Group Theory in Quantum Mechanics

Lecture 12.6 (3.3.15)

Symmetry and Dynamics of C_N cyclic systems(contd.)

(Geometry of U(2) characters - Ch. 6-9 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 3-7 of Ch. 2)

Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)
- Comparing spacetime uncertainty (Δx or Δt) with per-spacetime bandwidth ($\Delta\kappa$ or $\Delta\upsilon$)

Introduction to beat dynamics and "Revivals" due to Bohr-dispersion

Relating ∞-Square-well waves to Bohr rotor waves

∞-Square-well wave dynamics
- SinNx/x wavepacket bandwidth and uncertainty
  - ∞-Square-well revivals: SinNx/x packet explodes! (and then UNexplodes!)

Bohr-rotor wave dynamics
- Gaussian wave-packet bandwidth and uncertainty
  - Gaussian Bohr-rotor revivals and quantum fractals

Understanding fractals using geometry of fractions (Rationalizing rationals)
- Farey-Sums and Ford-products
- Discrete C_N beat phase dynamics (Characters gone wild!)
- The classical bouncing-ball Monster-Mash

Polygonal geometry of $U(2)\supset C_N$ character spectral function
- Algebra
- Geometry

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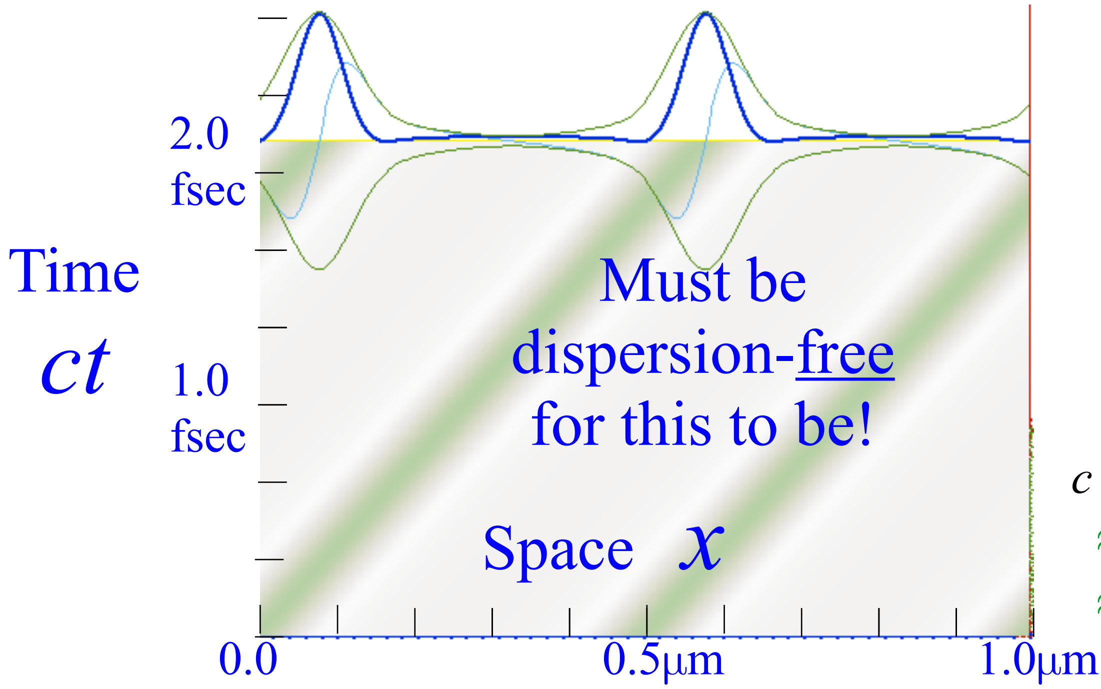
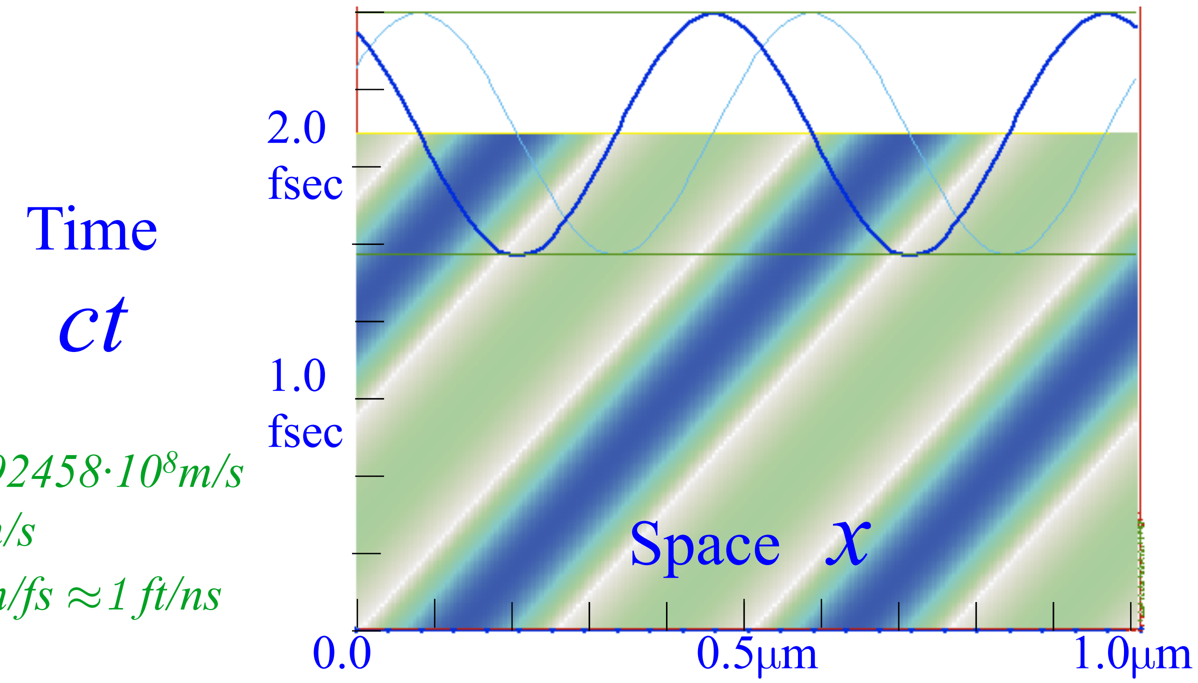
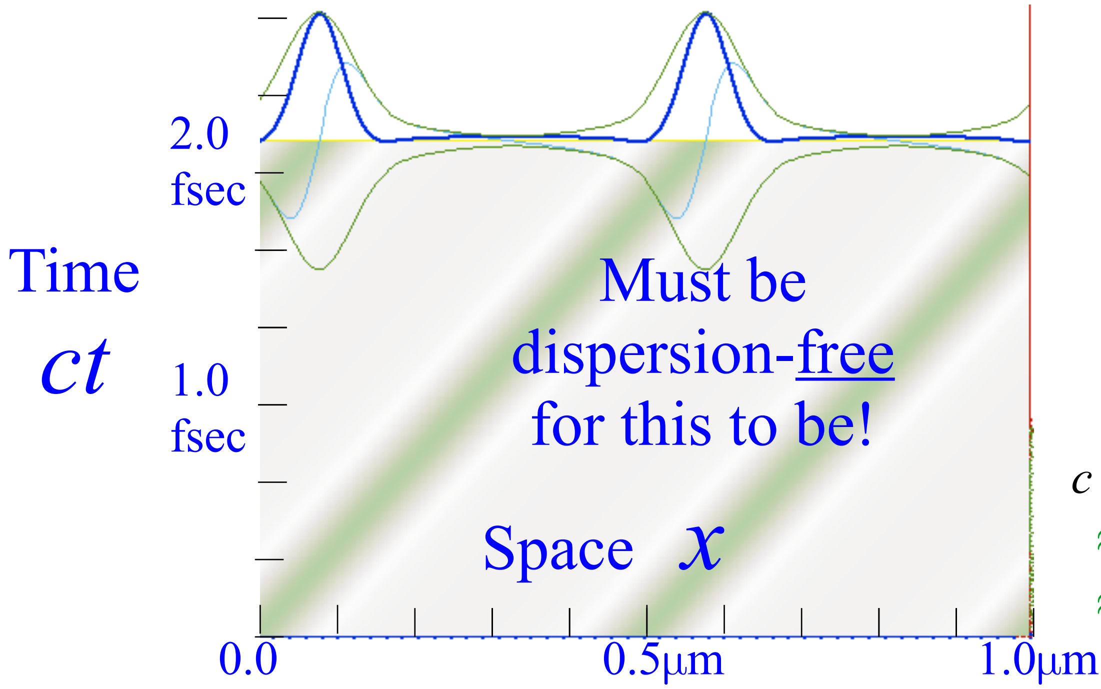
Geometry



$c = 2.99792458 \cdot 10^8 m/s$
$\approx 3 \cdot 10^8 m/s$
$\approx 0.3\ \mu m/fs \approx 1\ ft/ns$



It helps to introduce two *archetypes* of light waves and contrast them.



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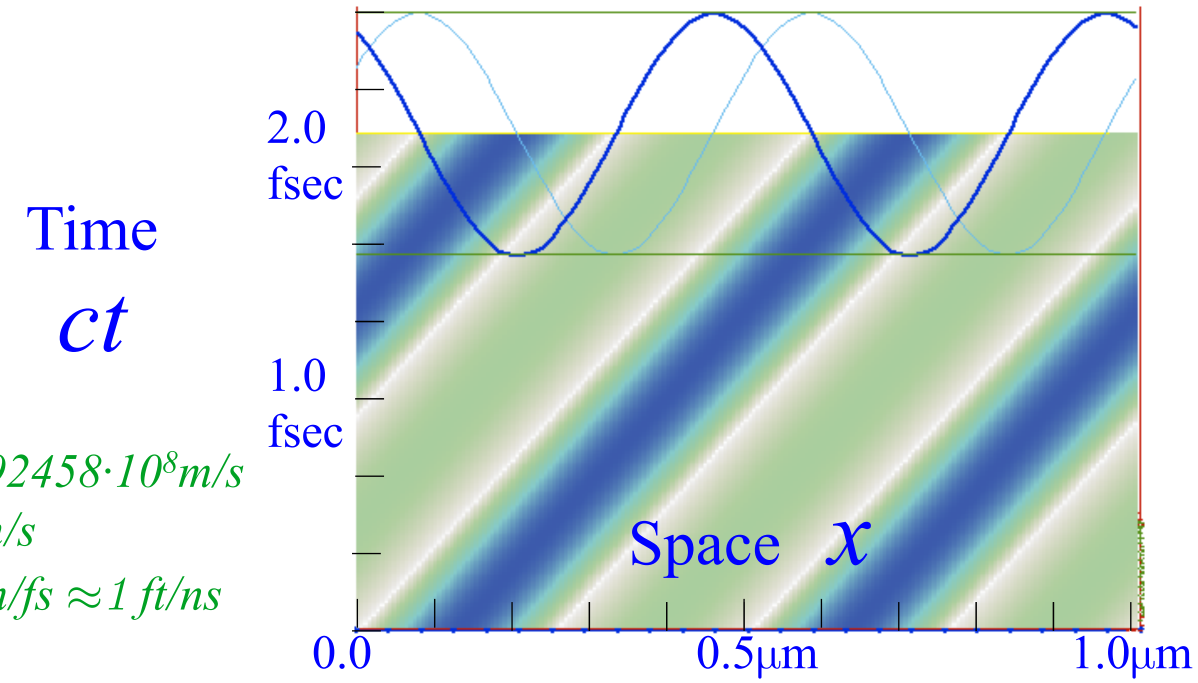
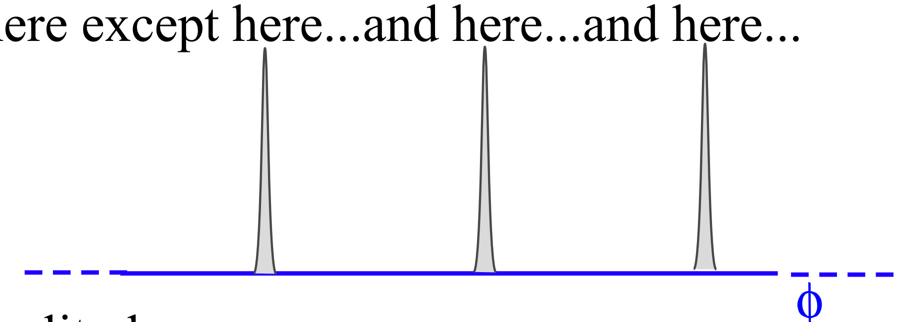


It helps to introduce two *archetypes* of light waves and contrast them.

The first *(PW)* is a *Particle-like Wave* or part of a *Pulse-Wave* train.
The second *(CW)* is a *Coherent Wave* or part of a *Continuous-Wave* train.
..or *Cosine Wave* ...or *Colored Wave*

(1) The PW archetype

PW amplitude is *ZERO*
everywhere except here...and here...and here...

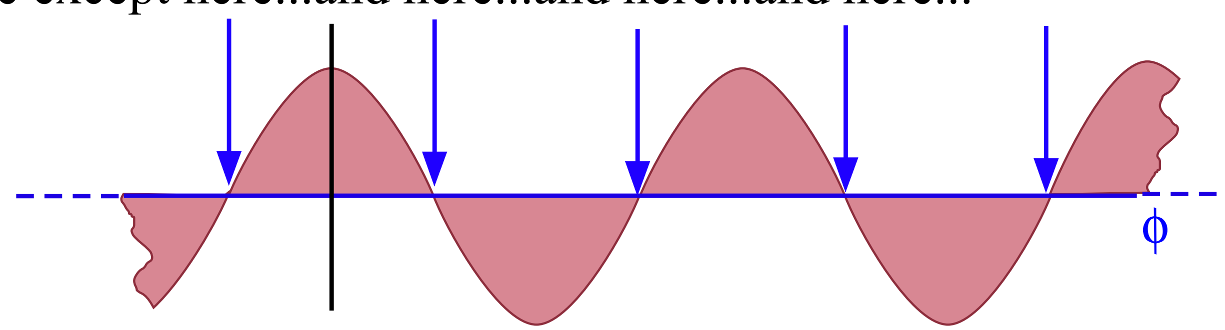


PW amplitude...
ZEROS.
...but has sharp *PEAKS*.
...is best defined by where it *IS*.

Ideal *PW* shape is a *Dirac Delta function*.

(2) The CW archetype

CW amplitude is *NON-zero*
everywhere except here...and here...and here...and here...

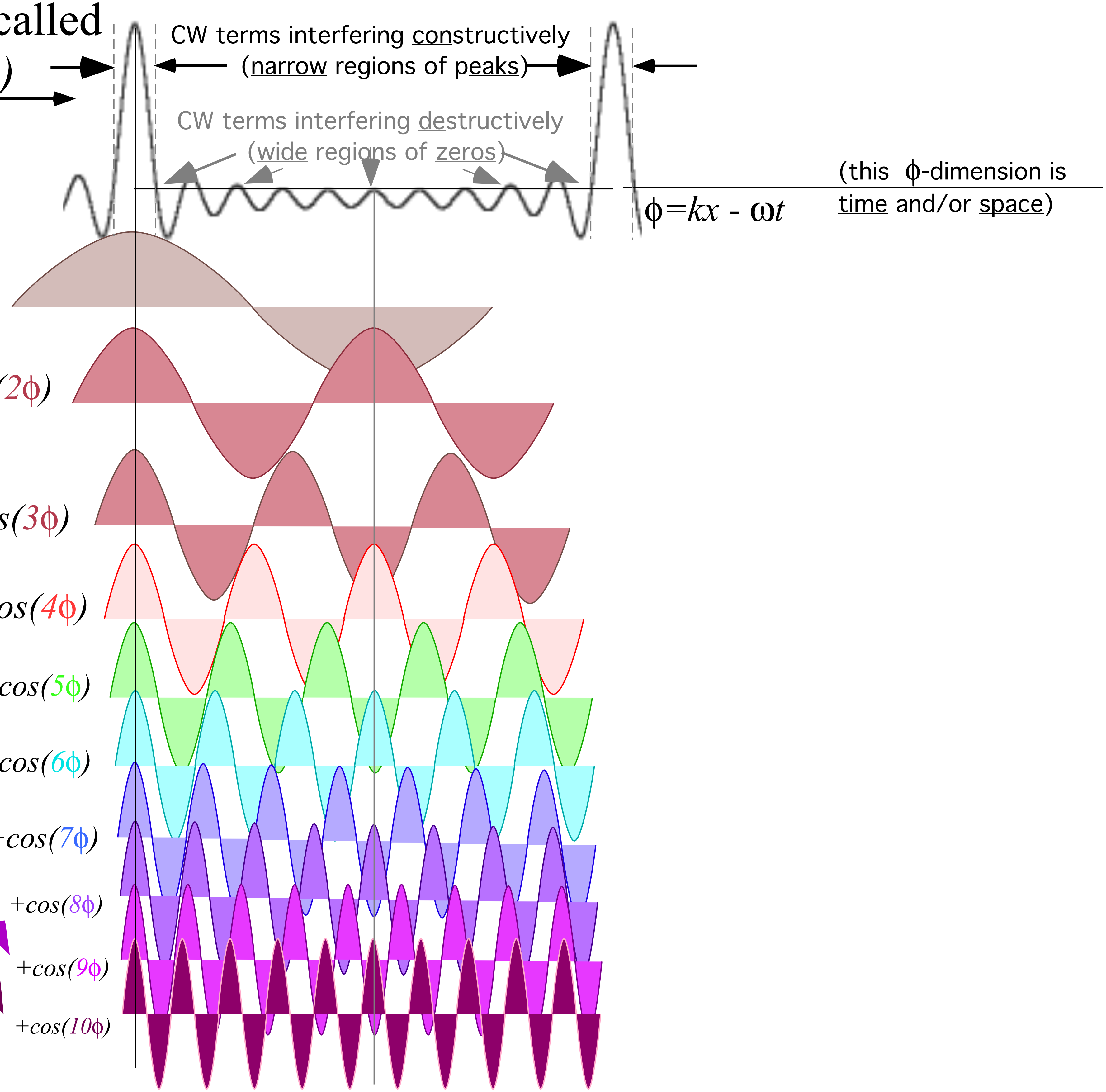


...is mostly *NON-zero* with rounded crests and troughs.
...but has sharp *ZEROS*.
...is best defined by where it *IS NOT*.

Ideal *CW* shape is a *cosine wave* *(cos(ϕ))*

PW forms are also called *Wave Packets (WP)* since they are interfering sums of many *CW* terms

(10-Cosine Waves make up this pulse)



CW terms are also called *Color Waves* or *Fourier Spectral Components*

PW forms are also called *Wave Packets (WP)* since they are interfering sums of many *CW* terms

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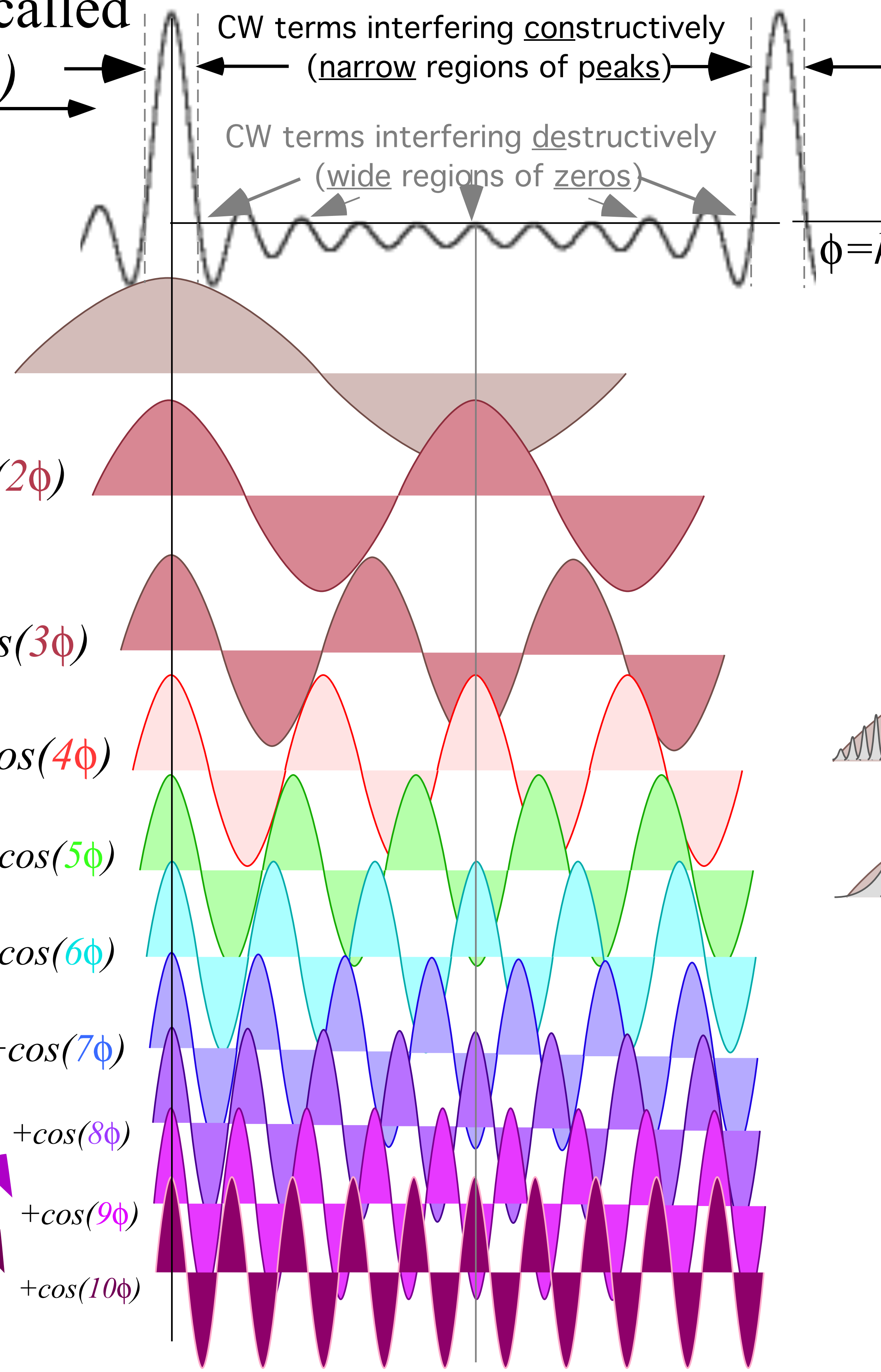
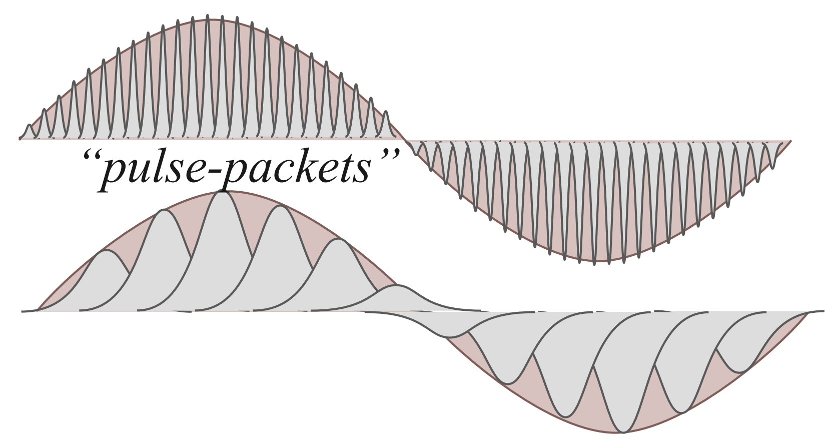


CW terms are also called *Color Waves* or *Fourier Spectral Components*

... and *vice-versa* ... *CW* forms can be made *artificially* from *PW* sums ...



(this is digital *sampling* or digital-to-analog *synthesis*.)

As we'll see, this is a ***terrible*** way to make ***quantum*** CW...

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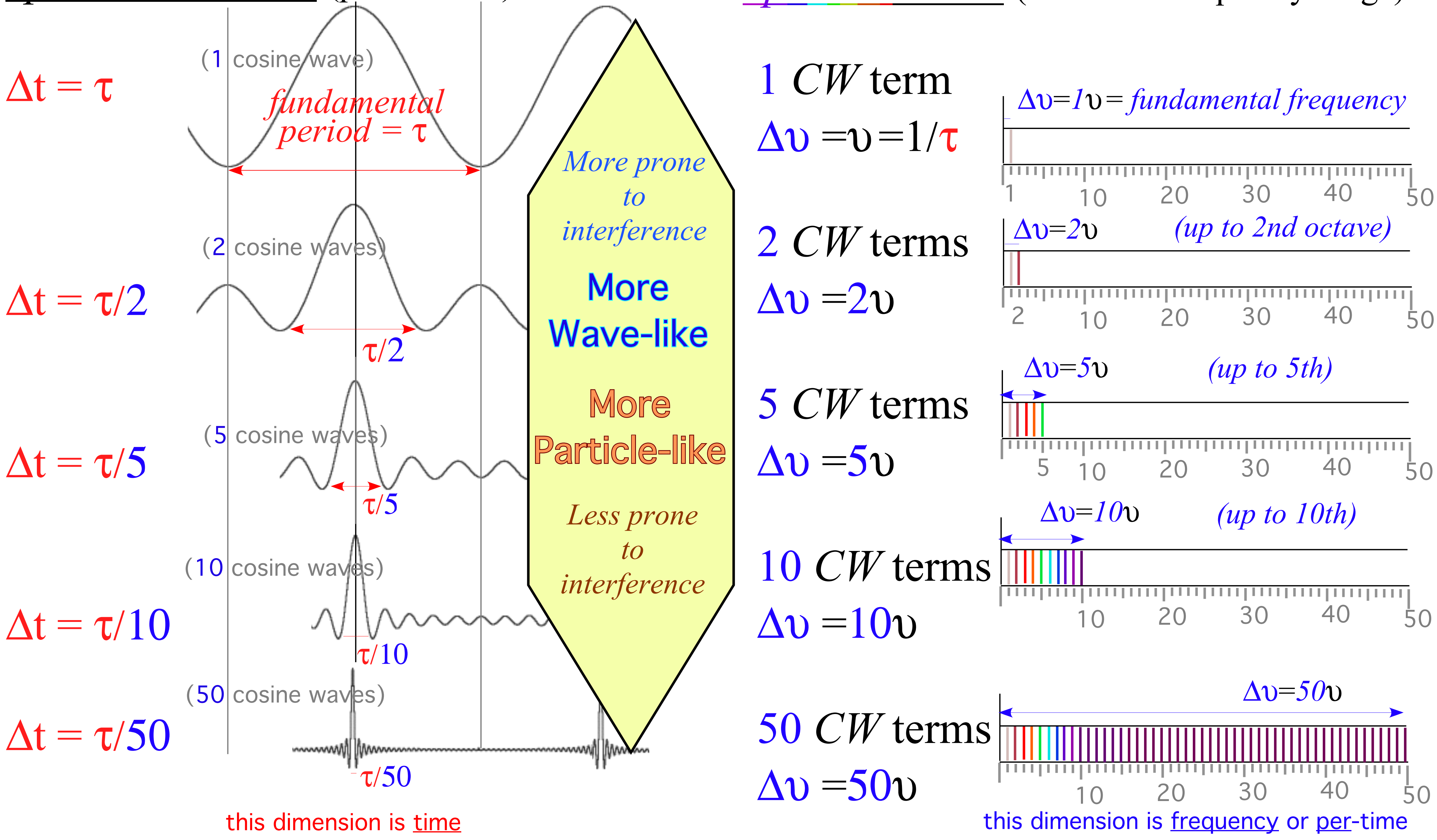
PW widths reduce proportionally with more *CW* terms (greater *Spectral* width)

Space-time width (pulse width) | *Spectral width* (harmonic frequency range)



Fourier-Heisenberg product: $\Delta t \cdot \Delta \upsilon = 1$ (time-frequency uncertainty relation)

PW widths reduce proportionally with more *CW* terms (greater *Spectral* width)

Space-time width (pulse width)

Spectral width (harmonic frequency range)

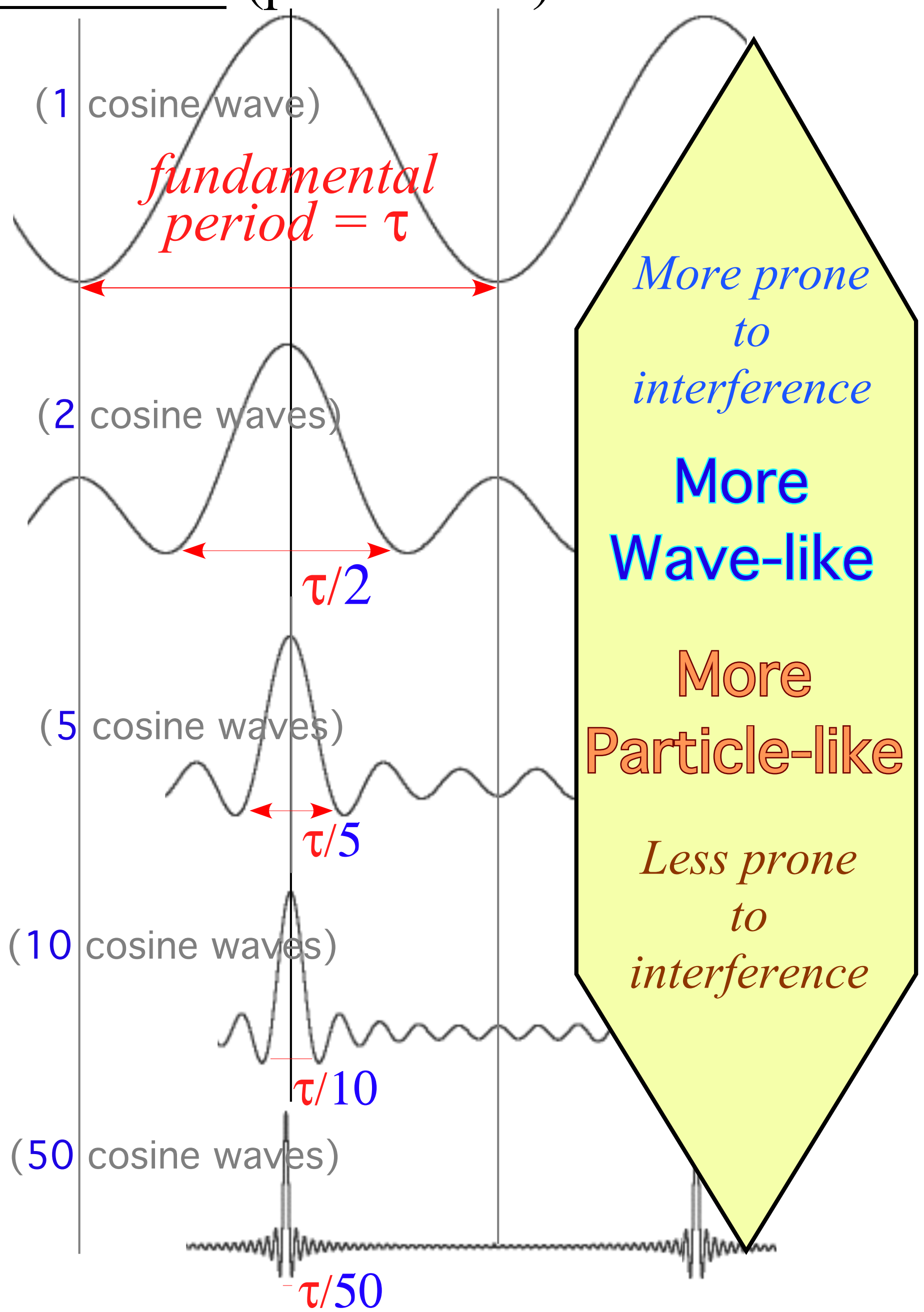
$\Delta t = \tau$ — 1 *CW* term, $\Delta\upsilon = \upsilon = 1/\tau$

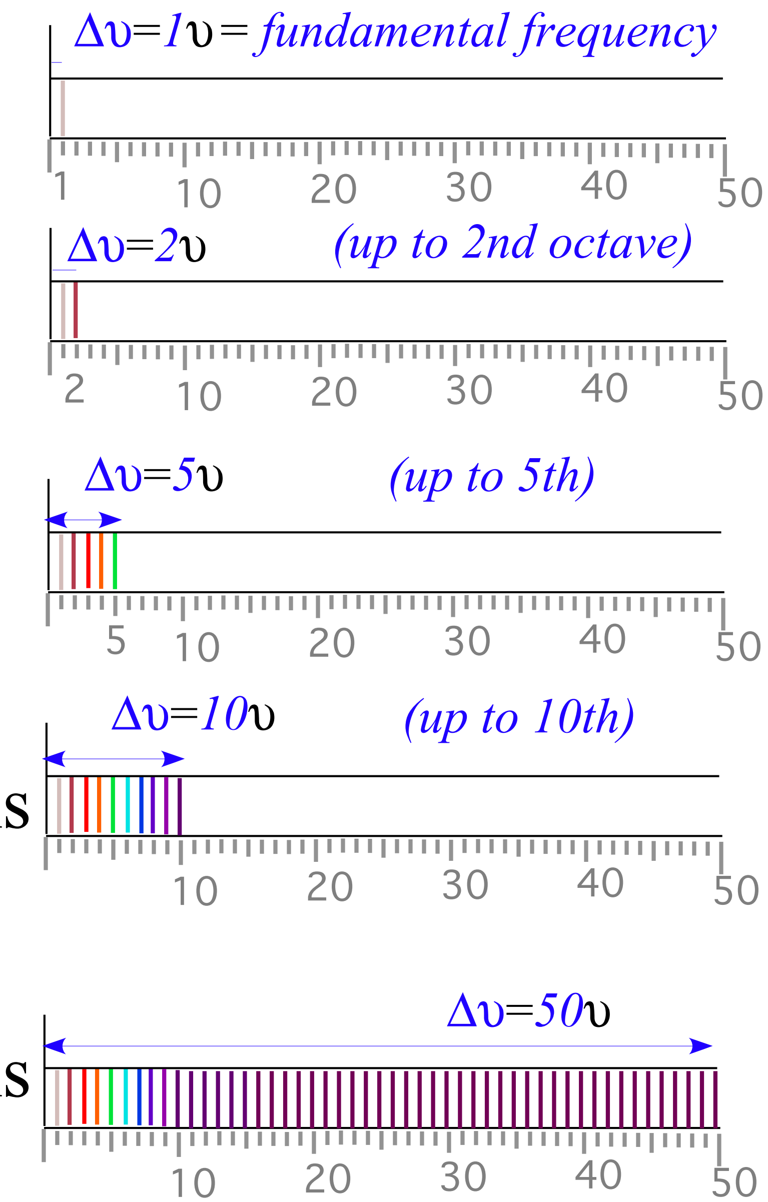
$\Delta t = \tau/2$ — 2 *CW* terms, $\Delta\upsilon = 2\upsilon$

$\Delta t = \tau/5$ — 5 *CW* terms, $\Delta\upsilon = 5\upsilon$

$\Delta t = \tau/10$ — 10 *CW* terms, $\Delta\upsilon = 10\upsilon$

$\Delta t = \tau/50$ — 50 *CW* terms, $\Delta\upsilon = 50\upsilon$





this dimension is time

this dimension is frequency or per-time

Fourier-Heisenberg product: $\Delta t \cdot \Delta\upsilon = 1$ (time-frequency uncertainty relation)

or this dimension is space...

if this dimension is wavenumber or per-space...

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∞-Square well PE versus Bohr rotor

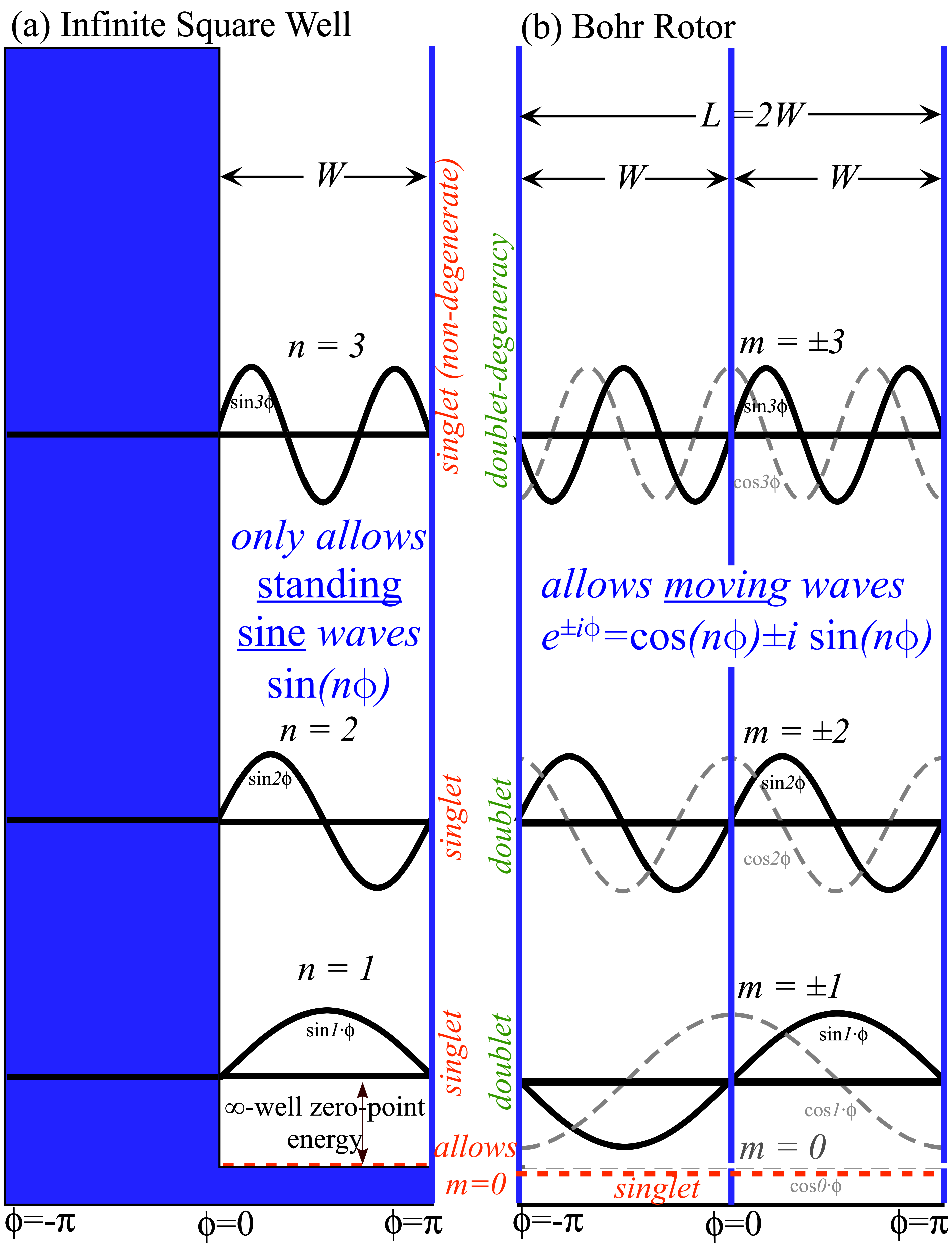

Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

From QTCA Unit 5 Ch. 12

$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

($k_m=m$ if: $L=2\pi$)

∞-Square well PE versus Bohr rotor

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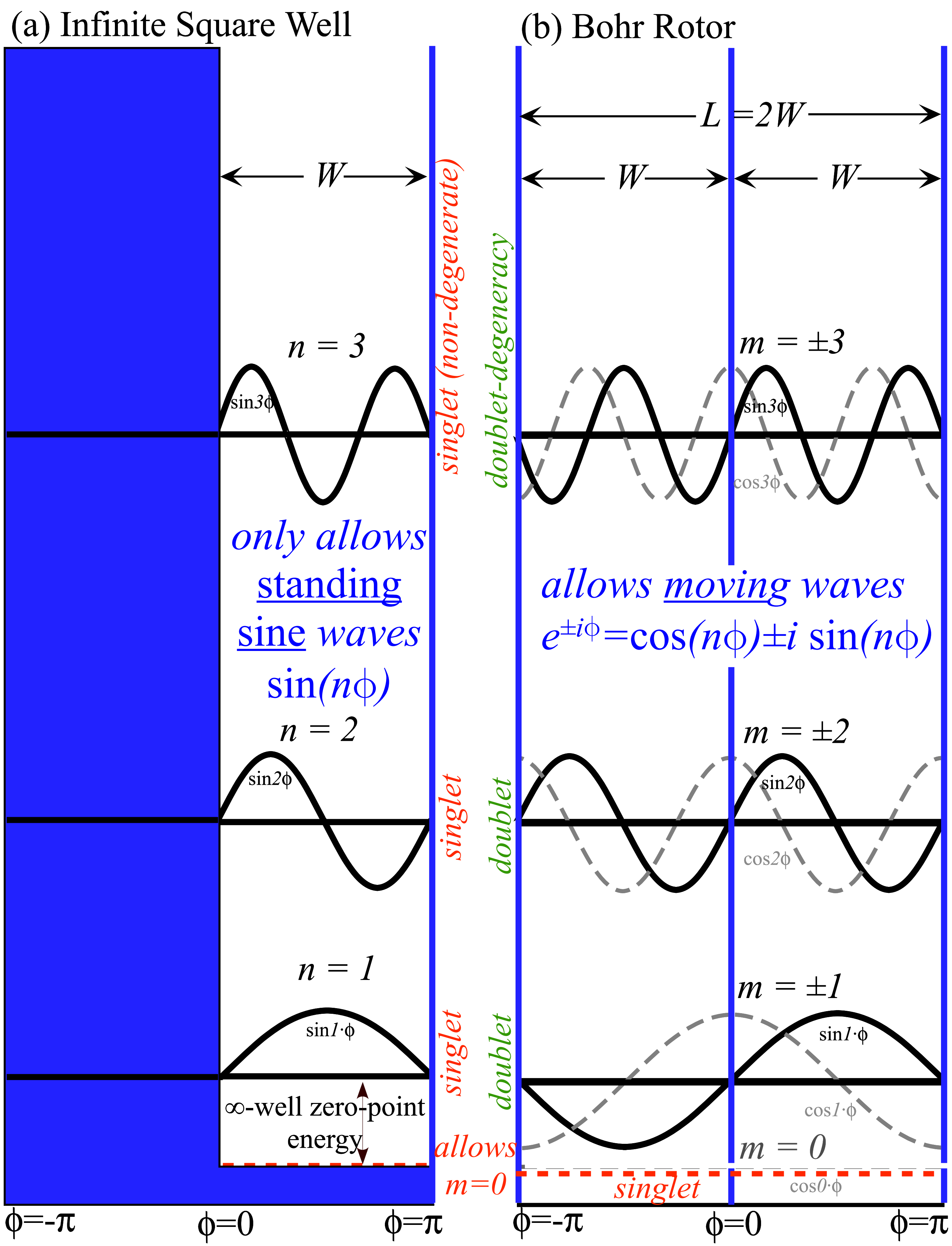

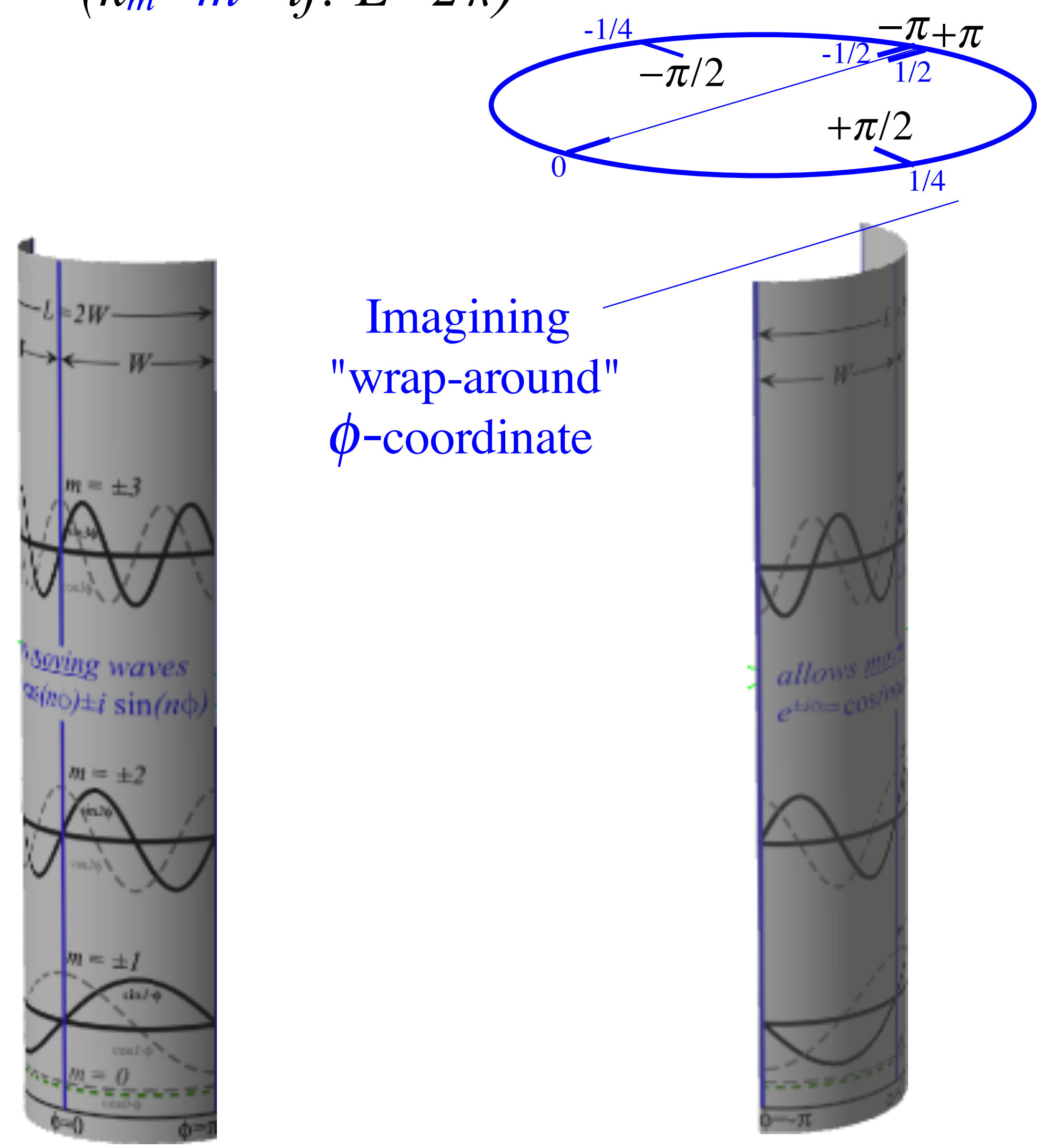

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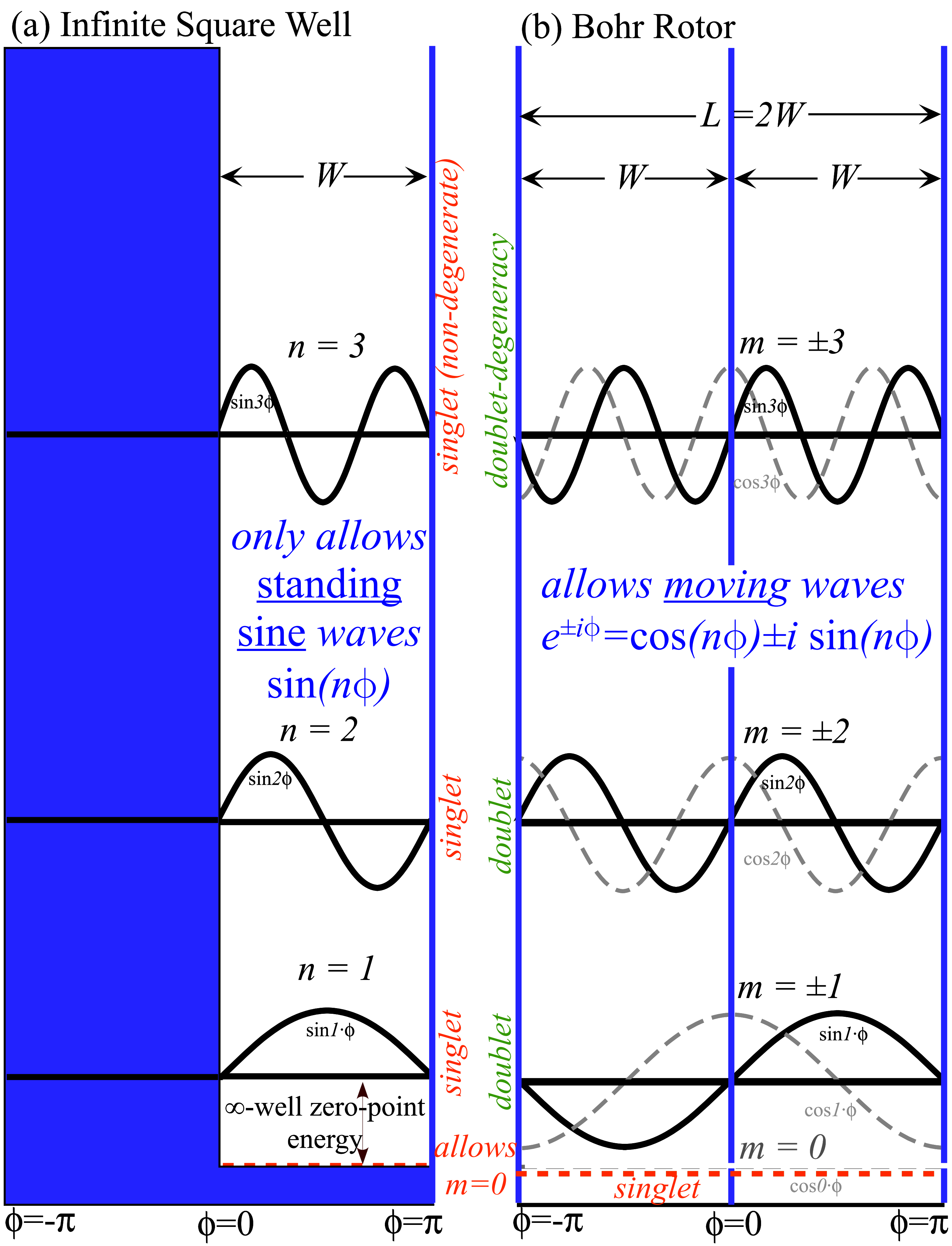

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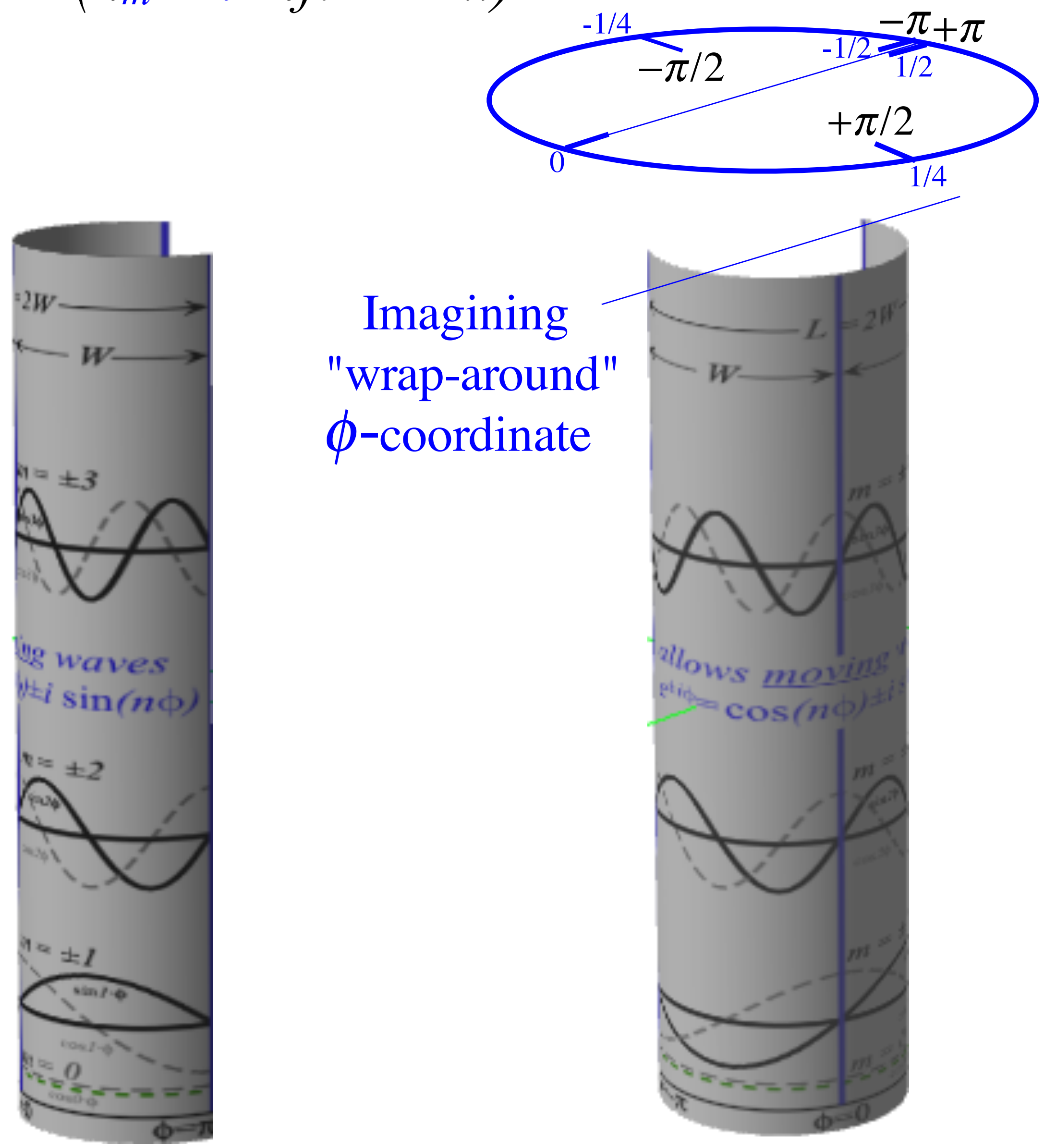

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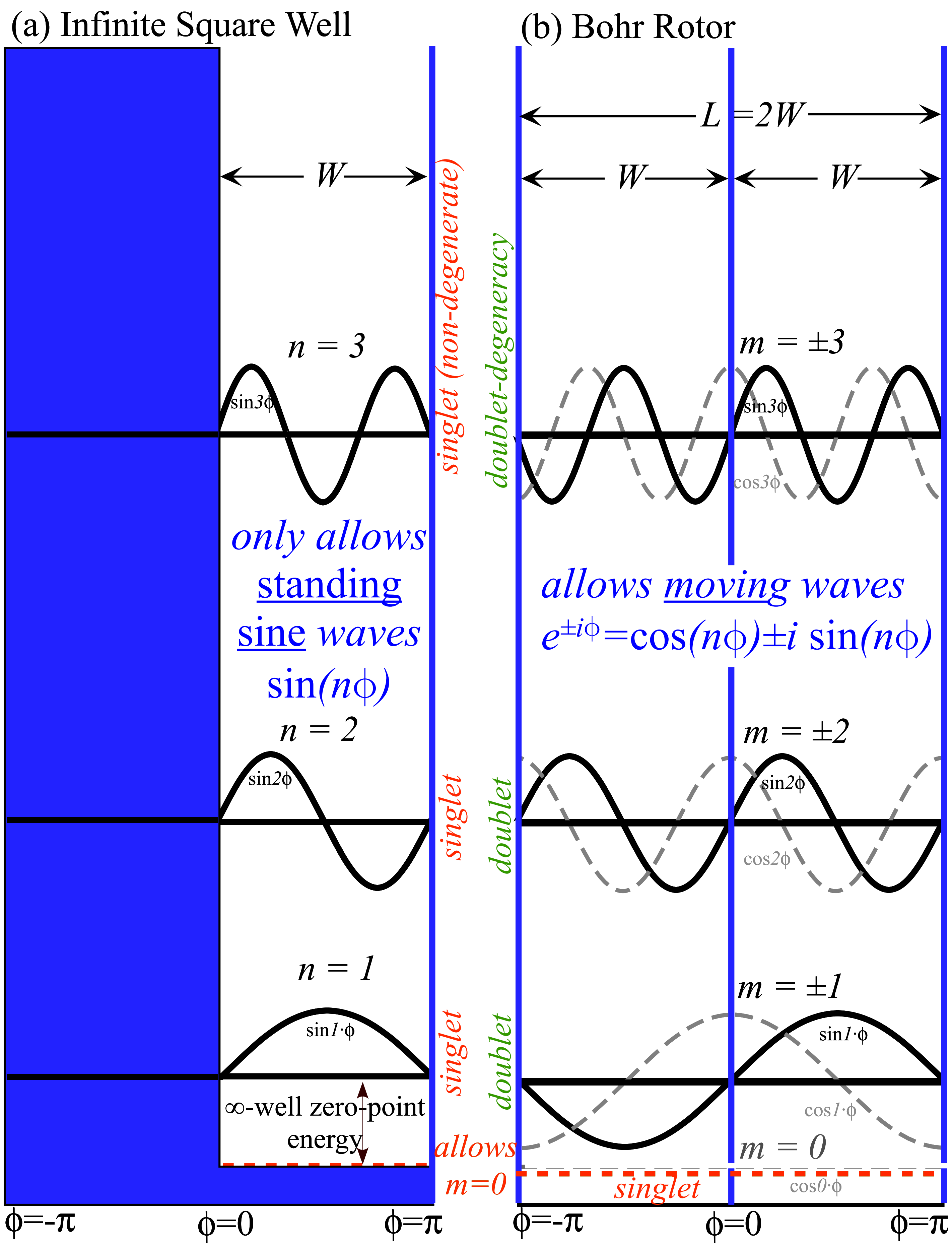

