Lecture 9
Thur. 9.22.2016

Equations of Lagrange and Hamilton mechanics in GeneralizedCurvilinear Coordinates (GCC)

(Ch. 12 of Unit 1 and Ch. 1-5 of Unit 2 and Ch. 1-5 of Unit 3)

Quick Review of Lagrange Relations in Lectures 7-8

Using differential chain-rules for coordinate transformations
- *Polar coordinate example of Generalized Curvilinear Coordinates (GCC)*
- *Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*
- *Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

How to say Newton's "F=ma" in Generalized Curvilinear Coords.
- *Use Cartesian KE quadratic form* $KE=T=1/2\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*
- *Lagrange GCC trickery gives Lagrange force equations*
- *Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

GCC Cells, base vectors, and metric tensors
- *Polar coordinate examples: Covariant* $\mathbf{E}_m$ *vs. Contravariant* $\mathbf{E}^m$
  - *Covariant* g_{mn} *vs. Invariant* δ_m^n *vs. Contravariant* g^{mn}

Lagrange prefers Covariant g_{mn} *with Contravariant velocity*
- *GCC Lagrangian definition*
- *GCC "canonical" momentum* p_m *definition*
- *GCC "canonical" force* F_m *definition*
  - *Coriolis "fictitious" forces (... and weather effects)*

Quick Review of Lagrange Relations in Lectures 7-8

→ 0th and 1st equations of Lagrange and Hamilton

Quick Review of Lagrange Relations in Lectures 7-8

0^{th} and 1^{st} equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian** and **Estrangian** have <u>no</u> explicit dependence on **momentum p

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian** and **Estrangian** have <u>no</u> explicit dependence on **velocity v

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian** and **Hamiltonian** have <u>no</u> explicit dependence on **speedinum V

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v} \bullet \mathbf{M} \bullet \mathbf{v}}{2}$$
$$= \mathbf{M} \bullet \mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1^{st} equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p} \bullet \mathbf{M}^{-1} \bullet \mathbf{p}}{2}$$
$$= \mathbf{M}^{-1} \bullet \mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1^{st} equation(s)

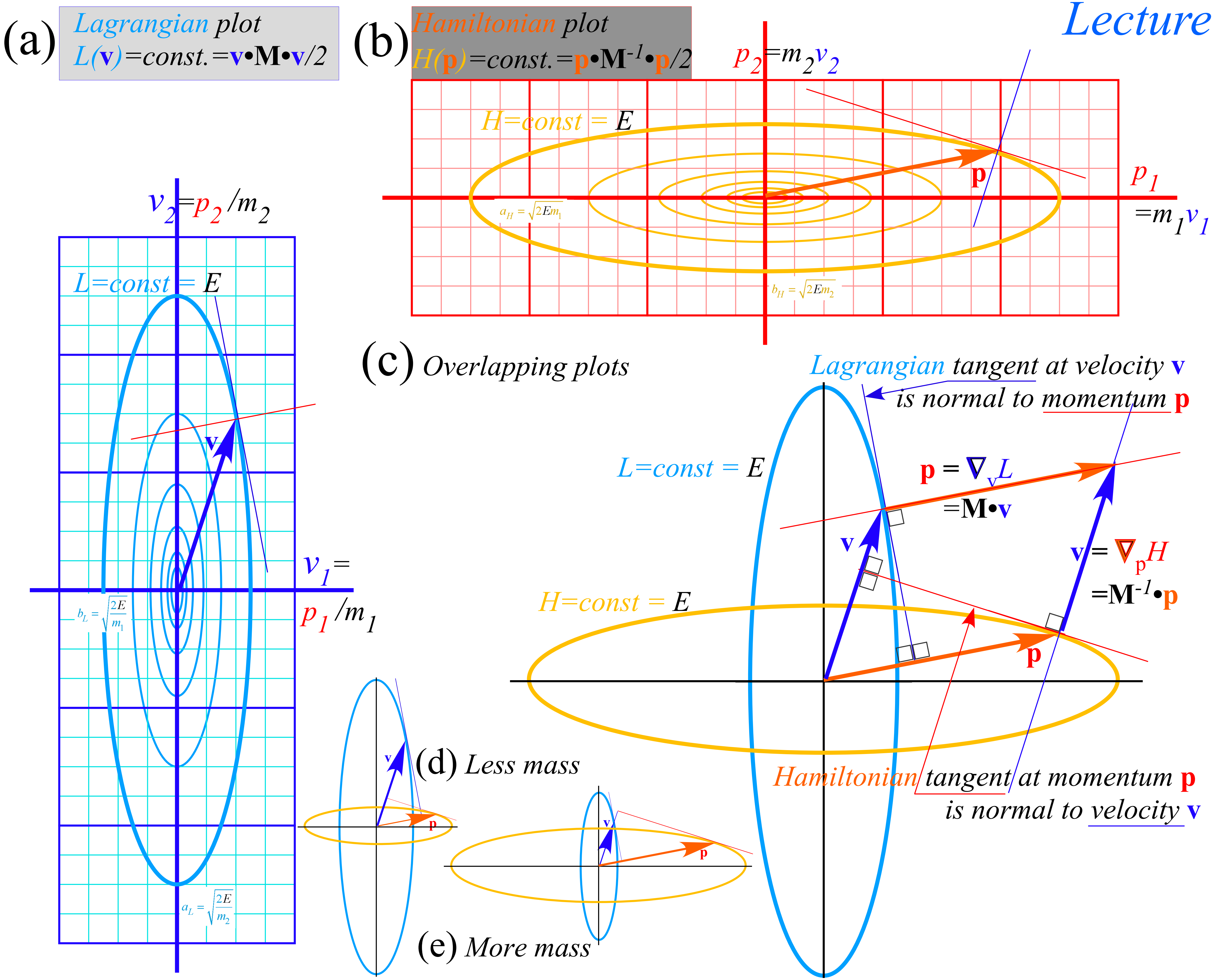
$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)

p. 25 of Lecture 8

Unit 1
Fig. 12.2

p. 28 of
Lecture 8



(a) *Lagrangian plot* $L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$

(b) *Hamiltonian plot* $H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$

(c) *Overlapping plots*

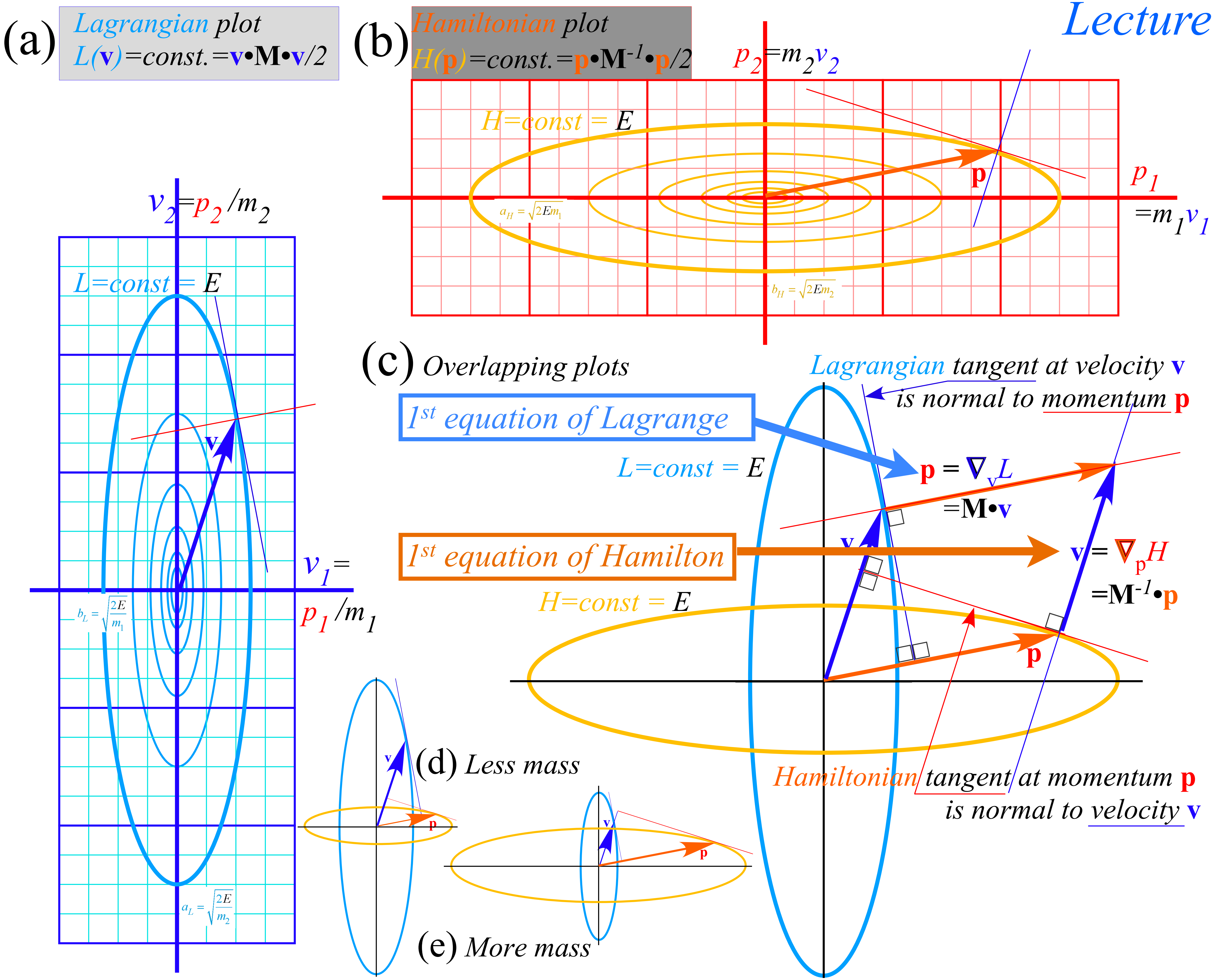
Lagrangian tangent at velocity $\mathbf{v}$ *is normal to momentum* $\mathbf{p}$

