11.7
in Unit 1

Introduction to Lagrangian-Hamiltonian duality

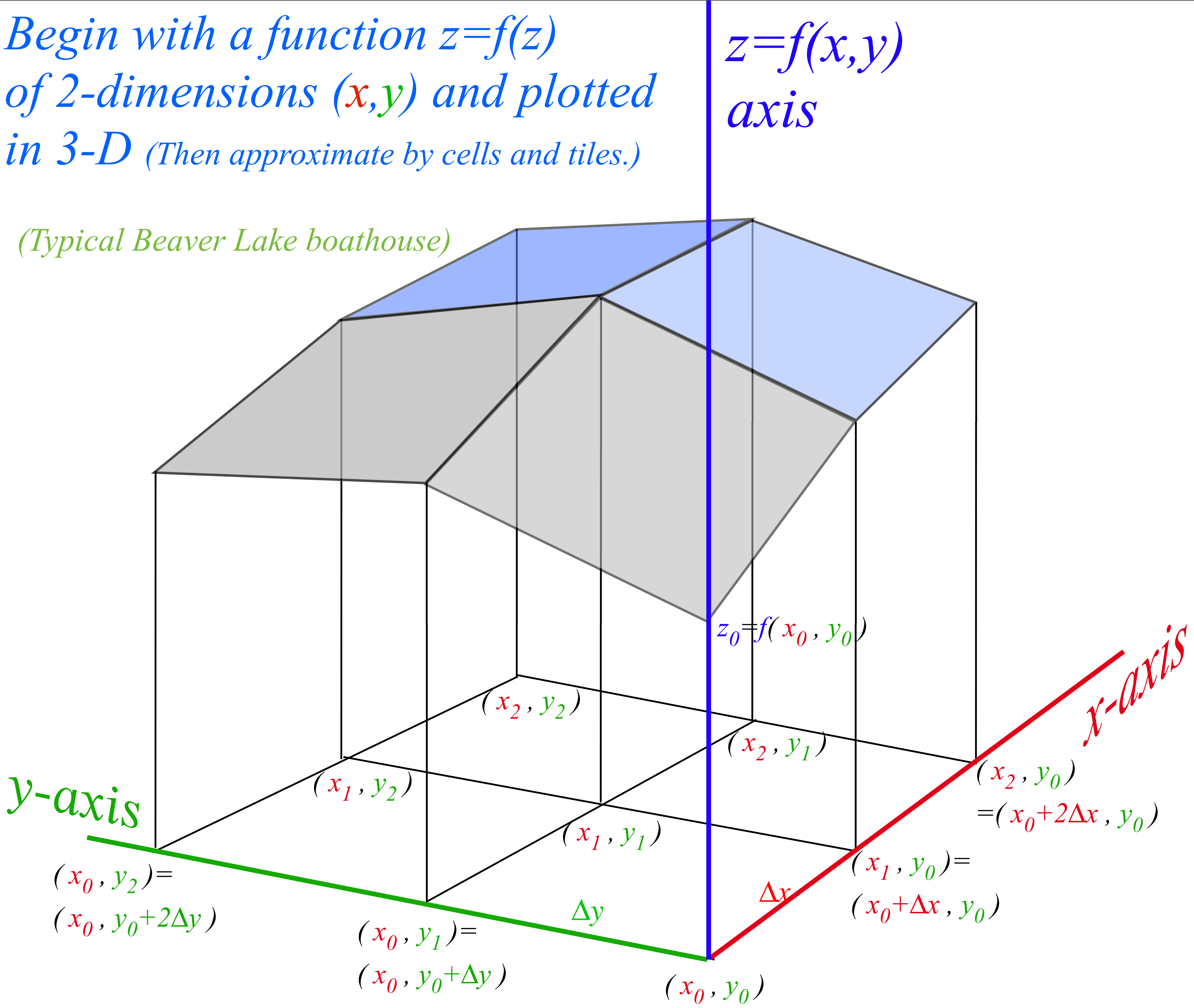
→ *Review of partial differential relations*

  Chain rule and order symmetry

Duality relations of Lagrangian and Hamiltonian ellipse

Introducing the 1st (partial $\frac{\partial ?}{\partial ?}$*) differential equations of mechanics*

Begin with a function z=f(z)
of 2-dimensions (x,y) and plotted
in 3-D (Then approximate by cells and tiles.)



