Lecture 18
Tue. 10.25.2012

Hamilton Equations for Trebuchet and Other Things (Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
- *Hamilton's equations and Poincare invariant relations*
- *Hamiltonian expression and contravariant γ^{mn} tensor*

Hamiltonian energy and momentum conservation and symmetry coordinates
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Chapter 1. The Trebuchet: A dream problem for Galileo?

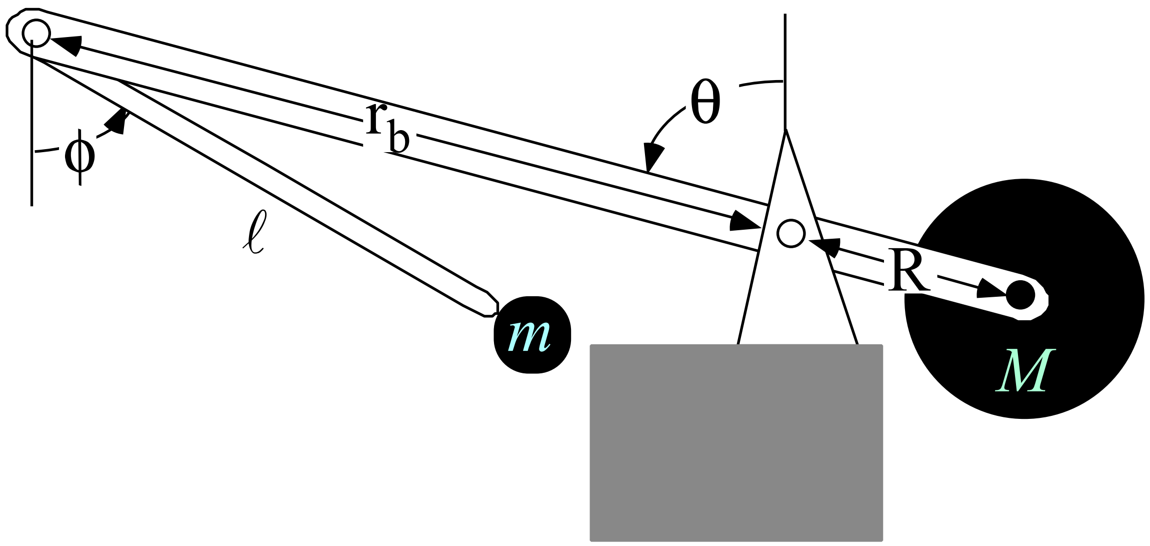
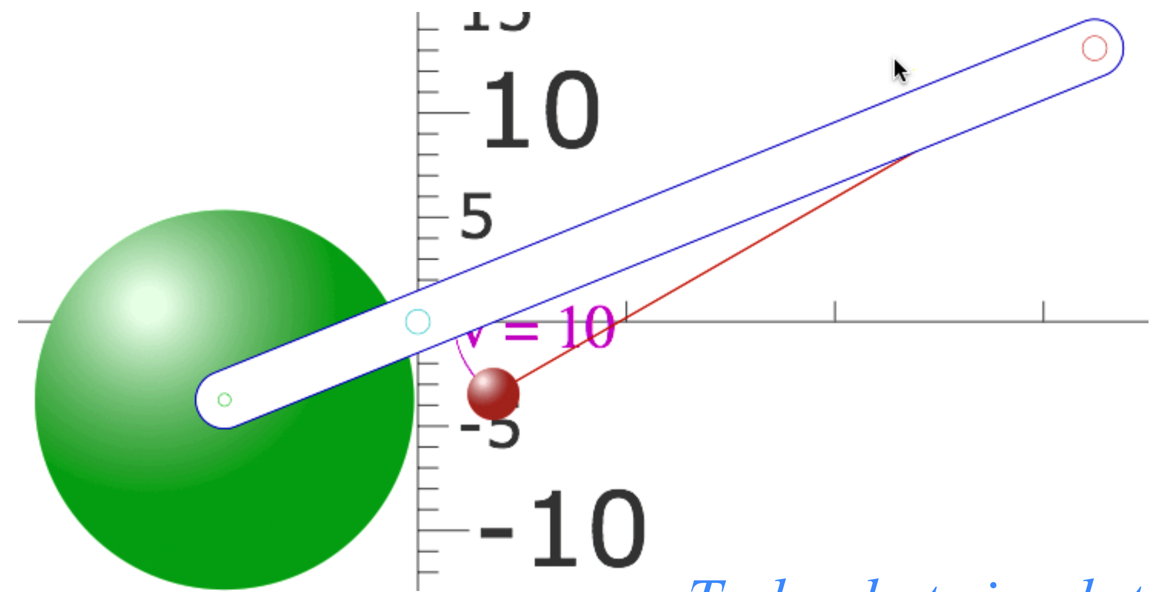


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/rso/modphys/testing/markup/TrebuchetWeb.html

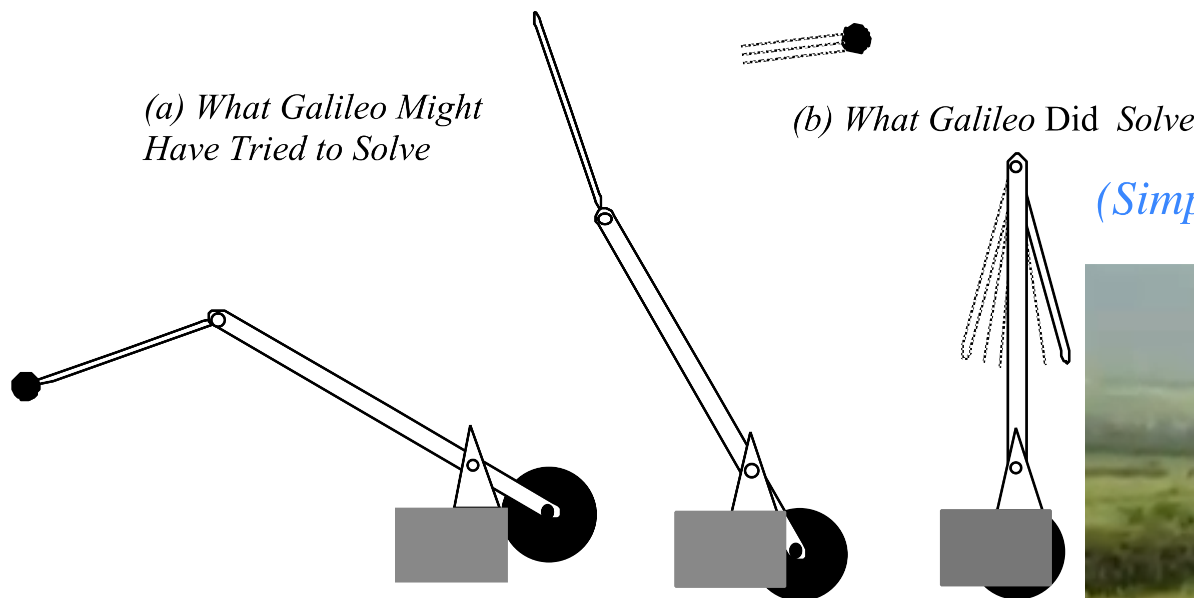
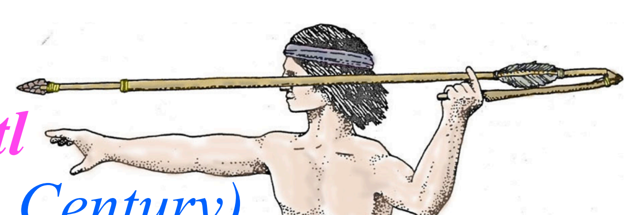


Fig. 2.1.2 Galileo's (supposed) problem

(Simple pendulum dynamics)



The Atlatl
(Cahokia, IL 12th Century)

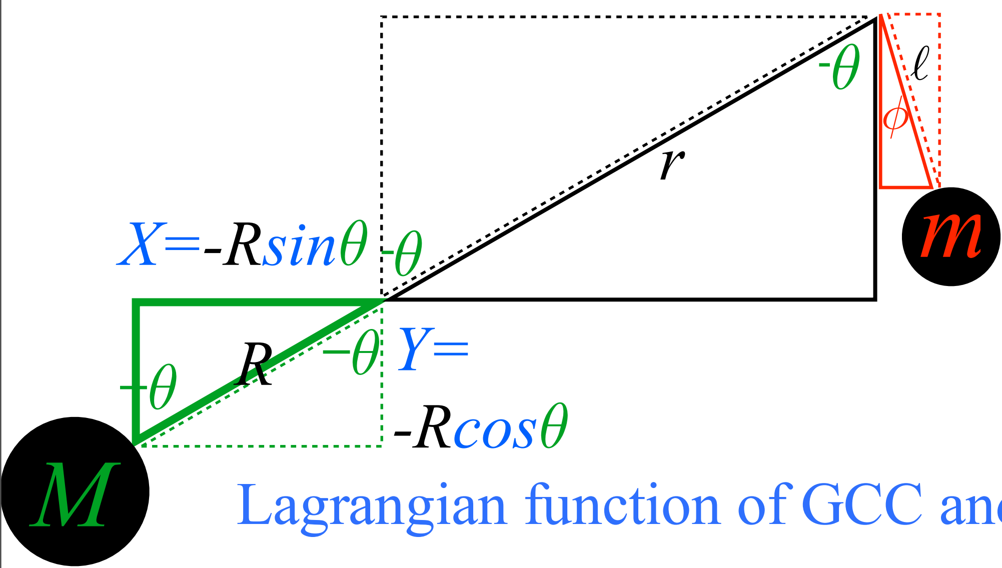
Review of Hamiltonian equation derivation (Elementary trebuchet)

→ *Hamiltonian definition from Lagrangian and γ_{mn} tensor*

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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1st differential chain

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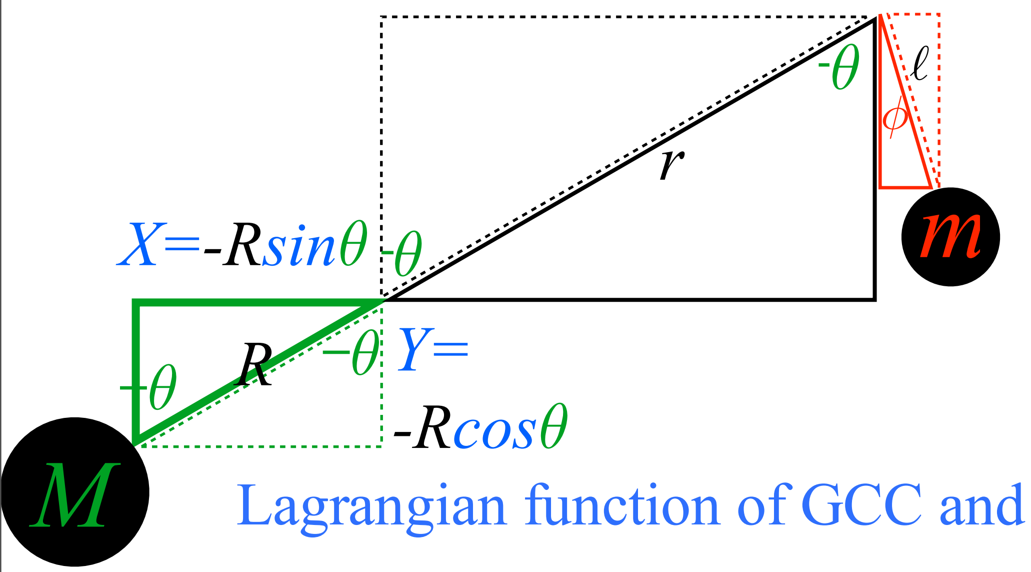

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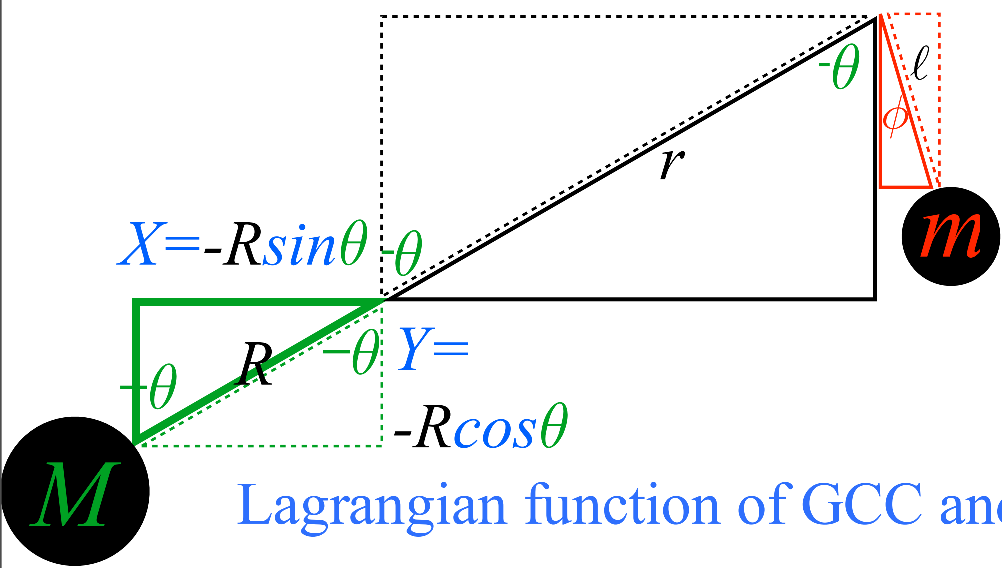
velocity chain



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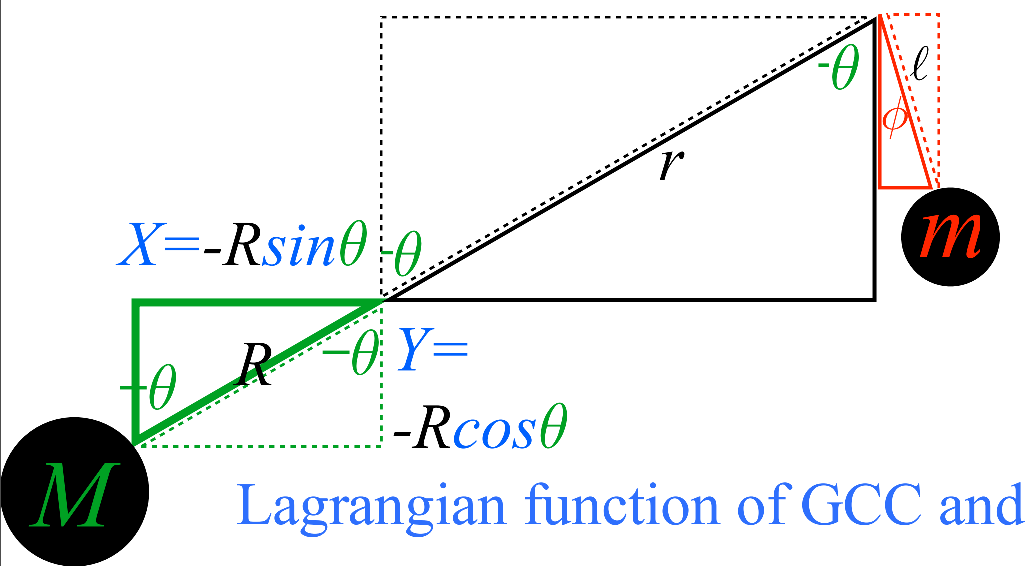
Lagrange equations



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(Consolidating)

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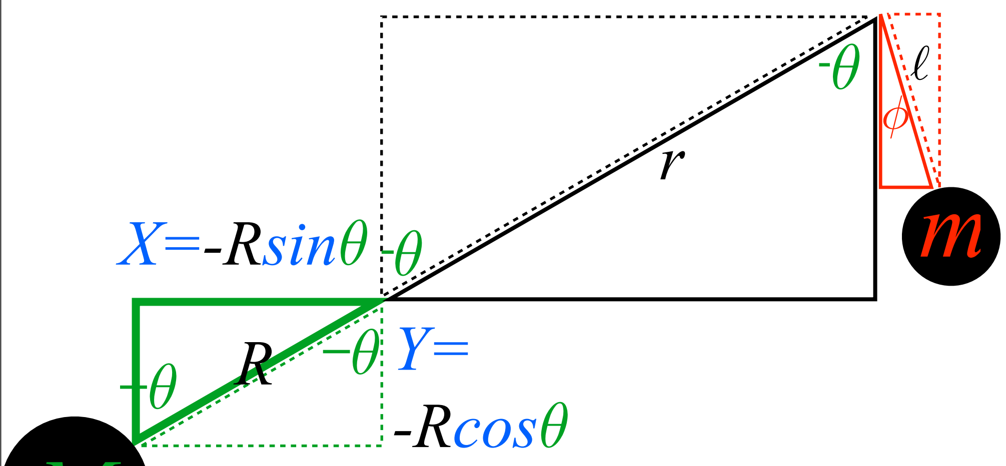

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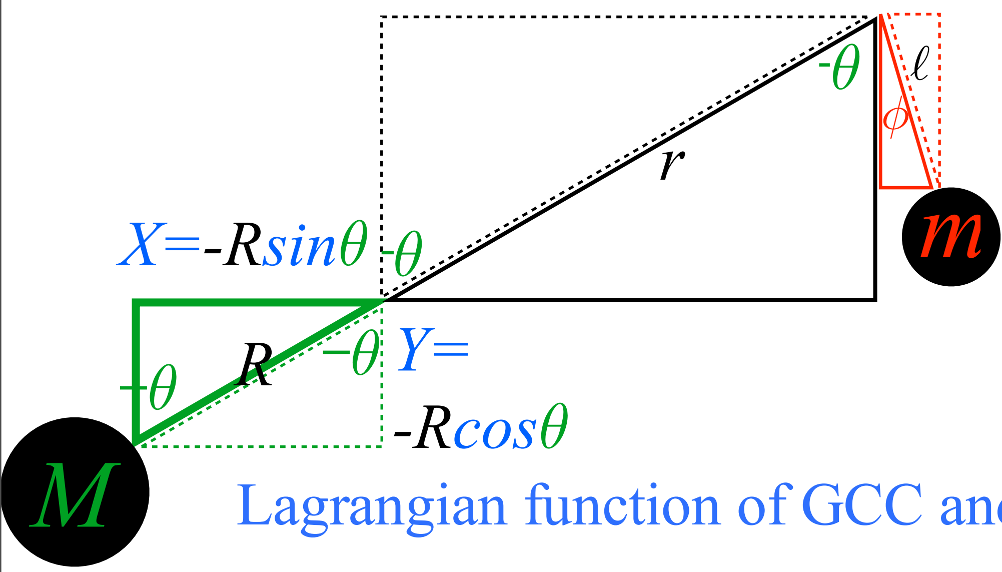

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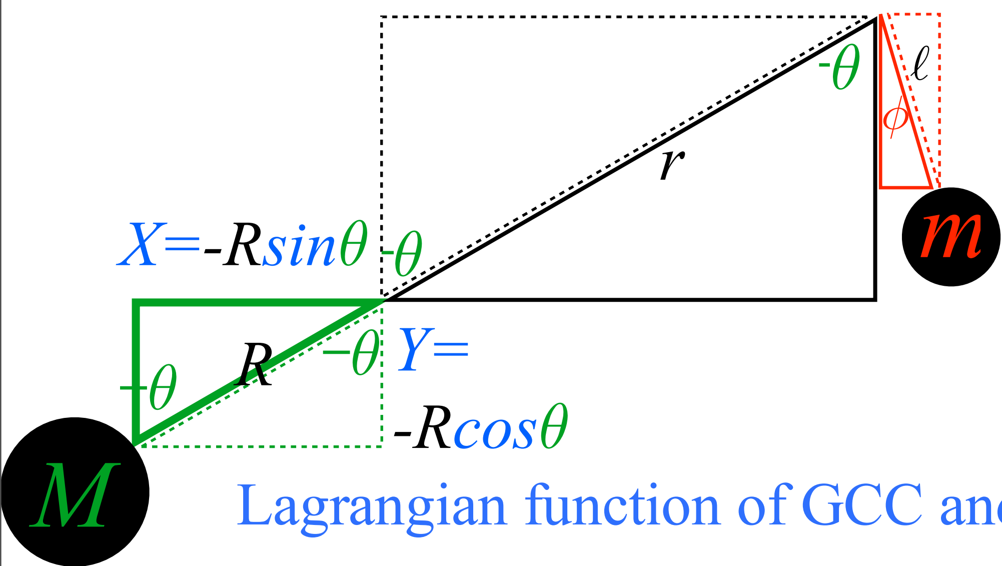

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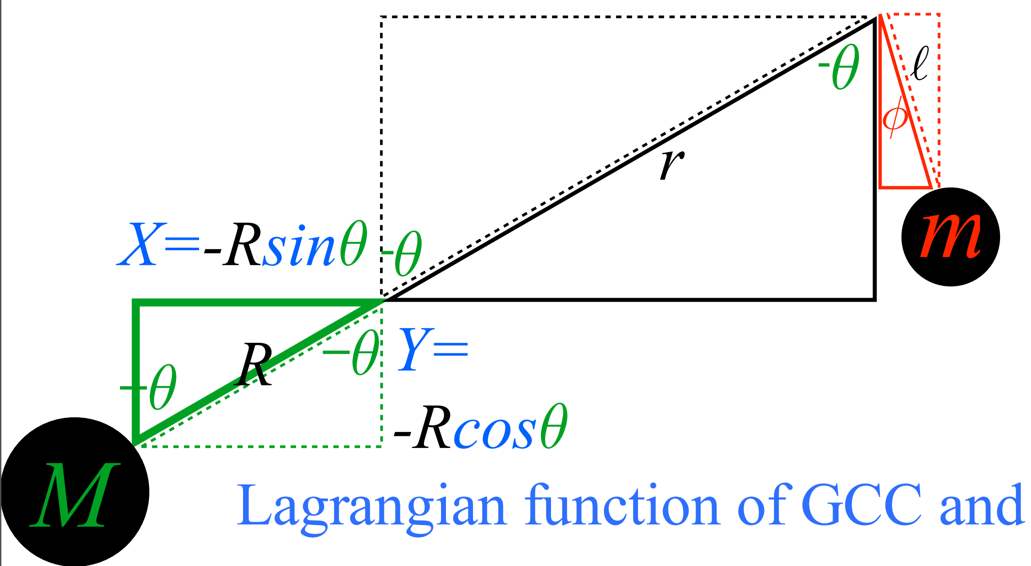

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Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta$$

by Lagrange equations

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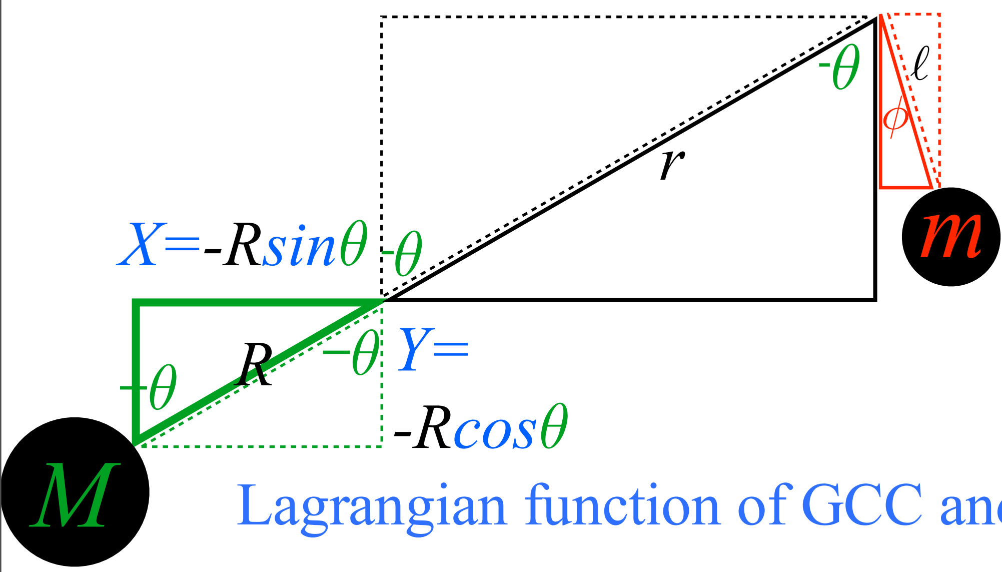

