Lecture 20
Tue. 11.02.2012

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential
Lagrangian for particle-in-($\mathbf{A}$,Φ)-potential
Hamiltonian for particle-in-($\mathbf{A}$,Φ)-potential

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits
Vector theory vs. complex variable theory
Mechanical analog of cyclotron and FBI rule
Cycloid geometry and flying sticks

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

Canonical momentum in ($\boldsymbol{A}$,Φ) potential

Hamiltonian formulation

Hamilton's equations

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^-19 Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \left(\nabla \times \mathbf{A}\right)\right]$$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij} v_i (\nabla\times\mathbf{A})_j$$

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

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\end{aligned}$$

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Vector analysis for particle-in-$(\mathbf{A},\Phi)$-potential

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Applying Levi-Civita ε-identity:

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Converting back to Gibbs's **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

Vector analysis for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's $F=ma$ equation for a mass m of charge e.

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Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero. $\partial_k v^j\equiv 0$ *iff* : $\nabla\mathbf{v}=\frac{\partial\mathbf{v}}{\partial\mathbf{r}}=\mathbf{0}$

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

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Summary of Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k}v_j = (\partial_k A_j)v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j\frac{\partial A_k}{\partial x_j} = (v_j\partial_j A_k)$$

$$\left[\nabla(\mathbf{A}\bullet\mathbf{v})\right]_k = \left[(\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}\right]_k$$

$$\partial_k\left(A_b v_b\right) = \left(\partial_k v_b\right)A_b - \left(\partial_k v_a\right)A_a$$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential*

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Lagrangian for particle-in-(A,Φ)-potential

So-called ***pondermotive*** form for Newton's $F=ma$ equation for a mass m of charge e.

electronic charge:
e =-1.602176·10^{-19}*Coulombs*

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Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

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Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*

Hamiltonian formulation

Hamilton's equations

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Canonical momentum is defined by L's $\mathbf{v}$-derivative

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<u>if</u> magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

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Otherwise vector potential term $-\mathbf{v}\cdot e\mathbf{A}$ leads to an extraordinary *canonical momentum:* $\mathbf{p} = m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.

Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* $\mathbf{p}$ as follows: $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

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- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- → *Hamiltonian formulation*
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Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right) \right)$$

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But Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

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The Hamiltonian function of the Legendre-Poincare form is the following.

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right) \right)$$

$$H = \frac{1}{2} m\mathbf{v} \bullet \mathbf{v} + e\Phi(\mathbf{r},t)$$

(Only correct numerically!)

Vector potential **A** seems to cancel out completely, leaving a familiar $H=T+V$ with only scalar $V=e\Phi$.

But Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

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Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- *Hamiltonian formulation*
- → *Hamilton's equations*

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...and now

we come back

full circle...

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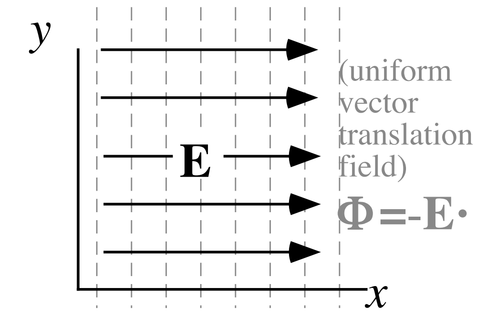
$$\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)=\mathbf{v}\bullet\left(\nabla\mathbf{A}\right)-\left(\mathbf{v}\bullet\nabla\right)\mathbf{A}$$ *for particle mechanics*

Crossed E and B field mechanics

A constant **E** field has a scalar potential field Φ with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E}\bullet\mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = \nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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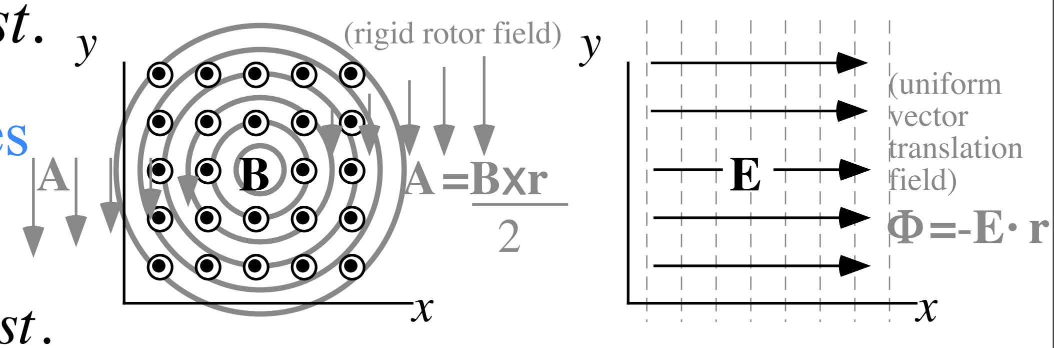

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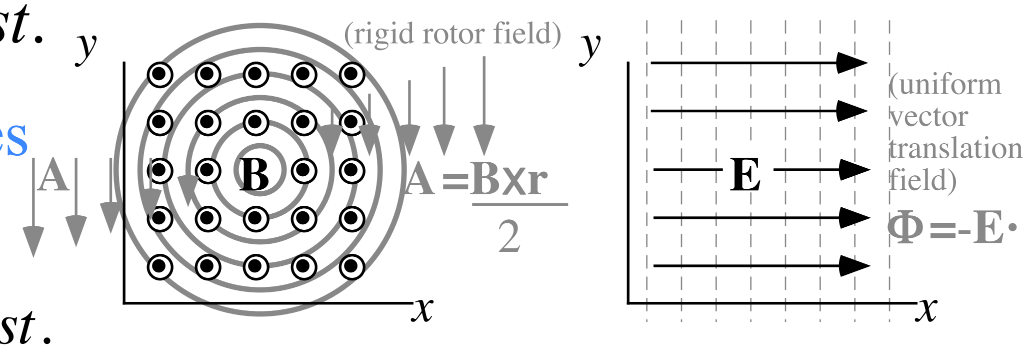

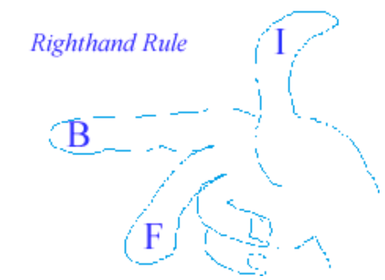

F=q**v**×**B**=**I**×**B**

Newtonian electromagnetic equations of motion:

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B}$$

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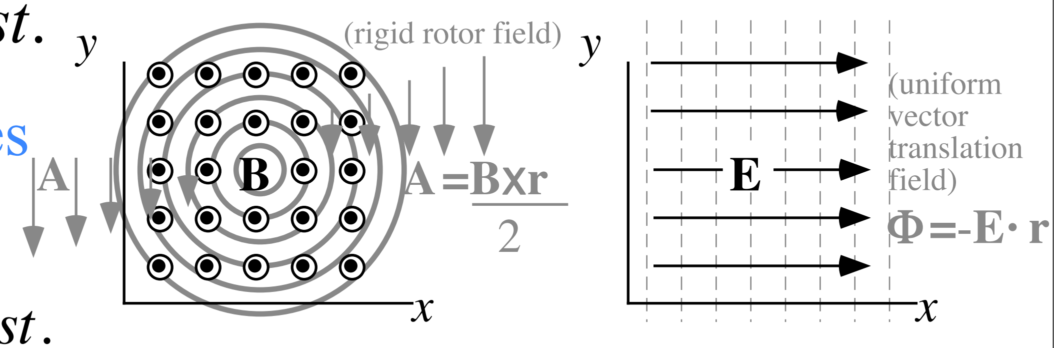

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$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \tfrac{e}{m}E_x \qquad \varepsilon_y = \tfrac{e}{m}E_y \qquad B = \tfrac{e}{m}B_z$$

Shorthand Labeling

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

→ *Vector theory vs. complex variable theory*

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

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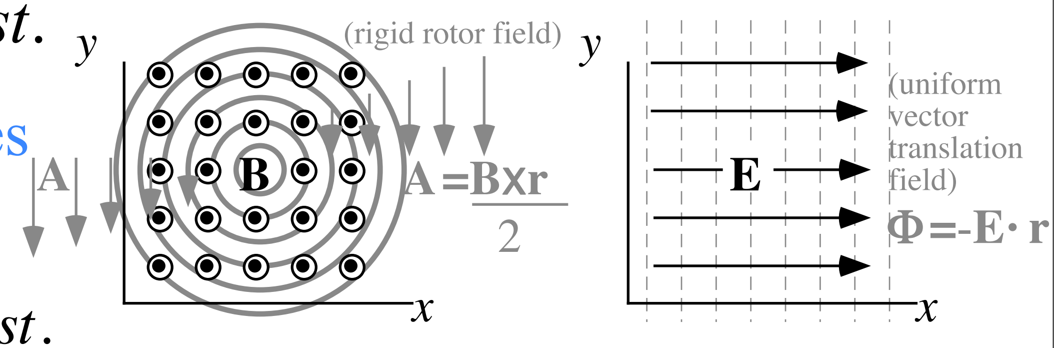
Fig. 2.4.1.



A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

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Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\boxed{\begin{array}{l} \dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling} \end{array}}$$

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

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Crossed E and B field mechanics

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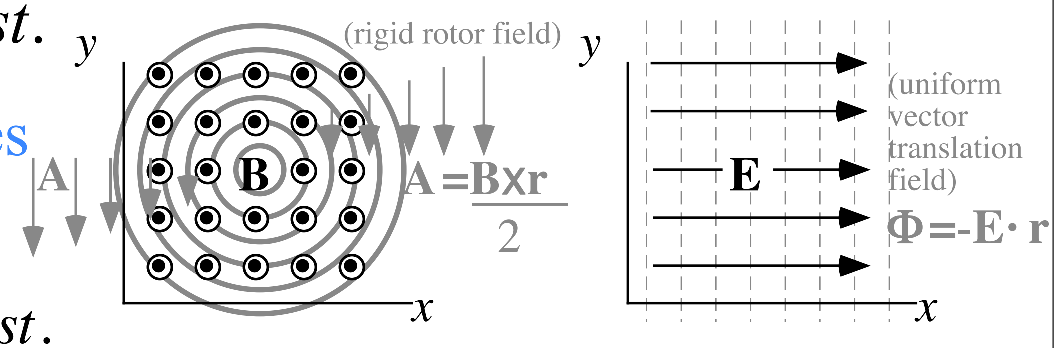

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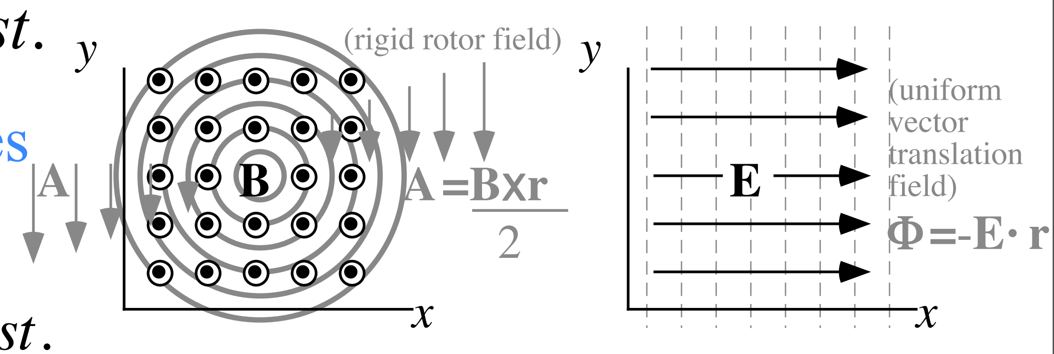