Begin with a function z=f(z)
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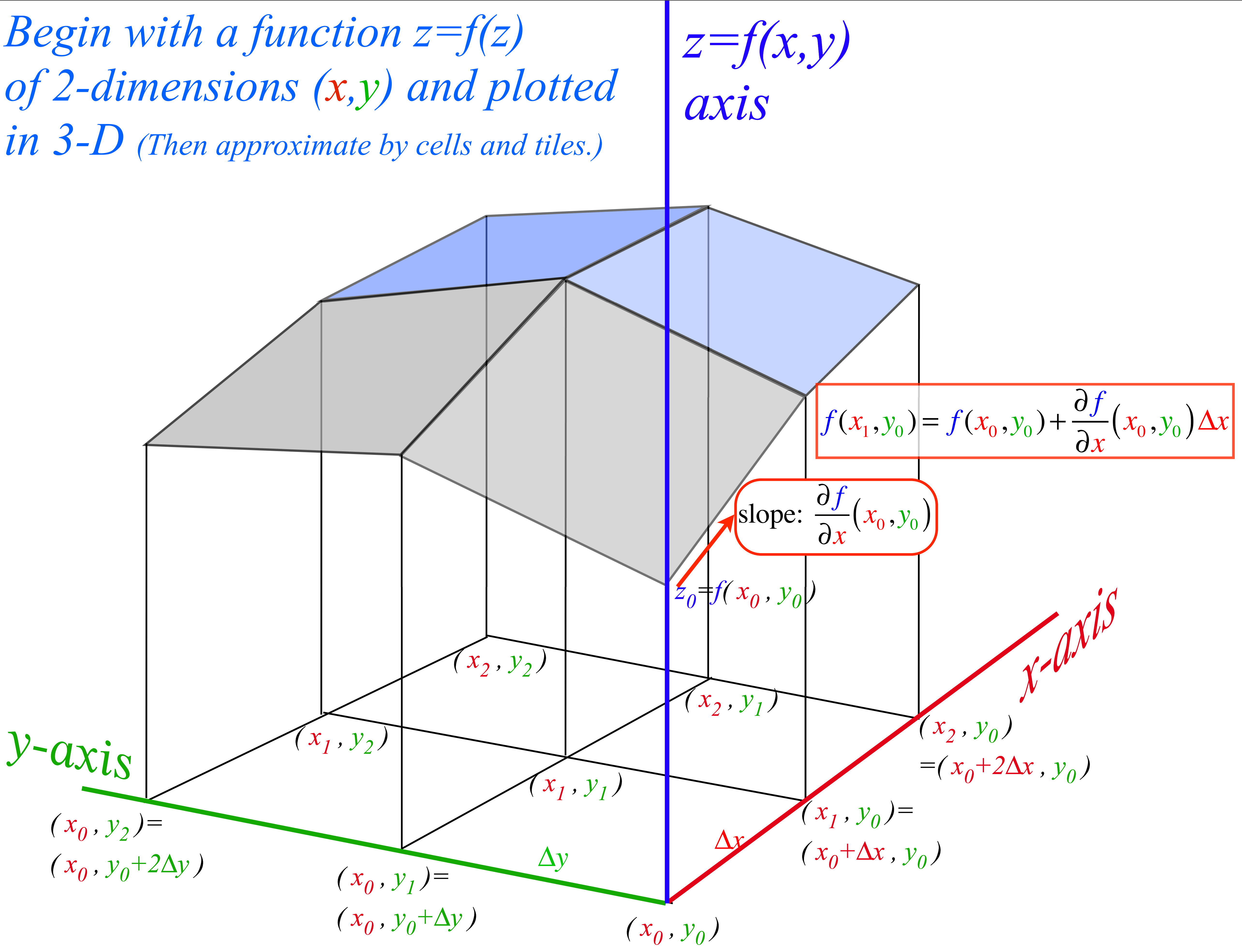
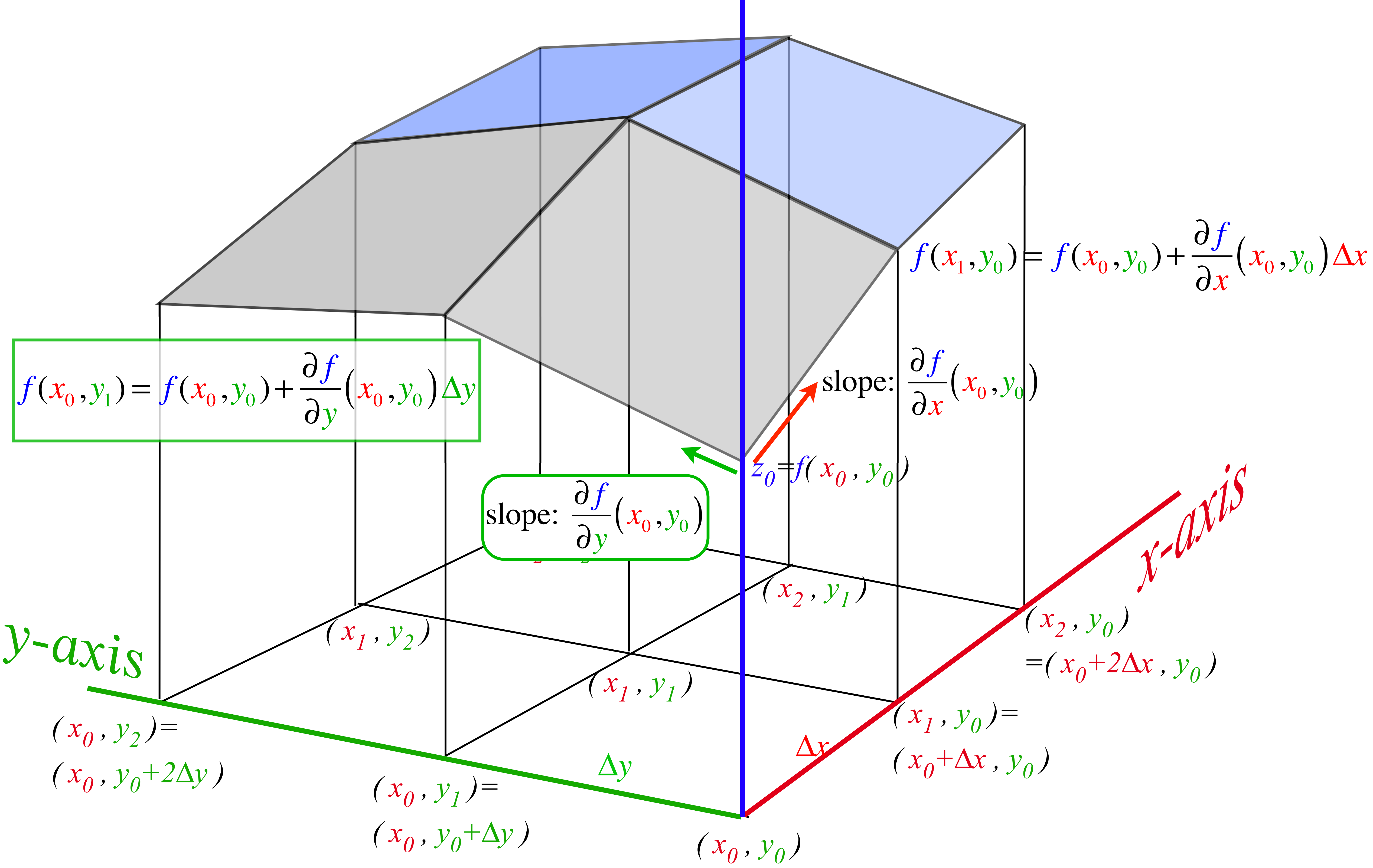


$$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

Begin with a function z=f(z)
of 2-dimensions (x,y) and plotted
in 3-D (Then approximate by cells and tiles.)

$z=f(x,y)$
axis



$$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$$

$$\boxed{f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y}$$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

$z_0=f(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

x-axis

y-axis

(x_2,y_1)

(x_2,y_0)
$=(x_0+2\Delta x,y_0)$

(x_1,y_2)

(x_1,y_1)