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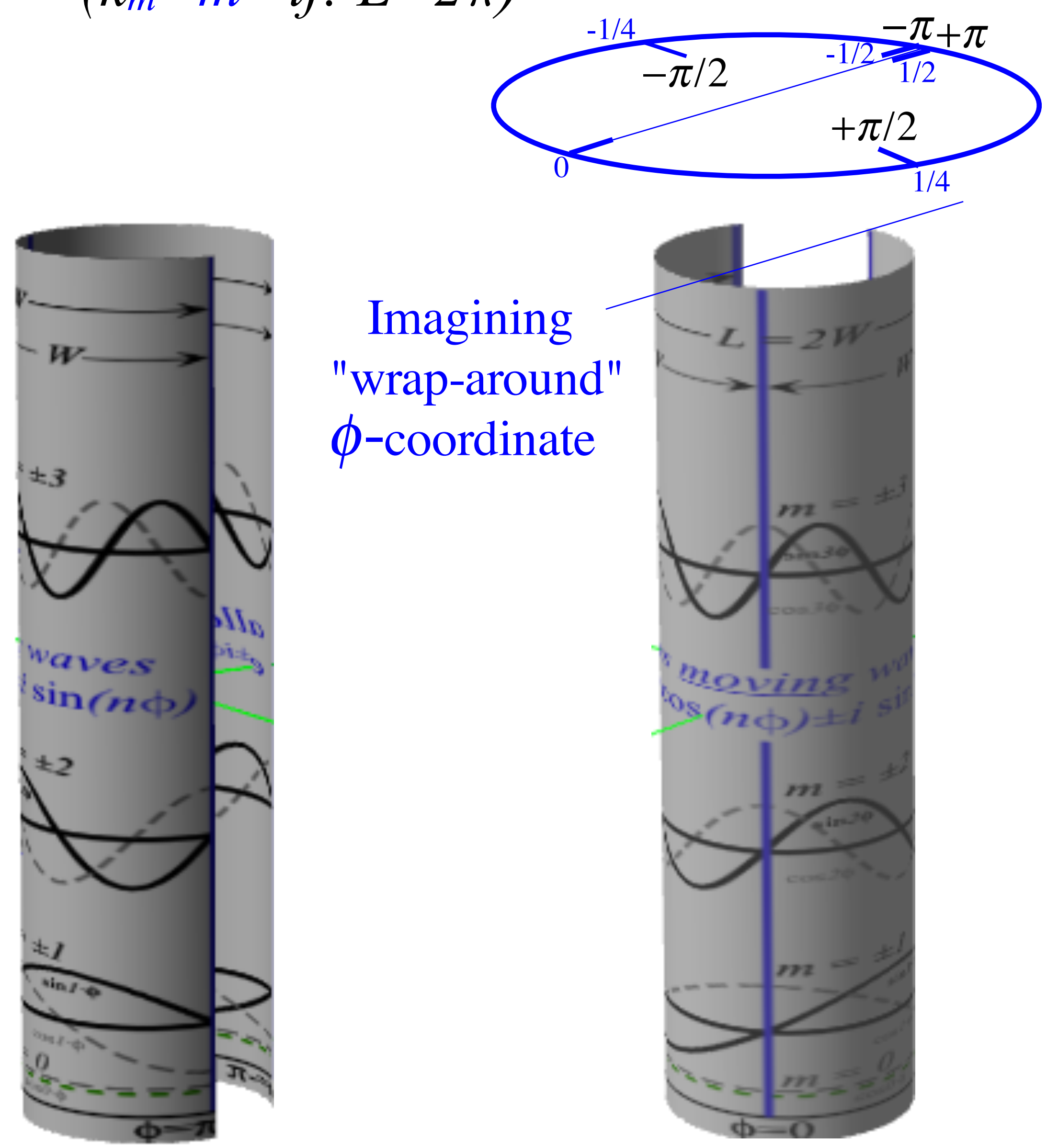

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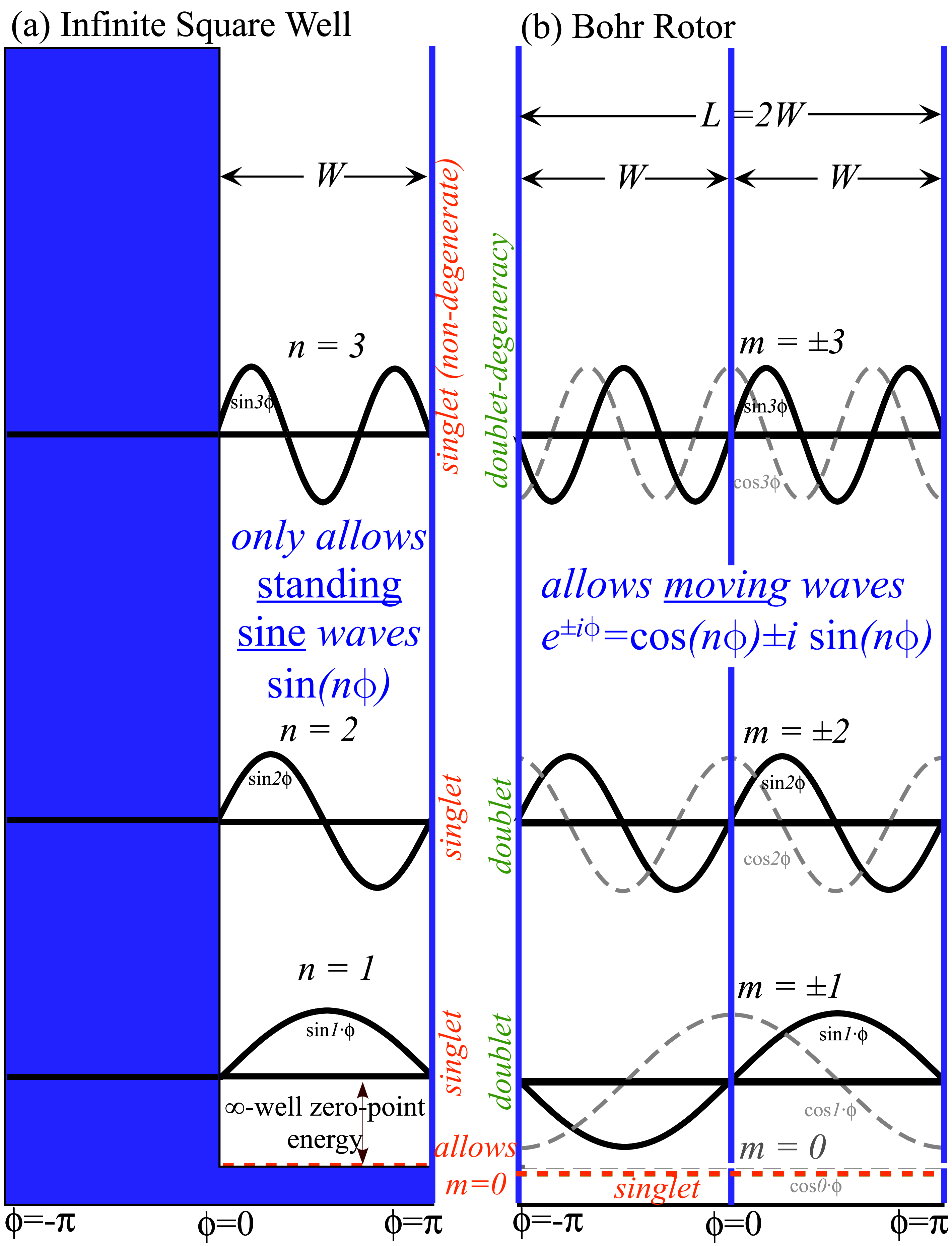

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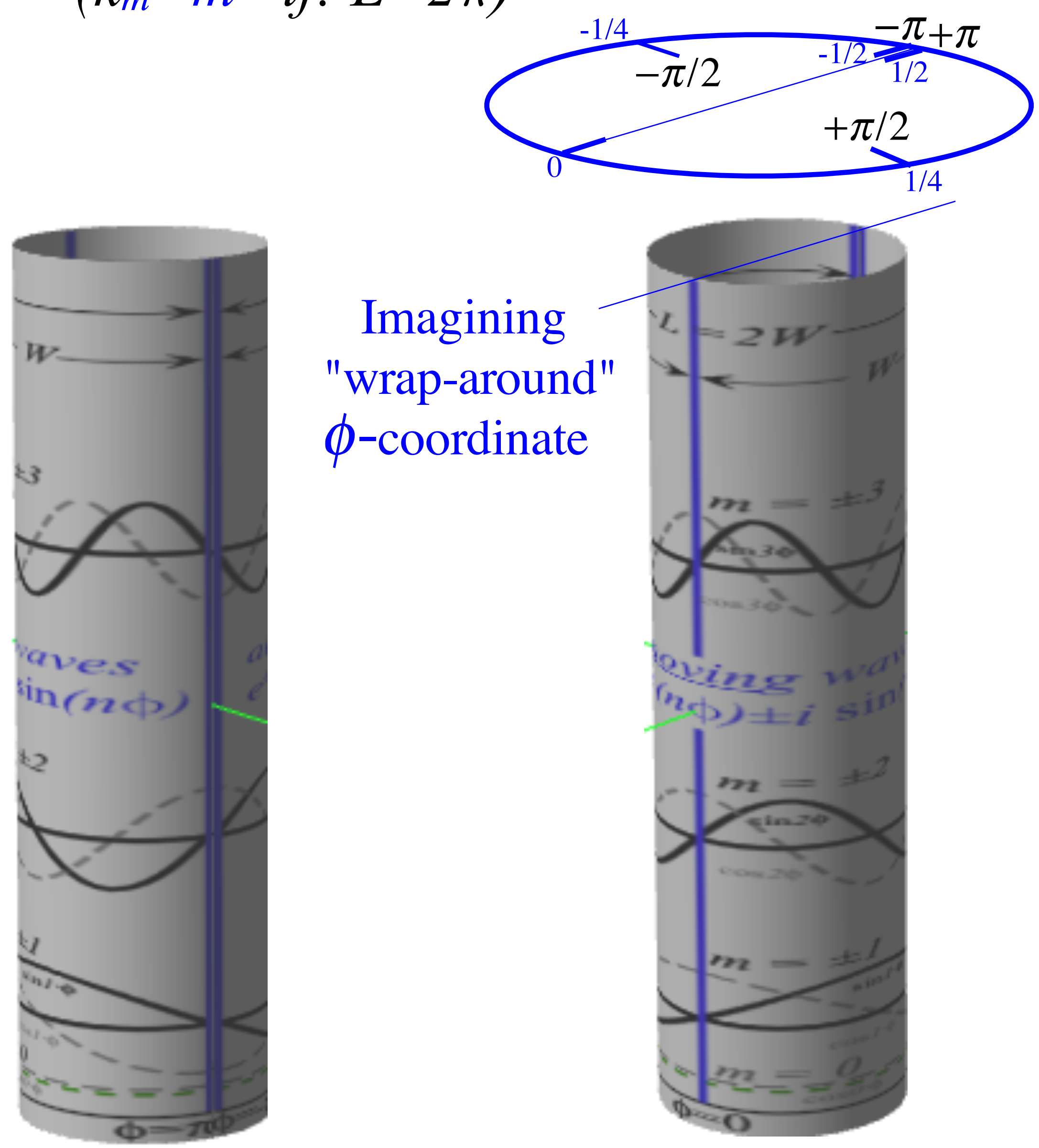

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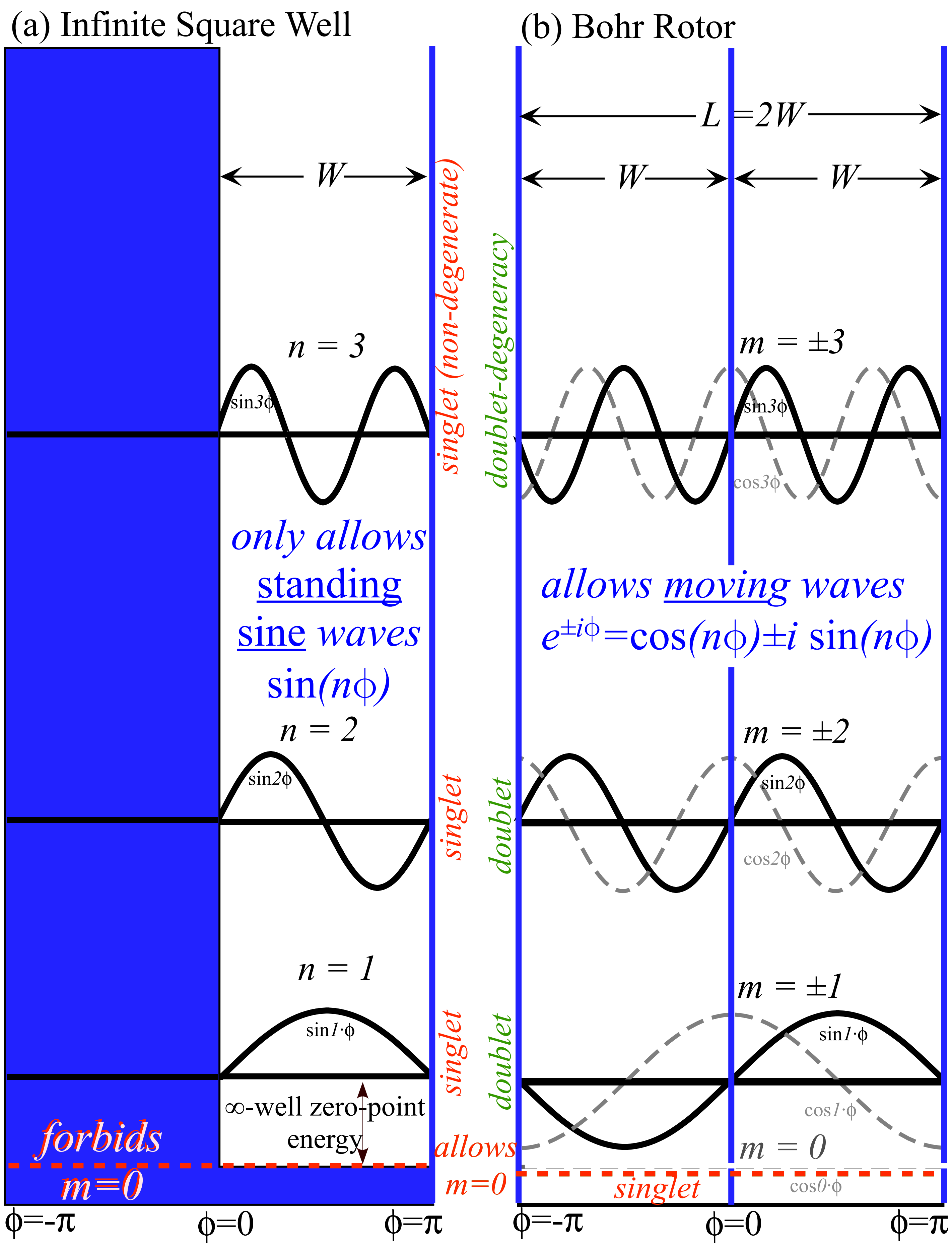

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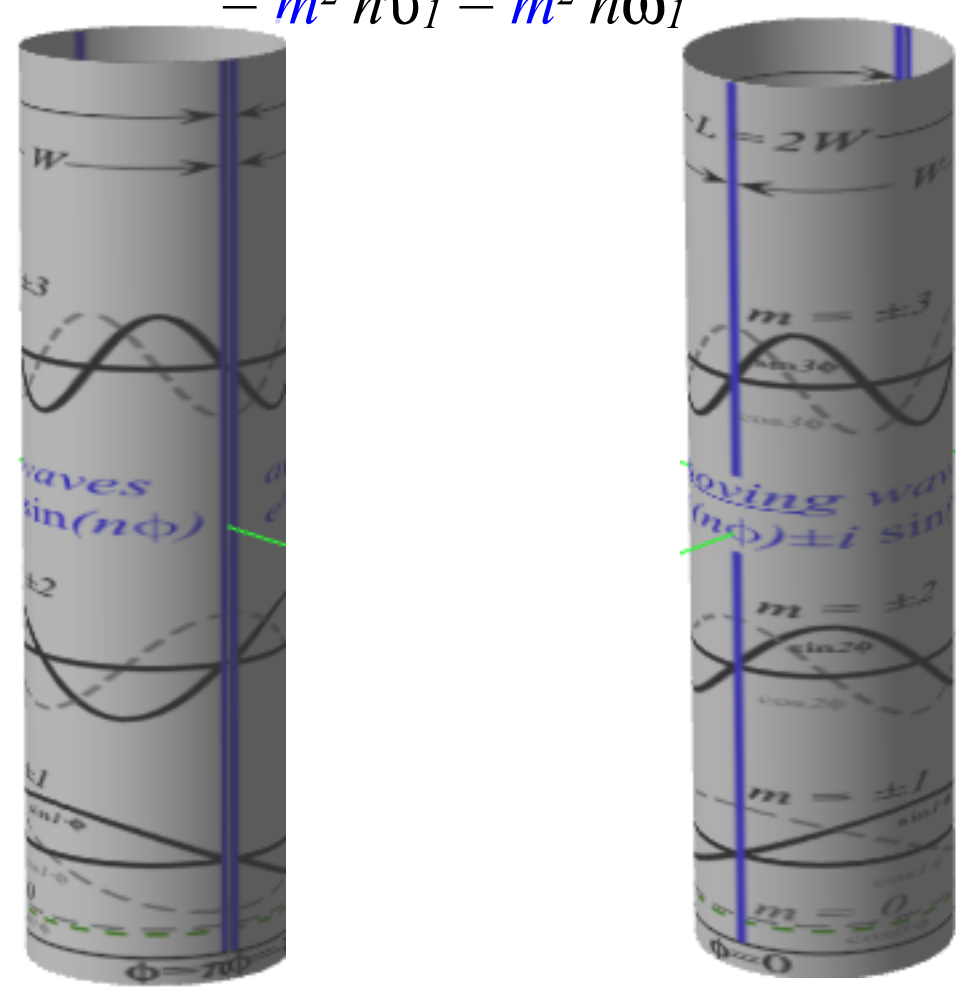
$$E_m= (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$
$$= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$



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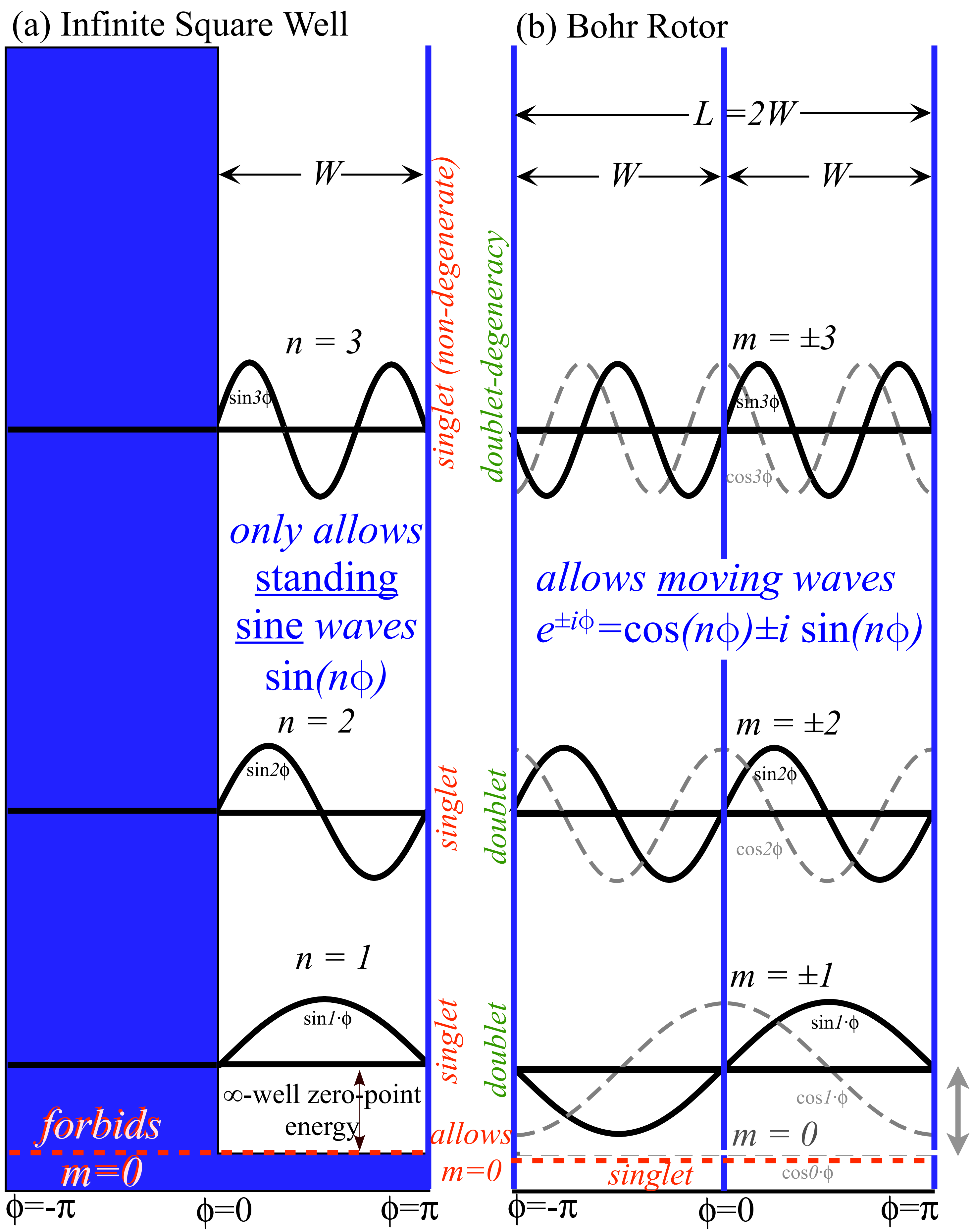

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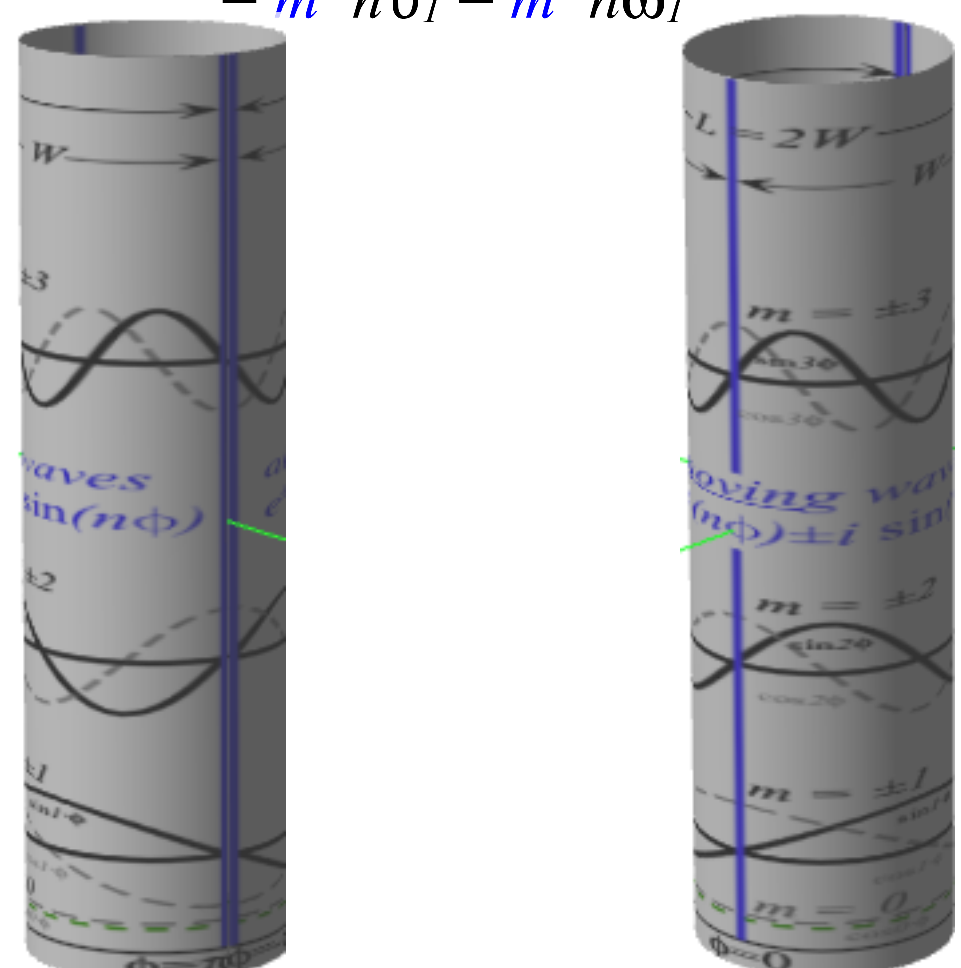

fundamental Bohr ∠−frequency

$\omega_1=2\pi\upsilon_1$

lowest transition (beat) frequency

$\upsilon_1 =(E_1-E_0)/h$ (E_0 is defined as zero)

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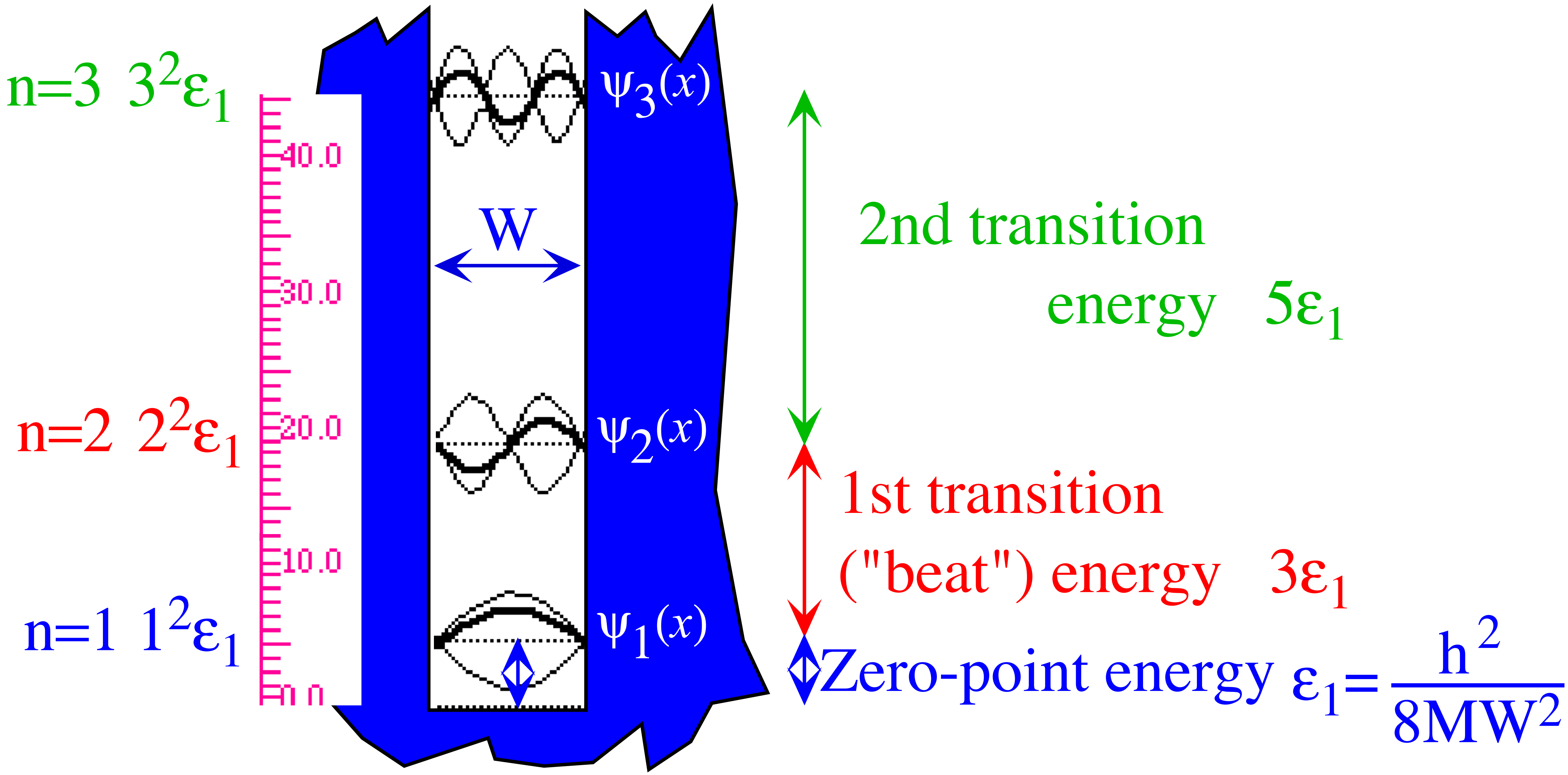
∞-Square well PE (The story of prisoner-M)

Boundary conditions: $k_n W = n\pi$ or: $k_n = n\pi/W$

Energy eigenfunctions: $\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi x}{W}\right)$ $(n=1,2,3,...\infty)$



From QTCA Unit 5 Ch. 12

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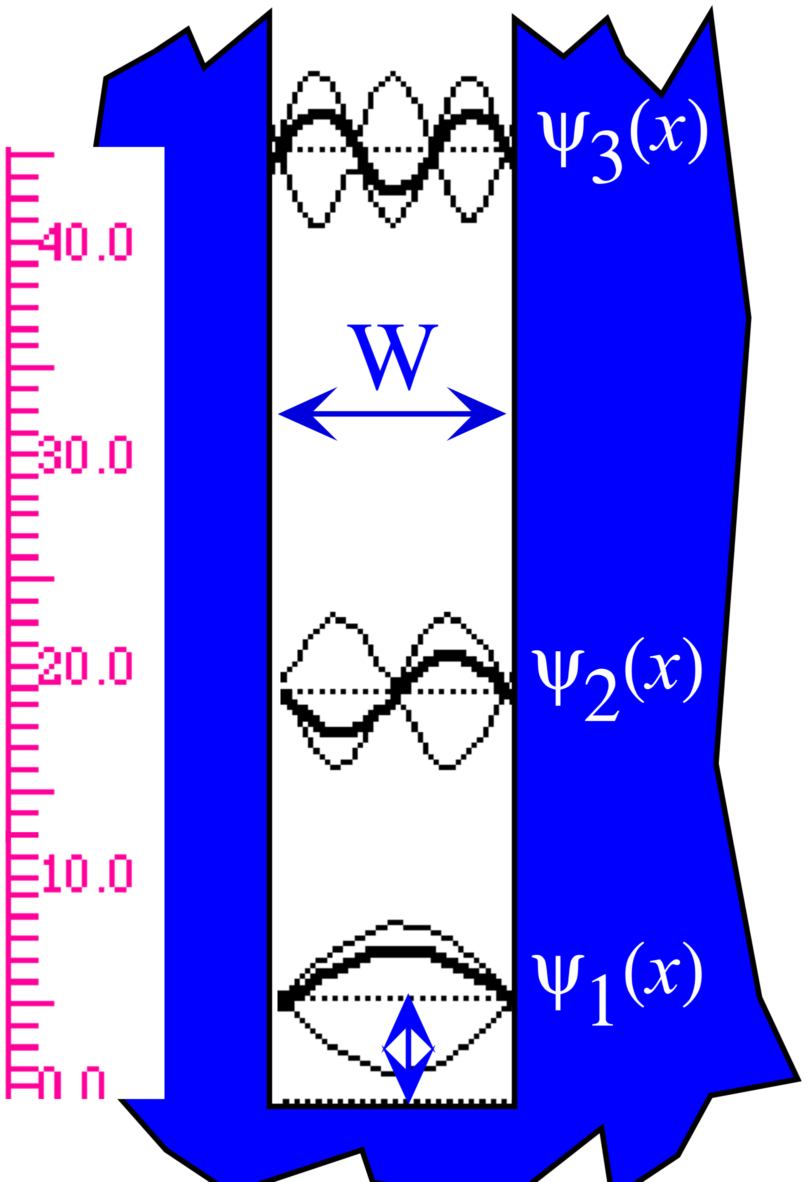
Energy eigenvalues: $\varepsilon_n = KE = p_n^2/2M = (\hbar k_n)^2/2M$

$$\varepsilon_n = \frac{\hbar^2}{2M}k_n^2 = \frac{\hbar^2 n^2 \pi^2}{2MW^2} = \left(1^2, 2^2, 3^2, ... \text{or } n^2\right)\frac{h^2}{8MW^2}$$

n=3 $3^2\varepsilon_1$

n=2 $2^2\varepsilon_1$

n=1 $1^2\varepsilon_1$



2nd transition energy $5\varepsilon_1$

1st transition ("beat") energy $3\varepsilon_1$

Zero-point energy $\varepsilon_1 = \frac{h^2}{8MW^2}$

fundamental Bohr ∠–frequency $\omega_1 = 2\pi\upsilon_1$

lowest *transition (beat) frequency*

$\upsilon_1 = (\varepsilon_1 - \varepsilon_0)/h$ *(ε_0 is defined as zero)*

$\omega_1 = 2\pi\upsilon_1 = 2\pi\varepsilon_1/h = 2\pi h/(8MW^2)$

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$$\omega_{beat} = \omega_2 - \omega_1 = \frac{\varepsilon_2 - \varepsilon_1}{\hbar} = \frac{2^2 - 1^2}{\hbar}\frac{h^2}{8MW^2} = 3\frac{2\pi h}{8MW^2} = 3\omega_1$$

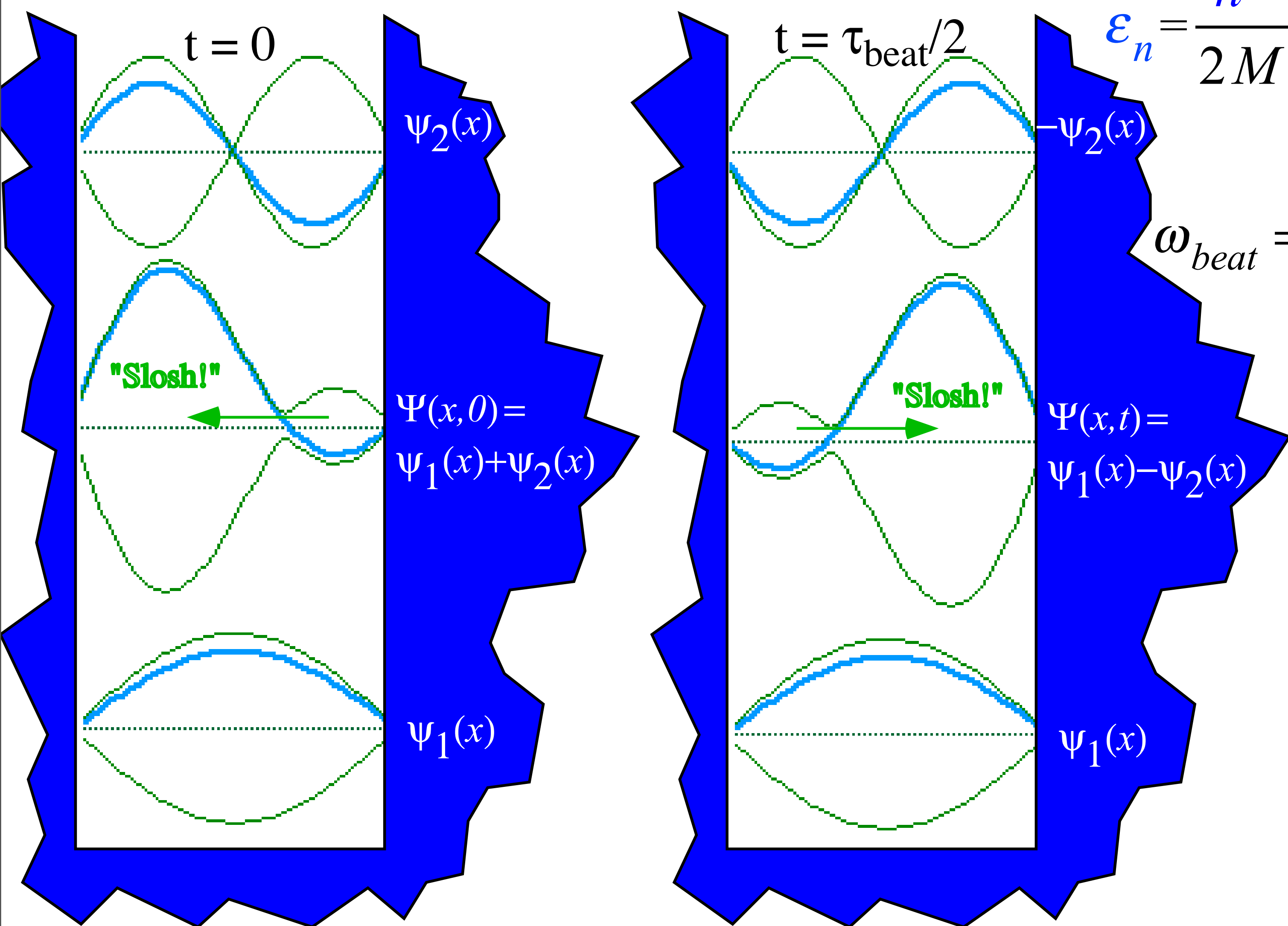

Fig. 12.1.2 Exercise in prison. Infinite square well eigensolution combination "sloshes" back and forth.

From QTCA Unit 5 Ch. 12

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$

lowest transition (beat) frequency

$\upsilon_1 = (\varepsilon_1 - \varepsilon_0)/h$ (ε_0 is defined as zero)

$\omega_1 = 2\pi\upsilon_1 = 2\pi\varepsilon_1/h = 2\pi h/(8MW^2)$

$\upsilon_1 = h/(8MW^2)$

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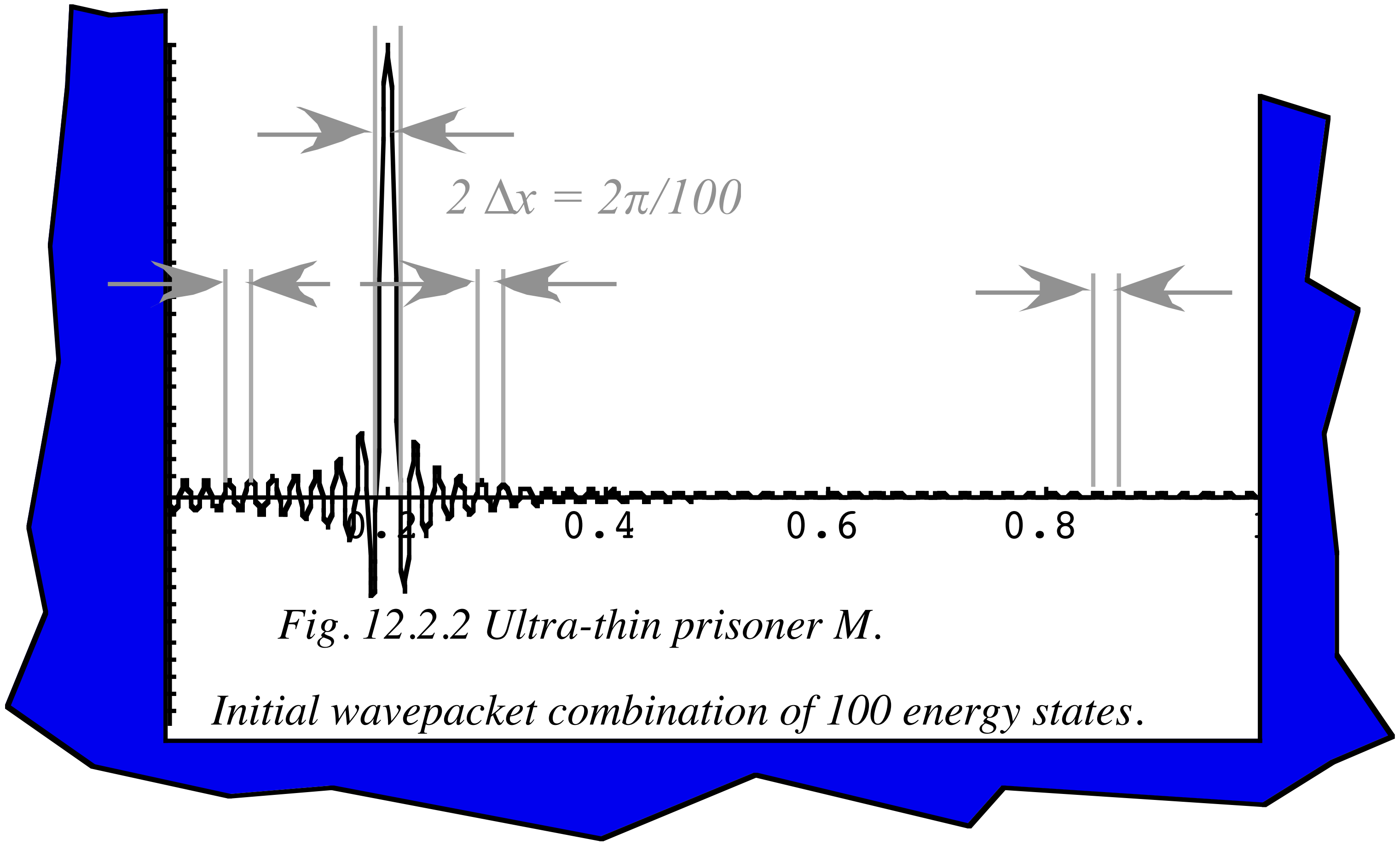
$$\delta(x-a) = \langle x | a \rangle$$



From QTCA Unit 5 Ch. 12

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle$$

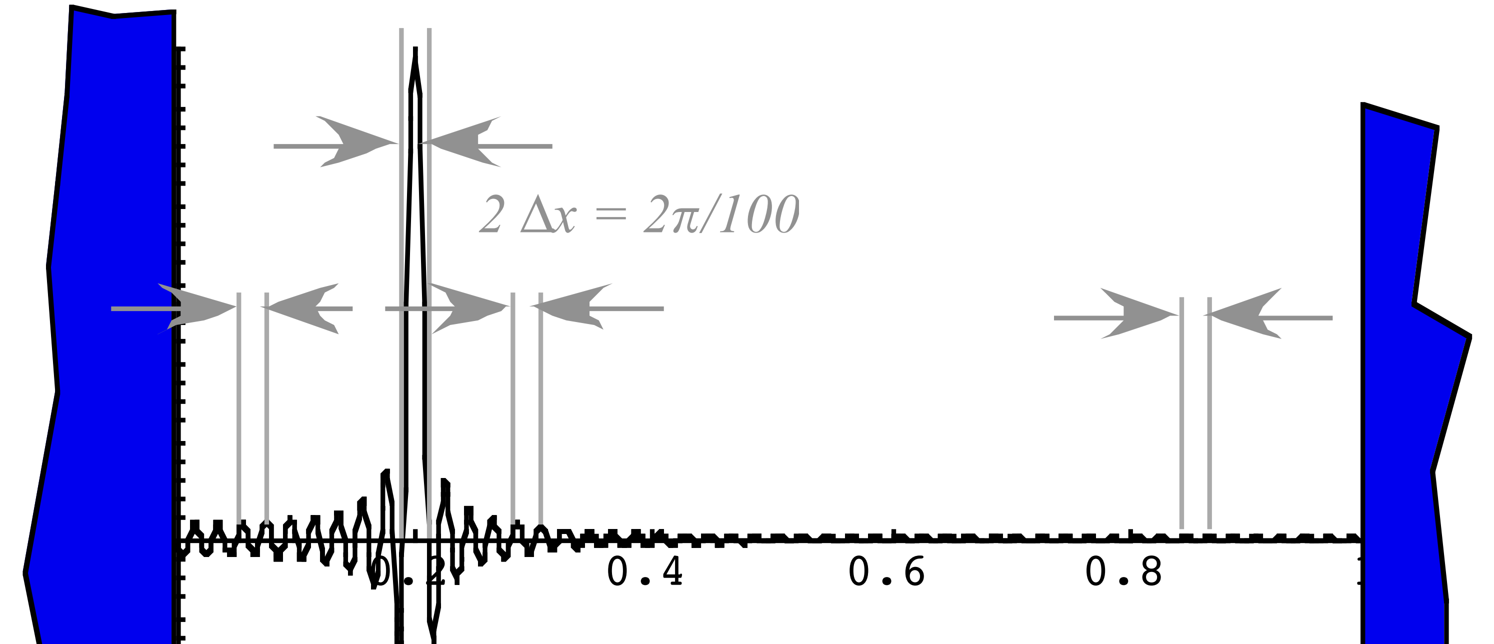


Fig. 12.2.2 Ultra-thin prisoner M.

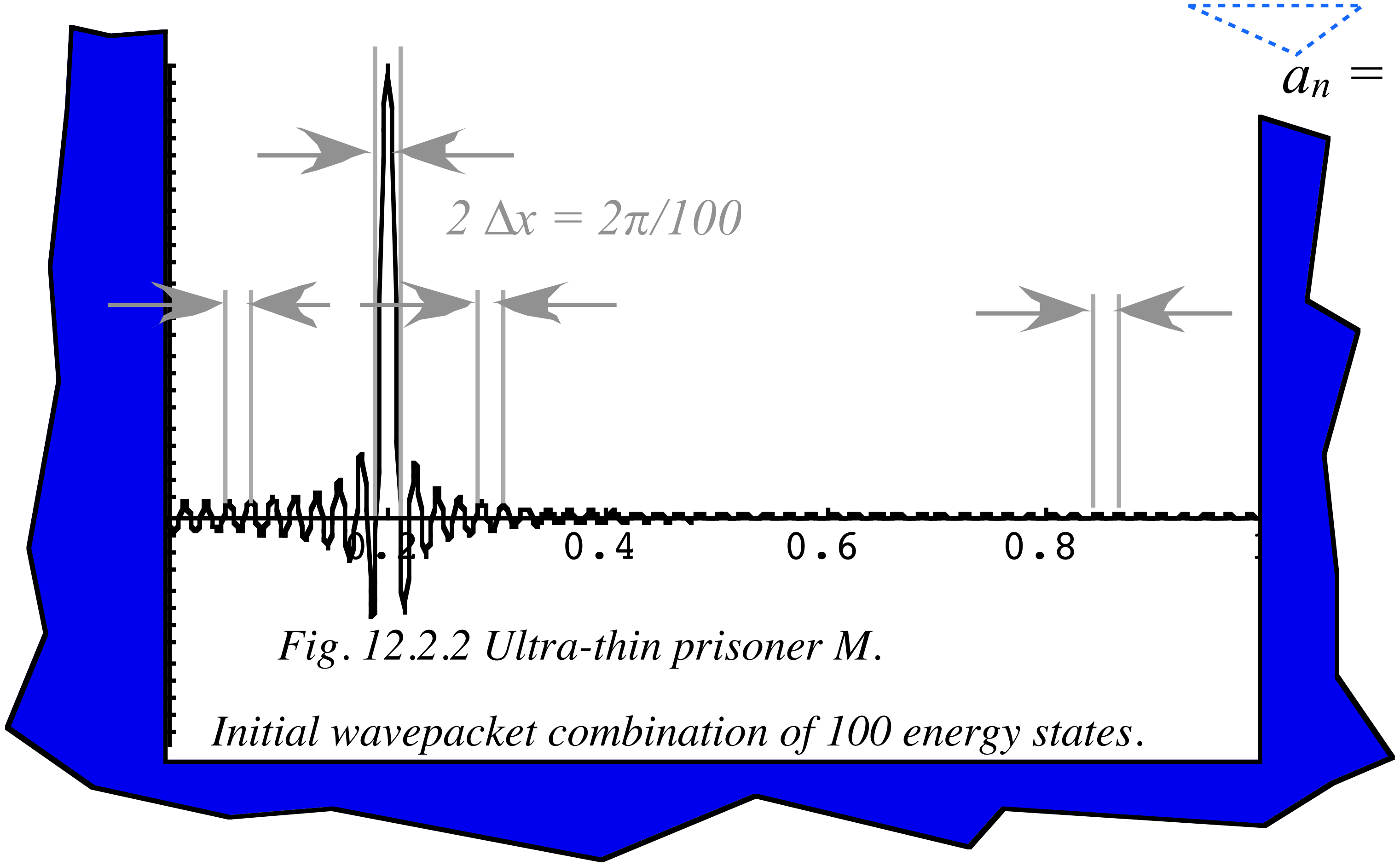
Initial wavepacket combination of 100 energy states.

From QTCA Unit 5 Ch. 12

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a) = \langle x|a\rangle = \sum_{n=1}^{\infty} \langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$

$a_n = \langle\varepsilon_n|a\rangle = (2/W) \sin k_n a \qquad (k_n = n\pi/W)$



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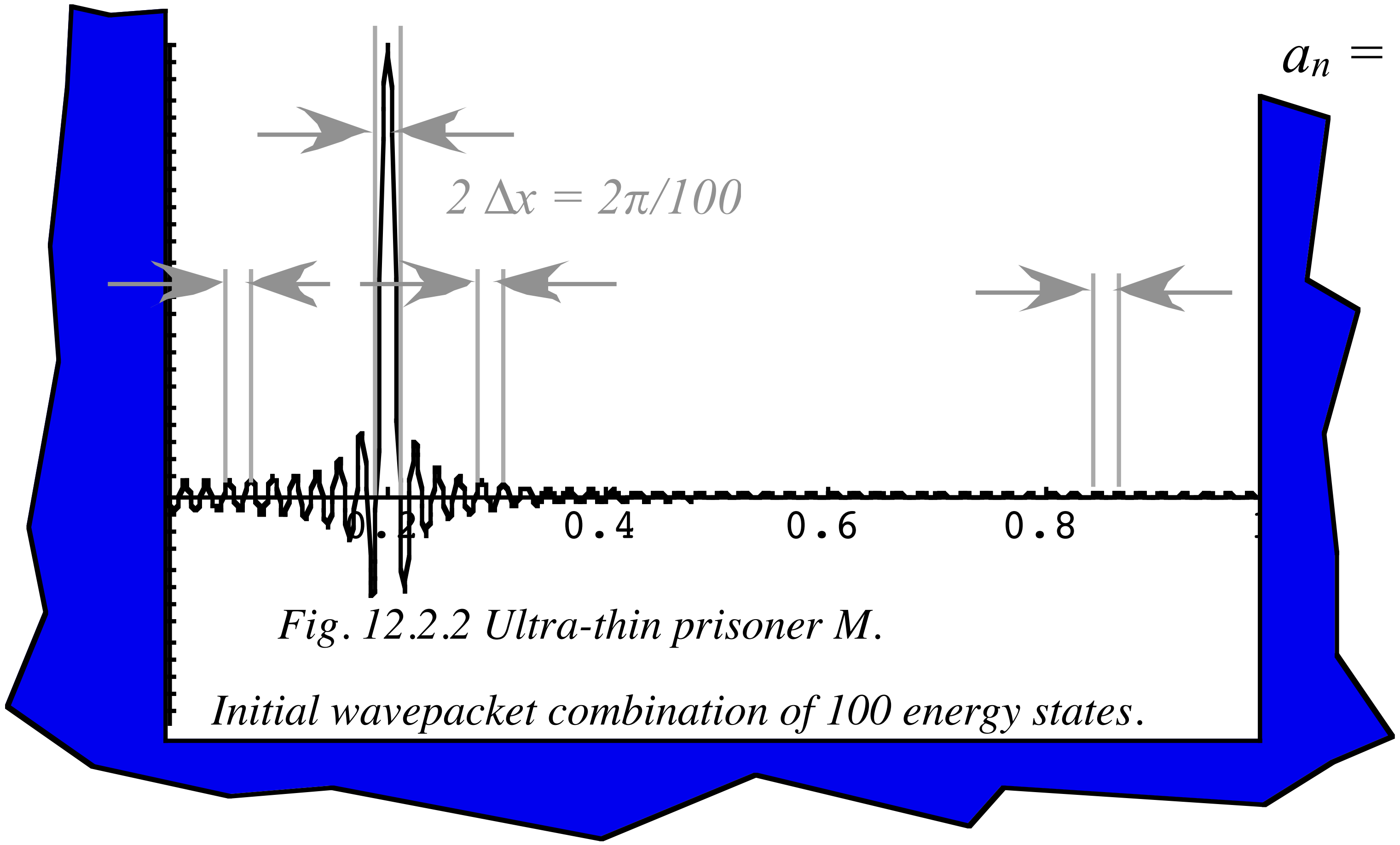

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From QTCA Unit 5 Ch. 12

SinNx/x wavepackets bandwidth and uncertainty

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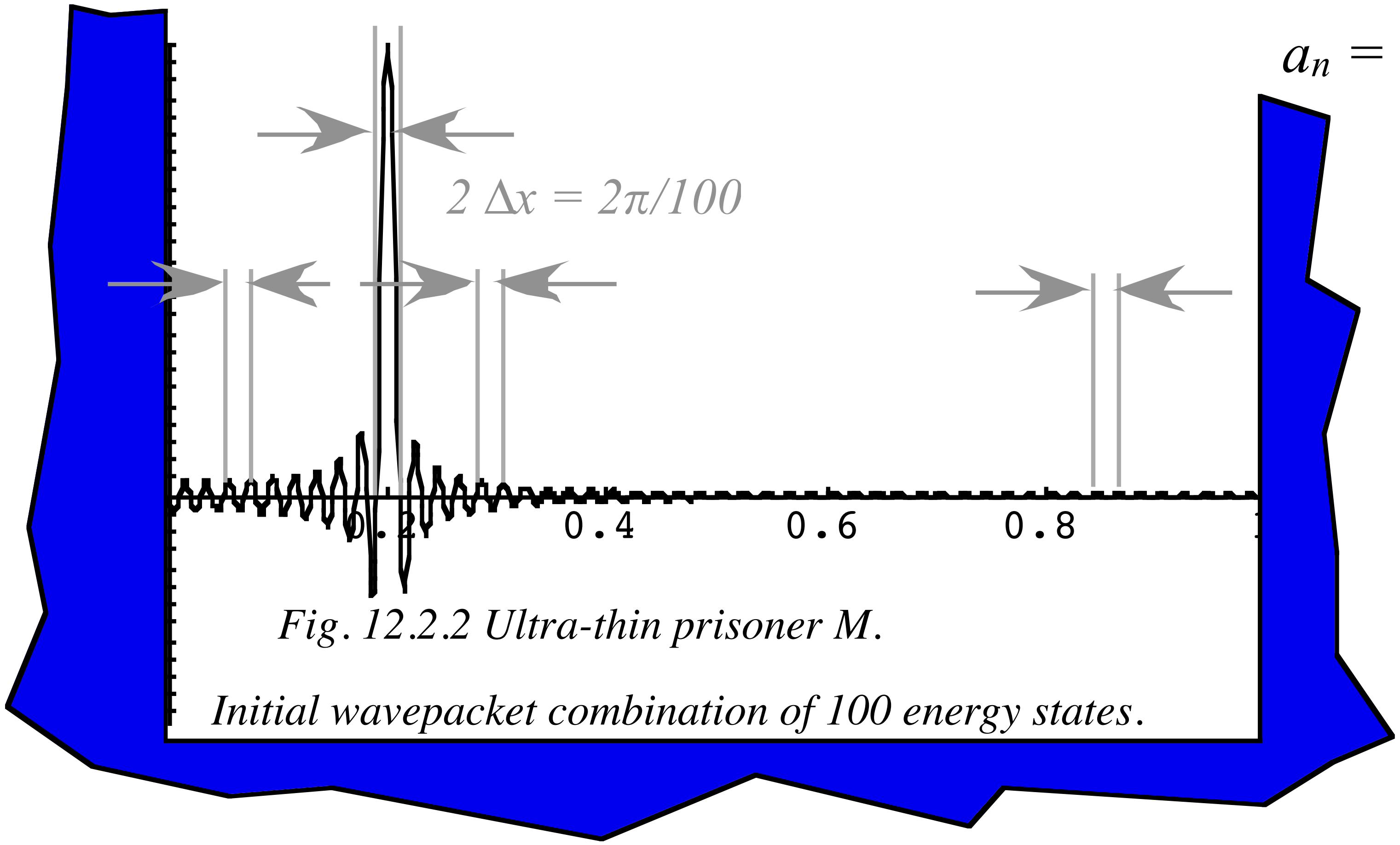