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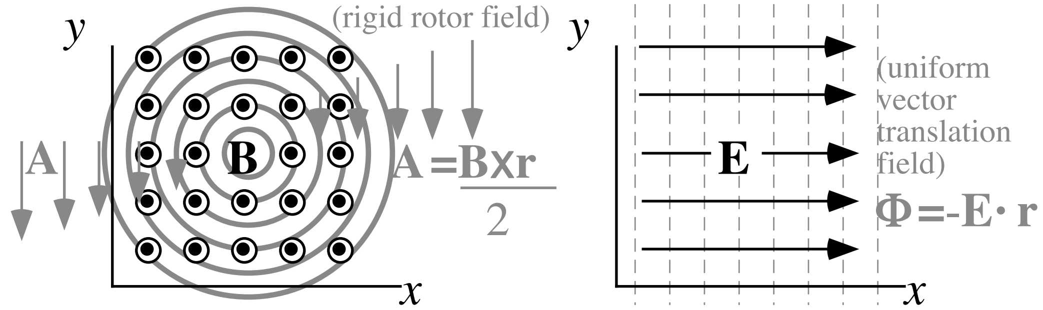

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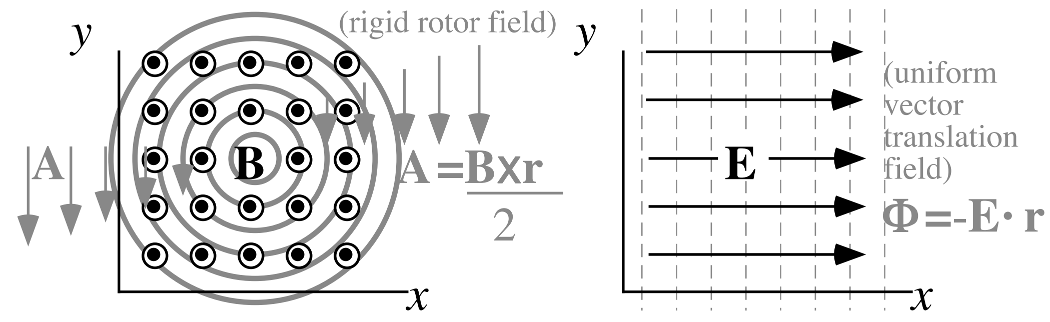

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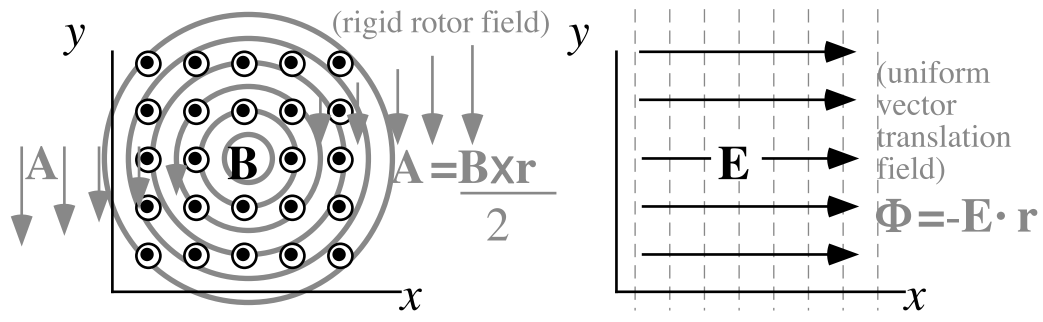

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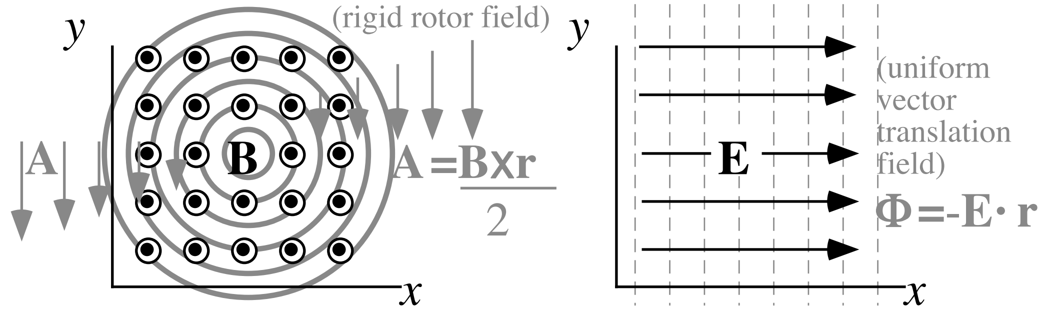

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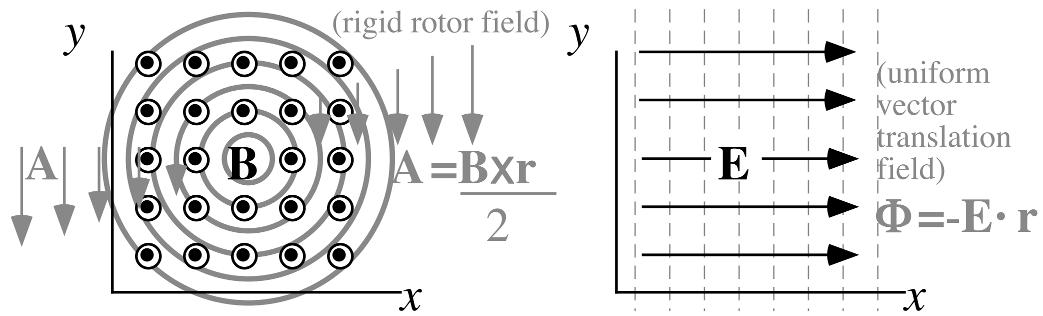

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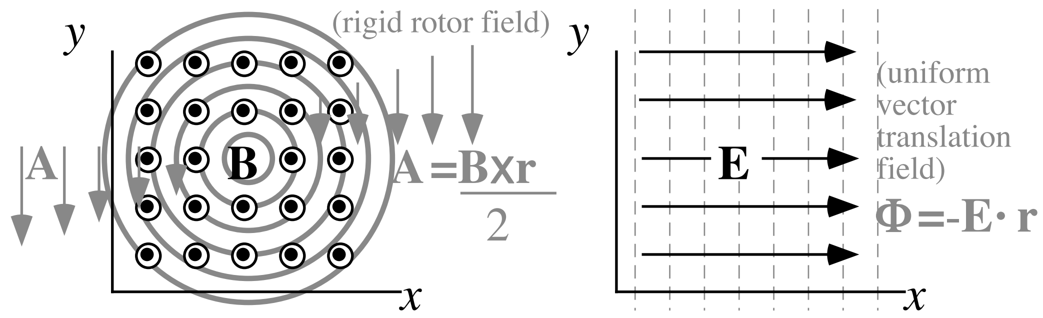

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

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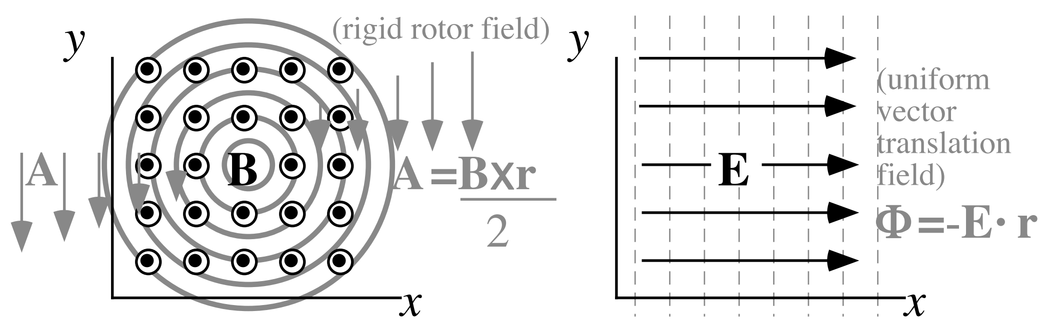

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A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V}=(const.)V}$

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$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

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Shorthand Labeling

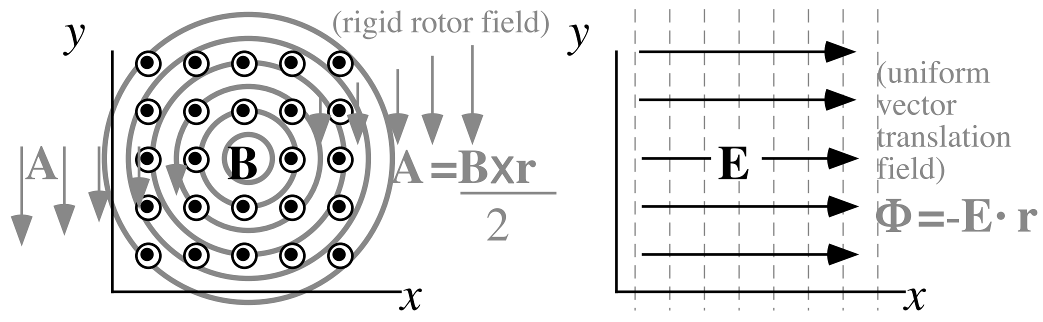

Fig. 2.4.1.

Complex variable ***velocity:*** $v=v_x+iv_y$ *and* ***electric field:*** $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_{\mathbf{x}} \to 1 \quad \text{and : } \hat{\mathbf{e}}_{\mathbf{y}} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \dot{v}(t) = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0)+\beta) - \beta = e^{-iB\cdot t}\left(v(0)+i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}$$

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}\left(v(0)+i\frac{\varepsilon}{B}\right) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - \left(\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2}\right)$$

$$x(t) + iy(t) = e^{-iB\cdot t}\left(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}\right) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t) \qquad \text{where}: \boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$$

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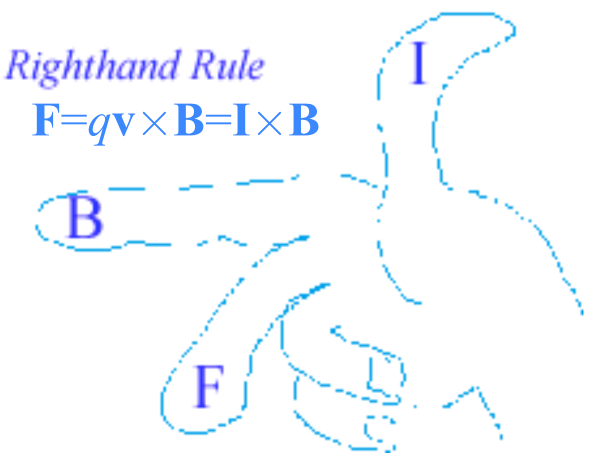
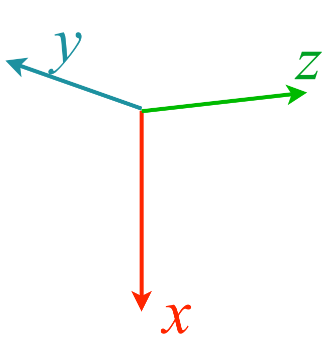
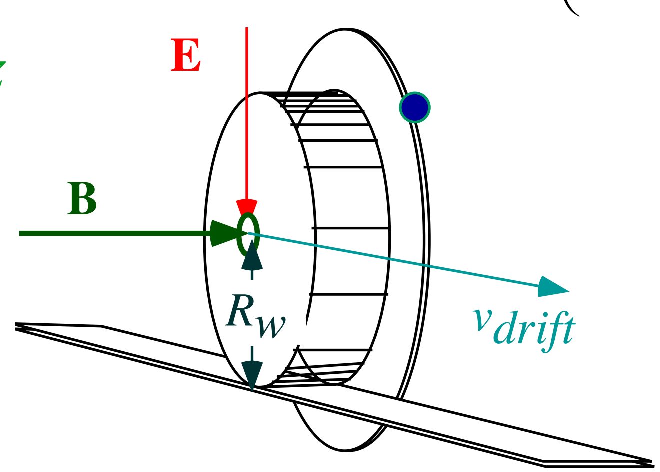
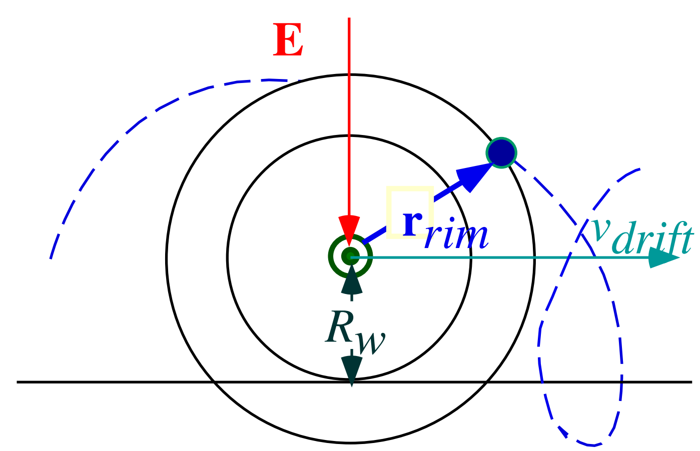
Righthand Rule