$(x_1,y_0)=$
$(x_0+\Delta x,y_0)$

Δx

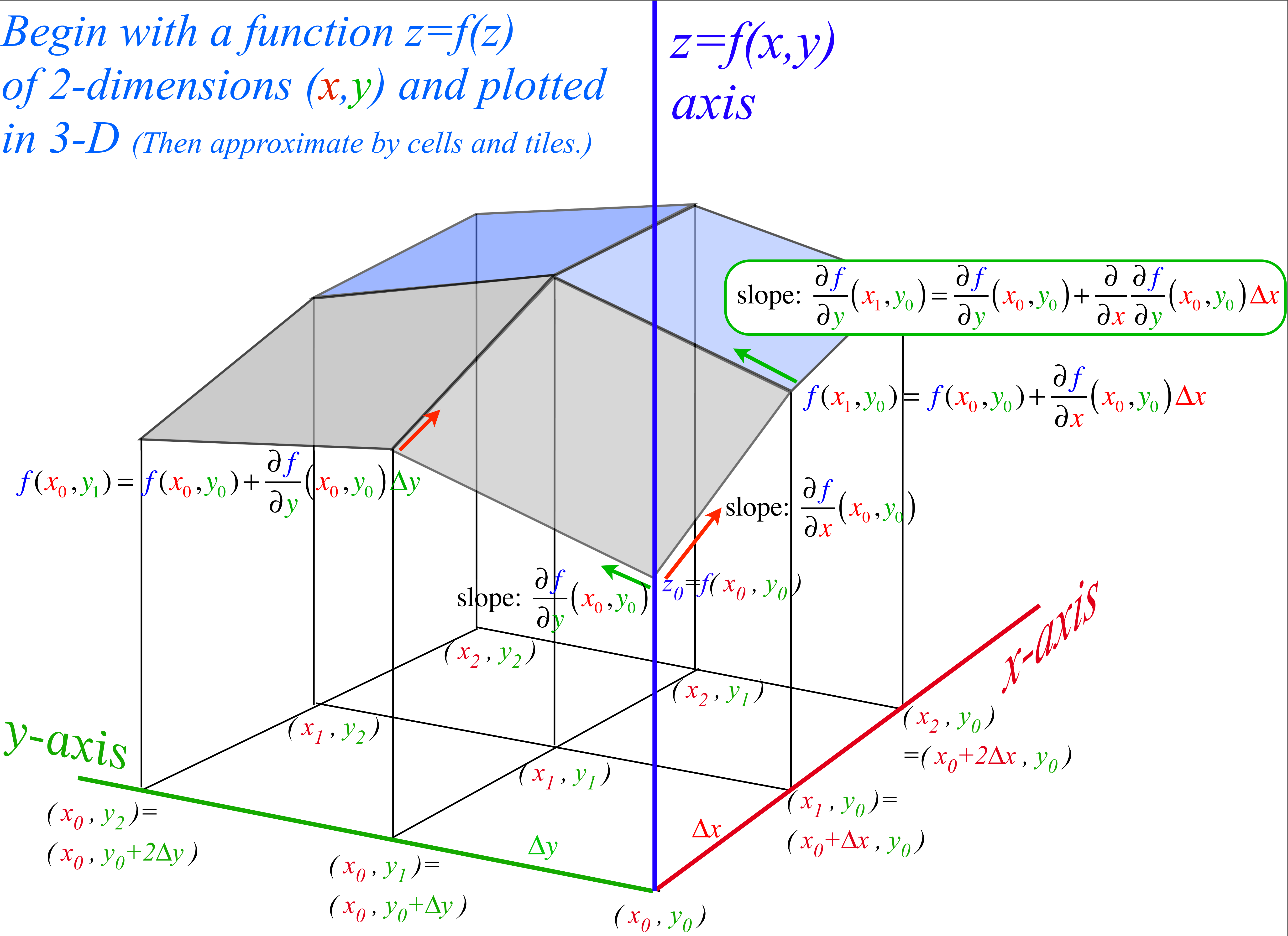
Δy

$(x_0,y_2)=$
$(x_0,y_0+2\Delta y)$

$(x_0,y_1)=$
$(x_0,y_0+\Delta y)$

(x_0,y_0)

Begin with a function z=f(z) of 2-dimensions (x,y) and plotted in 3-D *(Then approximate by cells and tiles.)*



slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

$$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$$

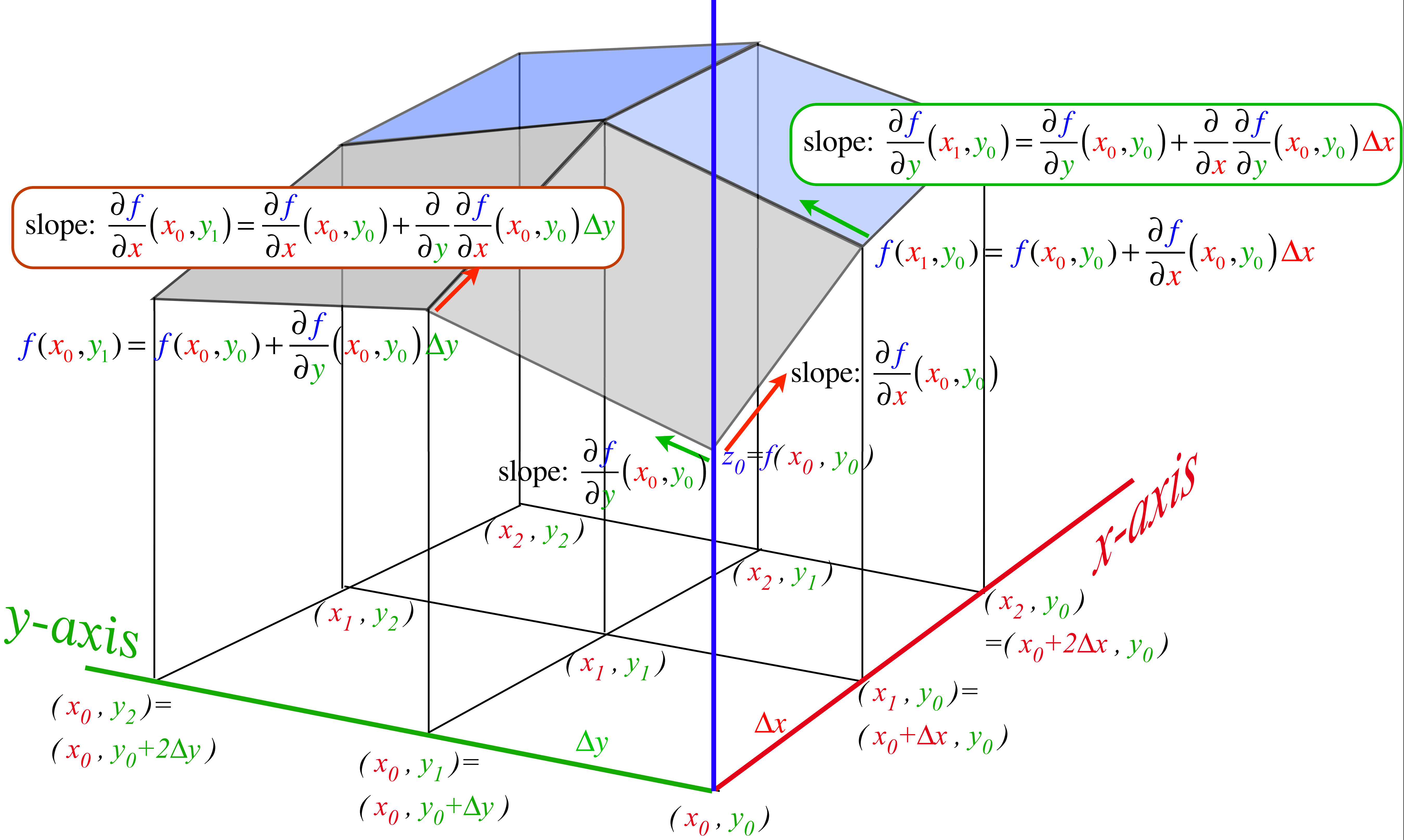
$$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