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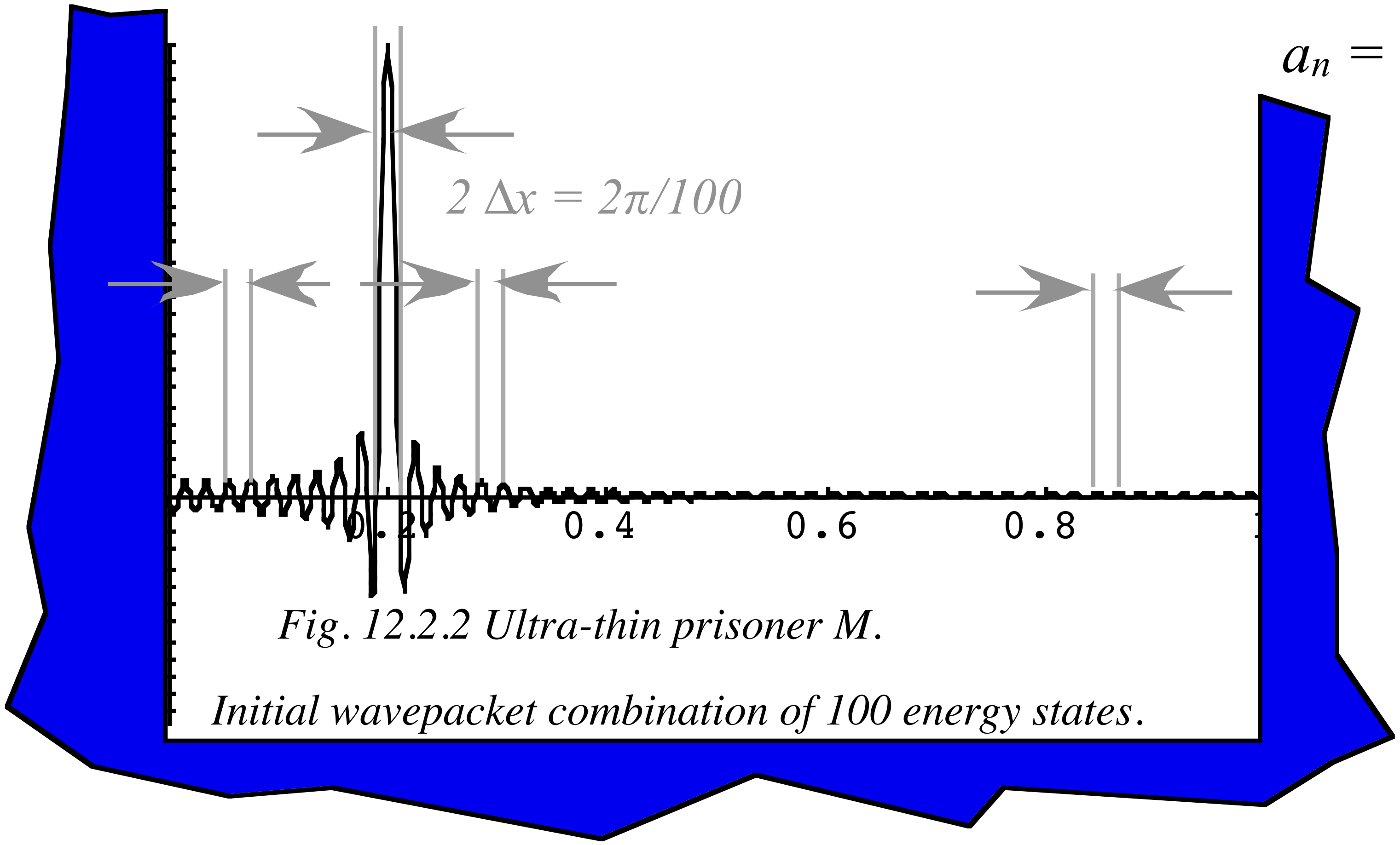

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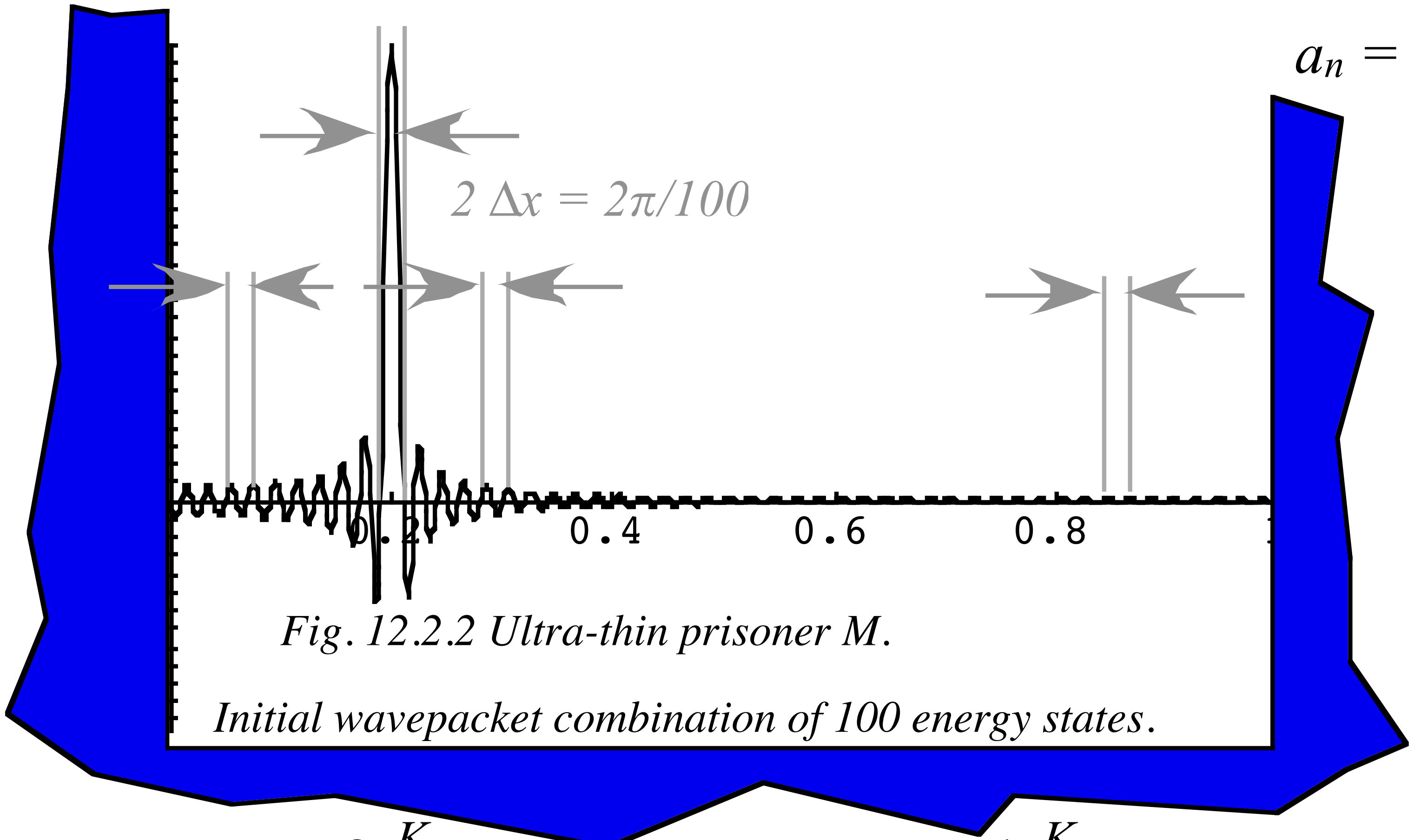

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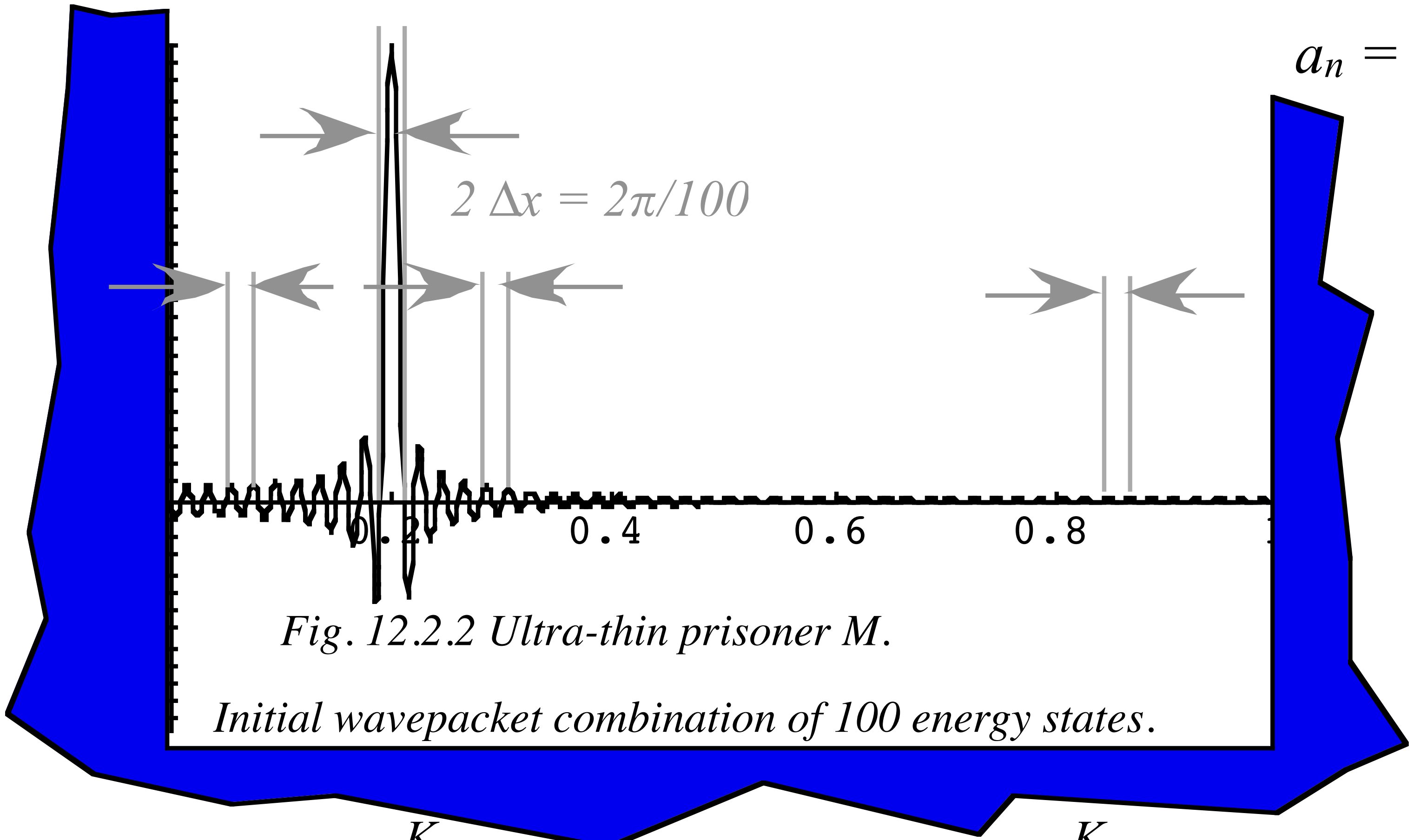

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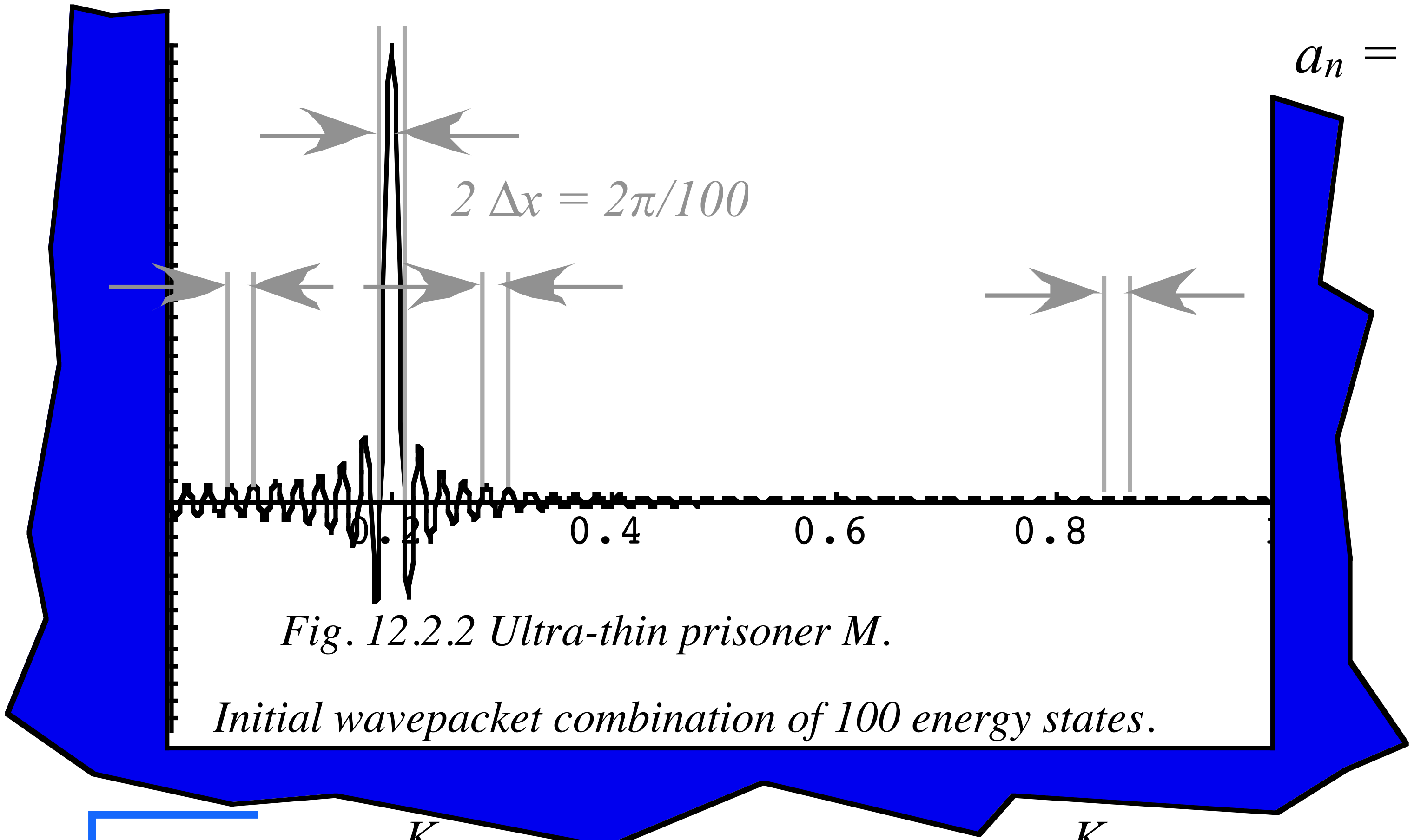

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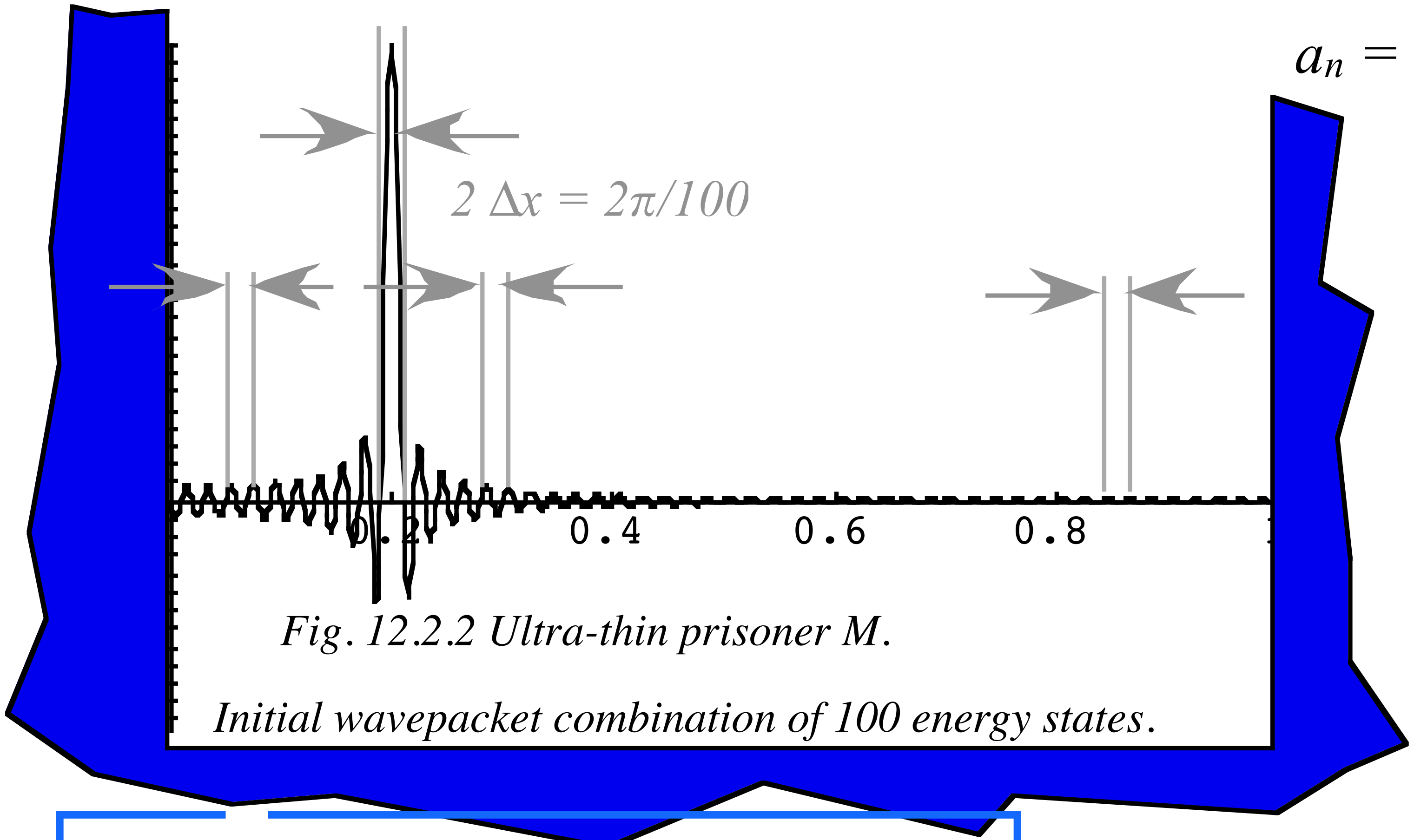

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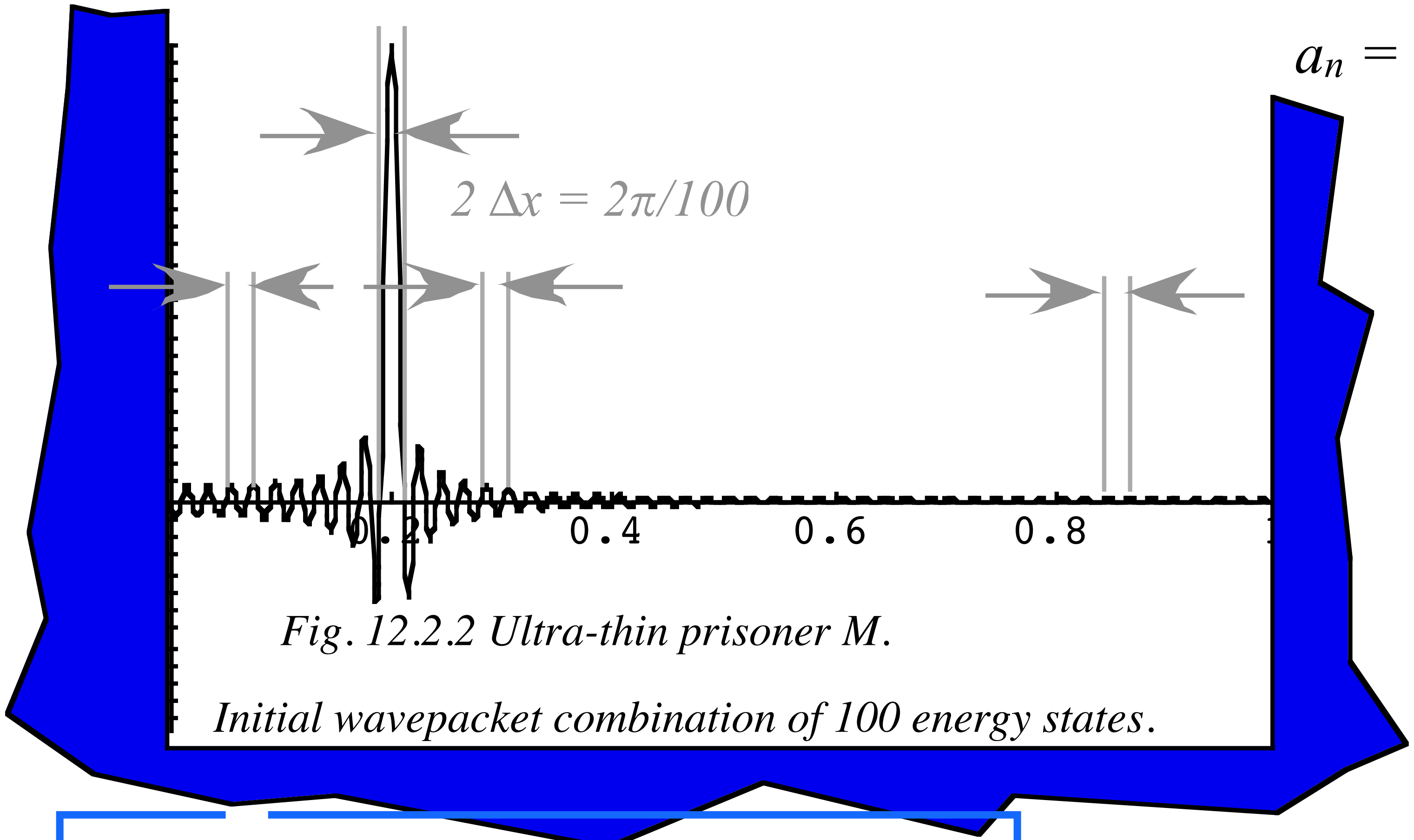

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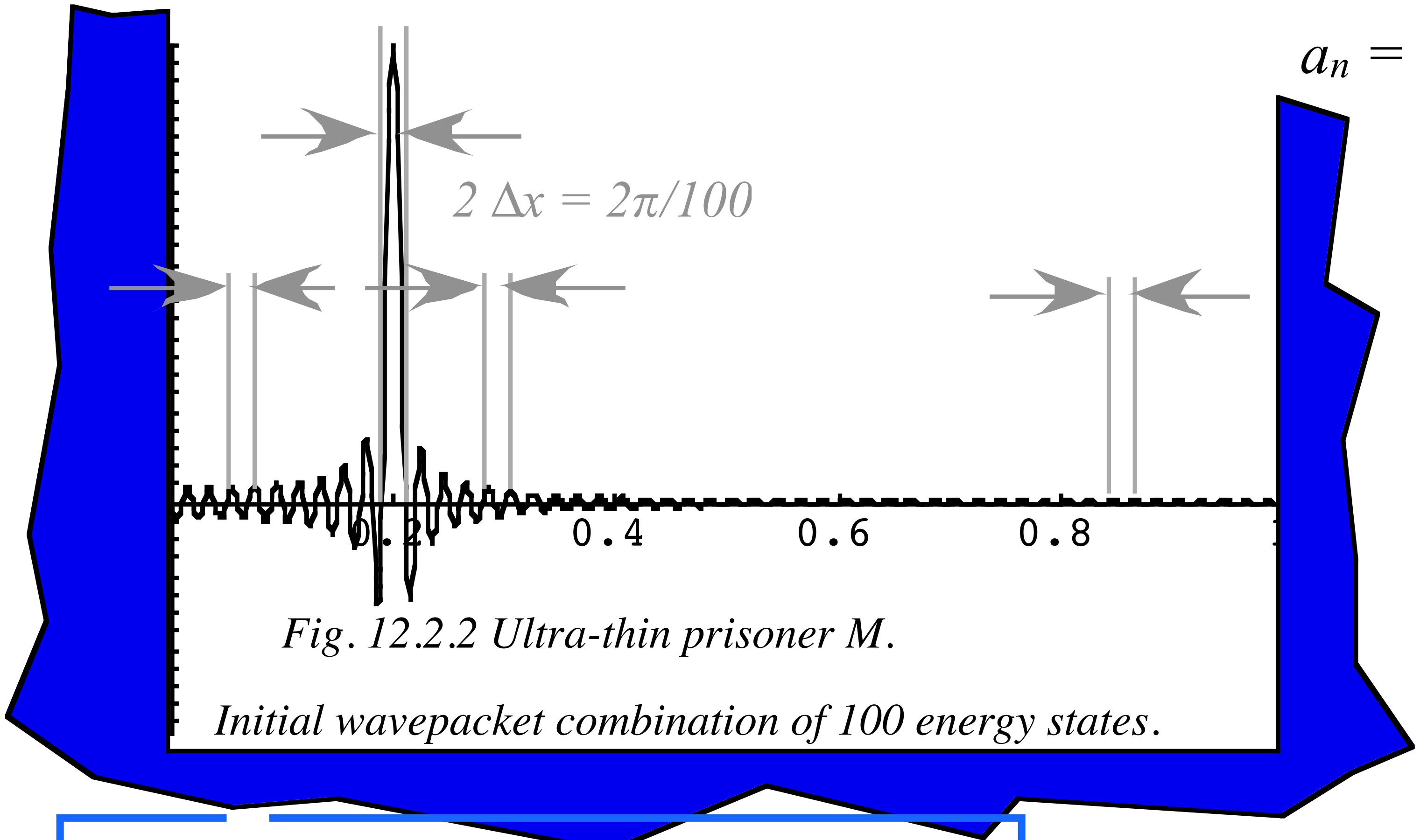

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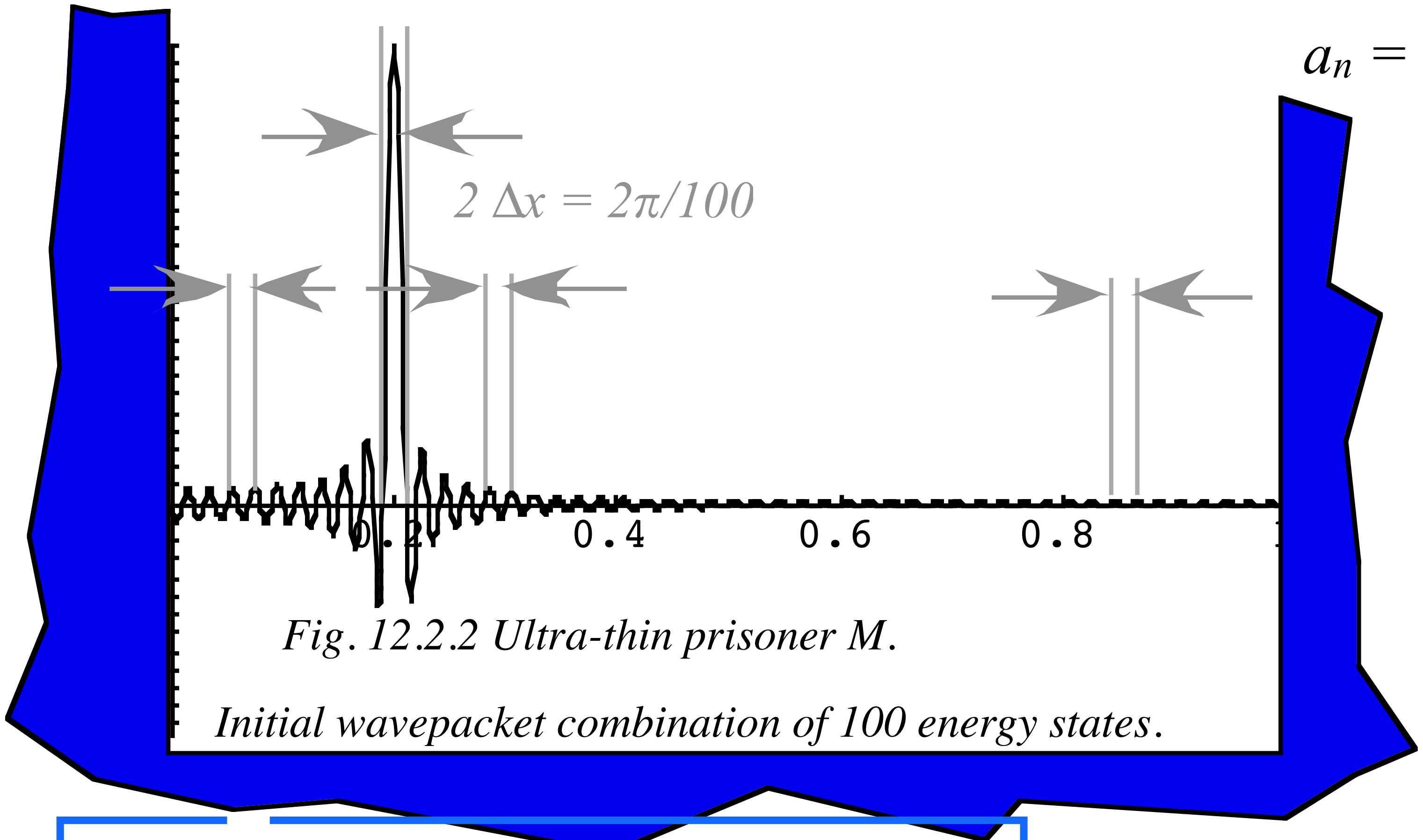

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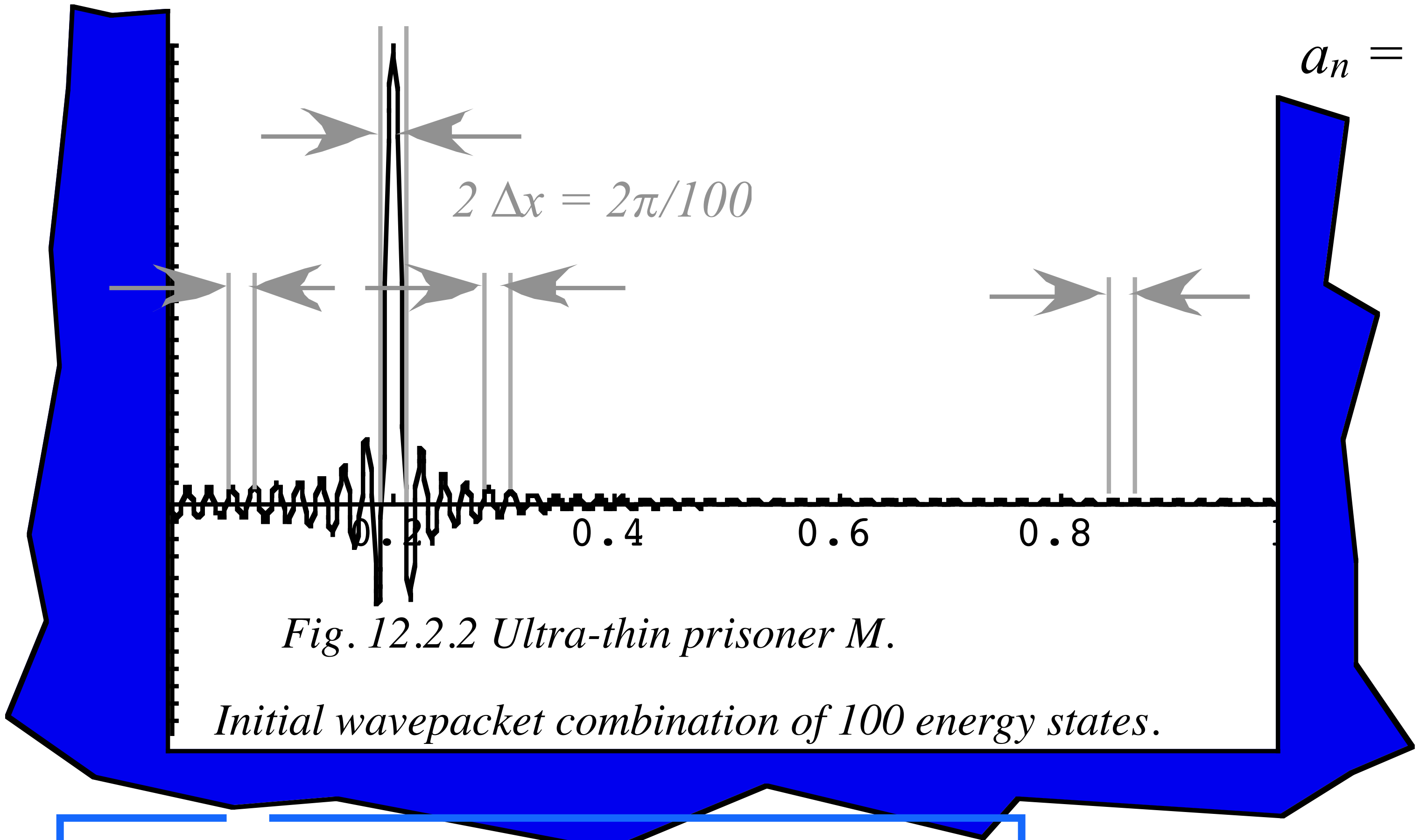

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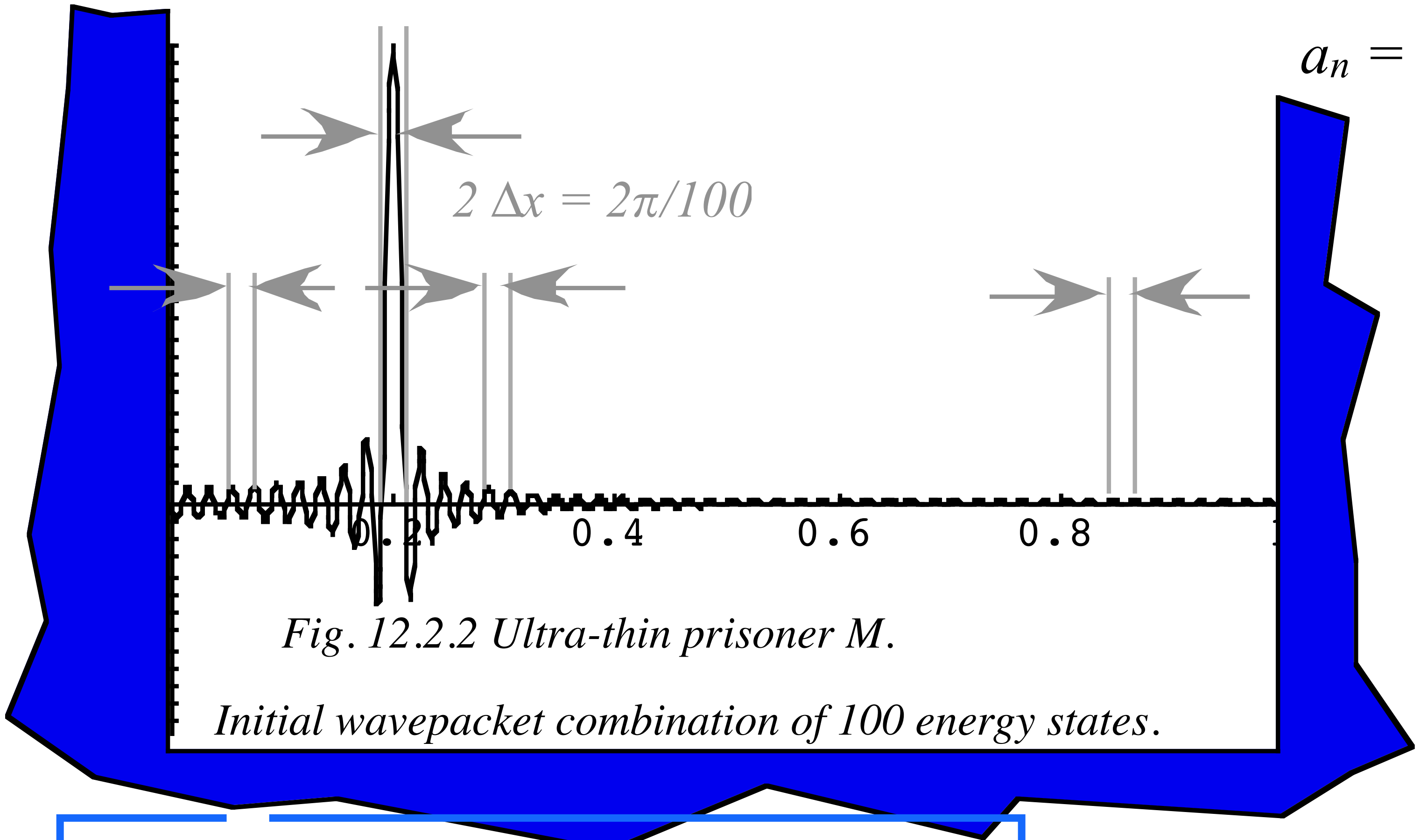

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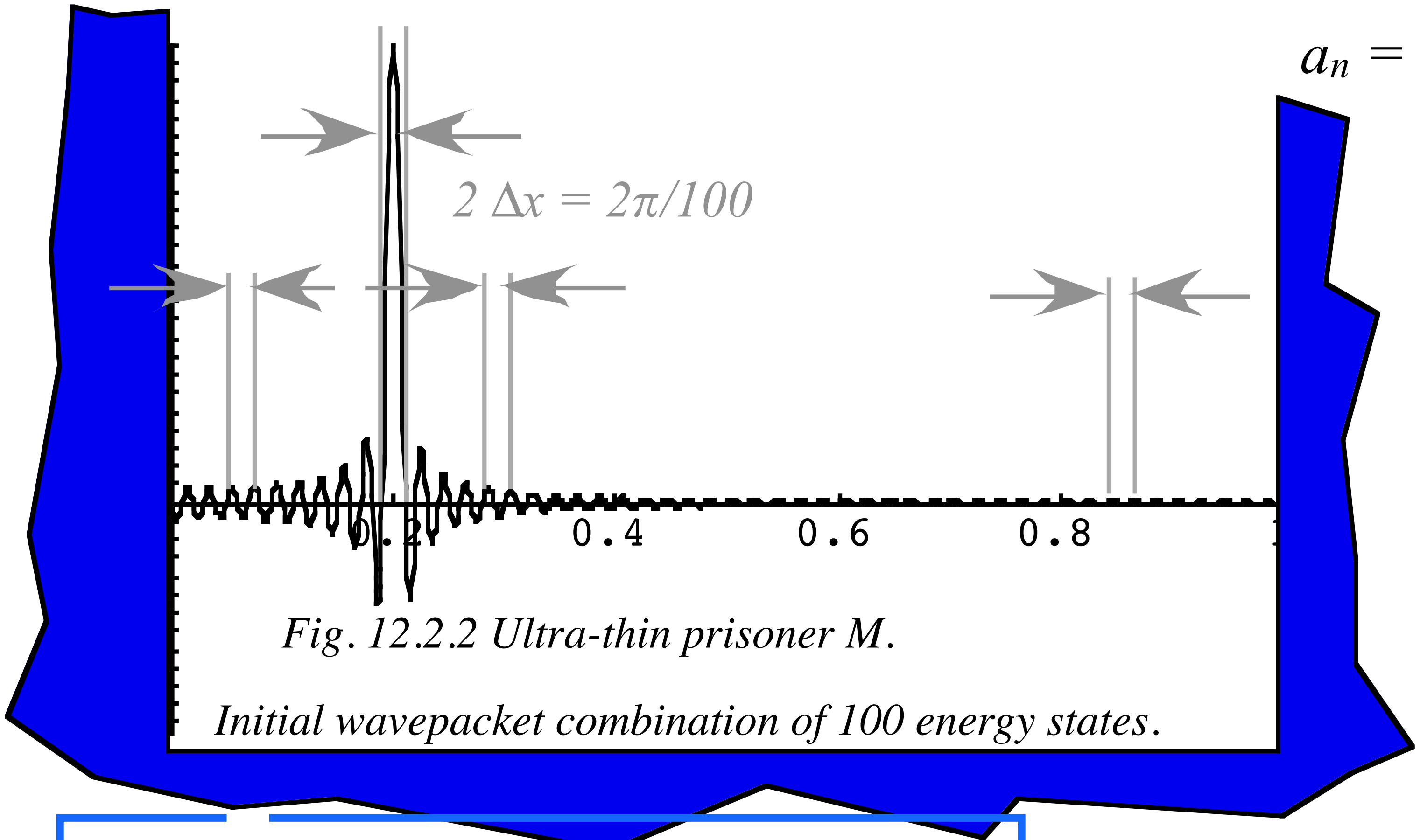

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$\Delta x \cdot \Delta\kappa = 1/2$ if $p = h\kappa$

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From QTCA Unit 5 Ch. 12

Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)

Comparing spacetime uncertainty (Δx or Δt) with per-spacetime bandwidth ($\Delta\kappa$ or $\Delta\upsilon$)

Introduction to beat dynamics and "Revivals" due to Bohr-dispersion

Relating ∞-Square-well waves to Bohr rotor waves

∞-Square-well wave dynamics

SinNx/x wavepacket bandwidth and uncertainty

→ *∞-Square-well revivals: SinNx/x packet explodes! (and then UNexplodes!)*

Bohr-rotor wave dynamics

Gaussian wave-packet bandwidth and uncertainty

Gaussian Bohr-rotor revivals and quantum fractals

Understanding fractals using geometry of fractions (Rationalizing rationals)

Farey-Sums and Ford-products

Discrete C_N beat phase dynamics (Characters gone wild!)

The classical bouncing-ball Monster-Mash

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Algebra

Geometry

Wavepacket explodes!

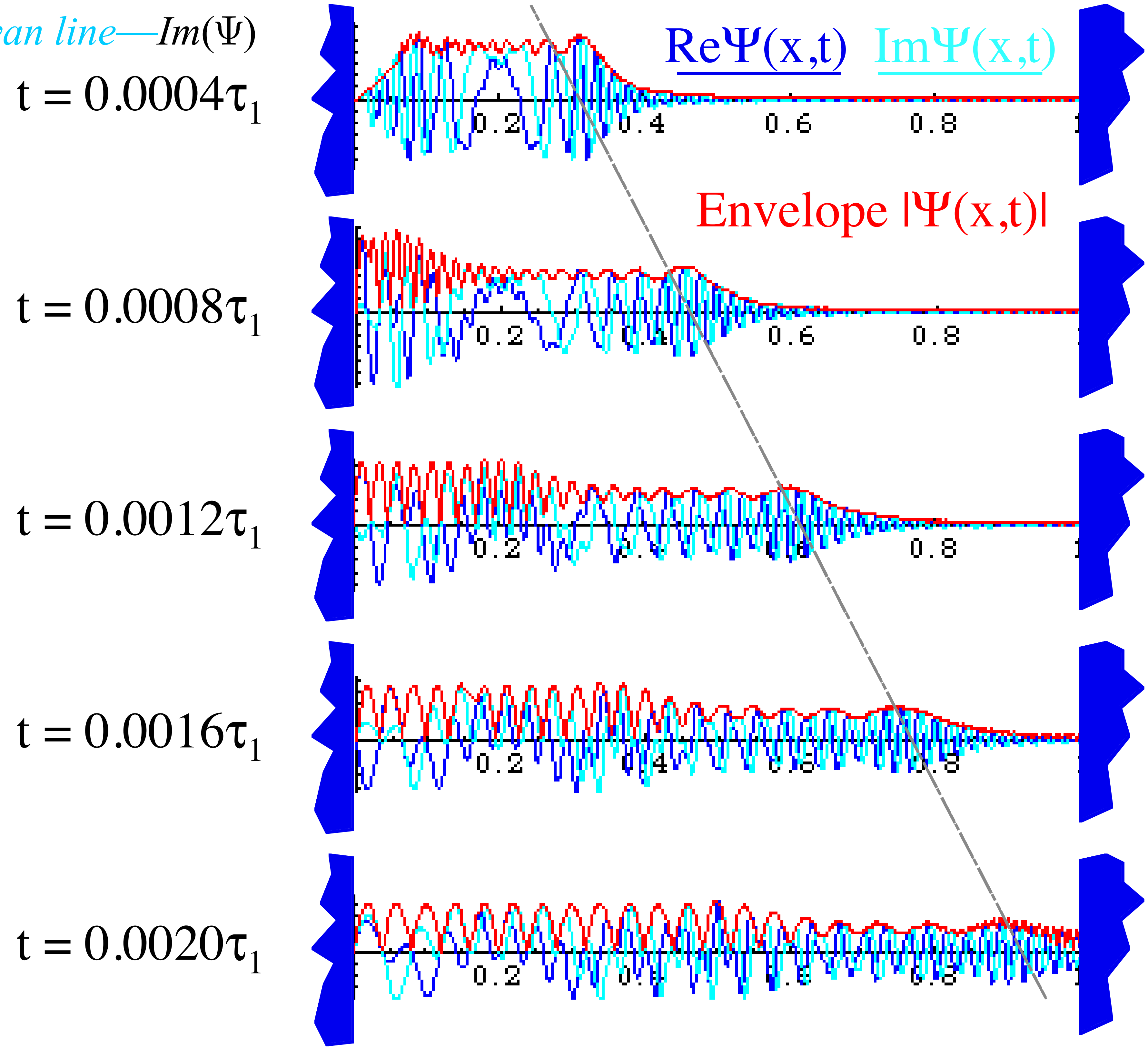
red line—— $|\Psi|$

blue line—$Re(\Psi)$

cyan line—$Im(\Psi)$

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$



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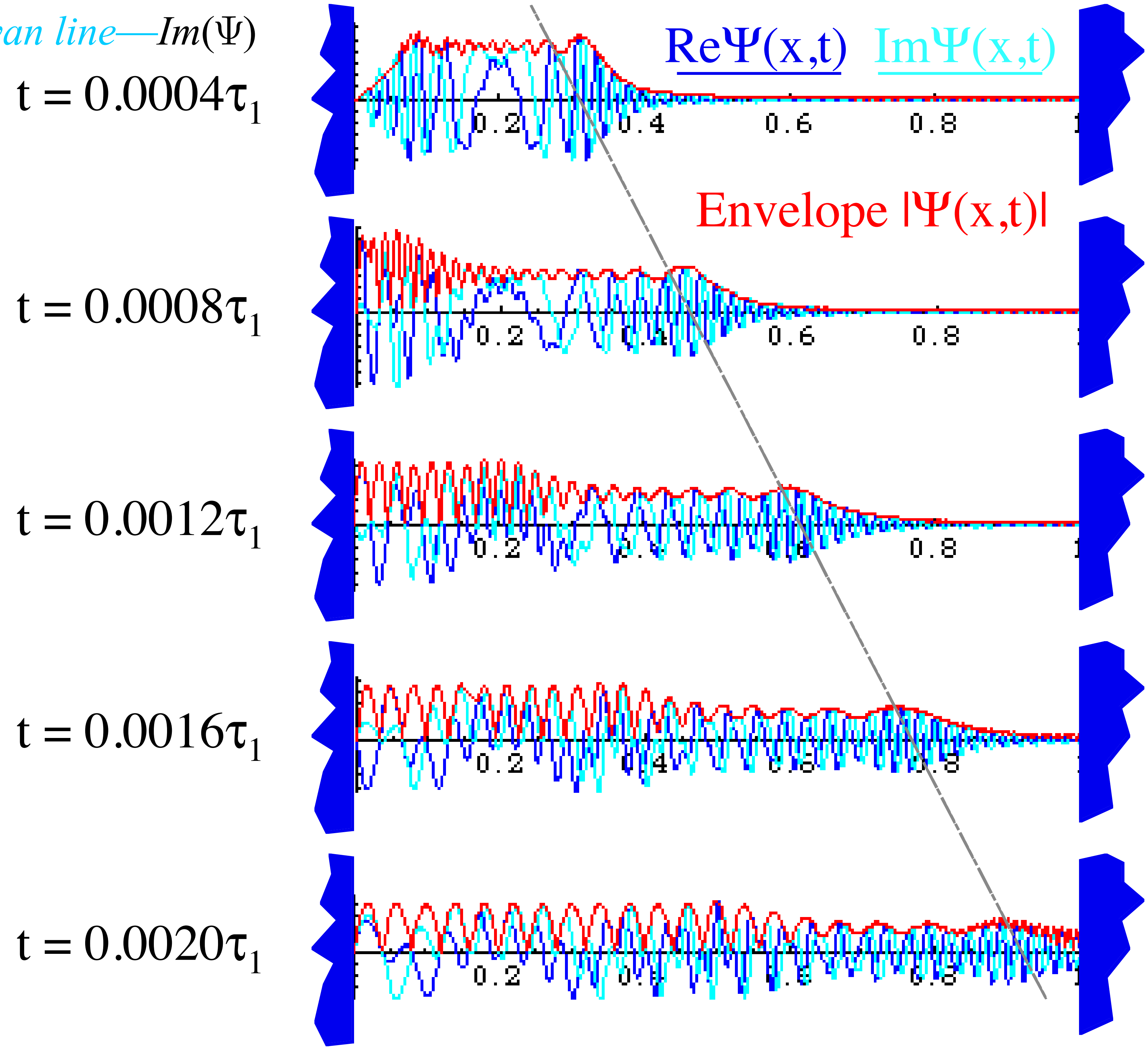

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ε_n-*level classical velocity*:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

$t = 0.0004\tau_1$

$t = 0.0008\tau_1$

$t = 0.0012\tau_1$

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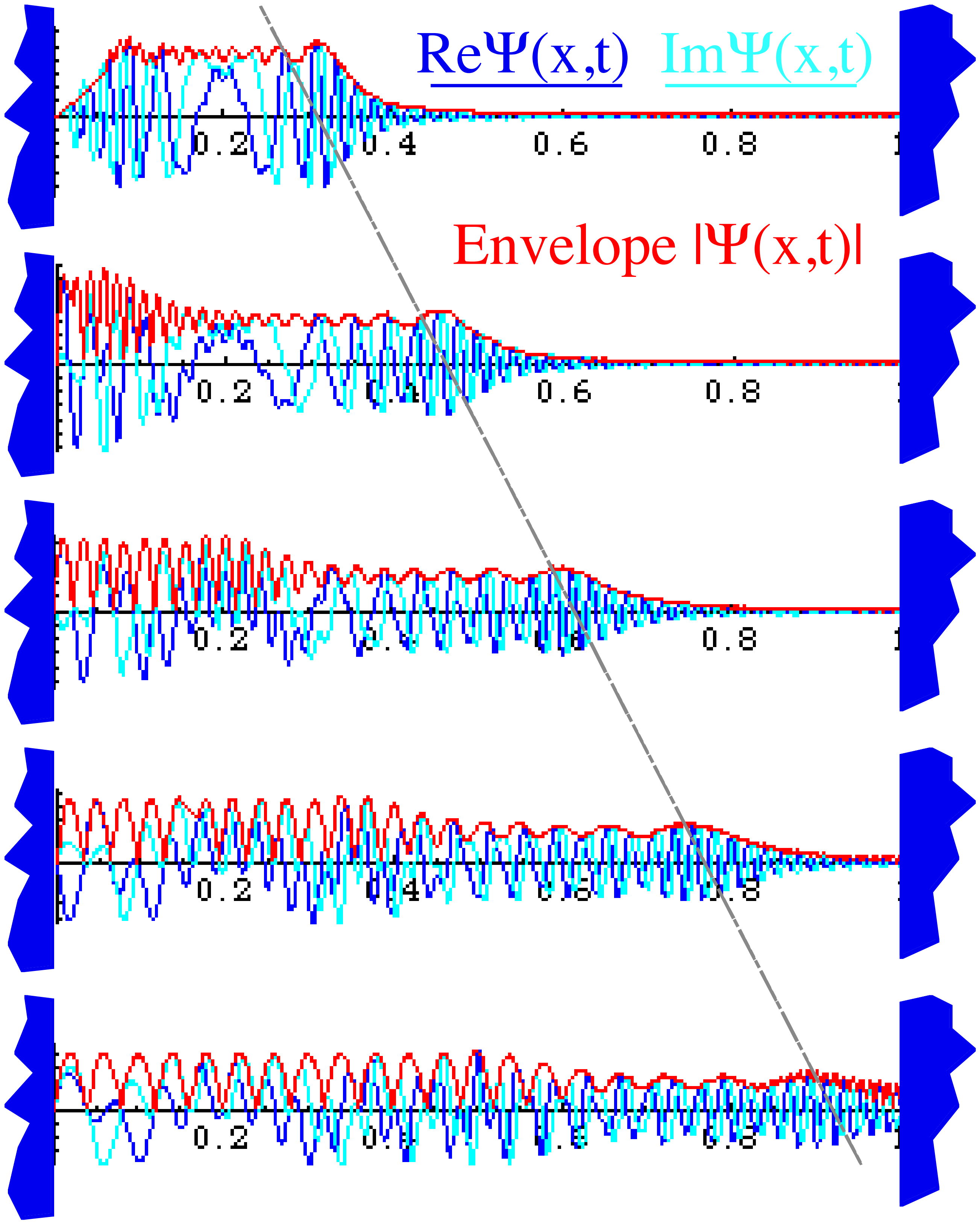

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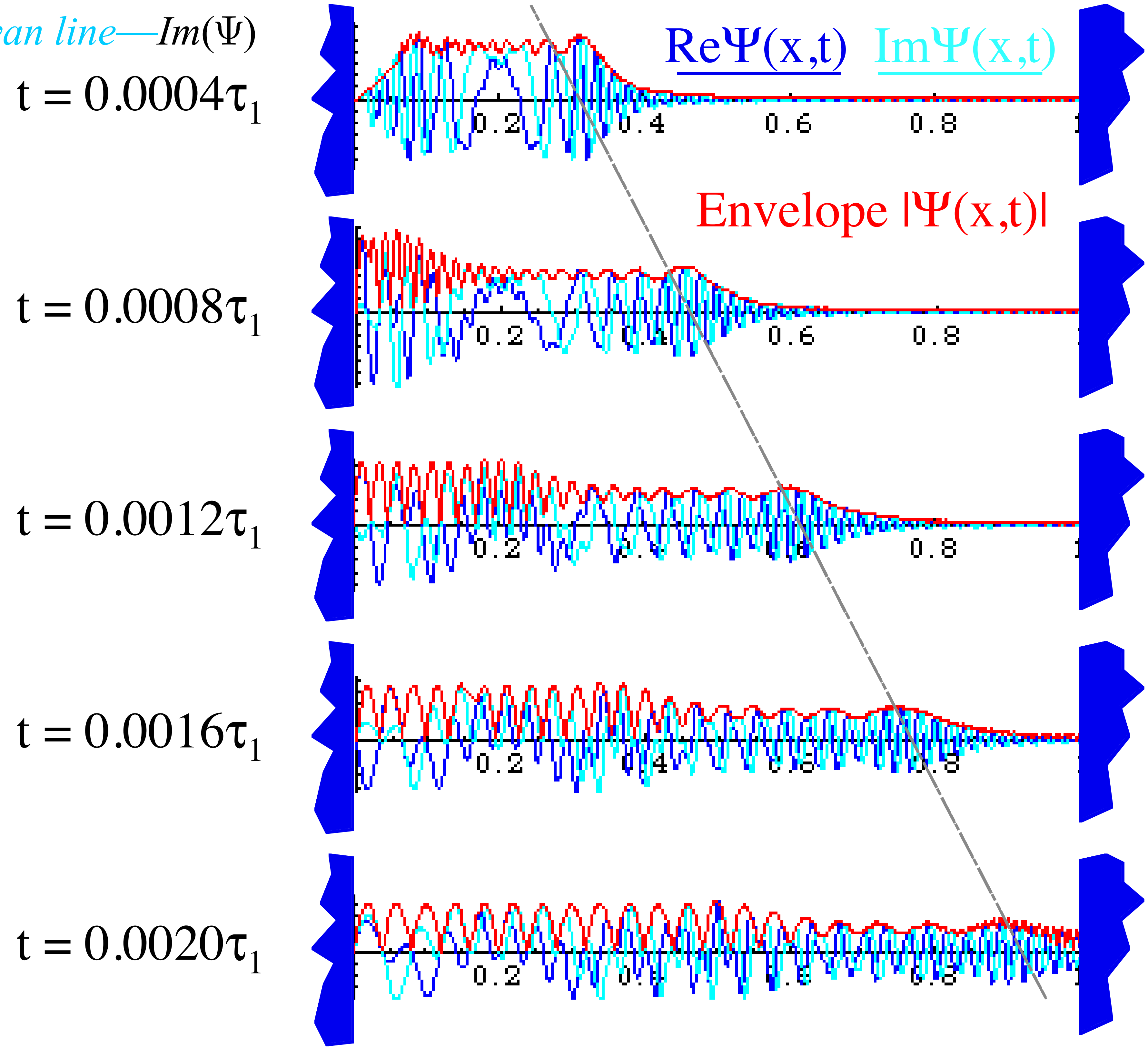

ε_n-level classical velocity:

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ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn} = \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