F=q**v**×**B**=**I**×**B**









$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos Bt & \sin Bt \\ -\sin Bt & \cos Bt \end{pmatrix} \begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix} + \begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

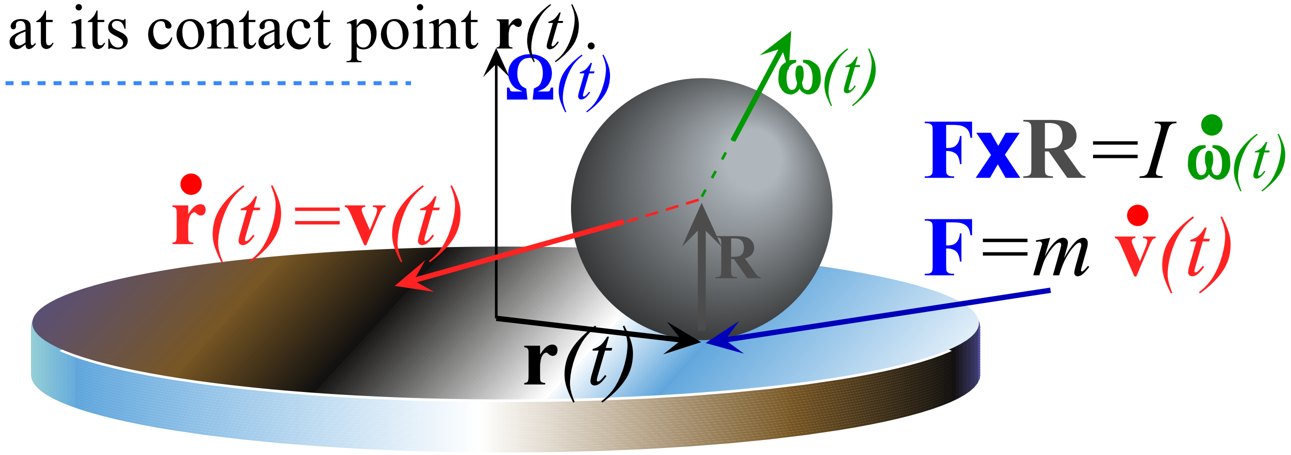
Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals

table surface velocity $\mathbf{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Turntable turning at constant angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$.

Mechanical analog of cyclotron and FBI rule

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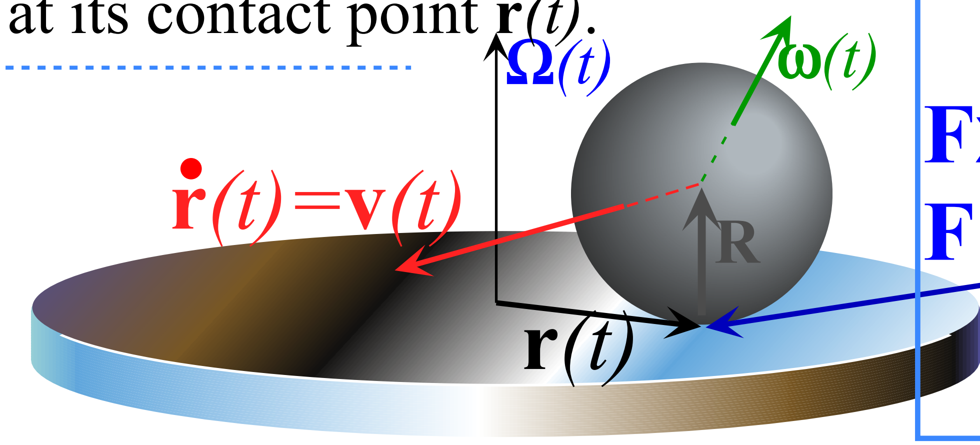

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Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\omega}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\omega}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

*Torque-and-*F=ma
equations of motion:

$$I\dot{\omega}(t) = \mathbf{F}(t)\times\mathbf{R}$$
$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
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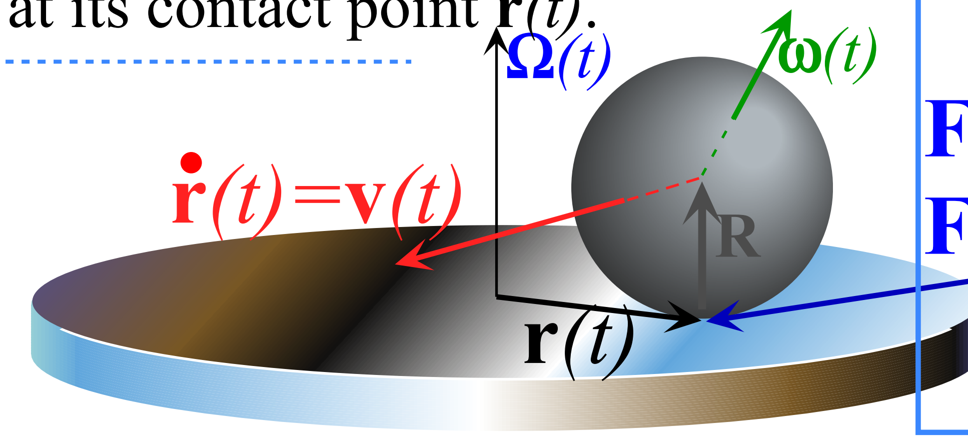
Rolling Constraint



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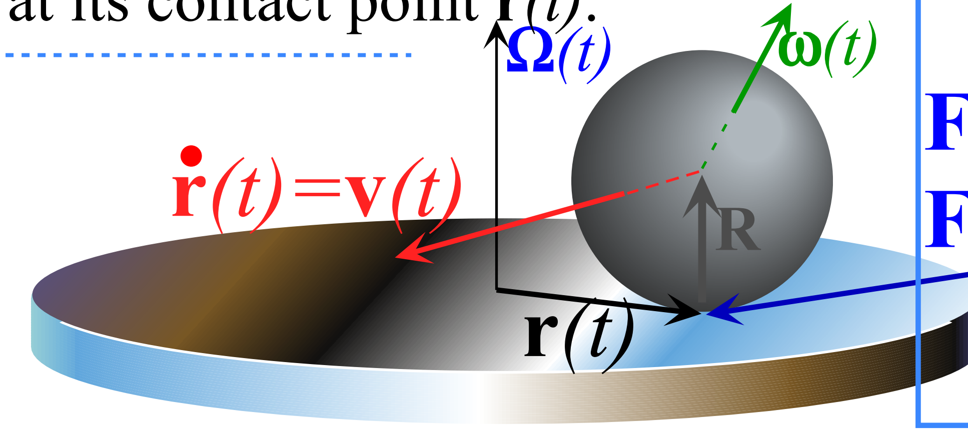

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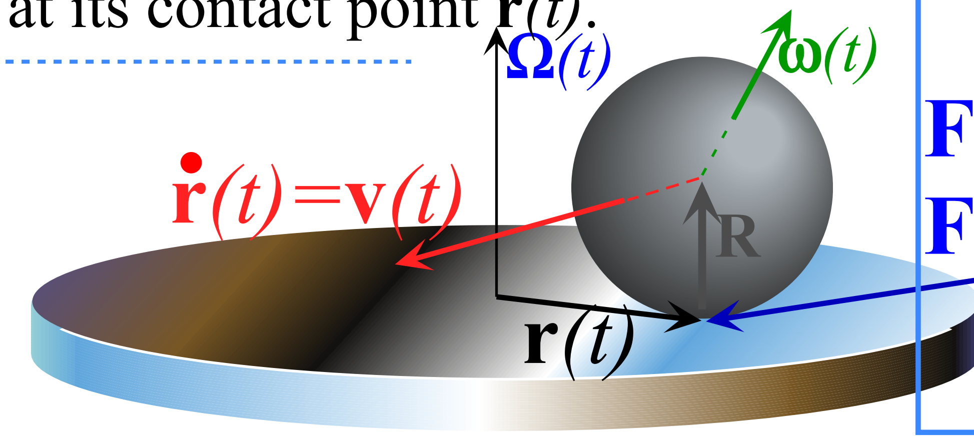

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$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad =\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

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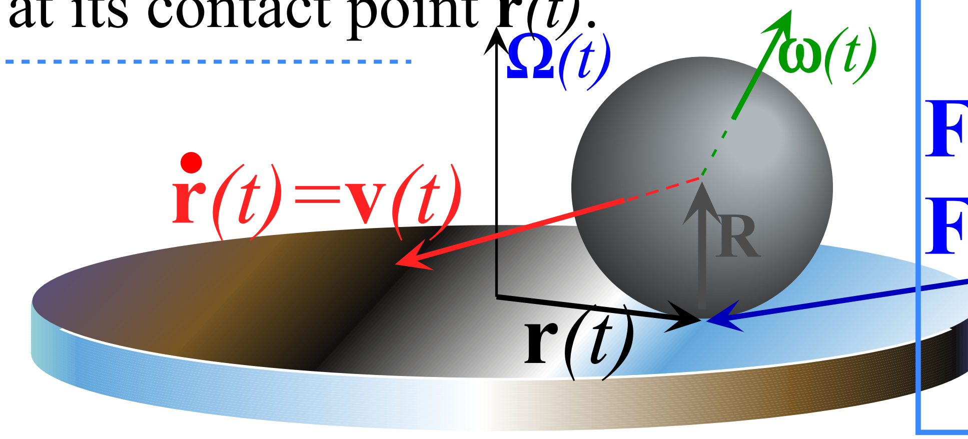

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$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\;\dot{\boldsymbol{\omega}}(t)\;\times\hat{\mathbf{z}}R$$

use: $$\dot{\boldsymbol{\omega}}(t)=\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$$

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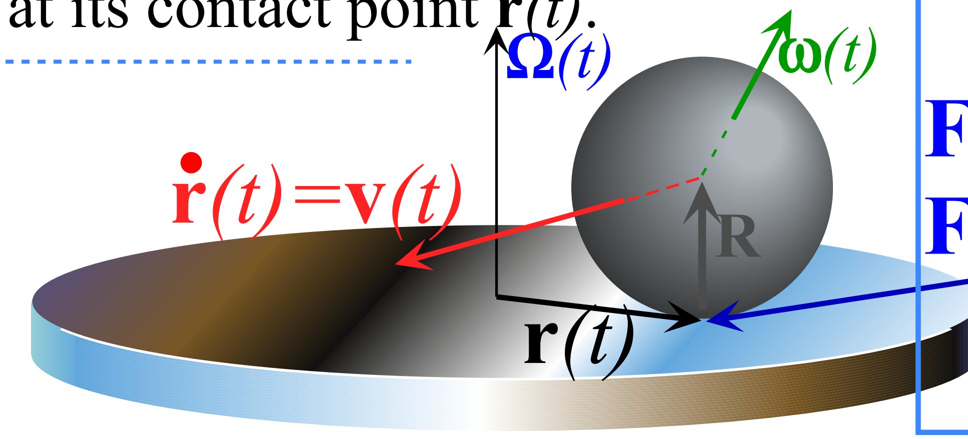

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Do time-derivative

*Torque-and-*F=ma *equations of motion:*

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$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$

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$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t) \times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t) = \dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$= \boldsymbol{\Omega}\times\mathbf{v}(t) + \frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A} = (\mathbf{A}\bullet\mathbf{B})\mathbf{C} - (\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