$z_0=f(x_0,y_0)$

Begin with a function z=f(z) of 2-dimensions (x,y) and plotted in 3-D *(Then approximate by cells and tiles.)*



slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$

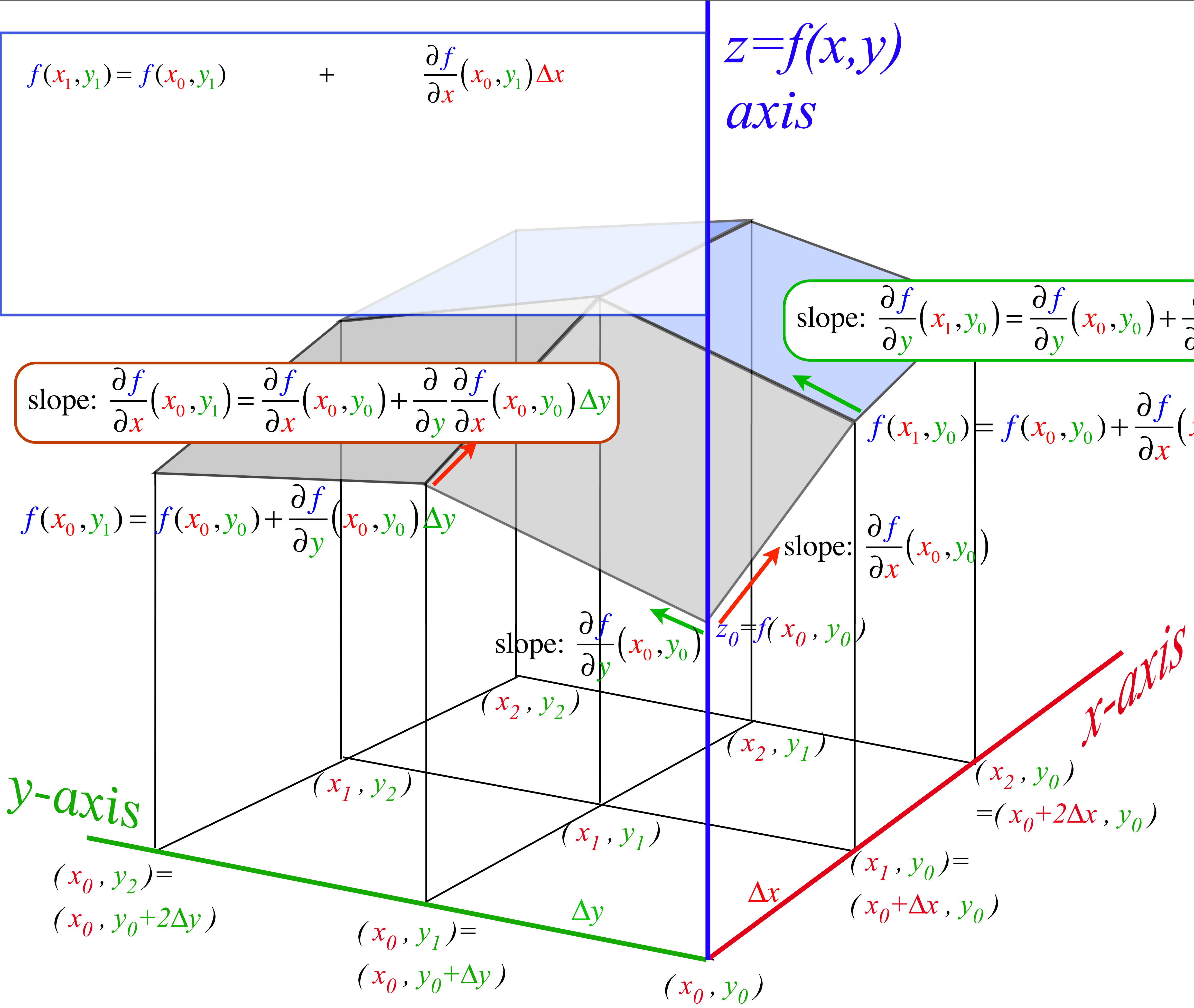
$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

$z_0=f(x_0,y_0)$

$$f(x_1,y_1)=f(x_0,y_1)\quad+\quad\frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$



slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$

$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

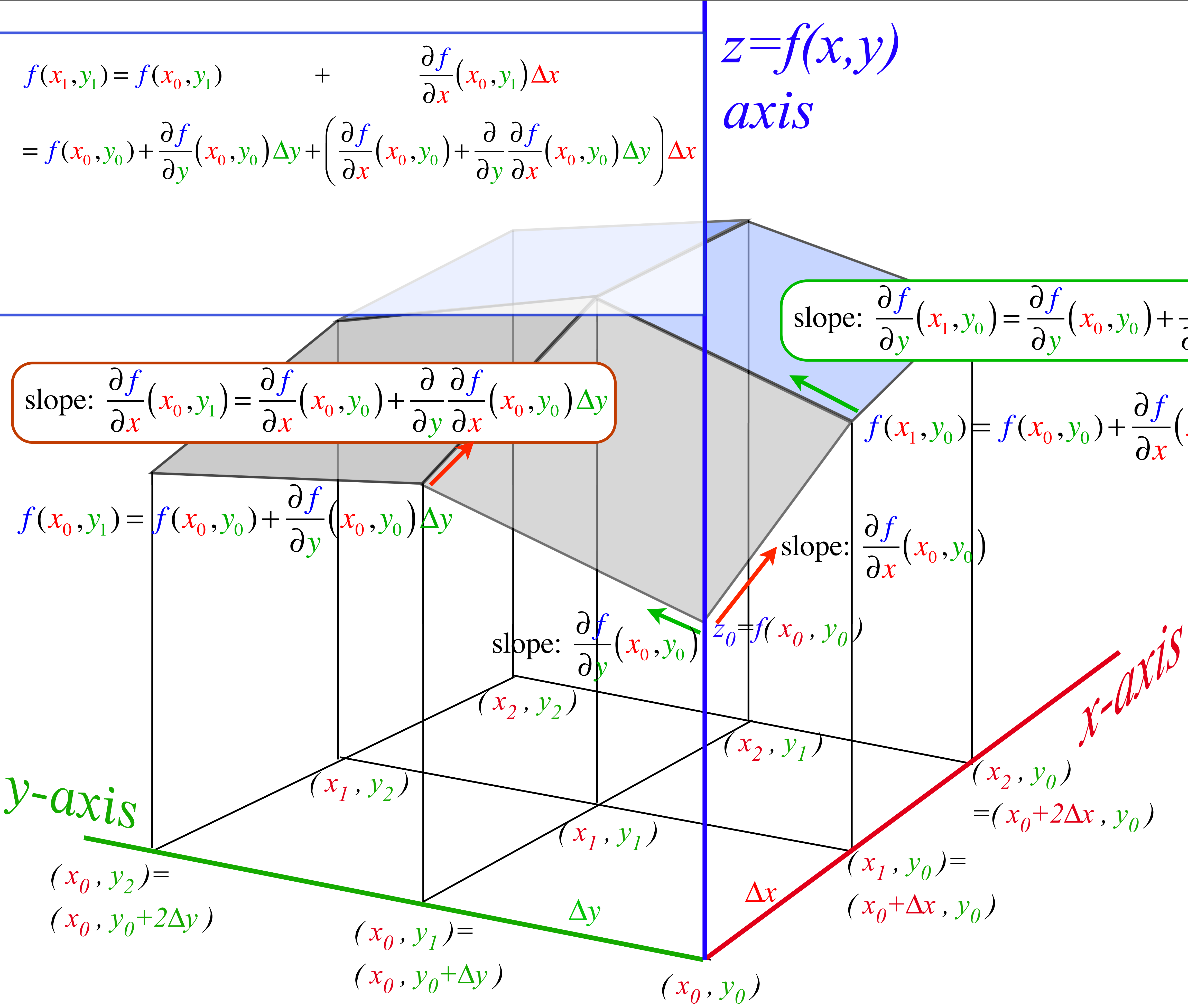
slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

