Group Theory in Quantum Mechanics

Lecture 12 (2.20.15)

Symmetry and Dynamics of C_N cyclic systems

(Geometry of U(2) characters - Ch. 6-9 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 3-7 of Ch. 2)

C_6 Spectral resolution: 6th roots of unity and higher
- Complete sets of coupling parameters and Fourier dispersion
- Gauge shifts due to complex coupling

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- Wave coordinates
- Pulse-waves (PW) vs Continuous -waves (CW)
- Wave coordinates for Linear Dispersion
- Wave coordinates for Bohr-Schrodinger Dispersion
- Einstein-Lorentz-Minkowski laser coordinates

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- ∞-Square well PE versus Bohr rotor
- SinNx/x wavepackets bandwidth and uncertainty
  - SinNx/x explosion and revivals
- Bohr-rotor dynamics
  - Gaussian wave-packet bandwidth and uncertainty
  - Gaussian Bohr-rotor revivals
- Farey-Sums and Ford-products
- Phase dynamics

C_3 **g**†**g**-*product-table and basic group representation theory*

C_3 **H**-and-**r**^{p}-*matrix representations and conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations

C_3 *character table and modular labeling*

Ortho-completeness ***inversion*** *for operators and states*

Modular quantum number arithmetic

C_3-group jargon and structure of various tables

C_3 Eigenvalues and wave dispersion functions

Standing waves vs Moving waves

→ *C_6 Spectral resolution: 6th roots of unity and higher*

Complete sets of coupling parameters and Fourier dispersion

Gauge shifts due to complex coupling

1st Step in Abelian symmetry analysis

*Expand C_6 symmetric **H** matrix using C_6 group table($\overset{gg^\dagger}{form}$)*

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + \ldots + r_{n-1}\mathbf{r}^{n-1} = \Sigma r_q \mathbf{r}^k$$

$g^\dagger$

	C_6	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	1	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{r}$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
	$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
g	$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$
	$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$
	$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + r_3\mathbf{r}^3 + r_4\mathbf{r}^4 + r_5\mathbf{r}^5$$

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

*C_6 group table gives **r**-matrices,...*

(known as a *regular* representation of the group)

1st Step in Abelian symmetry analysis

Expand C_6 symmetric $\mathbf{H}$ matrix using C_6 group table($gg^\dagger$ form)

$g^\dagger$

	C_6	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{r}$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
	$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
g	$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$
	$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$
	$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$

$$\mathbf{H}= r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2+\ldots+r_{n-1}\mathbf{r}^{n-1}=\Sigma r_q\mathbf{r}^k$$

$$\mathbf{H} \qquad = r_0\mathbf{r}^0 \qquad + r_1\mathbf{r}^1 \qquad + r_2\mathbf{r}^2 \qquad + r_3\mathbf{r}^3 \qquad + r_4\mathbf{r}^4 \qquad + r_5\mathbf{r}^5$$

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3 \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4 \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5 \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

C_6 group table gives **r***-matrices,...*

Put "1" wherever $\mathbf{r}^3$ appears in product-table

(known as a *regular* representation of the group)

1^{st} Step in Abelian symmetry analysis

Expand C_6 symmetric $\mathbf{H}$ matrix using C_6 group table($\overset{gg^\dagger}{form}$)

$g^\dagger$

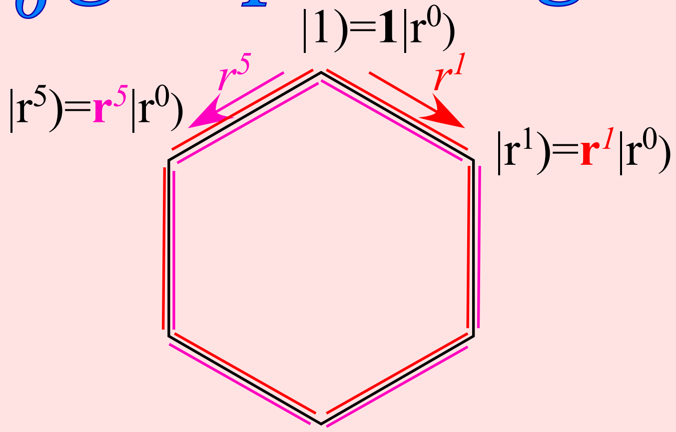
	C_6	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{r}$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
	$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
g	$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$
	$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$
	$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$