Hamiltonian tangent at momentum $\mathbf{p}$ *is normal to velocity* $\mathbf{v}$

(d) *Less mass*

(e) *More mass*

Unit 1
Fig. 12.2



Using differential chain-rules for coordinate transformations

→ Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1

Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2

Using differential chain-rules† *for coordinate transformations*

A pair of 2-variable functions f(x,y) and g(x,y) can define a coordinate system on (x,y)-space

for example: polar coordinates

$r^2(x,y)= x^2+y^2$ *and* $\theta(x,y)=atan2(y,x)$

(Not in text. Recall Lecture 8 p. 15-19)†

$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$dg(x,y)=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy$$

$$dr(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$$

$$d\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$$

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$$dr(x,y)=\frac{\partial r}{\partial x}dx+\frac{\partial r}{\partial y}dy$$

$$d\theta(x,y)=\frac{\partial \theta}{\partial x}dx+\frac{\partial \theta}{\partial y}dy$$

Easy to invert differential chain relations (even if functions are not easily inverted)

$$dx=\frac{\partial x}{\partial f}df+\frac{\partial y}{\partial g}dg$$

$$dy=\frac{\partial y}{\partial f}df+\frac{\partial y}{\partial g}dg$$

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial \theta}d\theta$$

$$dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial \theta}d\theta$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

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Notation for differential GCC (Generalized Curvilinear Coordinates $\{q^1, q^2, q^3, \ldots\}$)

$$dx^j=\frac{\partial x^j}{\partial q^m}dq^m\left(\equiv\sum_{m=1}^{N}\frac{\partial x^j}{\partial q^m}dq^m \quad \left\{\begin{matrix}\text{Defining a shorthand}\\ \text{dummy-index } m\text{-sum}\end{matrix}\right\}\right)$$

What does "*q*" stand for?
One guess: "*Queer*"
And they do get pretty queer!

These x^j are plain old CC (Cartesian Coordinates $\{dx^1=dx,\ dx^2=dy,\ dx^3=dx,\ dx^4=dt\}$)

Using differential chain-rules[†] for coordinate transformations

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$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

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for example: *polar coordinates*

$r^2(x,y)=x^2+y^2$ *and* $\theta(x,y)=\mathrm{atan2}(y,x)$

(Not in text. Recall Lecture 8 p. 15-19)[†]

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One guess: "*Queer*"
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Connection lines may help to indicate summation (OK on scratch paper...Difficult in text)

These x^j are plain old CC (Cartesian Coordinates $\{dx^1=dx,\ dx^2=dy,\ dx^3=dx,\ dx^4=dt\}$)

Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

→ *Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1*

Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2

Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

$$dx^j = \frac{\partial x^j}{\partial q^m} dq^m$$

Same kind of linear relation exists between CC velocity $v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt}$ and GCC velocity $\upsilon^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt}$

$$\dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m$$

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This is a key "lemma-1" for setting up mechanics:

or:

$$\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m}$$

lemma-1

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$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \textit{lemma-1}}$$

Jacobian J_m^j matrix gives each CCC differential dx^j or velocity $\dot{x}^j$ in terms of GCC dq^m or $\dot{q}^m$.

$$J_m^j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \left\{ \begin{array}{l} \text{Defining } \textit{Jacobian} \\ \text{matrix component} \end{array} \right\}$$

Recall polar coordinate transformation matrix:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

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{Defining *Jacobian* matrix component}

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Inverse (so-called) *Kajobian* K_j^m matrix is flipped partial derivatives of J_m^j.

$$K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j}$$

{Defining *"Kajobian"* (inverse to Jacobian)}

Polar coordinate inverse transformation matrix:

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix}$$

$$= \frac{\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}{(\det J = r)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

Defining 2x2 matrix inverse: *(always test inverse matrices!)*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}{AD - BC}$$

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Jacobian J_m^j matrix gives each CCC differential dx^j or velocity $\dot{x}^j$ in terms of GCC dq^m or $\dot{q}^m$.

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$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix}$$

Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

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Product of matrix J_m^j and K_j^m is a unit matrix by definition of partial derivatives.

(always test inverse matrices!)

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_j^m \cdot J_n^j \equiv \frac{\partial q^m}{\partial x^j} \cdot \frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial q^n} = \delta_n^m = \begin{cases} 1 & \textit{if } m = n \\ 0 & \textit{if } m \neq n \end{cases}$$

Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1

→ *Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2*

Getting the GCC ready for mechanics (2^{nd} part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m \right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m} \right)\dot{q}^m + \frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

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Apply derivative chain sum to Jacobian.

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n \partial q^m}\right)\frac{dq^n}{dt}$$

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(Not in text. Recall Lecture 9 p. 15-19)†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m\partial_n=\partial_n\partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^n\,\partial q^m}\right)\frac{dq^n}{dt} = \left(\frac{\partial^2 x^j}{\partial q^m\,\partial q^n}\right)\frac{dq^n}{dt} = \frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

Getting the GCC ready for mechanics (2^{nd} part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u\cdot v)=\frac{du}{dt}\cdot v+u\cdot\frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt}\dot{x}^j = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m\right)=\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)\dot{q}^m+\frac{\partial x^j}{\partial q^m}\ddot{q}^m$$

(Not in text. Recall Lecture 9 p. 15-19)†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m\partial_n=\partial_n\partial_m$

$$\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)=\frac{\partial}{\partial q^n}\left(\frac{\partial x^j}{\partial q^m}\right)\frac{dq^n}{dt}=\left(\frac{\partial^2 x^j}{\partial q^n\,\partial q^m}\right)\frac{dq^n}{dt}=\left(\frac{\partial^2 x^j}{\partial q^m\,\partial q^n}\right)\frac{dq^n}{dt}=\frac{\partial}{\partial q^m}\left(\frac{\partial x^j}{\partial q^n}\frac{dq^n}{dt}\right)$$

By chain-rule def. of CC velocity:

$$=\frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$$

Getting the GCC ready for mechanics (2^{nd} part)

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By chain-rule def. of CC velocity: $= \frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$

This is the key "lemma-2" for setting up Lagrangian mechanics .

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \textit{lemma 2}}$$

Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u\cdot v)=\frac{du}{dt}\cdot v+u\cdot\frac{dv}{dt}$

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By chain-rule def. of CC velocity: $= \frac{\partial}{\partial q^m}\left(\dot{x}^j\right)$

The "lemma-1" was in the GCC velocity analysis just before this one for acceleration.

$$\boxed{\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \qquad \textit{lemma 1}}$$

This is the key "lemma-2" for setting up Lagrangian mechanics .

$$\boxed{\frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right) = \frac{\partial \dot{x}^j}{\partial q^m} \qquad \textit{lemma 2}}$$

How to say Newton's "F=ma" in Generalized Curvilinear Coords.

→ *Use Cartesian KE quadratic form* $KE=T=1/2\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*

Lagrange GCC trickery gives Lagrange force equations

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)

Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$

$$T = \frac{1}{2} M_{jk}\, v^j v^k = \frac{1}{2} M_{jk}\, \dot{x}^j \dot{x}^k$$

where: M_{jk} are CC inertia constants

Multidimensional CC version of Newt-II (**F**=**M**•**a**) using M_{jk} *constants*

$$f_j = M_{jk}\, a^k = M_{jk}\, \ddot{x}^k$$

Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

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Multidimensional CC version of Newt-II (**F**=**M•a**) using M_{jk} *constants*

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Multidimensional CC version of work-energy differential (dW= **F**•d**x**). *Insert GCC differentials dq^m*

$$dW=f_j dx^j=f_j\left(\frac{\partial x^j}{\partial q^m}dq^m\right)=M_{jk}\,\ddot{x}^k\left(\frac{\partial x^j}{\partial q^m}dq^m\right)$$

(It's time to bring in the queer q^m !)