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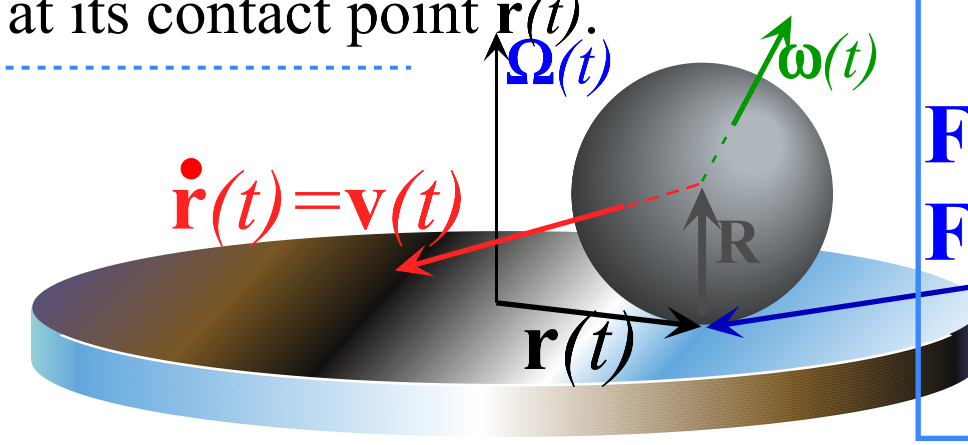

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Torque-and-F=ma equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$
$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\;\dot{\boldsymbol{\omega}}(t)\;\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

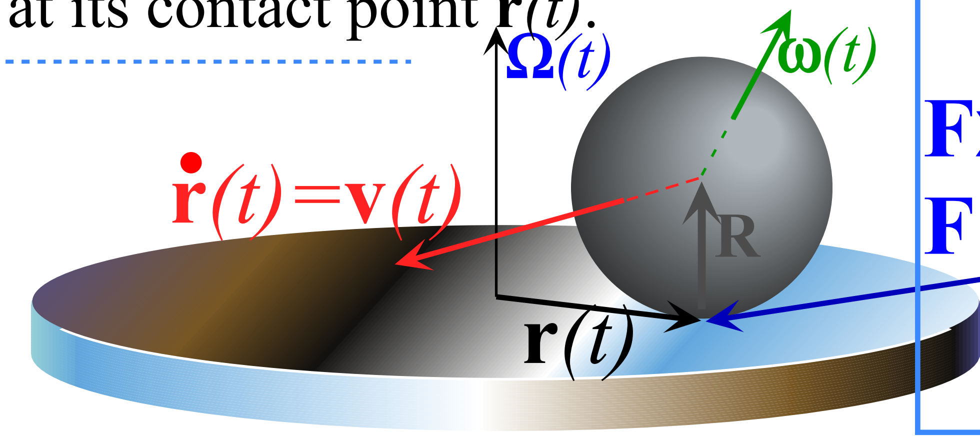
use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

$$=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

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$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t) = \boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R} \quad = \boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

*Torque-and-*F=ma *equations of motion:*

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$$= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

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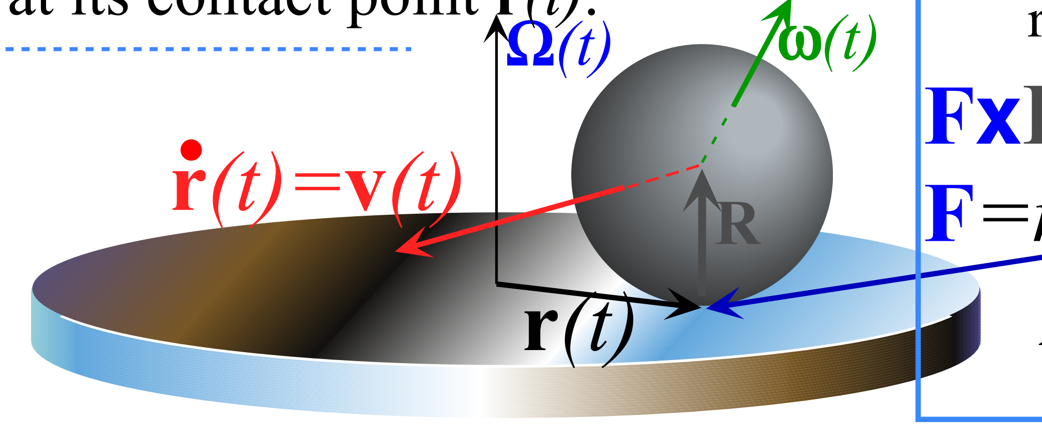

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use: $\dot{\boldsymbol{\omega}}(t) = \dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

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$$= \boldsymbol{\Omega}\times\mathbf{v}(t) + \quad 0 \quad - \frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\mathbf{v}(t)$$

or : $\dot{\mathbf{v}}(t) = \dfrac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}\times\mathbf{v}(t)$

Mechanical analog of cyclotron and FBI rule

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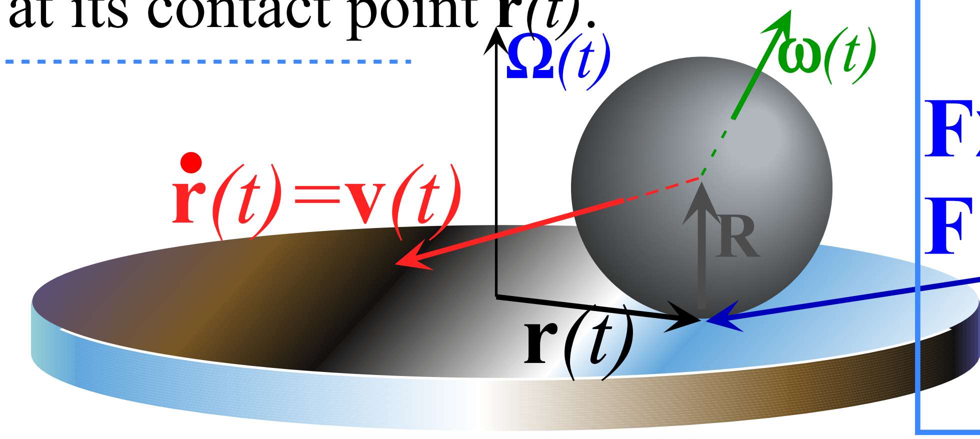

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$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

$\mathbf{F}=\mathbf{B}\times\mathbf{v}$ *mechanical analog:*

$$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\dfrac{mR^2}{I}}\times\mathbf{v}(t)$$

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

→ *Cycloid geometry and flying sticks*

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

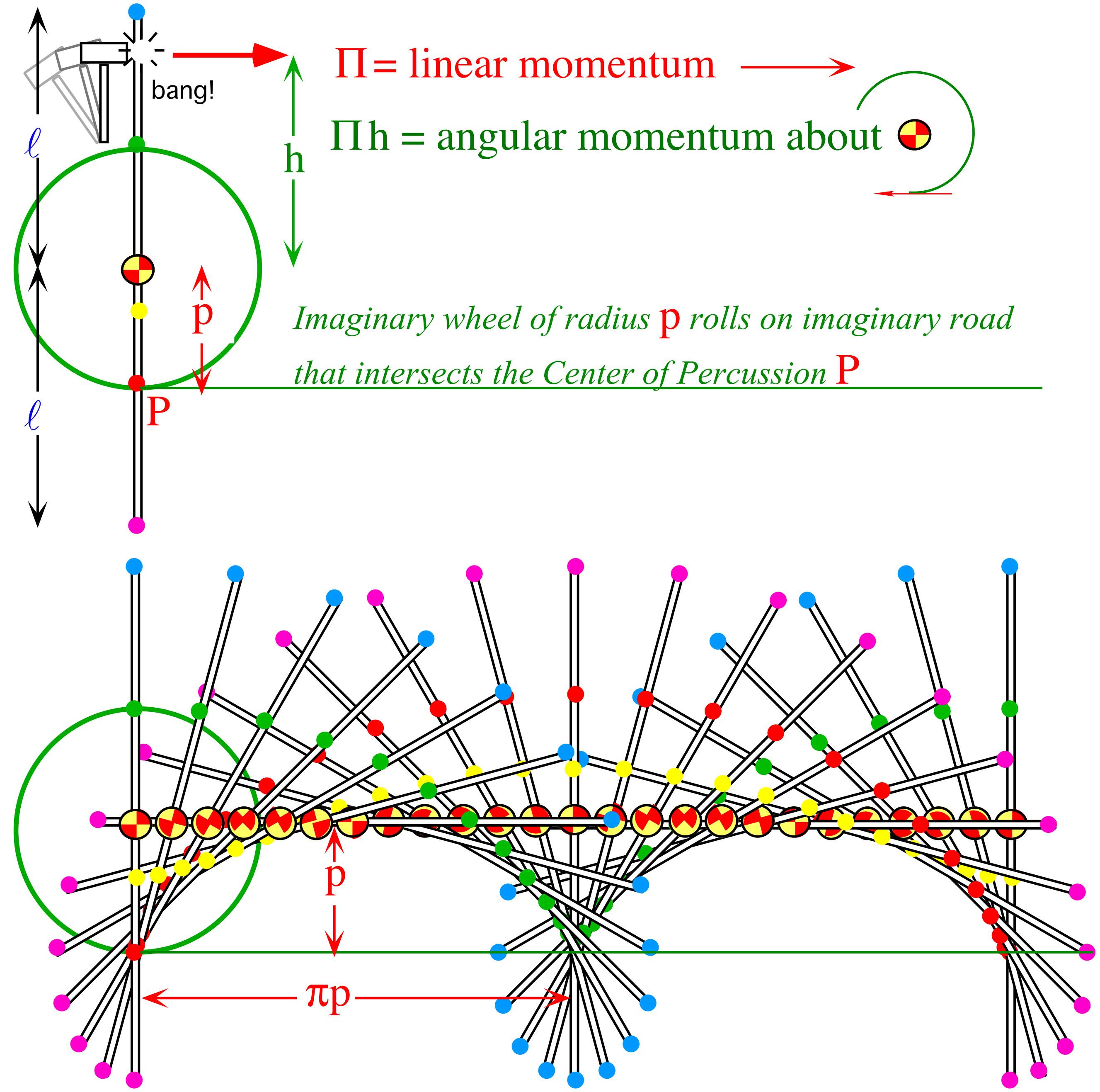
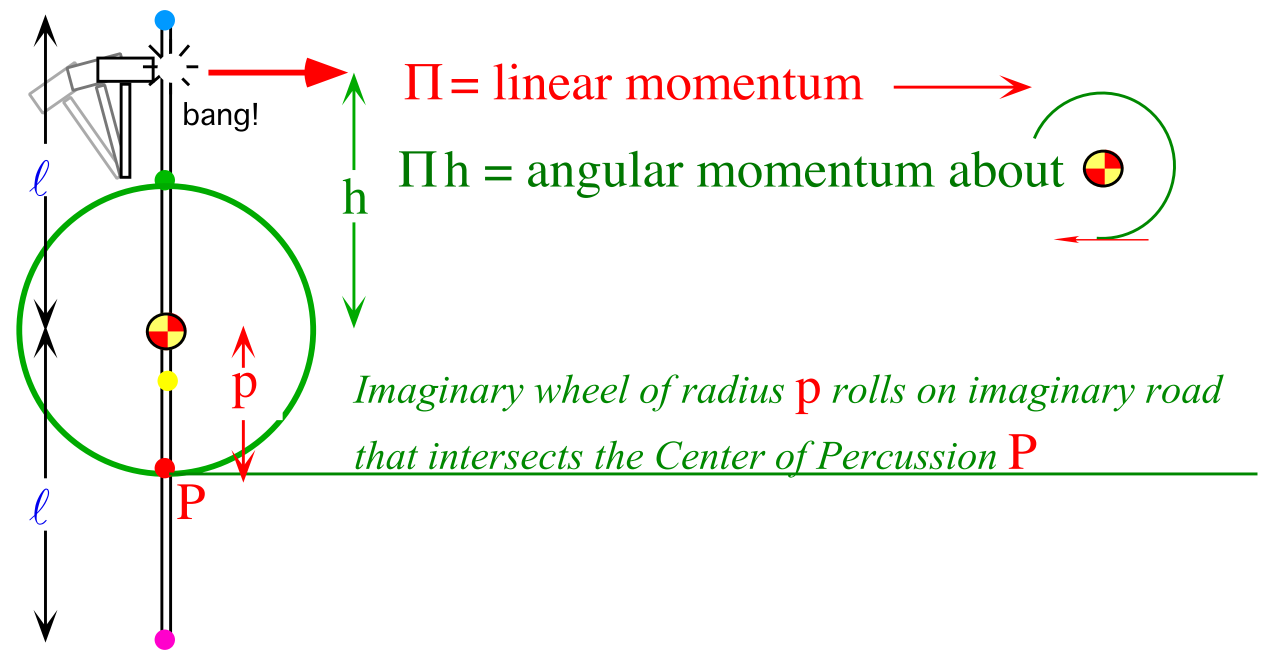
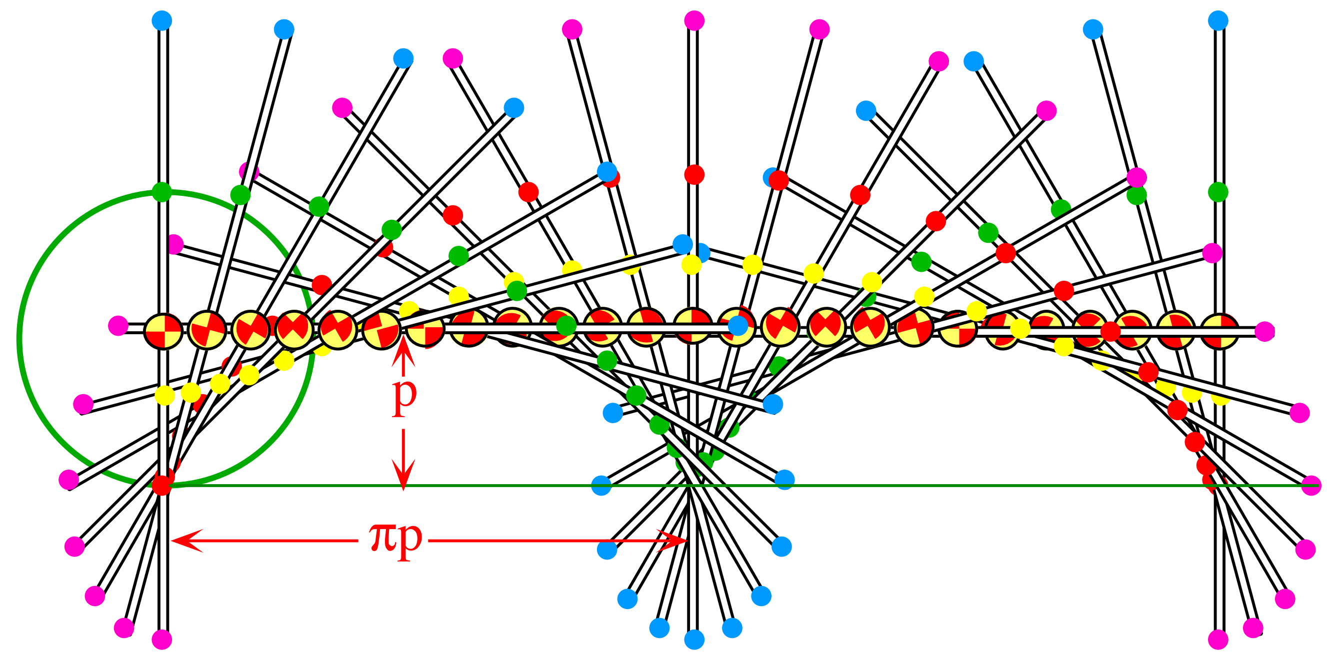