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Dynamic metric tensor γ_{mn} *in GCC* θ *and* ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

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$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$ *Lagrange equations*

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$ *(Consolidating)*

$$\frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right) = -\frac{\partial L}{\partial t}$$ *(Rearranging)*

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$ *Defining the Hamiltonian function*

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} = -\frac{\partial L}{\partial \theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \frac{\partial L}{\partial p_\theta} = \dot{\theta}$$

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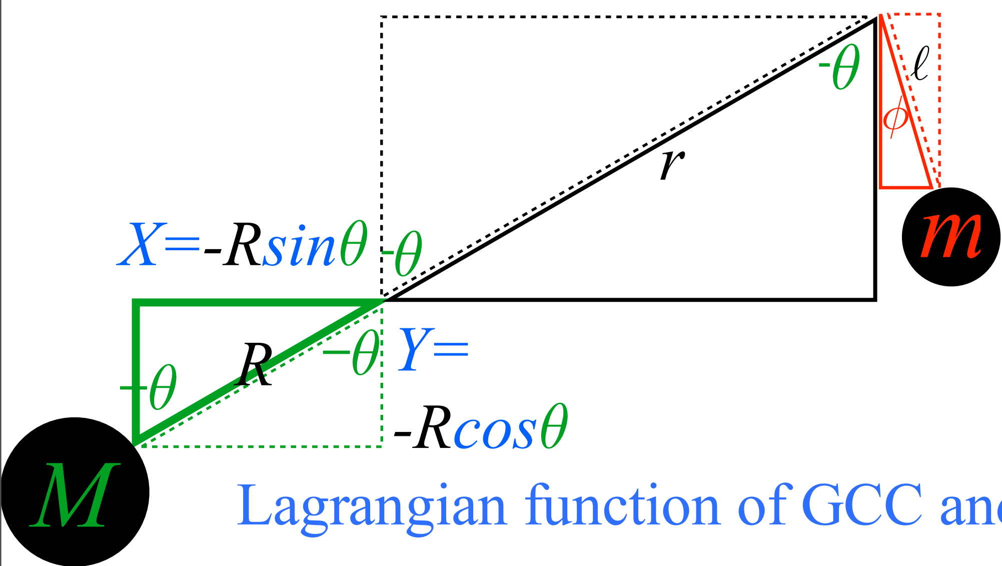

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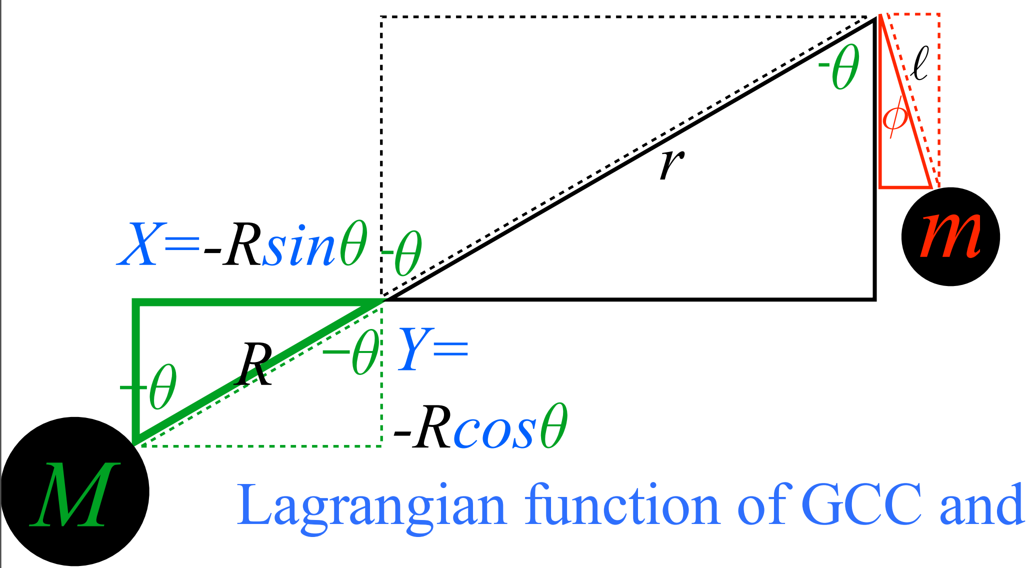

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$$\frac{\partial H}{\partial\theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \frac{\partial H}{\partial\dot{\theta}} \equiv 0 \qquad \frac{\partial H}{\partial\phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \frac{\partial H}{\partial\dot{\phi}} \equiv 0$$

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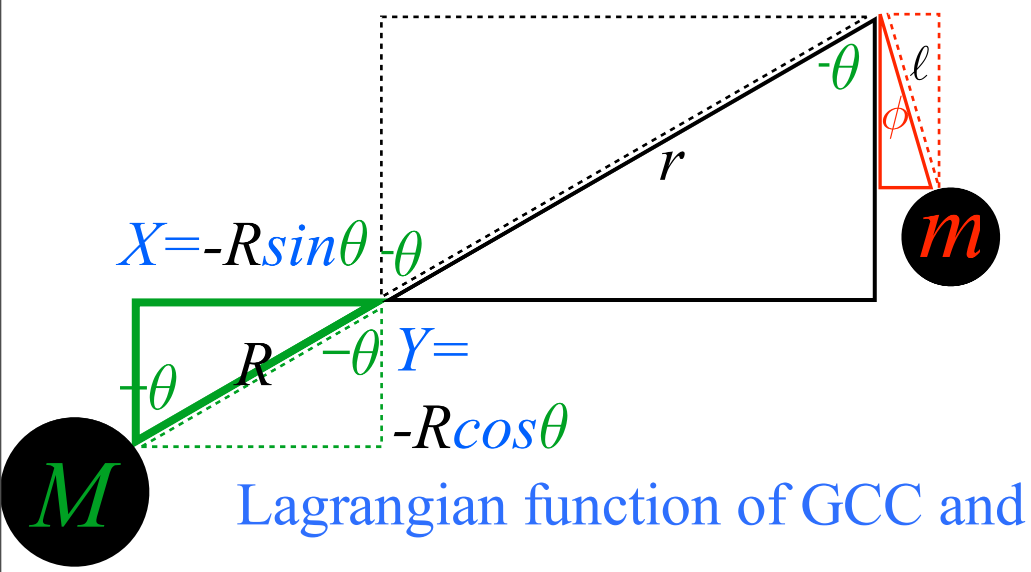

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(Consolidating)

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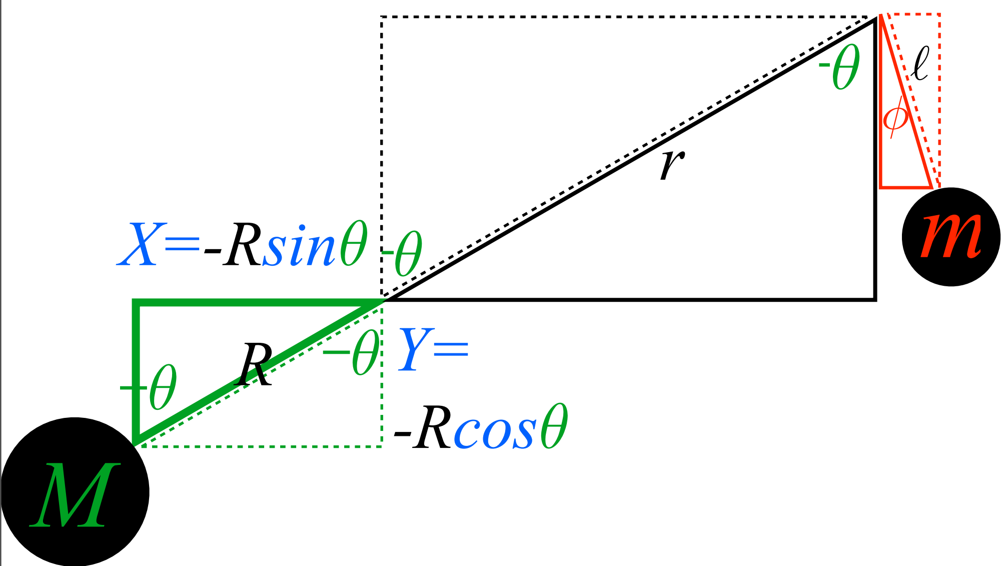
Hamilton's equations

Review of Hamiltonian equation derivation (Elementary trebuchet)

Hamiltonian definition from Lagrangian and γ_{mn} tensor

Hamilton's equations and Poincare invariant relations

→ *Hamiltonian expression and contravariant γ^{mn} tensor*



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Covariant metric tensor γ_{mn}

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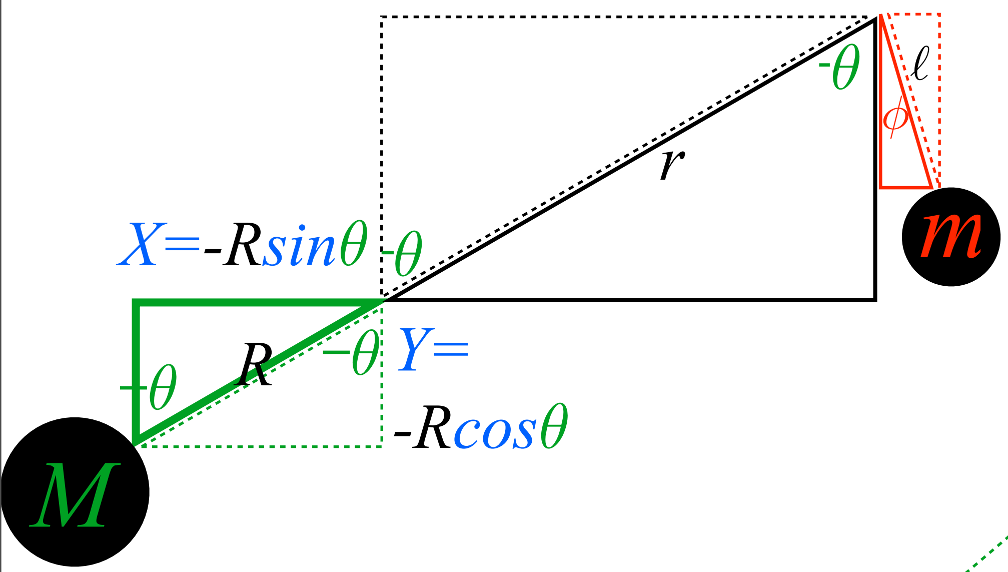

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(Only correct numerically!)

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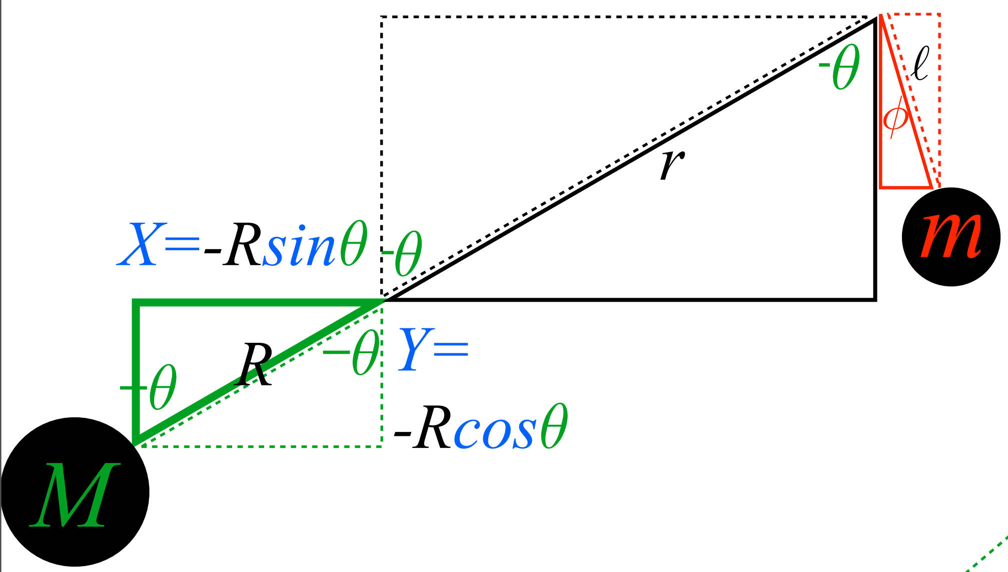

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Hamiltonian must be explicit in momenta p_m

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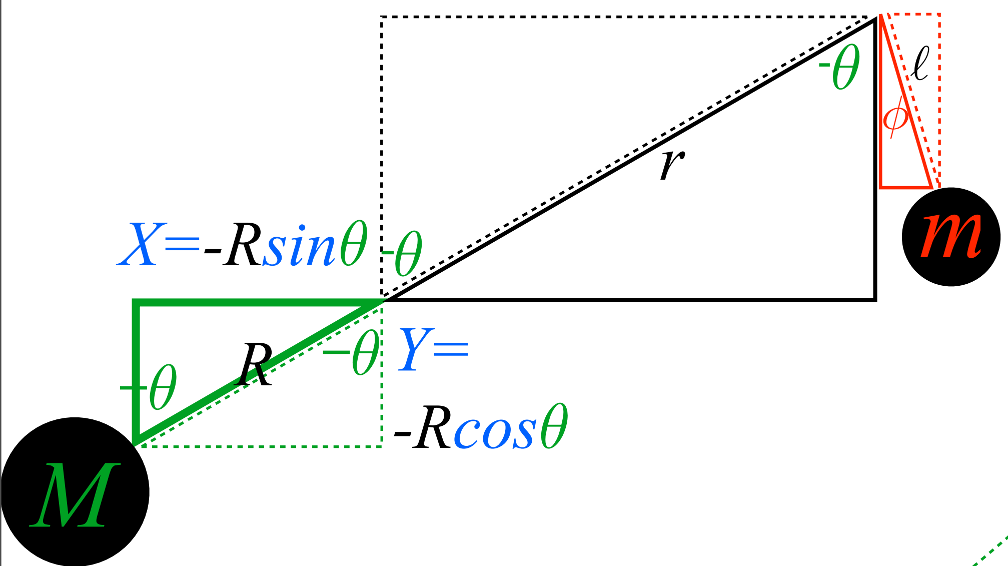

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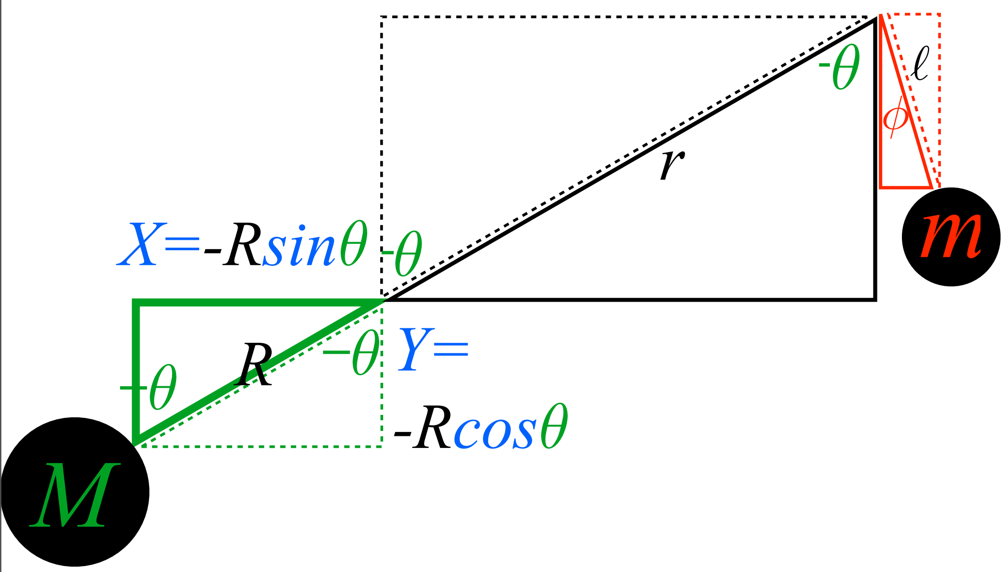
$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V = T + V \equiv E$$ (Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi) + V = T + V$$ (Correct formally and numerically)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V \quad = T + V \equiv E$$

(Only correct numerically!)

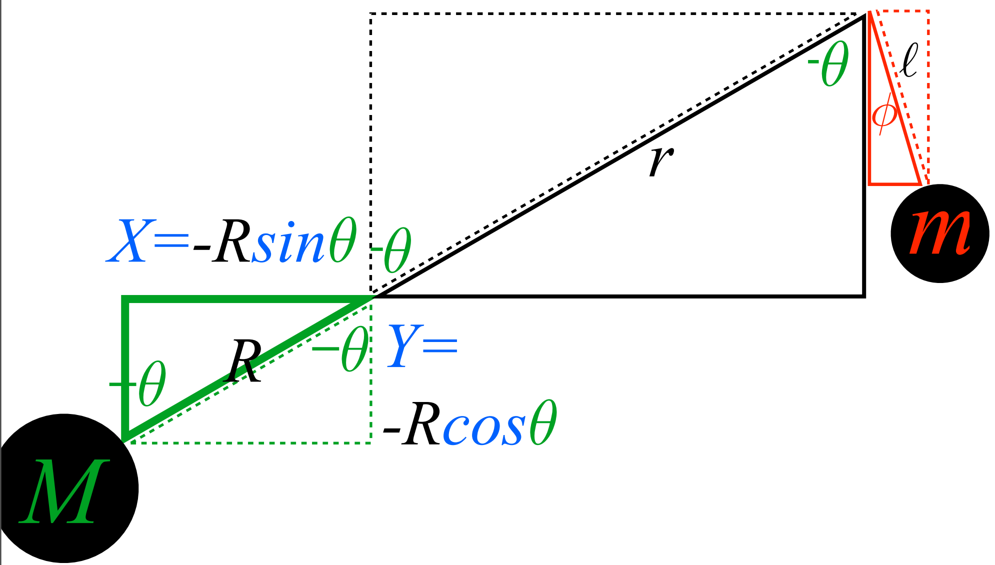
Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi) + V \quad = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta-\phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot{\theta} = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

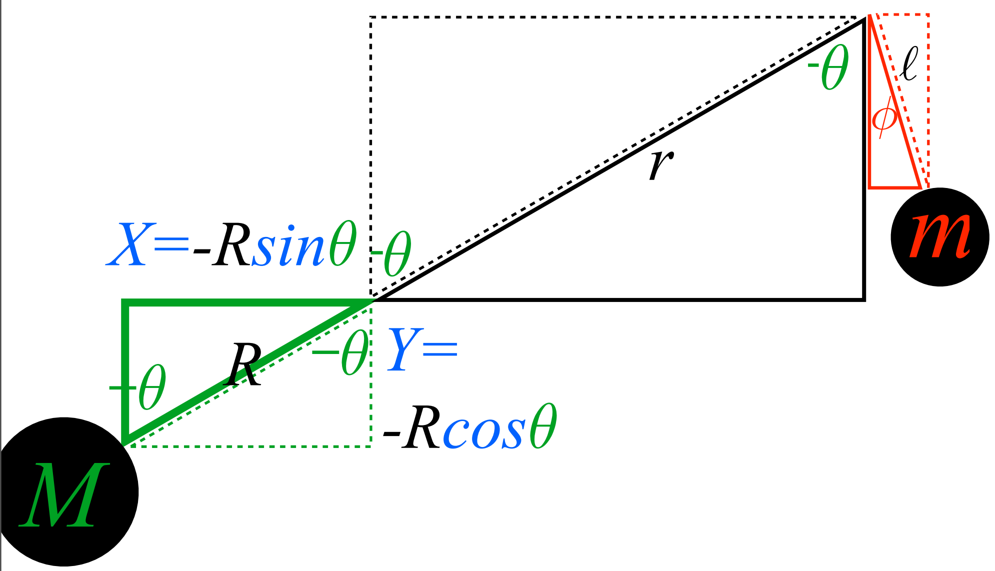
$$\dot{\phi} = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot{\phi} = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\dot{\theta} = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

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Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

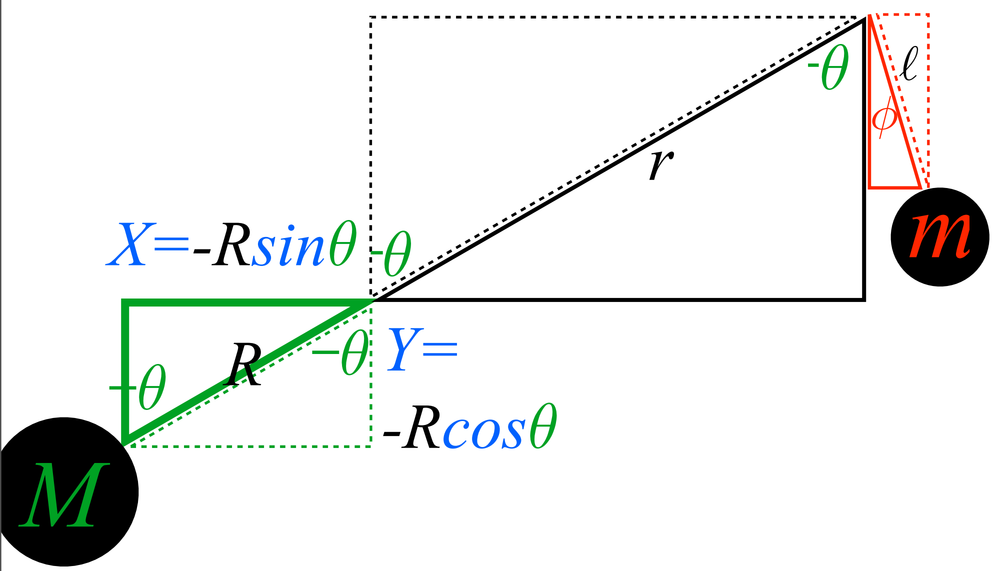
Momentum/force equations

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} = mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} = -mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\dot\theta = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

$$\dot\phi = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

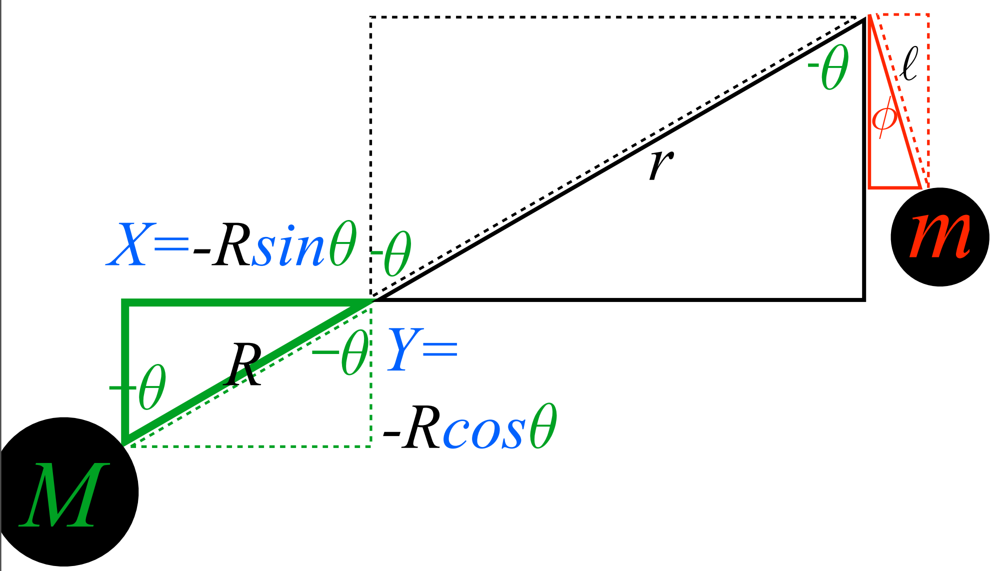
Momentum/force equations

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}$$
$$= mr\ell\dot\theta\dot\phi \sin(\theta-\phi) + F_\theta$$
$$= mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}$$
$$= -mr\ell\dot\theta\dot\phi \sin(\theta-\phi) + F_\phi$$
$$= -mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\dot\theta = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

$$\dot\phi = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

Momentum/force equations

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}$$

$$= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta} p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi} p_\phi^2)\sin(\theta-\phi) + F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