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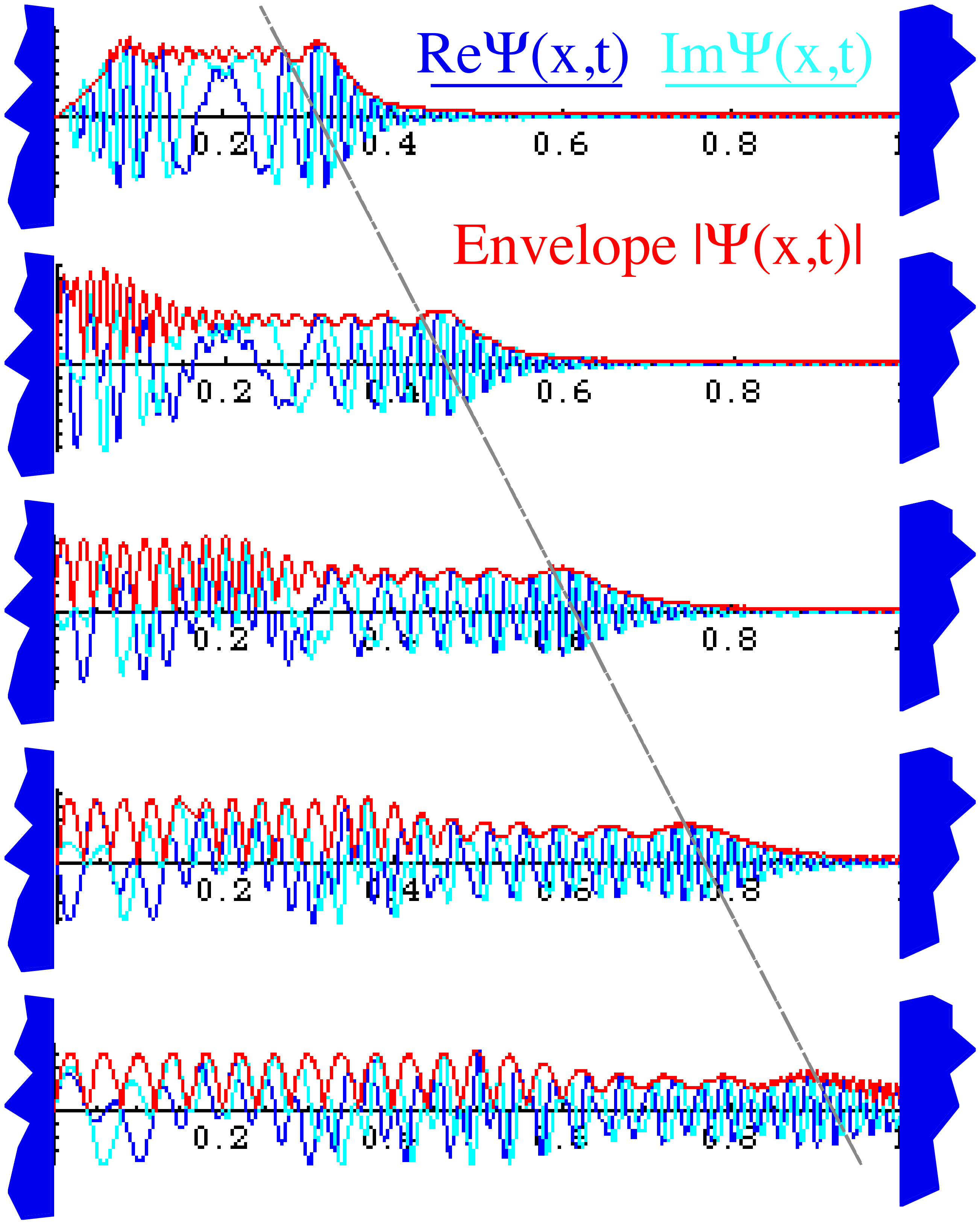

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$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk} = \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk} = \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn} = \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n\text{=}100\right)$$

Wavepacket explodes!

red line—— $|\Psi|$

blue line—*Re*(Ψ)

cyan line—*Im*(Ψ)

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1} = \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$

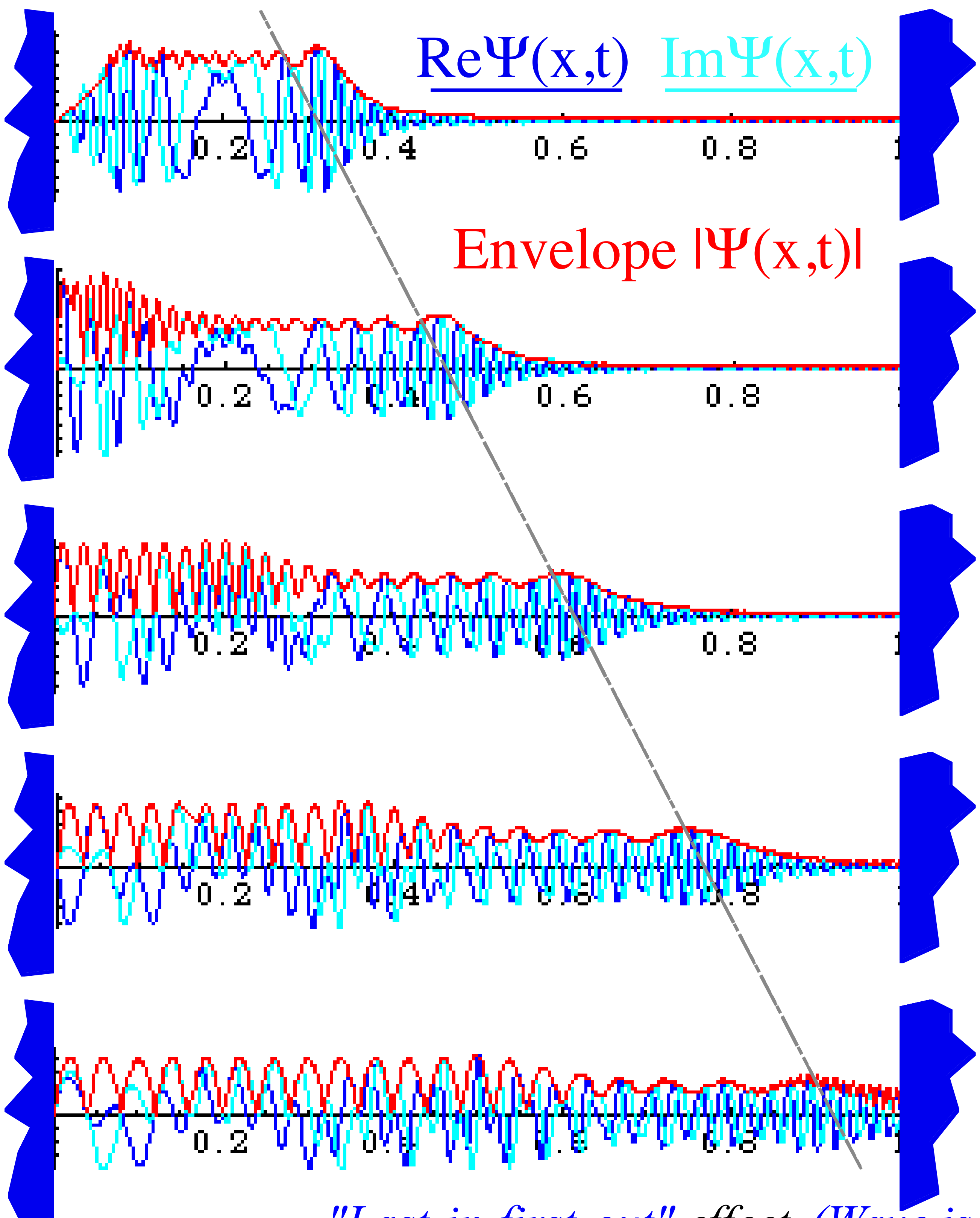
$t = 0.0004\tau_1$

$t = 0.0008\tau_1$

$t = 0.0012\tau_1$

$t = 0.0016\tau_1$

$t = 0.0020\tau_1$



ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n{=}100\right)$$

"Last-in-first-out" effect *(Wave is mostly wrinkles from n=100)*

Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)

Comparing spacetime uncertainty (Δx or Δt) with per-spacetime bandwidth ($\Delta\kappa$ or $\Delta\upsilon$)

Introduction to beat dynamics and "Revivals" due to Bohr-dispersion

Relating ∞-Square-well waves to Bohr rotor waves

∞-Square-well wave dynamics

SinNx/x wavepacket bandwidth and uncertainty

→ *∞-Square-well revivals: SinNx/x packet explodes! (and then UNexplodes!)*

Bohr-rotor wave dynamics

Gaussian wave-packet bandwidth and uncertainty

Gaussian Bohr-rotor revivals and quantum fractals

Understanding fractals using geometry of fractions (Rationalizing rationals)

Farey-Sums and Ford-products

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The classical bouncing-ball Monster-Mash

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Algebra

Geometry

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

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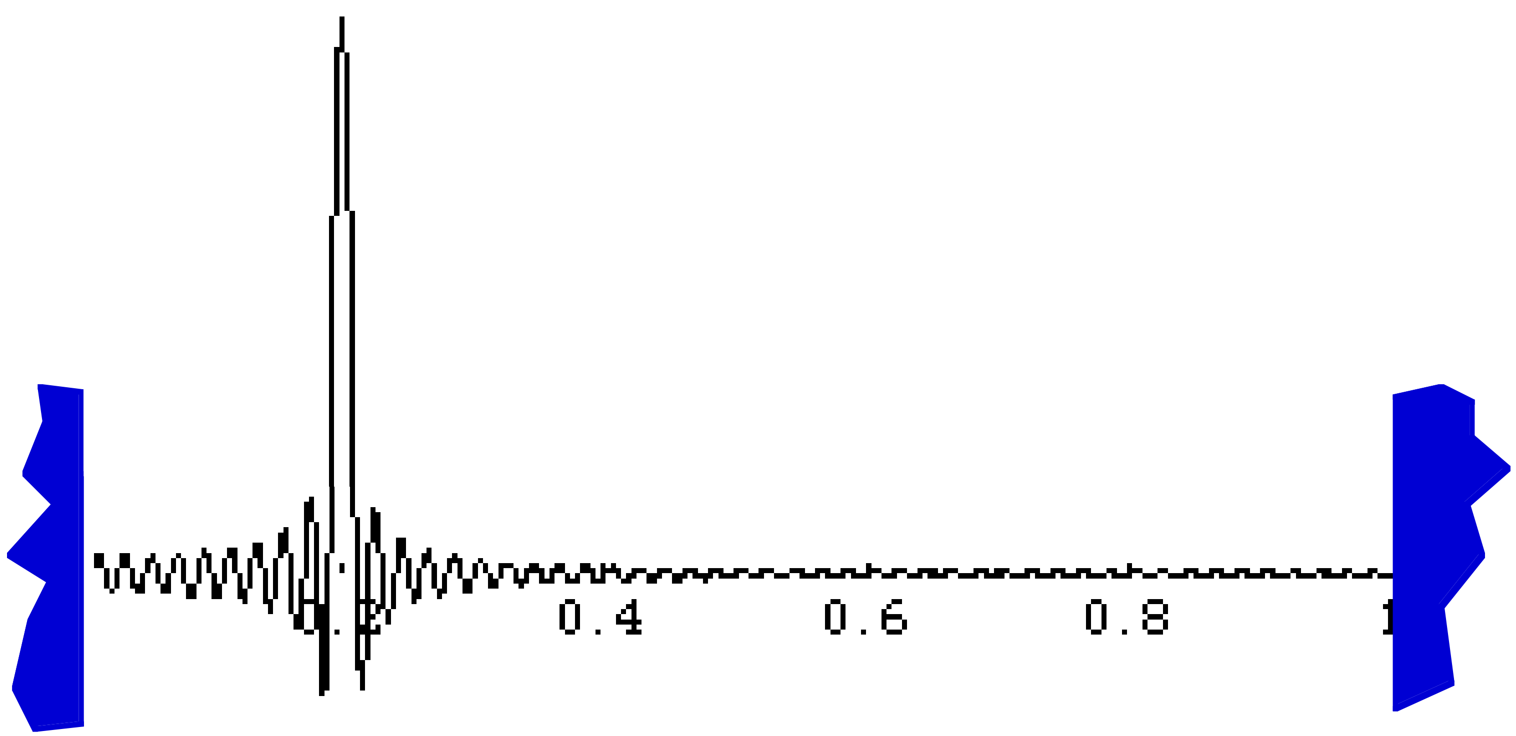
ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

$t = 1.0000\tau_1$

$= 3.0\tau_{beat}$



Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

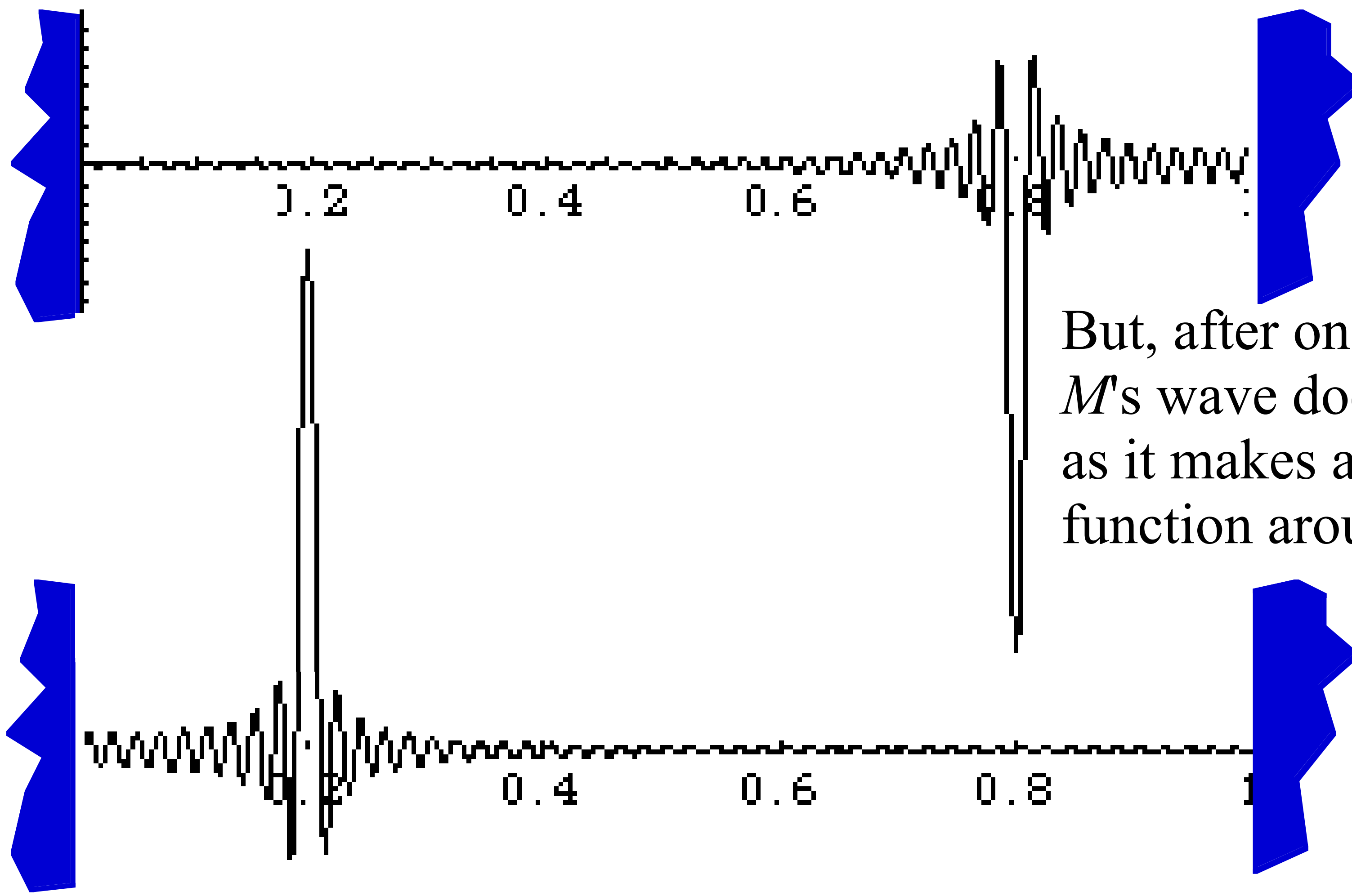
ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at *x=0.2W*,

$t = 0.5000\tau_1$

$= 1.5\tau_{beat}$



But, after only 50 round-trips *M*'s wave does a *partial revival* as it makes an upside down-delta function around *x=0.8W*.

$t = 1.0000\tau_1$

$= 3.0\tau_{beat}$

At fractional times τ_1/n *M* undergoes a number of *fractional revivals*

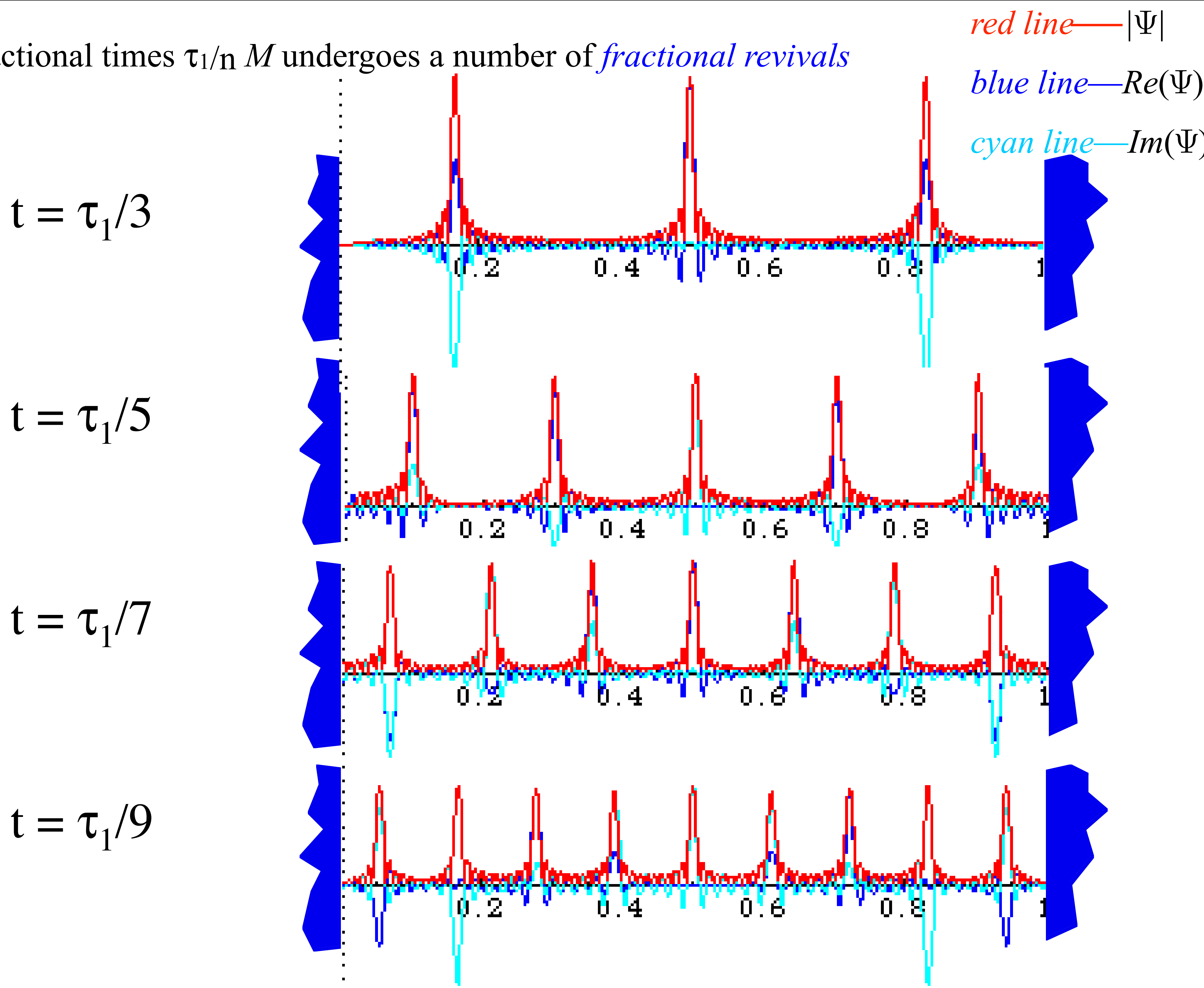


Fig. 12.2.5 The "Dance of the deltas." Mini-Revivals for prisoner M's wavepacket envelope function.

Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)

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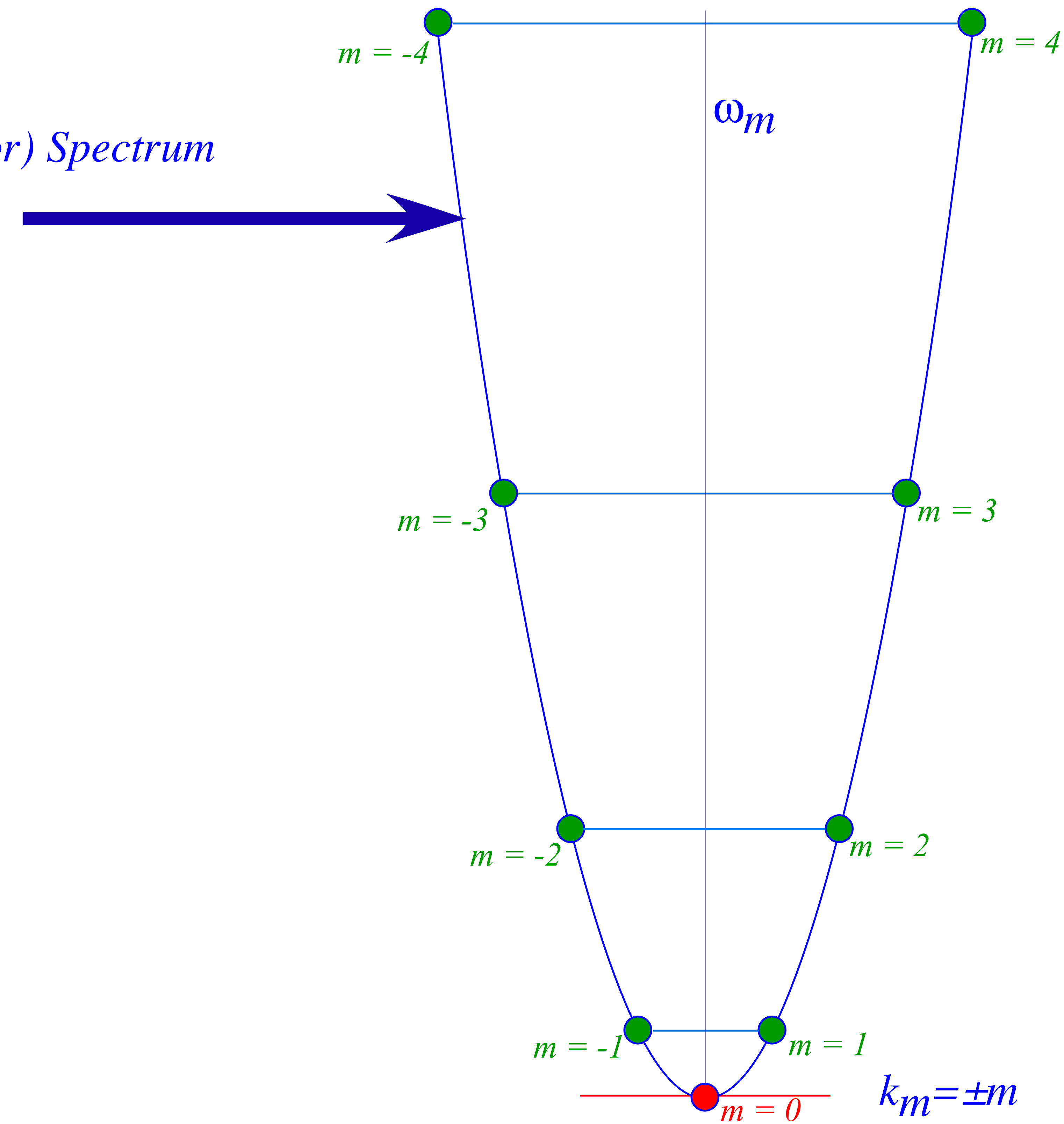
Geometry

Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$



Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

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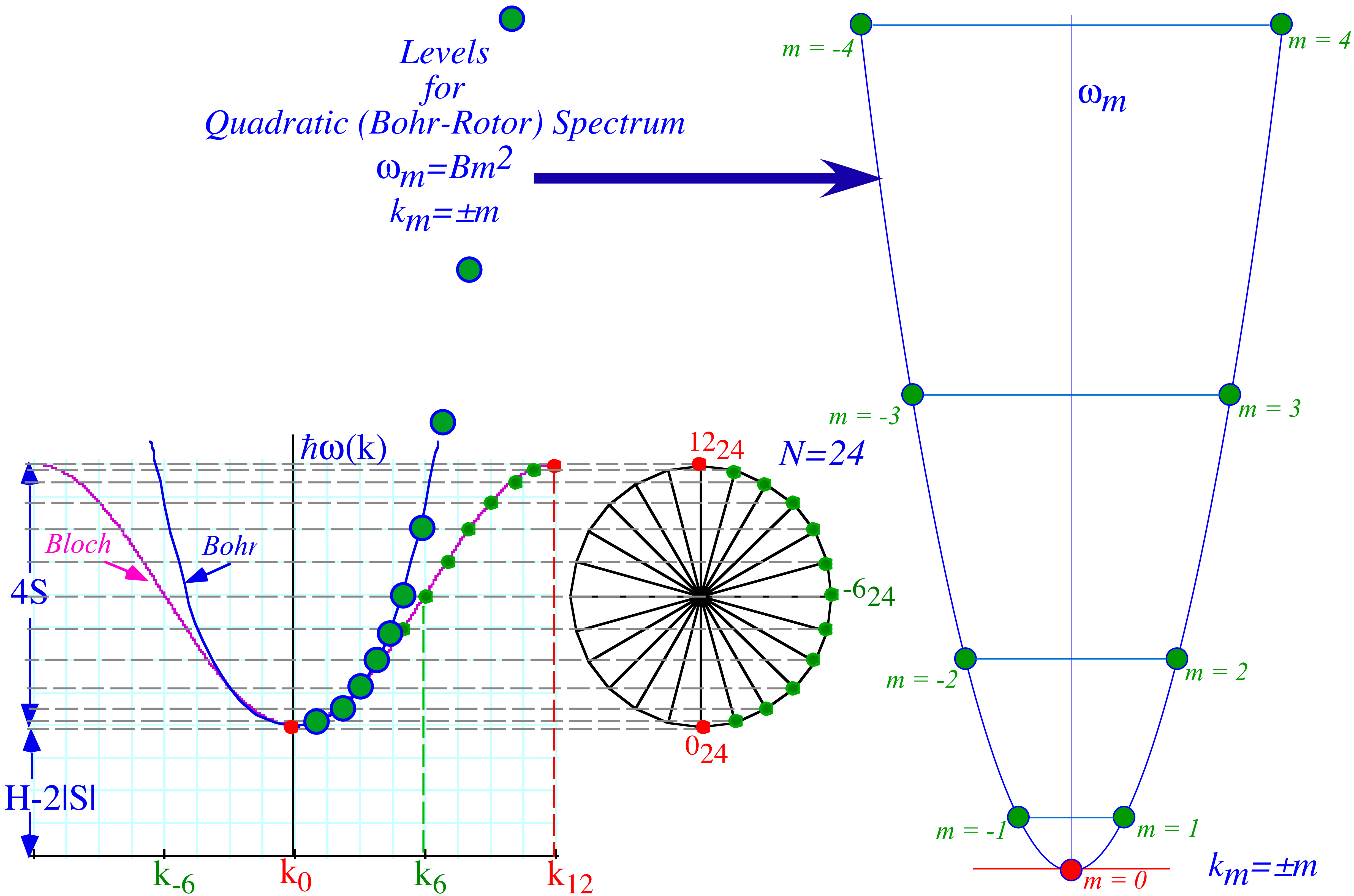
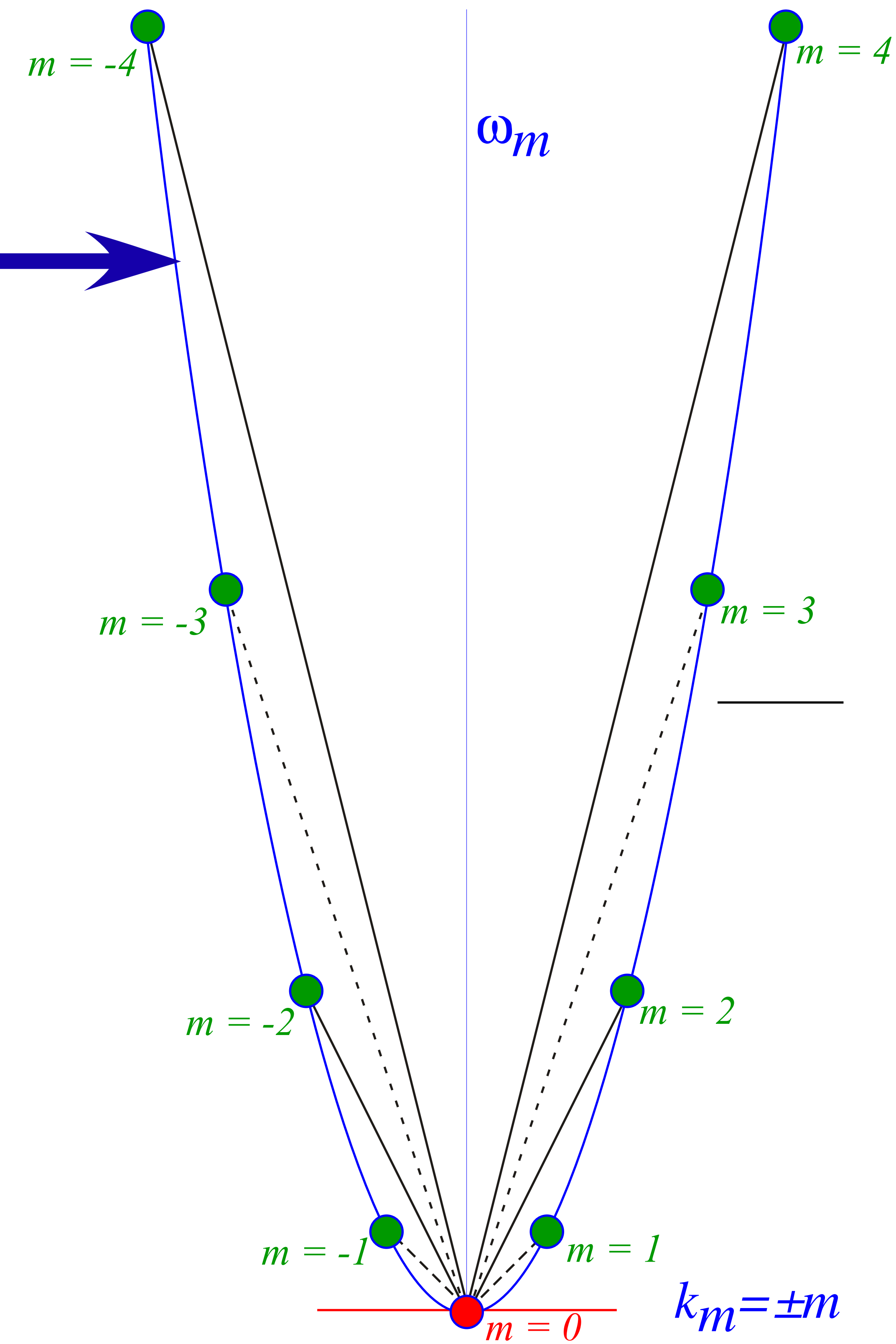


Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$
$$k_m = \pm m$$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



Possible wave velocities
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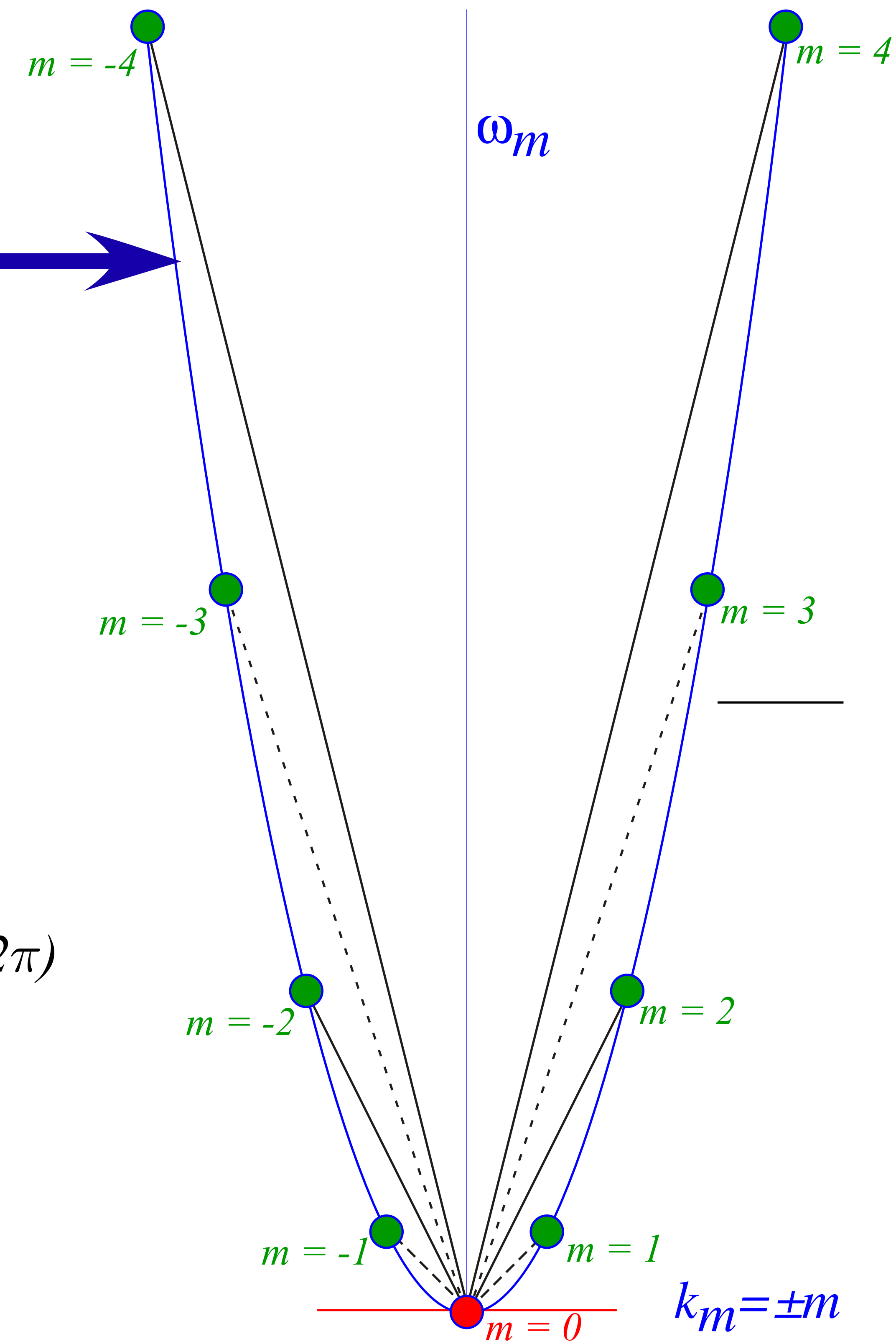


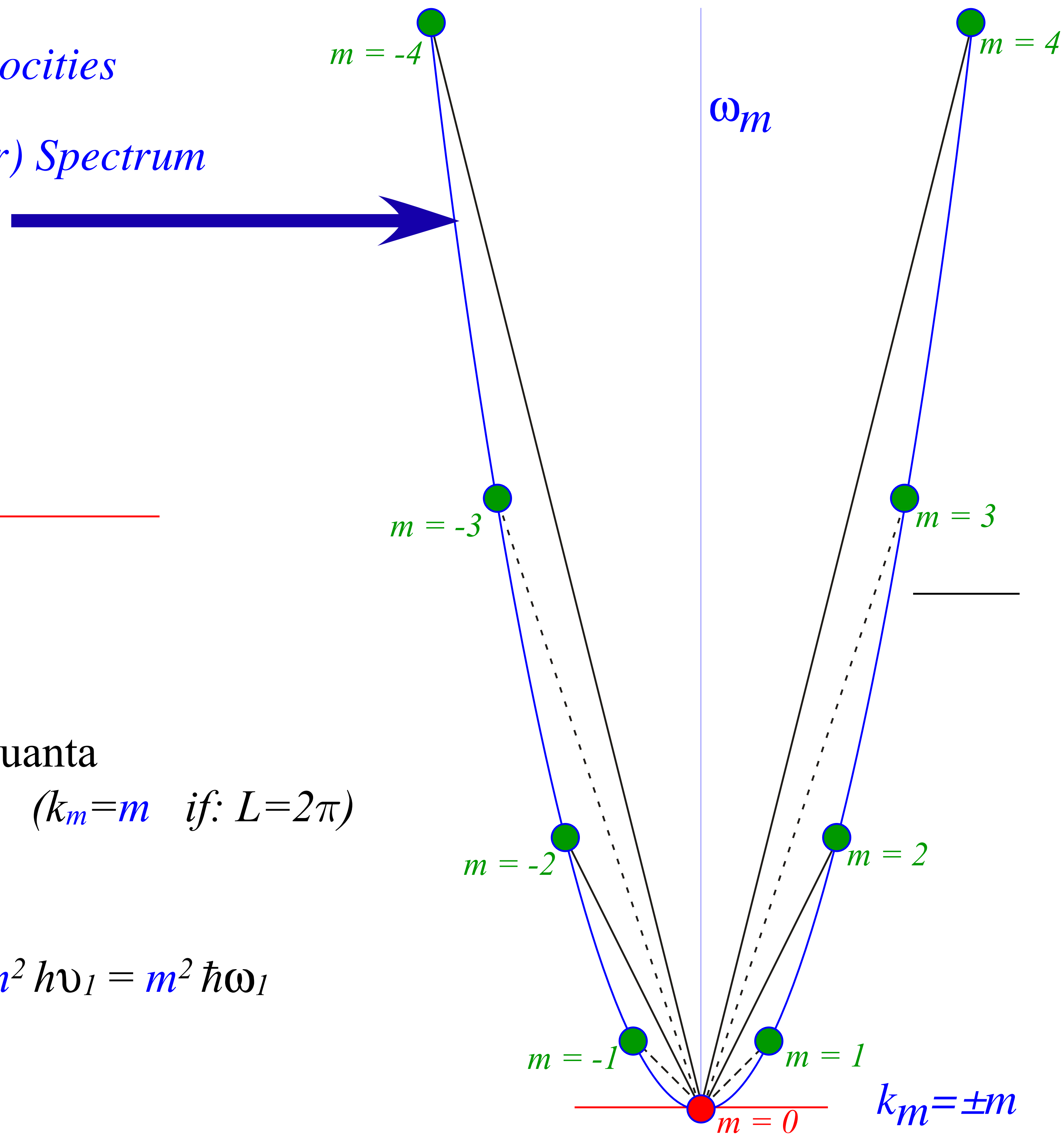
$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m \quad if: L = 2\pi)$

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$

$$k_m = \pm m$$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $(k_m = m$ *if:* $L = 2\pi)$

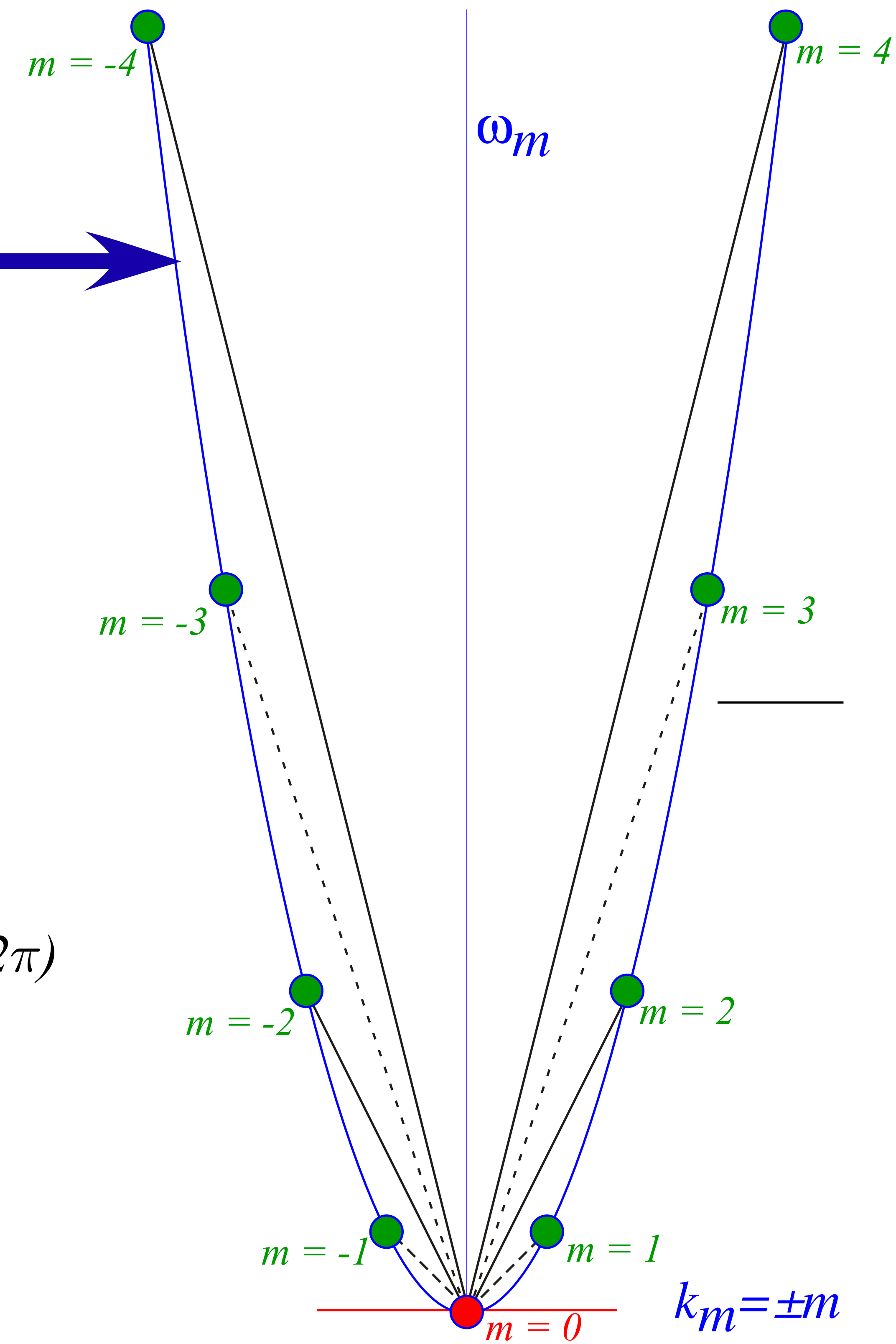
$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ ($k_m = m$ *if:* $L = 2\pi$)

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$
and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$

$V_{group} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{m^2 - n^2}{m \pm n} B = (m \pm n)B$

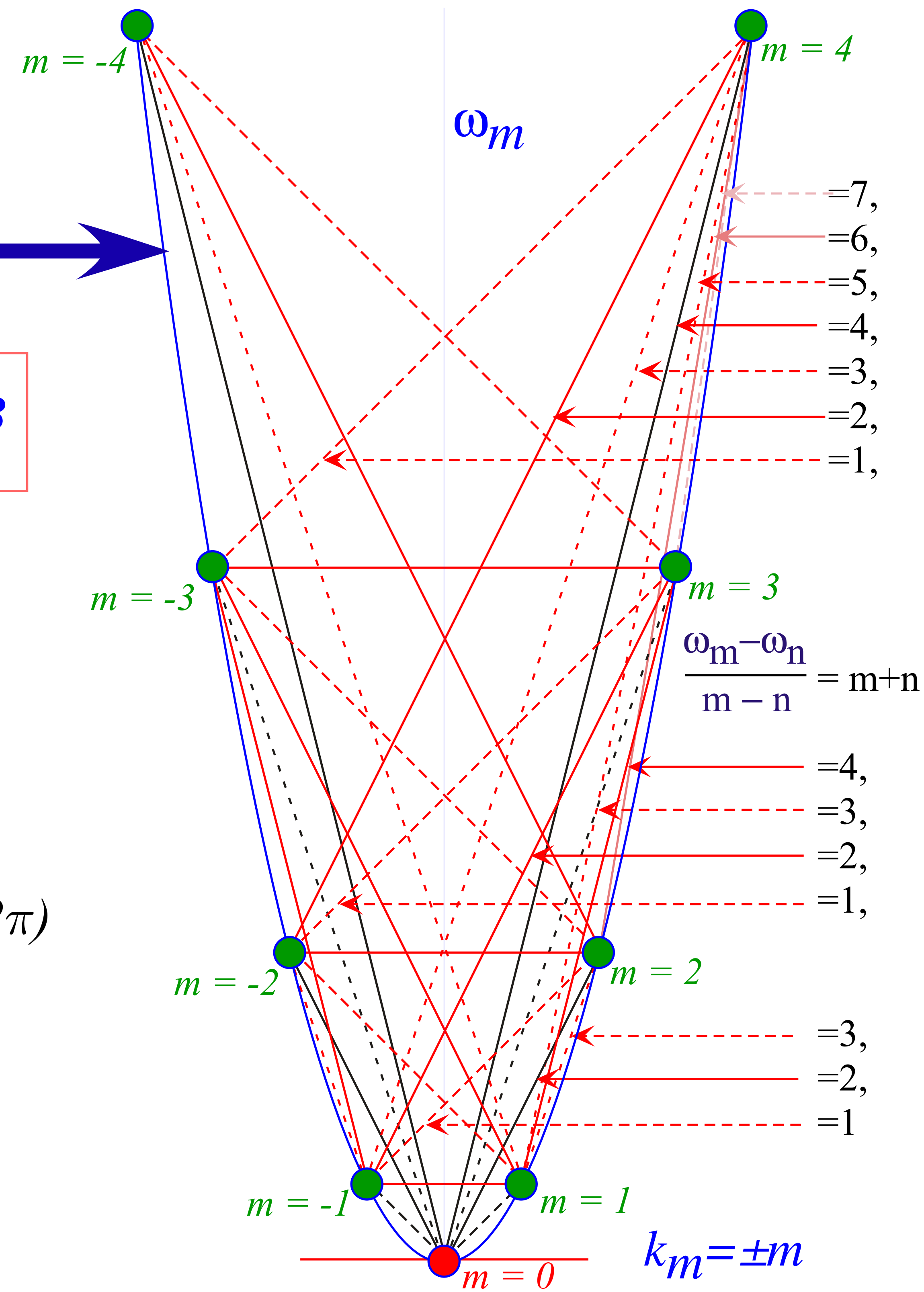
Note: V_{group} usually faster than V_{phase}
(That happens if we ignore Mc^2 !)

$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m \quad if: L = 2\pi)$

$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2 h\upsilon_1 = m^2 \hbar\omega_1$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$

and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$



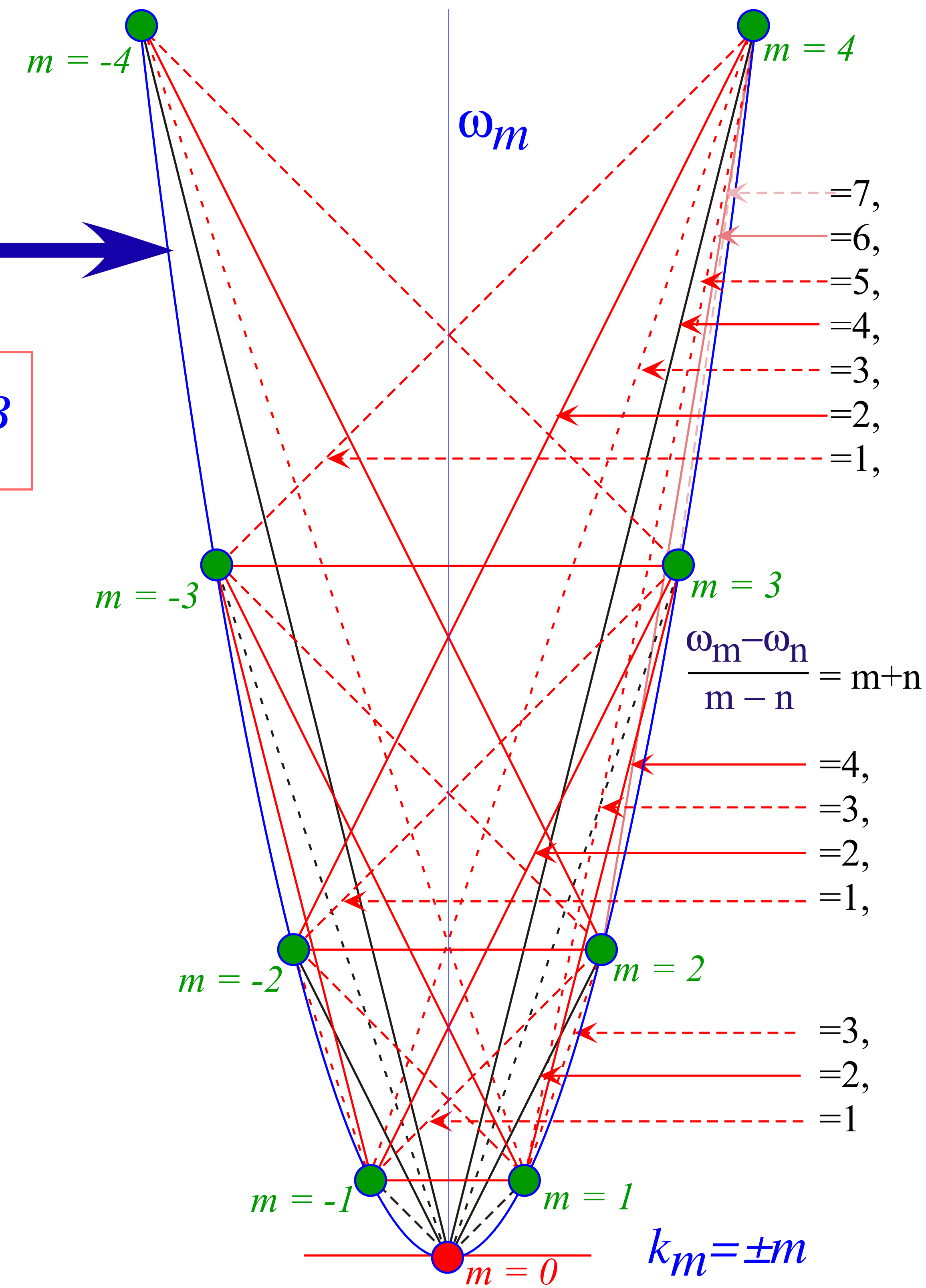
Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$

$$V_{group} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{m^2 - n^2}{m \pm n} B = (m \pm n)B$$

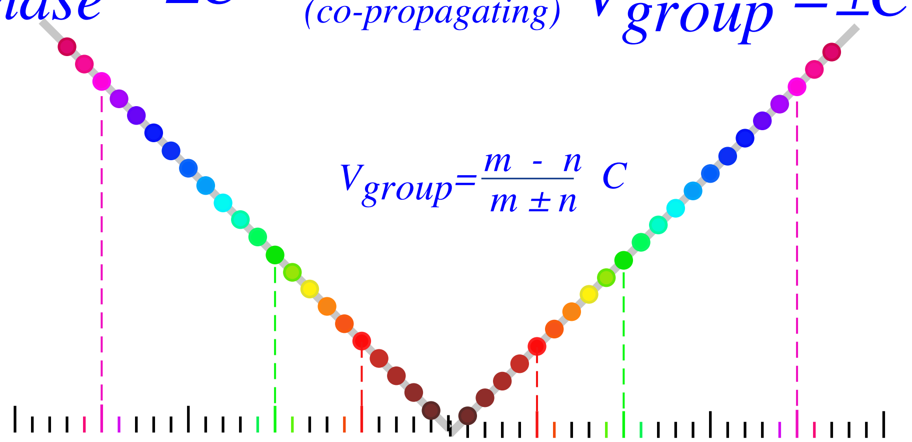


Possible wave velocities for Linear (Optical) Spectrum

$\omega_m = C|m|^1$

$k_m = m$

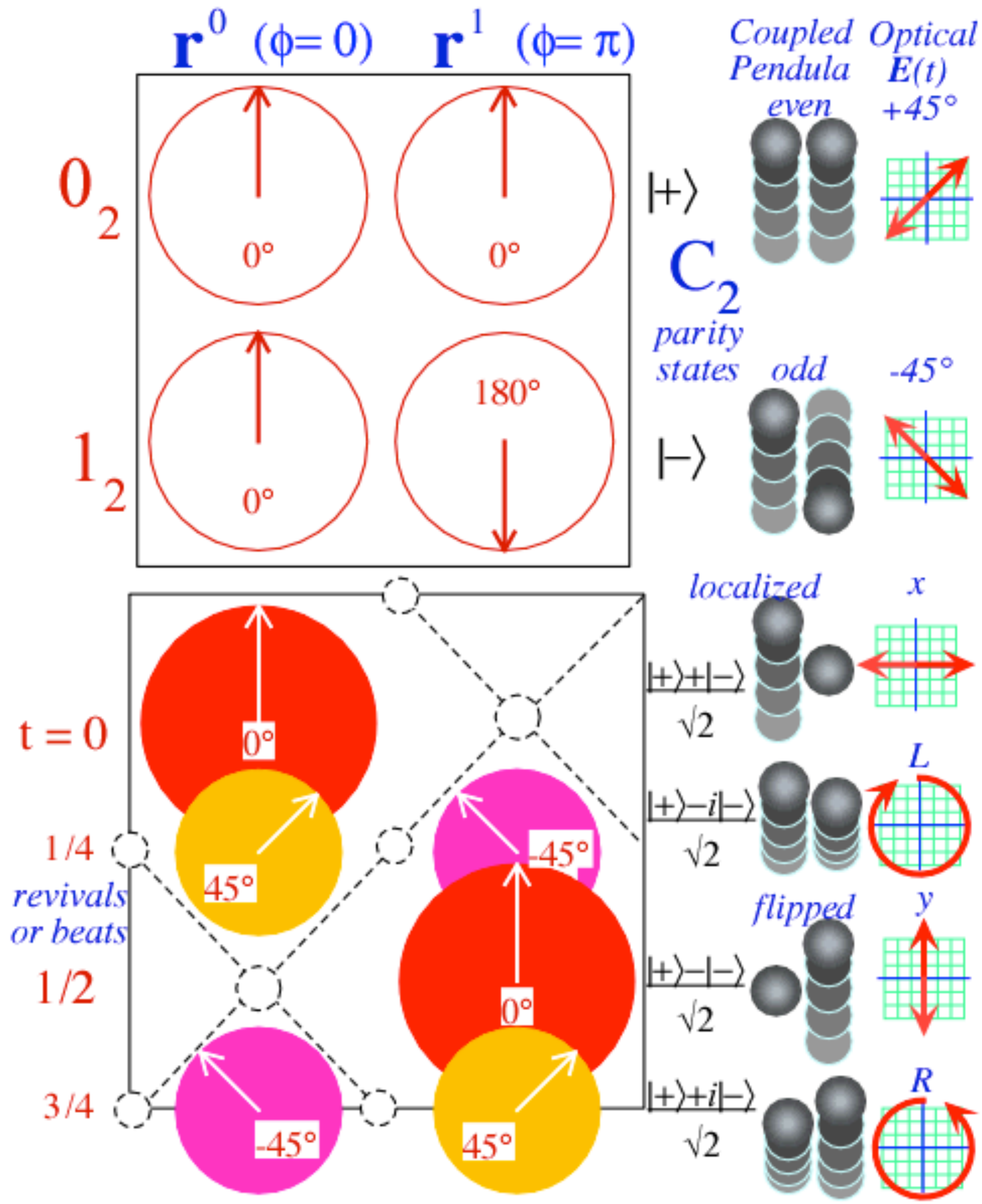
$V_{phase} = \pm C$ (co-propagating) $V_{group} = \pm C$