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center you give it some linear momentum Π and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.





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$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2) \text{ for stick})$$
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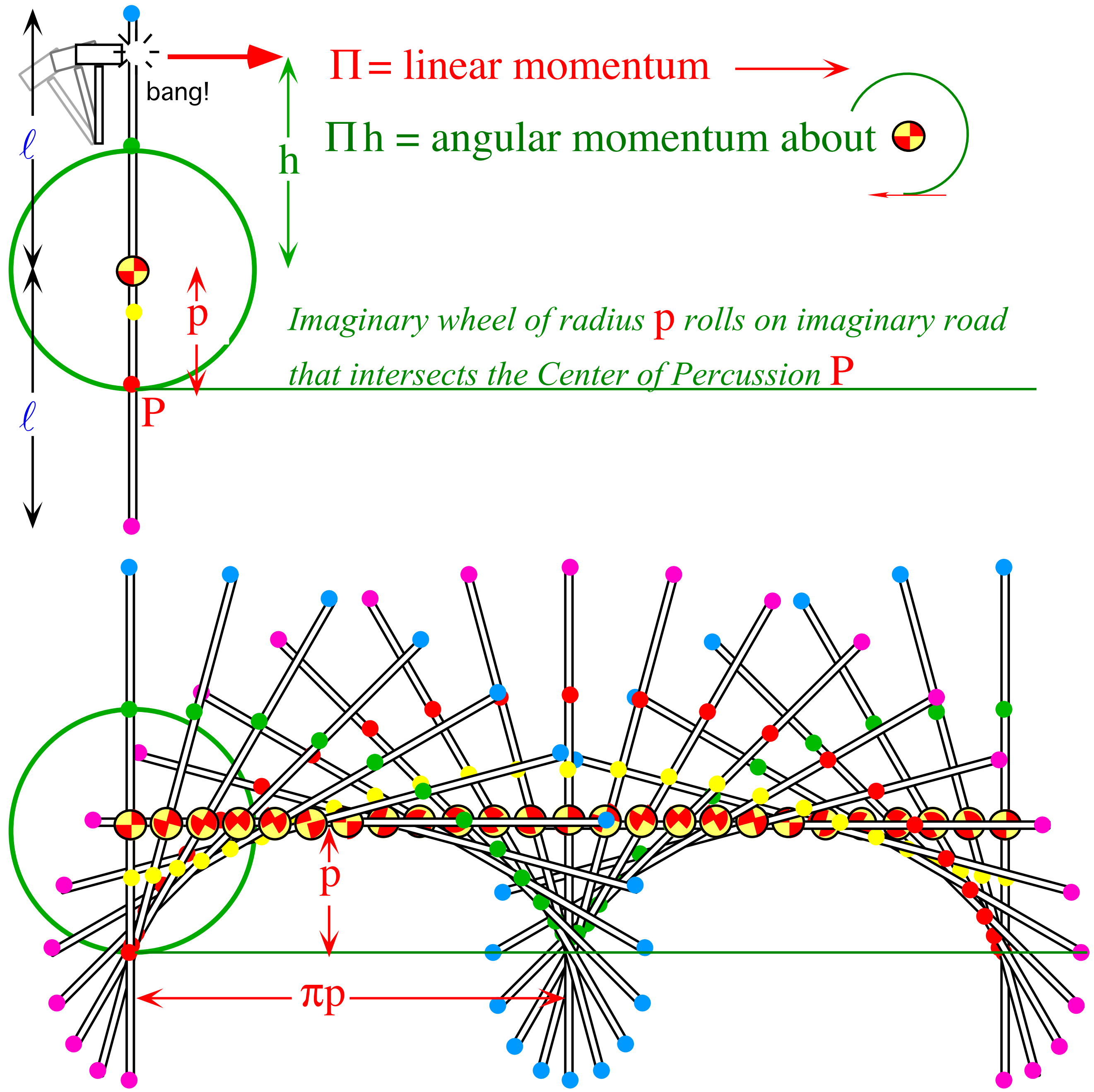

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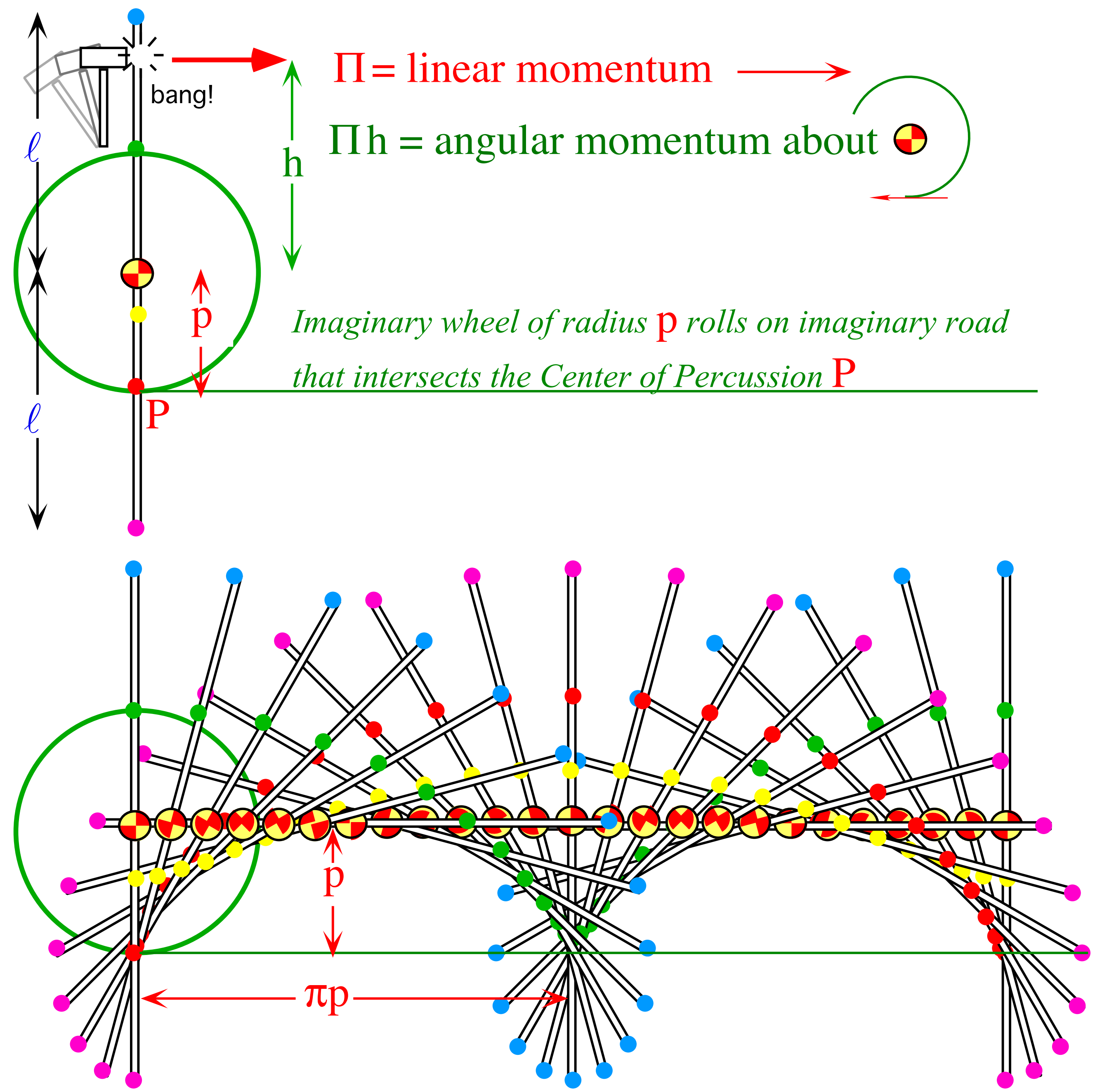
One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.