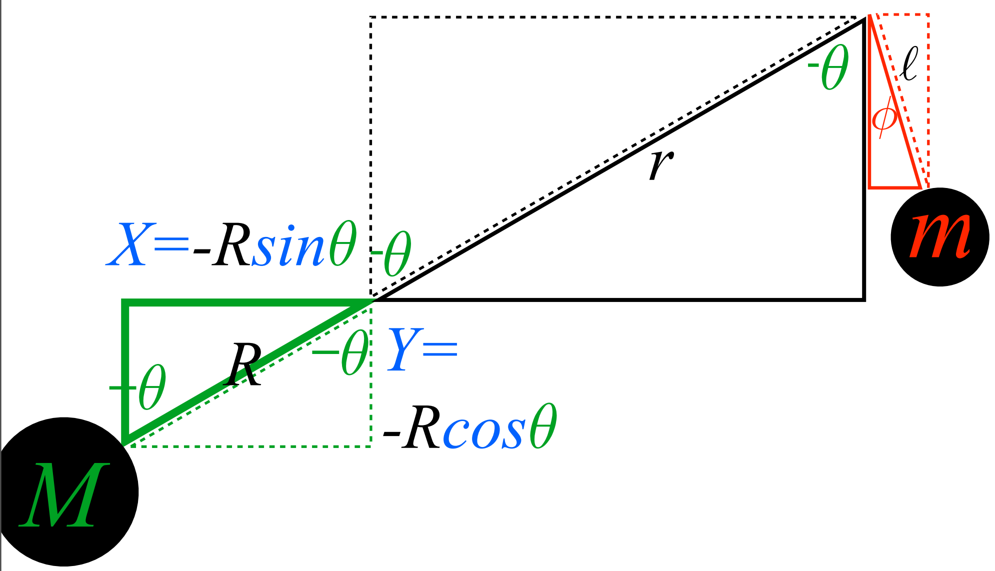
$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}$$

$$= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi$$

$$= -mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\phi$$

$$= -[\text{messy factor}]\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

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Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

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Momentum/force equations

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}$$

$$= mr\ell\dot\theta\dot\phi \sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi) \sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta} p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi} p_\phi^2) \sin(\theta-\phi) + F_\theta$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}$$

$$= \text{-}mr\ell\dot\theta\dot\phi \sin(\theta-\phi) + F_\phi$$

$$= \text{-}mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi) \sin(\theta-\phi) + F_\phi$$

$$= \text{-[messy factor]} \sin(\theta-\phi) + F_\phi$$

A lesson on Hamiltonian "elegance"...
...may be very elegant formally...but may <u>*not*</u> *be so elegant computationally!*

Hamiltonian energy and momentum conservation and symmetry coordinates

→ *Coordinate transformation helps reduce symmetric Hamiltonian*

Free-space trebuchet kinematics by symmetry

- *Algebraic approach*
- *Direct approach and Superball analogy*

Trebuchet vs Flinger and sports kinematics

- *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

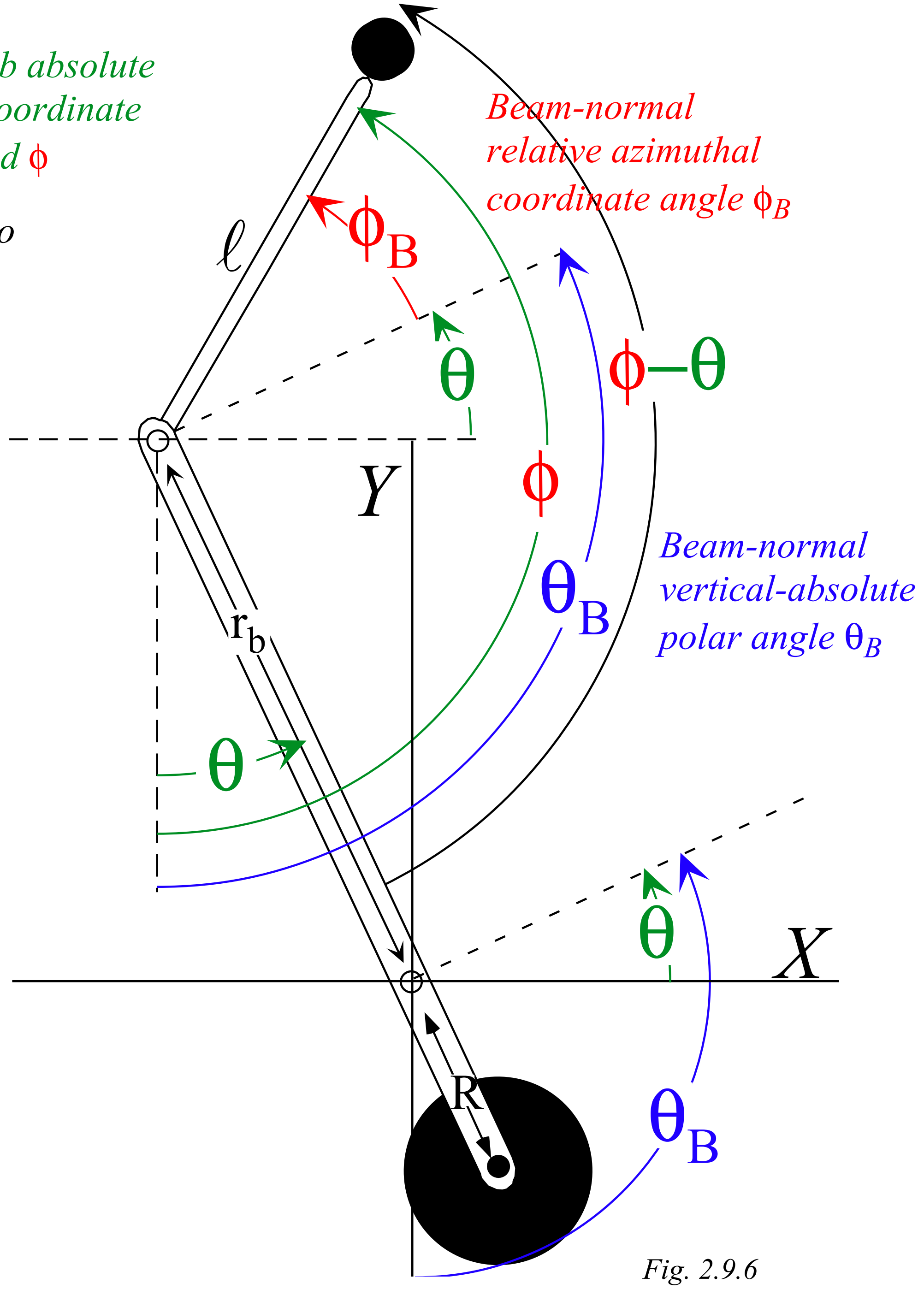

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \qquad -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

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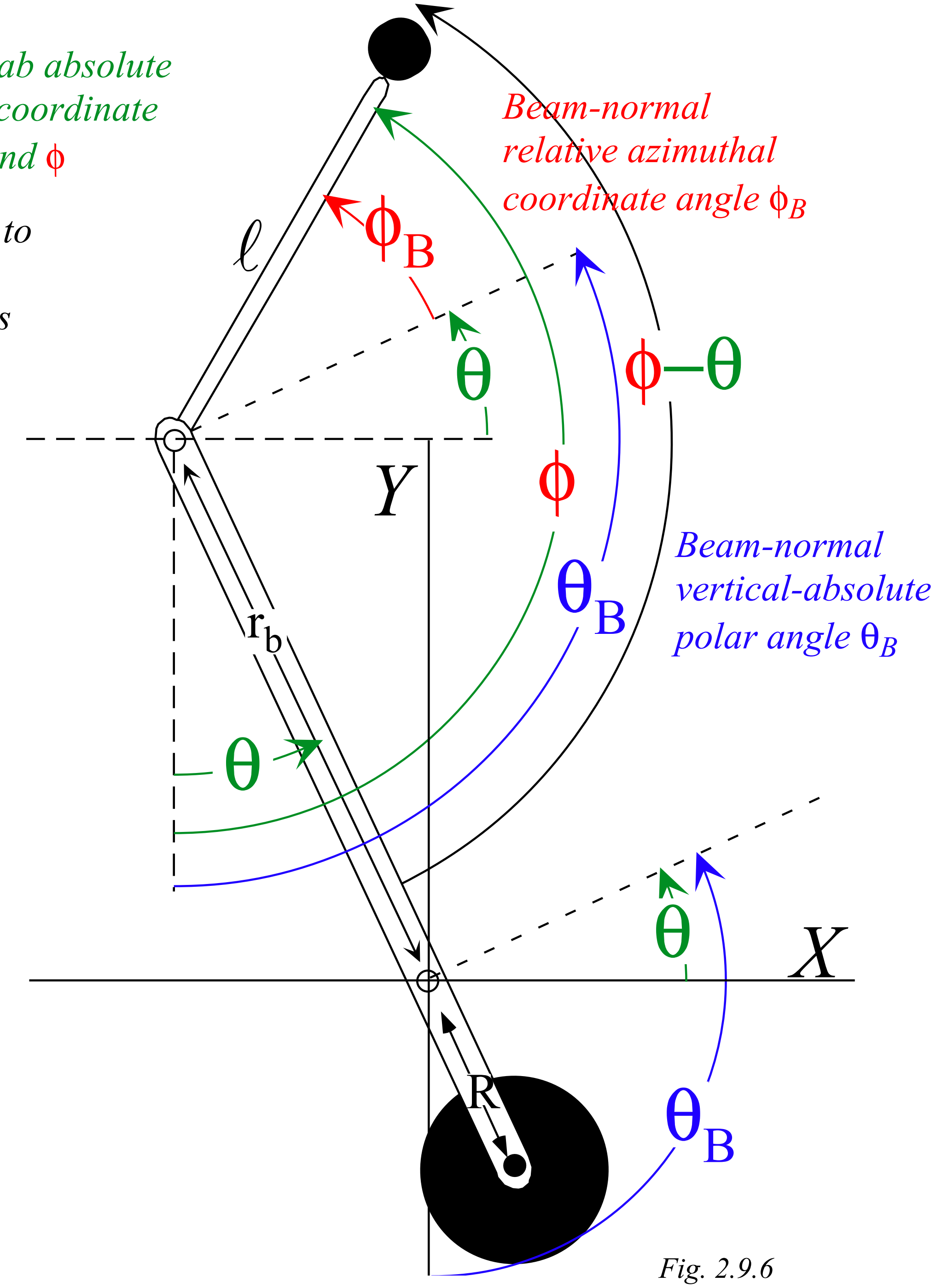

Fig. 2.9.6

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Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B - \pi/2$

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$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot{\theta} + p_\phi\dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot{\theta}_B + p_\phi^B\dot{\phi}_B$$

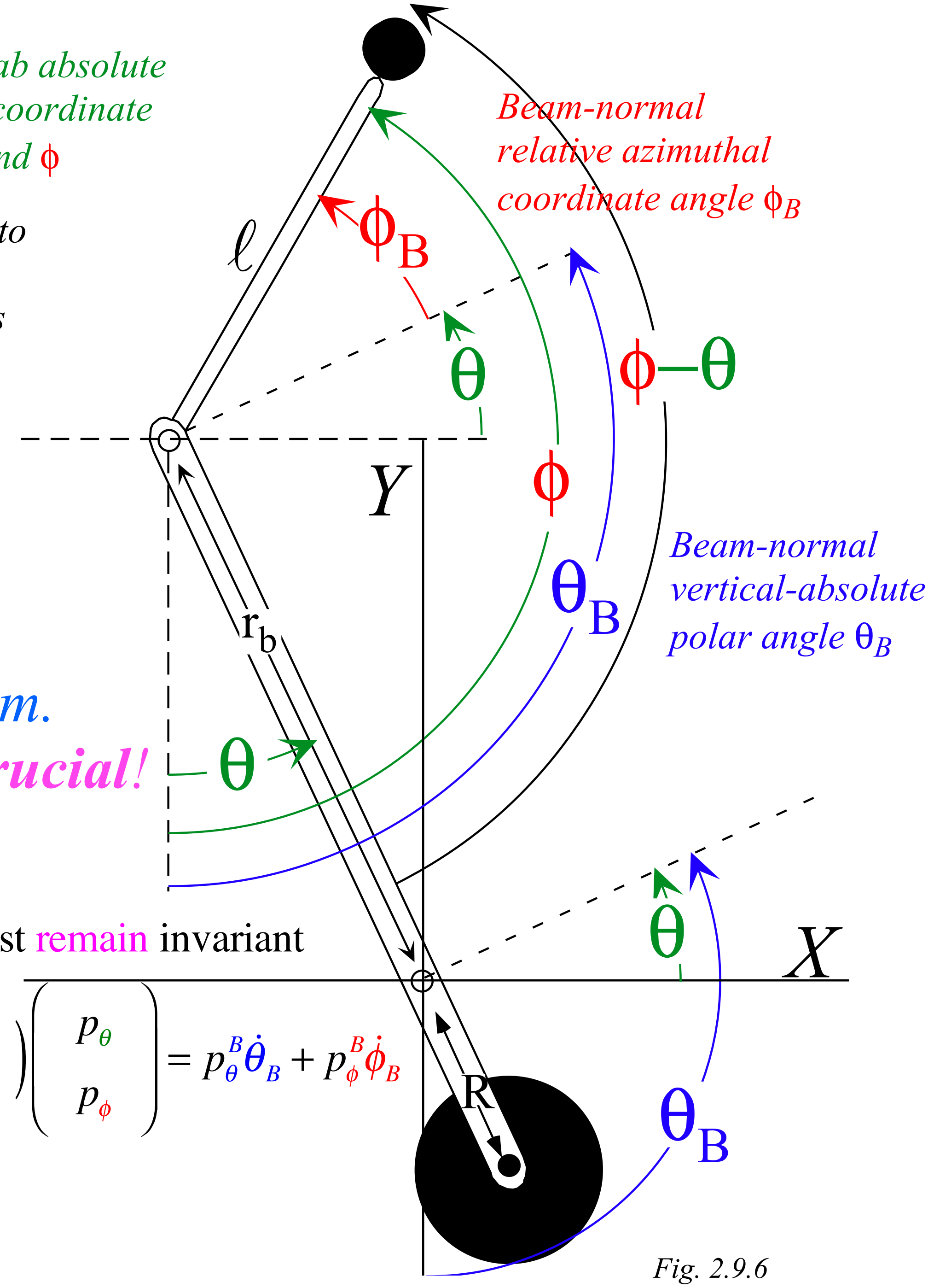

Fig. 2.9.6
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(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

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$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

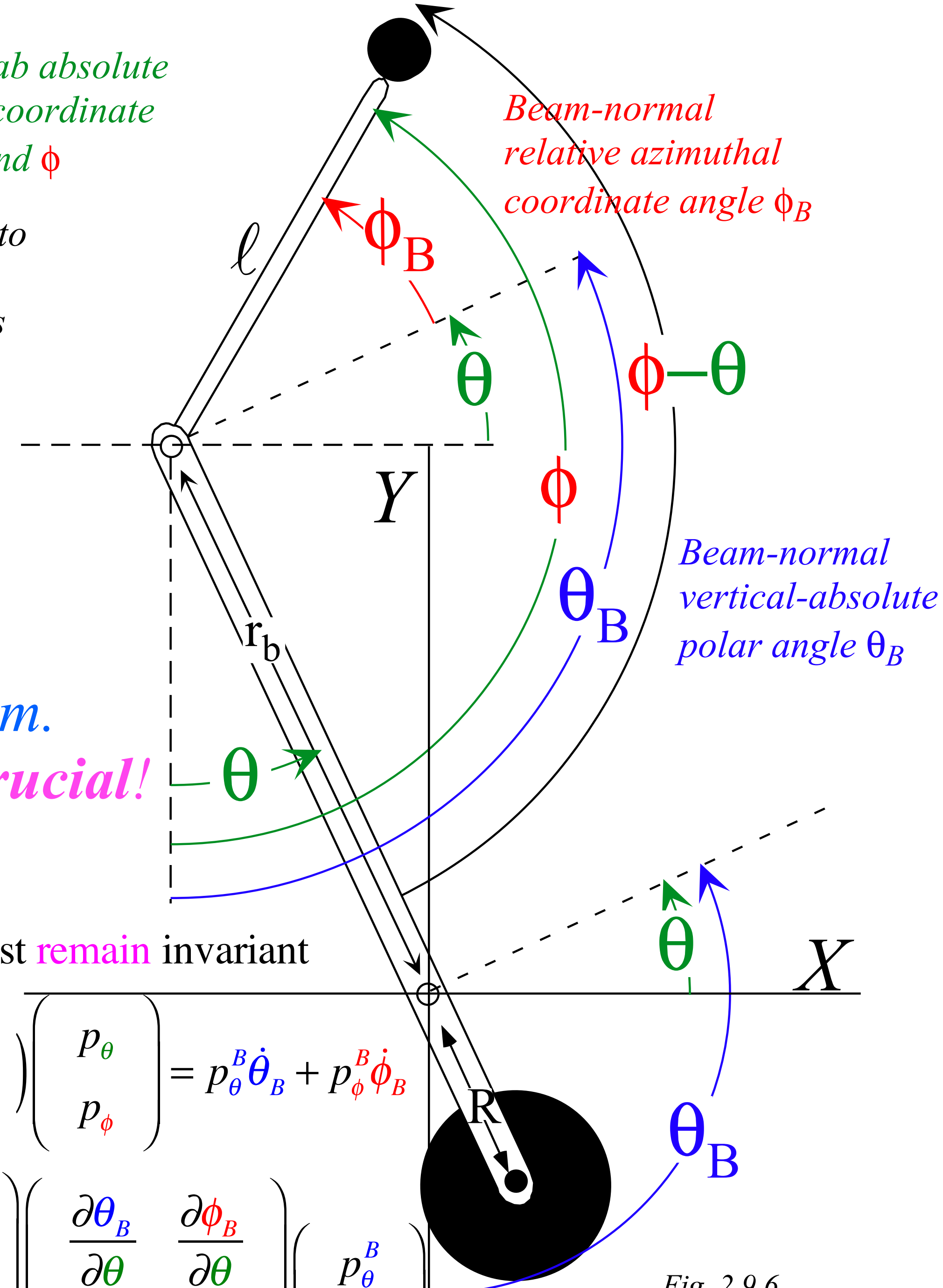


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,φ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\quad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

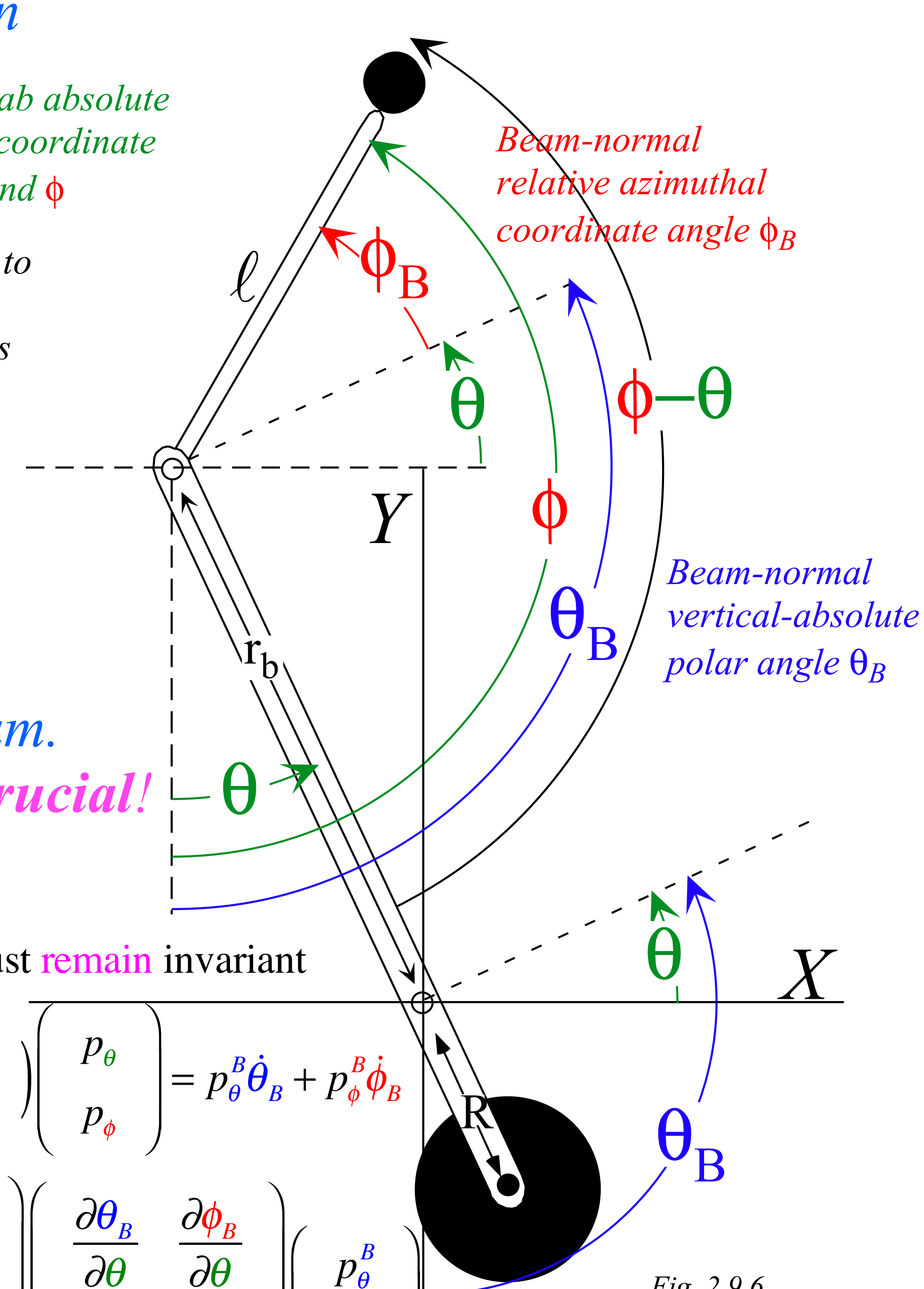
$p_\phi = \quad p^B_\phi$

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$



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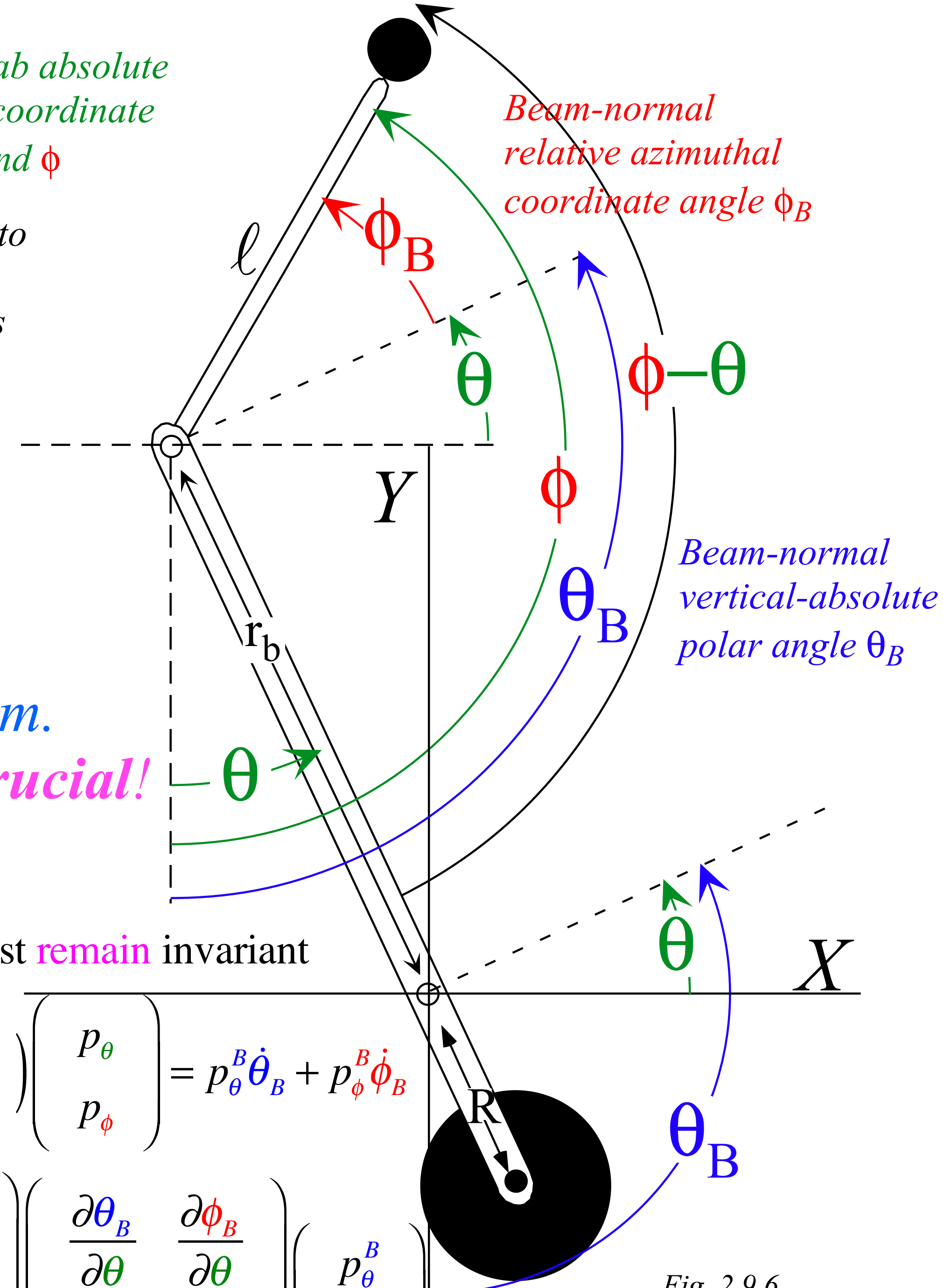
$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