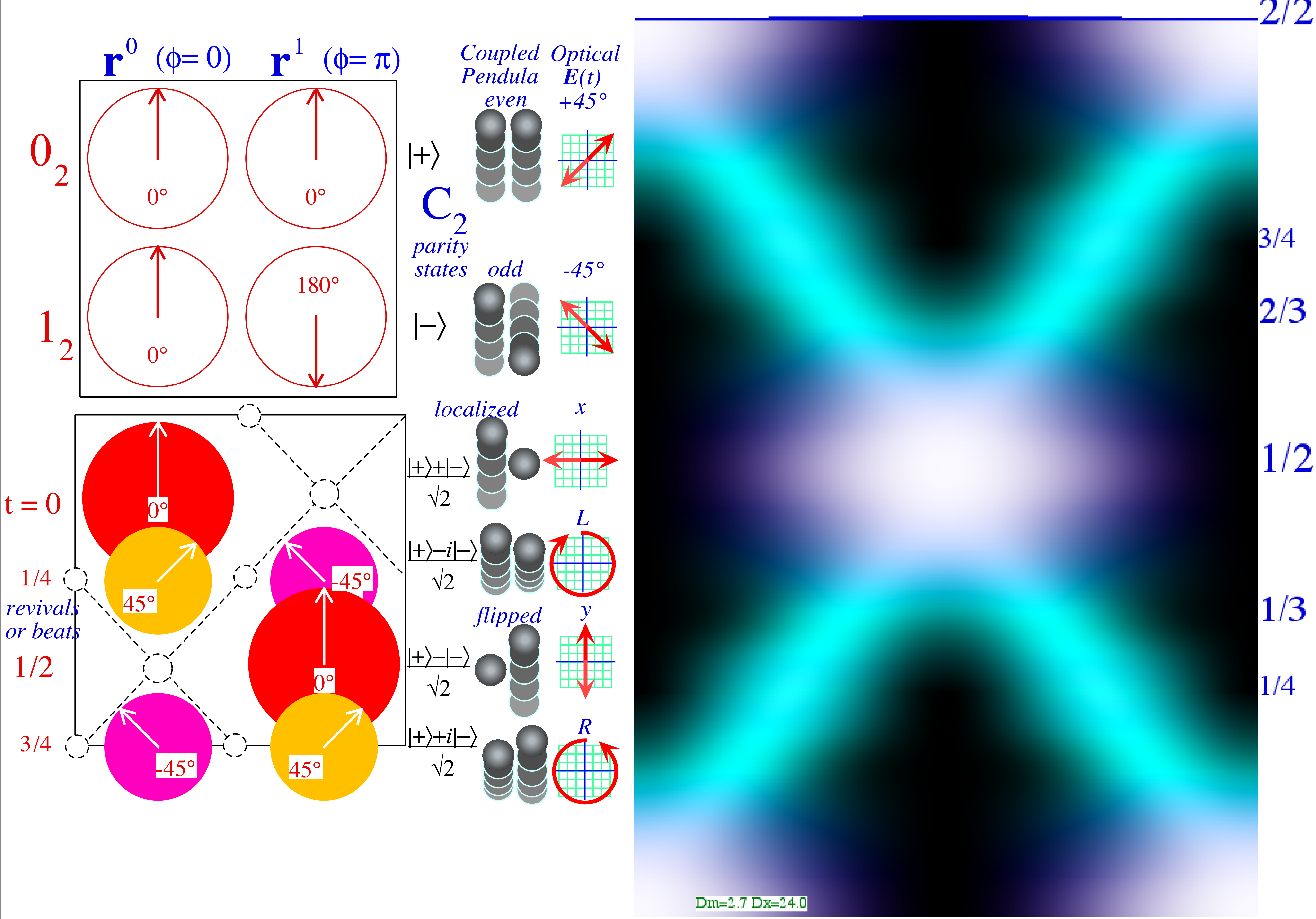
C_2
Fourier transformation matrix

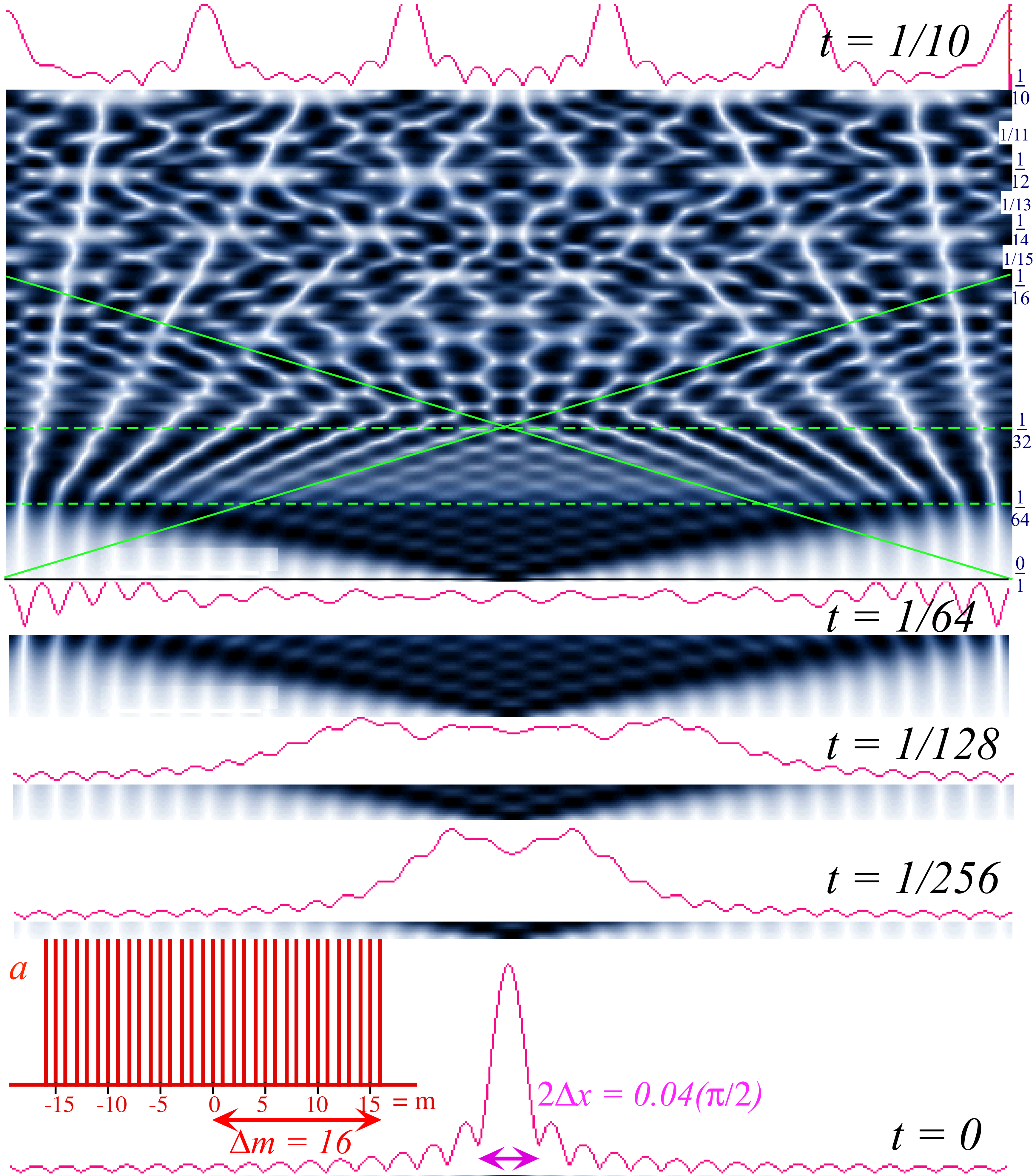
and

dynamics



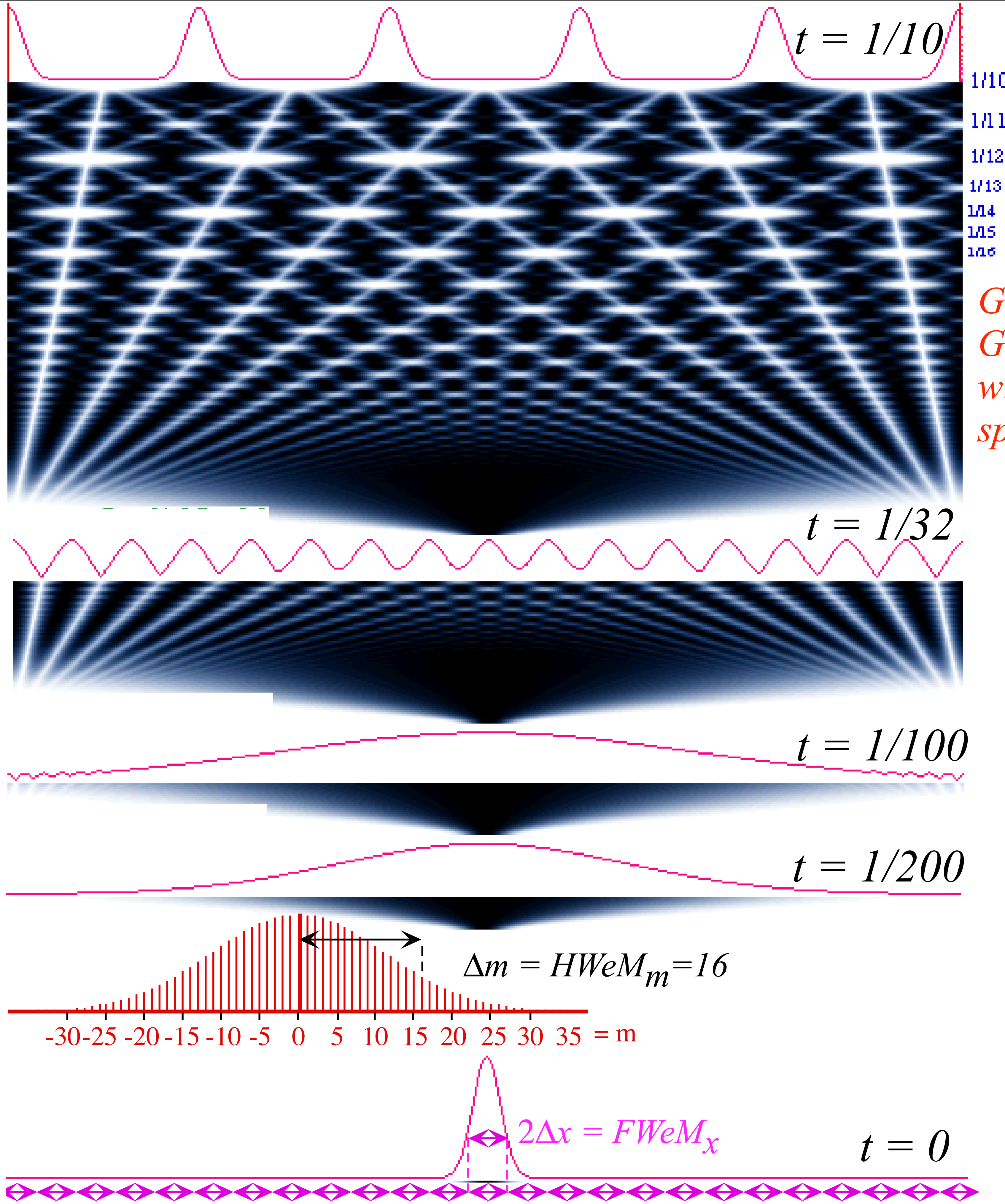
Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2





(sinNx)/x has a "boxcar spectrum" with very complicated space-time revival paths

Known as a "boxcar" spectrum



Gaussian wave has a Gaussian spectrum with comparatively simple space-time revival paths

(Gaussian wave properties are derived in several pages below…)

Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)

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➡ *Gaussian wave-packet bandwidth and uncertainty*

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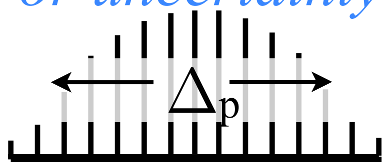
Geometry

Gaussian wave-packet bandwidth and uncertainty

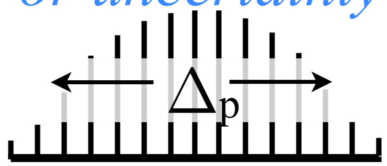
*Let constant Δ_p be momentum-**m** "spread" or uncertainty*

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:



$$\Psi(\phi,t=0) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi}$$

*Let constant Δ_p be momentum-**m** "spread" or uncertainty*



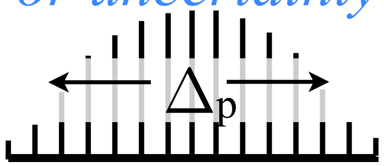
Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{-\left(\frac{m}{\Delta_p}\right)^2}e^{im\phi}=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{-\left(\frac{m}{\Delta_p}\right)^2+im\phi}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_p be momentum-**m** "spread" or uncertainty*



Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2 + im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2 + im\phi + \left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2}$$

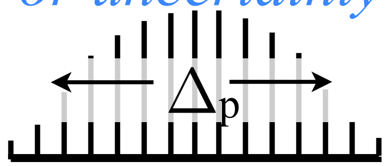
Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract : $\left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2$

in exponent…

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_p be momentum-**m** "spread" or uncertainty*



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$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2 + im\phi + \left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

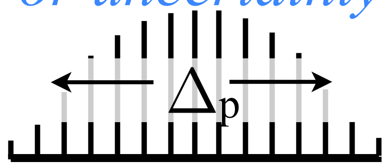
Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract : $\left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2$ *in exponent...*

Extract binomial : $-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2$

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_p be momentum-**m** "spread" or uncertainty*



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$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2+im\phi+\left(\frac{\Delta_p}{2}\phi\right)^2-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{A\left(\Delta_p,\phi\right)}{2\pi} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

where:

$$A\left(\Delta_p,\phi\right)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2}$$

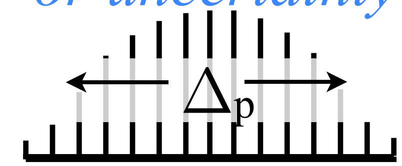
Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract : $\left(\frac{\Delta_p}{2}\phi\right)^2-\left(\frac{\Delta_p}{2}\phi\right)^2$ *in exponent...*

Extract binomial : $-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2$

Gaussian wave-packet bandwidth and uncertainty

Let constant Δ_p be momentum-m "spread" or uncertainty



Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

$$\Psi(\phi,t=0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2 + im\phi + \left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$= \frac{A(\Delta_p,\phi)}{2\pi} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract : $\left(\frac{\Delta_p}{2}\phi\right)^2 - \left(\frac{\Delta_p}{2}\phi\right)^2$ *in exponent...*

Extract binomial : $-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2$

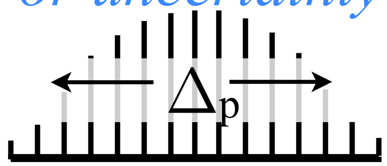
where:

$$A(\Delta_p,\phi) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2} \xrightarrow[\Delta_p>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2}$$

$m=0, \pm1, \pm2, \pm3,\ldots$ are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad(k_m=m$ *if:* $L=2\pi)$

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_p be momentum-**m** "spread" or uncertainty*



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$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2+im\phi+\left(\frac{\Delta_p}{2}\phi\right)^2-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{A(\Delta_p,\phi)}{2\pi} e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract in exponent... : $\left(\frac{\Delta_p}{2}\phi\right)^2-\left(\frac{\Delta_p}{2}\phi\right)^2$

Extract binomial : $-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2$

where:

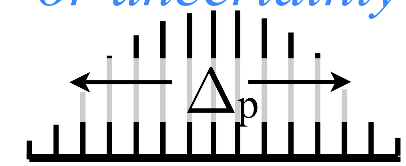
$$A(\Delta_p,\phi)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2} \xrightarrow[\Delta_p>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_p}-i\frac{\Delta_p}{2}\phi \quad \text{so: } dk=\Delta_p dK\right]$$

$m=0, \pm 1, \pm 2, \pm 3,\ldots$ are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Let constant Δ_p be momentum-m "spread" or uncertainty



Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2+im\phi+\left(\frac{\Delta_p}{2}\phi\right)^2-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}-i\frac{\Delta_p}{2}\phi\right)^2} \quad e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

$$=\frac{A(\Delta_p,\phi)}{2\pi} \quad e^{-\left(\frac{\Delta_p}{2}\phi\right)^2}$$

Gaussian integral:

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx}\sqrt{\int_{-\infty}^{\infty} e^{-y^2}dy}=\sqrt{\iint e^{-(x^2+y^2)}\,dx\,dy}$$

$$=\sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta}=\sqrt{2\pi\int_0^{\infty} e^{-r^2}\frac{dr^2}{2}}=\sqrt{\pi}$$

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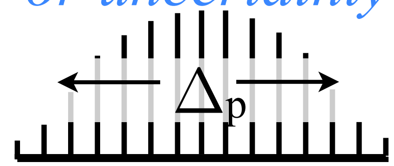

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It is a Gaussian distribution, too

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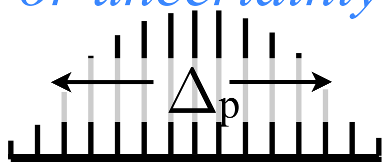

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It is a Gaussian distribution, too

$$\Psi(\phi,t{=}0)\simeq\frac{\Delta_p}{2\sqrt{\pi}} e^{-\left(\frac{\phi}{\Delta_\phi}\right)^2}$$

where: $\Delta_\phi=\dfrac{2}{\Delta_p}$ or: $\Delta_\phi\Delta_p=2$

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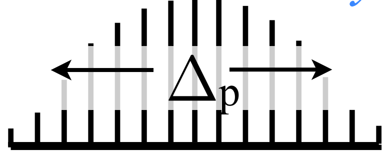

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Gaussian uncertainty relation

(Compare to $\Delta x \cdot \Delta k = \pi$ *for ∞-Well)*

where:

$$A\left(\Delta_p,\phi\right) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2} \xrightarrow[\Delta_p>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_p} - i\frac{\Delta_p}{2}\phi\right)^2}$$

$\left[\text{let: } K = \dfrac{k}{\Delta_p} - i\dfrac{\Delta_p}{2}\phi \quad \text{so: } dk = \Delta_p dK\right]$ then: $A\left(\Delta_p,\phi\right) \simeq \Delta_p \int_{-\infty}^{\infty} dK\, e^{-(K)^2} = \Delta_p\sqrt{\pi}$

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$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$ and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

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∞-Square-well revivals: SinNx/x packet explodes! (and then UNexplodes!)

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Gaussian wave-packet bandwidth and uncertainty

→ *Gaussian Bohr-rotor revivals and quantum fractals*

Understanding fractals using geometry of fractions (Rationalizing rationals)

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Polygonal geometry of $U(2)\supset C_N$ character spectral function

Algebra

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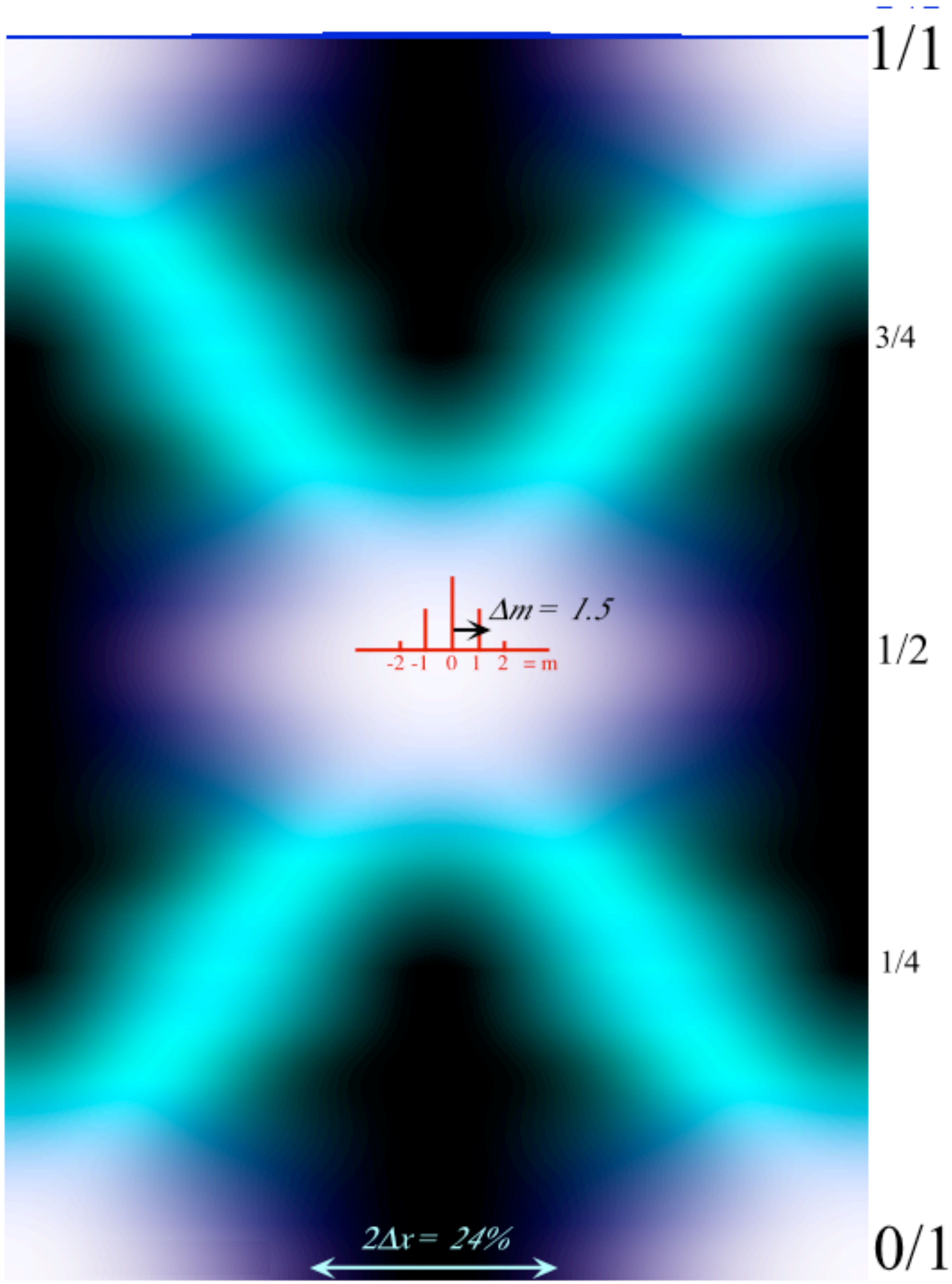
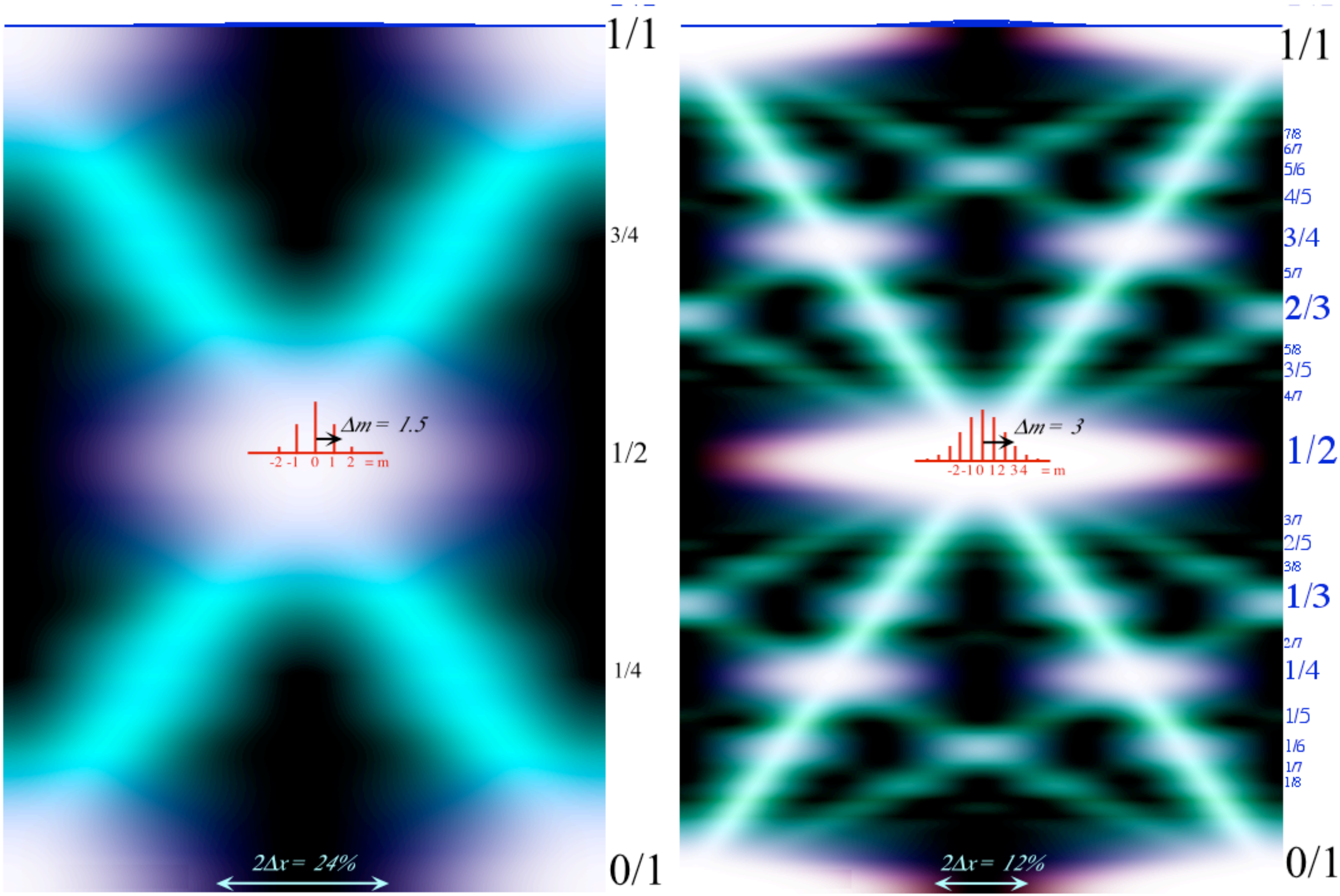
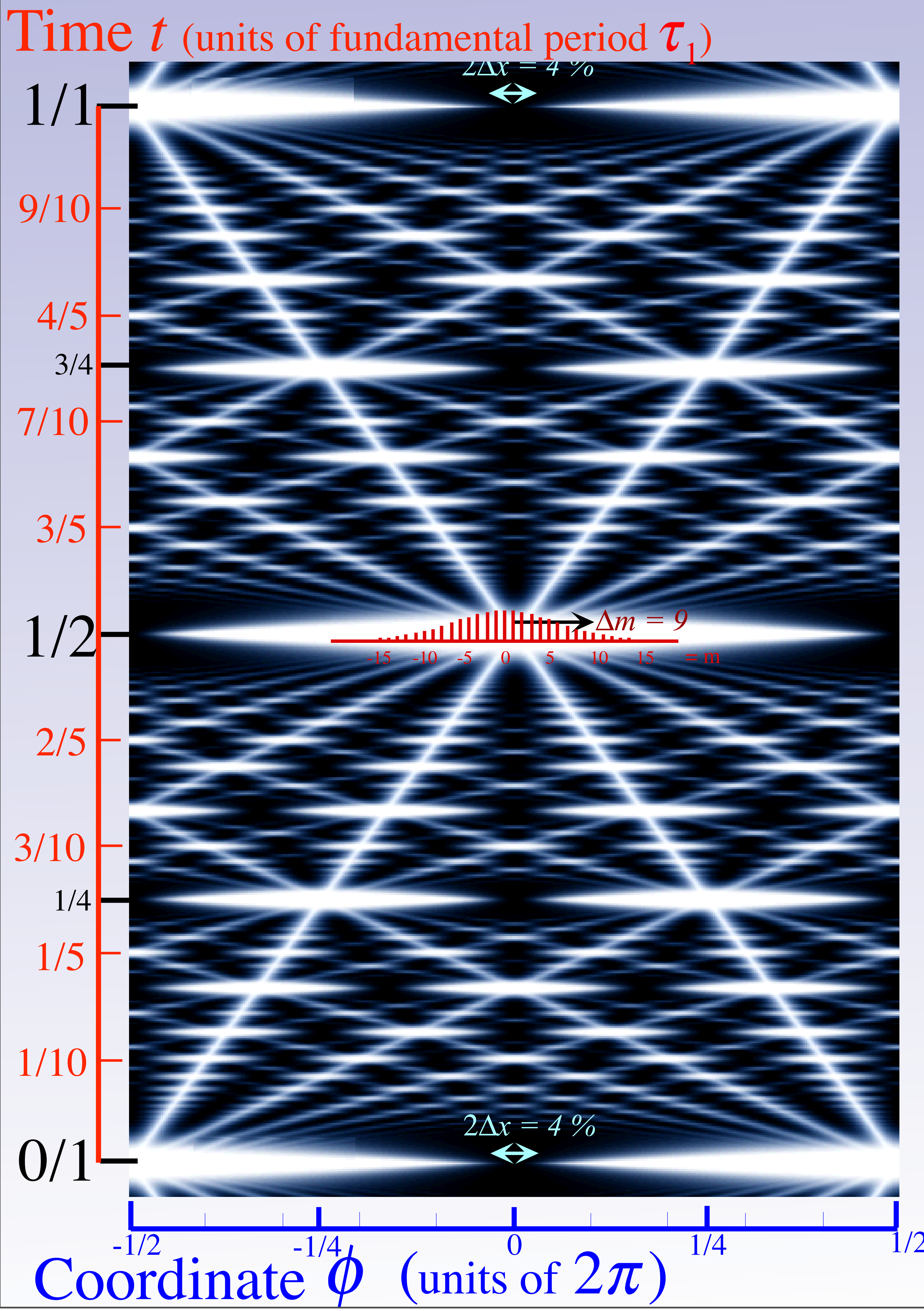
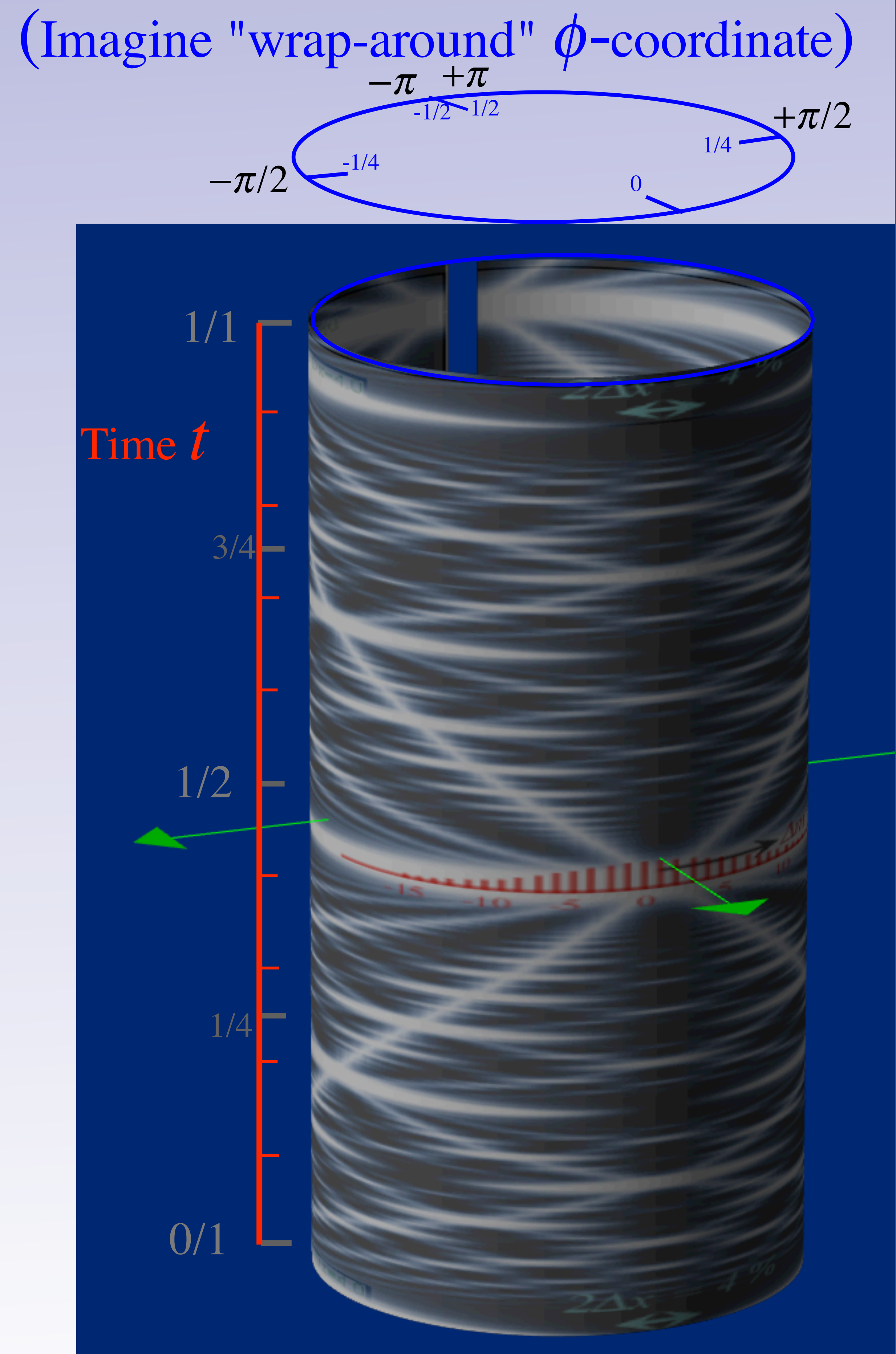

1/1
3/4
Δm = 1.5
-2 -1 0 1 2 = m
1/2
1/4
2Δx = 24%
0/1



1/1
3/4
1/2
1/4
0/1
Δm = 1.5
-2 -1 0 1 2 = m
2Δx = 24%
1/1
7/8
6/7
5/6
4/5
3/4
5/7
2/3
5/8
3/5
4/7
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
0/1
Δm = 3
-2-1 0 1 2 3 4 = m
2Δx = 12%





[Harter, J. Mol. Spec. 210, 166-182 (2001)]

Web simulation *or:* http://www.uark.edu/ua/modphys/markup/WaveItWeb.html

http://www.uark.edu/ua/modphys/markup/WaveItWeb.html?scenario=Quantum%20Carpet

Also, try testing *or else* markup

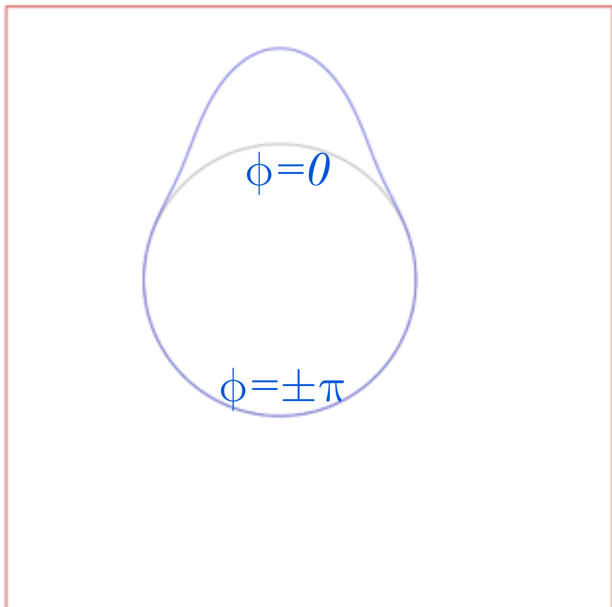
Click here....

Launch | Fourier Control | Scenarios | Pause | Set T=0 | Zero Amps | T-Scale= 1

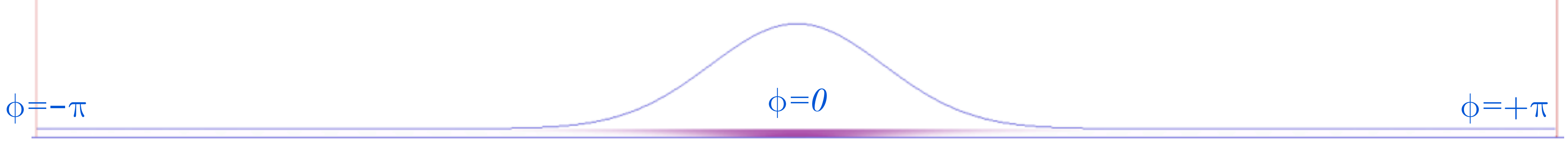
Twelve (n=12) oscillator
Twelve (n=12) oscillator
Twelve (n=12) oscillator
C(n) Character Table

..then here.... Quantum Carpet

$\phi=-\pi$ $\phi=0$ $\phi=+\pi$



Starts with Gaussian Ψ(φ,t)
at φ=0 on Bohr wave ring
that expands and "beats"



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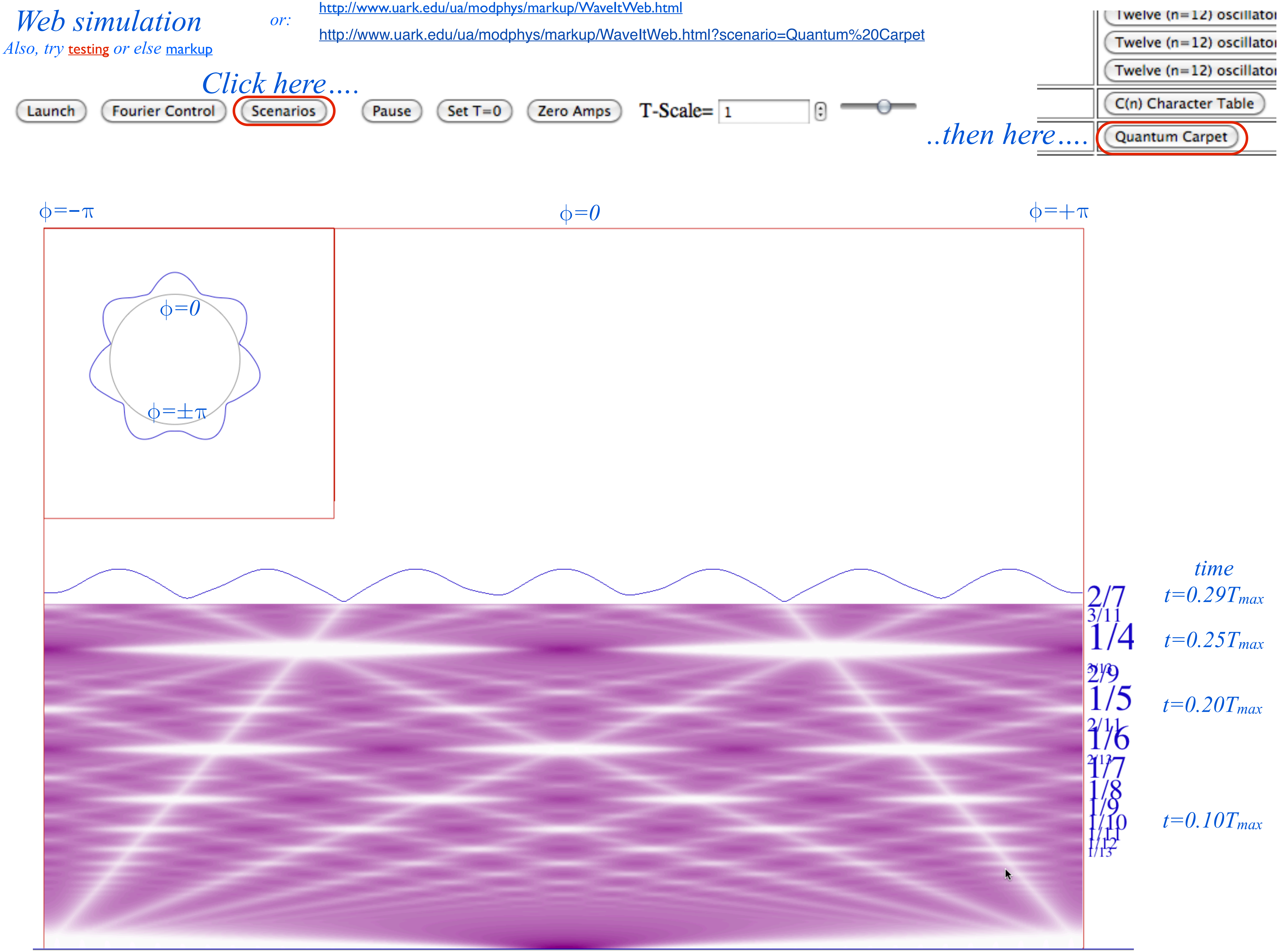
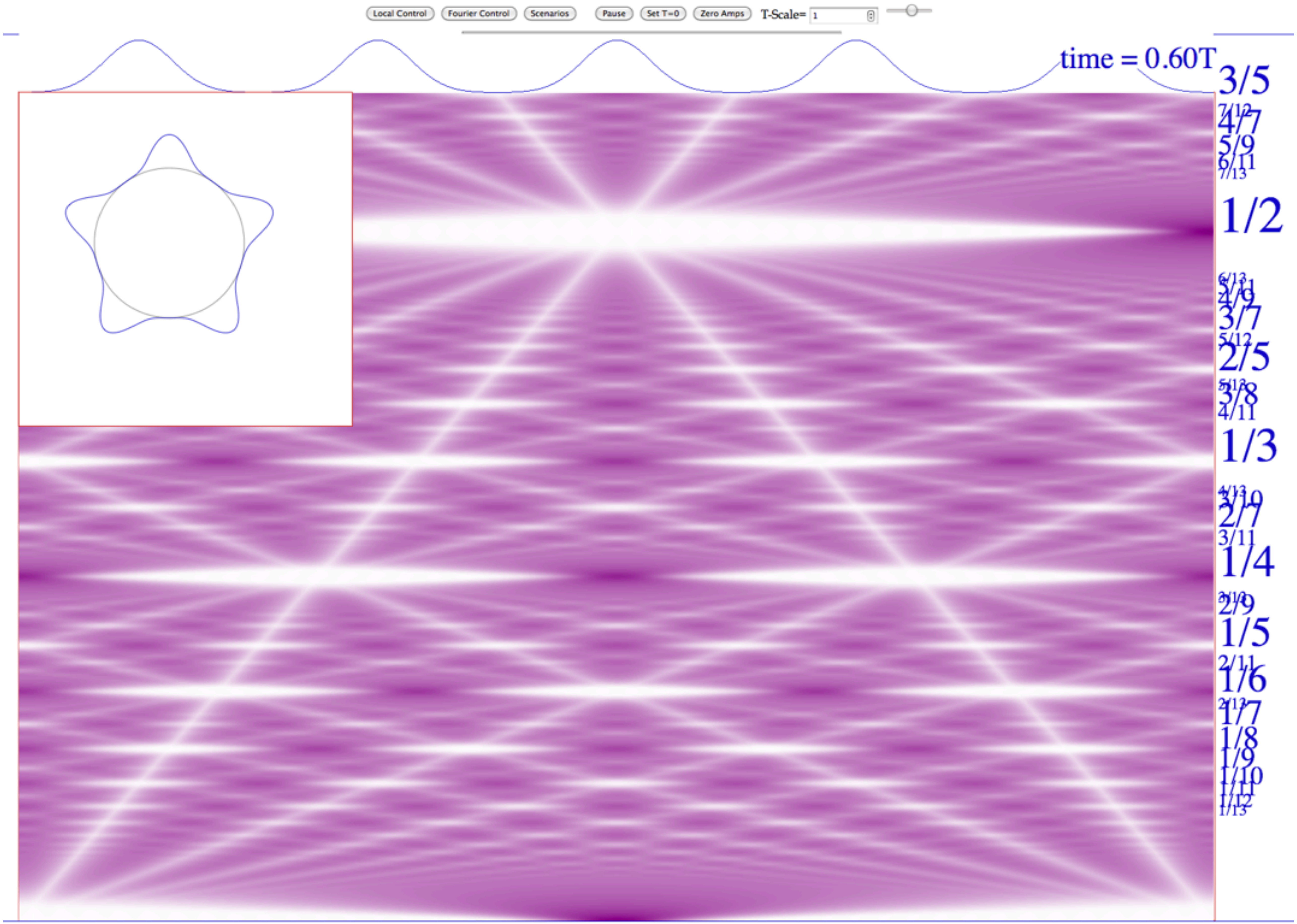




Local Control
Fourier Control
Scenarios
Pause
Set T=0
Zero Amps
T-Scale=
time = 0.60T
3/5
7/12
4/7
5/9
6/11
7/13
1/2
6/13
5/11
4/9
3/7
5/12
2/5
5/13
3/8
4/11
1/3
4/13
3/10
2/7
3/11
1/4
3/13
2/9
1/5
2/11
1/6
2/13
1/7
1/8
1/9
1/10
1/11
1/12
1/13

Launch | Fourier Control | Scenarios | Pause | Set T=0 | Zero Amps | T-Scale= 1

Set this and then click here....

Type: Quantum Carpet

Time Behavior: Pause at End

Time Start (% Period) = 0

Time End (% Period)= 60

Del-x Width (% L) = 4

Excitation (Max n) = 20

Left (% L) = 0

Right (% L)= 100

n-Mean (% Max n)= 0

Peak1 Mean (% L)= 50

OverAll Scale = 1

Peak2 Mean (% L)= 0

Peak2 Amp (% Peak1)= 0

Draw Ring ☑ m/n Labels ☑

m-Boxcar ☐

Draw m-Bars ☑ m-Bars Max = 30

Aspect Ratio {W/H} = 1.5

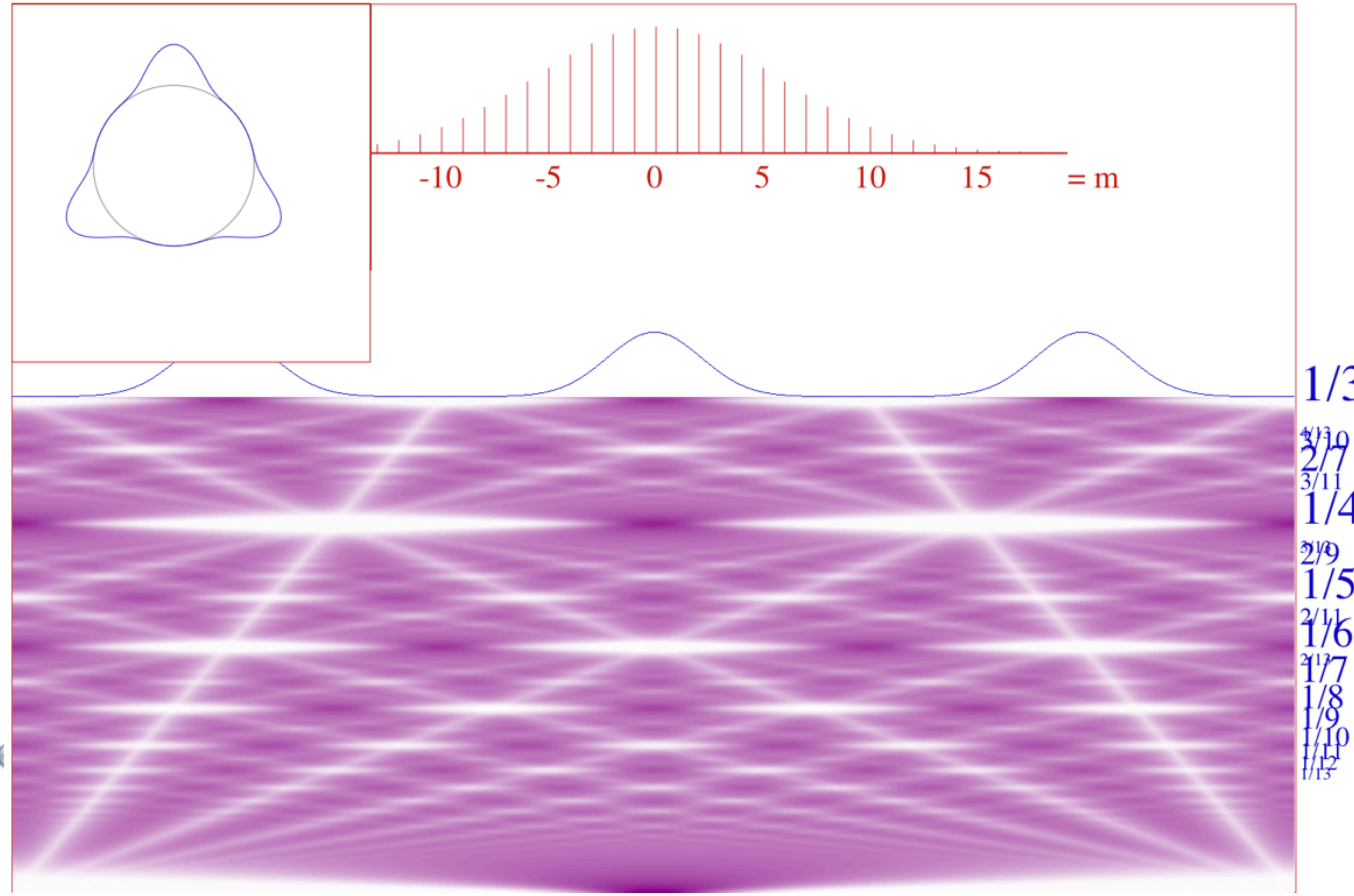
Red Level = 128

Green Level = 0

Blue Level = 128

Alpha Level = 1

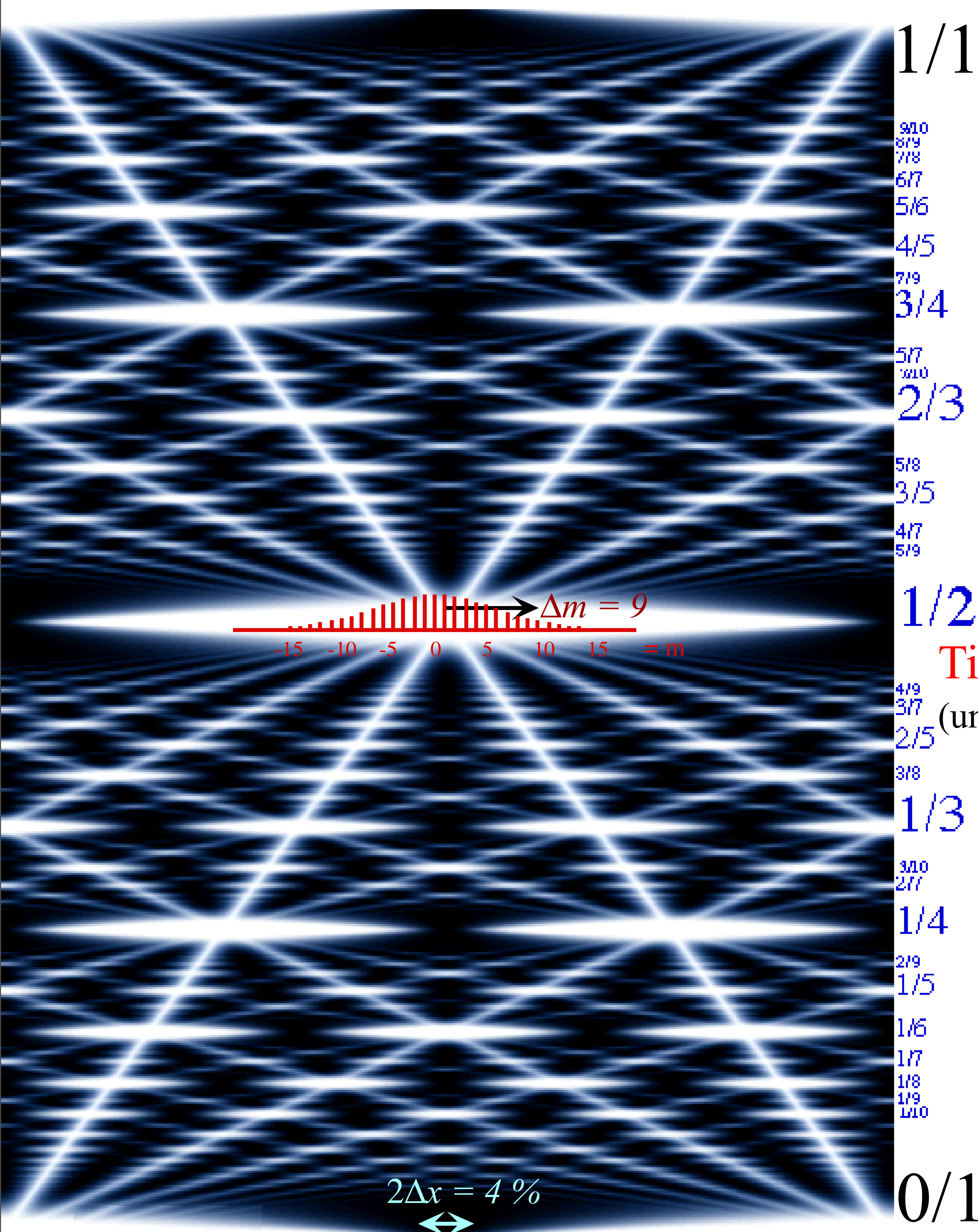
Definition Level = 0.5



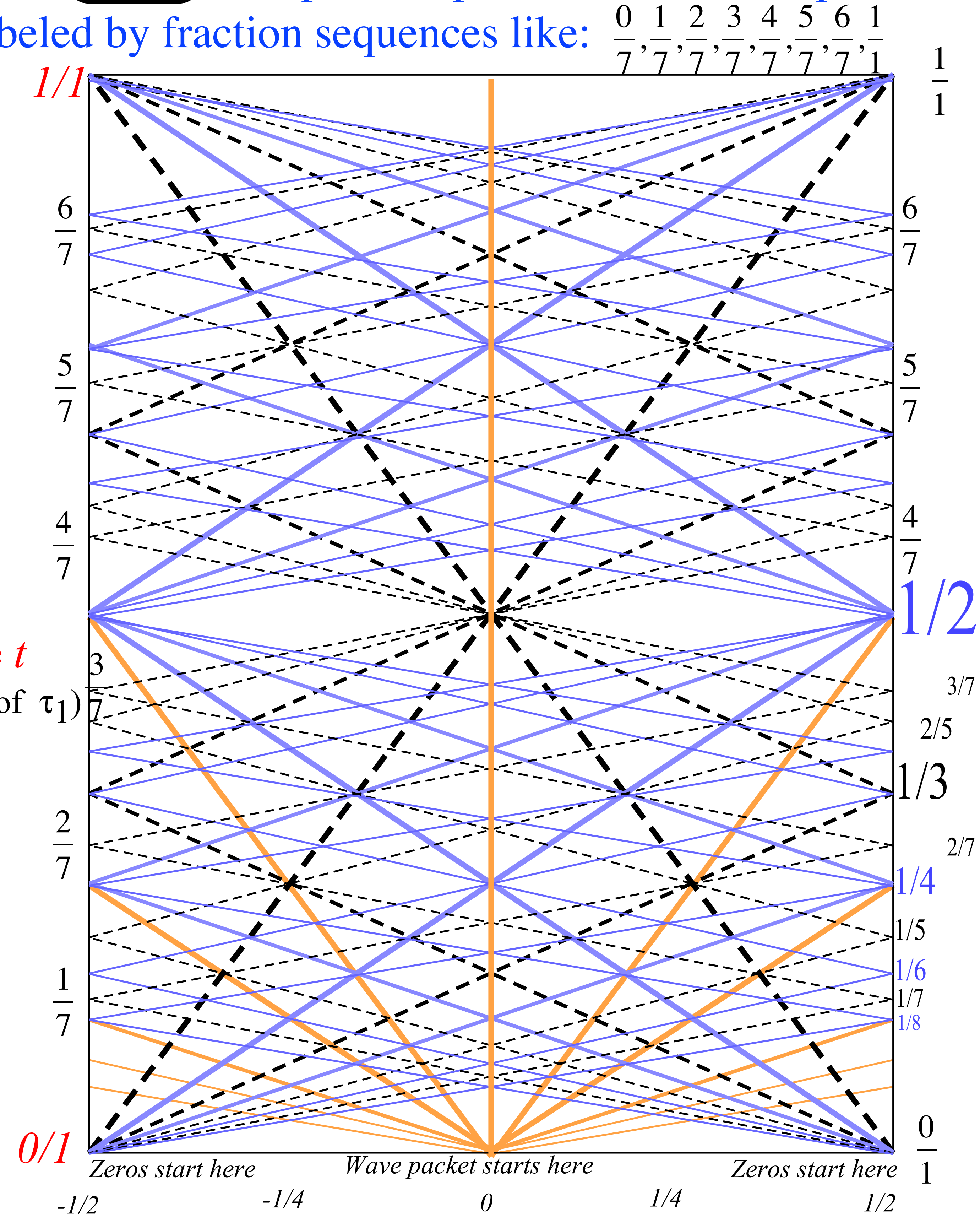
N-level-system and revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$



Time t (units of τ_1)



Coordinate ϕ (units of 2π)

[Harter, J. Mol. Spec. 210, 166-182 (2001)]

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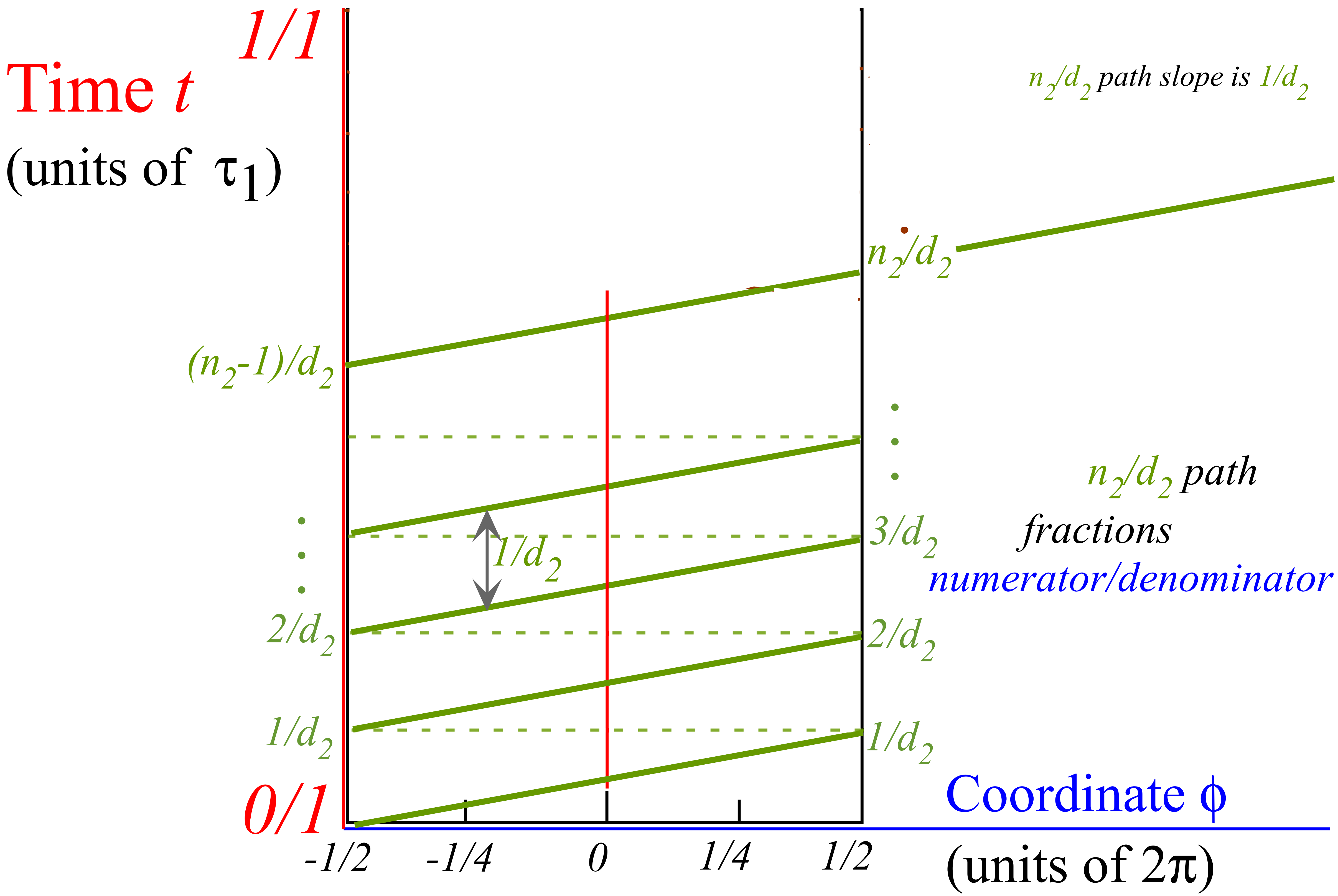
Label by *numerators N* and *denominators D* of rational fractions *N/D*