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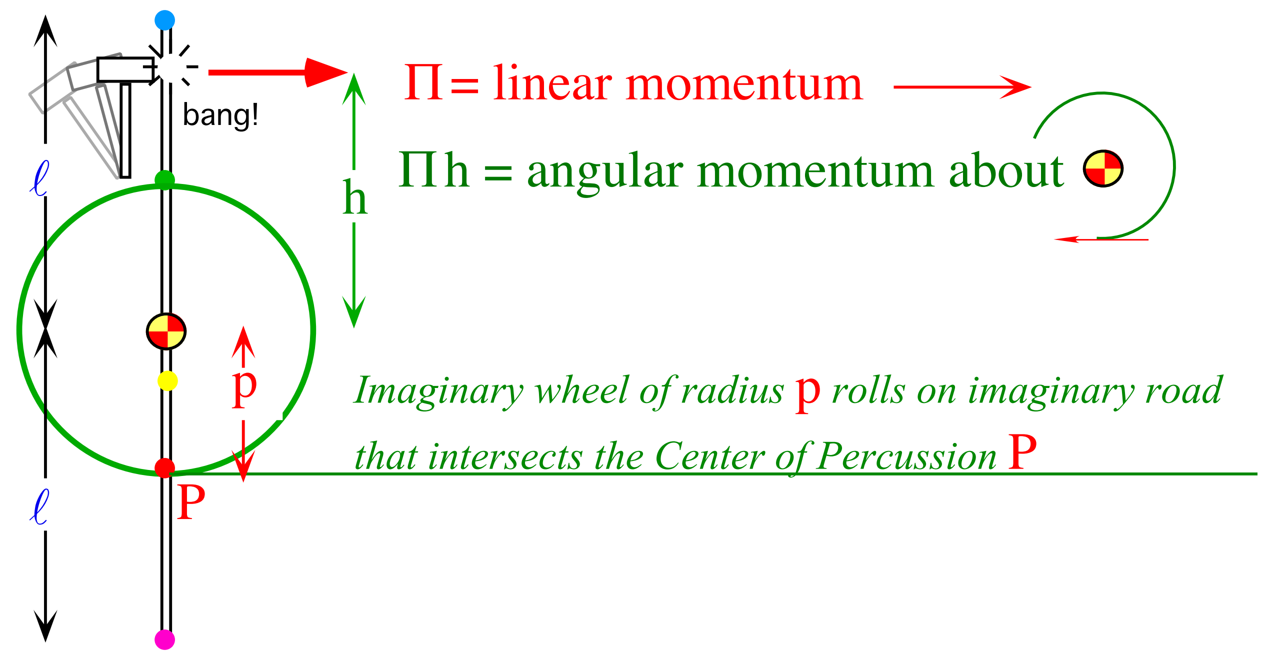

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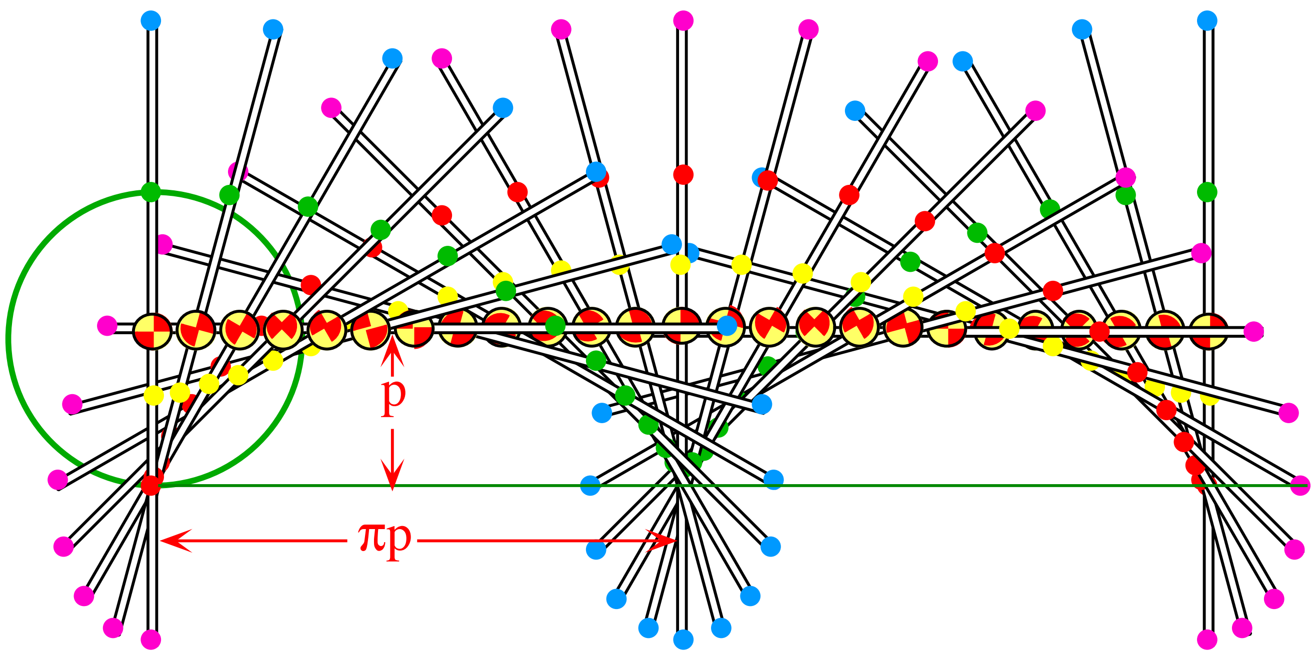

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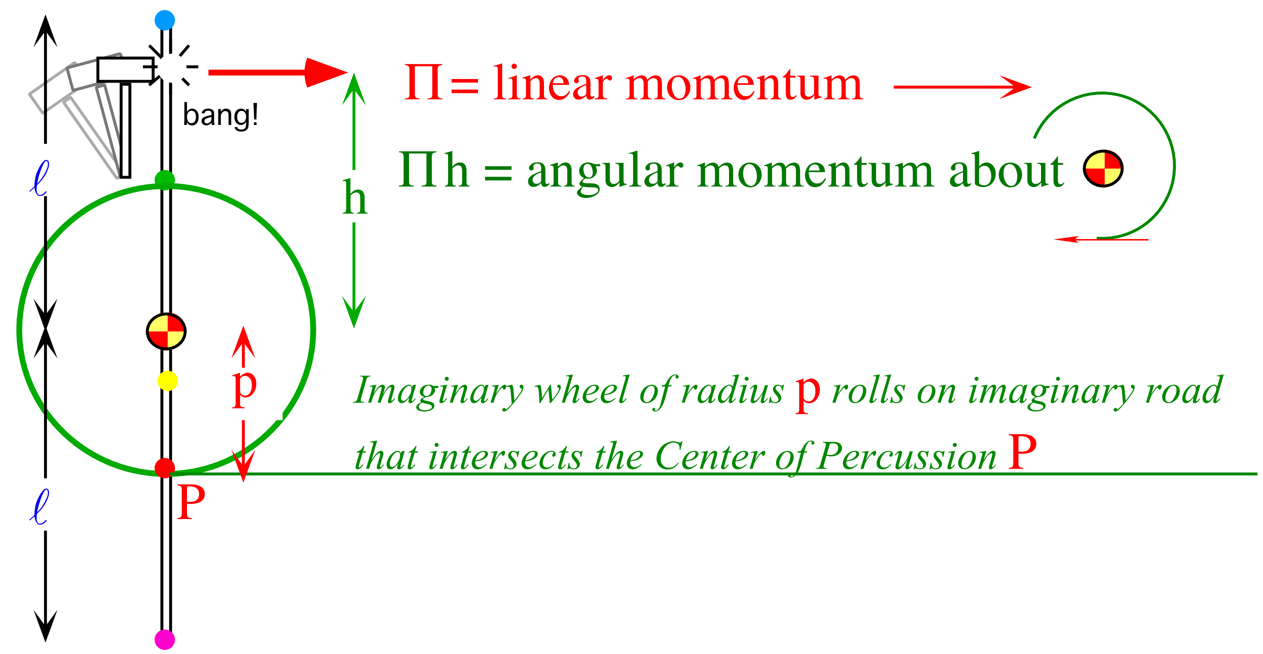

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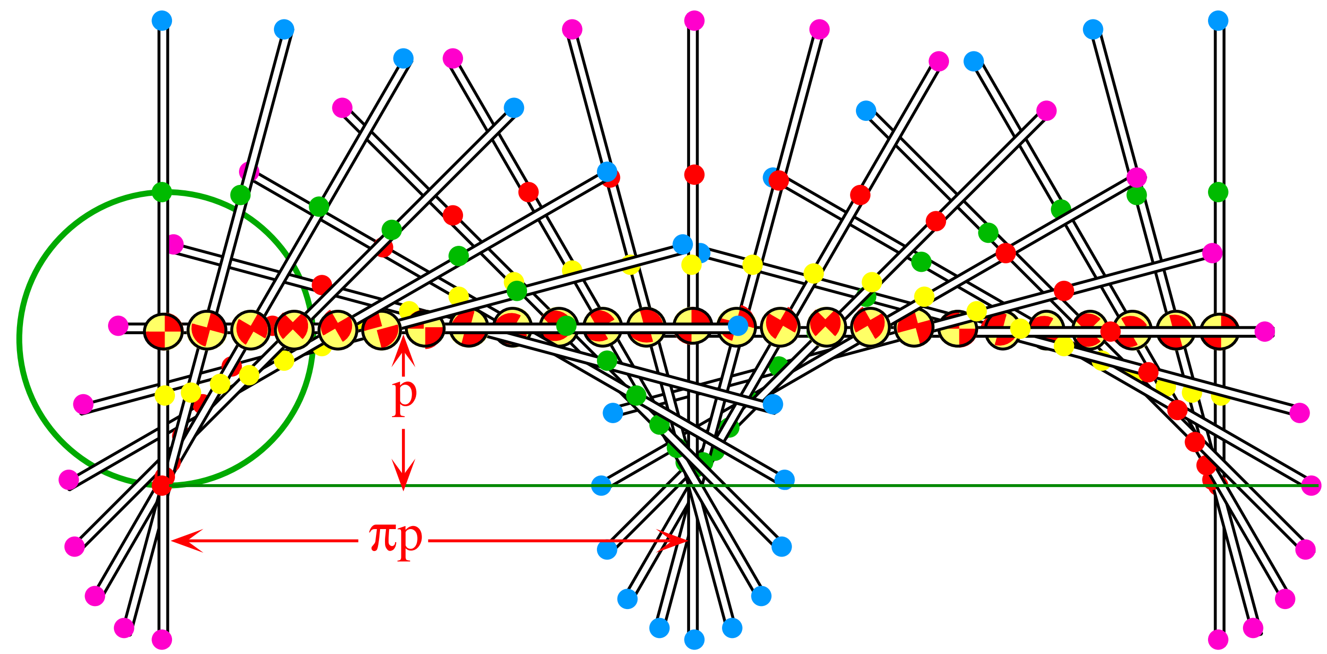
P follows a normal cycloid made by a circle of radius $p=I/(Mh)$ rolling on an imaginary road thru point P in direction of Π.



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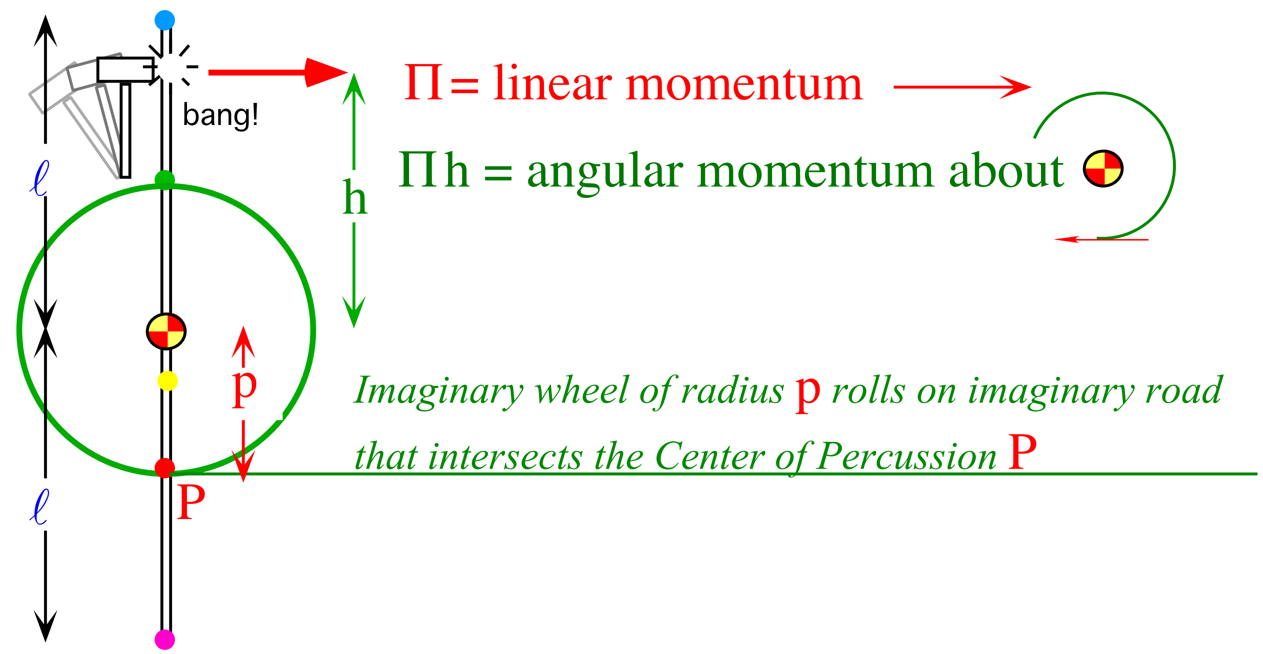