$$\mathbf{H}= r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2+\ldots+r_{n-1}\mathbf{r}^{n-1}=\Sigma\, r_q\mathbf{r}^k$$

$$\mathbf{H} \qquad = r_0\mathbf{r}^0 \qquad + r_1\mathbf{r}^1 \qquad + r_2\mathbf{r}^2 \qquad + r_3\mathbf{r}^3 \qquad + r_4\mathbf{r}^4 \qquad + r_5\mathbf{r}^5$$

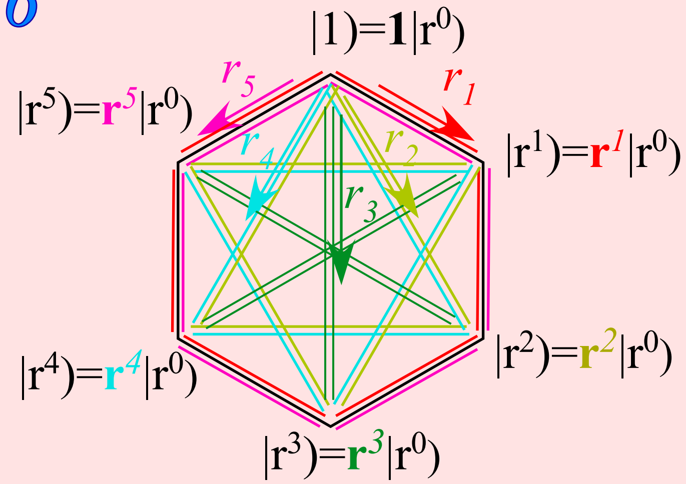
$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3 \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4 \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5 \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

C_6 group table gives $\mathbf{r}$-matrices,...C_6-allowed $\mathbf{H}$-matrices...



Nearest neighbor coupling

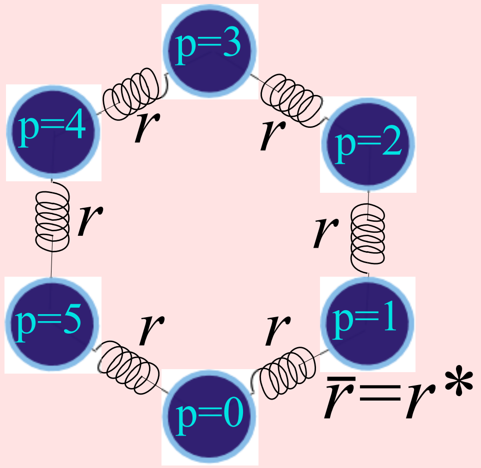
$$\begin{pmatrix} r_0 & r_5 & & & & r_1 \\ r_1 & r_0 & r_5 & & & \\ & r_1 & r_0 & r_5 & & \\ & & r_1 & r_0 & r_5 & \\ & & & r_1 & r_0 & r_5 \\ r_5 & & & & r_1 & r_0 \end{pmatrix}$$