p_m transform is TRANSPOSE INVERSE to q^m

Be careful with momentum.
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$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

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Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

$p_\phi = \quad p^B_\phi$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta - \phi)\right]} \qquad + V$$

Original (φ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

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p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

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Previous lab absolute trebuchet coordinate angles θ and φ

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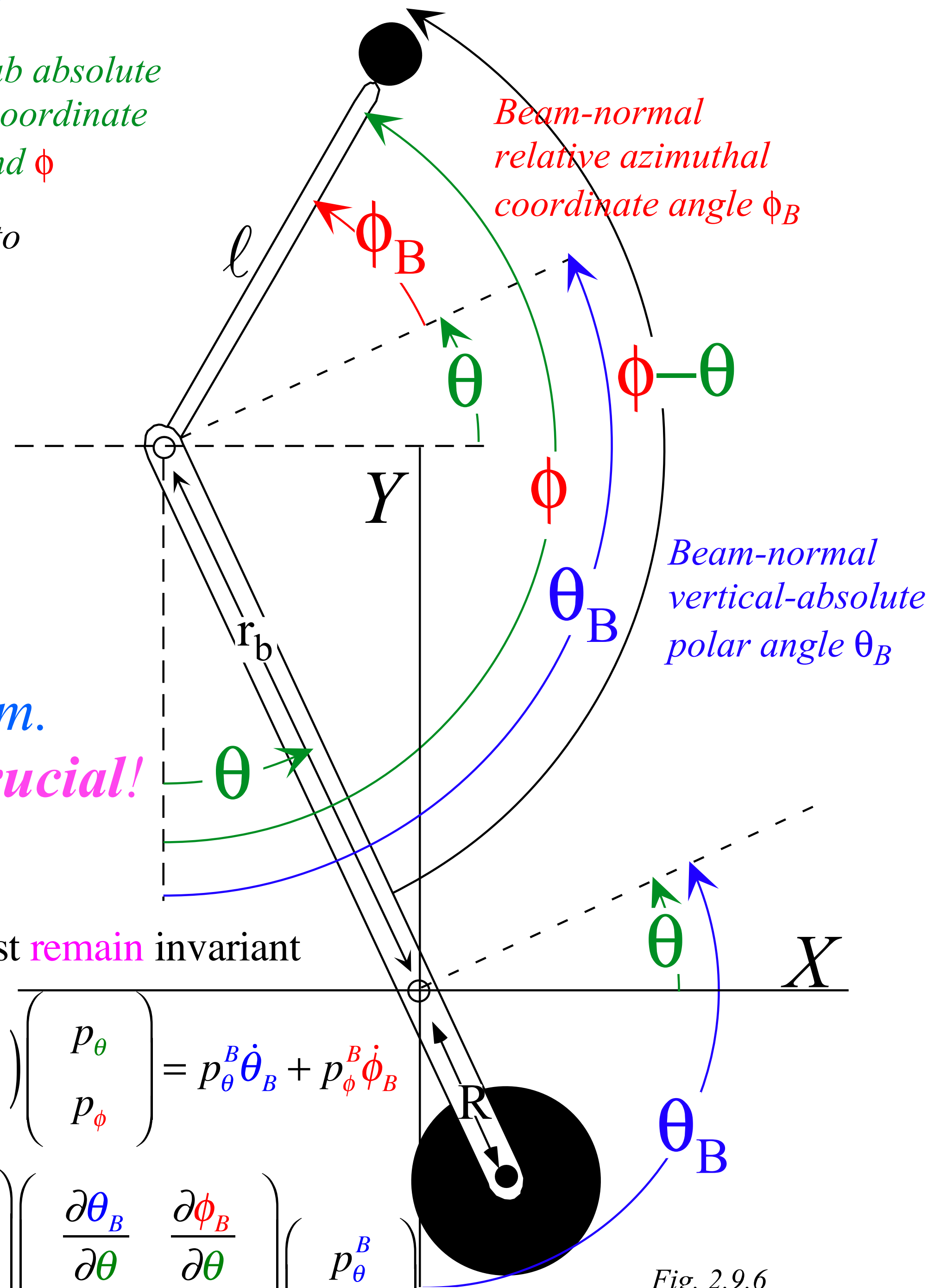

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
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Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta-\phi)}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} + V$$

Original (φ,θ) Hamiltonian

$$H = \frac{m\ell^2\left(p^B_\theta - p^B_\phi\right)^2 + \left(MR^2+mr^2\right)\left(p^B_\phi\right)^2 - 2mr\ell p^B_\phi\left(p^B_\theta - p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} + V$$

Transformed (ϕ_B,θ_B) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

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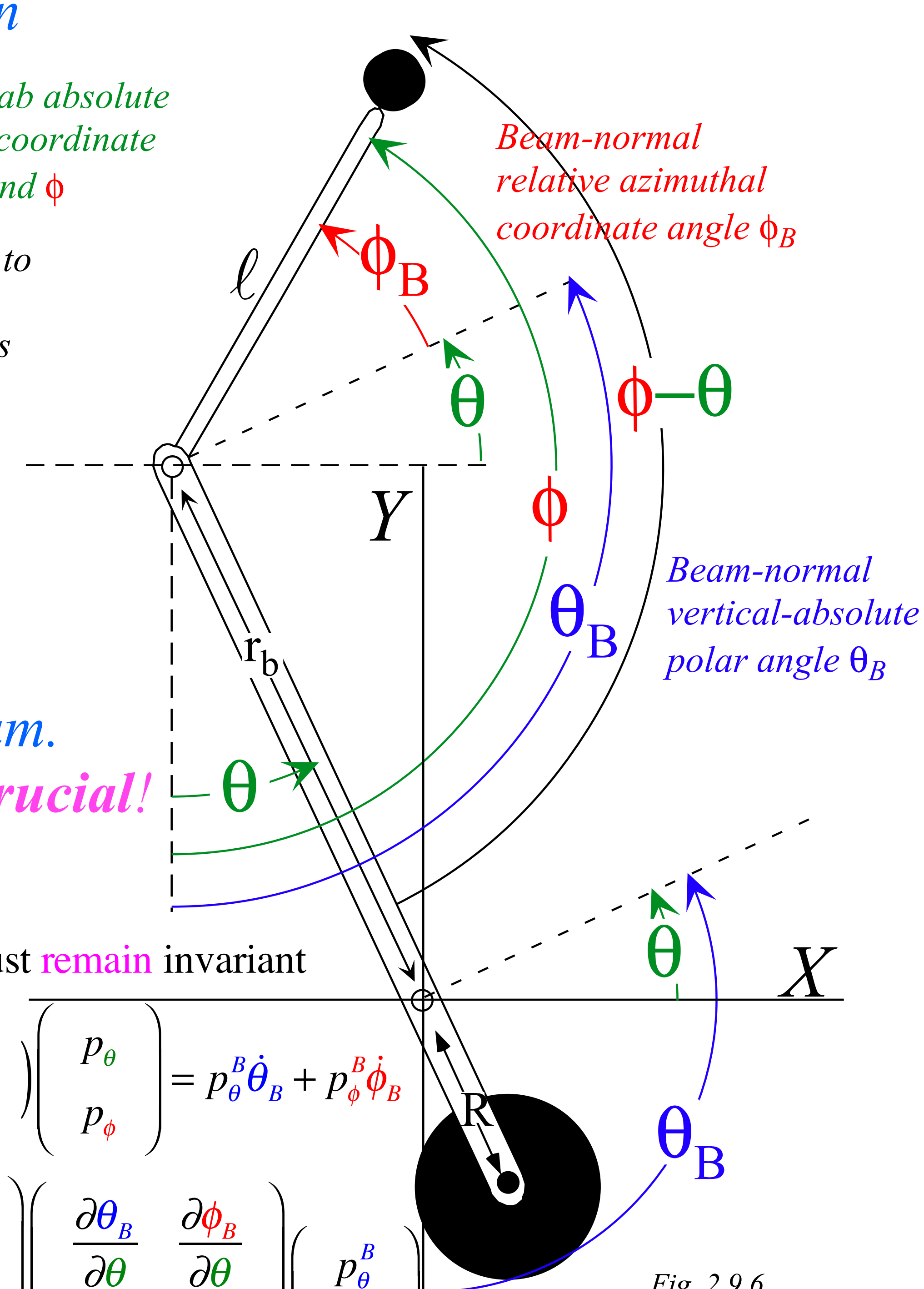


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta - \phi) \right]} + (MR - mr) g \cos\theta + mg\ell \cos\phi$$

$F_\theta = -MgR\sin\theta + mgr\sin\theta$

$F_\phi = -mg\ell\sin\phi$

$$H = \frac{m\ell^2 \left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B \left(p_\theta^B - p_\phi^B\right) \sin\phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2\phi_B \right]} - \left(MR - mr\right) g \sin\theta_B \; - mg\ell\cos\left(\phi_B + \theta_B\right)$$

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

Free-space trebuchet kinematics by symmetry

→ *Algebraic approach*

Direct approach and Superball analogy

Trebuchet vs Flinger and sports kinematics

Many approaches to Mechanics

Free-space trebuchet kinematics by symmetry: Algebraic approach

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

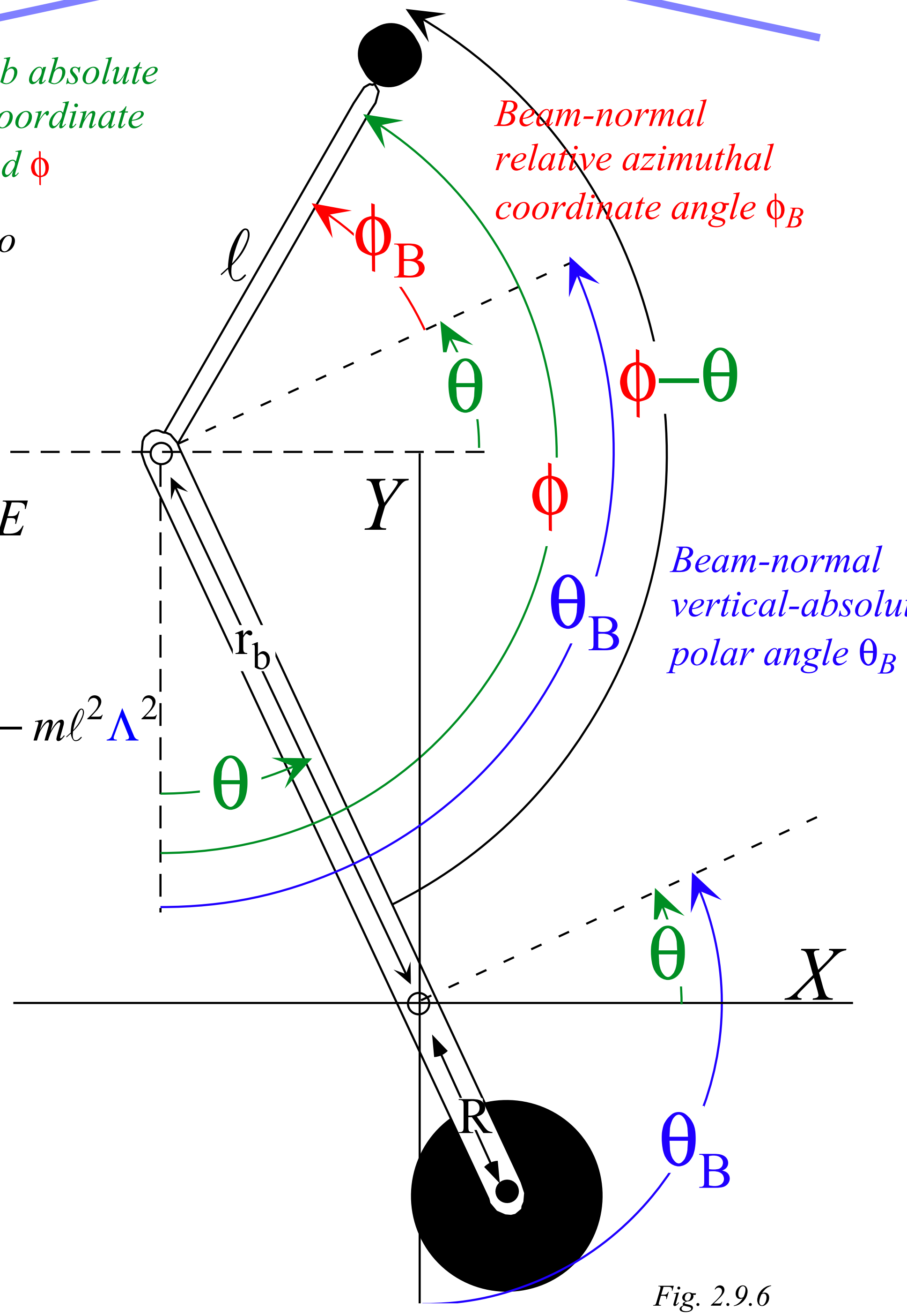
$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+I\right)\left(p_\phi^B\right)^2+2\Lambda\left(mr\ell\sin\phi_B-m\ell^2\right)p_\phi^B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]-m\ell^2\Lambda^2$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



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(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} - \left(MR - mr\right)g\sin\theta_B - mg\ell\cos\left(\phi_B + \theta_B\right)$$

For zero-gravity H is not a function of θ_B

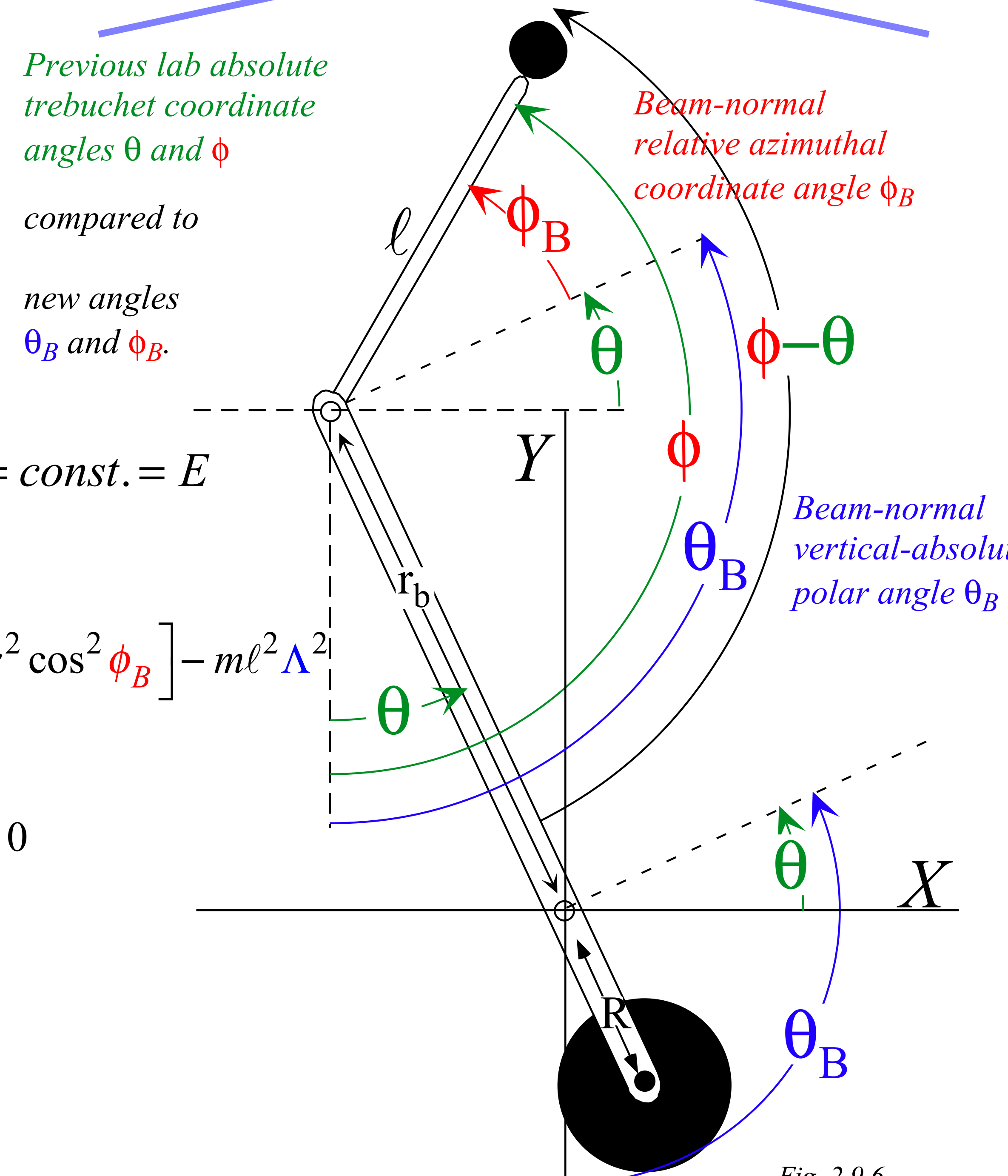
so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ *and :* $p_\theta^B = \Lambda = const.$

$$H = \frac{m\ell^2\left(\Lambda - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(\Lambda - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} = const. = E$$

$$\left(m\ell^2 + 2mr\ell\sin\phi_B + I\right)\left(p_\phi^B\right)^2 + 2\Lambda\left(mr\ell\sin\phi_B - m\ell^2\right)p_\phi^B = Em\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right] - m\ell^2\Lambda^2$$

$$\left(1 - 2\frac{r}{\ell}\sin\phi_B + J\right)\left(p_\phi^B\right)^2 + 2\Lambda\left(\frac{r}{\ell}\sin\phi_B - 1\right)p_\phi^B + \Lambda^2 - E\left[I - mr^2\sin^2\phi_B\right] = 0$$

where: $I = MR^2 + mr^2 = Jm\ell^2$



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(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)$$

For zero-gravity H is not a function of θ_B

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Previous lab absolute trebuchet coordinate angles θ *and* ϕ

compared to

new angles θ_B *and* ϕ_B.

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+I\right)\left(p_\phi^B\right)^2+2\Lambda\left(mr\ell\sin\phi_B-m\ell^2\right)p_\phi^B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]-m\ell^2\Lambda^2$$

$$\left(1-2\frac{r}{\ell}\sin\phi_B+J\right)\left(p_\phi^B\right)^2+2\Lambda\left(\frac{r}{\ell}\sin\phi_B-1\right)p_\phi^B+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

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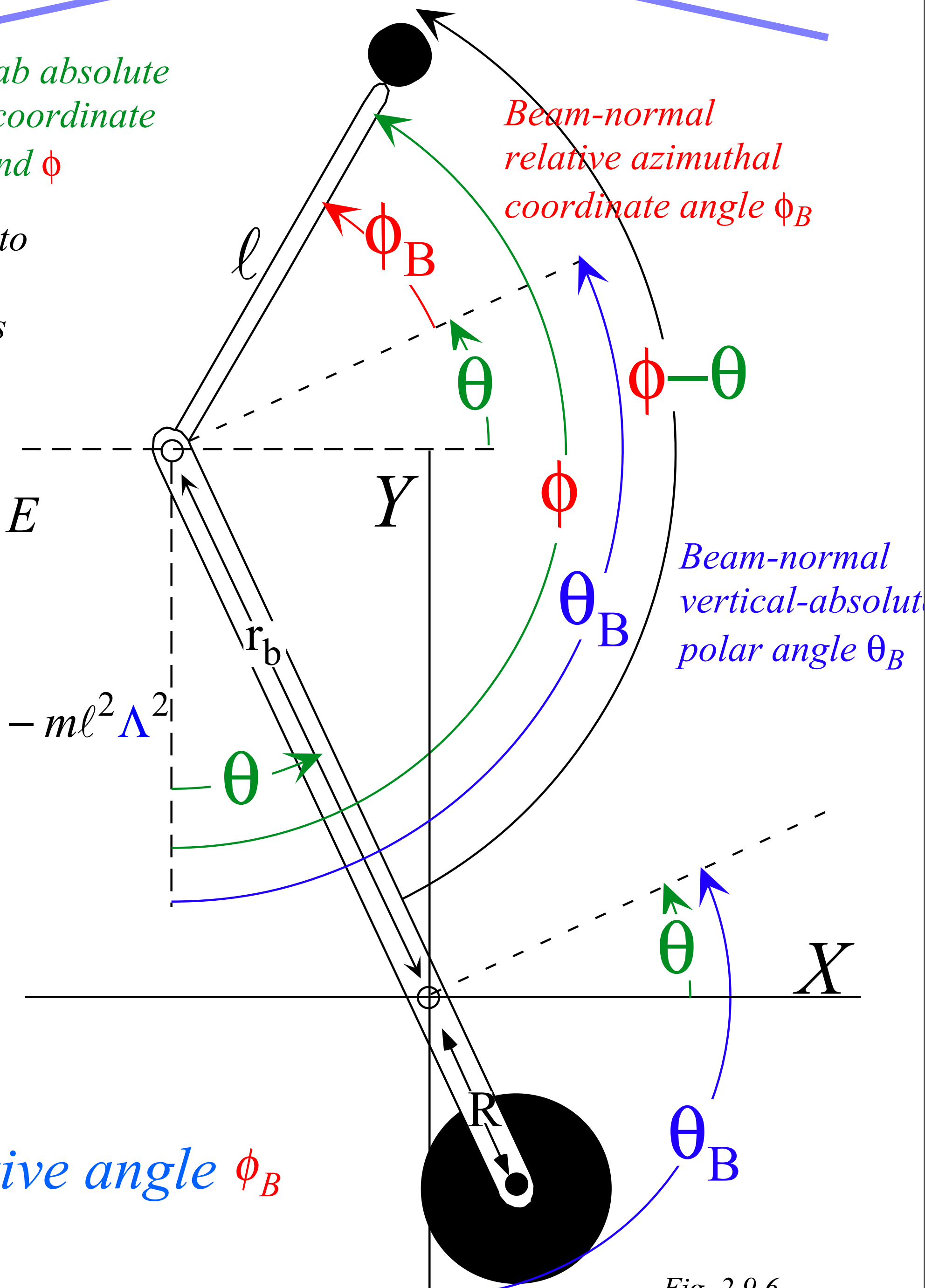


Fig. 2.9.6
Lab (θ,ϕ) *and beam-normal* (θ_B,ϕ_B) *relative coordinates for trebuchet.*
(Each value is positive.)