$z_0=f(x_0,y_0)$

$$f(x_1,y_1)=f(x_0,y_1)+\frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$



$z=f(x,y)$ axis

slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$

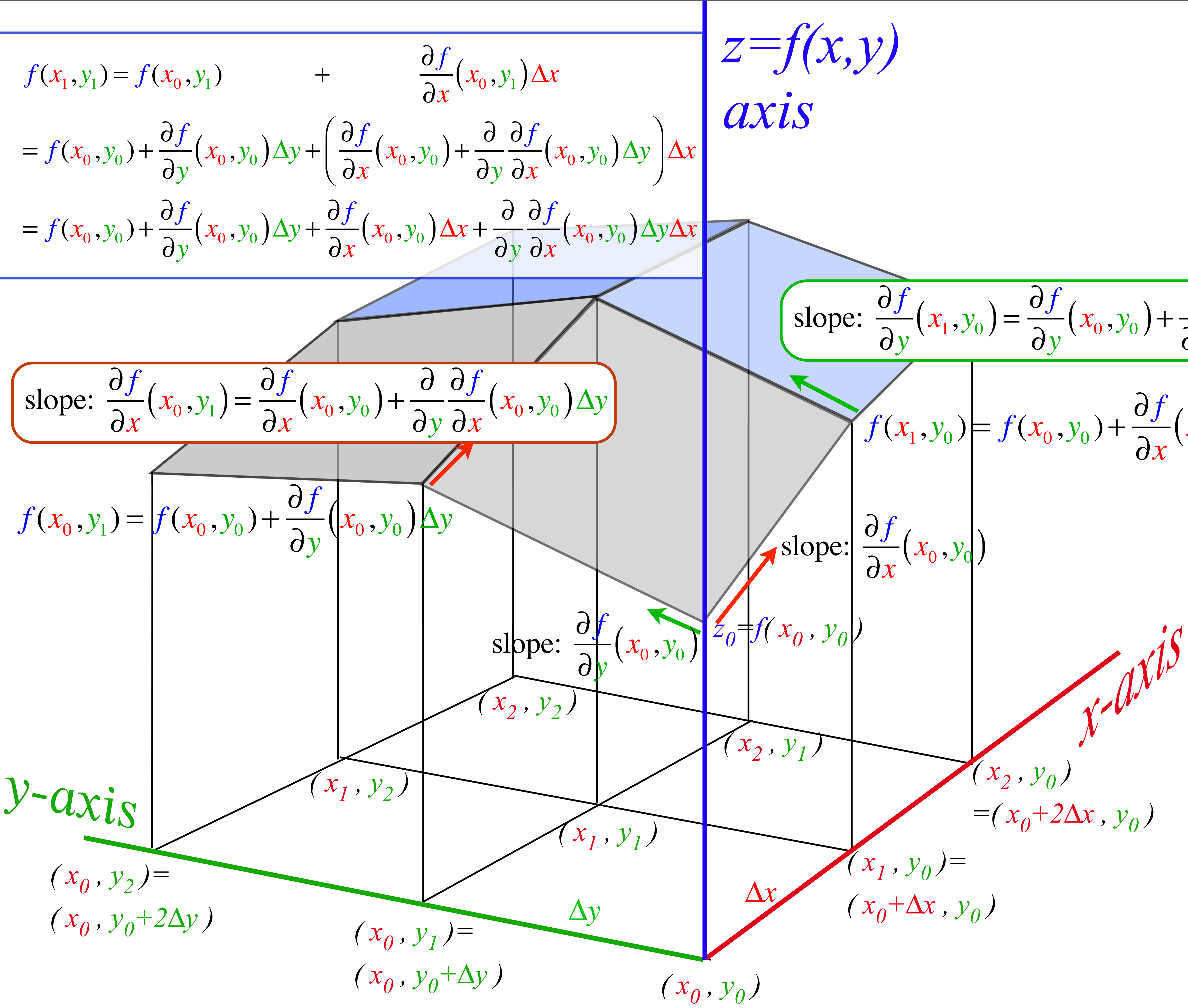
$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

$z_0=f(x_0,y_0)$

$$
\begin{aligned}
f(x_1,y_1) &= f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x \\
&= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x \\
&= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x
\end{aligned}
$$



$z=f(x,y)$ axis

slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$

$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

$z_0=f(x_0,y_0)$

(x_2,y_2)

(x_2,y_1)

(x_1,y_2)

(x_1,y_1)

x-axis

(x_2,y_0)
$=(x_0+2\Delta x,y_0)$

y-axis

$(x_0,y_2)=$
$(x_0,y_0+2\Delta y)$

Δx

Δy

$(x_1,y_0)=$
$(x_0+\Delta x,y_0)$

$(x_0,y_1)=$
$(x_0,y_0+\Delta y)$

(x_0,y_0)

$$f(x_1,y_1)=f(x_0,y_1) \qquad + \qquad \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$

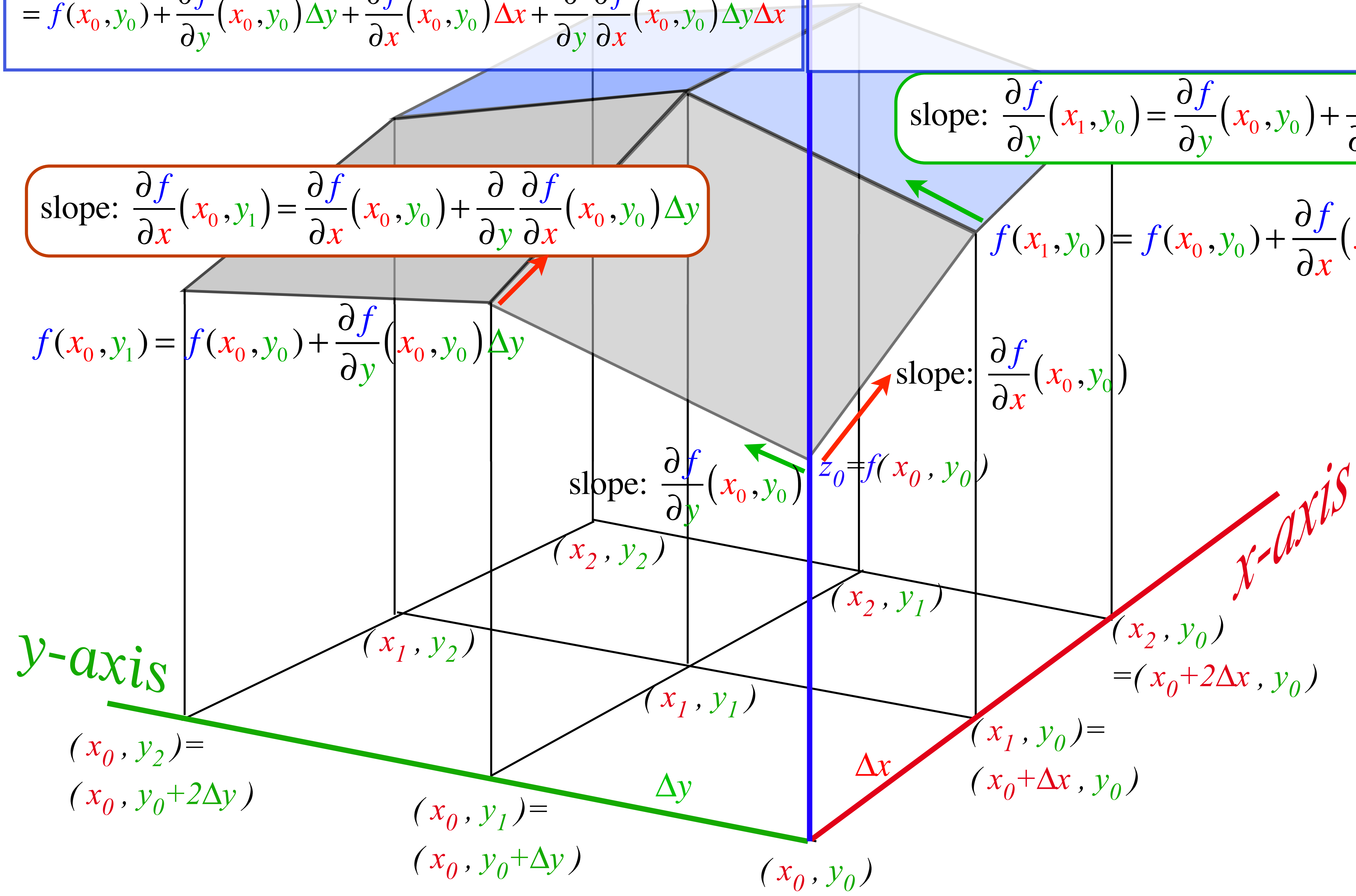
$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$