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dq^{m} are independent so dq^{m}-sum is true term-by-term.

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Here *generalized GCC force component F_m* is defined:

$$\textit{where}: \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\, \ddot{x}^k \frac{\partial x^j}{\partial q^m}$$

How to say Newton's "F=ma" in Generalized Curvilinear Coords.

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→ *Lagrange GCC trickery gives Lagrange force equations*

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)

Now Lagrange GCC trickery begins

Obvious stuff...(sort of, if you've looked at it for a century!)

Lagrange's clever end game: First set $A = M_{jk}\ddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\left[\ddot{A}B = \frac{d}{dt}\left(\dot{A}B\right) - \dot{A}\dot{B}\right]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \overset{\ddot{A}B}{M_{jk}\ddot{x}^k \frac{\partial x^j}{\partial q^m}} = \frac{d}{dt}\overset{\left(\dot{A}B\right)}{\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right)} - \overset{\dot{A}\dot{B}}{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}$$

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Cartesian M_{jk}
must be constant
for this to work
(Bye, Bye relativistic mechanics or QM!)

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$\ddot{A}B$ $\qquad$ $\left(\dot{A}B\right)$ $\qquad$ $\dot{A}\dot{B}$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right) - M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)$$

Then convert ∂x^j to $\partial \dot{x}^j$ by *Lemma 1* and *Lemma 2* on 2nd term.

$$F_m = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right) - M_{jk}\dot{x}^k \left(\frac{\partial \dot{x}^j}{\partial q^m}\right)$$

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$$F_m = f_j \frac{\partial x^j}{\partial q^m} = \overset{\ddot{A}B}{M_{jk}\,\ddot{x}^k \frac{\partial x^j}{\partial q^m}} = \frac{d}{dt}\overset{\left(\dot{A}B\right)}{\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right)} - \overset{\dot{A}\dot{B}}{M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)}$$

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$$F_m = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m}\right) - M_{jk}\dot{x}^k\left(\frac{\partial \dot{x}^j}{\partial q^m}\right)$$

Simplify using: $\left[M_{ij}v^i \frac{\partial v^j}{\partial q} = M_{ij}\frac{\partial}{\partial q}\frac{v^i v^j}{2}\right]$ where q may be $\dot{q}^m$ or q^m

$$F_m = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right) - \frac{\partial}{\partial q^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right)$$

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$$\overset{\ddot{A}B}{}\qquad\qquad\overset{\left(\dot{A}B\right)}{}\qquad\qquad\overset{\dot{A}\dot{B}}{}$$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk}\,\ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left(M_{jk}\dot{x}^k \frac{\partial x^j}{\partial q^m}\right) - M_{jk}\dot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)$$

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Simplify using: $\left[M_{ij} v^i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \frac{v^i v^j}{2}\right]$ where q may be $\dot{q}^m$ or q^m

$$F_m = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right) - \frac{\partial}{\partial q^m}\left(\frac{M_{jk}\dot{x}^k\dot{x}^j}{2}\right)$$

The result is *Lagrange's GCC force equation* in terms of *kinetic energy* $T = \frac{1}{2} M_{jk}\,\dot{x}^j \dot{x}^k$

$$\boxed{F_m = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}}$$

or: $\mathbf{F} = \frac{d}{dt}\frac{\partial T}{\partial \mathbf{v}} - \frac{\partial T}{\partial \mathbf{r}}$

How to say Newton's "F=ma" in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE=T=1/2\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}$ *and* $\mathbf{F}=\mathbf{M}\bullet\mathbf{a}$ *to get GCC force*

Lagrange GCC trickery gives Lagrange force equations

→ *Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)*

But, Lagrange GCC trickery is not yet done...

(Still another trick-up-the-sleeve!)

If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

But, Lagrange GCC trickery is not yet done...

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If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes *Lagrange's GCC potential equation* with a new definition for the *Lagrangian: L=T-U*.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}$$

$$L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

U(r) has
NO explicit
velocity
dependence!

But, Lagrange GCC trickery is not yet done...

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If the force is conservative it's a gradient $\mathbf{F} = -\nabla U$ In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes *Lagrange's GCC potential equation* with a new definition for the *Lagrangian: L=T-U*.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m} \qquad L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$ *U(r) has* *NO explicit velocity dependence!*

Lagrange's 1st GCC equation
(Defining GCC momentum)

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = \frac{\partial L}{\partial q^m}$$

Recall :

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

Lagrange's 2nd GCC equation
(Change of GCC momentum)

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

GCC Cells, base vectors, and metric tensors

→ *Polar coordinate examples:* Covariant $\mathbf{E}_m$ *vs.* Contravariant $\mathbf{E}^m$

Covariant g_{mn} *vs.* Invariant $\delta_m{}^n$ *vs.* Contravariant g^{mn}

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r \;\; \mathbf{E}_2=\mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r \;\; \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J\rangle = \begin{array}{cc} \begin{pmatrix} \dfrac{\partial x^1}{\partial q^1} & \dfrac{\partial x^1}{\partial q^2} \\ \dfrac{\partial x^2}{\partial q^1} & \dfrac{\partial x^2}{\partial q^2} \end{pmatrix} \\ \uparrow\mathbf{E}_1 \quad \uparrow\mathbf{E}_2 \end{array} = \begin{array}{cc} \begin{pmatrix} \dfrac{\partial x}{\partial r}=\cos\phi & \dfrac{\partial x}{\partial \phi}=-r\sin\phi \\ \dfrac{\partial y}{\partial r}=\sin\phi & \dfrac{\partial y}{\partial \phi}=r\cos\phi \end{pmatrix} \\ \uparrow\mathbf{E}_r \qquad\qquad \uparrow\mathbf{E}_\phi \end{array}$$

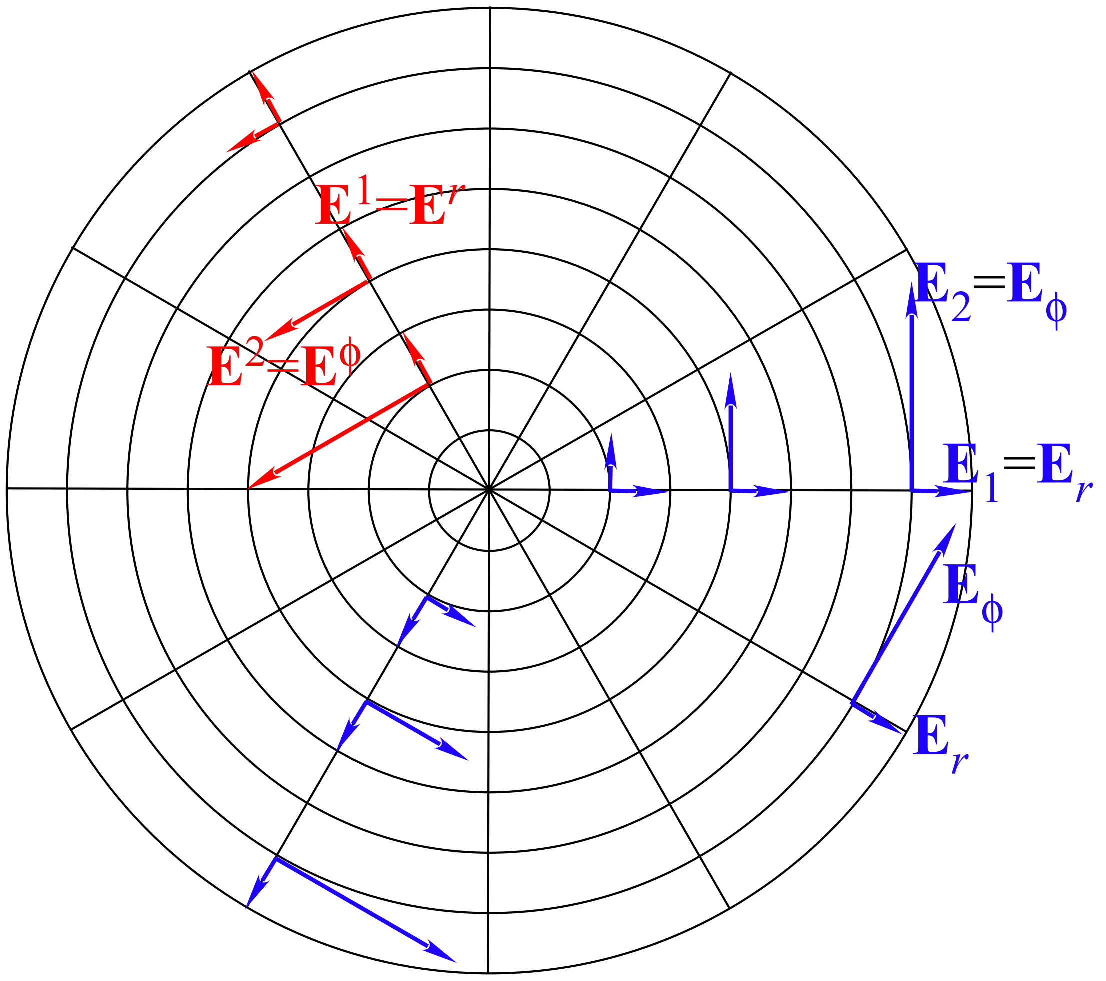
$$\langle K\rangle = \langle J^{-1}\rangle = \begin{pmatrix} \dfrac{\partial r}{\partial x}=\cos\phi & \dfrac{\partial r}{\partial y}=\sin\phi \\ \dfrac{\partial \phi}{\partial x}=\dfrac{-\sin\phi}{r} & \dfrac{\partial \phi}{\partial y}=\dfrac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Derived from polar definition: $x=r\cos\phi$ *and* $y=r\sin\phi$