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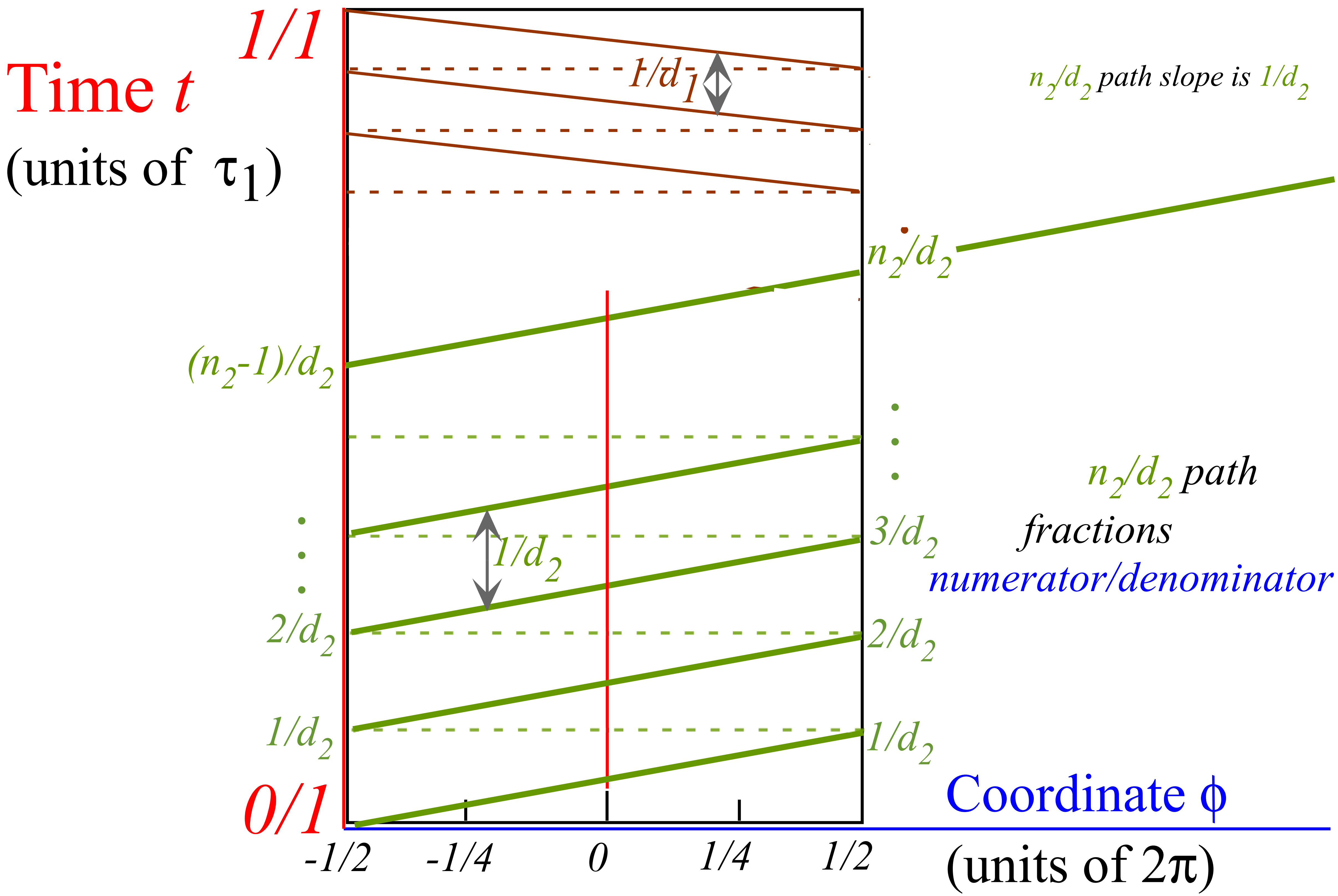

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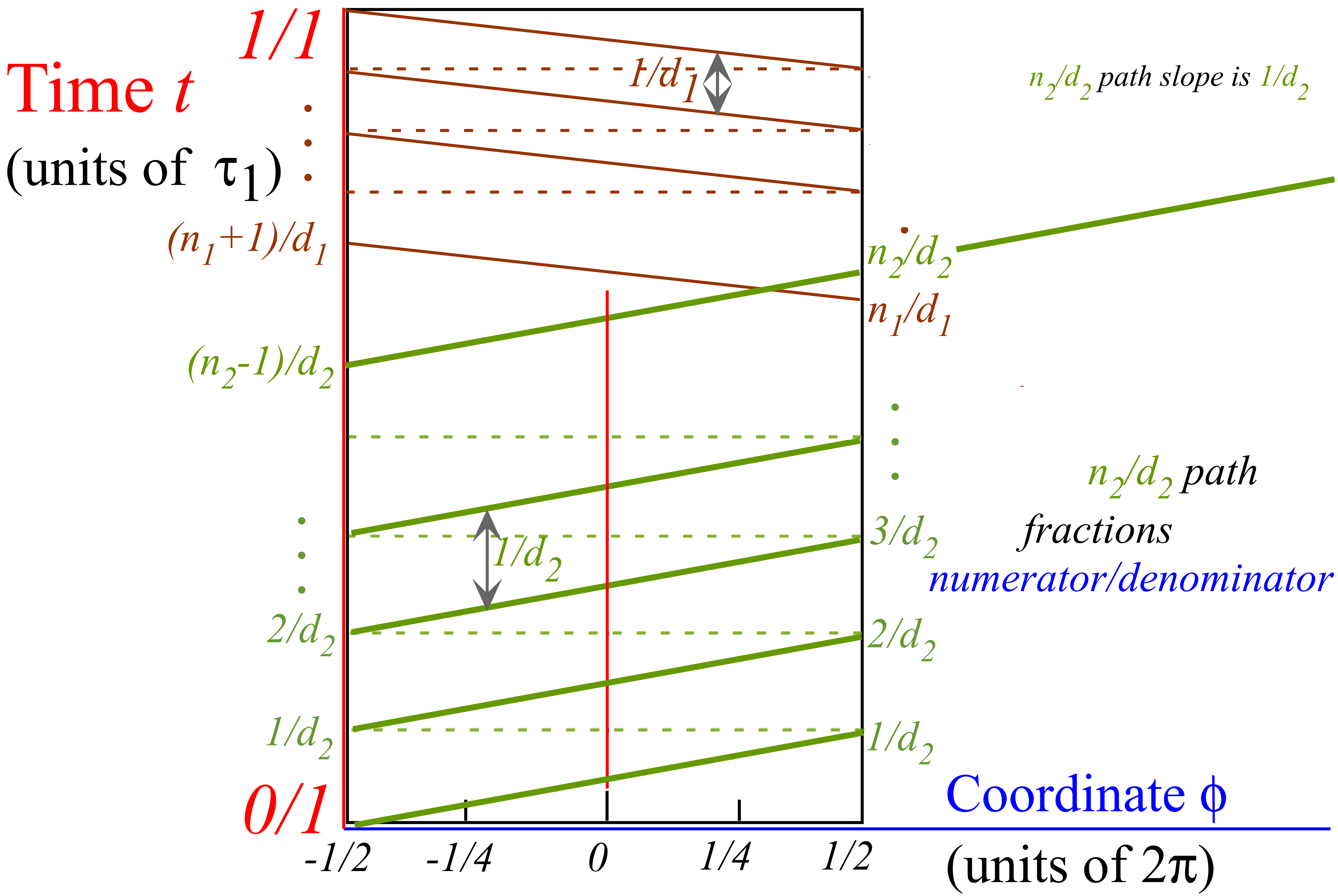

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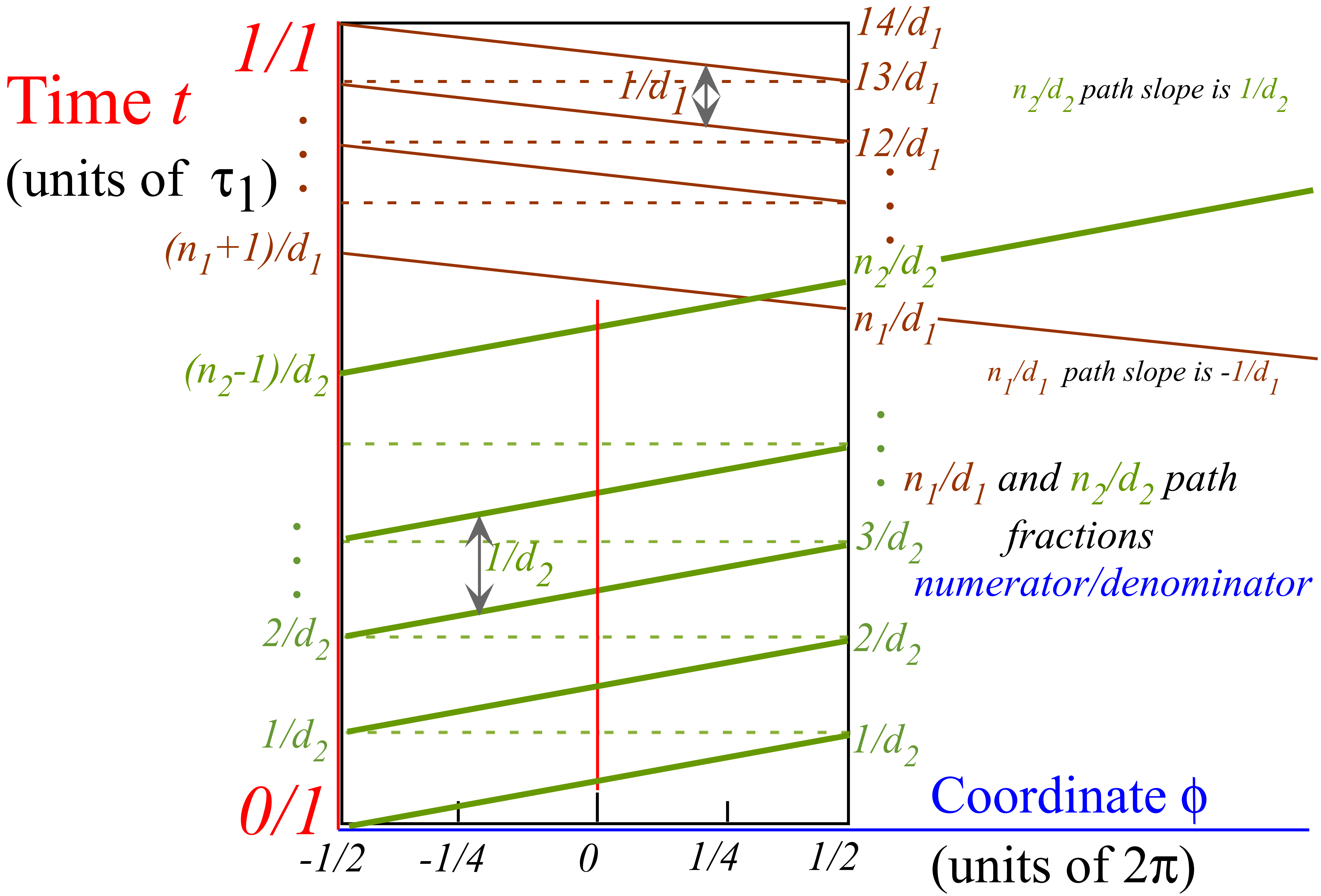

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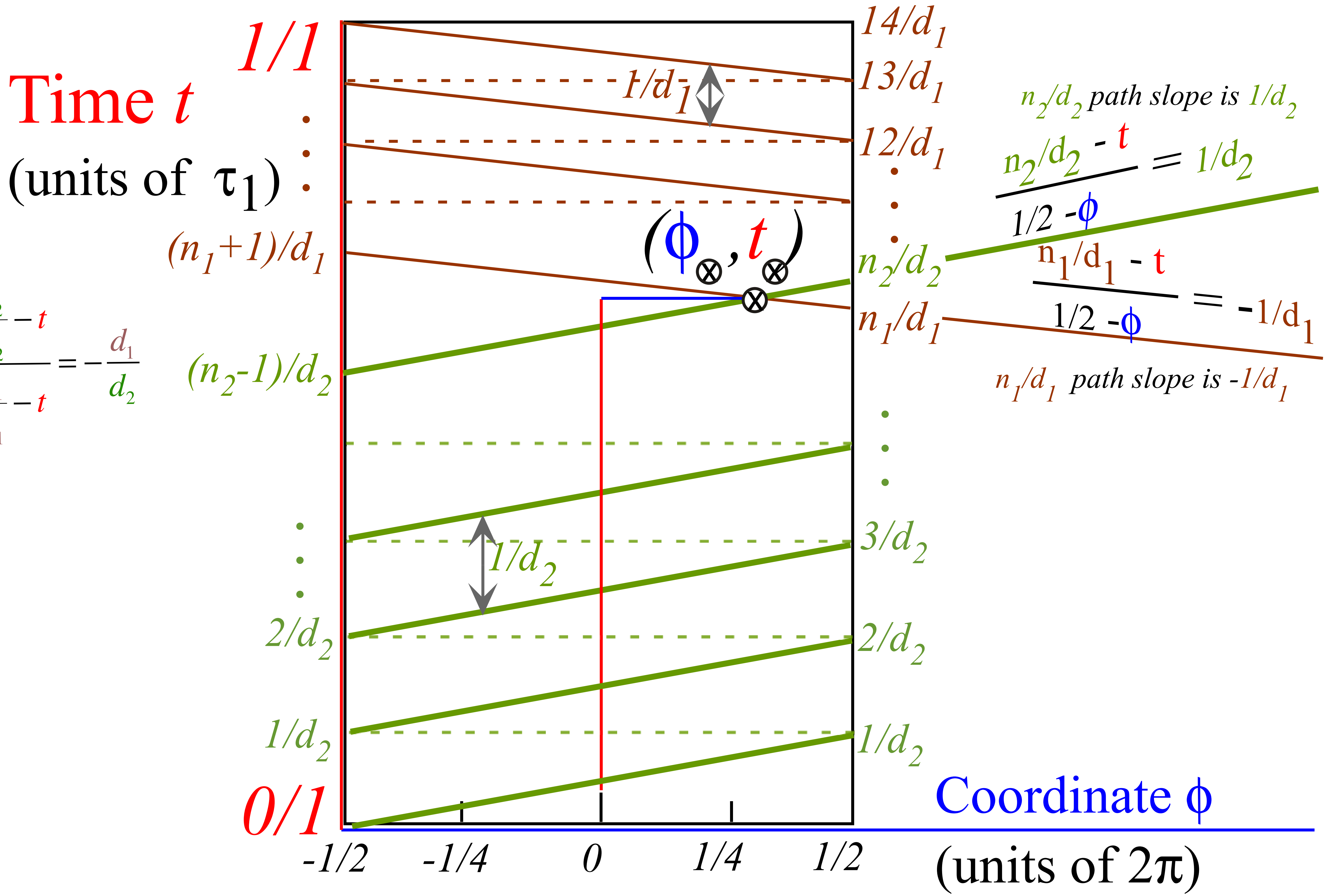

Solving: $\frac{\frac{n_2}{d_2}-t}{\frac{n_1}{d_1}-t}=-\frac{d_1}{d_2}$

$\frac{n_2/d_2 - t}{1/2-\phi} = 1/d_2$

$\frac{n_1/d_1 - t}{1/2-\phi} = -1/d_1$

[*John Farey, Phil. Mag.(1816)*]

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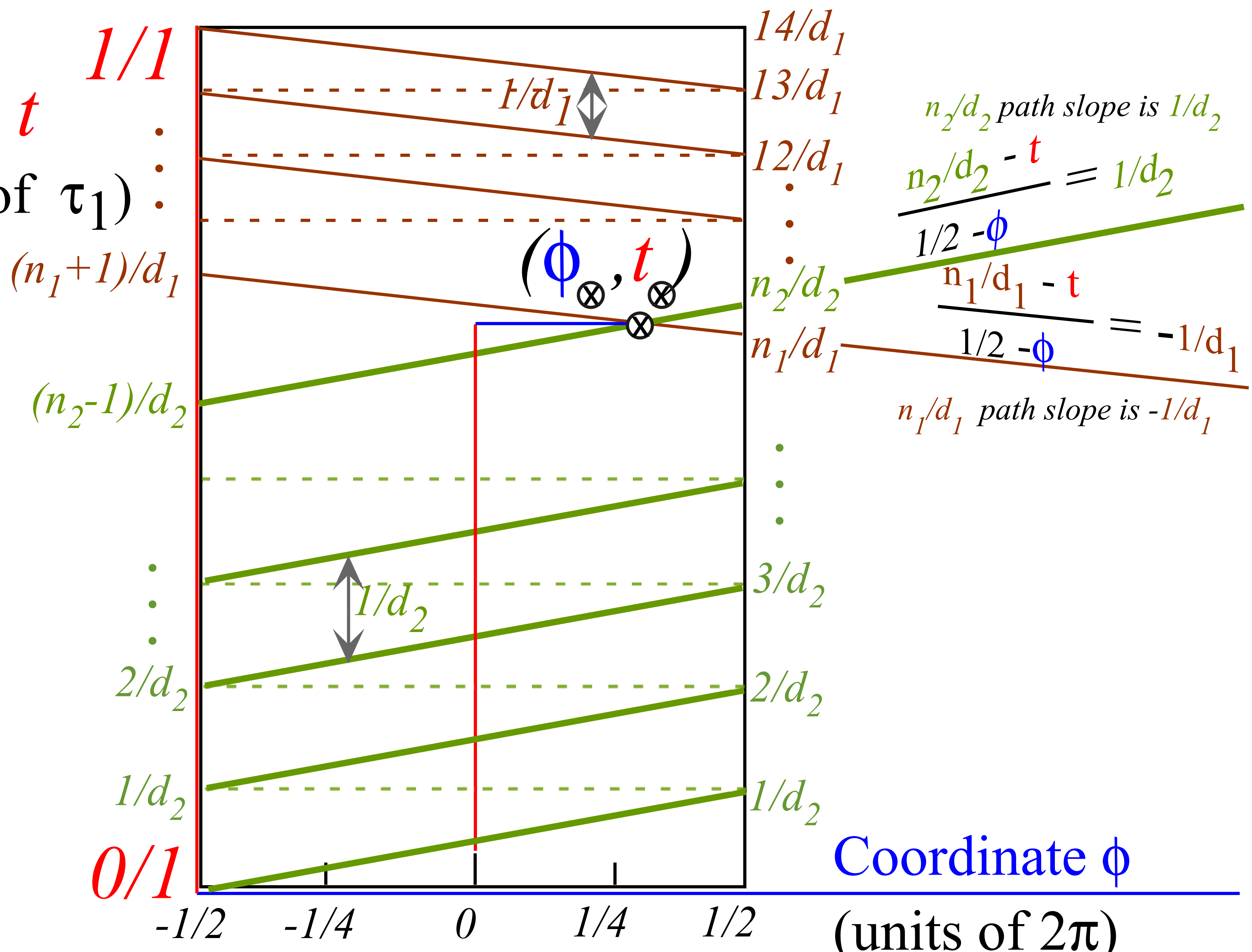
$\dfrac{n_2}{d_2}-t=-\dfrac{d_1}{d_2}\dfrac{n_1}{d_1}+\dfrac{d_1}{d_2}t$



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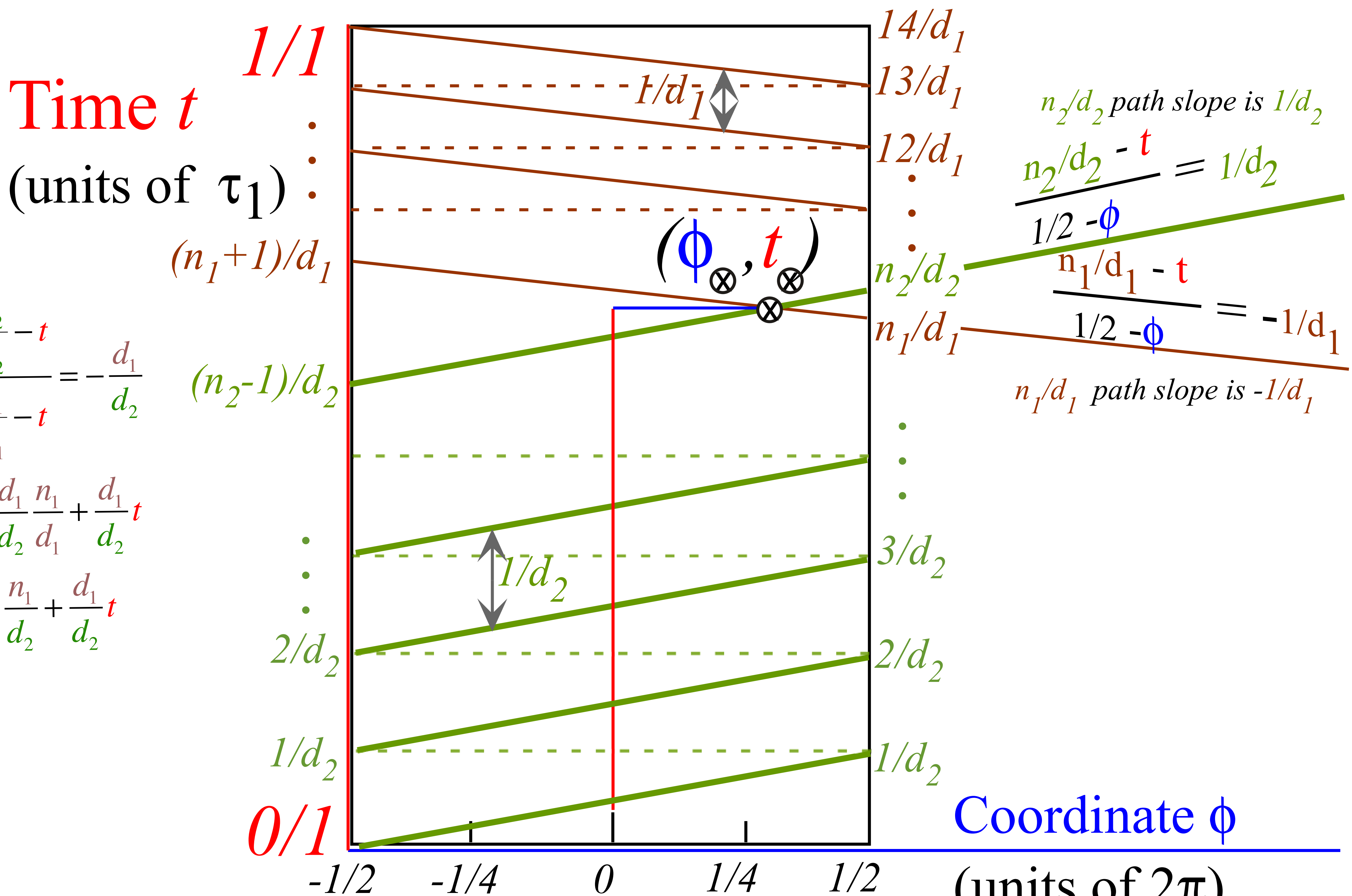

Solving: $\dfrac{\frac{n_2}{d_2} - t}{\frac{n_1}{d_1} - t} = -\dfrac{d_1}{d_2}$

$$\frac{n_2}{d_2} - t = -\frac{d_1}{d_2}\frac{n_1}{d_1} + \frac{d_1}{d_2}t$$

$$= -\frac{n_1}{d_2} + \frac{d_1}{d_2}t$$

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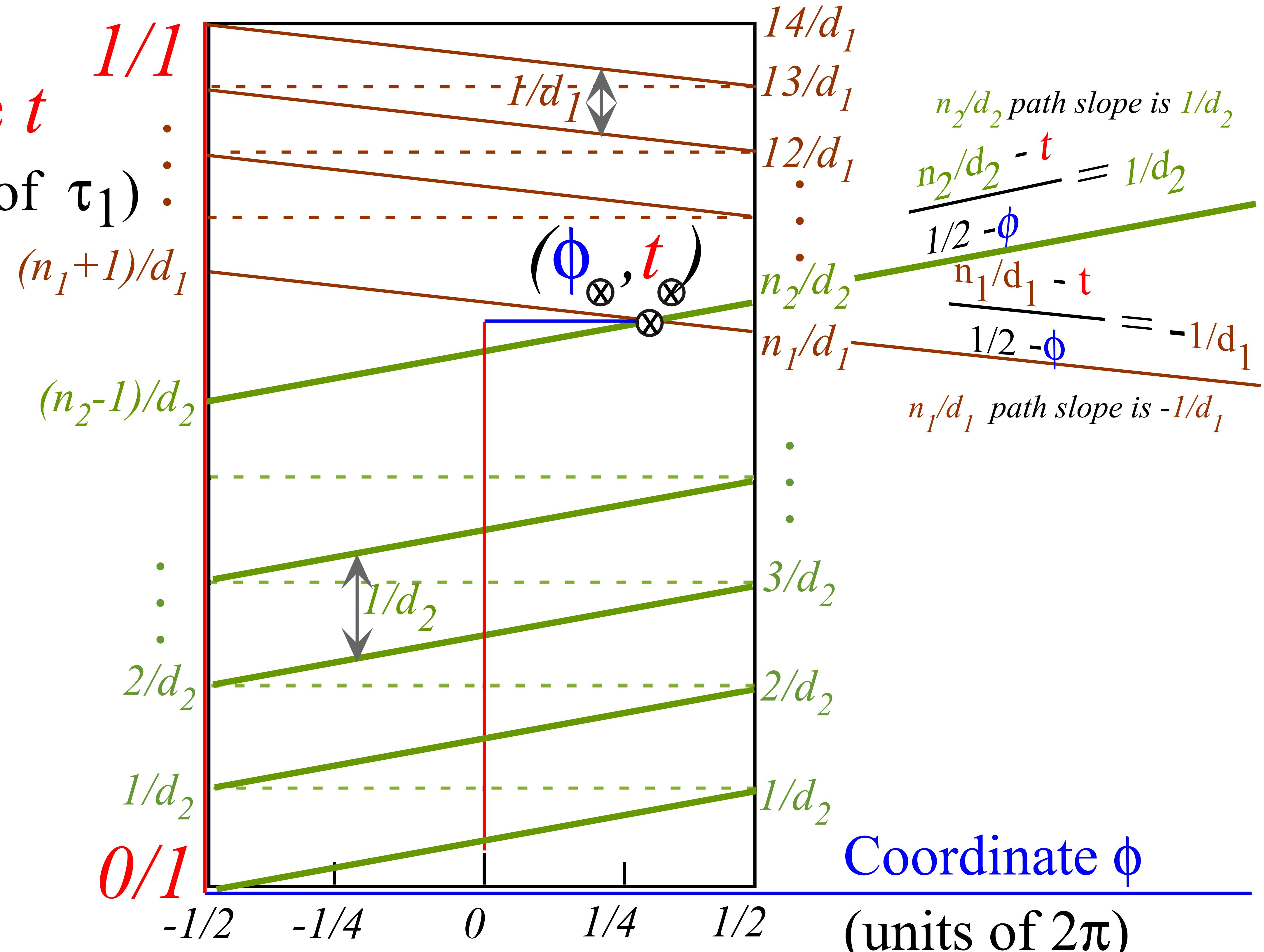
Time *t*

(units of τ_1)



[John Farey, Phil. Mag.(1816)]

Solving: $\dfrac{\frac{n_2}{d_2}-t}{\frac{n_1}{d_1}-t} = -\dfrac{d_1}{d_2}$

$$\frac{n_2}{d_2}-t = -\frac{d_1}{d_2}\frac{n_1}{d_1}+\frac{d_1}{d_2}t$$

$$= -\frac{n_1}{d_2}+\frac{d_1}{d_2}t$$

$$n_2+n_1 = d_2 t + d_1 t$$

Farey Sum algebra of revival-beat wave dynamics

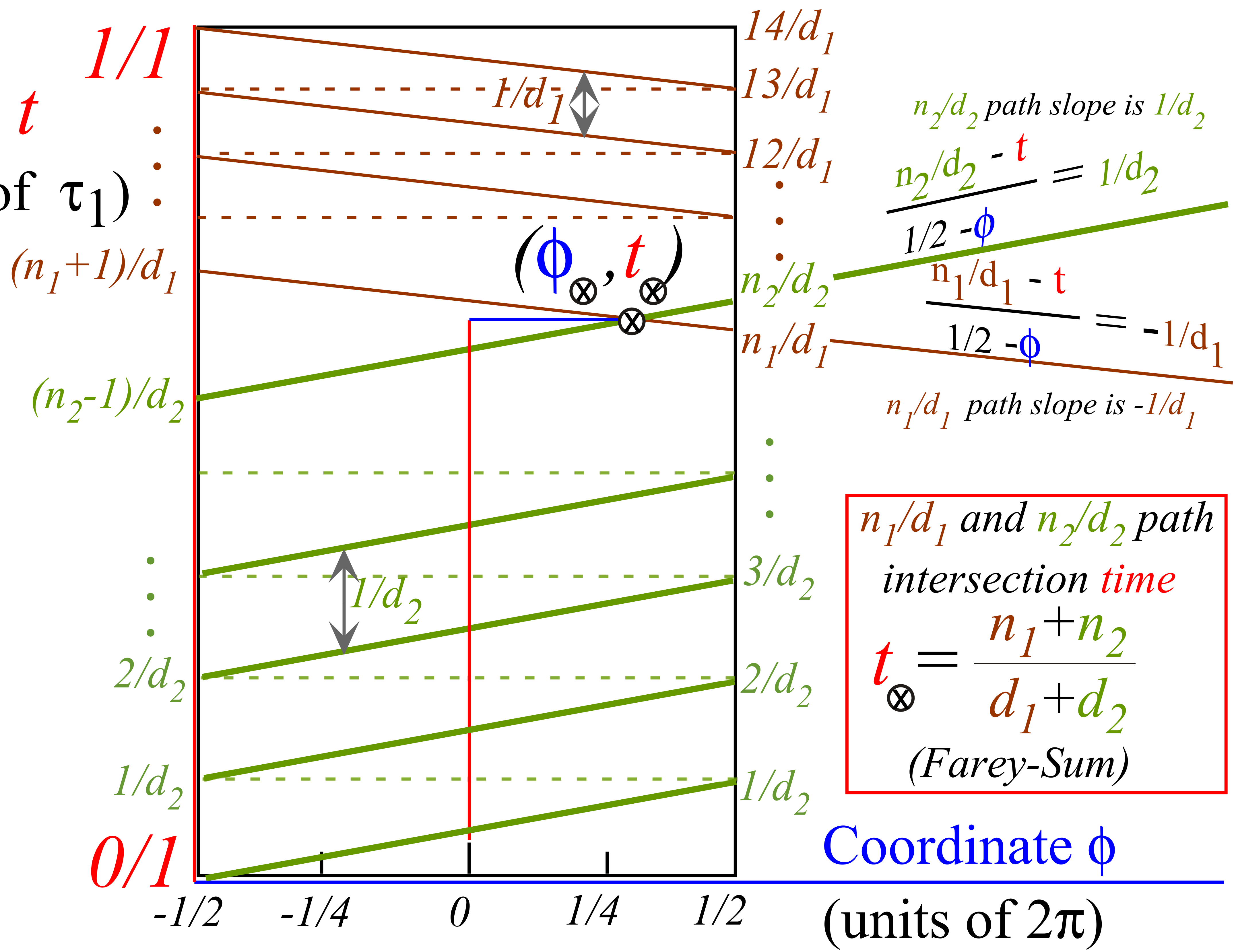
Label by *numerators N* and *denominators D* of rational fractions *N/D*

Solving: $\dfrac{\frac{n_2}{d_2}-t}{\frac{n_1}{d_1}-t}=-\dfrac{d_1}{d_2}$

$$\frac{n_2}{d_2}-t=-\frac{d_1}{d_2}\frac{n_1}{d_1}+\frac{d_1}{d_2}t$$

$$=-\frac{n_1}{d_2}+\frac{d_1}{d_2}t$$

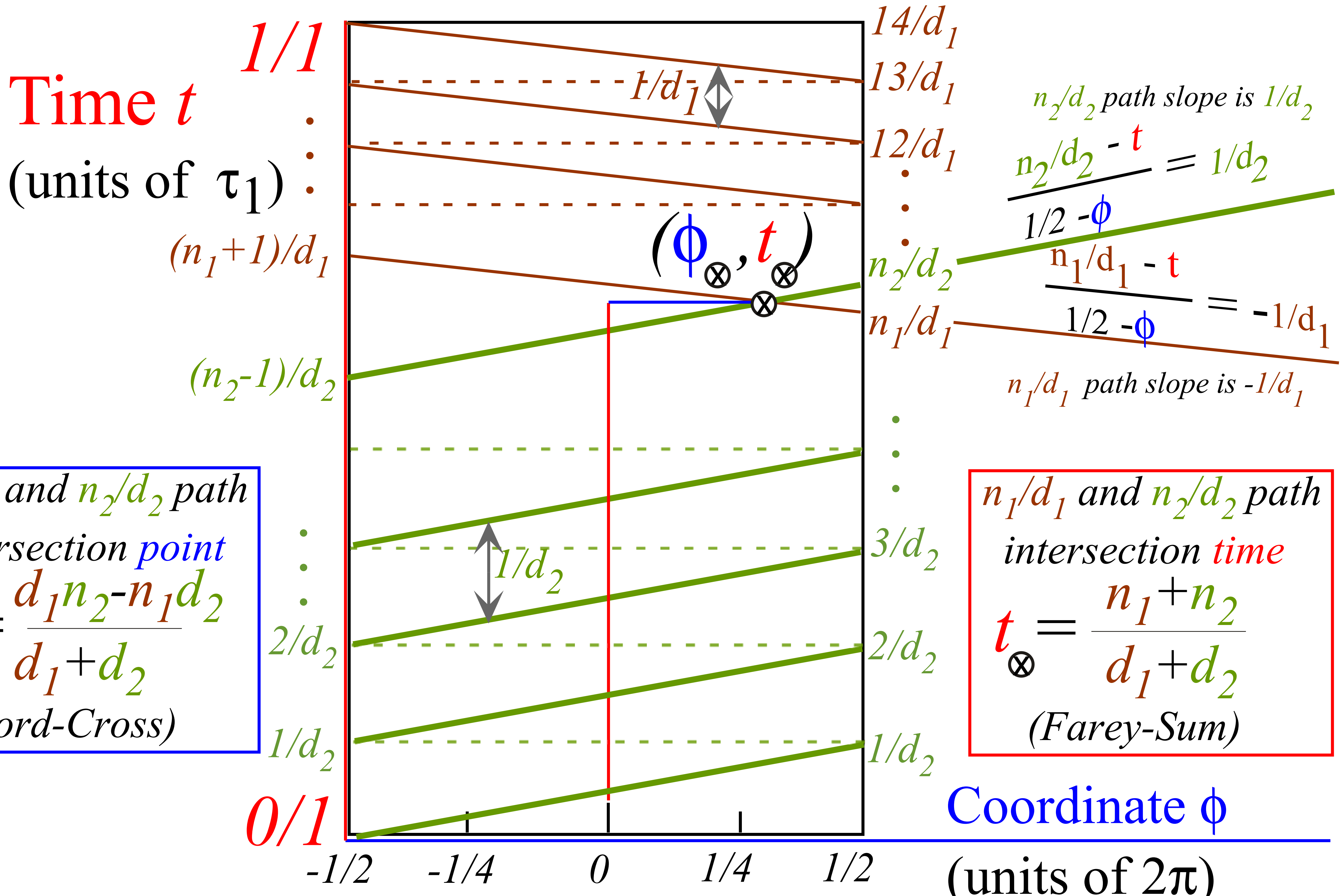
$$n_2+n_1=d_2t+d_1t$$



[John Farey, Phil. Mag.(1816)]

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

(Farey-Sum)

[*Lester. R. Ford, Am. Math. Monthly 45,586(1938)*]

[*John Farey, Phil. Mag.(1816)*]

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Polygonal geometry of $U(2) \supset C_N$ character spectral function

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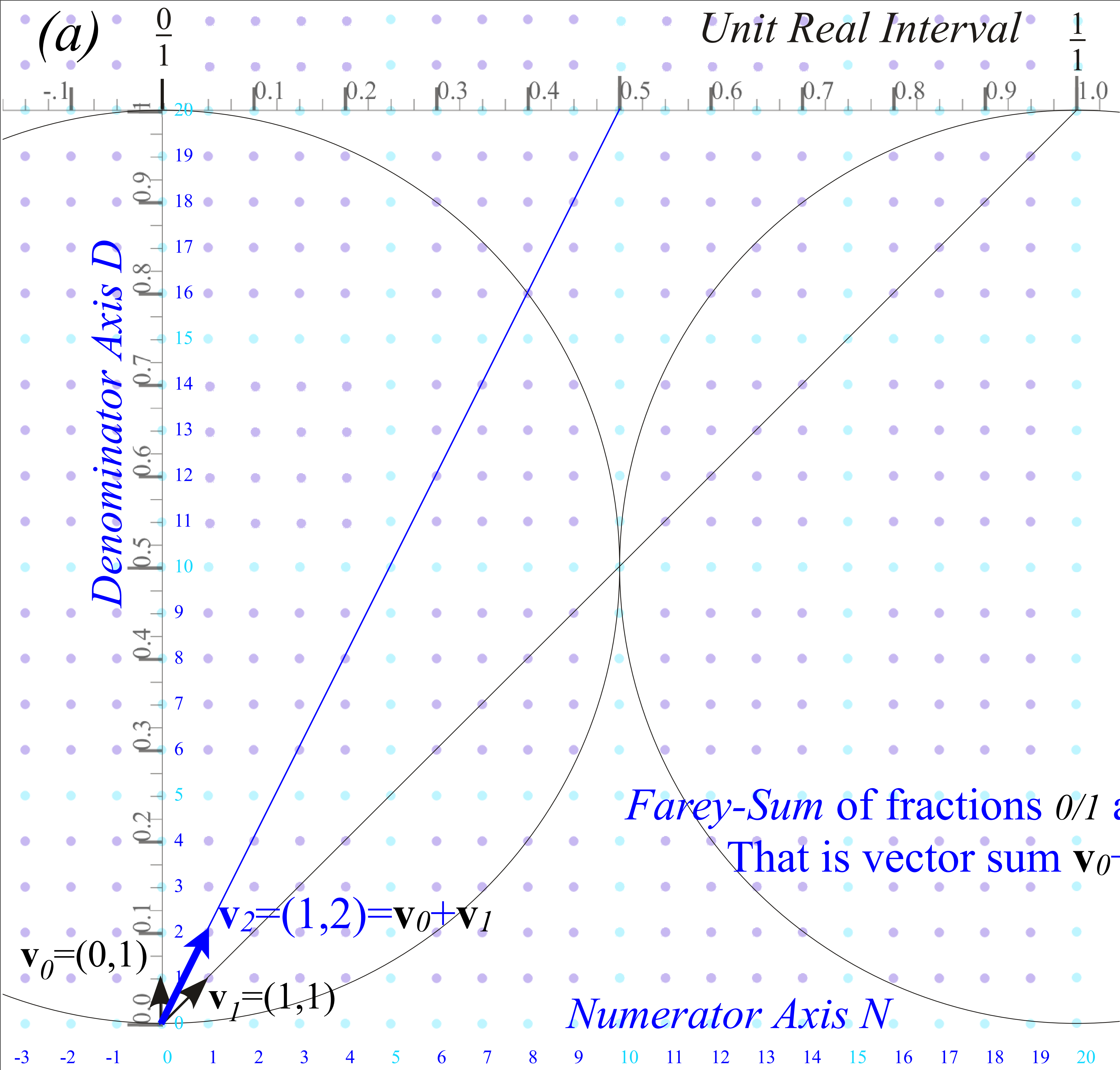
Geometry

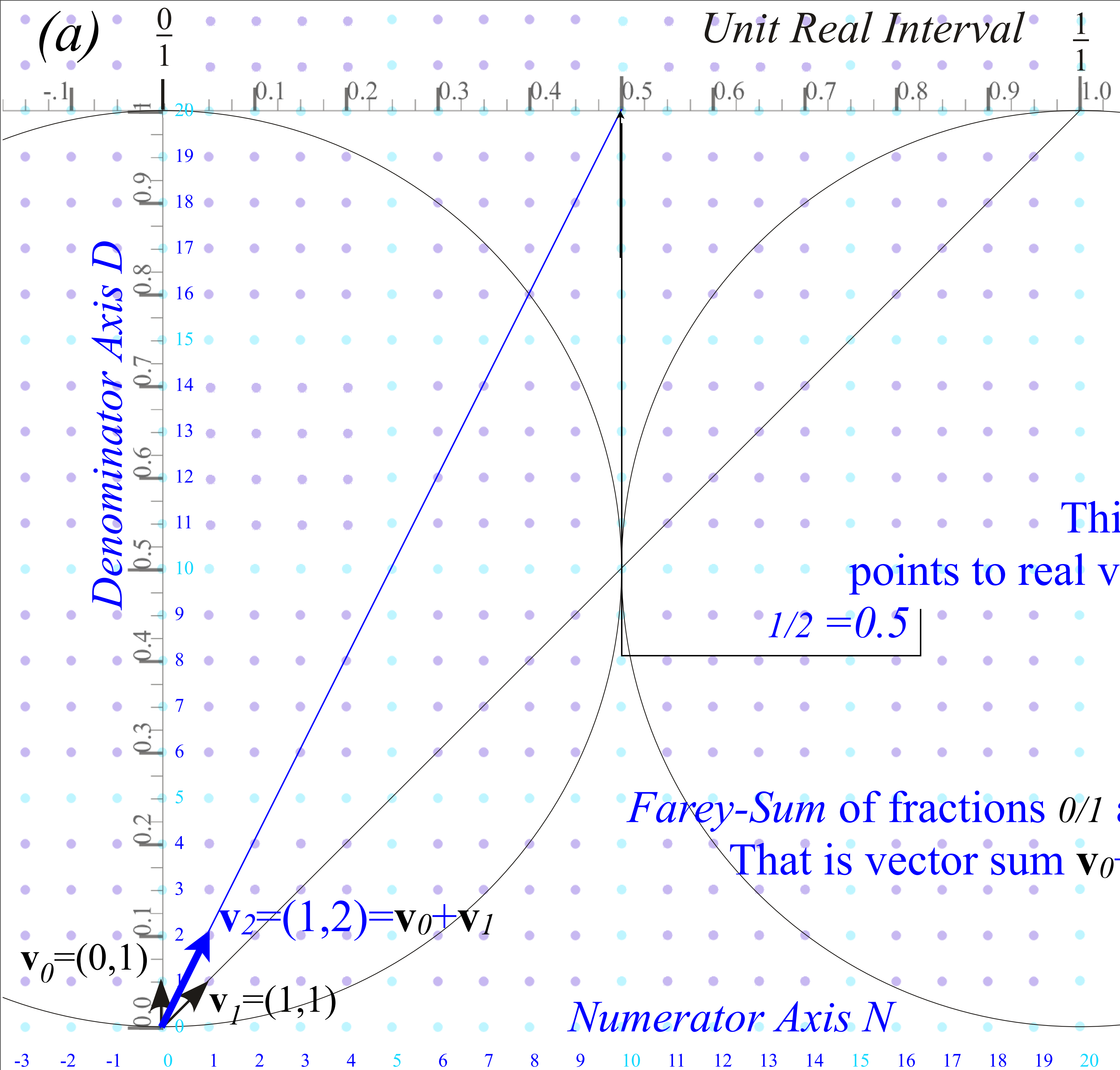


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

Farey Sum related to vector sum and *Ford Circles*

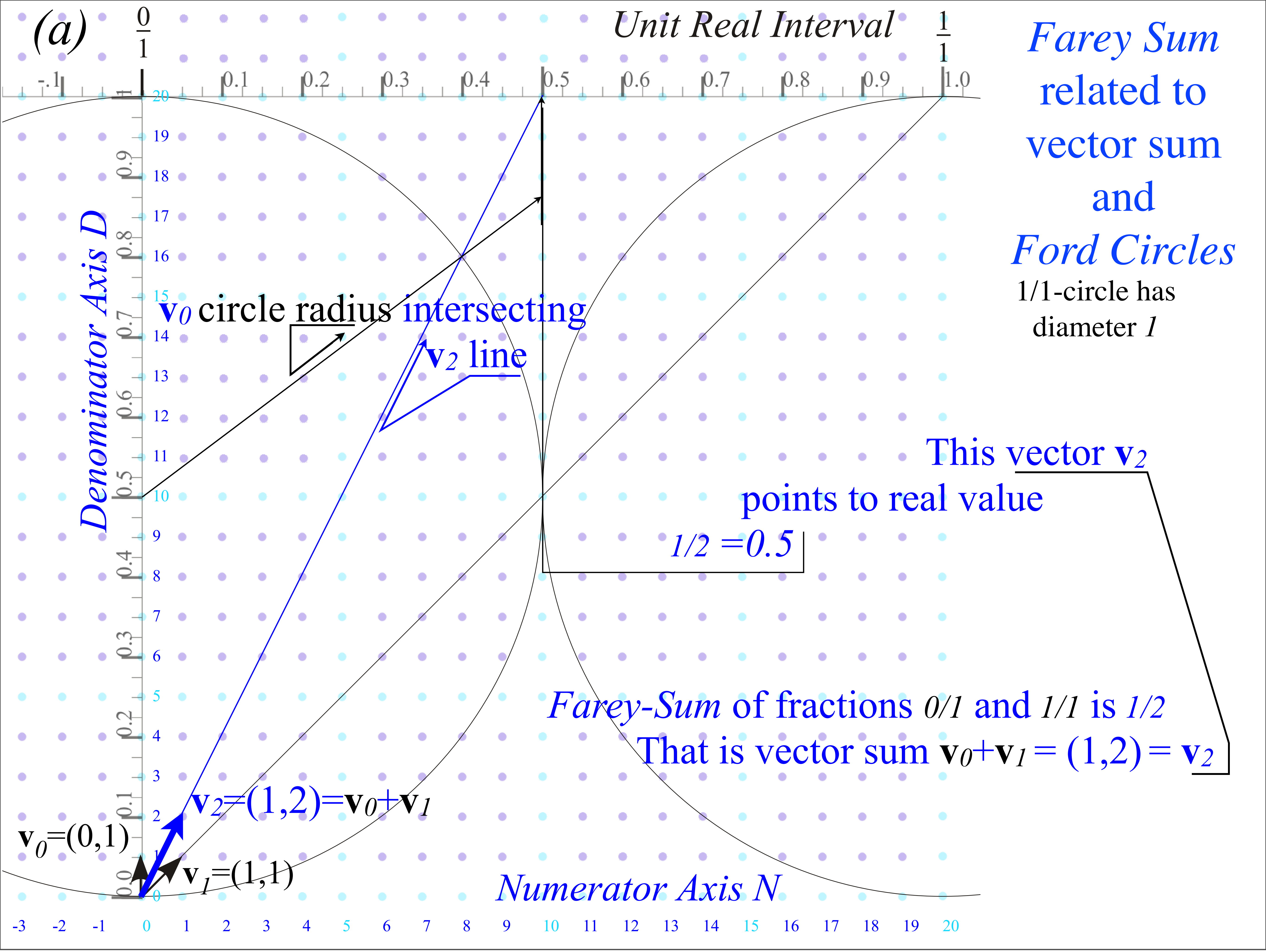
1/1-circle has diameter *1*



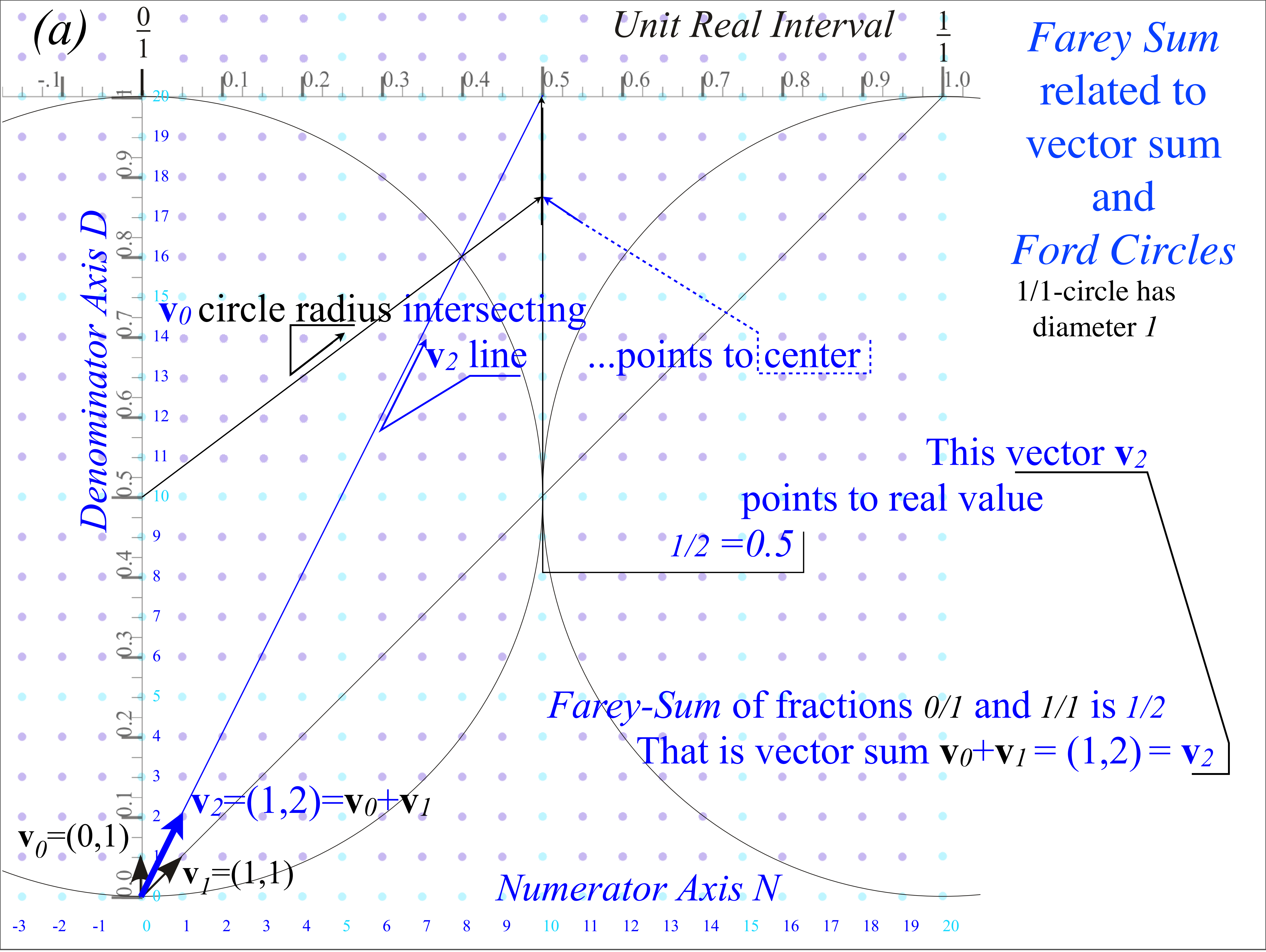
This vector $\mathbf{v}_2$ points to real value *1/2 =0.5*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

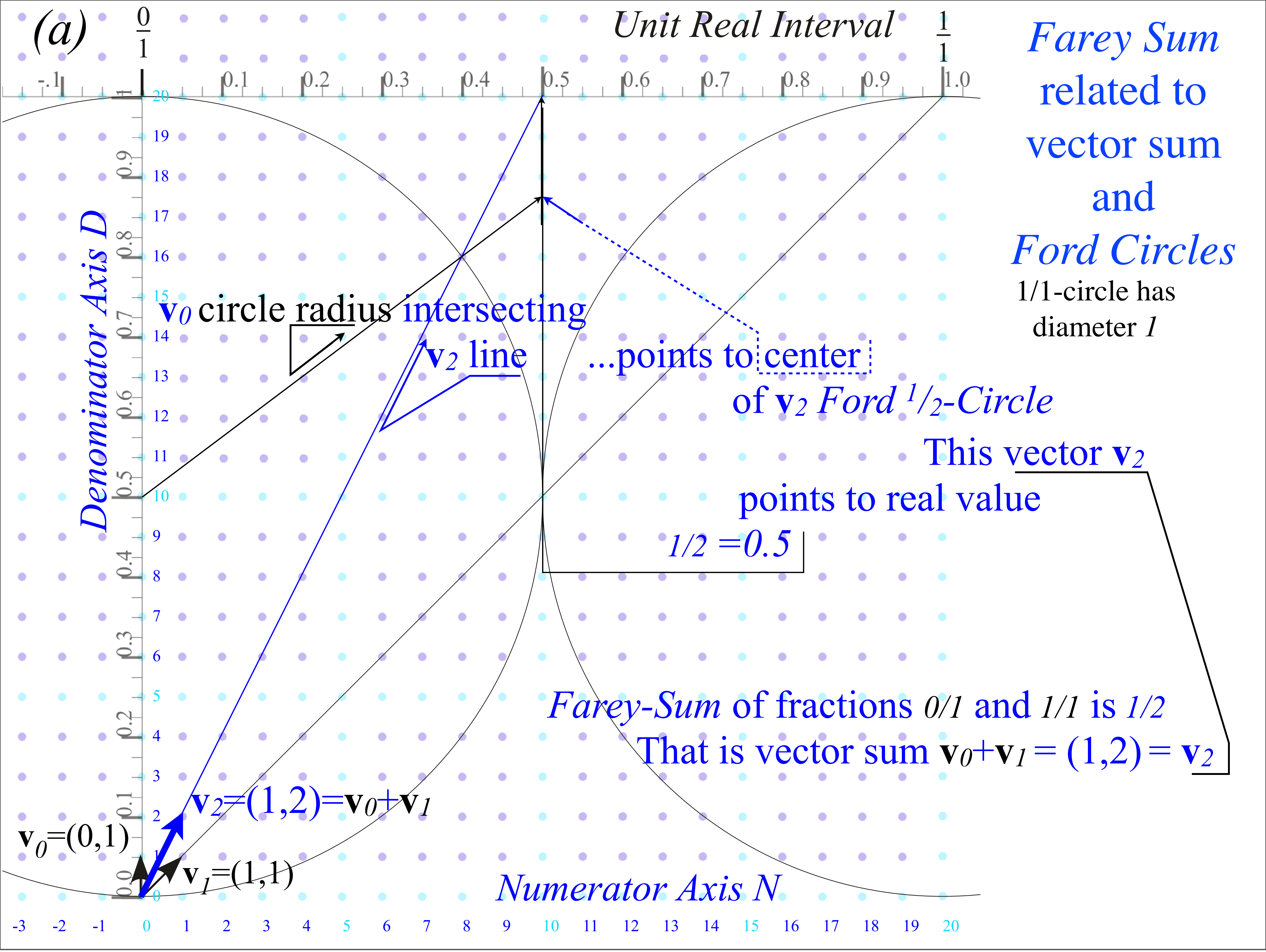
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

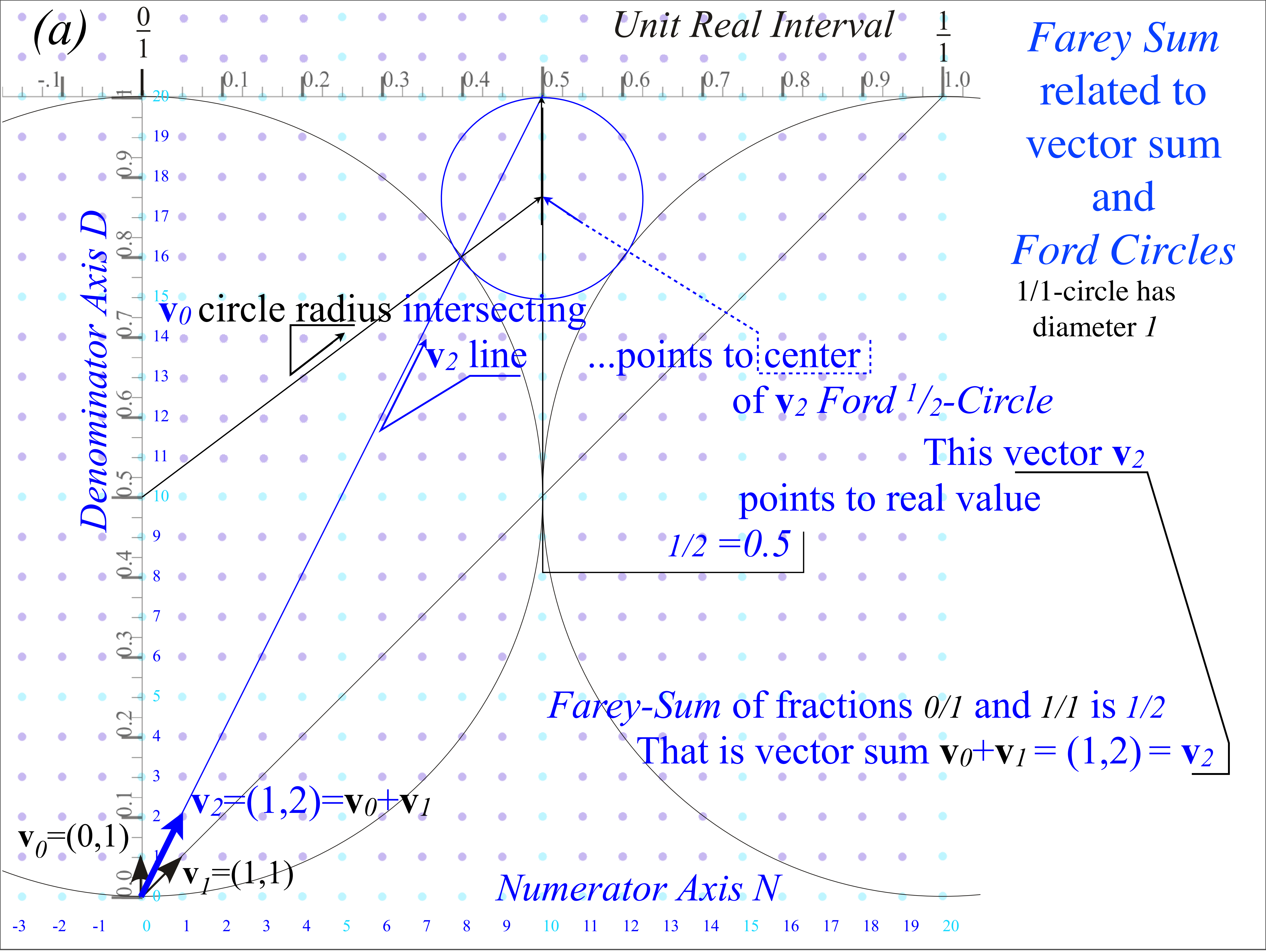


(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
v0 circle radius intersecting v2 line
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Numerator Axis N