$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$

$$f(x_1,y_1)=f(x_1,y_0) \qquad + \qquad \frac{\partial f}{\partial y}(x_1,y_0)\Delta y$$

slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

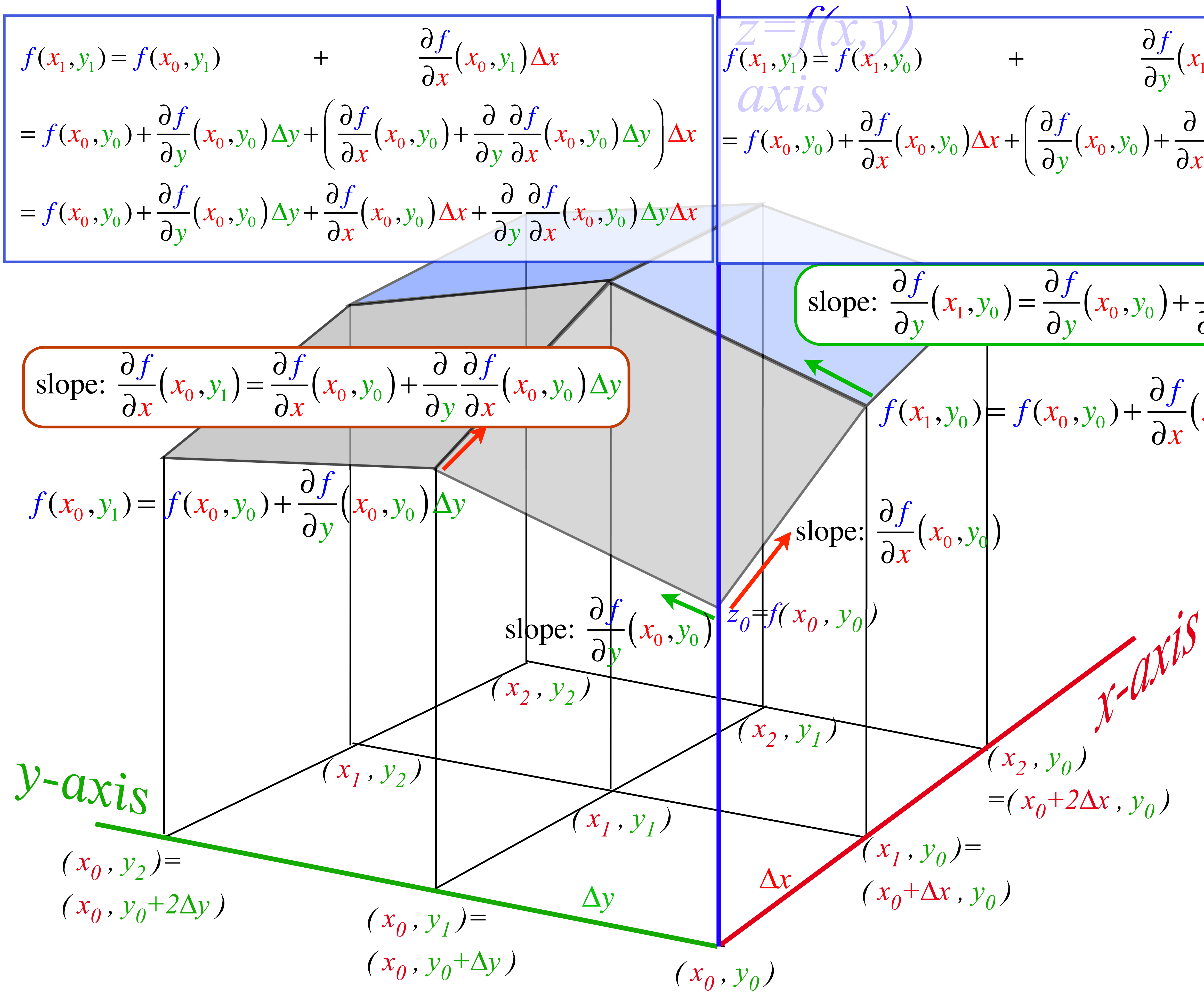


$$
\begin{aligned}
f(x_1,y_1) &= f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x \\
&= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x \\
&= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x
\end{aligned}
$$

$$
\begin{aligned}
f(x_1,y_1) &= f(x_1,y_0) + \frac{\partial f}{\partial y}(x_1,y_0)\Delta y \\
&= f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\left(\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\right)\Delta y
\end{aligned}
$$