ALL neighbor coupling

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$

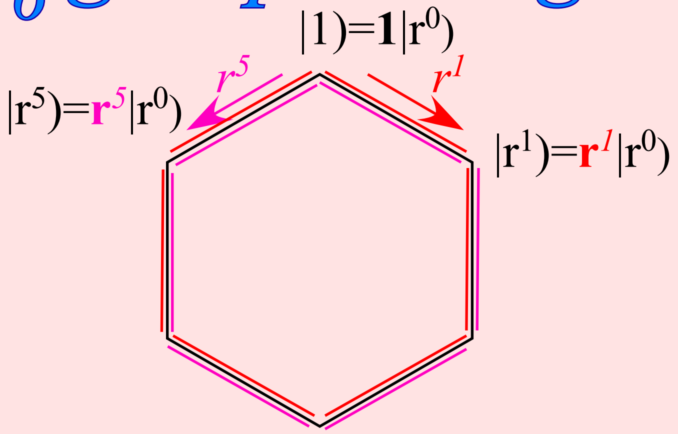
(a) 1^{st} Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c}p=0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{pmatrix} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

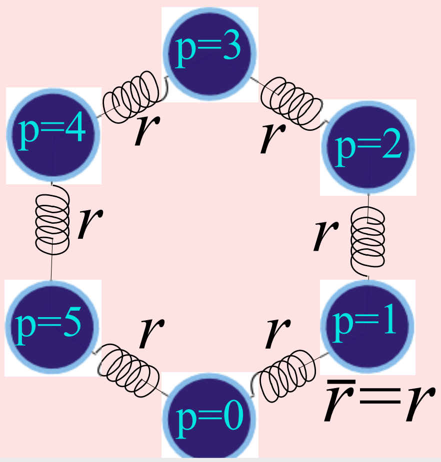
C_6 group table gives $\mathbf{r}$-matrices,..



Nearest neighbor coupling

$$\begin{pmatrix} r_0 & r_5 & & & & r_1 \\ r_1 & r_0 & r_5 & & & \\ & r_1 & r_0 & r_5 & & \\ & & r_1 & r_0 & r_5 & \\ & & & r_1 & r_0 & r_5 \\ r_5 & & & & r_1 & r_0 \end{pmatrix}$$

(a) 1^{st} Neighbor C_6



$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^1-r\mathbf{r}^{-1}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}$$

Conjugation symmetry

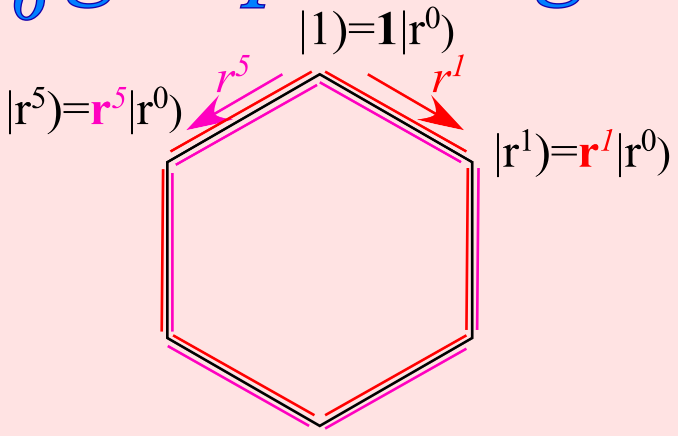
Hermitian Hamiltonian $(\mathbf{H}^*_{jk}=\mathbf{H}_{kj})$ requires $r_0^*=r_0$ and $r_1=r_5^*$.

Elementary Bloch model assumes both are real

$(\,r_1=-r=r_5^*)$

r_1 equals conjugate of r_5: $(\,r_1=r_5^*)$

C_6 *group table gives* **r***-matrices,..*

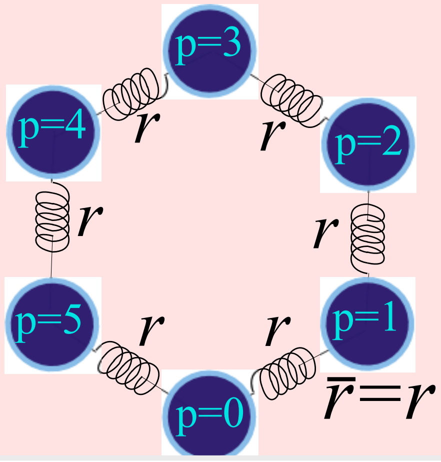


Elementary - Bloch - Model : Nearest neighbor coupling:

$$\mathbf{H}^{B1(6)}=r_0\mathbf{1}+r_1\mathbf{r}^1+r_5\mathbf{r}^5 \quad = 2r\mathbf{1}-r\mathbf{r}^1+-r\mathbf{r}^{-1}$$

$$\begin{pmatrix}
r_0 & r_5 & \cdot & \cdot & \cdot & r_1 \\
r_1 & r_0 & r_5 & \cdot & \cdot & \cdot \\
\cdot & r_1 & r_0 & r_5 & \cdot & \cdot \\
\cdot & \cdot & r_1 & r_0 & r_5 & \cdot \\
\cdot & \cdot & \cdot & r_1 & r_0 & r_5 \\
r_5 & \cdot & \cdot & \cdot & r_1 & r_0
\end{pmatrix}
=
\left|\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}\right.$$

(a) 1^{st} Neighbor C_6

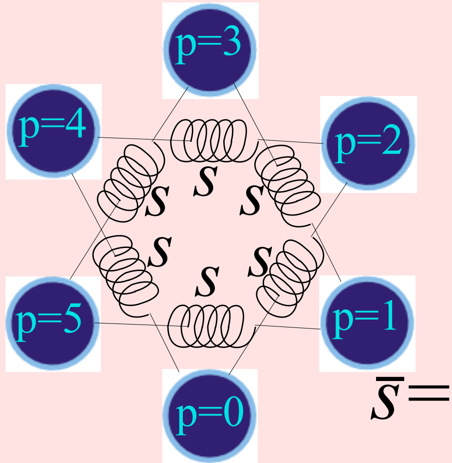


$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^{1}-r\mathbf{r}^{-1}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}$$

(b) 2^{nd} Neighbor C_6



$\bar{s}=s^*=s$

$$\mathbf{H}^{B2(6)}=H_2\mathbf{1}-s\mathbf{r}^{2}-s\mathbf{r}^{-2}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\
\cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\
-s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\
\cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\
-s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\
\cdot & -s & \cdot & -s & \cdot & H_2 & 5
\end{array}$$

Conjugation symmetry

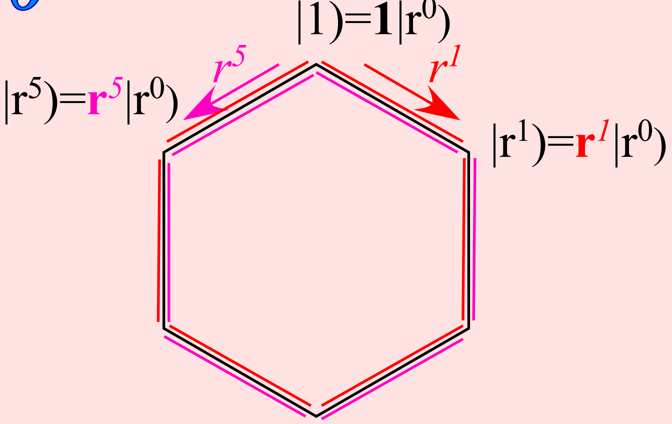
$(\mathbf{H}^*_{jk}=\mathbf{H}_{kj})$

requires $r_0^*=r_0$ and $r_2=r_4^*$.

r_1 equals conjugate of r_5: ($r_1=r_5^*=-r$)