Inverse polar definition:

$r^2=x^2+y^2$ *and* $\phi=\text{atan2}(y,x)$

(a) Polar coordinate bases



Unit 1
Fig. 12.10

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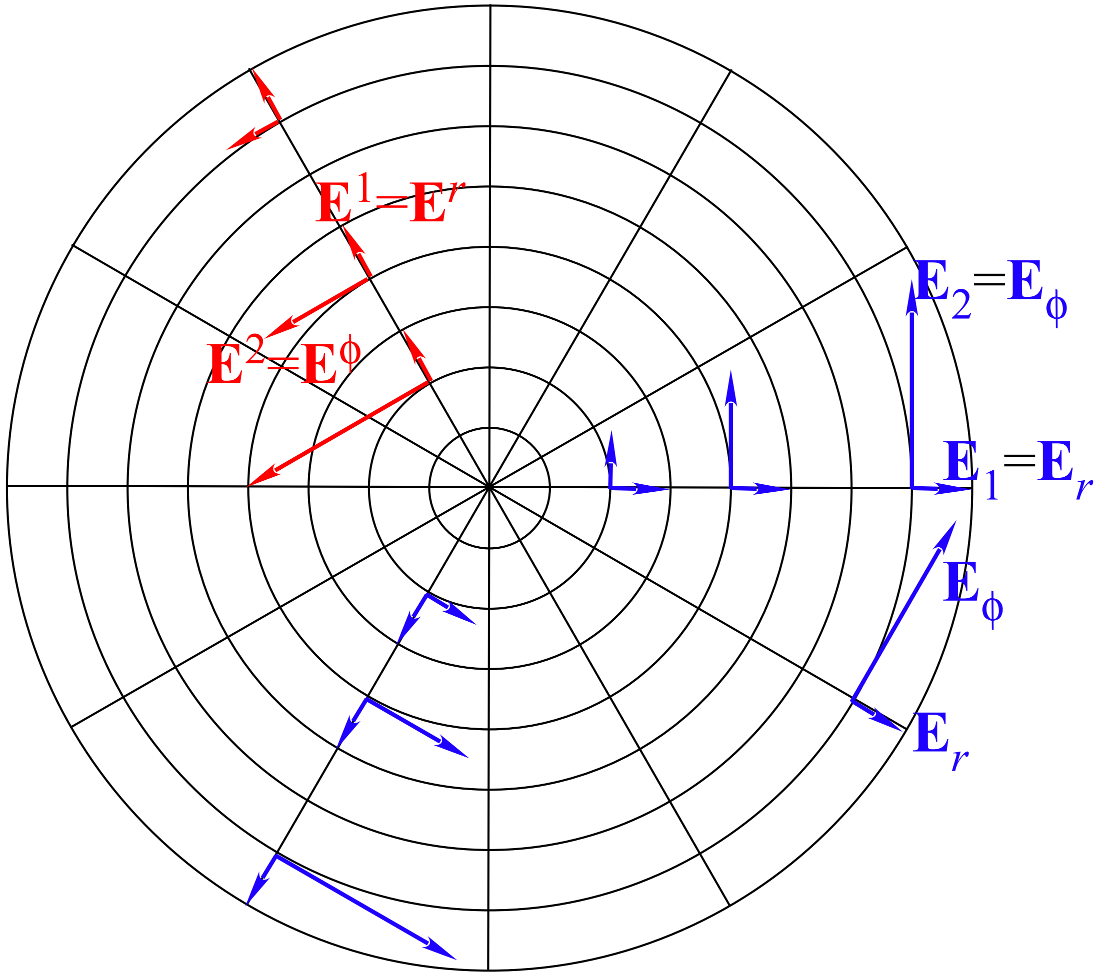
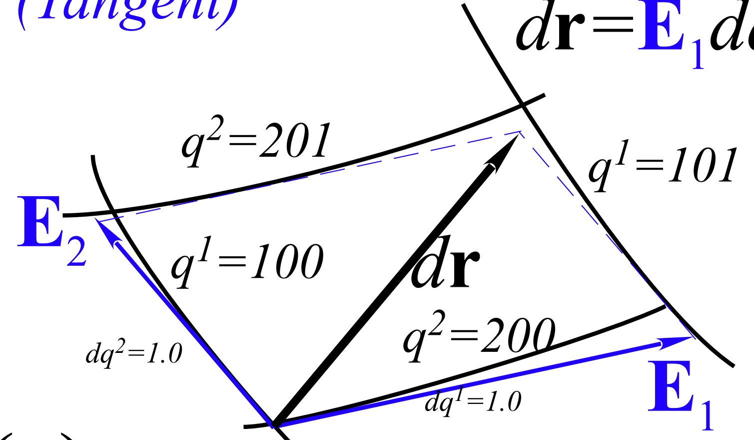


(b) *Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$

(Tangent)

$d\mathbf{r}=\mathbf{E}_1 dq^1+\mathbf{E}_2 dq^2$

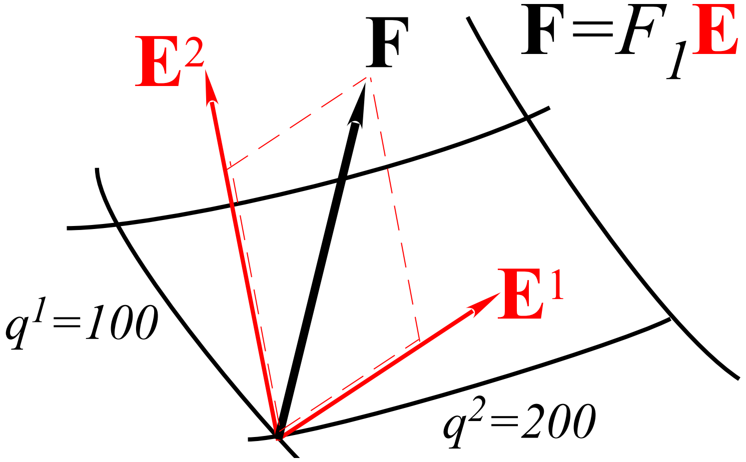


NOTE:These are 2D drawings! No 3D *perspective*

(c) *Contra*variant bases $\{\mathbf{E}^1\ \mathbf{E}^2\}$

(Normal)

$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$



Unit 1
Fig. 12.10

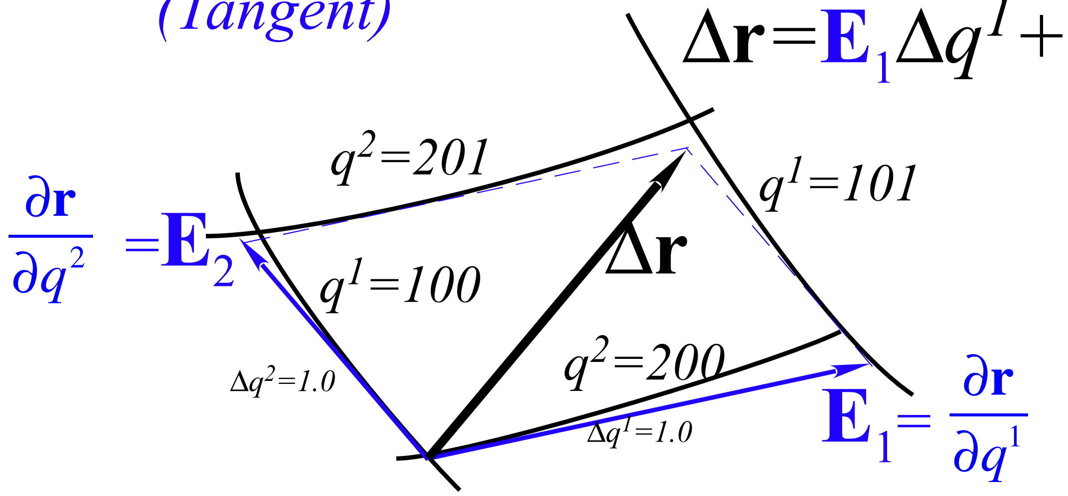
Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\overset{\text{geometric unit}}{\wedge}$ cell walls