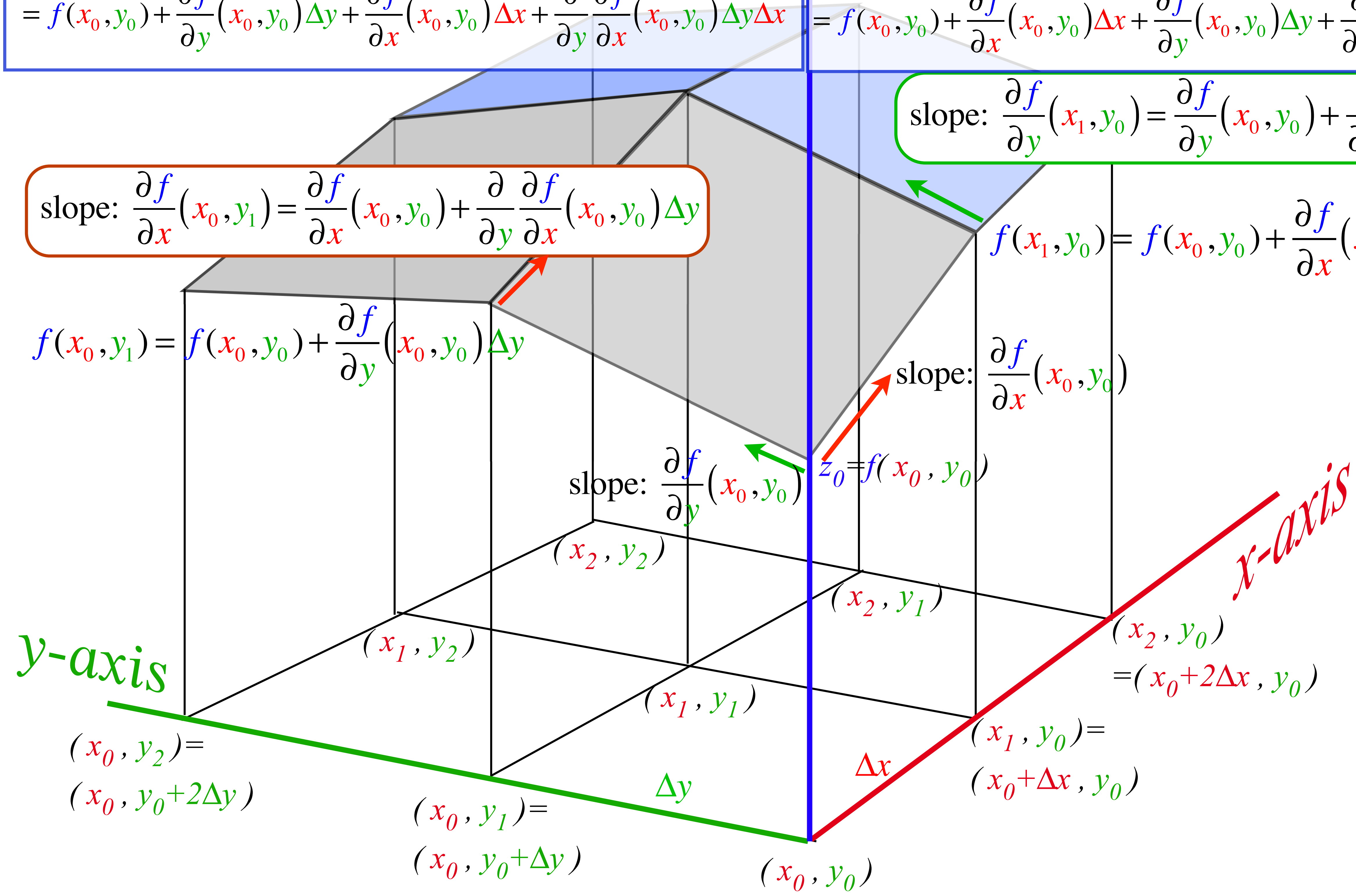
slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$



$$f(x_1,y_1) = f(x_0,y_1) + \frac{\partial f}{\partial x}(x_0,y_1)\Delta x$$
$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\left(\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\right)\Delta x$$
$$= f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y\Delta x$$

$$f(x_1,y_1) = f(x_1,y_0) + \frac{\partial f}{\partial y}(x_1,y_0)\Delta y$$
$$= f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\left(\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\right)\Delta y$$
$$= f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x\Delta y$$



slope: $\frac{\partial f}{\partial y}(x_1,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta x$

slope: $\frac{\partial f}{\partial x}(x_0,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta y$

$f(x_1,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x$

$f(x_0,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y$

slope: $\frac{\partial f}{\partial x}(x_0,y_0)$

slope: $\frac{\partial f}{\partial y}(x_0,y_0)$

Introduction to Lagrangian-Hamiltonian duality

Review of partial differential relations

→ *Chain rule and order symmetry*

Duality relations of Lagrangian and Hamiltonian ellipse

Introducing the 1^{st} *(partial* $\frac{\partial ?}{\partial ?}$ *) differential equations of mechanics*

What the geometry indicates....(Two important results)

$$f(x_1,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y\Delta x$$

What the geometry indicates....(Two important results)

$$f(x_1,y_1) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y\Delta x$$

1. Chain rules

$$\left[f(x_1,y_1) - f(x_0,y_0)\right] = df = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy \ldots_{(keep\ 1^{st} - order\ terms\ only!)}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}$$

(*shorthand notation*)

What the geometry indicates....(Two important results)

$$f(x_1,y_1)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

$$=f(x_0,y_0)+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y\Delta x$$

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$$\frac{df}{dt}=\frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt}+\frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f}=\frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial y}\dot{y}$$

(*shorthand notation*)

$$=\partial_x f\,\dot{x}+\partial_y f\,\dot{y}$$

2. Symmetry of partial deriv. ordering

(*pay attention to the* 2^{nd} *–order terms, too!*)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\quad \text{or:}\quad \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}\quad \text{or:}\quad \partial_y\partial_x f=\partial_x\partial_y f$$

(*shorthand notation*)

What the geometry indicates....(Two important results)

$$f(x_1,y_1) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)\Delta x\Delta y$$

$$= f(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0)\Delta y\Delta x$$

1. Chain rules

$$\left[f(x_1,y_1) - f(x_0,y_0)\right] = df = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy \ldots_{(\textit{keep } 1^{st}-\textit{order terms only!})}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)\frac{dx}{dt} + \frac{\partial f}{\partial y}(x_0,y_0)\frac{dy}{dt}$$

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} \qquad (\textit{shorthand notation}) \qquad = \partial_x f\,\dot{x} + \partial_y f\,\dot{y}$$

2. Symmetry of partial deriv. ordering

(*pay attention to the* 2^{nd} *−order terms, too!*)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{or:} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or:} \quad \partial_y \partial_x f = \partial_x \partial_y f$$

(*shorthand notation*)

$$\textit{Let}: \vec{\nabla} = \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \qquad \textit{so}: \vec{\nabla} f \bullet \mathbf{dr} = \begin{pmatrix} \partial_x f & \partial_y f \end{pmatrix} \bullet \begin{pmatrix} dx \\ dy \end{pmatrix} = \partial_x f\, dx + \partial_y f\, dy = df$$

Introduction to Lagrangian-Hamiltonian duality

- *Review of partial differential relations*
  - *Chain rule and order symmetry*
- → *Duality relations of Lagrangian and Hamiltonian ellipse*
- *Introducing the 1st (partial* $\frac{\partial ?}{\partial ?}$ *) differential equations of mechanics*

Three ways to express energy: *Consider kinetic energy (KE) first*

*1. **Lagrangian** is explicit function of **velocity**:* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$L(v_k\ldots) = \tfrac{1}{2}\,(m_1 v_1^2 + m_2 v_2^2 + \ldots) = L(\mathbf{v}\ldots) = \tfrac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \ldots = \tfrac{1}{2}\begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \ldots$$