$(r_2=r_4^*=-s)$ *We assume both are real*

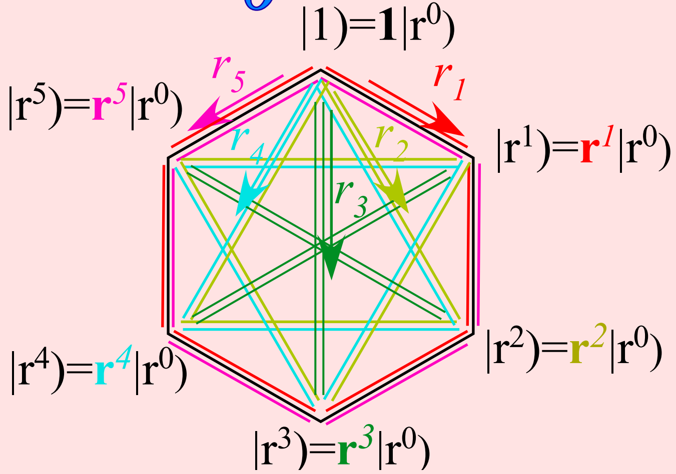
C_6 group table gives **r**-*matrices,..., and all C_6-allowed* **H**-*matrices...*



2^{nd} Nearest neighbor coupling:

$$\mathbf{H}^{B1(6)}=r_0\mathbf{1}+r_2\mathbf{r}^2+r_4\mathbf{r}^4$$

$$\begin{pmatrix}
r_0 & \cdot & r_4 & \cdot & r_2 & \cdot \\
\cdot & r_0 & \cdot & r_4 & \cdot & r_2 \\
r_2 & \cdot & r_0 & \cdot & r_4 & \cdot \\
\cdot & r_2 & \cdot & r_0 & \cdot & r_4 \\
r_4 & \cdot & r_2 & \cdot & r_0 & \cdot \\
\cdot & r_4 & \cdot & r_2 & \cdot & r_0
\end{pmatrix}$$

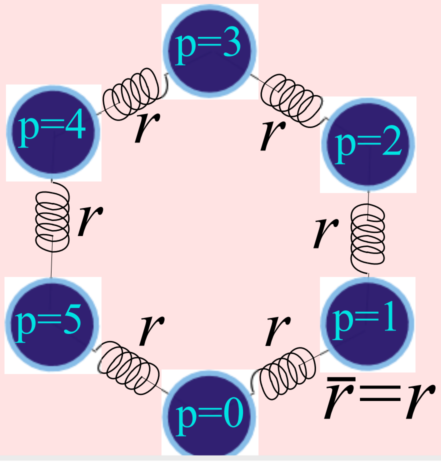


All–neighbor coupling:

$$\mathbf{H}^{A(6)}=r_0\mathbf{1}+r_1\mathbf{r}^1+r_2\mathbf{r}^2+r_3\mathbf{r}^3+r_4\mathbf{r}^4+r_5\mathbf{r}^5$$

$$\begin{pmatrix}
r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\
r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\
r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\
r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\
r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\
r_5 & r_4 & r_3 & r_2 & r_1 & r_0
\end{pmatrix}$$

(a) 1^{st} Neighbor C_6

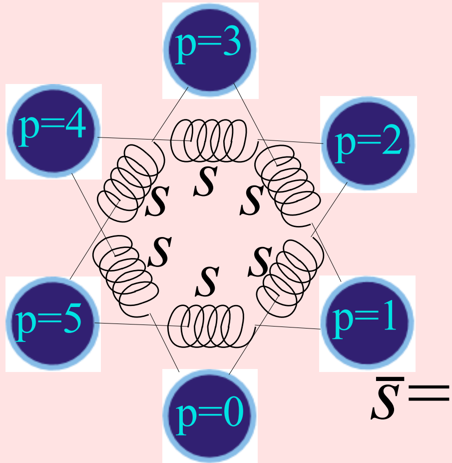


$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^{1}-r\mathbf{r}^{-1}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}$$