(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



NOTE:These
are 2D drawings!
No 3D *perspective*

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match *geometric unit* cell walls

(Tangent)

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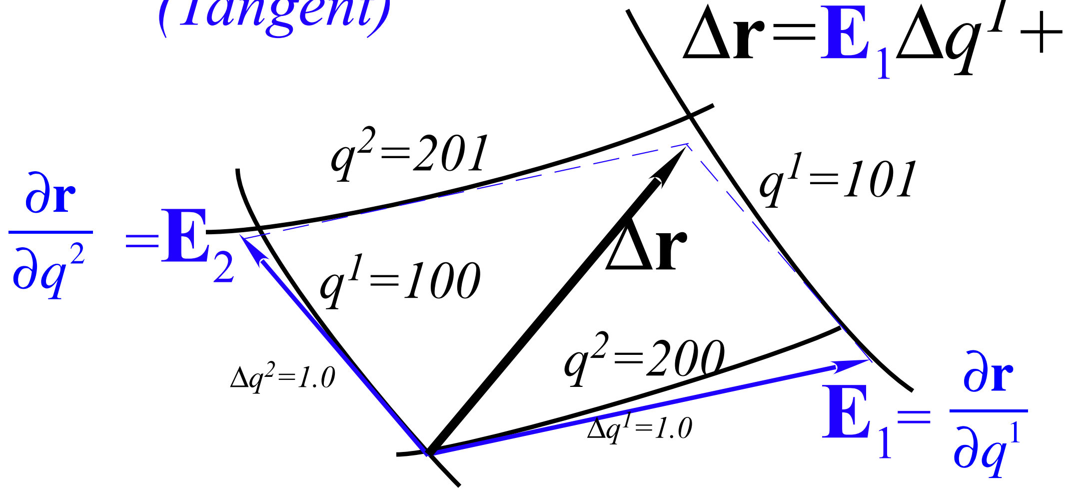


$\mathbf{E}_1$ *follows tangent to* q^2=*const.* ...
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... *remain constant*

NOTE:These
are 2D drawings!
No 3D *perspective*

Comparison: Covariant $\mathbf{E}_m=\frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m=\frac{\partial q^m}{\partial \mathbf{r}}=\nabla q^m$

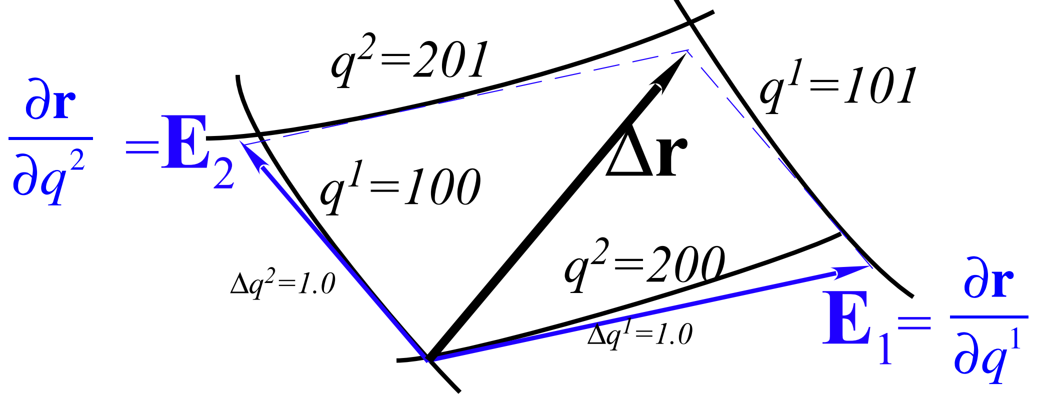
geometric unit

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match ^ cell walls

(Tangent)

$$\Delta\mathbf{r}=\mathbf{E}_1\Delta q^1+\mathbf{E}_2\Delta q^2$$

is based on chain rule: $d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial q^1}dq^1+\frac{\partial \mathbf{r}}{\partial q^2}dq^2=\mathbf{E}_1dq^1+\mathbf{E}_2dq^2$



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$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V}=V^1\mathbf{E}_1+V^2\mathbf{E}_2=V^1\frac{\partial \mathbf{r}}{\partial q^1}+V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

NOTE:These
are 2D drawings!
No 3D perspective

Comparison: Covariant $\mathbf{E}_m=\frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m=\frac{\partial q^m}{\partial \mathbf{r}}=\nabla q^m$

geometric unit

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match ^ cell walls

(Tangent)

$$\Delta\mathbf{r}=\mathbf{E}_1\Delta q^1+\mathbf{E}_2\Delta q^2$$

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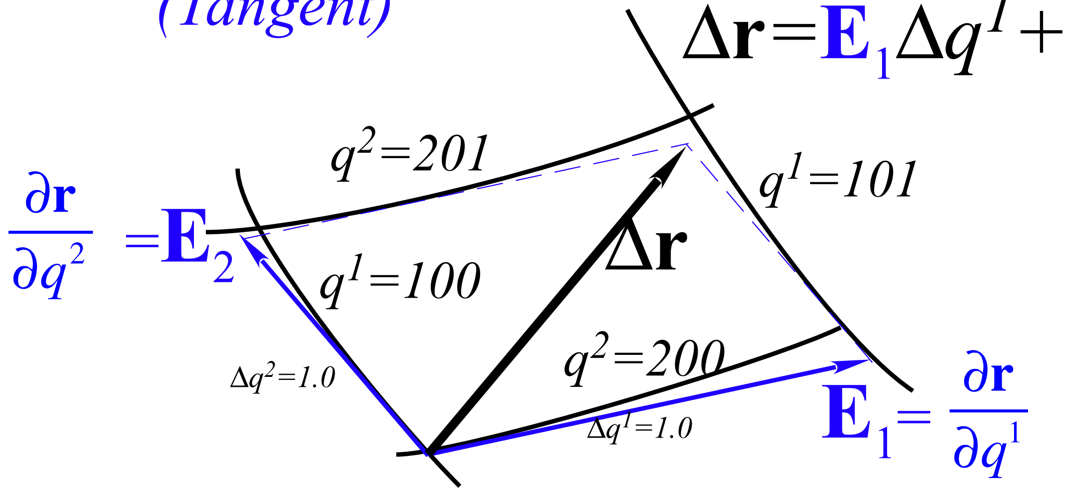

$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,*... remain constant*

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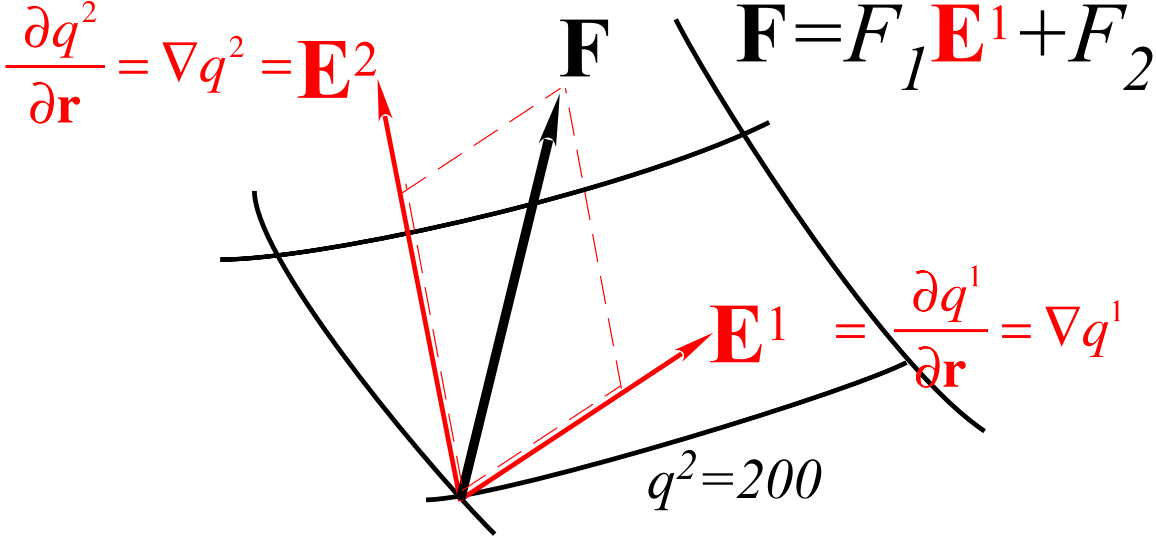
$$\mathbf{V}=V^1\mathbf{E}_1+V^2\mathbf{E}_2=V^1\frac{\partial \mathbf{r}}{\partial q^1}+V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$match reciprocal cells