(a)
0/1
Unit Real Interval
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
v0 circle radius intersecting
v2 line
...points to center
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Numerator Axis N


(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
v0 circle radius intersecting v2 line
...points to center of v2 Ford 1/2-Circle
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Denominator Axis D
Numerator Axis N


(a)
Unit Real Interval
0/1
1/1
Farey Sum
related to
vector sum
and
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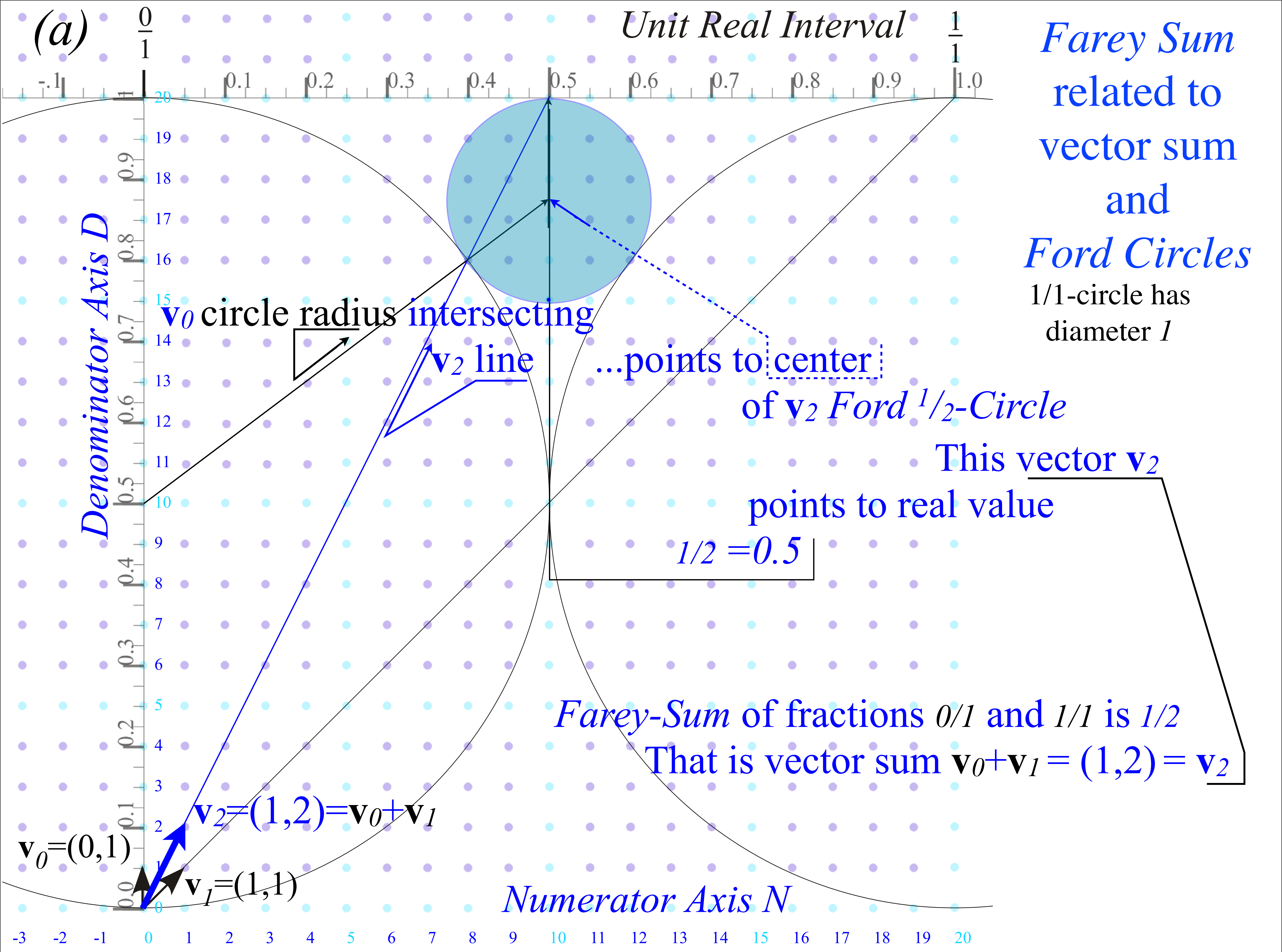

(a)
0/1
Unit Real Interval
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
v0 circle radius intersecting
v2 line
...points to center of v2 Ford 1/2-Circle
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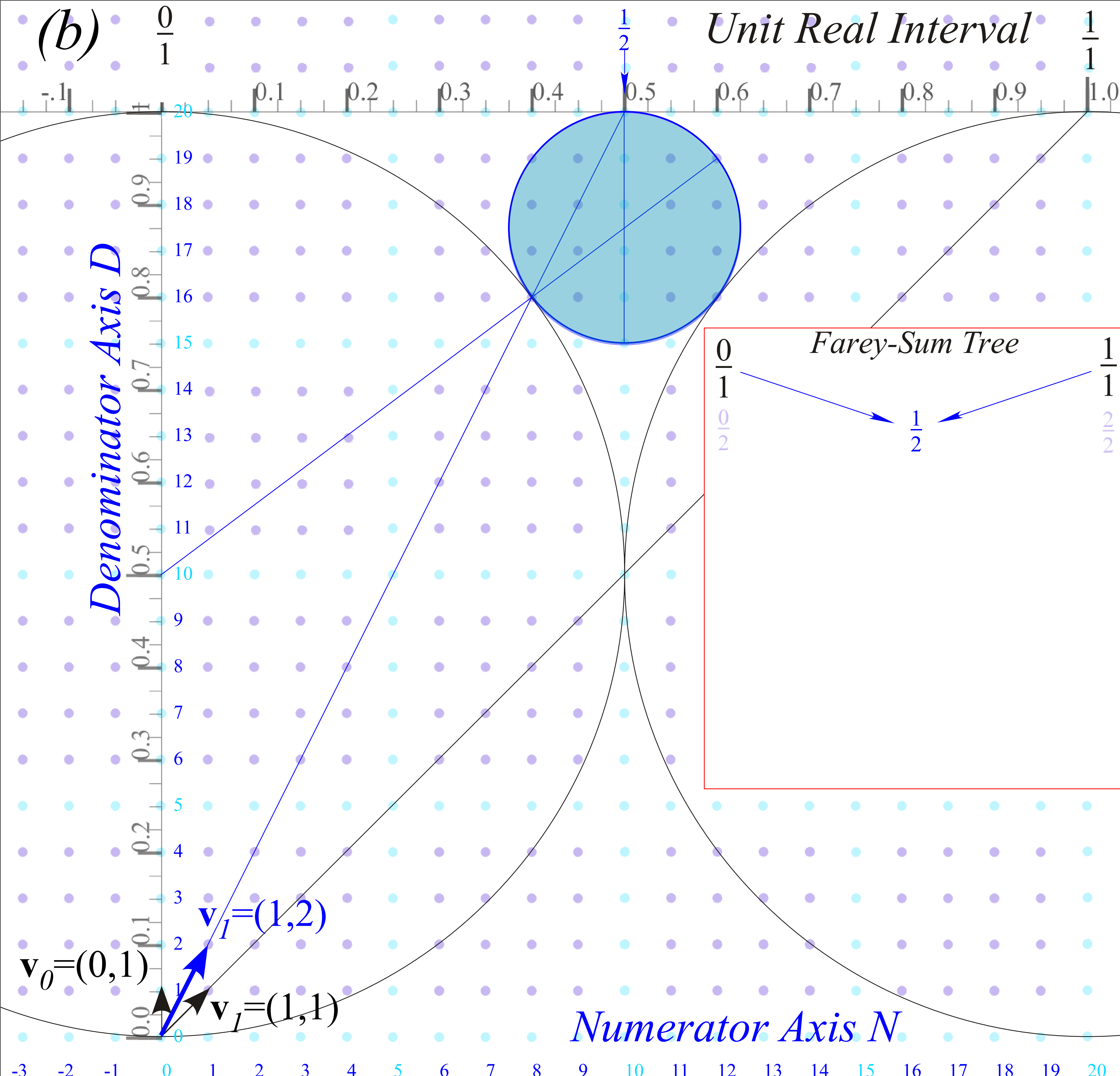
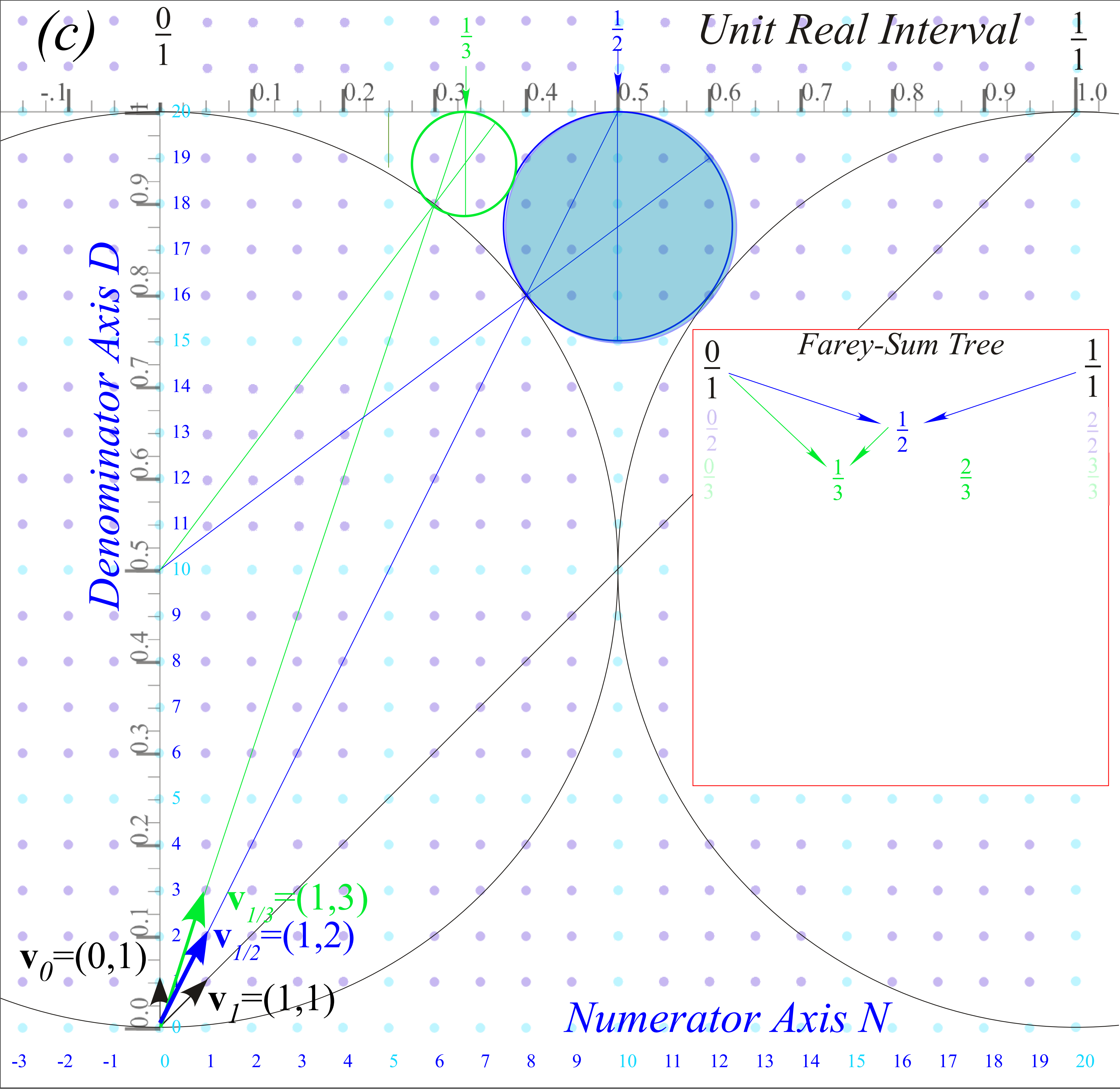


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

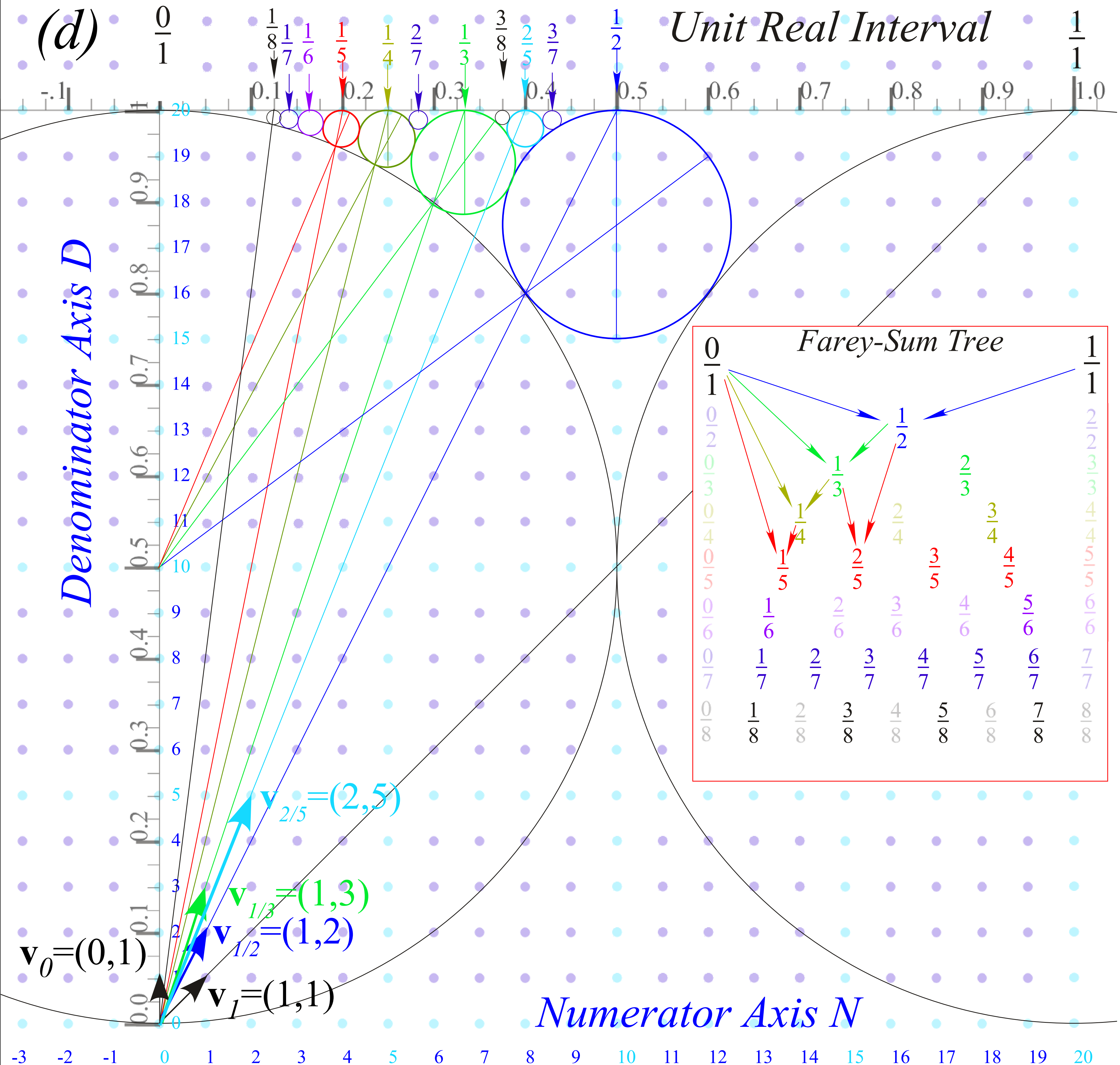
1/2-circle has diameter $1/2^2=1/4$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

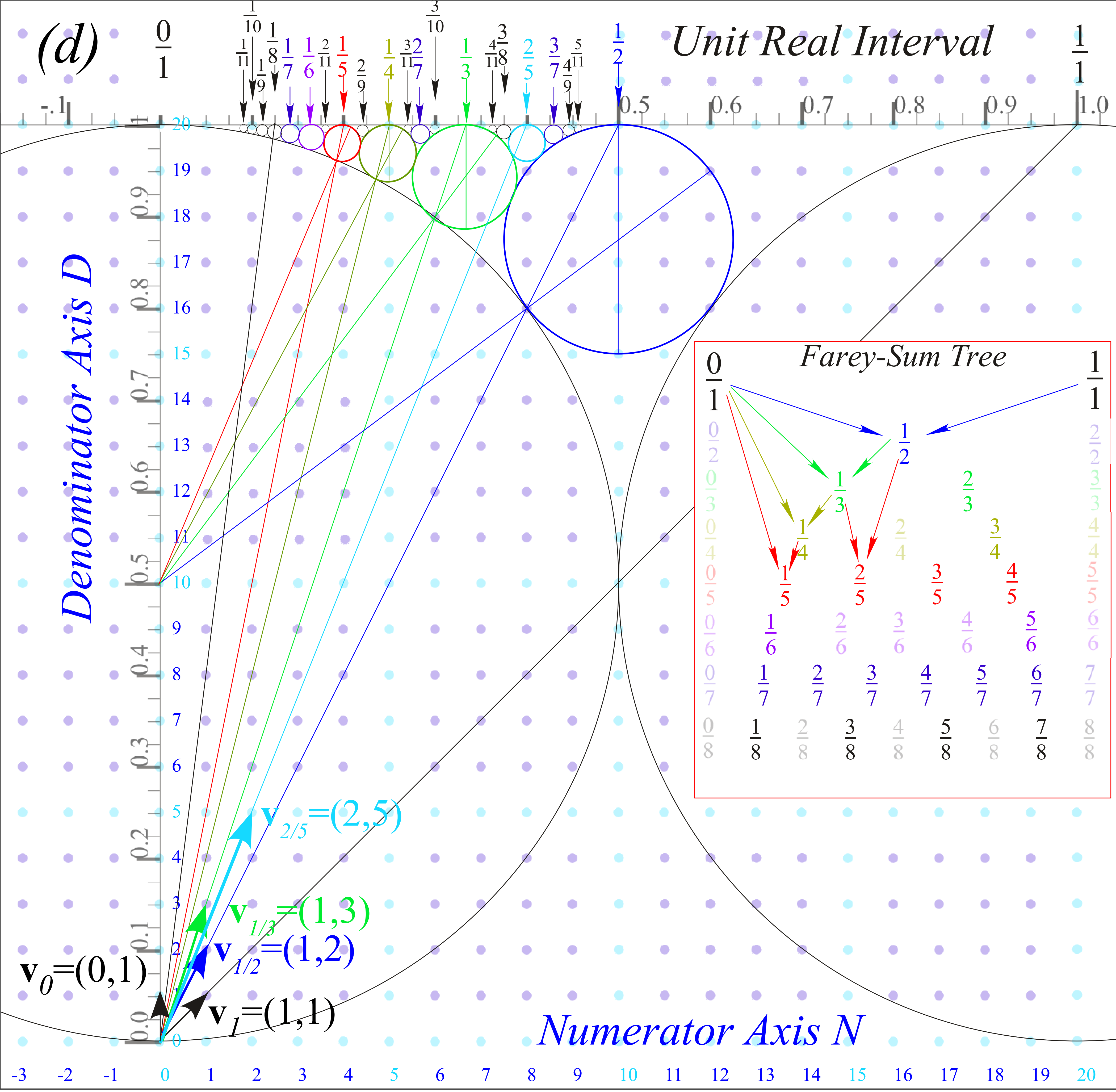


Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

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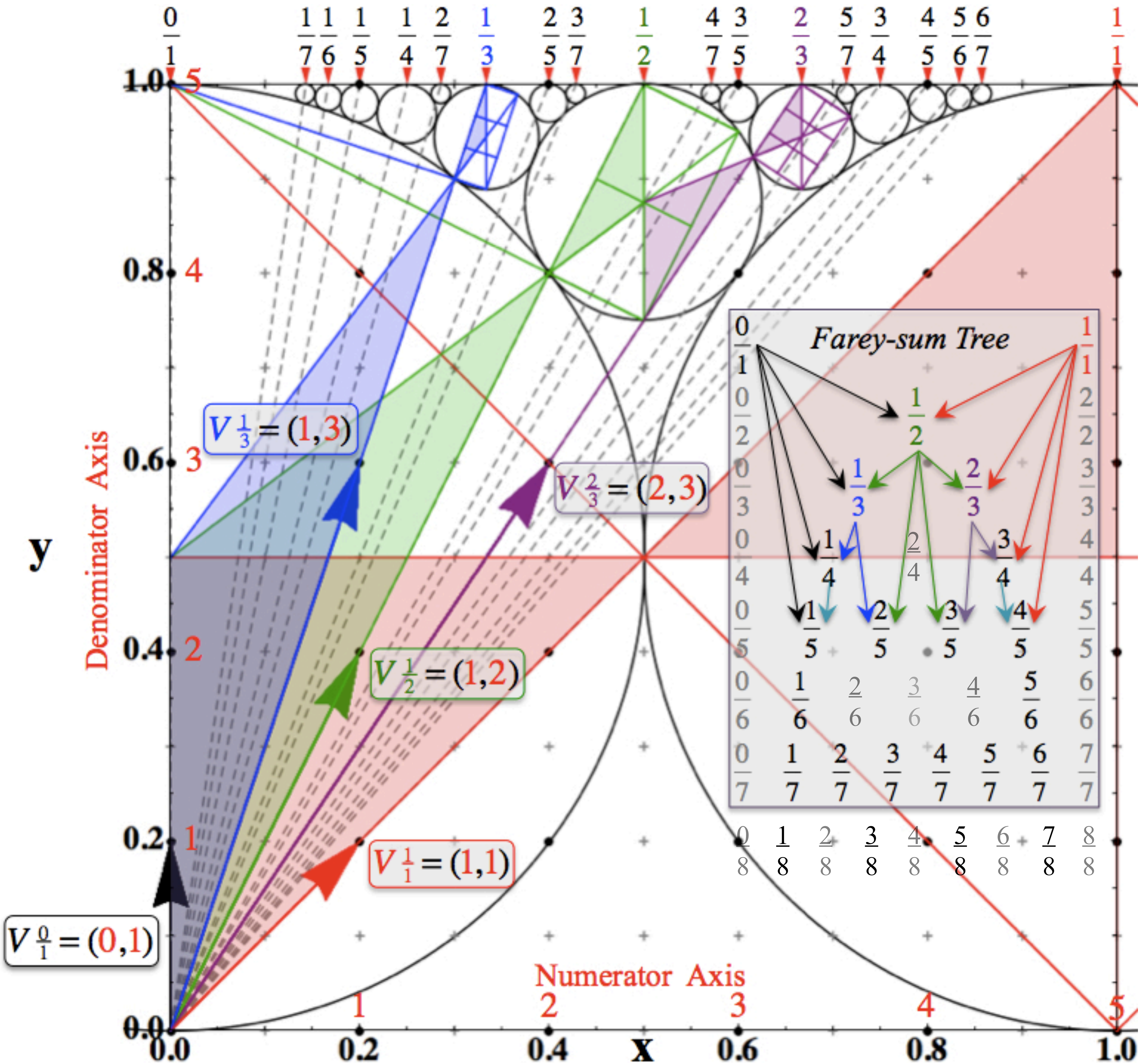
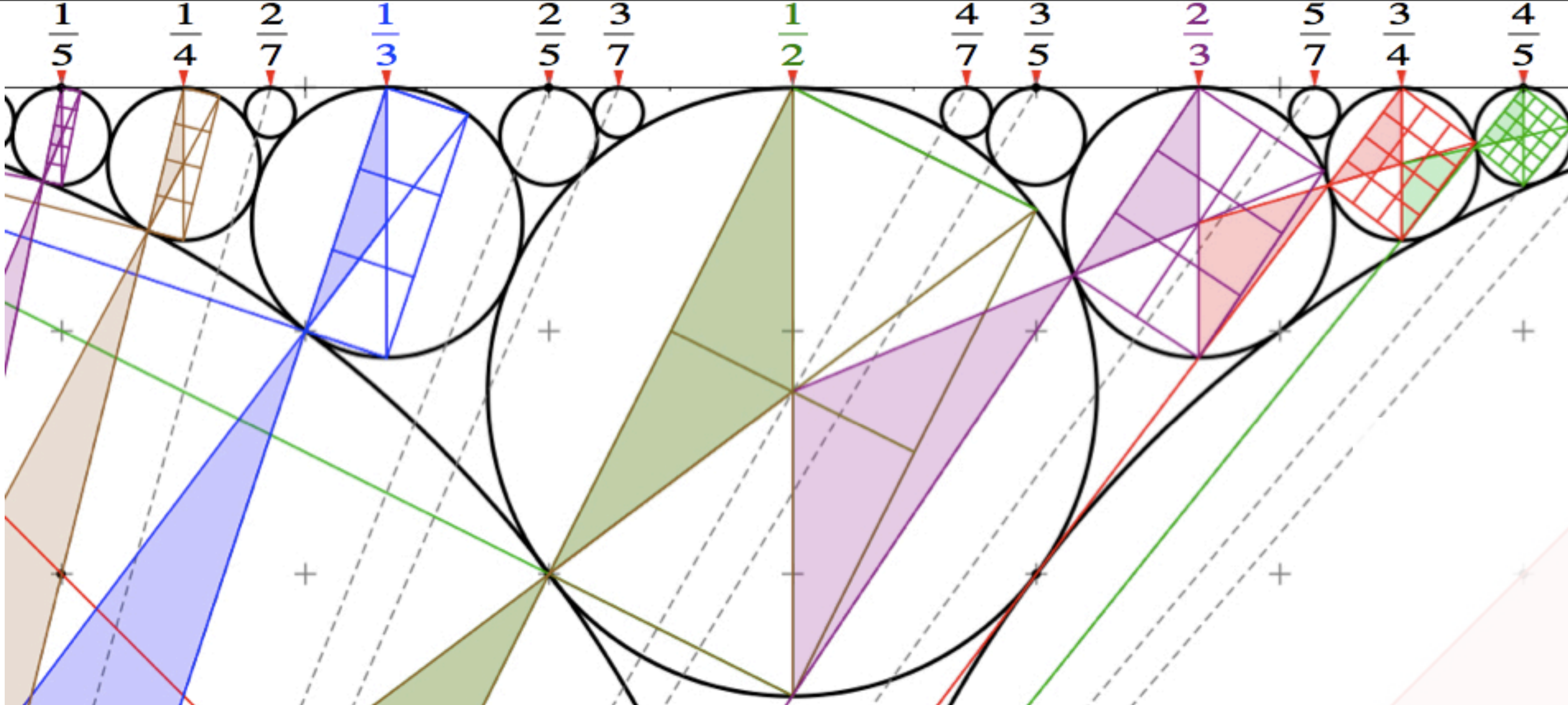


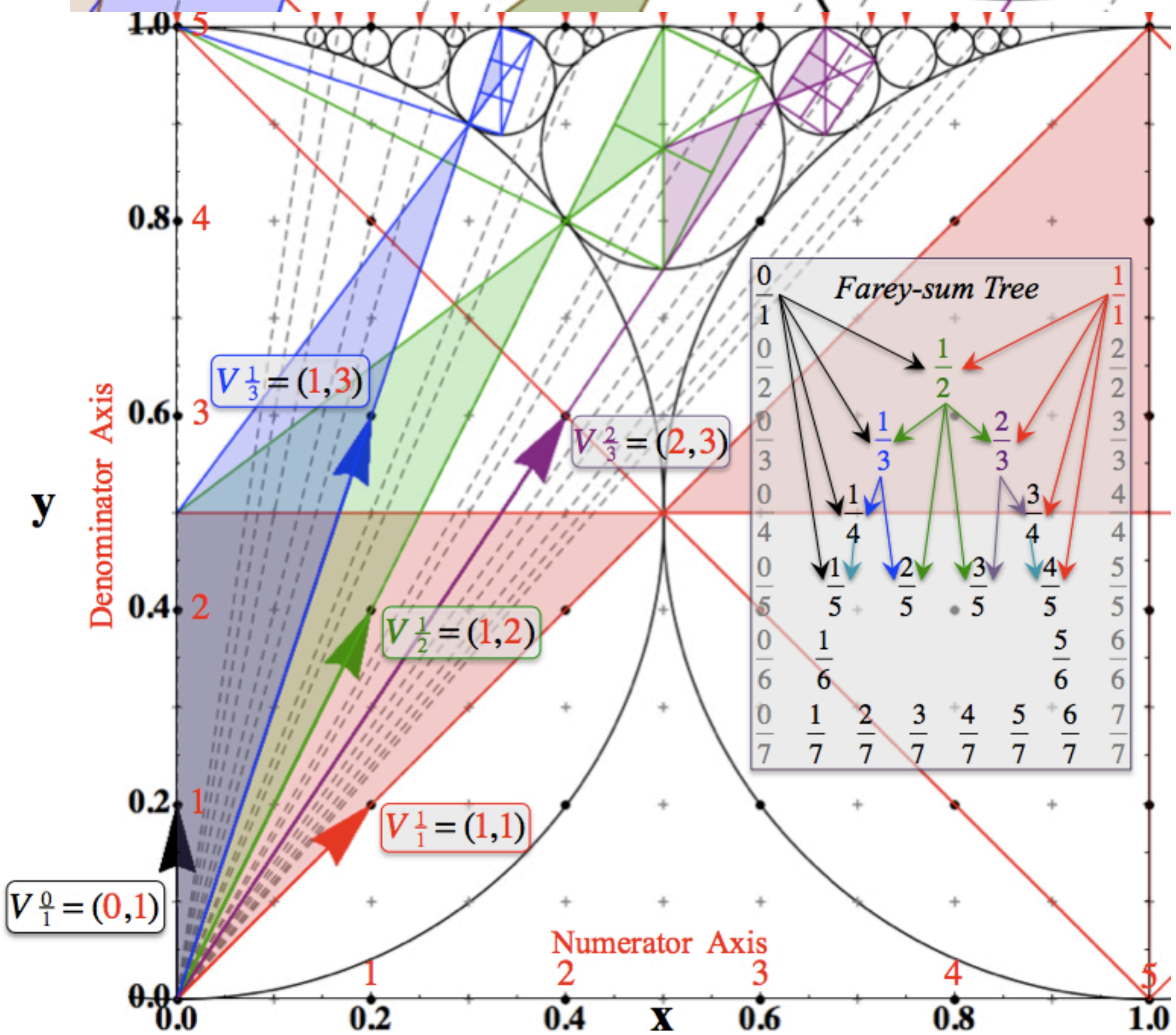
Thales Rectangles

provide
analytic geometry
of
fractal structure



"Quantized" Thales Rectangles

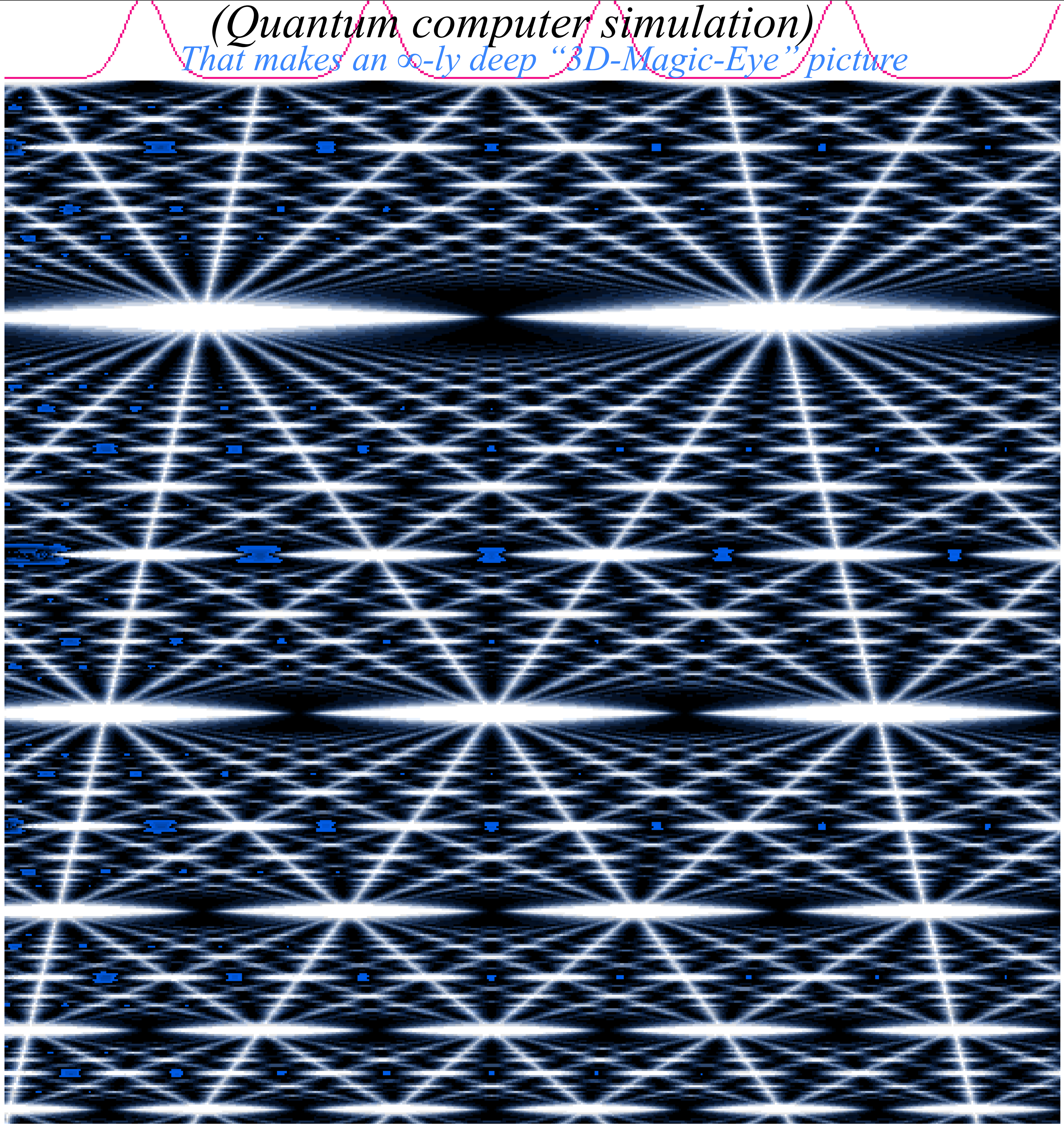
provide
analytic geometry
of
fractal structure



$D \le 1$	$\mathbf{\frac{0}{1}}$																						$\mathbf{\frac{1}{1}}$
$D \le 2$	$\frac{0}{1}$											$\mathbf{\frac{1}{2}}$											$\frac{1}{1}$
$D \le 3$	$\frac{0}{1}$							$\mathbf{\frac{1}{3}}$				$\frac{1}{2}$				$\mathbf{\frac{2}{3}}$							$\frac{1}{1}$
$D \le 4$	$\frac{0}{1}$					$\mathbf{\frac{1}{4}}$		$\frac{1}{3}$				$\frac{1}{2}$				$\frac{2}{3}$		$\mathbf{\frac{3}{4}}$					$\frac{1}{1}$
$D \le 5$	$\frac{0}{1}$				$\mathbf{\frac{1}{5}}$	$\frac{1}{4}$		$\frac{1}{3}$		$\mathbf{\frac{2}{5}}$		$\frac{1}{2}$		$\mathbf{\frac{3}{5}}$		$\frac{2}{3}$		$\frac{3}{4}$	$\mathbf{\frac{4}{5}}$				$\frac{1}{1}$
$D \le 6$	$\frac{0}{1}$			$\mathbf{\frac{1}{6}}$	$\frac{1}{5}$	$\frac{1}{4}$		$\frac{1}{3}$		$\frac{2}{5}$		$\frac{1}{2}$		$\frac{3}{5}$		$\frac{2}{3}$		$\frac{3}{4}$	$\frac{4}{5}$	$\mathbf{\frac{5}{6}}$			$\frac{1}{1}$
$D \le 7$	$\frac{0}{1}$		$\mathbf{\frac{1}{7}}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\mathbf{\frac{2}{7}}$	$\frac{1}{3}$		$\frac{2}{5}$	$\mathbf{\frac{3}{7}}$	$\frac{1}{2}$	$\mathbf{\frac{4}{7}}$	$\frac{3}{5}$		$\frac{2}{3}$	$\mathbf{\frac{5}{7}}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\mathbf{\frac{6}{7}}$		$\frac{1}{1}$
$D \le 8$	$\frac{0}{1}$	$\mathbf{\frac{1}{8}}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\mathbf{\frac{3}{8}}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\mathbf{\frac{5}{8}}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\mathbf{\frac{7}{8}}$	$\frac{1}{1}$

(Quantum computer simulation)

That makes an ∞-ly deep "3D-Magic-Eye" picture



Two wave archetypes: Pulse-Wave (PW) versus Continuous-Wave (CW)

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➡ *Discrete C_N beat phase dynamics (Characters gone wild!)*

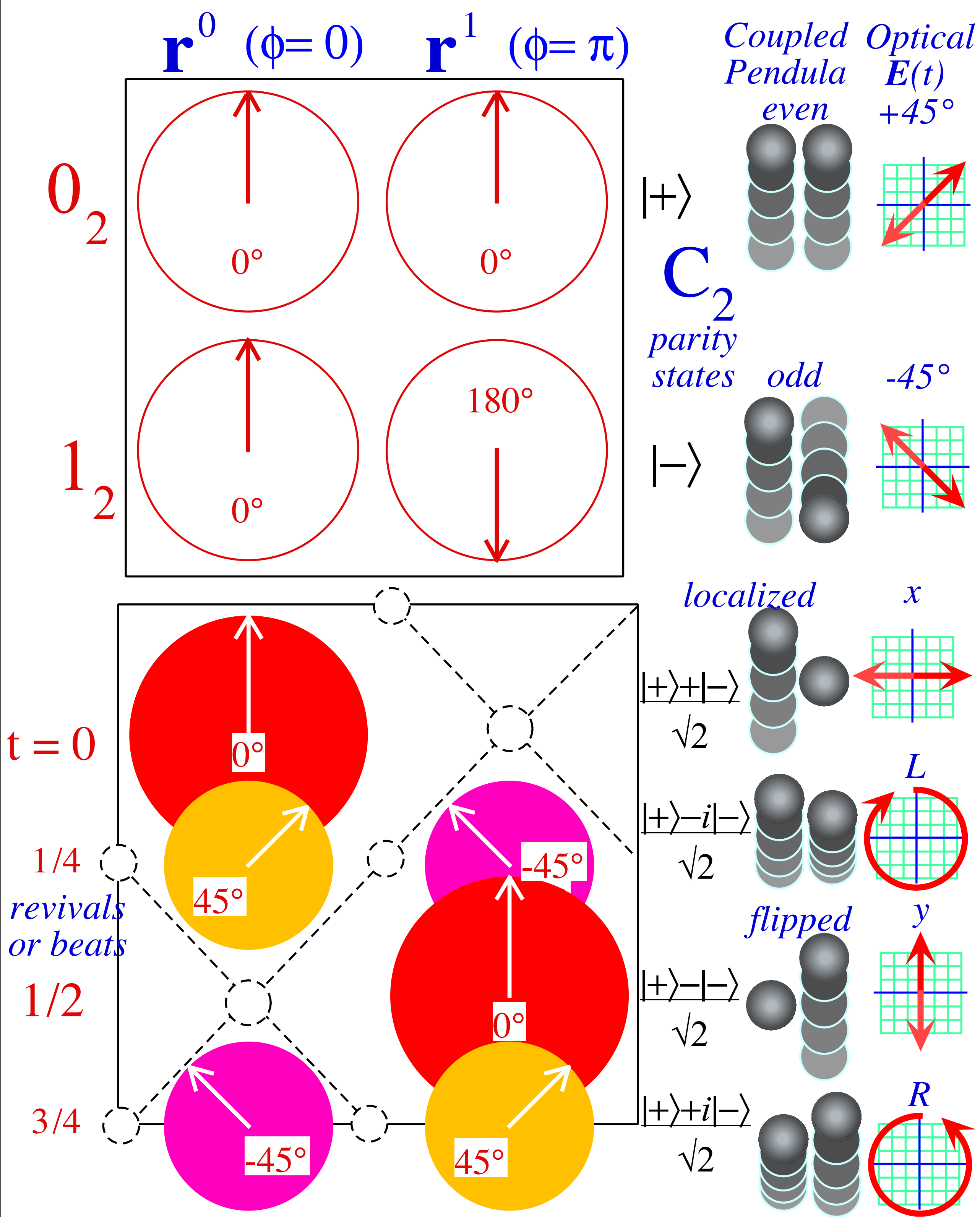
The classical bouncing-ball Monster-Mash

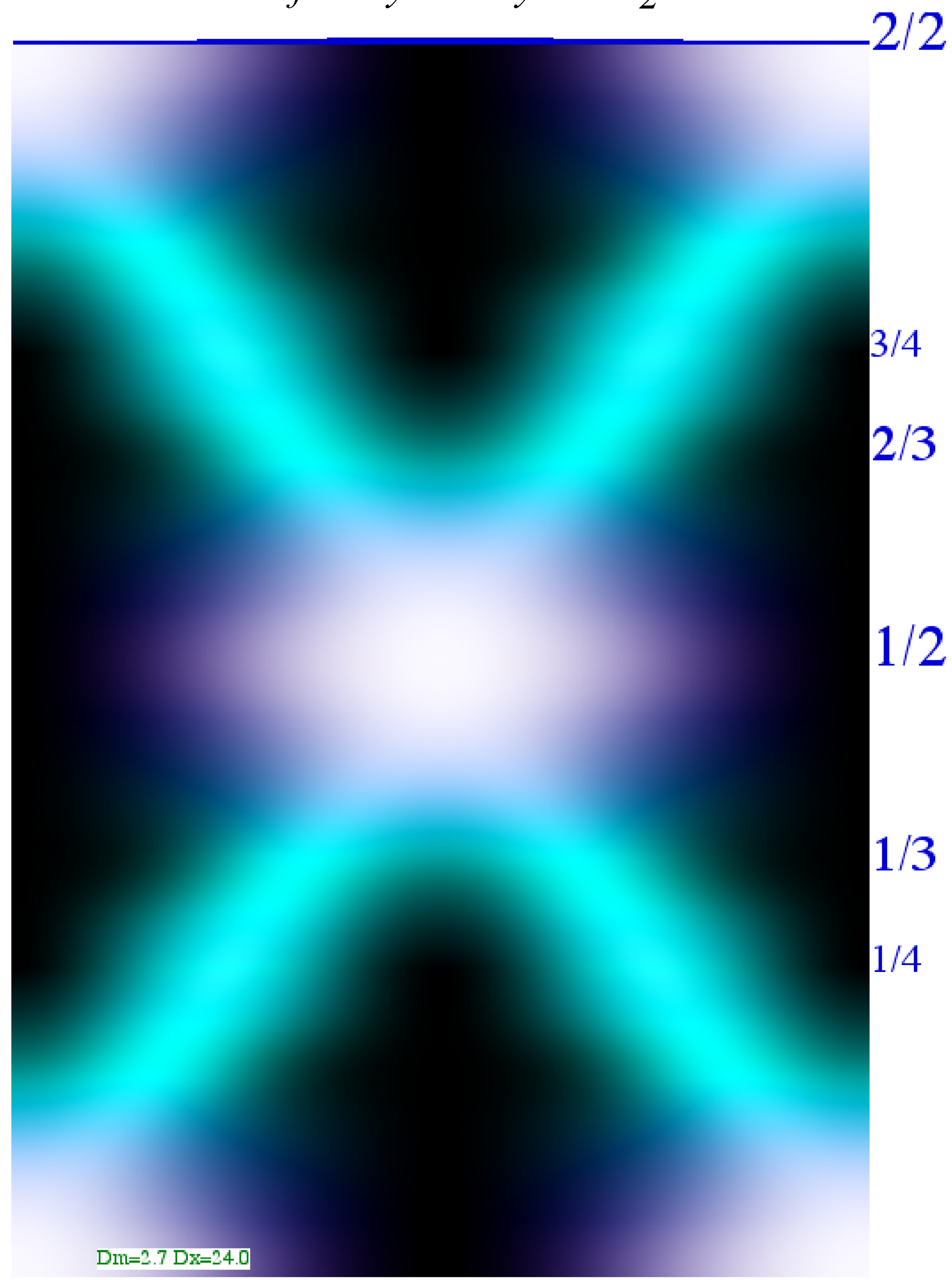
Polygonal geometry of $U(2) \supset C_N$ character spectral function

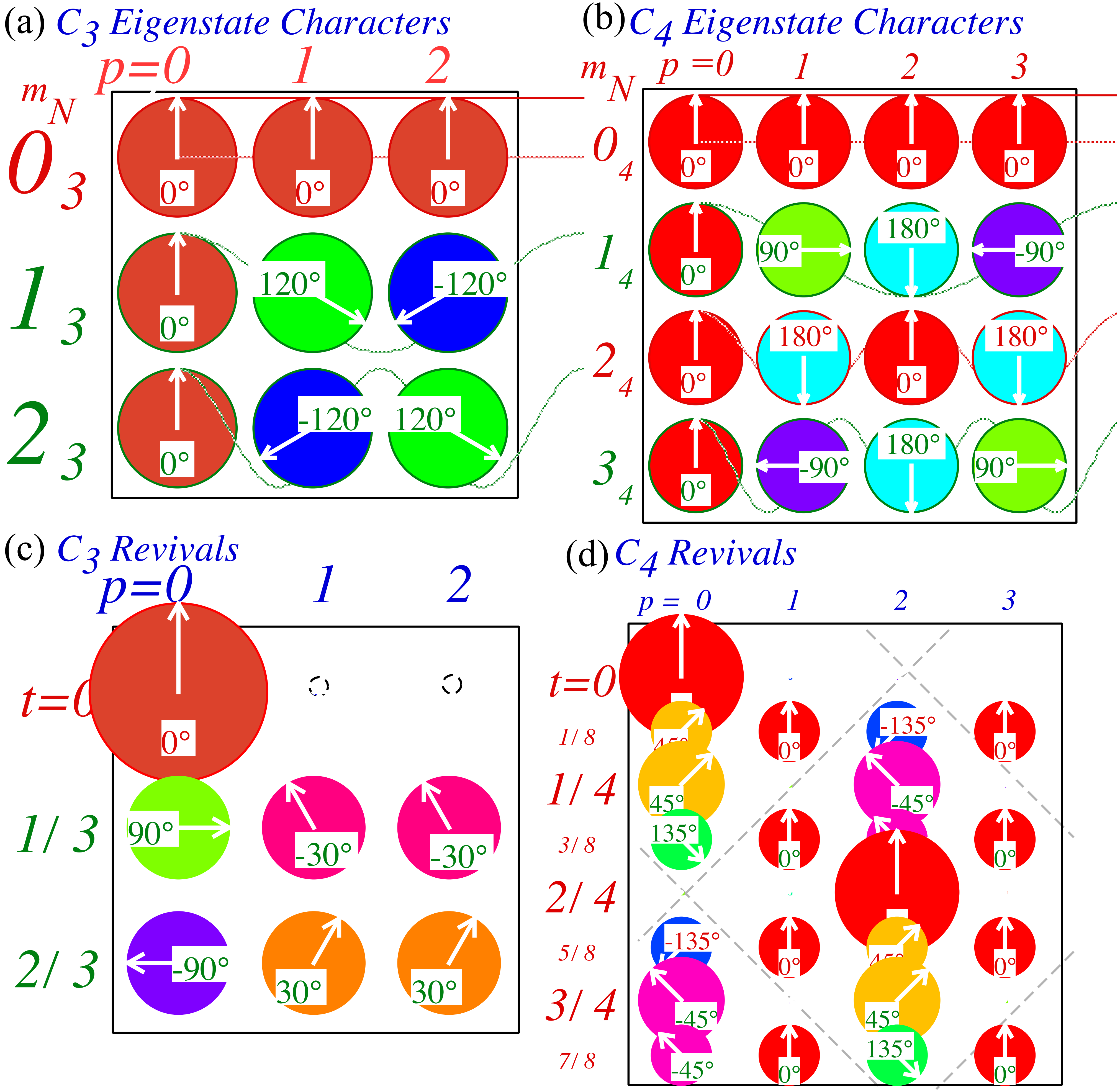
Algebra

Geometry

Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2

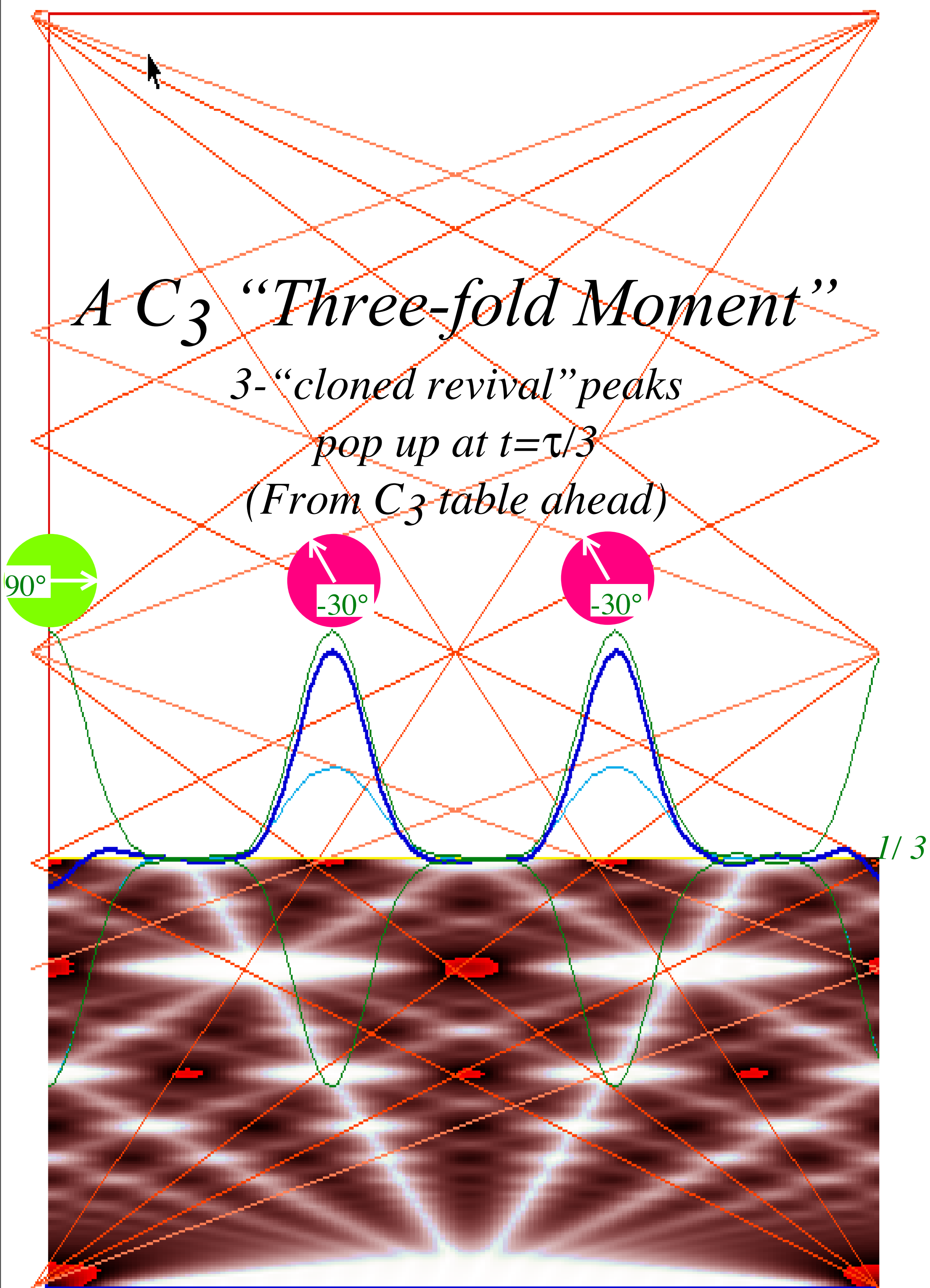


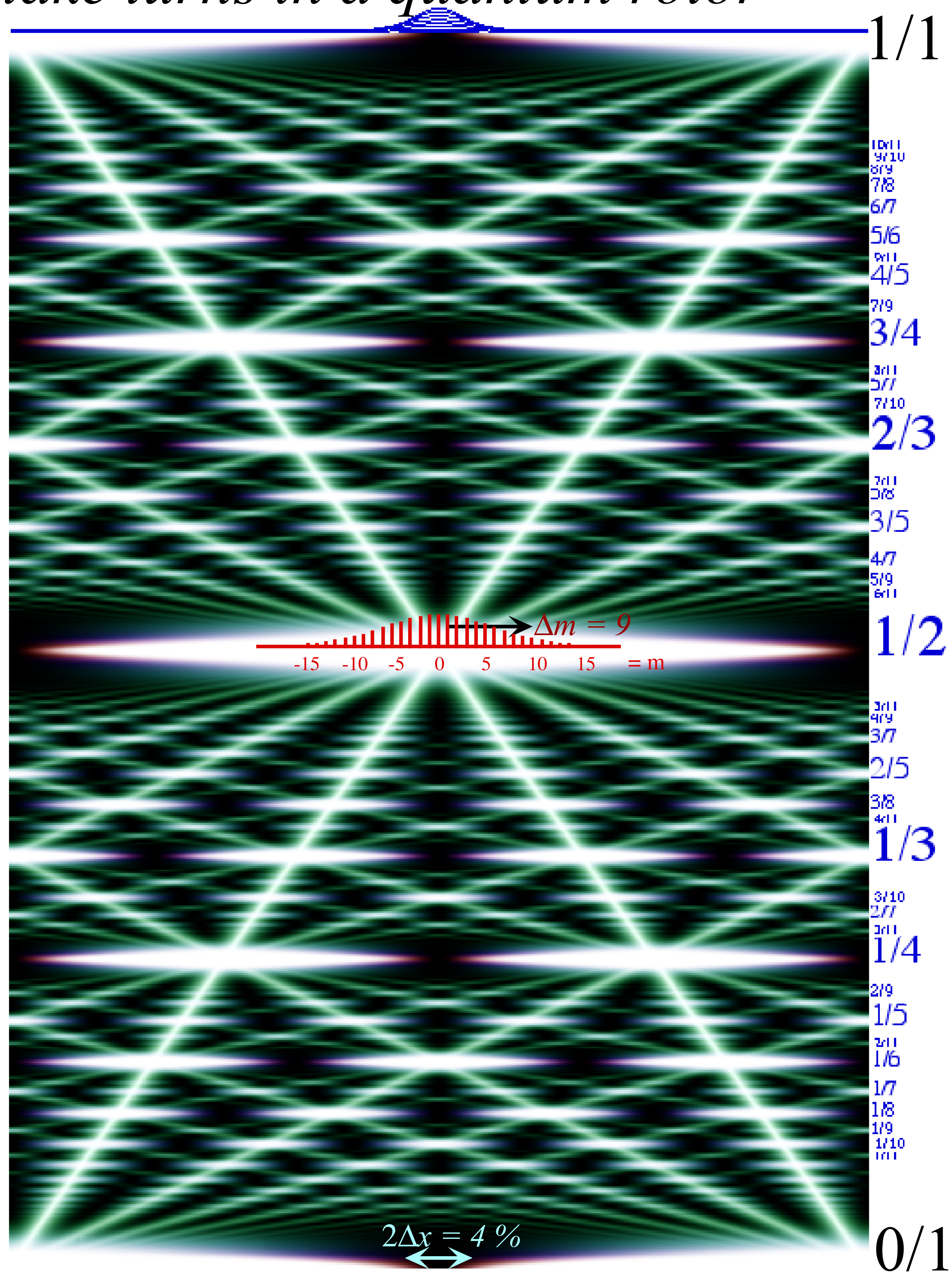




(a) C3 Eigenstate Characters
(b) C4 Eigenstate Characters
(c) C3 Revivals
(d) C4 Revivals
mN
p=0
1
2
3
03
13
23
04
14
24
34
t=0
1/8
1/4
3/8
2/4
5/8
3/4
7/8
1/3
2/3
0°
120°
-120°
90°
-90°
180°
-30°
30°
45°
-45°
135°
-135°