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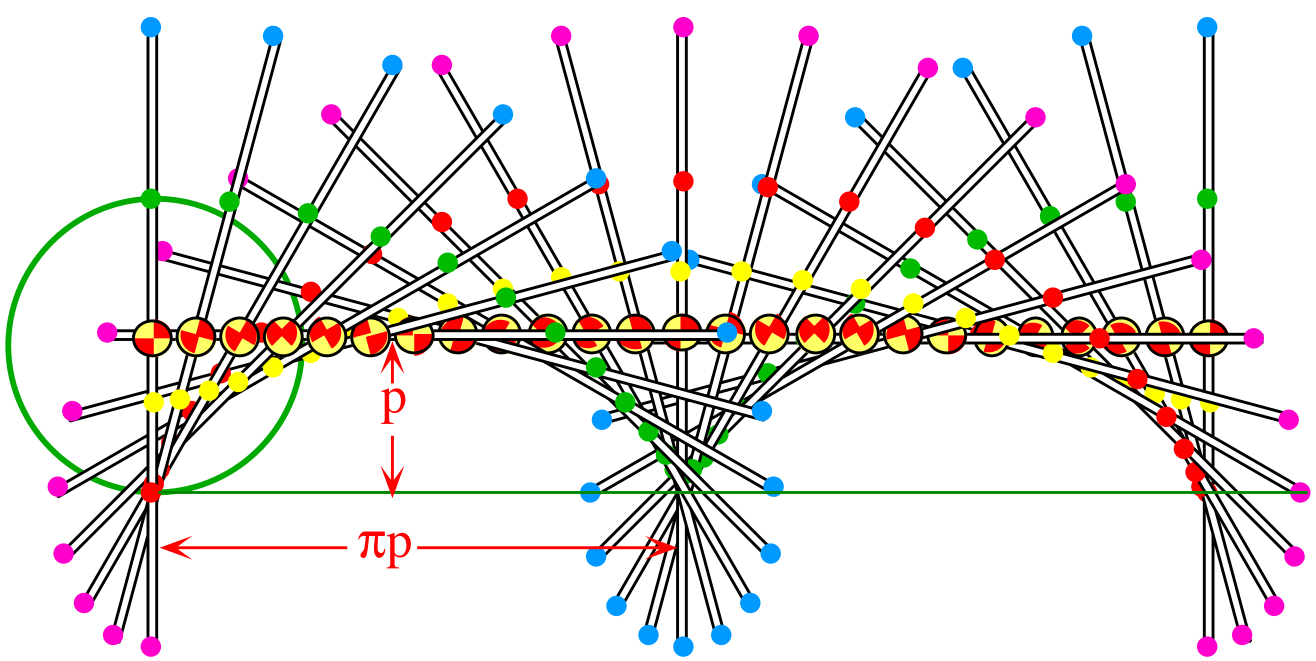

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of radius $p = I/(Mh)$ rolling on an imaginary road
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The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

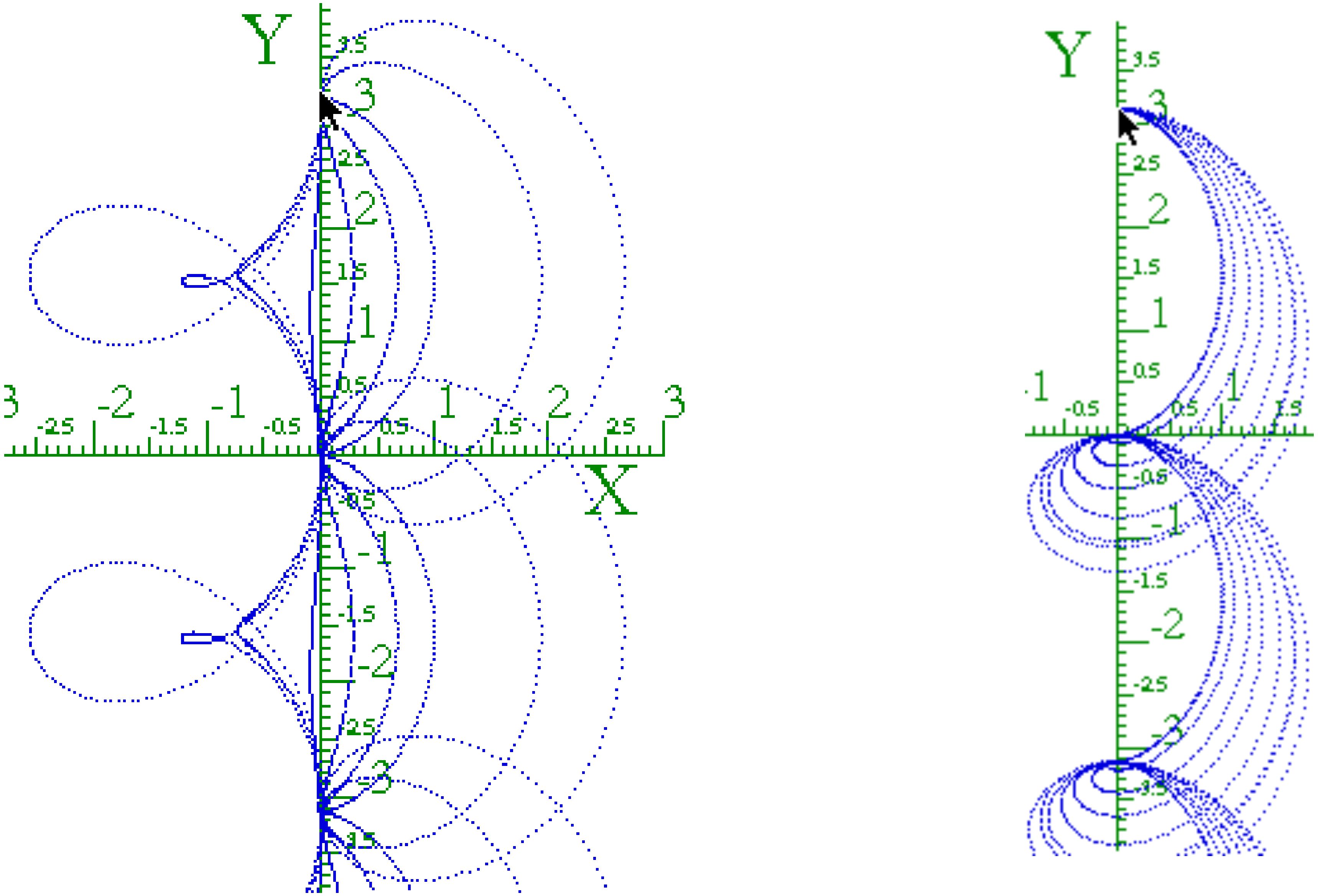
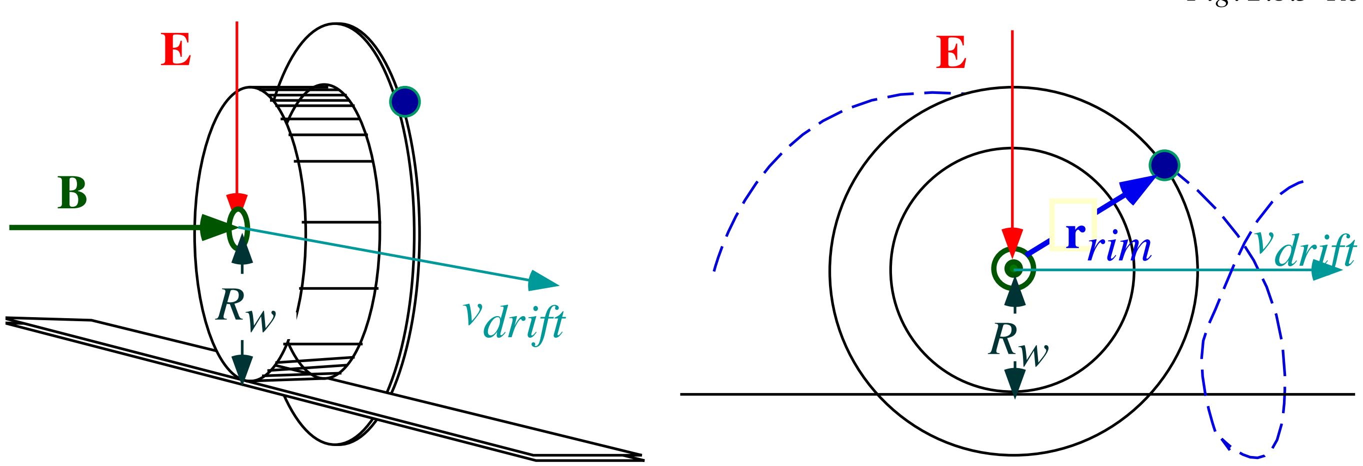


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (*E=1/2, B=1*)

Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits



Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0\ , \quad \text{with: } \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0\ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

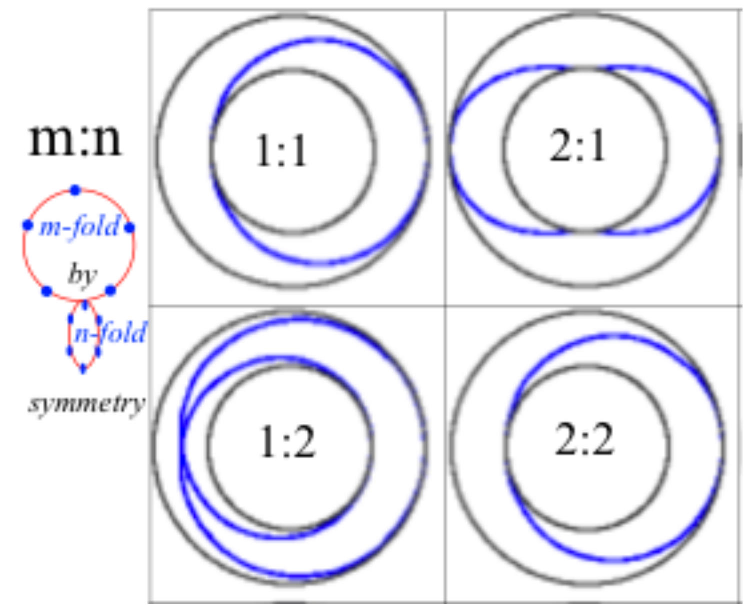
An effective "spring constant" at the stable point giving approximate frequency of oscillation.