Throwing momentum p_ϕ^B *as a function of beam-relative angle* ϕ_B

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - → *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Beam-relative coordinate transformation

$\theta = \theta_B \quad -\pi/2$	$\theta_B = \theta \quad +\pi/2$
$\phi = \theta_B + \phi_B$	$\phi_B = -\theta + \phi \quad -\pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = p_\theta + p_\phi = \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$

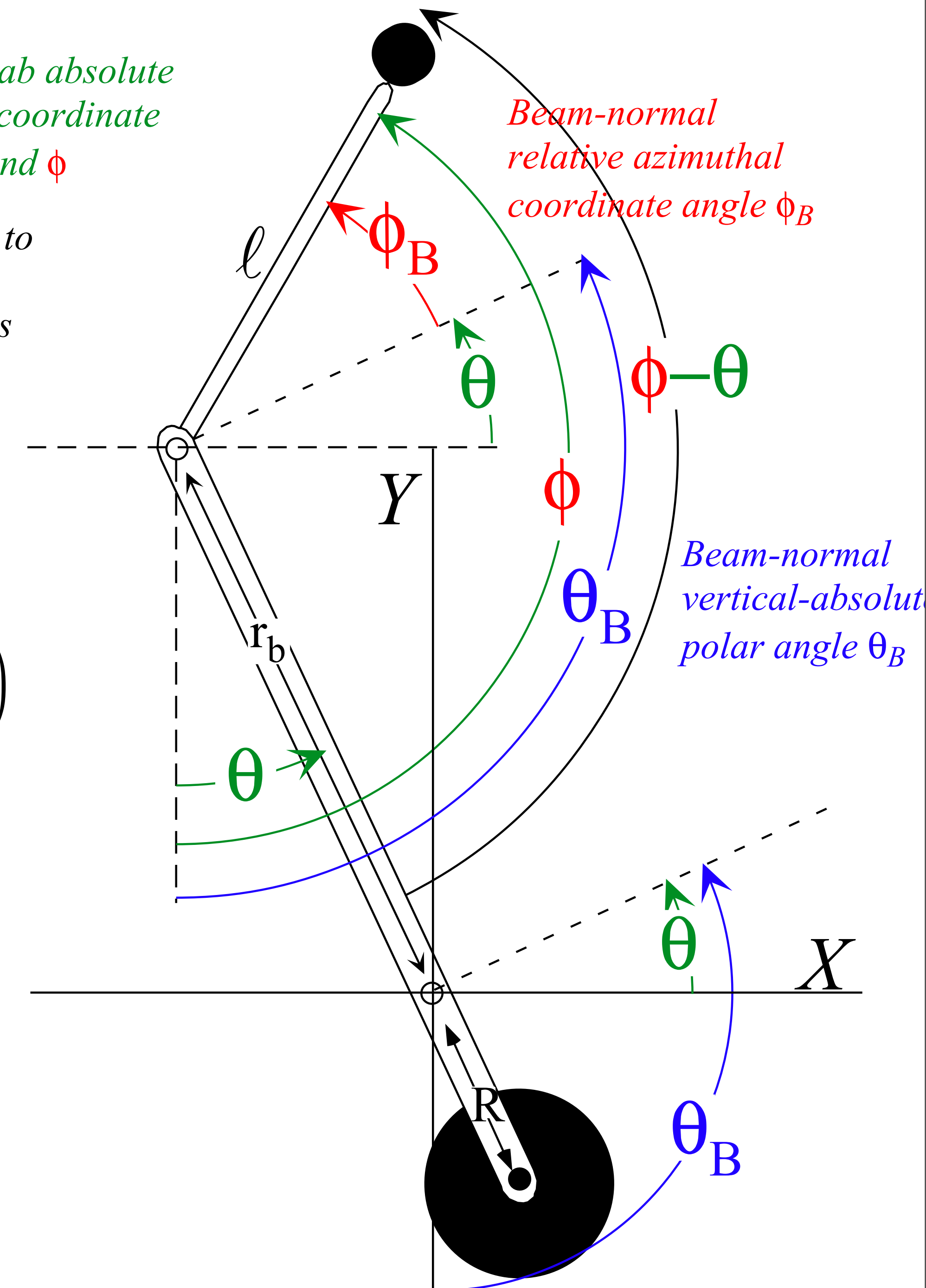
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

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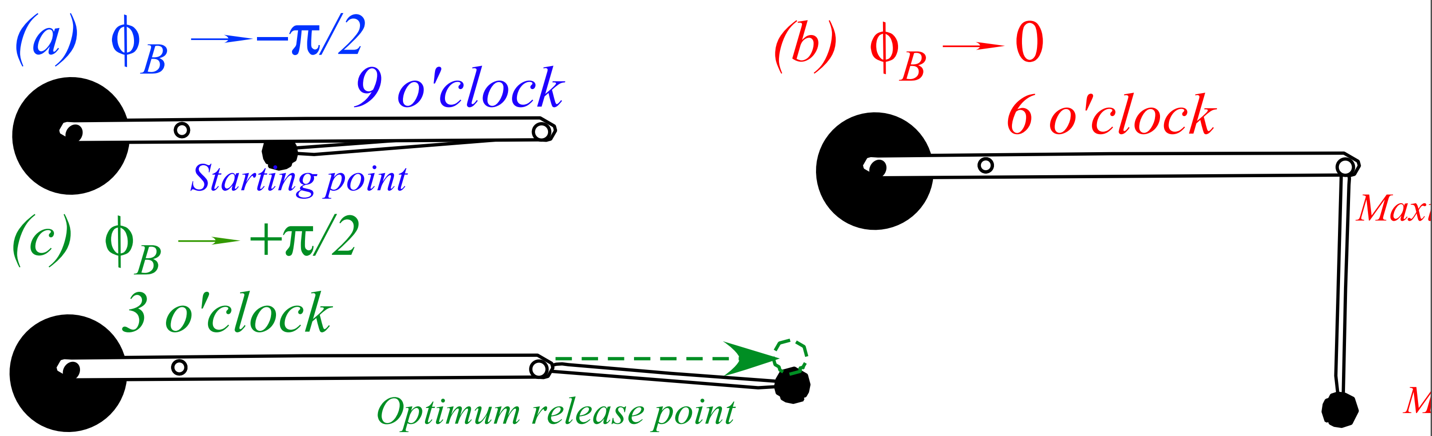

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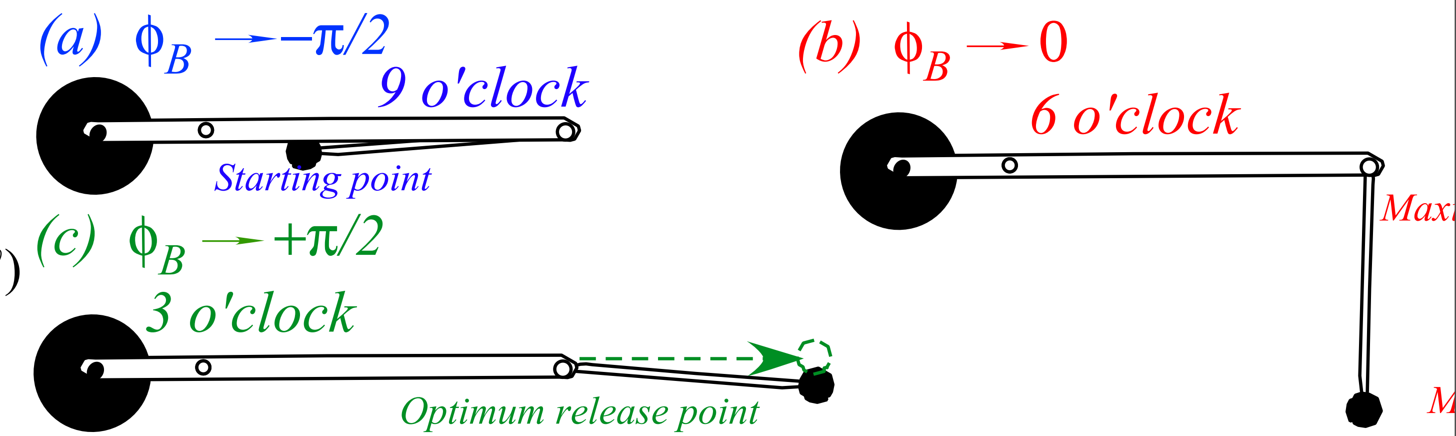

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \begin{cases} 2E = MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{cases} \quad or: \begin{cases} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

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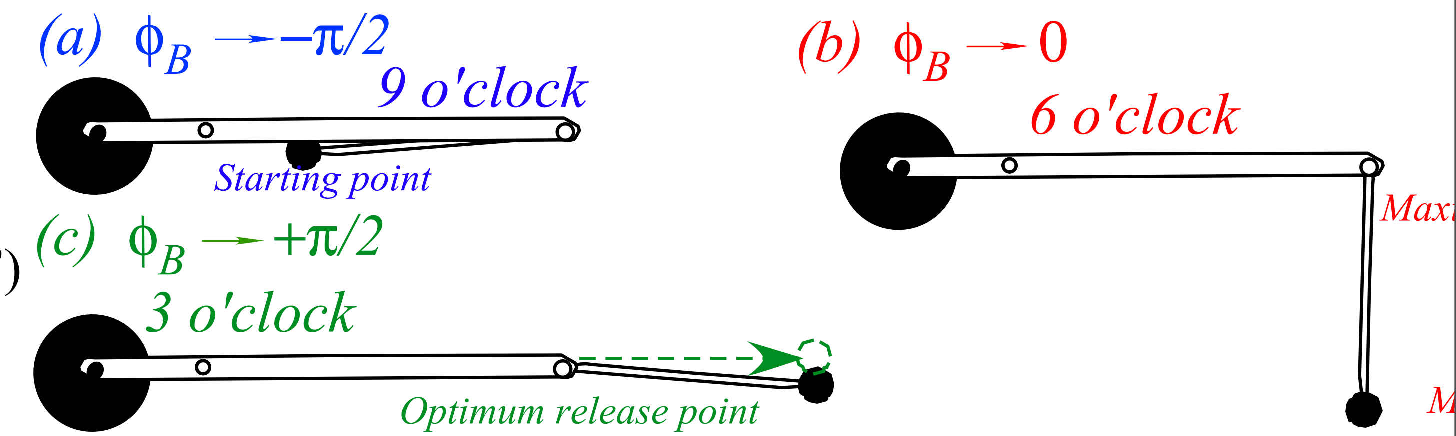

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Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

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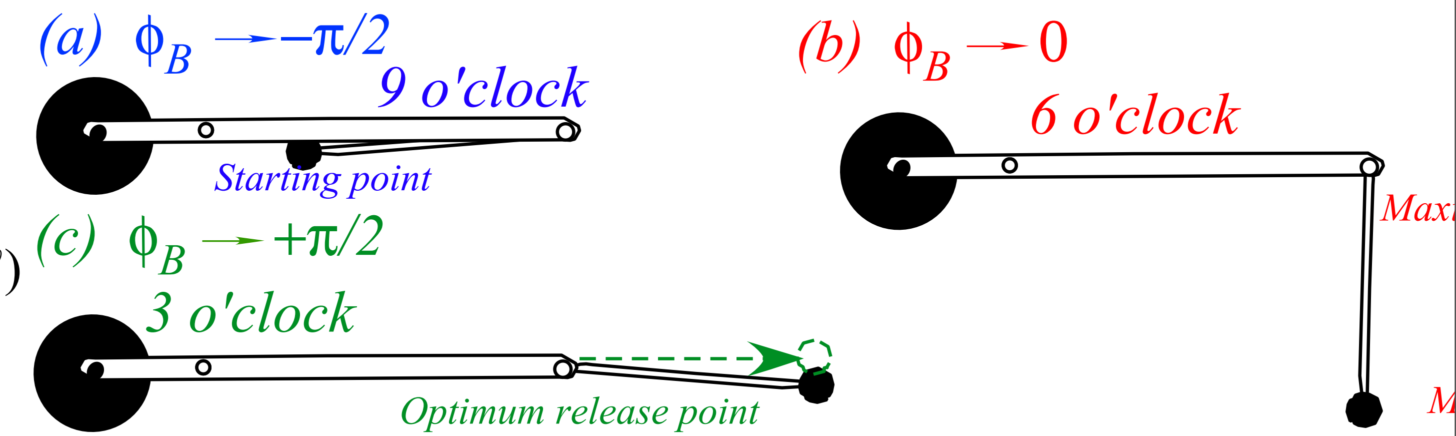

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Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \begin{cases} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{cases}$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

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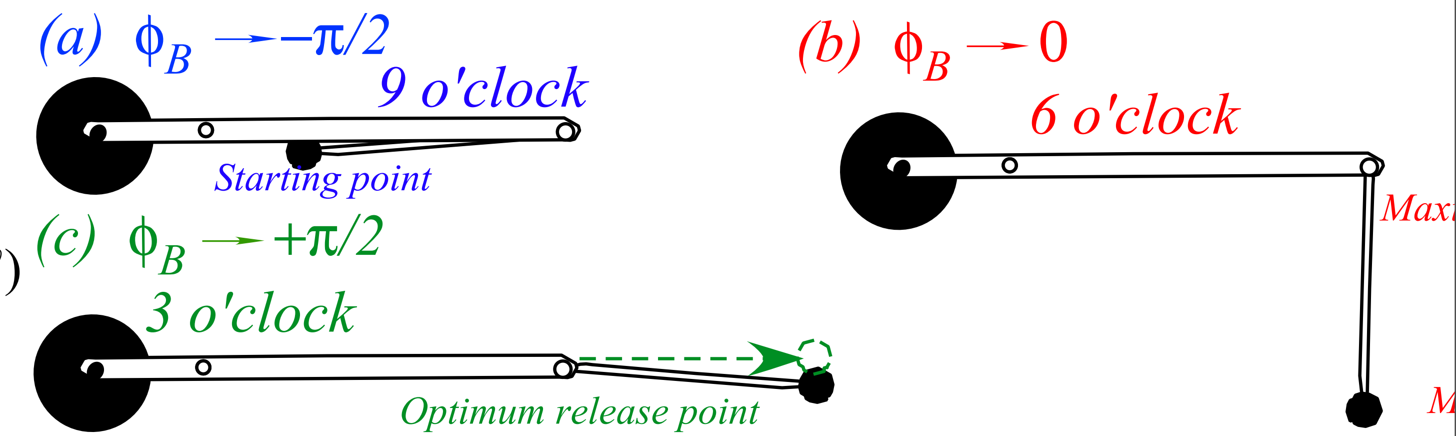
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



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Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \begin{cases} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{cases}$$

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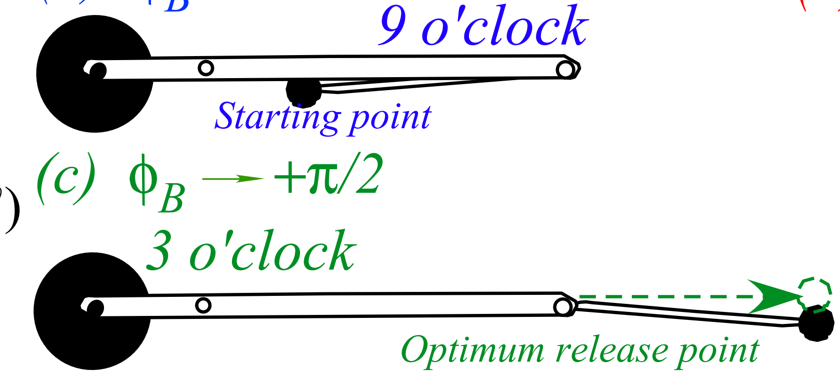
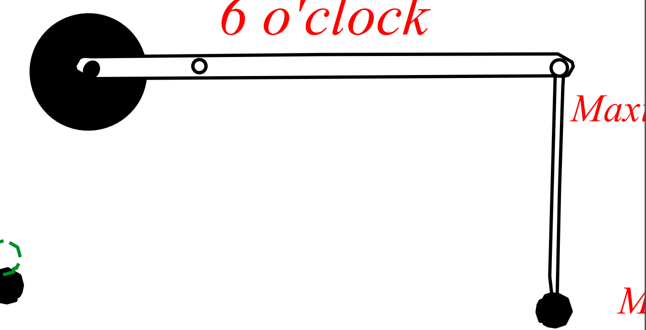
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy





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Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \begin{cases} 2E = MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right)\end{cases}$$

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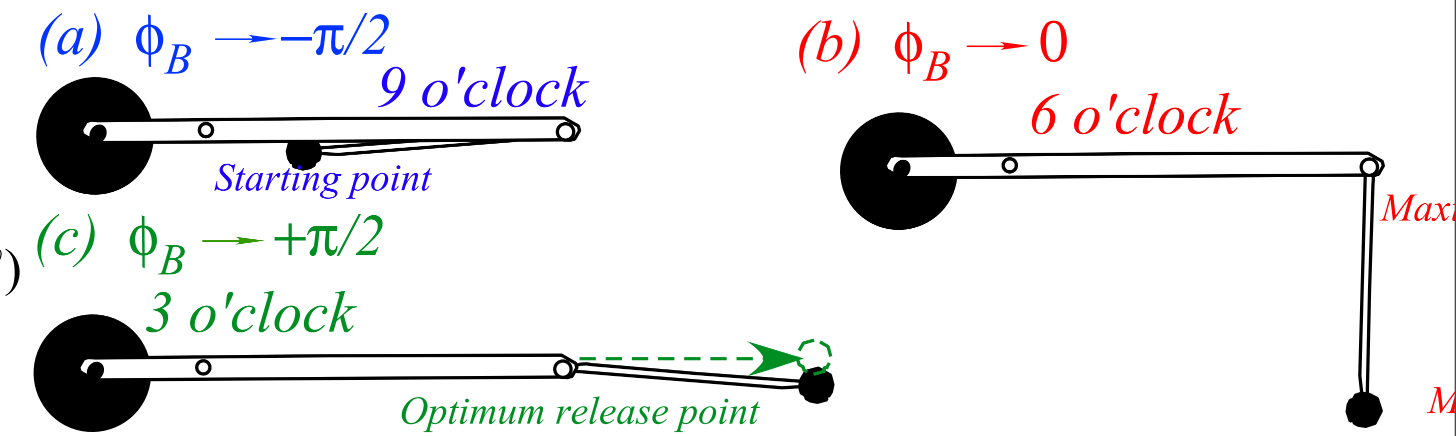

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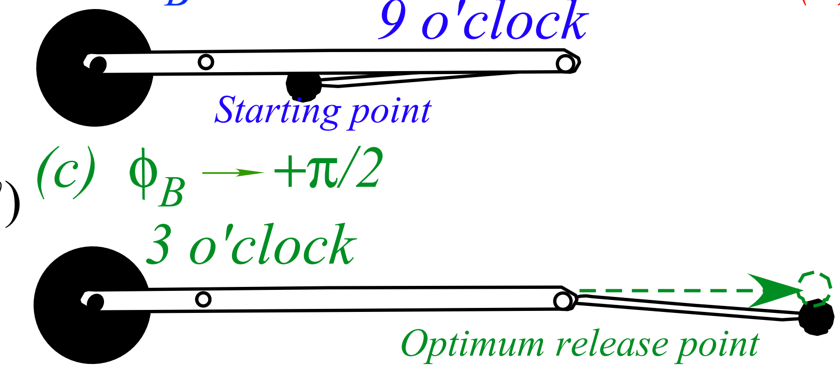
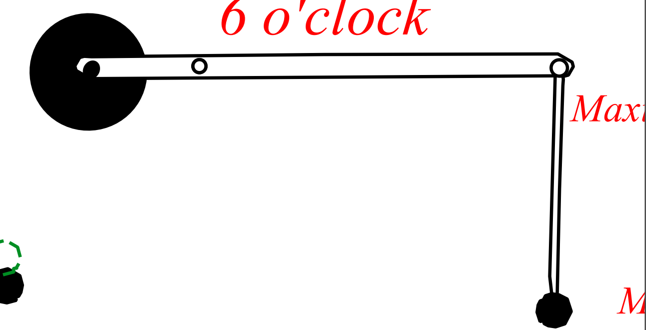
(a) $\phi_B \to -\pi/2$ 9 o'clock

(c) $\phi_B \to +\pi/2$ 3 o'clock



(b) $\phi_B \to 0$ 6 o'clock



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$$(\omega + \dot\theta_{\pi/2}) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

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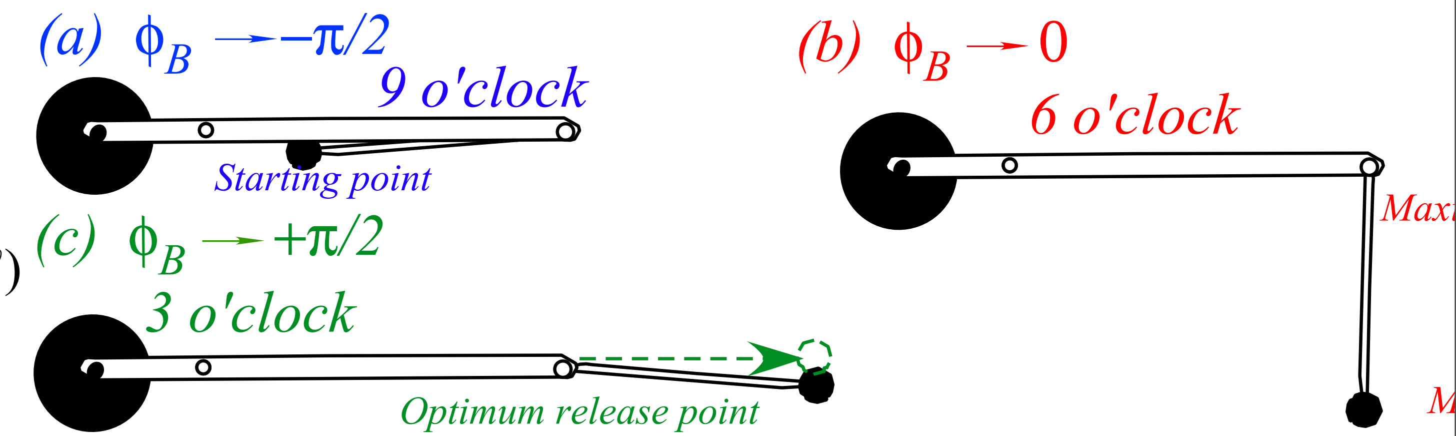