*2. "**Estrangian**" is explicit function of **R**-rescaled velocity:*

(or l'Estrangian

*or: "**speedinum**" V* $\mathbf{V} = \mathbf{R}\bullet\mathbf{v}$ or: $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$E(V_k\ldots) = \tfrac{1}{2}\,(V_1^2 + V_2^2 + \ldots) = E(\mathbf{V}\ldots) = \tfrac{1}{2}\mathbf{V}\bullet\mathbf{1}\bullet\mathbf{V} + \ldots = \tfrac{1}{2}\begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \ldots$$

*3. **Hamiltonian** is explicit function of* $\mathbf{M}=\mathbf{R}^2$*-rescaled velocity:*

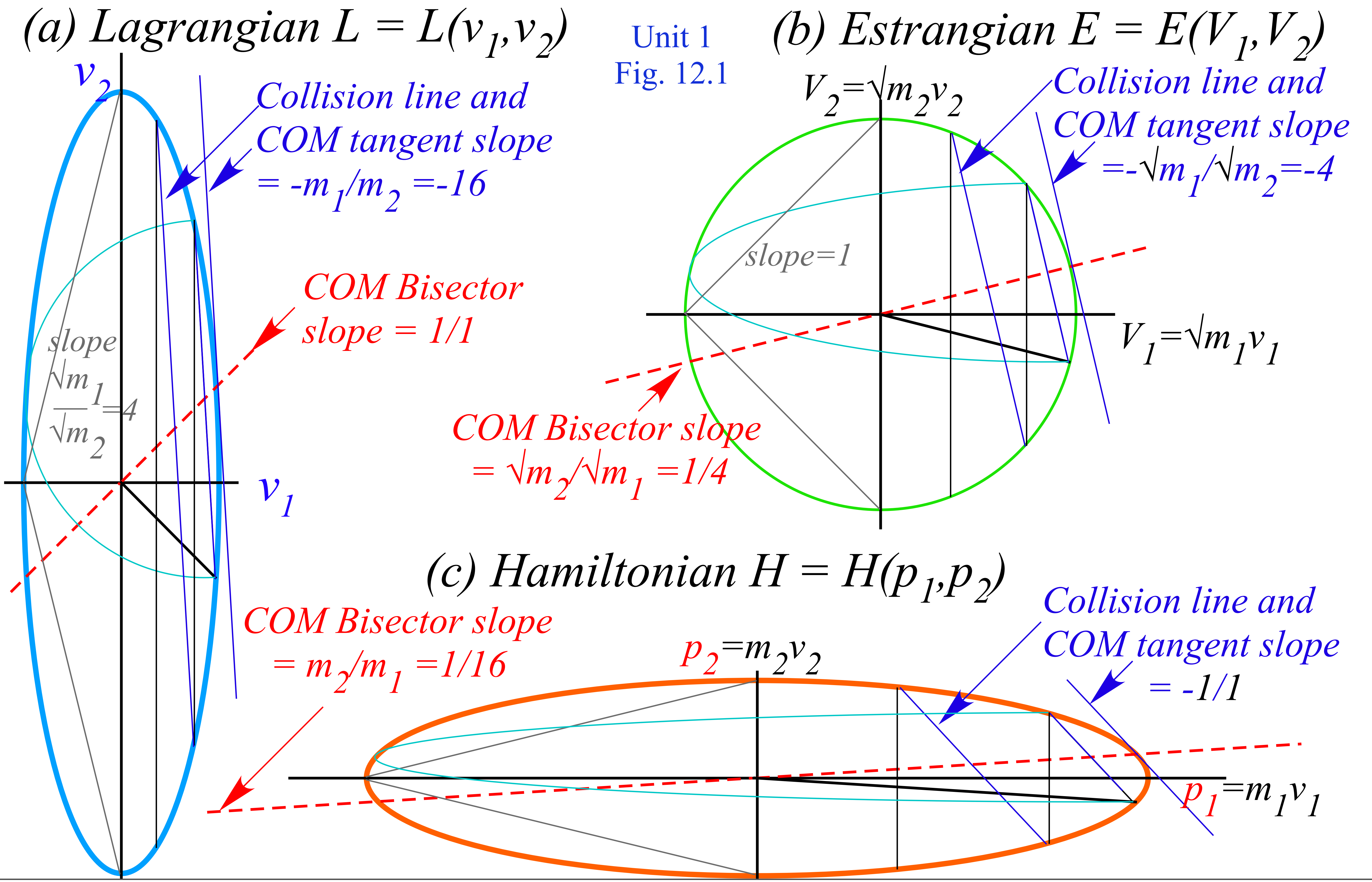
*or: **momentum p*** $\mathbf{p} = \mathbf{M}\bullet\mathbf{v}$ or: $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 v_1 \\ m_2 v_2 \end{pmatrix}$

$$H(p_k\ldots) = \tfrac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \ldots\right) = H(\mathbf{p}\ldots) = \tfrac{1}{2}\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p} + \ldots = \tfrac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \ldots$$

The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$:

Like Q=R²: $\mathbf{M}=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}=\mathbf{R}^2$ *Like √Q=R:* $\sqrt{\mathbf{M}}=\begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix}=\mathbf{R}$ *Like Q⁻¹=R⁻²:* $\mathbf{M}^{-1}=\begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix}=\mathbf{R}^{-2}$

Unit 1
Fig. 12.1



(a) Lagrangian $L = L(v_1,v_2)$

Collision line and COM tangent slope $= -m_1/m_2 = -16$

COM Bisector slope = 1/1

(b) Estrangian $E = E(V_1,V_2)$

Collision line and COM tangent slope $=-\sqrt{m_1}/\sqrt{m_2}=-4$

COM Bisector slope $= \sqrt{m_2}/\sqrt{m_1} = 1/4$

(c) Hamiltonian $H = H(p_1,p_2)$

COM Bisector slope $= m_2/m_1 = 1/16$

Collision line and COM tangent slope = -1/1

Introduction to Lagrangian-Hamiltonian duality

- *Review of partial differential relations*
    - *Chain rule and order symmetry*
- *Duality relations of Lagrangian and Hamiltonian ellipse*
- → *Introducing the* 1^{st} *(partial* $\frac{\partial ?}{\partial ?}$*) differential equations of mechanics*

Introducing the (partial$\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian***
have no explicit dependence
on ***momentum*** **p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian***
have no explicit dependence
on ***velocity*** **v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian***
have no explicit dependence
on ***speedinum*** **V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Introducing the (partial$\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian***
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$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

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have no explicit dependence
on ***velocity*** **v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian***
have no explicit dependence
on ***speedinum*** **V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2}$$
$$= \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2}$$
$$= \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Introducing the (partial$\frac{\partial ?}{\partial ?}$) differential equations of mechanics

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian *and* ***Estrangian***
have *<u>no</u>* *explicit dependence*
on ***momentum*** **p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian *and* ***Estrangian***
have *<u>no</u>* *explicit dependence*
on ***velocity*** **v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian *and* ***Hamiltonian***
have *<u>no</u>* *explicit dependence*
on ***speedinum*** **V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}}{2}$$
$$= \mathbf{M} \bullet \mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2}$$
$$= \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1st equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)

Unit 1
Fig. 12.2