Neighbor C_6

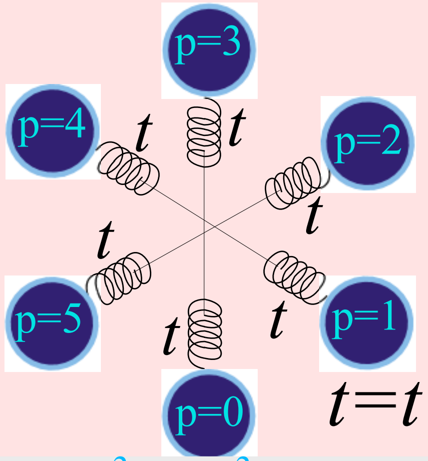


$\bar{s}=s^*=s$

$$\mathbf{H}^{B2(6)}=H_2\mathbf{1}-s\mathbf{r}^{2}-s\mathbf{r}^{-2}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\
\cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\
-s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\
\cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\
-s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\
\cdot & -s & \cdot & -s & \cdot & H_2 & 5
\end{array}$$

(c) 3^{rd} Neighbor C_6



$t=t^*$

$$\mathbf{H}^{B3(6)}=H_3\mathbf{1}-t\mathbf{r}^{3}-t\mathbf{r}^{-3}$$

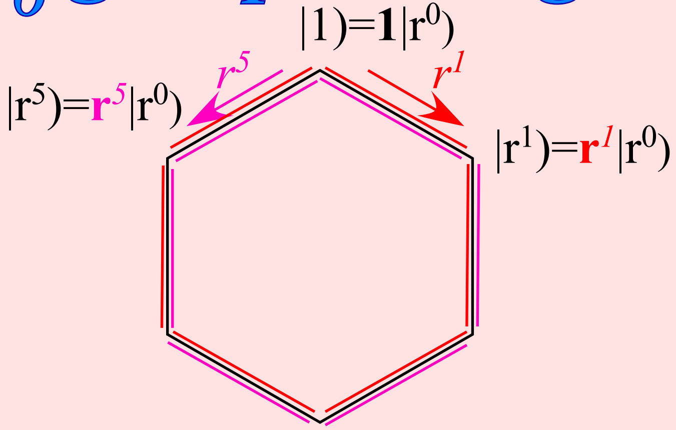
$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_3 & \cdot & \cdot & -t & \cdot & \cdot & 0 \\
\cdot & H_3 & \cdot & \cdot & -t & \cdot & 1 \\
\cdot & \cdot & H_3 & \cdot & \cdot & -t & 2 \\
-t & \cdot & \cdot & H_3 & \cdot & \cdot & 3 \\
\cdot & -t & \cdot & \cdot & H_3 & \cdot & 4 \\
\cdot & \cdot & -t & \cdot & \cdot & H_3 & 5
\end{array}$$

r_1 equals conjugate of r_5: ($r_1=r_5^*=-r$)

$(r_2=r_4^*=-s)$

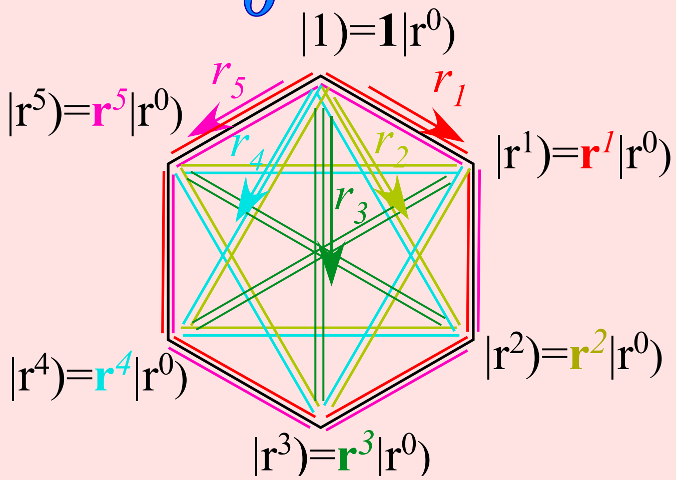
$(r_3=r_3^*=t)$ *must* be real

C_6 *group table gives* **r**-*matrices,..., and all* C_6-*allowed* **H**-*matrices...*