(Normal)



$$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$$

NOTE:These
are 2D drawings!
No 3D *perspective*

$\mathbf{E}^1$ *is normal to* q^1*=const. since*
gradient *of* q^1*is vector sum* $\nabla q^1=\begin{pmatrix}\frac{\partial q^1}{\partial x}\\ \frac{\partial q^1}{\partial y}\end{pmatrix}$
of all its partial derivatives

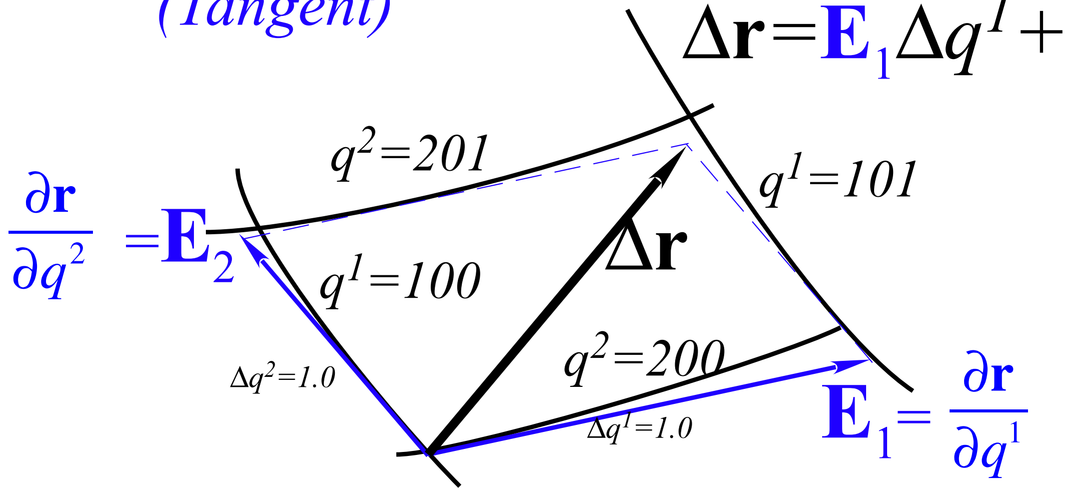
Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\overset{\text{geometric unit}}{\wedge}$ cell walls

(Tangent)

$$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



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since only q^1 varies in $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... remain constant

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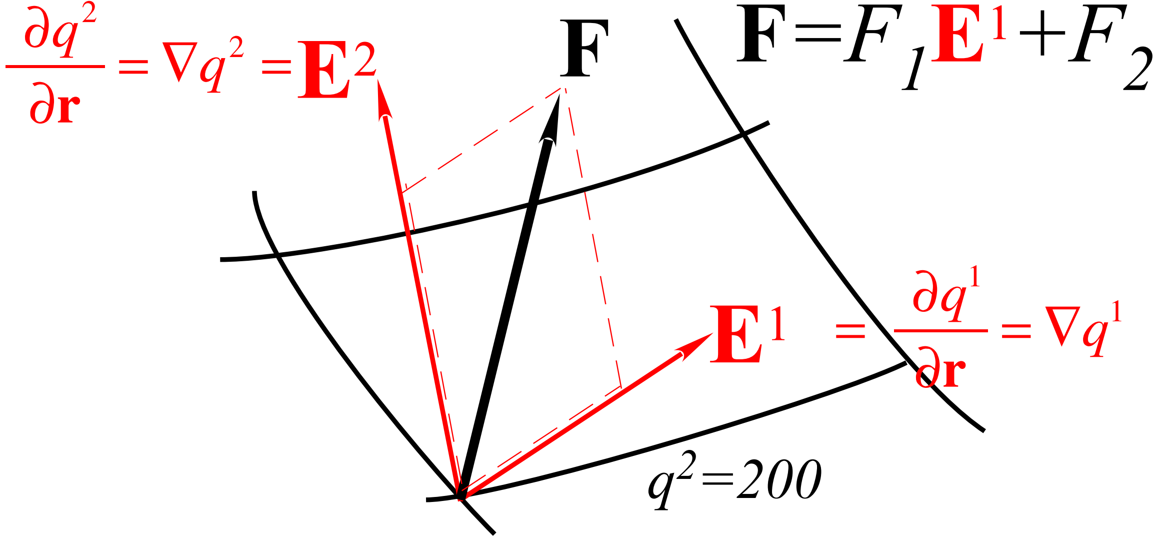
$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1\frac{\partial \mathbf{r}}{\partial q^1} + V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

Contravariant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

NOTE:These
are 2D drawings!
No 3D *perspective*

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2$$



$\mathbf{E}^1$ is normal to q^1=const. since
gradient *of q^1 is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
of all its partial derivatives

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F} = F_1\mathbf{E}^1 + F_2\mathbf{E}^2 = F_1\frac{\partial q^1}{\partial \mathbf{r}} + F_2\frac{\partial q^2}{\partial \mathbf{r}} = F_1\nabla q^1 + F_2\nabla q^2$$

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match $\overset{\text{geometric unit}}{\wedge}$ cell walls

(Tangent)

$$\Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$$

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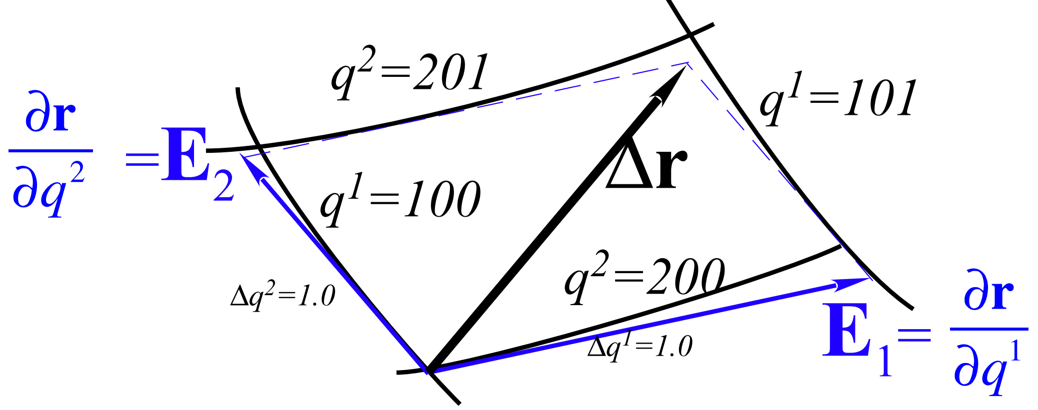

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$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

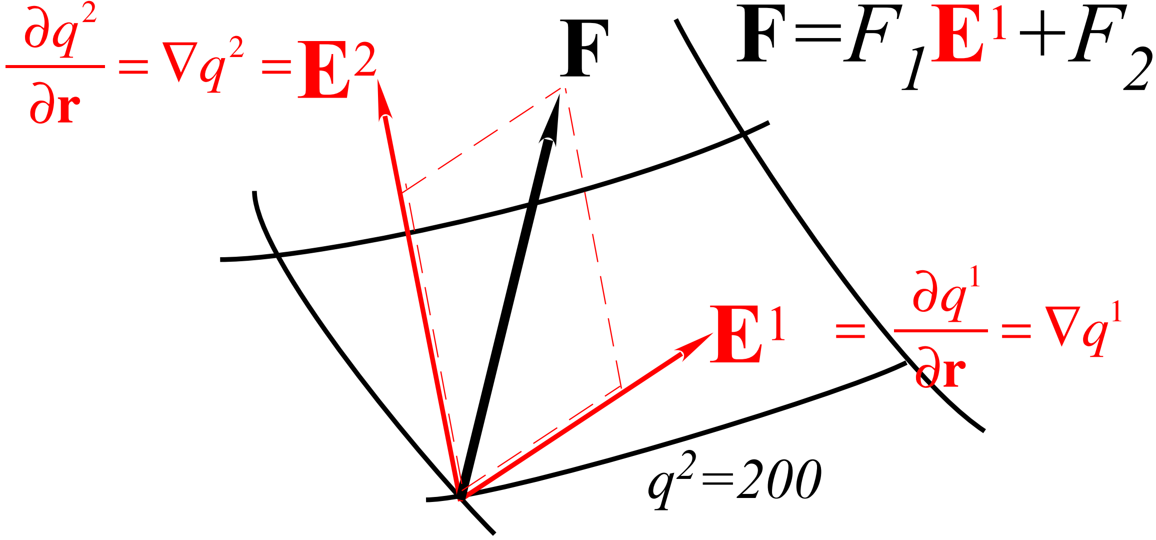
Co-Contra dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ are orthonormal:

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta^n_m$$

Contravariant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$



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GCC Cells, base vectors, and metric tensors

Polar coordinate examples: Covariant $\mathbf{E}_m$ *vs.* Contravariant $\mathbf{E}^m$

→ Covariant g_{mn} *vs.* Invariant $\delta_m{}^n$ *vs.* Contravariant g^{mn}

Covariant g_{mn} vs. Invariant $\delta_m{}^n$ vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & if\ m = n \\ 0 & if\ m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Covariant g_{mn} vs. Invariant $\delta_m{}^n$ vs. Contravariant g^{mn}

$$\mathbf{E}_m\bullet\mathbf{E}_n=\frac{\partial\mathbf{r}}{\partial q^m}\bullet\frac{\partial\mathbf{r}}{\partial q^n}\equiv g_{mn}$$

$$\mathbf{E}_m\bullet\mathbf{E}^n=\frac{\partial\mathbf{r}}{\partial q^m}\bullet\frac{\partial q^n}{\partial\mathbf{r}}=\delta_m^n$$

$$\mathbf{E}^m\bullet\mathbf{E}^n=\frac{\partial q^m}{\partial\mathbf{r}}\bullet\frac{\partial q^n}{\partial\mathbf{r}}\equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n\equiv\begin{cases}1 & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}$$

Contravariant
metric tensor
g^{mn}

Polar coordinate examples (again):

$$\langle J\rangle=\begin{pmatrix}\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2}\\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial r}=\cos\phi & \frac{\partial x}{\partial\phi}=-r\sin\phi\\ \frac{\partial y}{\partial r}=\sin\phi & \frac{\partial y}{\partial\phi}=r\cos\phi\end{pmatrix}$$

$$\uparrow\mathbf{E}_1\quad\uparrow\mathbf{E}_2\qquad\qquad\uparrow\mathbf{E}_r\qquad\qquad\uparrow\mathbf{E}_\phi$$

$$\langle K\rangle=\langle J^{-1}\rangle=\begin{pmatrix}\frac{\partial r}{\partial x}=\cos\phi & \frac{\partial r}{\partial y}=\sin\phi\\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{r} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{r}\end{pmatrix}\begin{array}{l}\leftarrow\mathbf{E}^r=\mathbf{E}^1\\ \\ \leftarrow\mathbf{E}^\phi=\mathbf{E}^2\end{array}$$

Covariant g_{mn} vs. Invariant $\delta_m{}^n$ vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn} \qquad \mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n \qquad \mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Polar coordinate examples (again):

$$\langle J \rangle = \begin{array}{c} \begin{pmatrix} \dfrac{\partial x^1}{\partial q^1} & \dfrac{\partial x^1}{\partial q^2} \\ \dfrac{\partial x^2}{\partial q^1} & \dfrac{\partial x^2}{\partial q^2} \end{pmatrix} \\ \begin{array}{cc} \uparrow \mathbf{E}_1 & \uparrow \mathbf{E}_2 \end{array} \end{array} = \begin{array}{c} \begin{pmatrix} \dfrac{\partial x}{\partial r} = \cos\phi & \dfrac{\partial x}{\partial \phi} = -r\sin\phi \\ \dfrac{\partial y}{\partial r} = \sin\phi & \dfrac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix} \\ \begin{array}{cc} \uparrow \mathbf{E}_r & \uparrow \mathbf{E}_\phi \end{array} \end{array}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \dfrac{\partial r}{\partial x} = \cos\phi & \dfrac{\partial r}{\partial y} = \sin\phi \\ \dfrac{\partial \phi}{\partial x} = \dfrac{-\sin\phi}{r} & \dfrac{\partial \phi}{\partial y} = \dfrac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

→ *GCC Lagrangian definition*

GCC "canonical" momentum p_m definition

GCC "canonical" force F_m definition

Coriolis "fictitious" forces (... and weather effects)

Lagrange prefers Co*variant* g_{mn} *with* Contra*variant velocity*

Lagrangian L=KE-U is supposed to be explicit function of velocity.

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

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Use polar coordinate Covariant g_{mn} metric (page 53)
$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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Use polar coordinate Covariant g_{mn} metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

- *GCC Lagrangian definition*
- → *GCC "canonical" momentum p_m definition*
- *GCC "canonical" force F_m definition*
  - *Coriolis "fictitious" forces (… and weather effects)*

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Use polar coordinate Covariant g_{mn} metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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(From preceding page)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

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Use polar coordinate Covariant g_{mn} metric (page 53)

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This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising;
radial momentum p_r has the
usual linear $M\cdot v$ form

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

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Use polar coordinate Covariant g_{mn} metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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(Defining GCC momentum)

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Recall :

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Lagrange's 2nd GCC equation
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$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m definition

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Torque relates to two distinct parts: Coriolis and angular acceleration

$$=0-\frac{\partial U}{\partial\phi}$$

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Rewriting GCC Lagrange equations :

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Conventional forms

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Field-free (U=0)

radial acceleration: $\ddot{r} = r\,\dot{\phi}^2$

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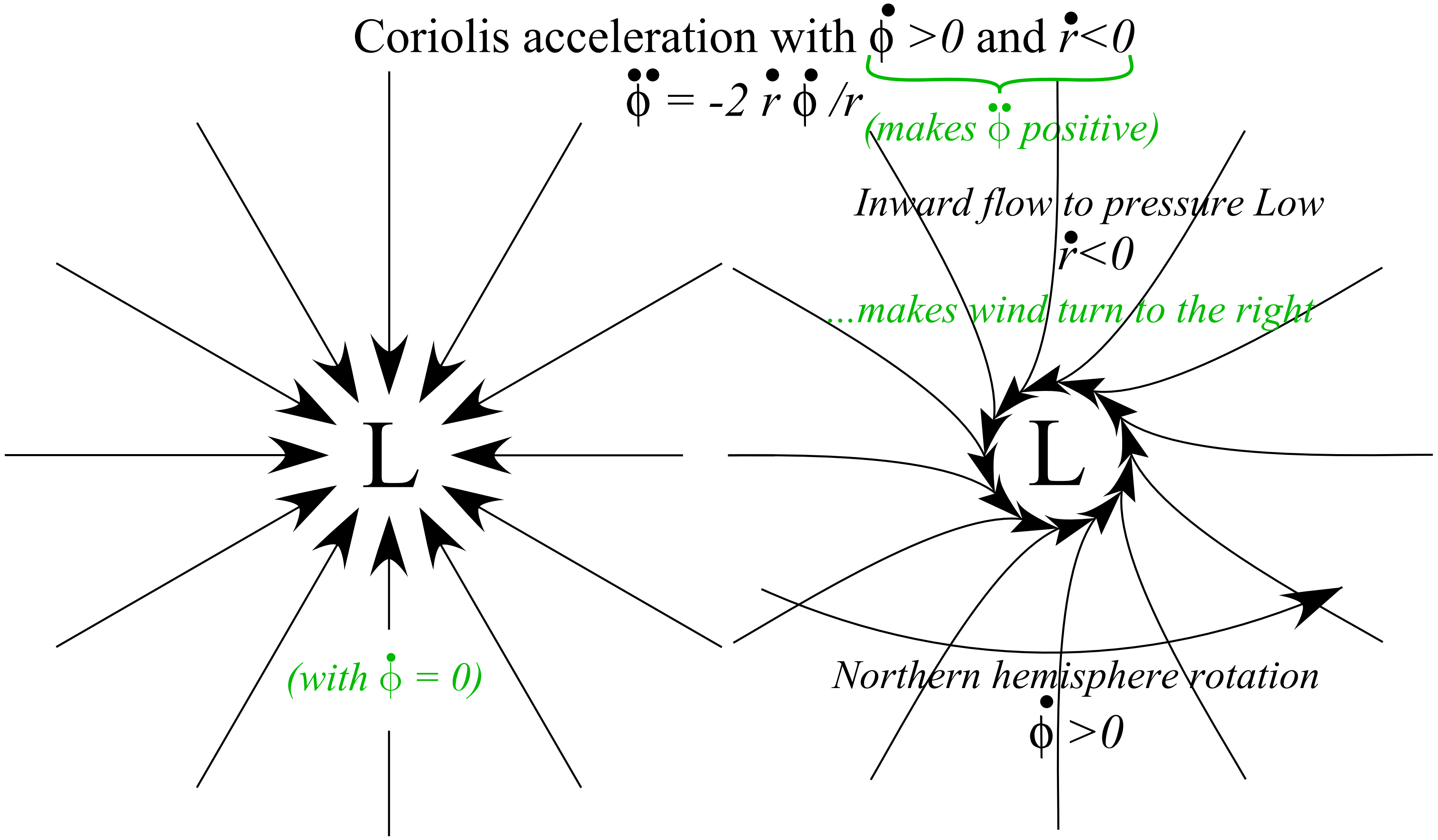

Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}_{}$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$

(makes $\ddot{\phi}$ positive)

Inward flow to pressure Low
$\dot{r} < 0$

...makes wind turn to the right

(with $\dot{\phi} = 0$)

Northern hemisphere rotation
$\dot{\phi} > 0$

Effect on
<u>Northern</u>
Hemisphere
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Cyclonic flow
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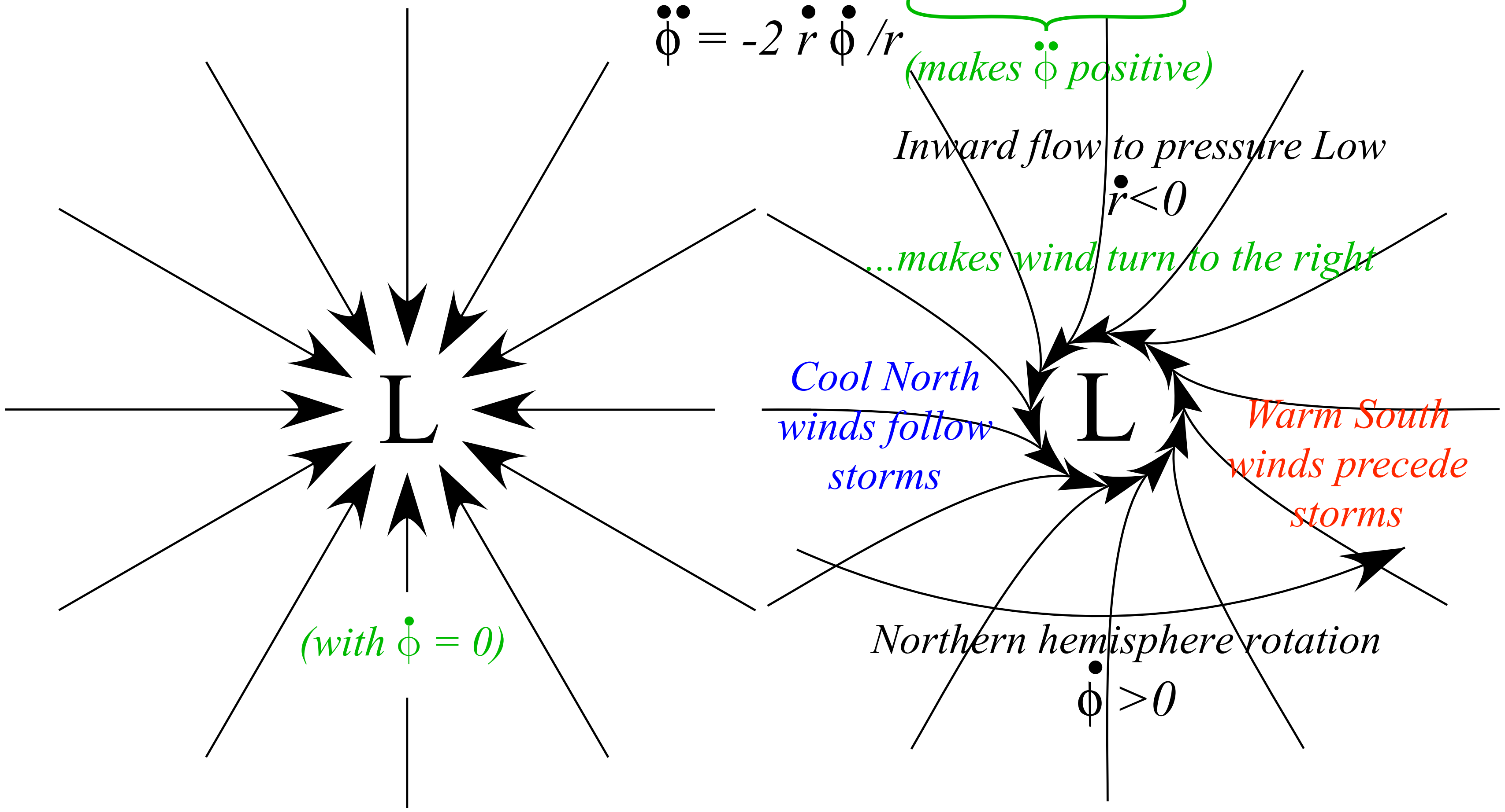

Effect on *__Northern__* *Hemisphere local weather*

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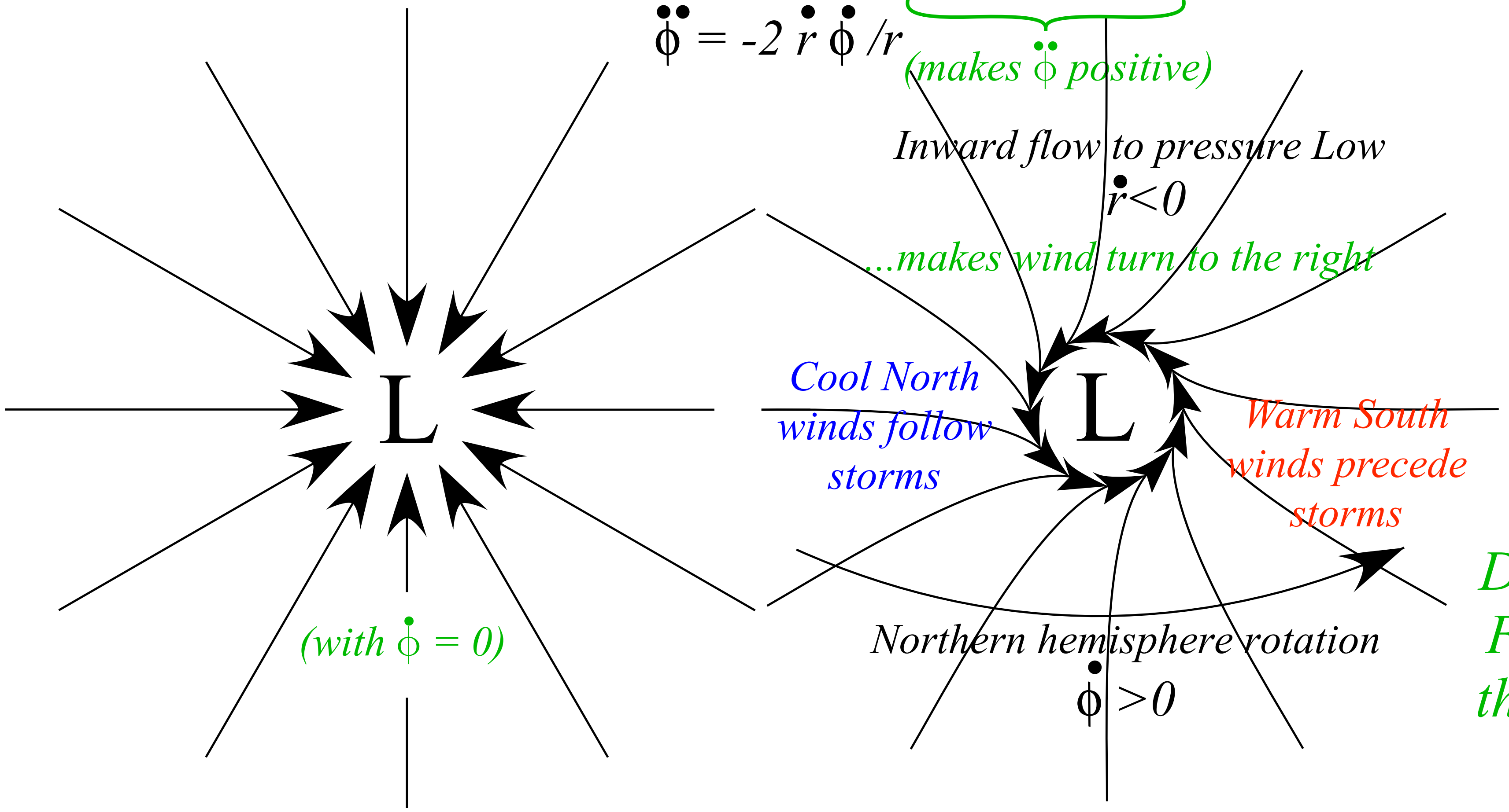

Inward flow to pressure Low
$\dot{r}<0$

...makes wind turn to the right

Cool North winds follow storms

Warm South winds precede storms

Northern hemisphere rotation
$\dot{\phi} > 0$

Effect on
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Deep quantum rule:
Flow tries to mimic
the external rotation
(least relative v)

Lecture 9 ends here
Thur. 9.22.2016