Revivals: All excited transitions take turns in a quantum rotor

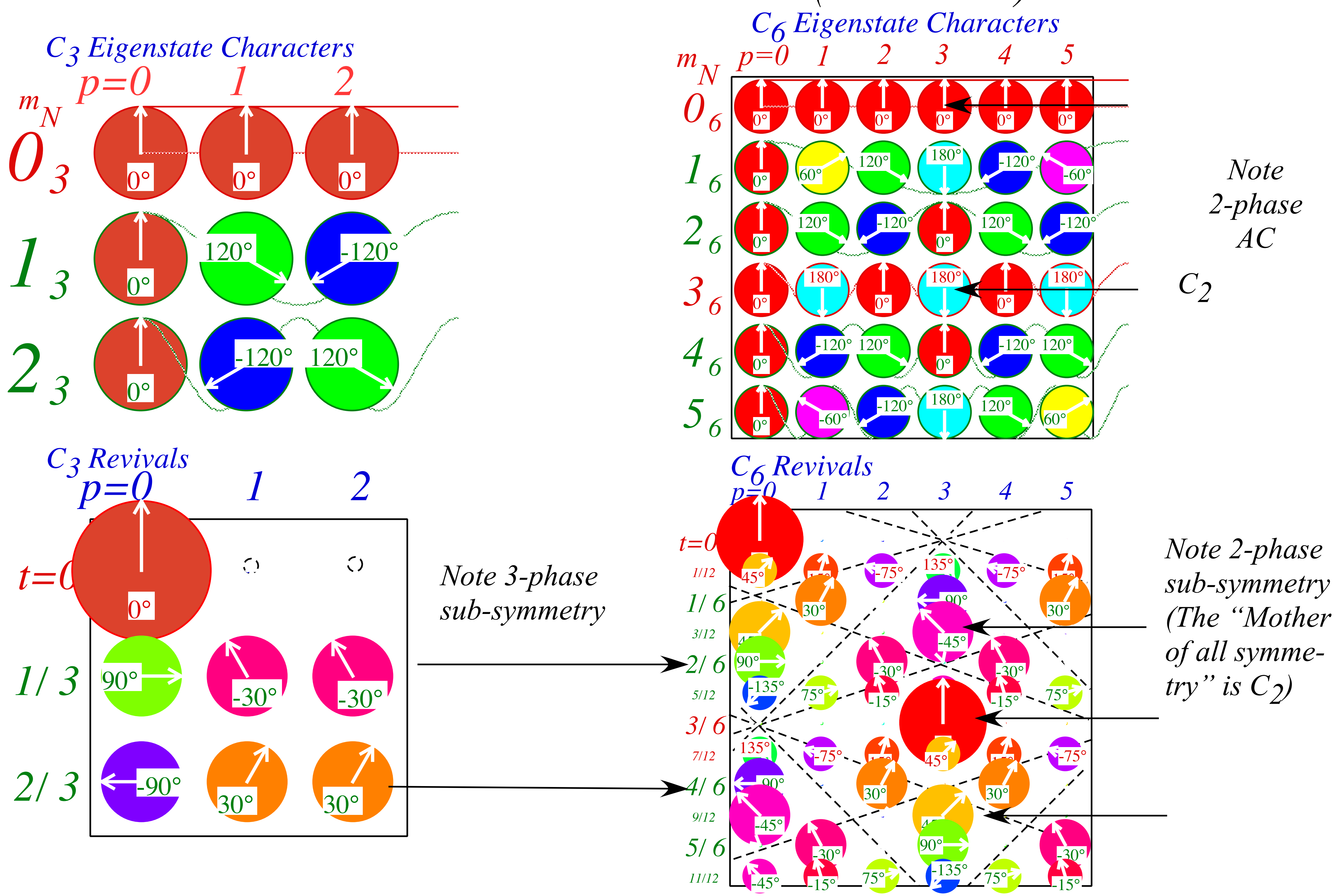


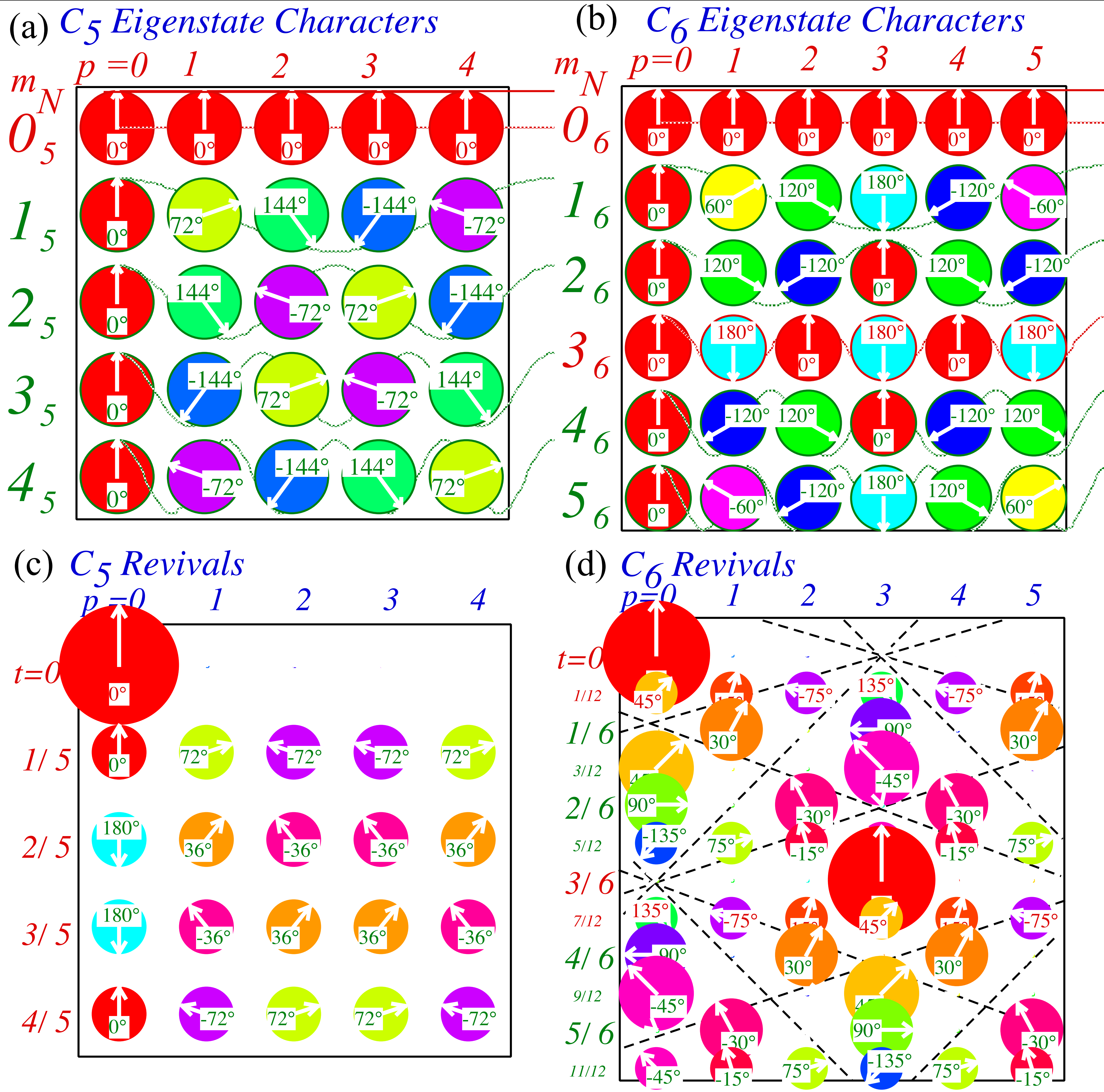


Simulating Complex Systems With Simpler Ones

Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

Discrete 6-State or Hexagonal System (6-Phase AC)




(a) C5 Eigenstate Characters
(b) C6 Eigenstate Characters
(c) C5 Revivals
(d) C6 Revivals
p =0 1 2 3 4
p=0 1 2 3 4 5
mN
05 15 25 35 45
06 16 26 36 46 56
t=0 1/5 2/5 3/5 4/5
t=0 1/12 1/6 3/12 2/6 5/12 3/6 7/12 4/6 9/12 5/6 11/12

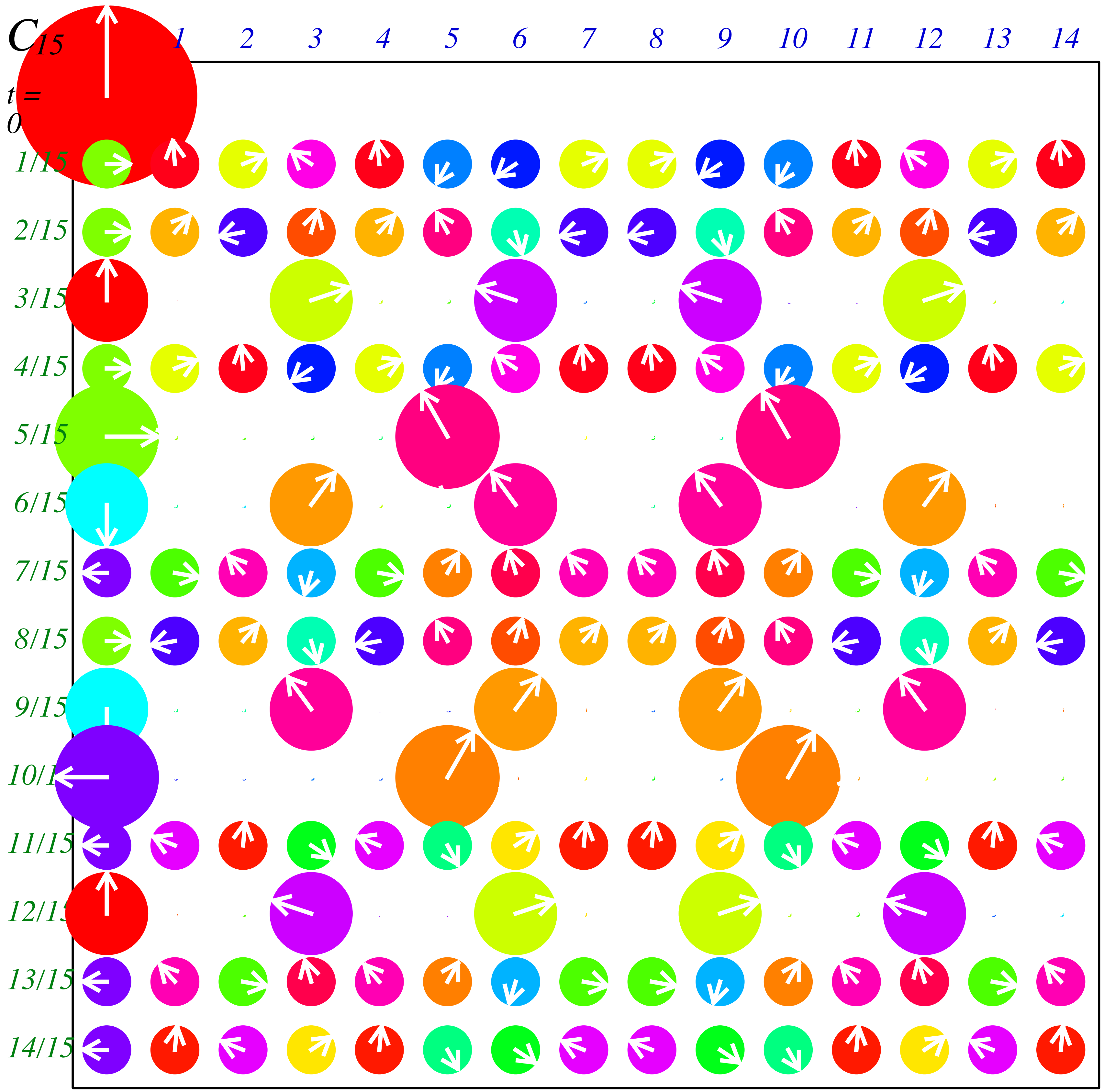
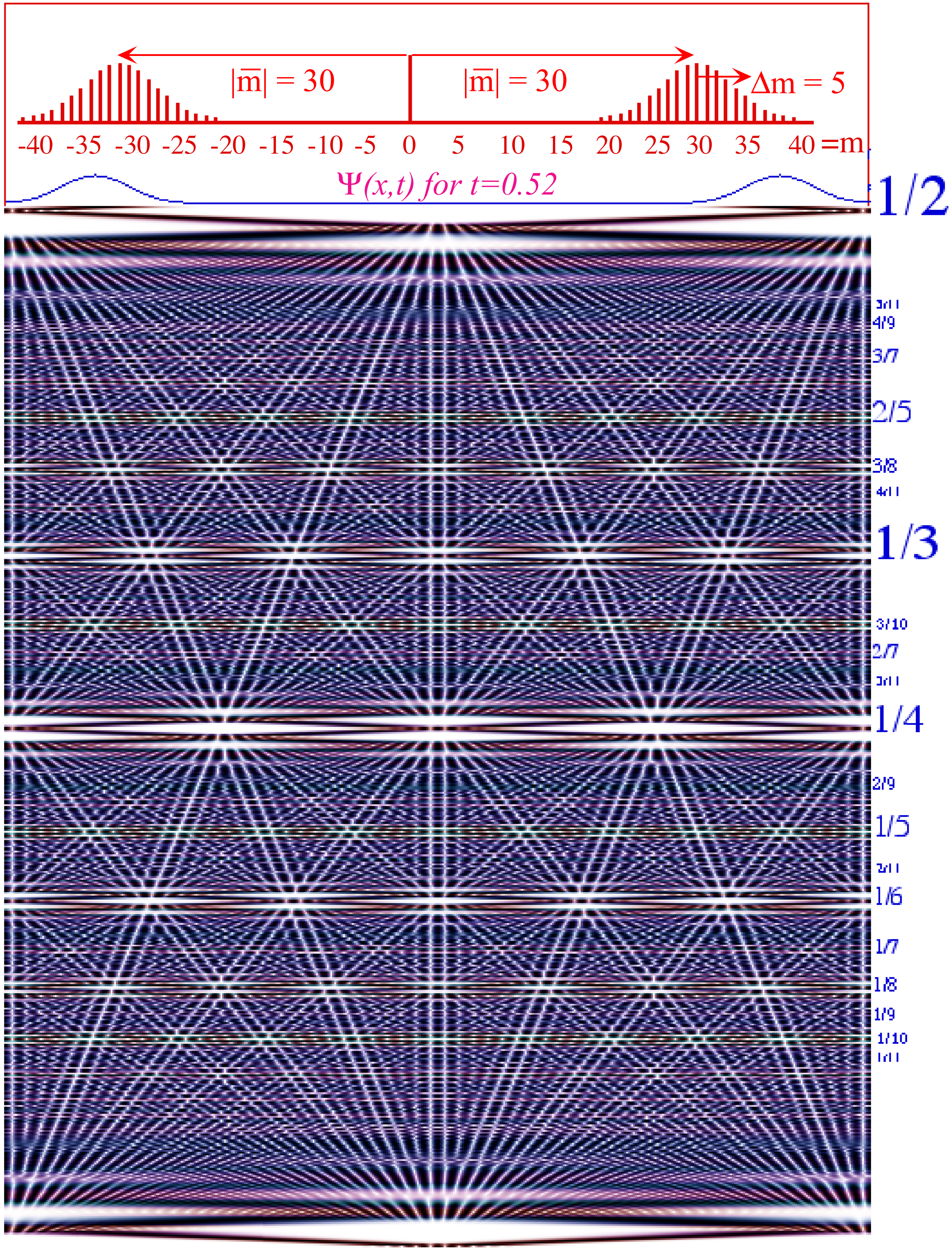


Fig. 9.4.4 Bohr space-time revival pattern for C_{15} Bohr system.



|m̄| = 30
|m̄| = 30
Δm = 5
-40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40 =m
Ψ(x,t) for t=0.52
1/2
4/9
3/7
2/5
3/8
1/3
3/10
2/7
1/4
2/9
1/5
1/6
1/7
1/8
1/9
1/10

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→ *The classical bouncing-ball Monster-Mash*

Polygonal geometry of $U(2)\supset C_N$ character spectral function

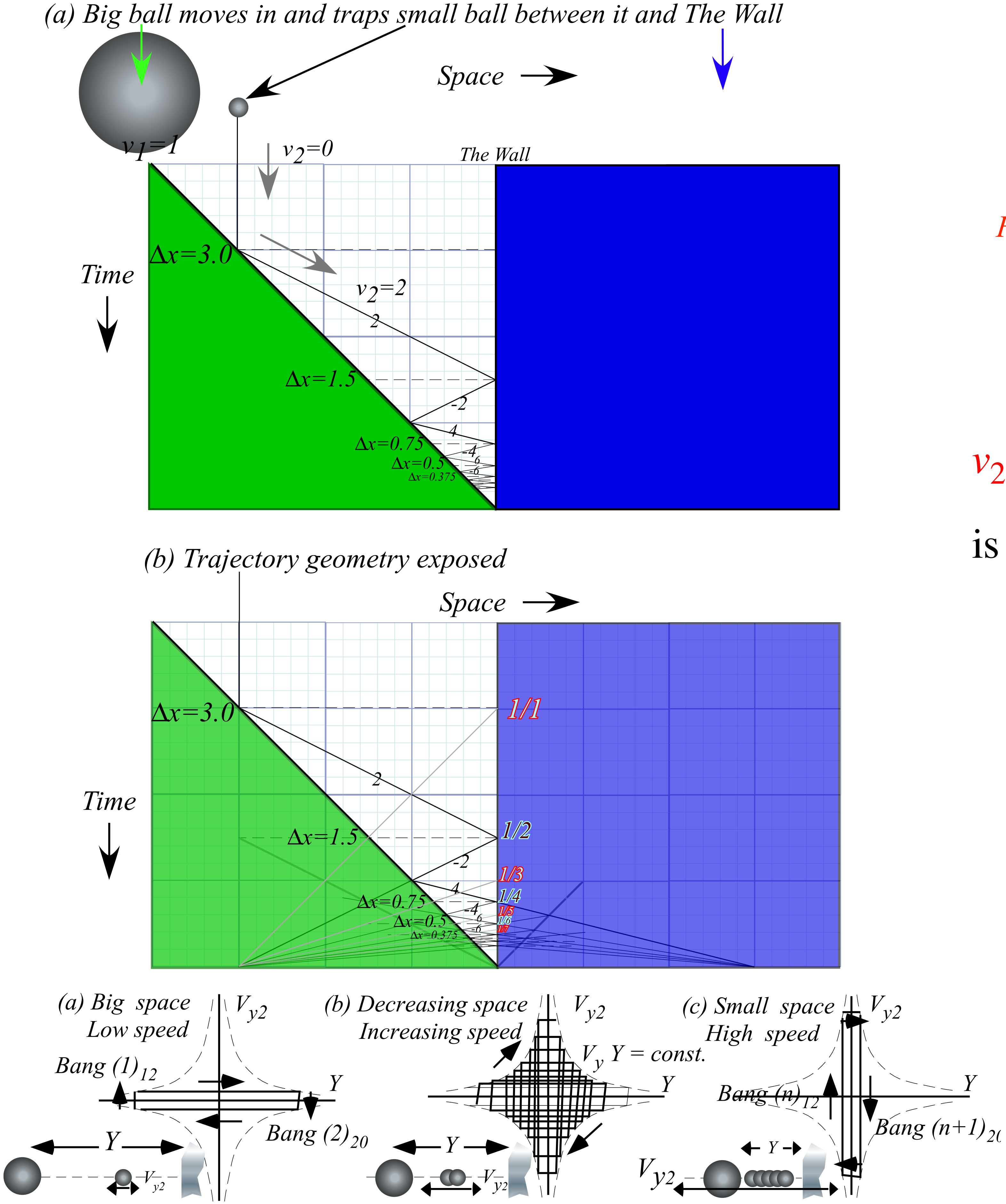
Algebra

Geometry

The Classical "Monster Mash"

Classical introduction to

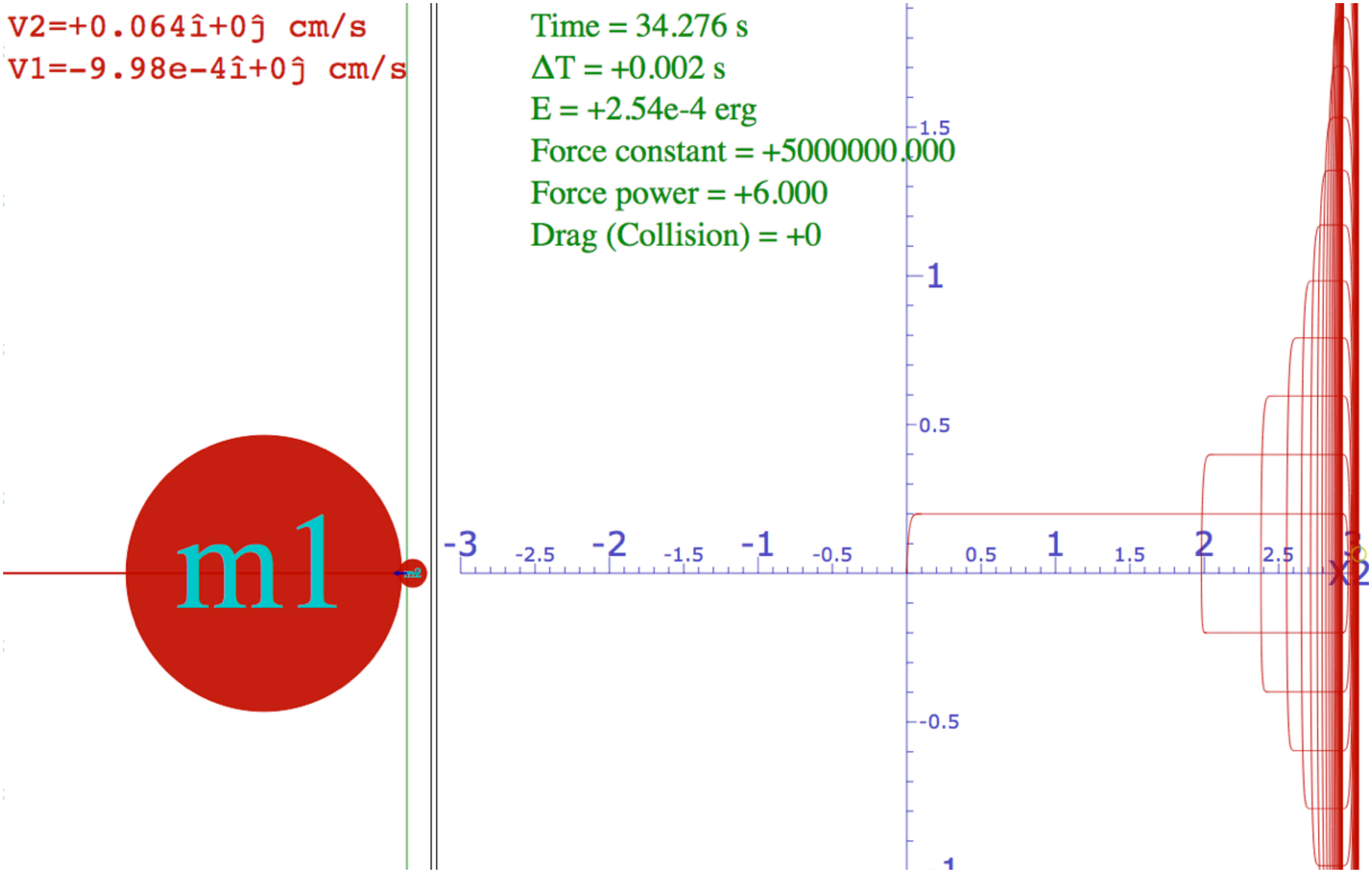
Heisenberg "Uncertainty" Relations



$$v_2 = \frac{const.}{Y} \quad \text{or:} \quad Y \cdot v_2 = const.$$

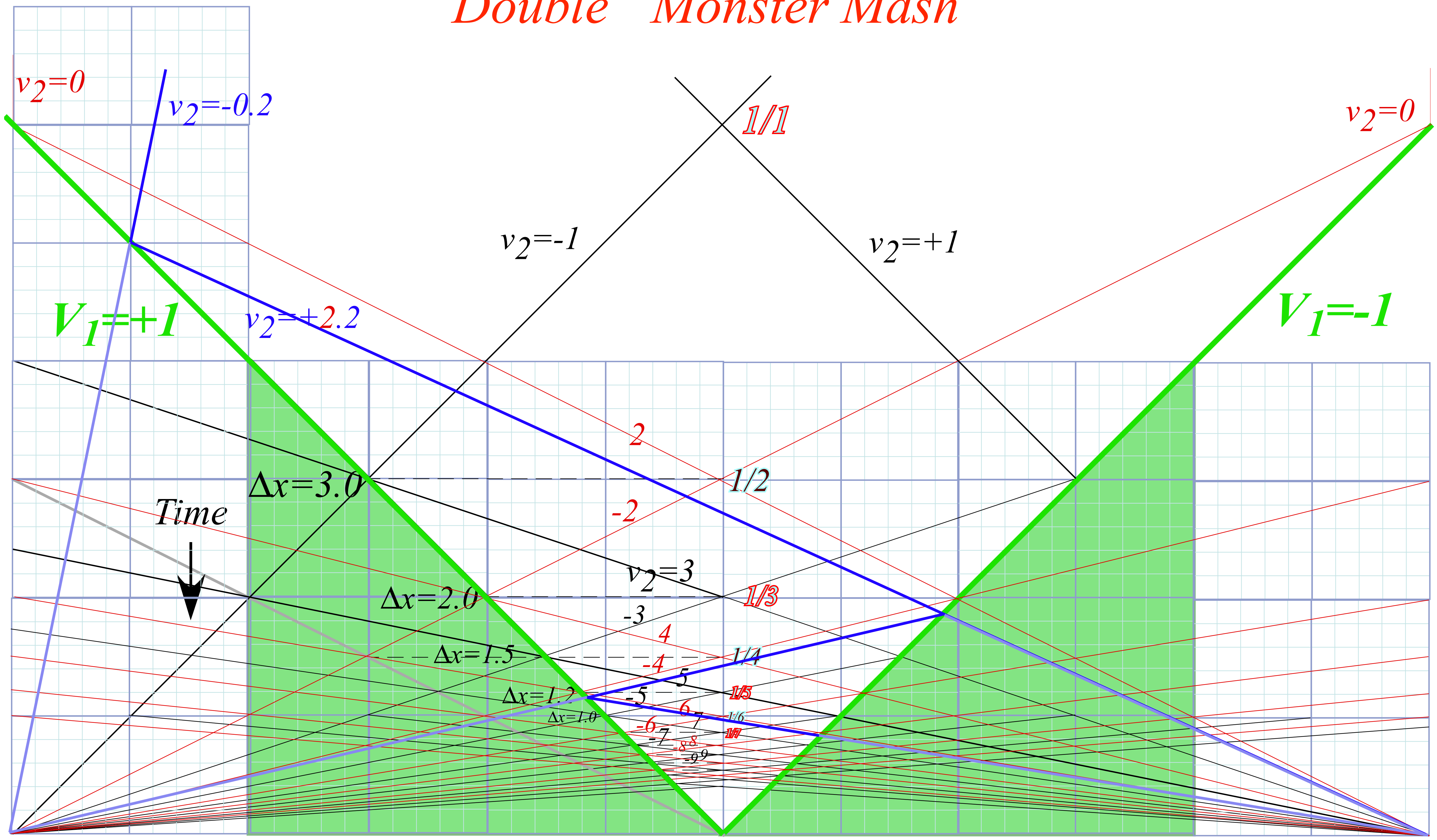
is analogous to: $\Delta x \cdot \Delta p = N \cdot \hbar$

From CMwBang!
Unit 1
Fig. 6.4



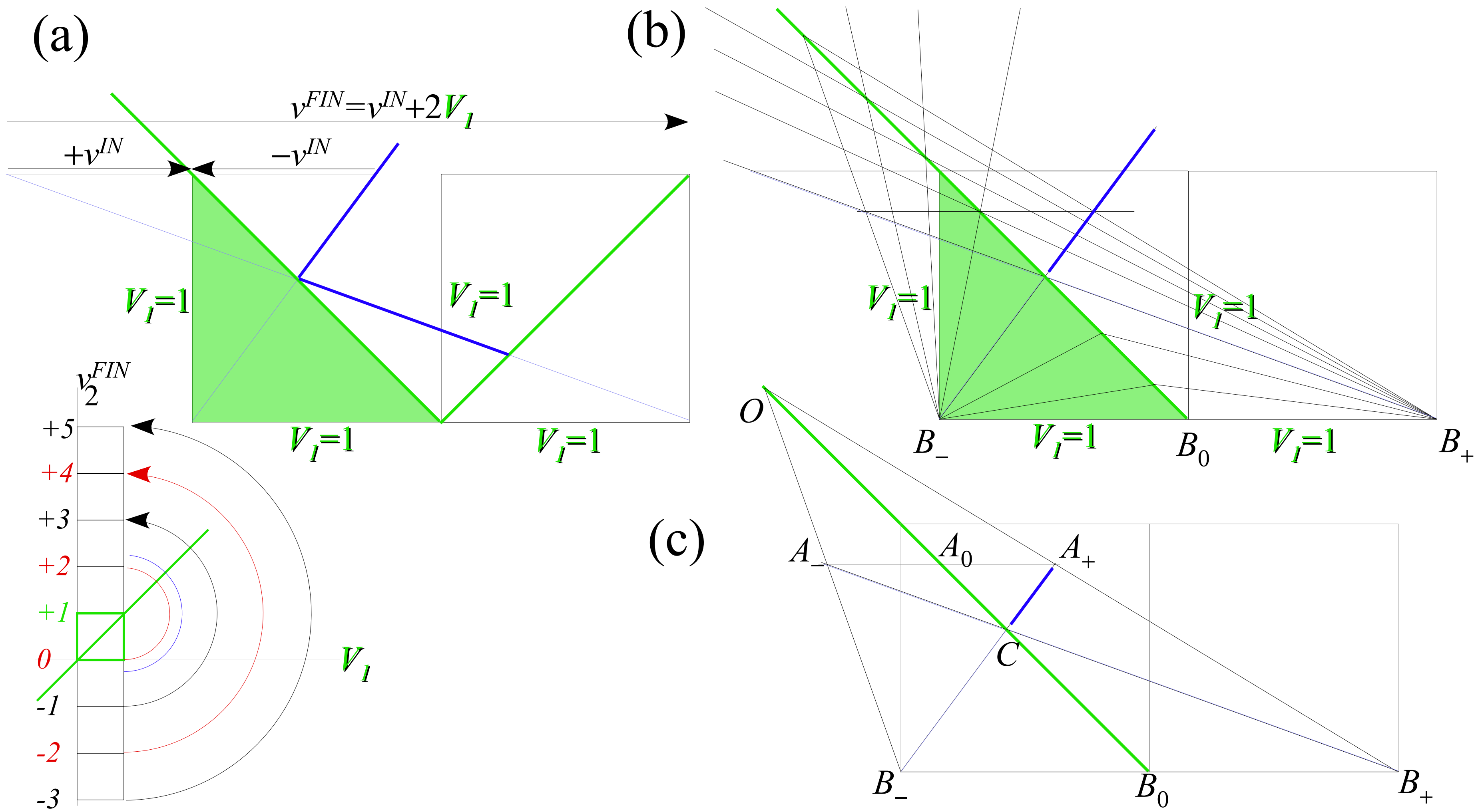
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=3000

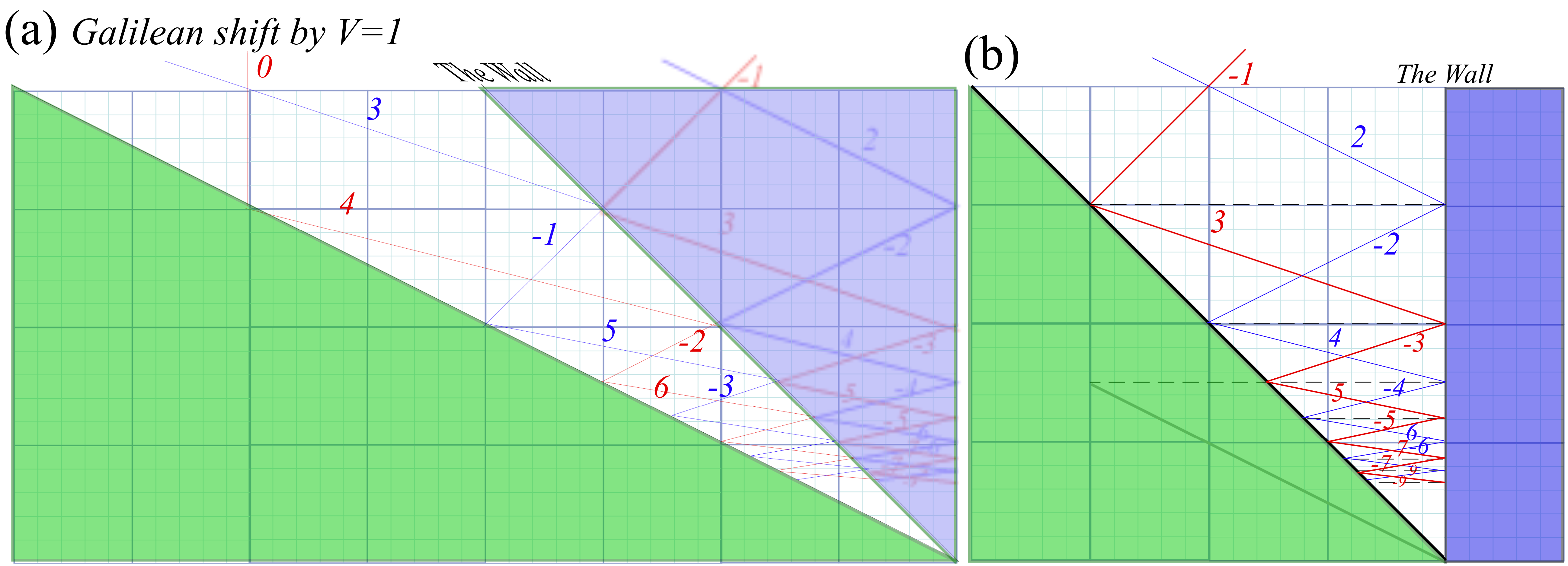
Double "Monster Mash"



From CMwBang! Unit 1
Fig. 6.5

From CMwBang Unit 1 Fig. 6.6 and Fig. 6.7





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➡ *Polygonal geometry of U(2)⊃C_N character spectral function*

Algebra

Geometry

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta) = trace D^{1/2}(\Theta) = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta) = trace D^{1}(\Theta) = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

Polygonal geometry of U(2)⊃C_N character spectral function

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(spinor-j=1/2)

$$\chi^{1}(\Theta) = traceD^{1}(\Theta) = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta) = e^{-i\frac{\Theta}{2}} + e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2}$$ *(spinor-j=1/2)*

$$\chi^{3/2}(\Theta) = e^{-i\frac{3\Theta}{2}} + \ldots + e^{i\frac{3\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

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Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

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$$\chi^{1/2}(\Theta) = trace D^{1/2}(\Theta) = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta) = trace D^{1}(\Theta) = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta) = e^{-i\frac{\Theta}{2}} + e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2}$$ *(spinor-j=1/2)*

$$\chi^{3/2}(\Theta) = e^{-i\frac{3\Theta}{2}} + \ldots + e^{i\frac{3\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta) = e^{-i\Theta\cdot 0} = 1$$ *(scalar-j=0)*

$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} = 1 + 2\cos\Theta$$ *(vector-j=1)*

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(spinor-j=1/2)

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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \quad =1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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$$\chi^{3/2}(\Theta) = e^{-i\frac{3\Theta}{2}} + \ldots + e^{i\frac{3\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2}$$

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$$\chi^j(\Theta) = TraceD^{(j)}(\Theta) = e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)} + e^{+i\Theta j}$$

$$\chi^j(\Theta)e^{-i\Theta} = e^{-i\Theta(j+1)} + e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)}$$

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$$\chi^{1/2}(\Theta) = e^{-i\frac{\Theta}{2}} + e^{i\frac{\Theta}{2}} = 2\cos\frac{\Theta}{2} \quad \textit{(spinor-j=1/2)}$$

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$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^j(\Theta) = TraceD^{(j)}(\Theta) = e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)} + e^{+i\Theta j}$$

$$\chi^j(\Theta)e^{-i\Theta} = e^{-i\Theta(j+1)} + e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^j(\Theta)(1-e^{-i\Theta}) = -e^{-i\Theta(j+1)} + e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^j(\Theta) = \frac{e^{+i\Theta j} - e^{-i\Theta(j+1)}}{1-e^{-i\Theta}} = \frac{e^{+i\Theta(j+\frac{1}{2})} - e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}} - e^{-i\frac{\Theta}{2}}} = \frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

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$$\chi^{j}(\Theta) = \frac{e^{+i\Theta j} - e^{-i\Theta(j+1)}}{1 - e^{-i\Theta}} = \frac{e^{+i\Theta(j+\frac{1}{2})} - e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}} - e^{-i\frac{\Theta}{2}}} = \frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

$$\chi^{j}(\frac{2\pi}{n}) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

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→ *Algebra*

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Character Spectral Function
where: $\ell^j=2j+1$
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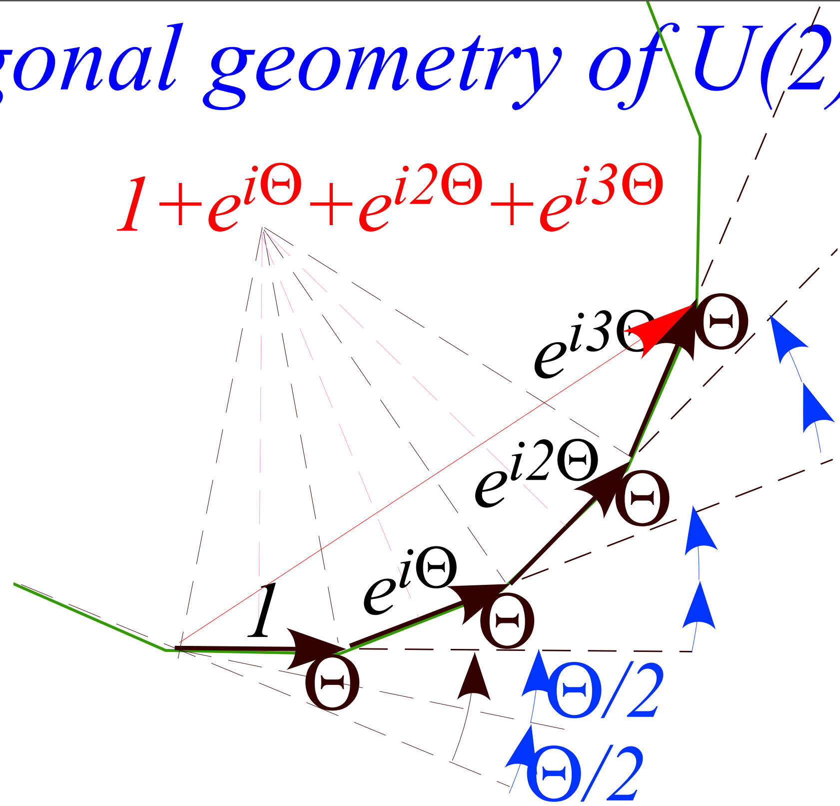
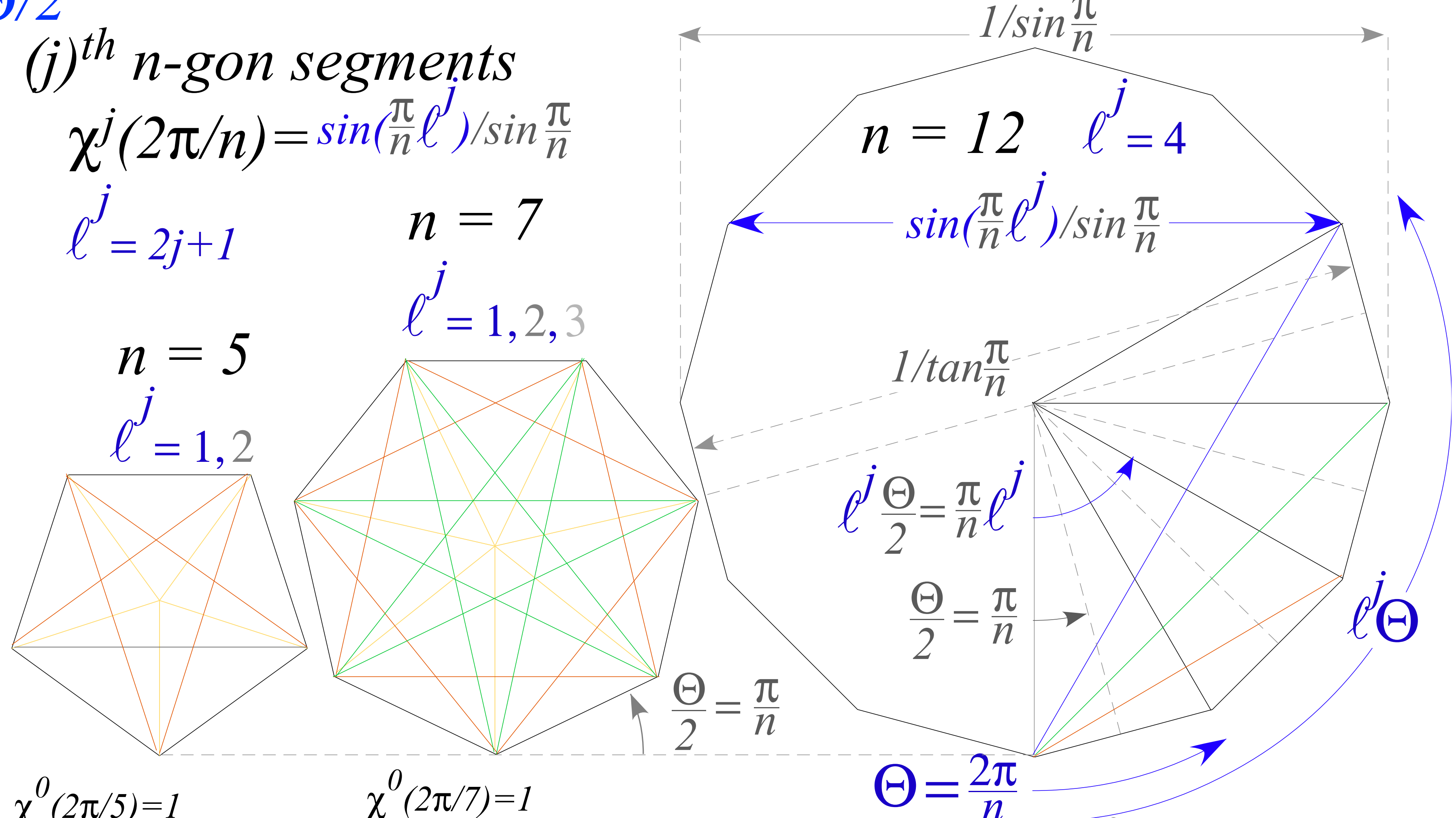


$(j)^{th}$ n-gon segments

$\chi^j(2\pi/n)=\sin(\frac{\pi}{n}\ell^j)/\sin\frac{\pi}{n}$

$\ell^j=2j+1$



$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of U(2)⊃C_N character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
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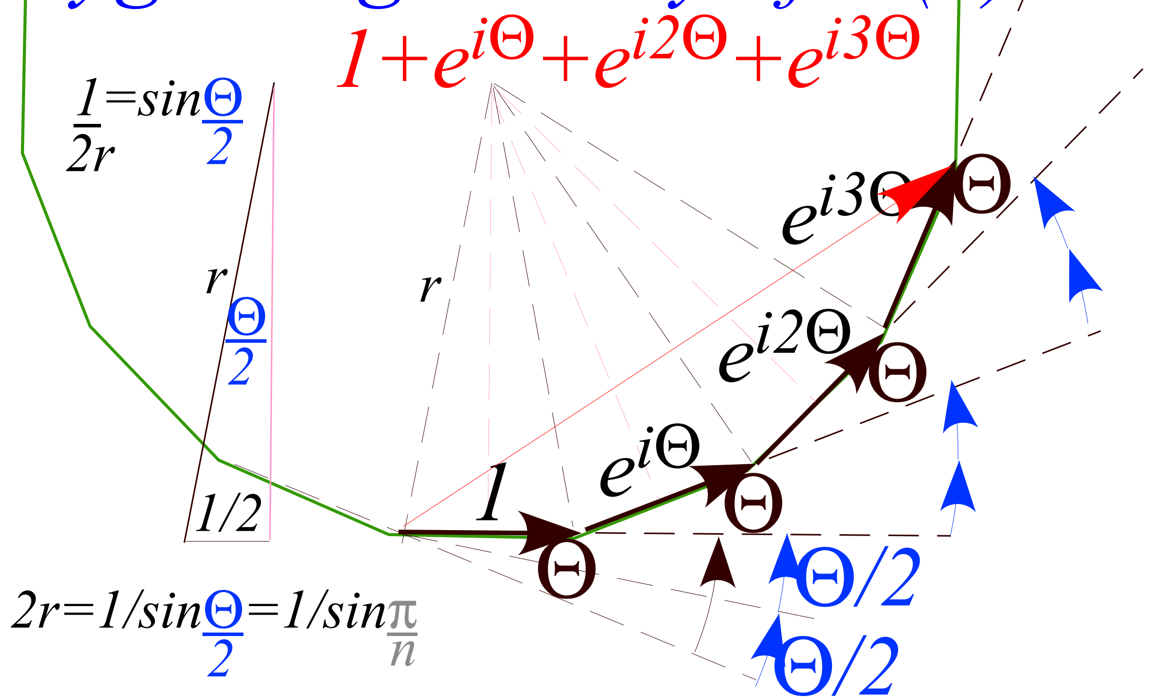

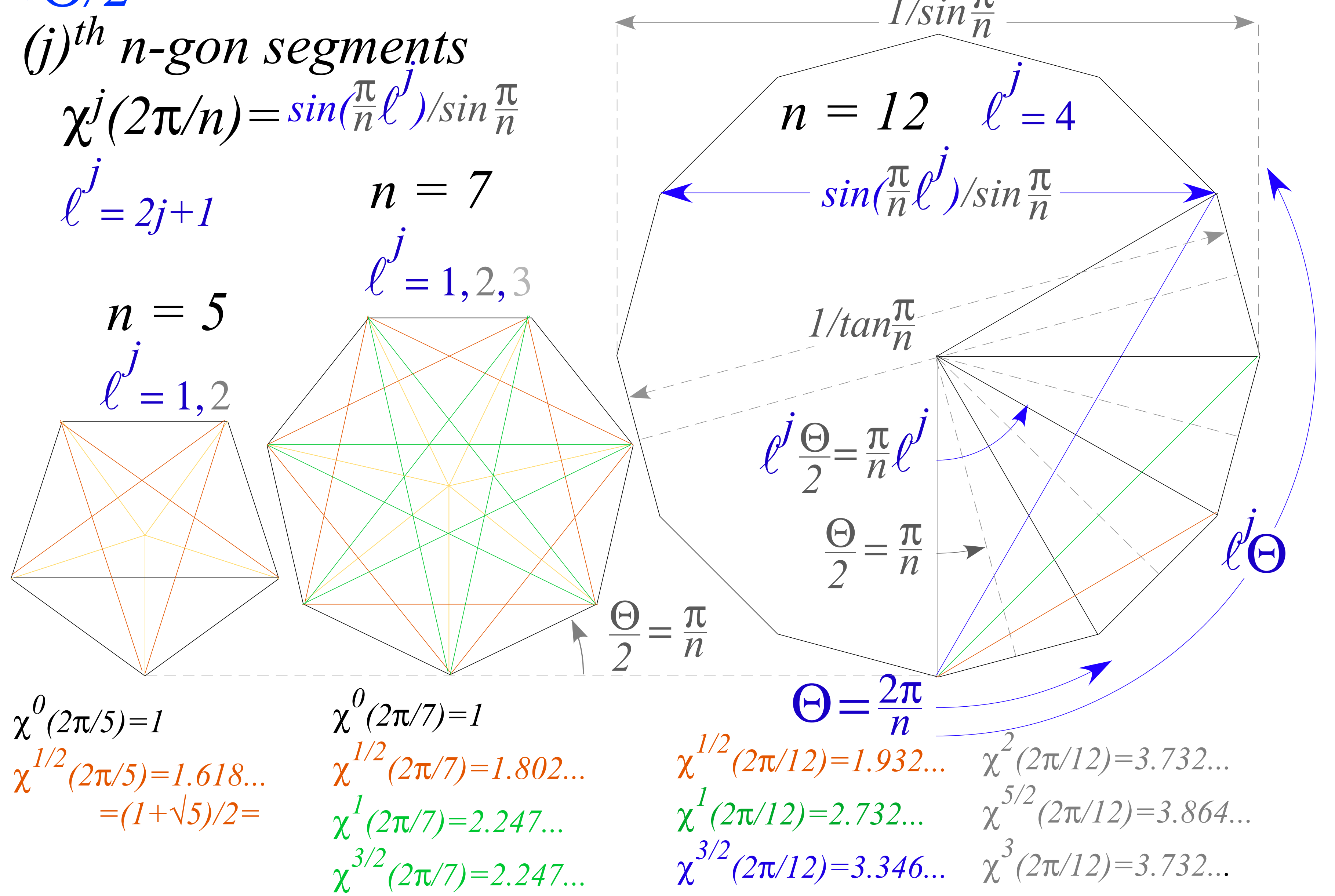

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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$(j)^{th}$ n-gon segments

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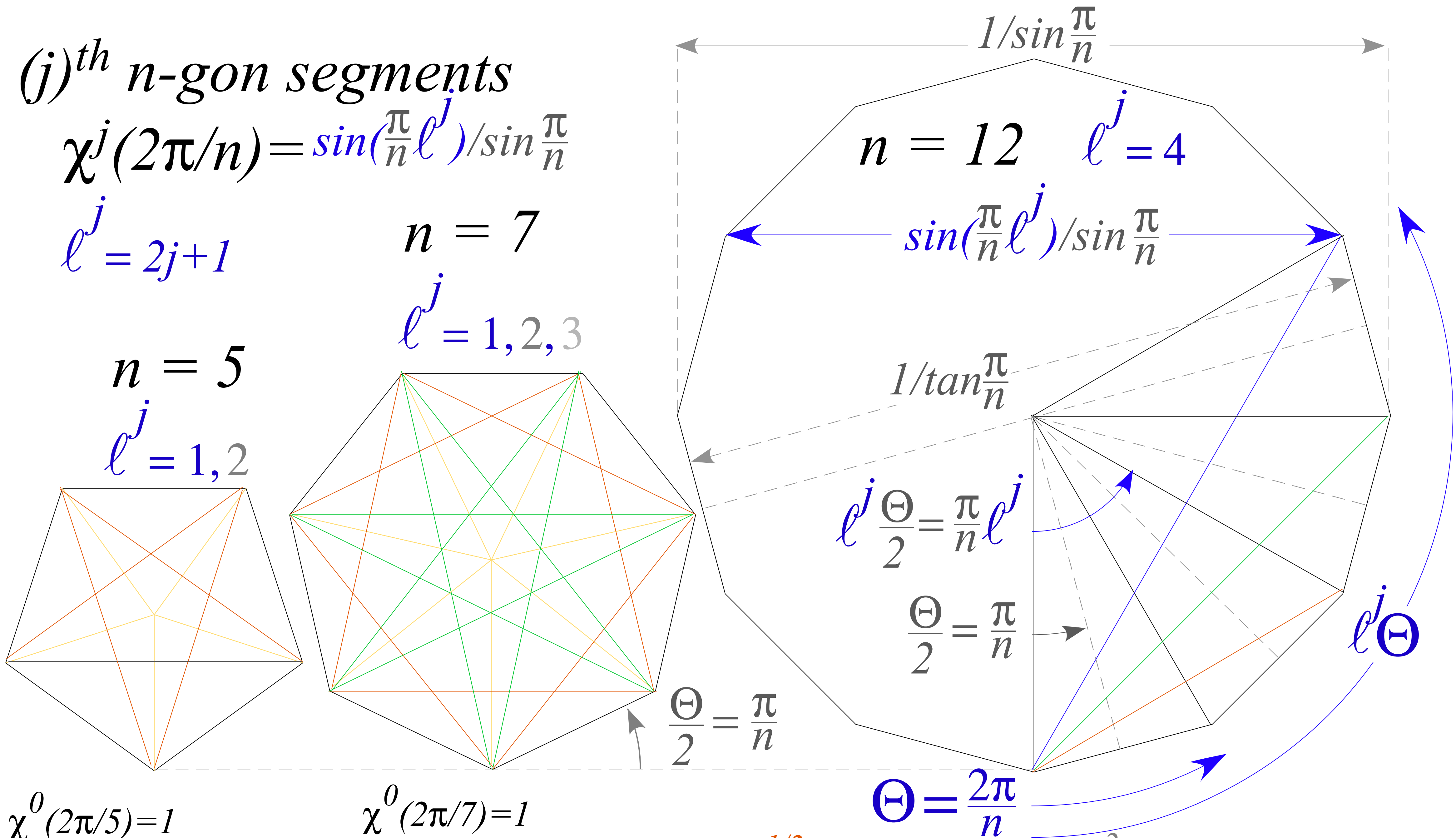
$$\ell^j=2j+1$$



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Character Spectral Function
where: $\ell^j=2j+1$
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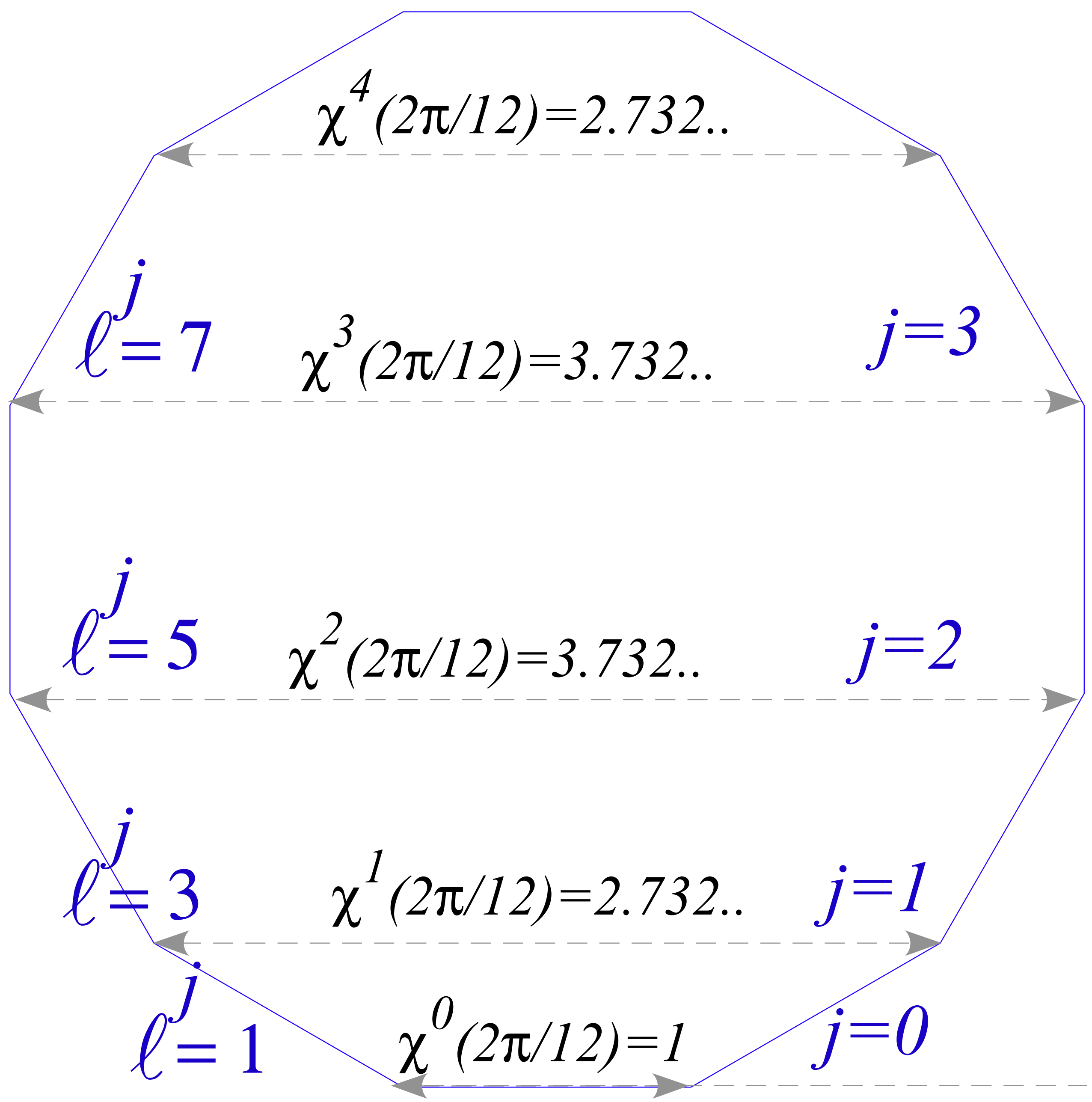
Integer j for n=12



Polygonal geometry of U(2)⊃C_N character spectral function

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