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$$\rightarrow \left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

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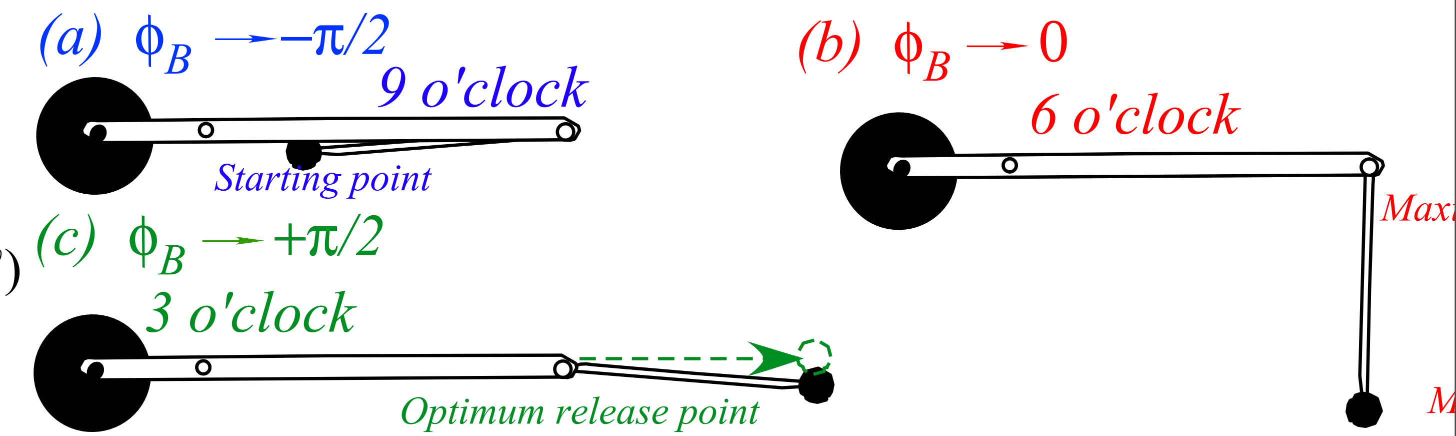
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$



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Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

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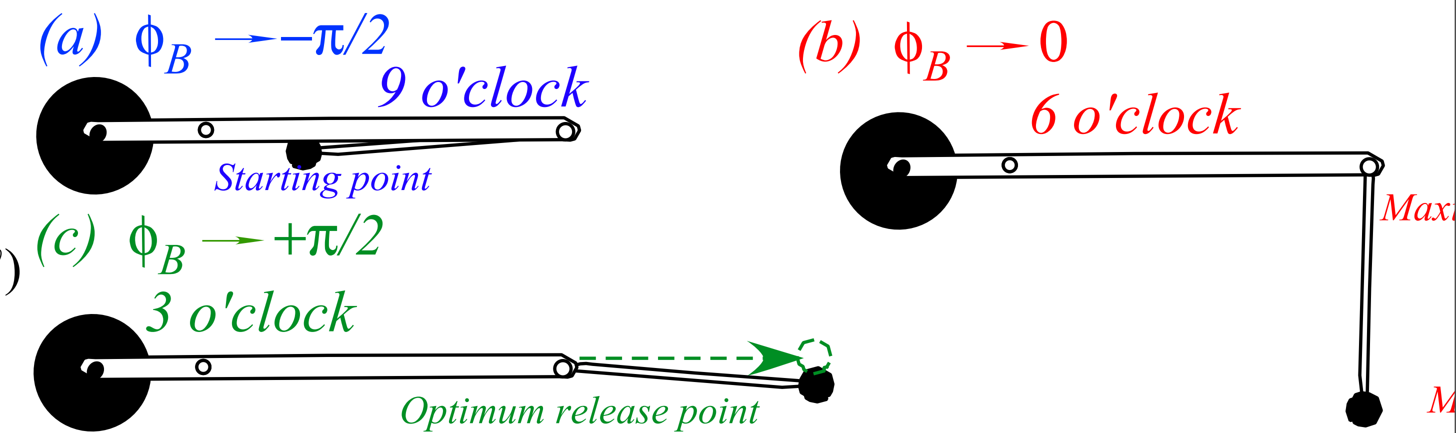

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Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

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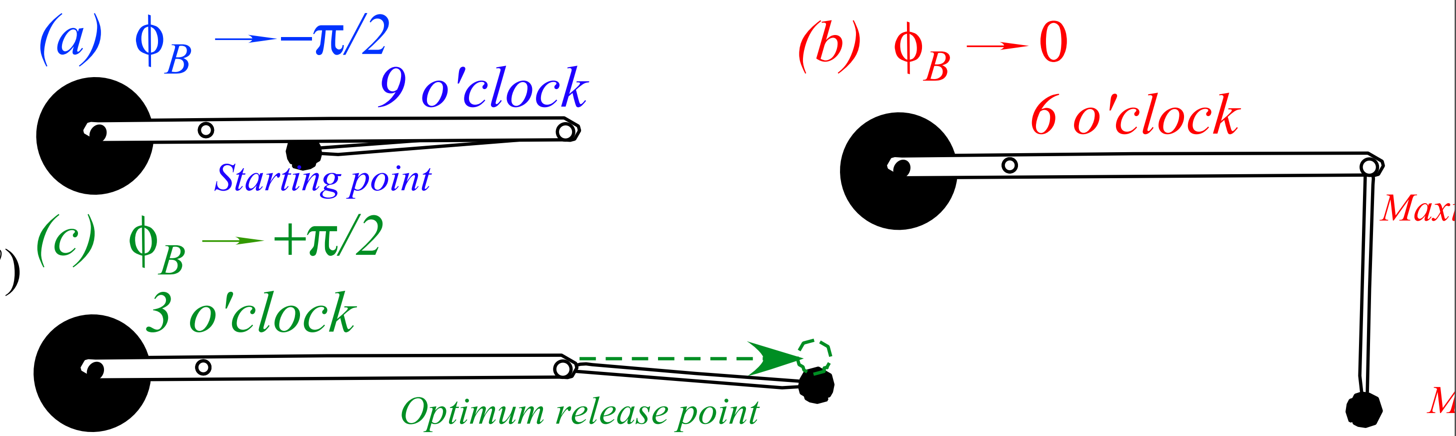

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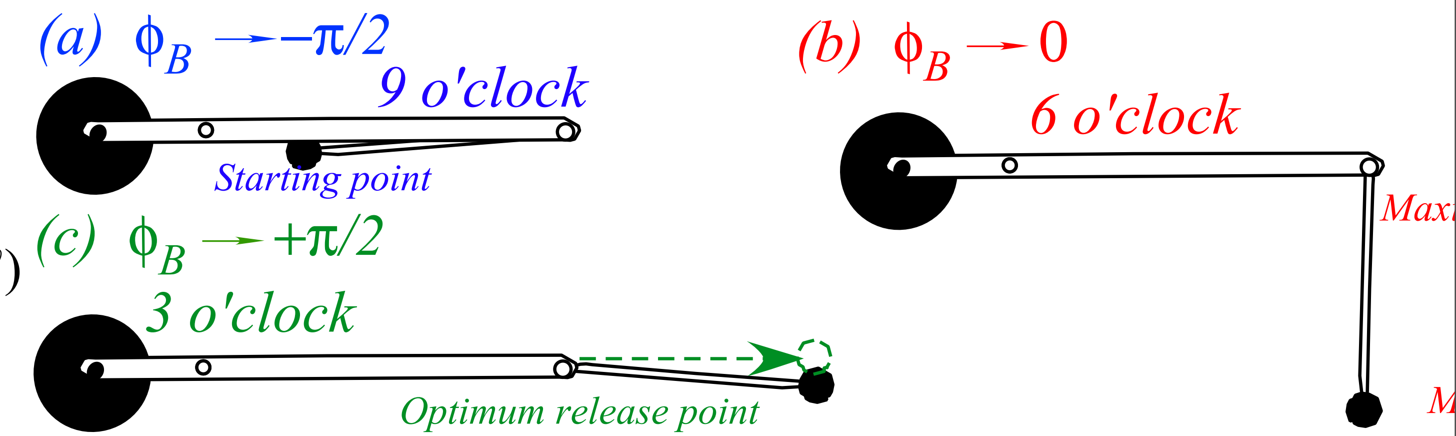
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)



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Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

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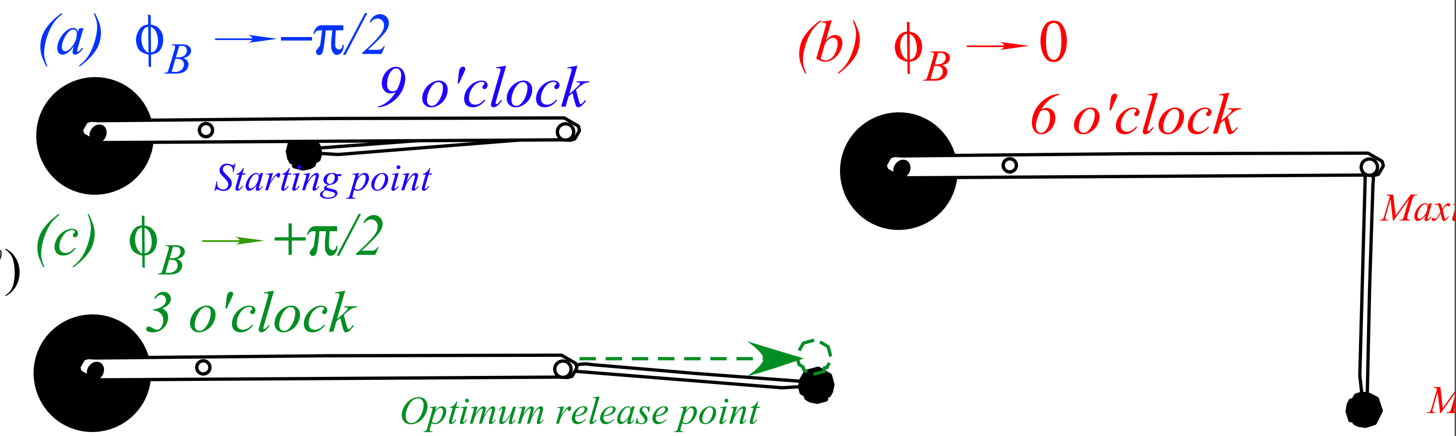
Large $M \gg m$ case

$$\boxed{\dot{\phi}_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot{\theta}_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



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Large $M \gg m$ case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot\phi_{\pi/2} = 0 + 2\omega = 2\omega}$$

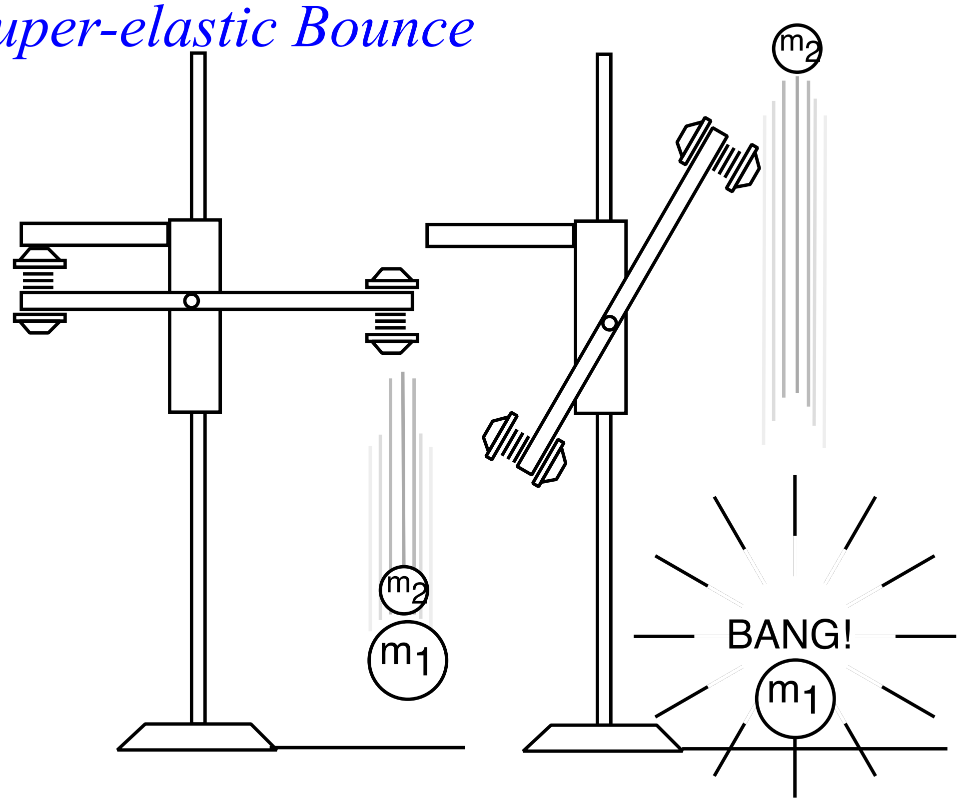
$$\boxed{\dot\theta_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Analogous Superball Models

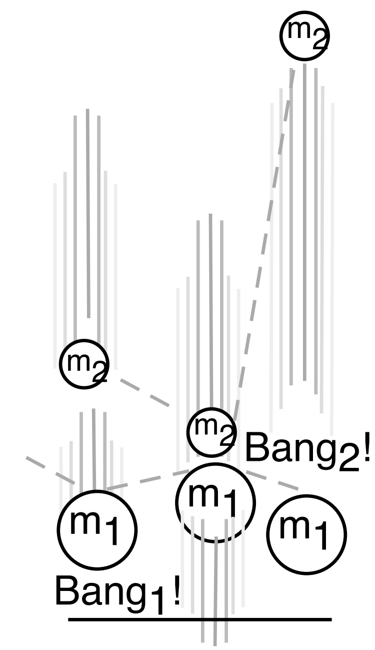
Similar in some ways to trebuchet models

Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. Phys. **39**, 656 (1971)
(A class project)

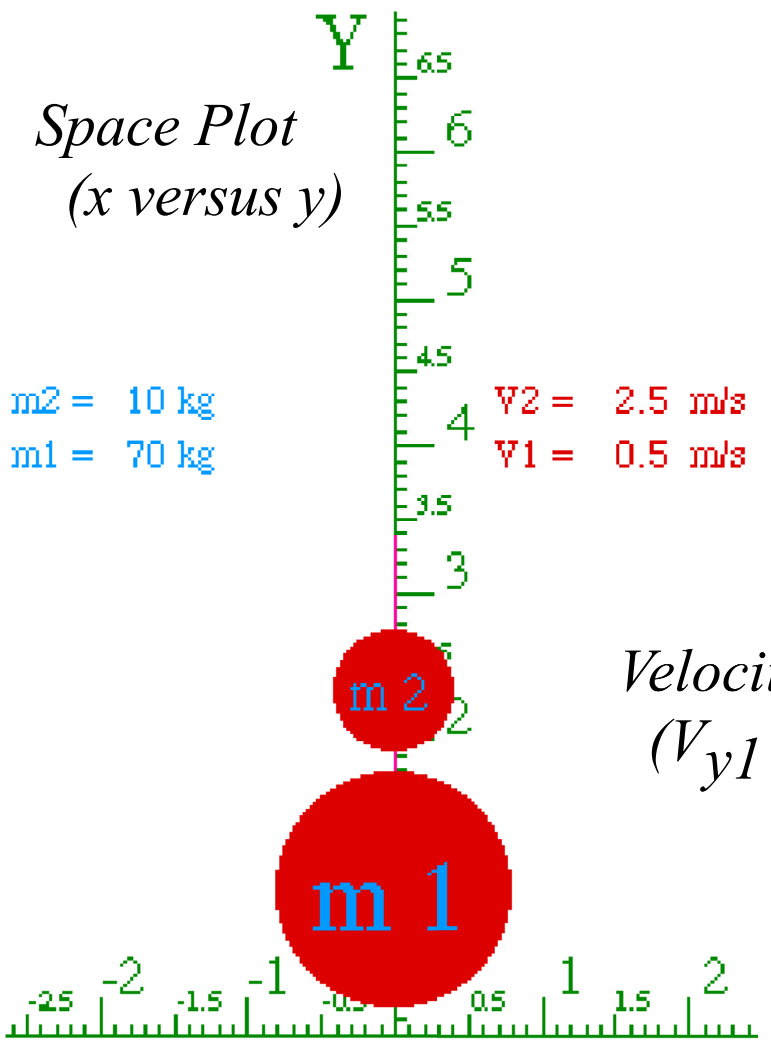
Super-elastic Bounce



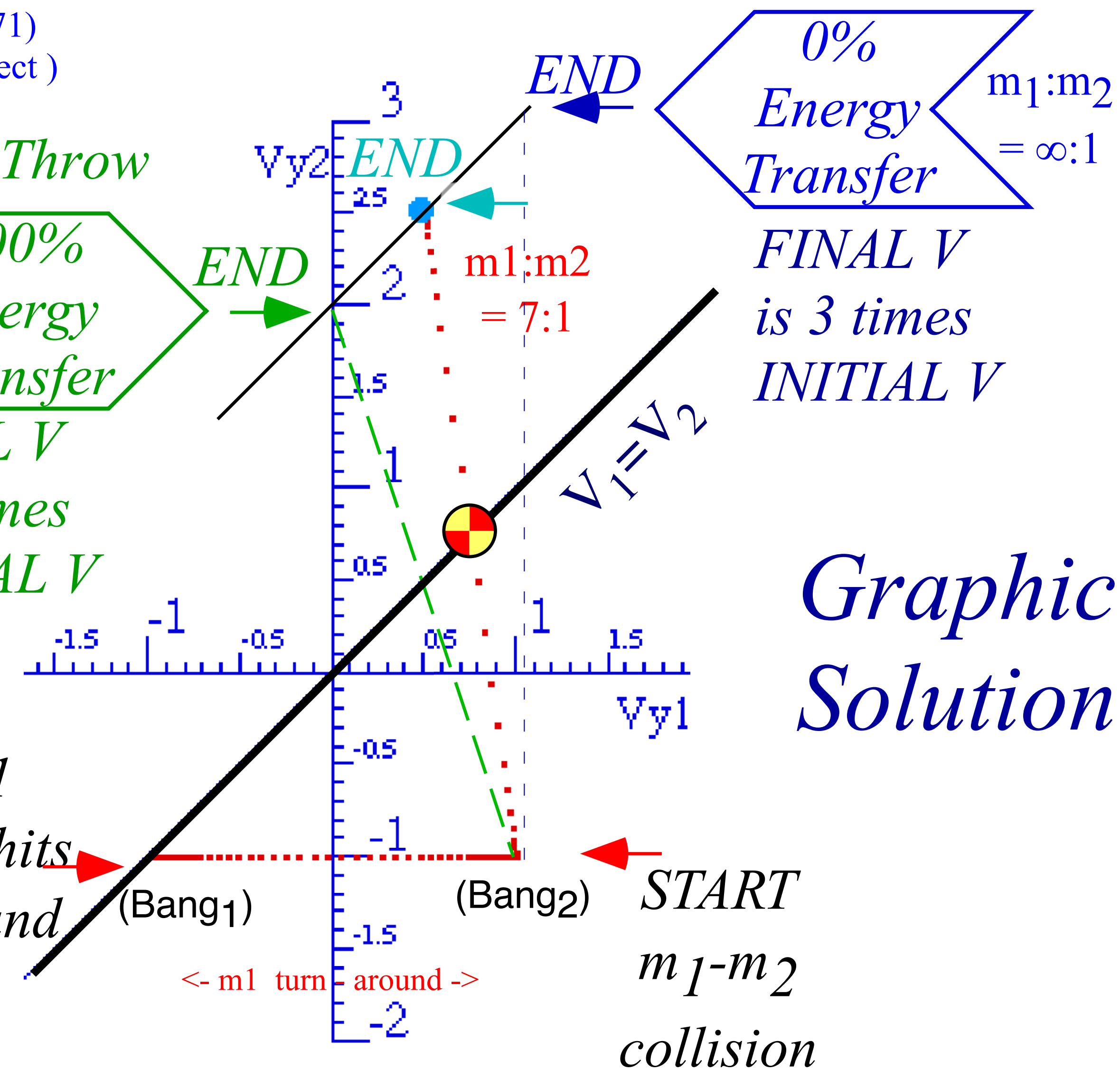
2-Bang Model



Space Plot
(x versus y)



Velocity Plot →
(V_{y1} versus V_{y2})



Graphic Solution

Hamiltonian energy and momentum conservation and symmetry coordinates

Coordinate transformation helps reduce symmetric Hamiltonian

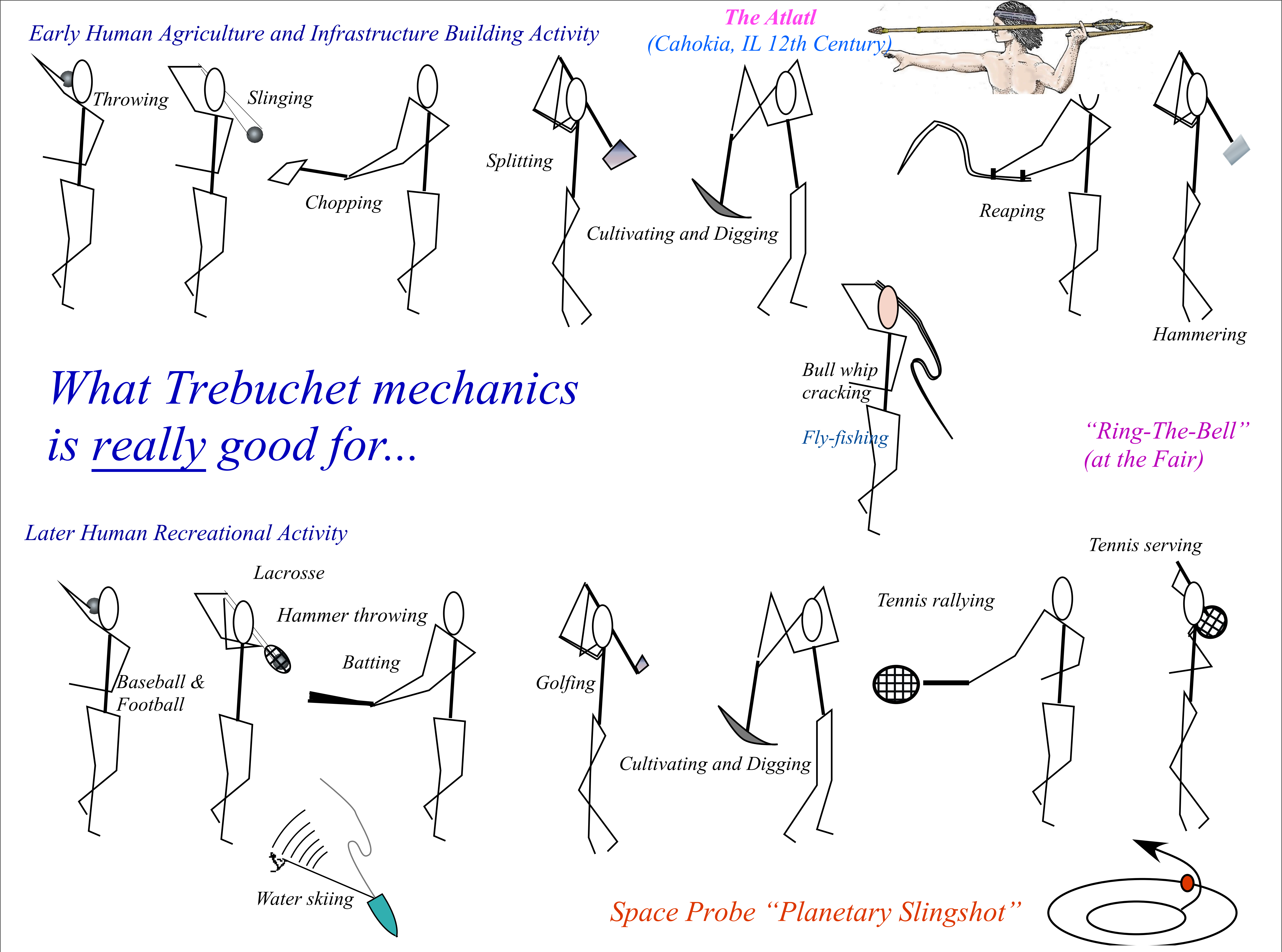
Free-space trebuchet kinematics by symmetry