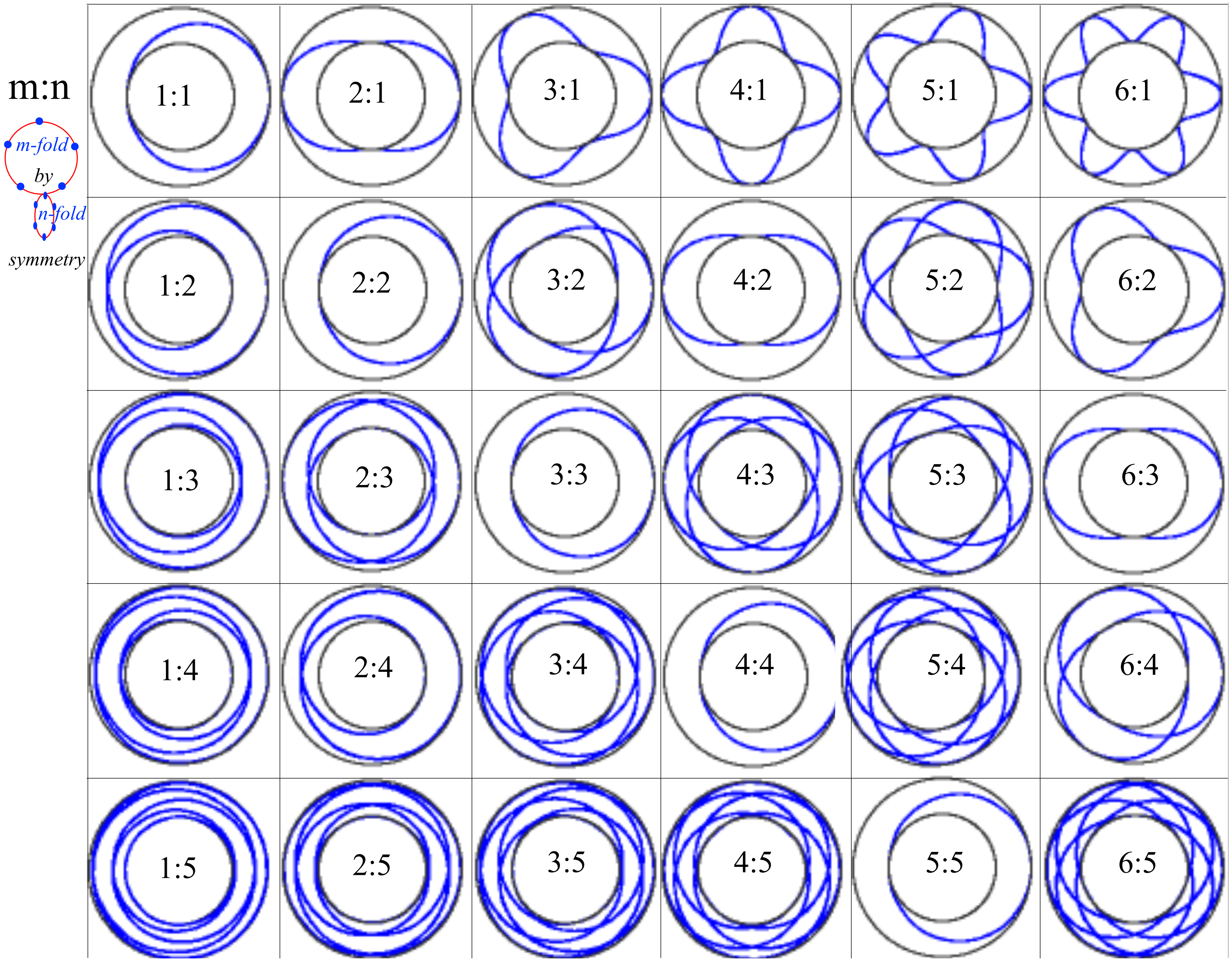
$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_{\phi}} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic}$$

Some generic shapes resulting from various ratios $n\rho : n\phi$




m:n
m-fold
by
n-fold
symmetry
1:1
2:1
3:1
4:1
5:1
6:1
1:2
2:2
3:2
4:2
5:2
6:2
1:3
2:3
3:3
4:3
5:3
6:3
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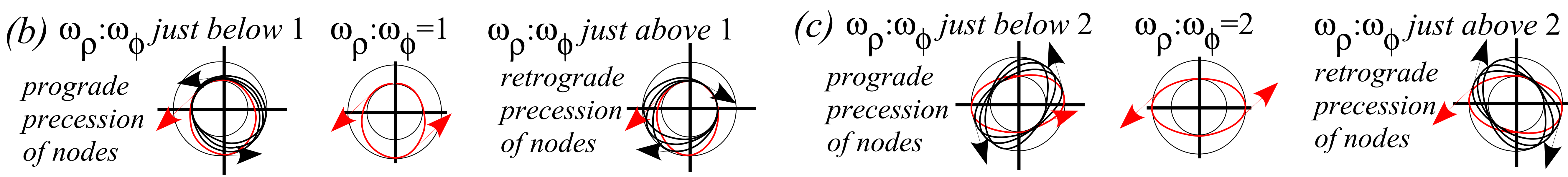
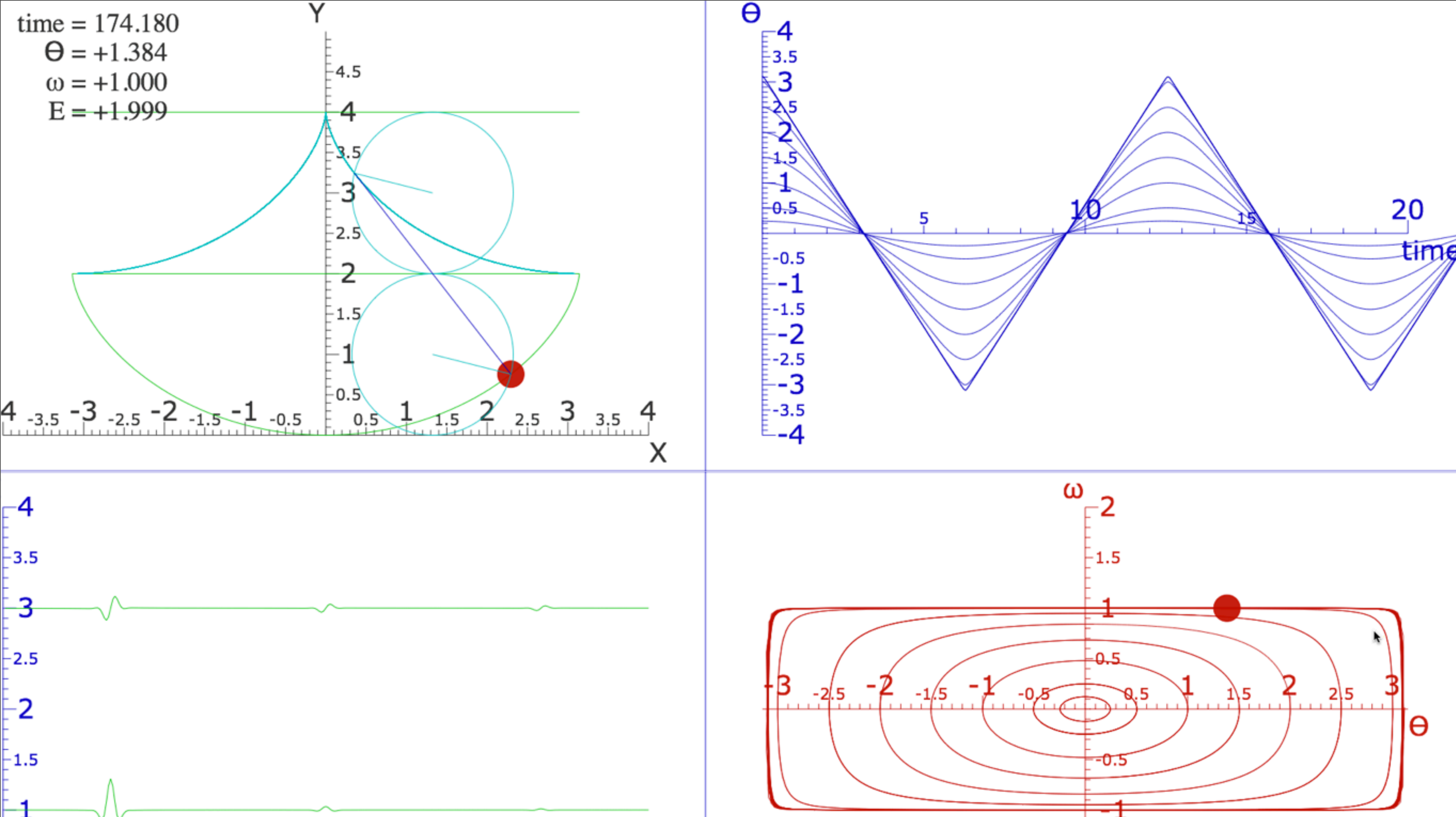
2:5
3:5
4:5
5:5
6:5

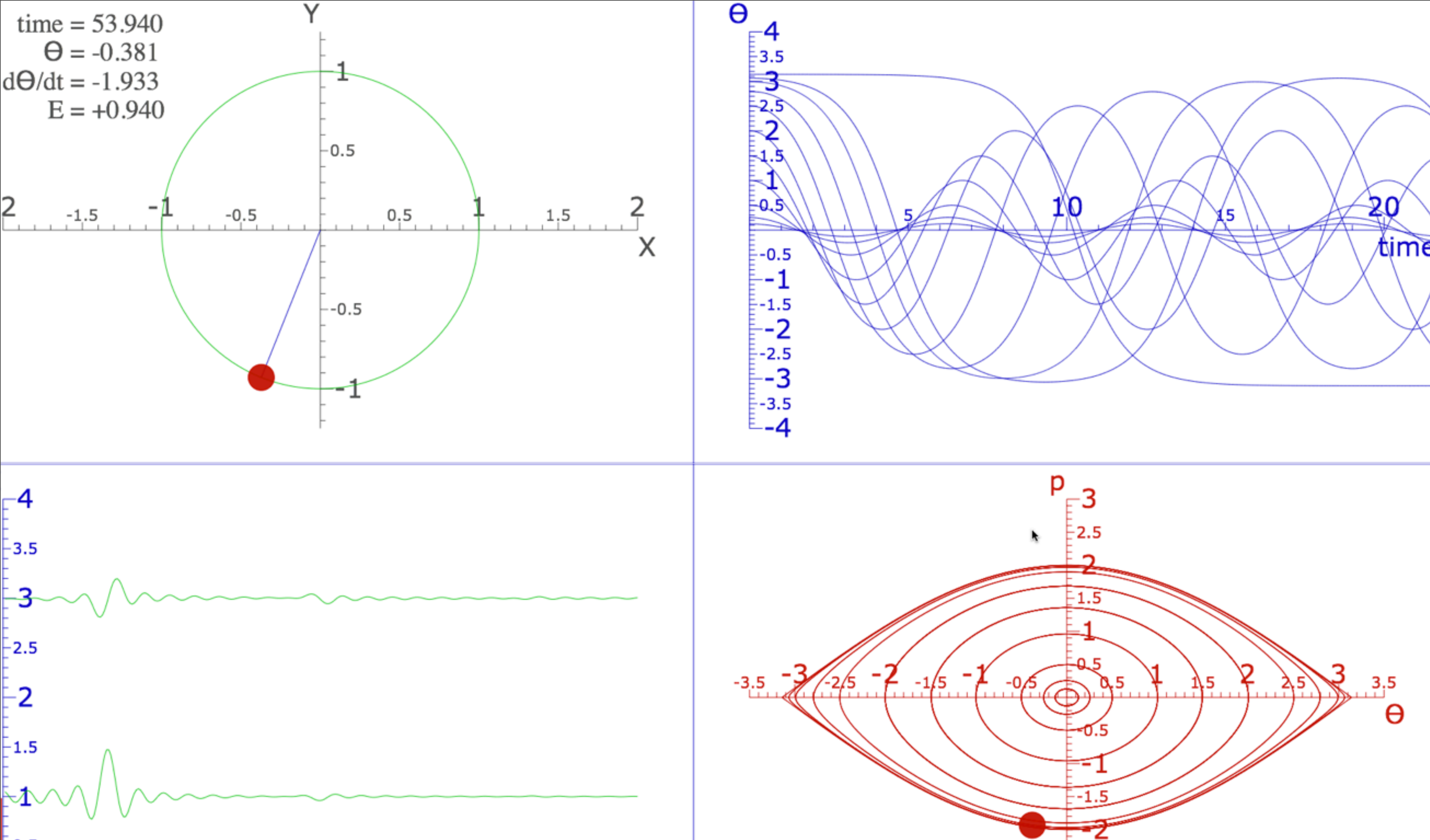

(b) $\omega_\rho:\omega_\phi$ just below 1
prograde precession of nodes
$\omega_\rho:\omega_\phi=1$
$\omega_\rho:\omega_\phi$ just above 1
retrograde precession of nodes
(c) $\omega_\rho:\omega_\phi$ just below 2
prograde precession of nodes
$\omega_\rho:\omega_\phi=2$
$\omega_\rho:\omega_\phi$ just above 2
retrograde precession of nodes

Separation of GCC Equations: Effective Potentials

Small radial oscillations

Cycloid vs Pendulum


time = 174.180
Θ = +1.384
ω = +1.000
E = +1.999
Y
X
Θ
time
ω
Θ


time = 53.940
Θ = -0.381
dΘ/dt = -1.933
E = +0.940
Y
X
Θ
time
p
Θ