Algebraic approach

Direct approach and Superball analogy

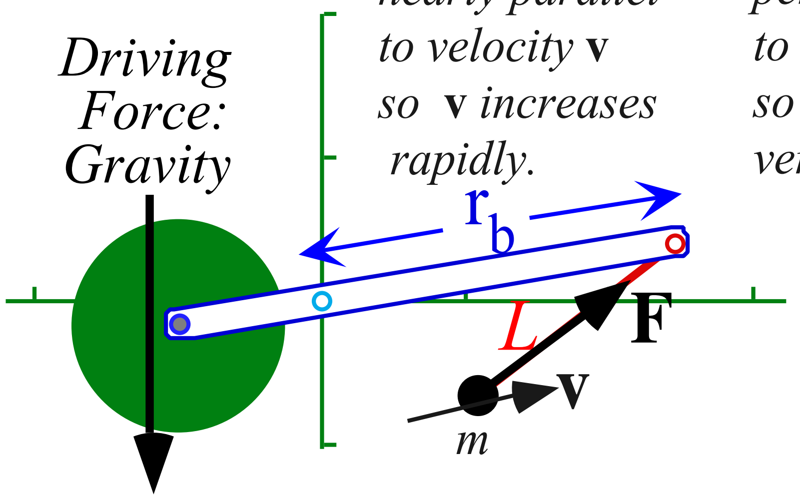
→ *Trebuchet vs Flinger and sports kinematics*

Many approaches to Mechanics


Early Human Agriculture and Infrastructure Building Activity
The Atlatl
(Cahokia, IL 12th Century)
Throwing
Slinging
Chopping
Splitting
Cultivating and Digging
Reaping
Hammering
Bull whip
cracking
Fly-fishing
What Trebuchet mechanics is really good for...
"Ring-The-Bell"
(at the Fair)
Later Human Recreational Activity
Lacrosse
Hammer throwing
Batting
Baseball &
Football
Golfing
Cultivating and Digging
Tennis rallying
Tennis serving
Water skiing
Space Probe "Planetary Slingshot"

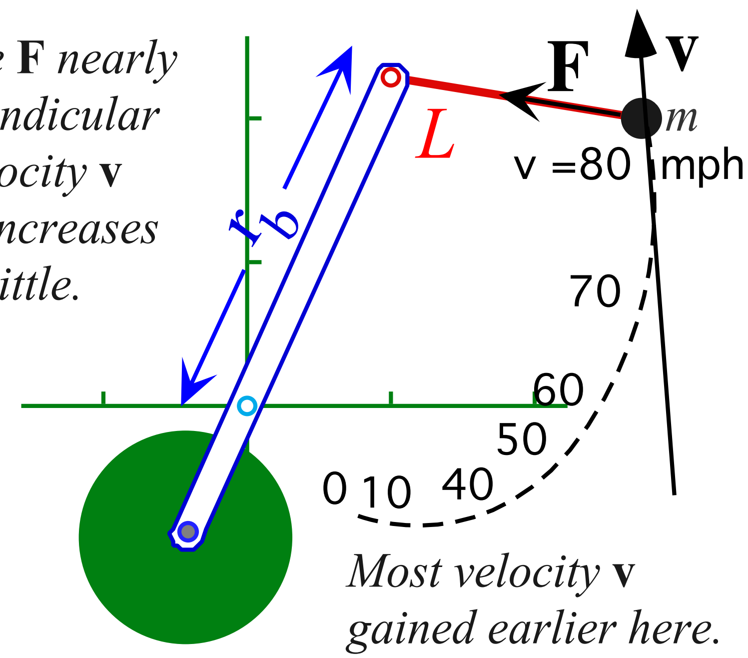
Trebuchet analogy with racquet swing - What we learn

Driving Force: Gravity



Large force **F** *nearly parallel to velocity* **v** *so* **v** *increases rapidly.*

Force **F** *nearly perpendicular to velocity* **v** *so* **v** *increases very little.*



F *mostly serves to steer m here.*

Most velocity **v** *gained earlier here.*

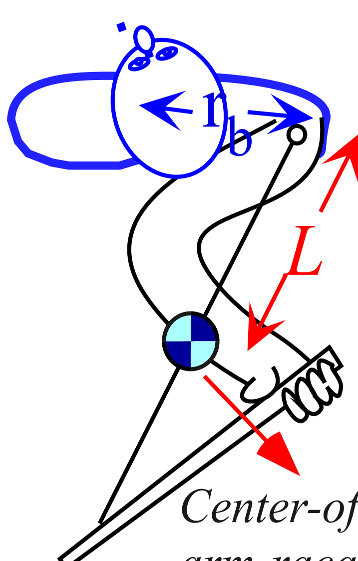
Early on

(Gain the energy/momentum)

Later on

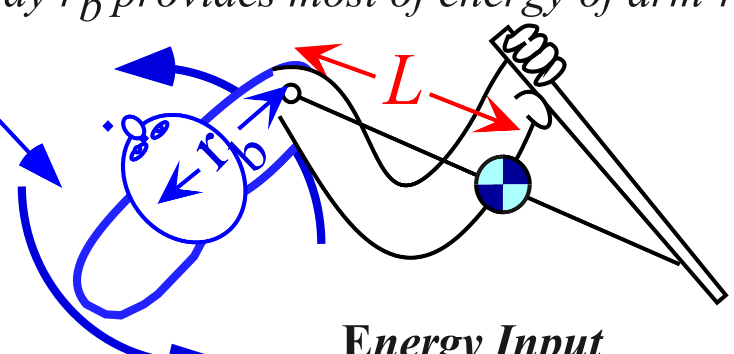
(Steer or guide)

Rotation of body r_b provides most of energy of arm-racquet lever L.



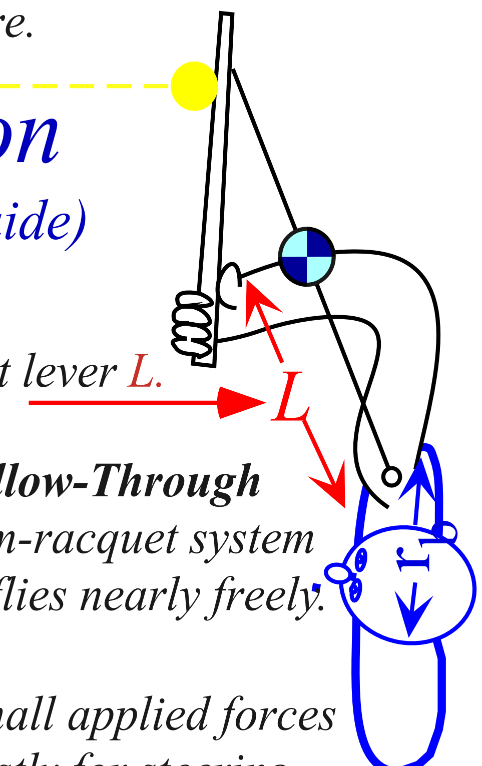
Preparation

Center-of-mass for semi-rigid arm-racquet system L is "cocked."



Energy Input

Most of speed gained early by arm-racquet system L.



Follow-Through

Arm-racquet system L flies nearly freely.

Small applied forces mostly for steering.

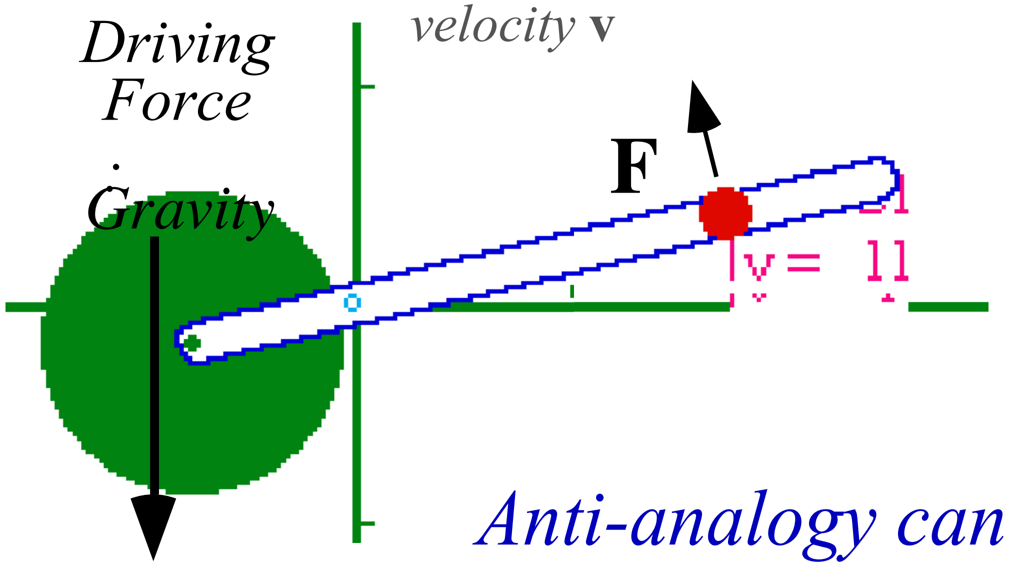
Ball hit occurs.

An Opposite to Trebuchet Mechanics- The "Flinger"

Early on

(Not much happening)

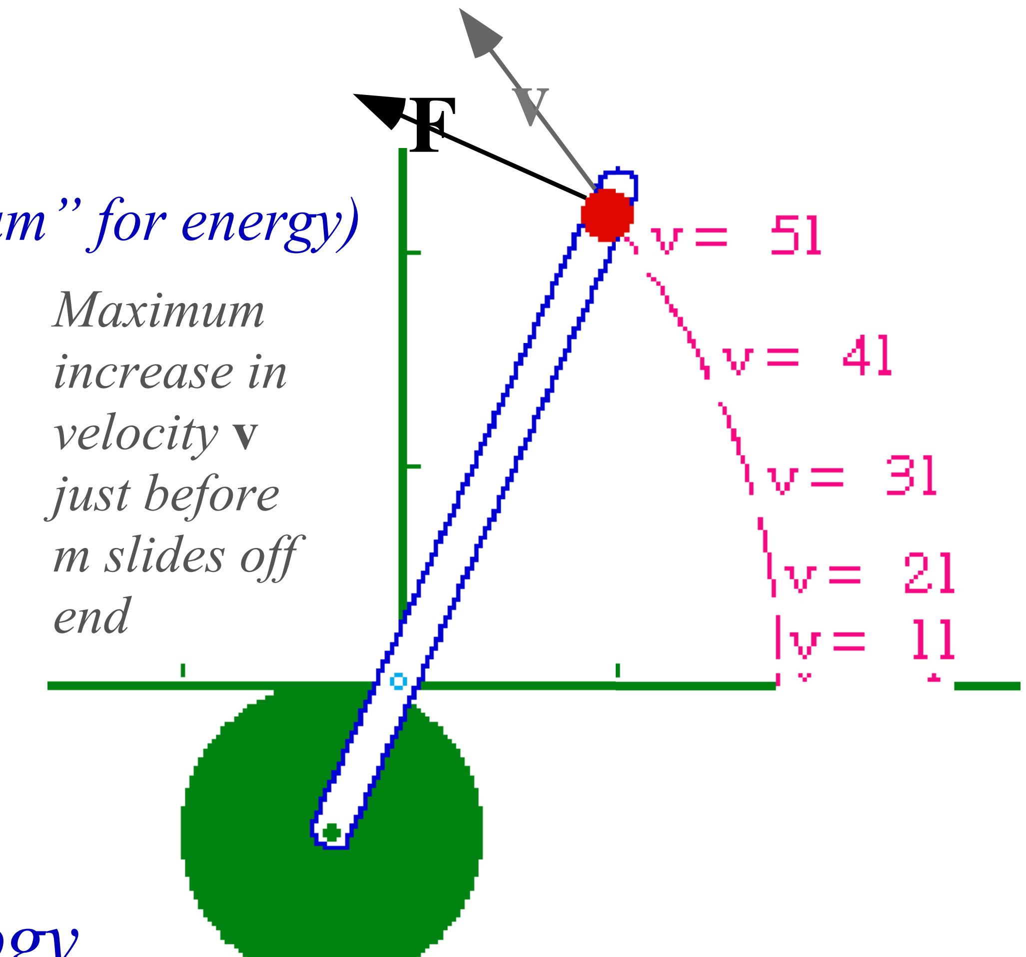
Not much increase in velocity **v**



Later on

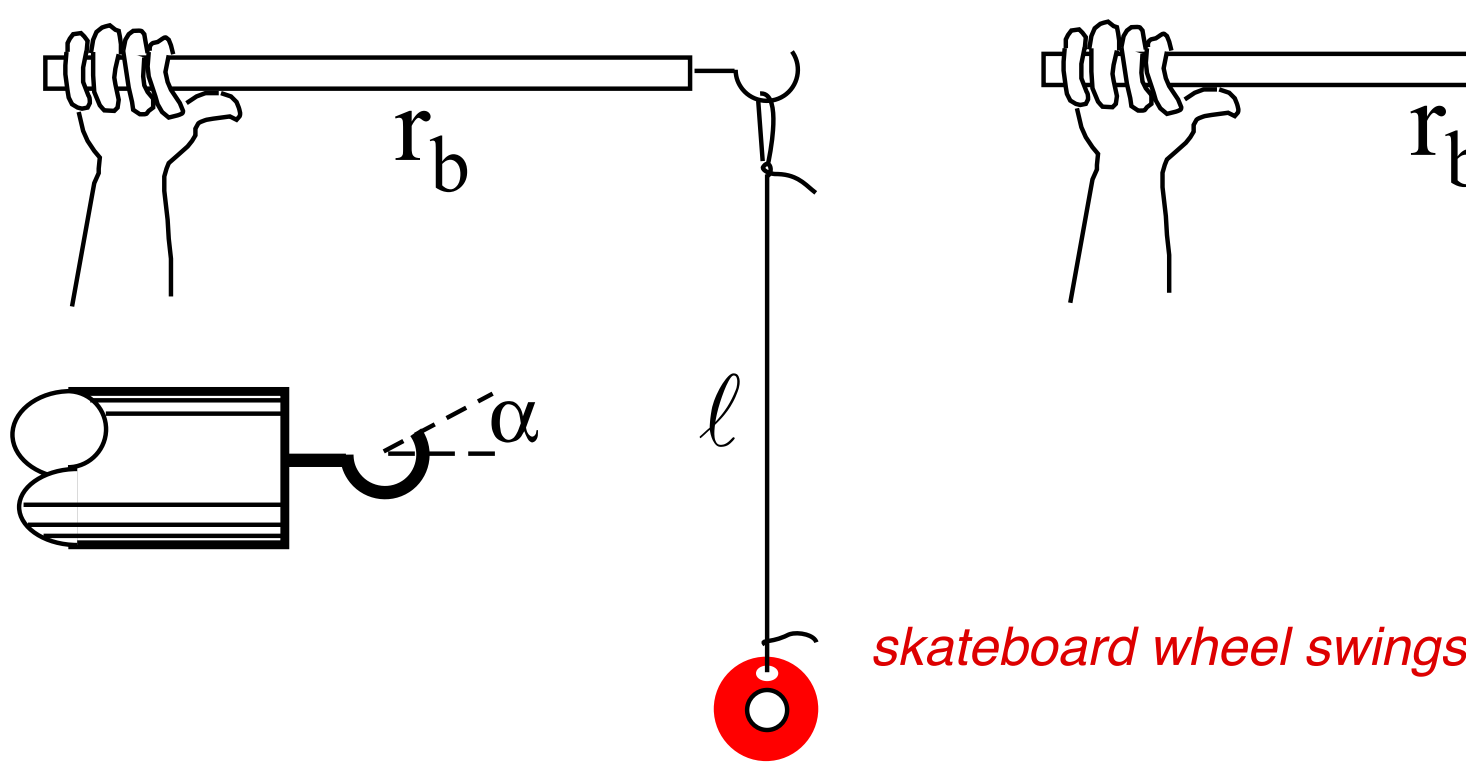
(Last-minute "cram" for energy)

Maximum increase in velocity **v** *just before m slides off end*

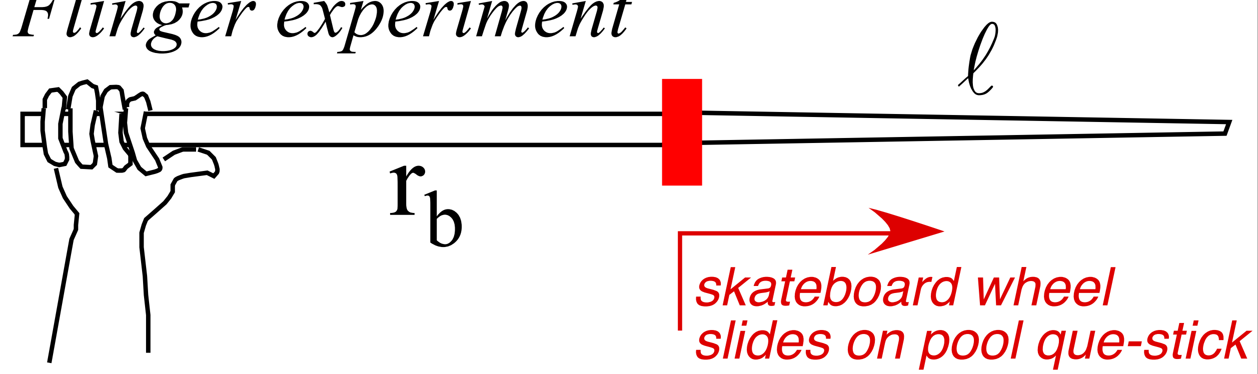


Anti-analogy can be useful pedagogy

Trebuchet-like experiment



Flinger experiment

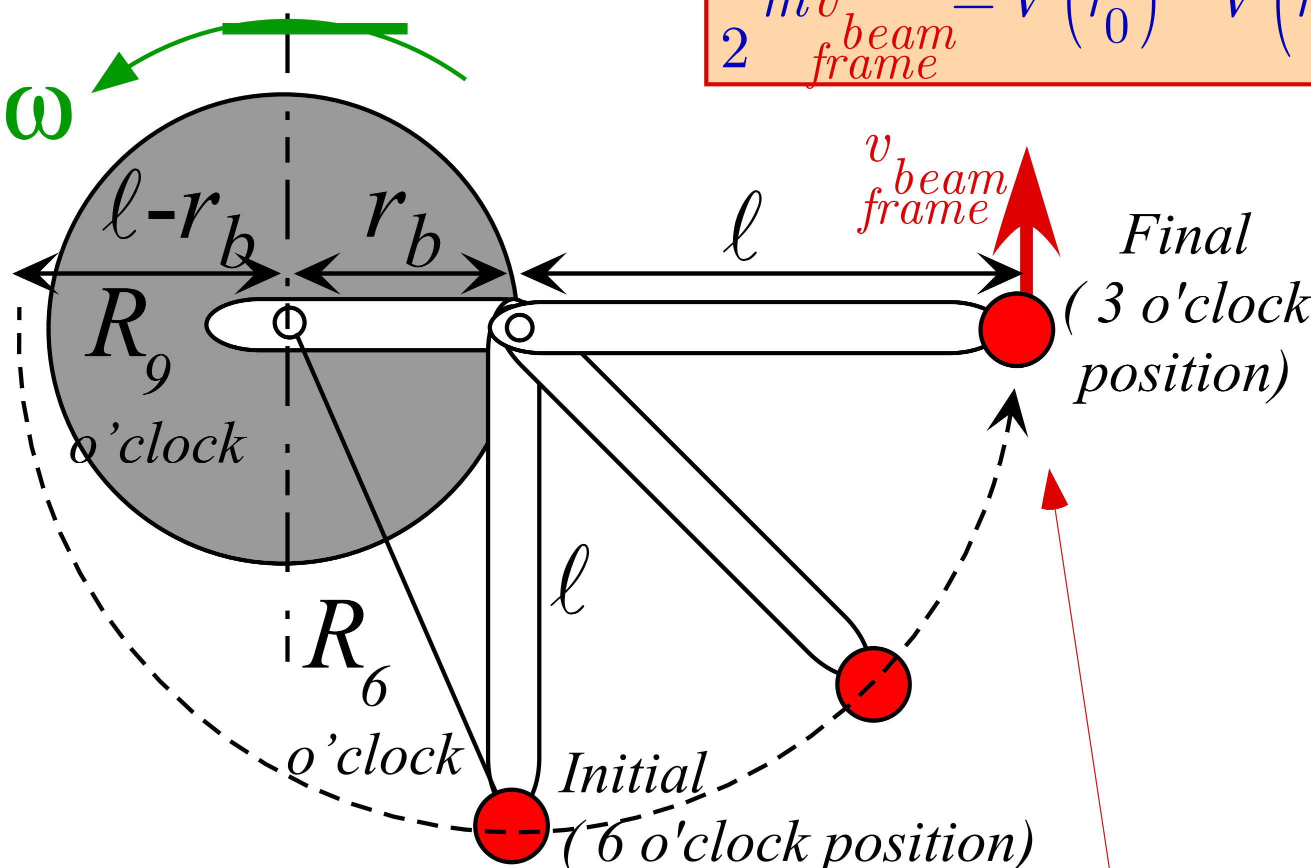


Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2} m v^2_{\substack{beam\\frame}} = V(r_0) \quad V(r_f) = \frac{1}{2} m \quad {}^2 r_f^2 \quad \frac{1}{2} m \quad {}^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2} m v^2_{\substack{beam\\frame}} (trebuchet) = \textit{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\frac{1}{2} m \quad {}^2 \left(r_b + \ell\right)^2 \quad \frac{1}{2} m \quad {}^2 \left(r_b^2 + \ell^2\right) = \frac{1}{2} m \quad {}^2 \left(2 r_b \ \ell\right)$$

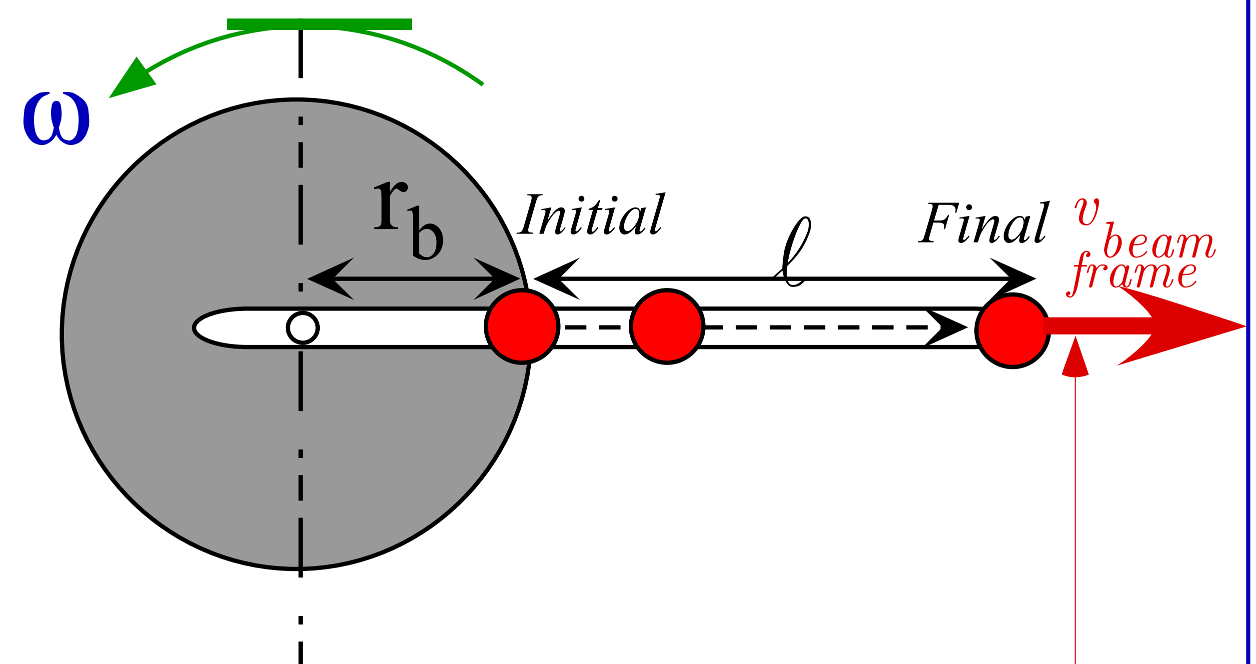
Final 3 o'clock — Initial 6 o'clock

$$R^2_{\substack{6\\o'clock}} = r_b^{\ 2} + \ell^2$$

Initial 9 o'clock $= \frac{1}{2} m \quad {}^2 \left(4 r_b \ \ell\right)$

$$R^2_{\substack{9\\o'clock}} = r_b^{\ 2} + \ell^2 - 2 r_b \ell$$

Assume: Constant beam ω



$$\frac{1}{2} m v^2_{\substack{beam\\frame}} (flinger) = \textit{Final Flinger } KE_{\substack{beam\\frame}}$$

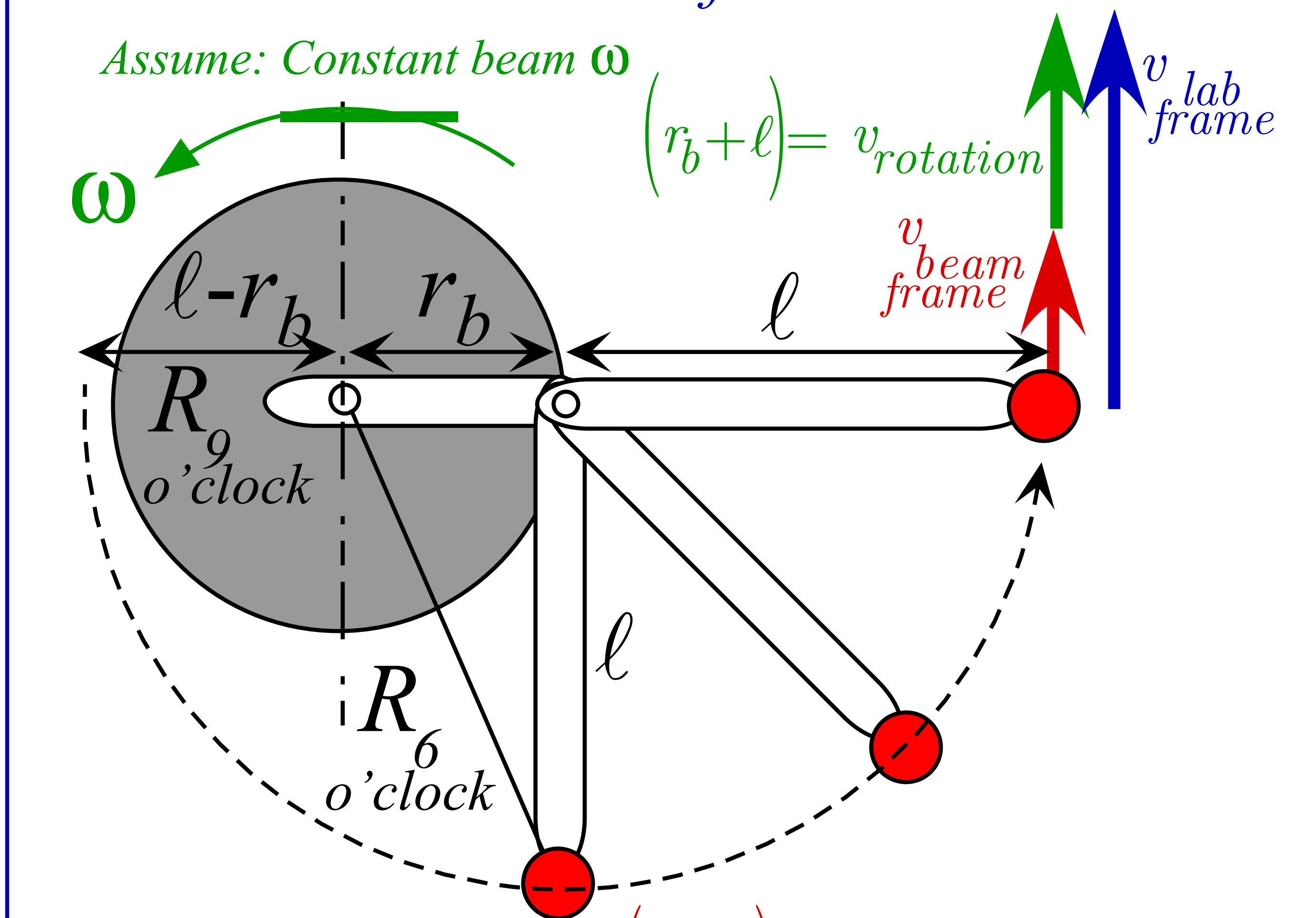
$$\frac{1}{2} m \quad {}^2 \left(r_b + \ell\right)^2 \quad \frac{1}{2} m \quad {}^2 r_b^2 = \frac{1}{2} m \quad {}^2 \ell \left(2 r_b + \ell\right)$$

Final 3 o'clock — Initial 3 o'clock

Flinger KE is $\frac{m}{2} \ {}^2 \ \ell^2$ more than 6 o'clock trebuchet but misdirected

Flinger KE is $\frac{m}{2} \ {}^2 \left(2 r_b \ell \quad \ell^2\right)$ less than 9 o'clock trebuchet and misdirected

Trebuchet model in lab frame



$$v^2_{beam\ frame}(\text{trebuchet}) = \begin{cases} {}^2\left(2r_b\,\ell\right) & \text{half-cocked 6 o'clock} \\ {}^2\left(4r_b\,\ell\right) & \text{fully-cocked 9 o'clock} \end{cases}$$

$$v_{lab\ \ frame}\left(\text{trebuchet}\right) =$$

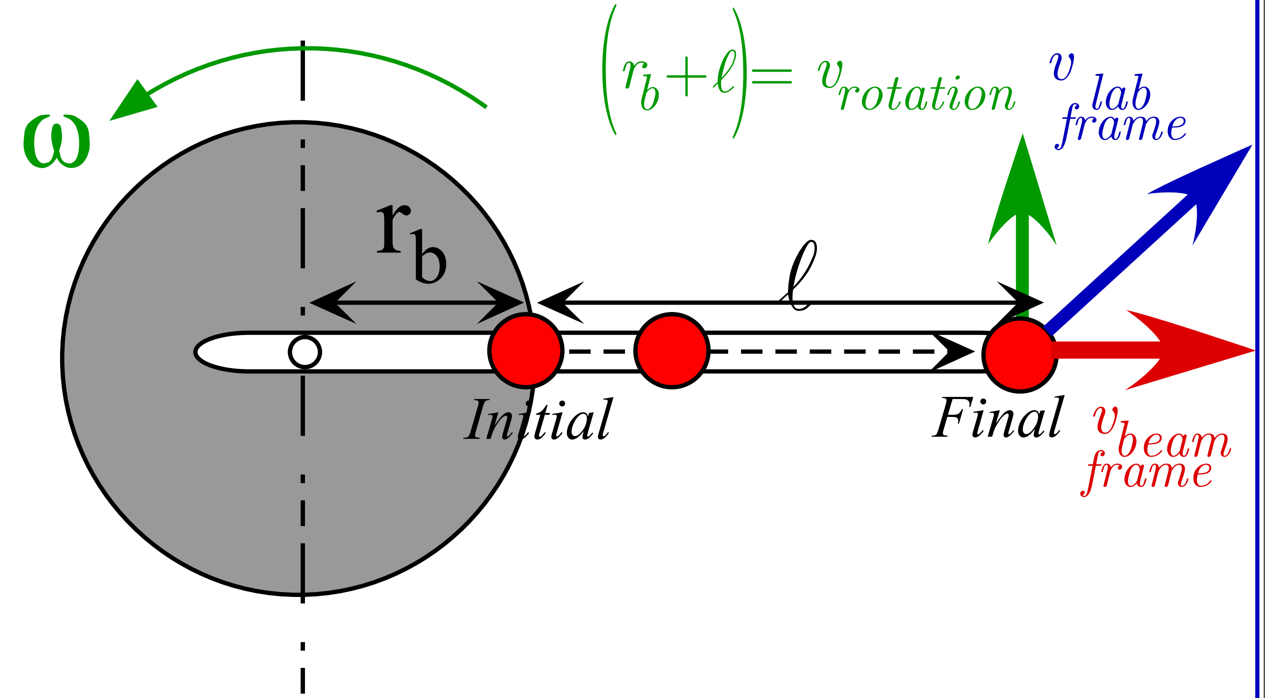
$$\left(r_b + \ell + \sqrt{2\ell r_b}\right) \quad \text{half-cocked 6 o'clock}$$

$$\left(r_b + \ell + 2\sqrt{\ell r_b}\right) \quad \text{fully-cocked 9 o'clock}$$

$$= \begin{matrix} 5.00 \\ 5.82 \end{matrix} \qquad = \begin{matrix} 5.16 \\ 6.00 \end{matrix} \qquad = \begin{matrix} 5.00 \\ 5.82 \end{matrix}$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Flinger model in lab frame



$$v^2_{beam\ frame}(\text{flinger}) = {}^2\ell\left(2r_b + \ell\right)$$

$$v_{lab\ \ frame}\left(\text{flinger}\right) =$$

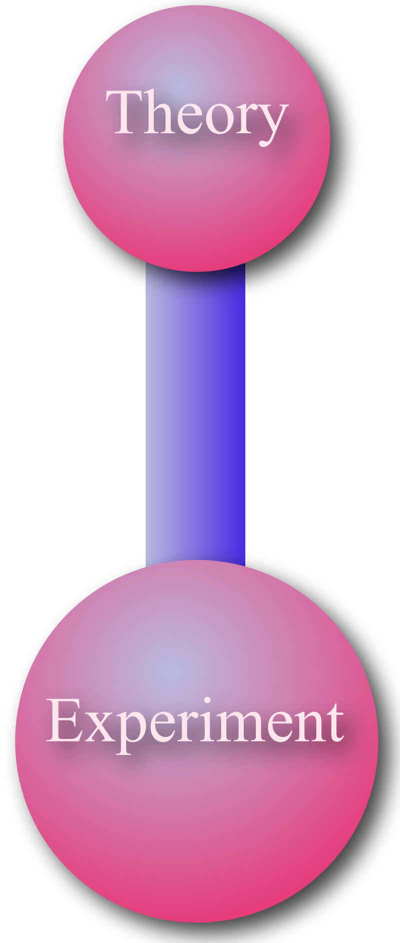
$$= \sqrt{\left(r_b + \ell\right)^2 + \ell\left(2r_b + \ell\right)} = \sqrt{2\left(r_b + \ell\right)^2 \quad r_b^{\ 2}}$$

(compare)

$$= 3.74 \qquad = 3.96 \qquad = 4.12$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty
Newton F=Ma Equations
Cartesian coordinates

- French Approach

Tres elegant
Lagrange Equations
in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} \quad \frac{\partial T}{\partial q^\ell}$$

- German Approach

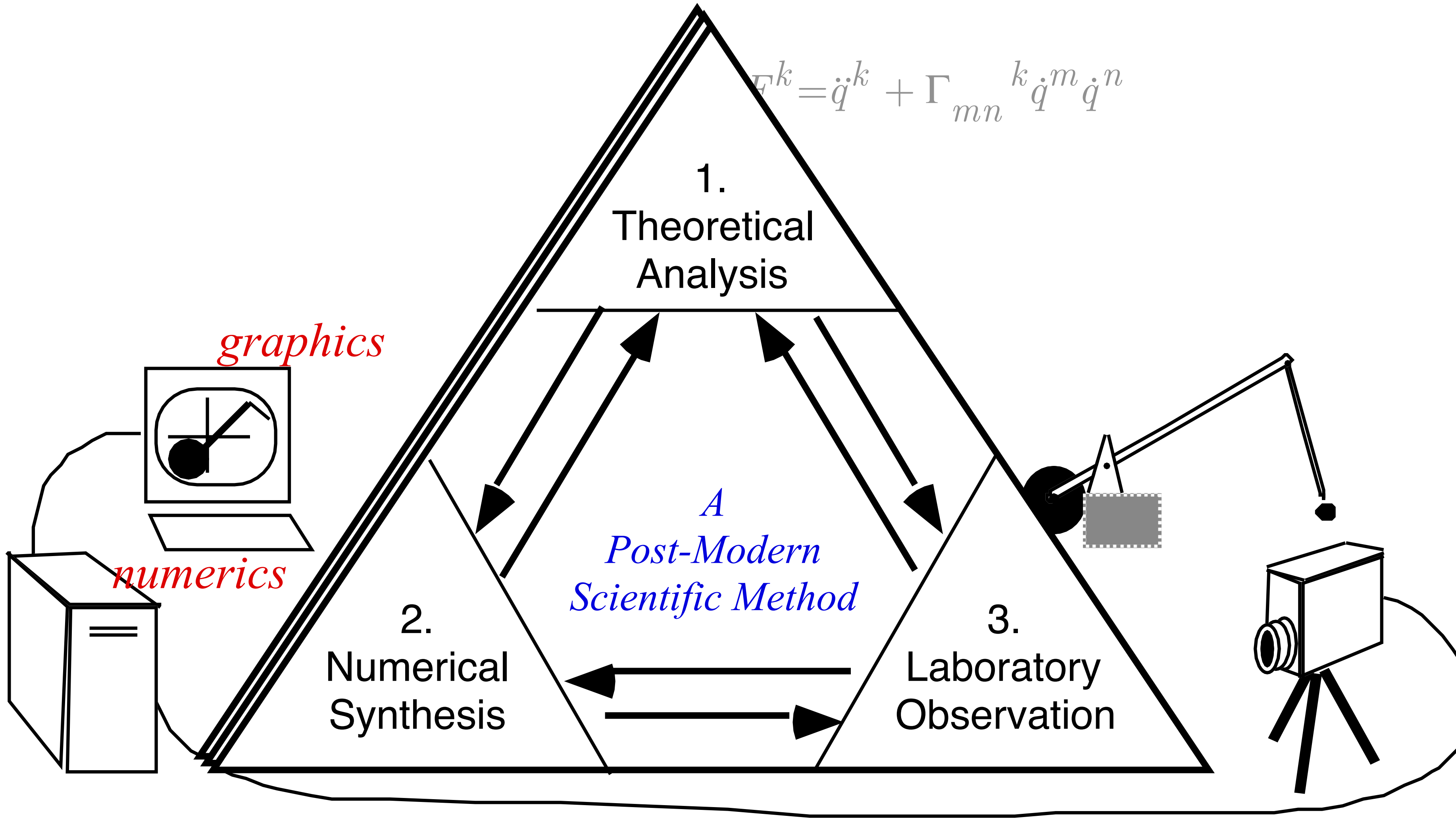
Pride and Precision
Riemann Christoffel Equations
in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

- Anglo-Irish Appproach

Powerfully Creative
Hamilton's Equations
Phase Space $\dot{p}_j = \frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach



All approaches have one thing in common:
The *Art of Approximation*
Physics lives and dies by the art of approximate models and analogs.

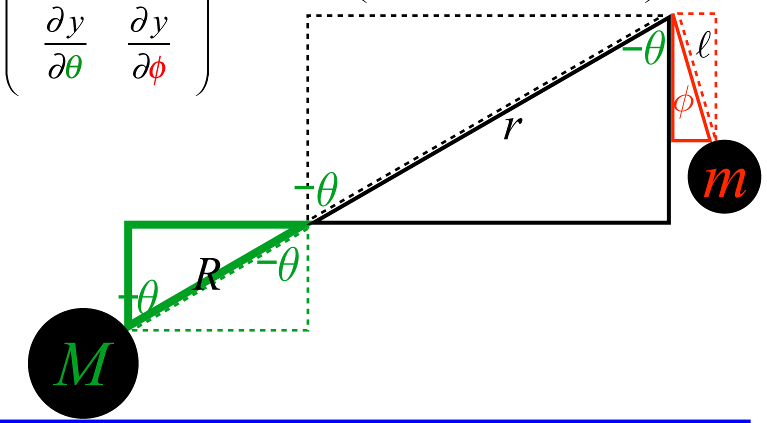
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\,dX \quad = \quad M\ddot{X}\,dX \quad = \quad F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi \quad = \quad M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\,dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\,dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\,dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable* θ *and* ϕ *for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ *and* ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given

($F_X=0$, $F_Y=-Mg$)

($F_x=0$, $F_y=-mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

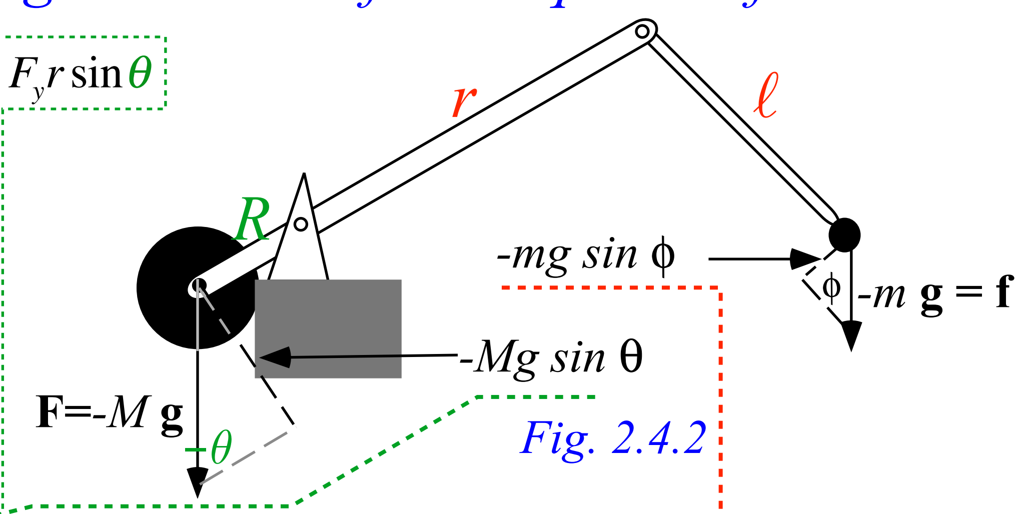


Fig. 2.4.2

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given

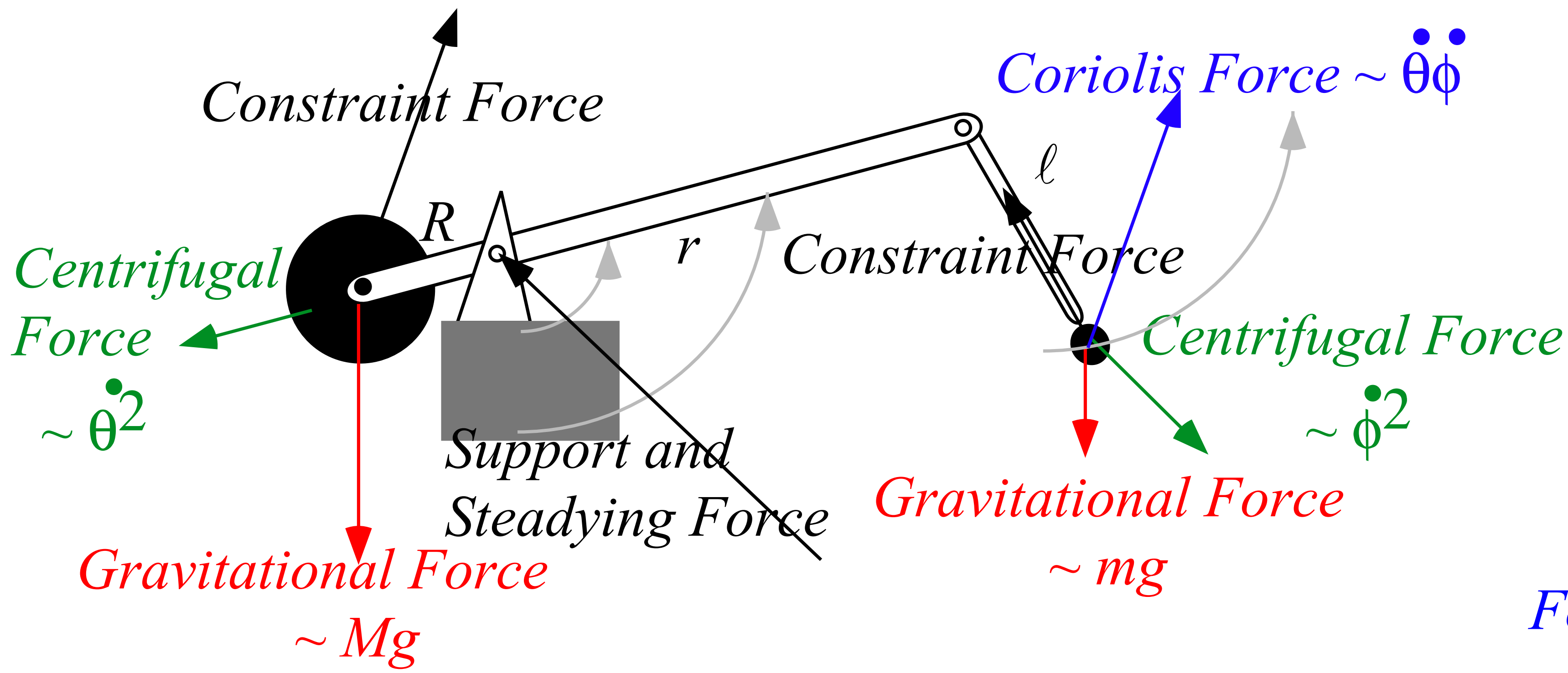
($F_X=0$, $F_Y=-Mg$)

($F_x=0$, $F_y=-mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Forces: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot{\phi}} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(derived on p. 26)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
$L = T - V$

Fig. 2.5.2 (modified)