Nearest neighbor coupling

$$\begin{pmatrix}
r_0 & r_5 & & & & r_1 \\
r_1 & r_0 & r_5 & & & \\
 & r_1 & r_0 & r_5 & & \\
 & & r_1 & r_0 & r_5 & \\
 & & & r_1 & r_0 & r_5 \\
r_5 & & & & r_1 & r_0
\end{pmatrix}$$



ALL neighbor coupling

$$\begin{pmatrix}
r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\
r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\
r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\
r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\
r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\
r_5 & r_4 & r_3 & r_2 & r_1 & r_0
\end{pmatrix}$$

2^{nd} Step

H diagonalized by spectral resolution of $r, r^2,\ldots,r^6 = 1$

All $x=r^p$ satisfy $x^6= 1$ and use **6th-roots-of-1** *for eigenvalues*

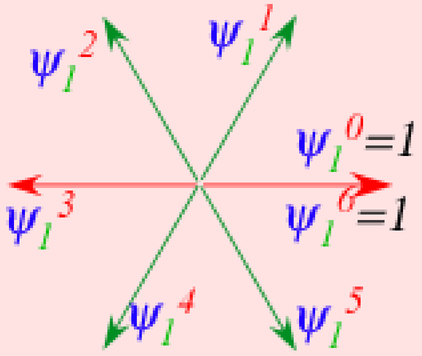
$\psi_1{}^0=1$
$\psi_1{}^1=e^{2\pi i/6}$
$\psi_1{}^2=\psi_2{}^1=e^{4\pi i/6}$
$\psi_1{}^3=\psi_3{}^1=-1$
$\psi_1{}^4=\psi_4{}^1=\psi_1{}^{-2}=e^{-4\pi i/6}$
$\psi_1{}^5=\psi_5{}^1=\psi_1{}^{-1}=e^{-2\pi i/6}$

$D^m(\mathbf{r})=e^{-2\pi i m/6}=\chi_1{}^m=\psi_1{}^{m*}$

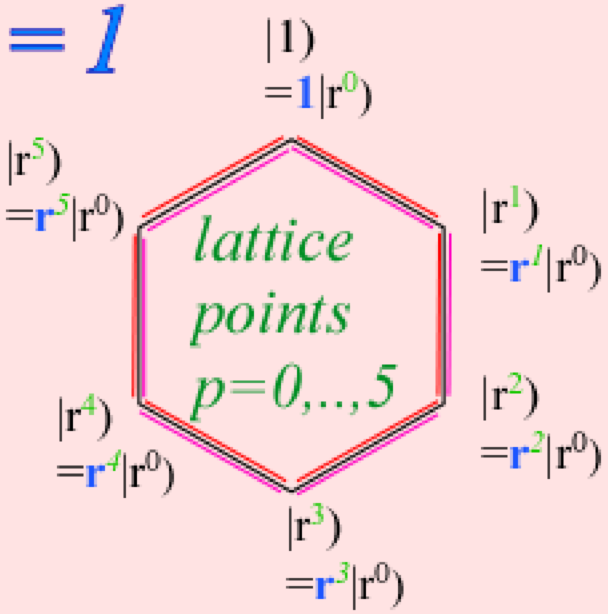
$D^m(\mathbf{r}^p)=e^{-2\pi i m\cdot p/6}=\chi_p{}^m=\psi_p{}^{m*}$

p=power (exponent)
or position point

m = momentum
or wave-number



6^{th}-roots of 1
m=0,...,5



Groups "know" their roots and will tell you them if you ask nicely!
You efficiently get:

- *invariant projectors*
- *irreducible projectors*
- *irreducible representations (irreps)*
- *H eigenvalues*
- *H eigenvectors*
- *T matrices*
- *dispersion functions*

2nd Step (contd.)

H diagonalized by spectral resolution of $r, r^2, \ldots, r^6 = 1$

All $x = r^p$ *satisfy* $x^6 = 1$ *and use* *6th-roots-of-1* *for eigenvalues*

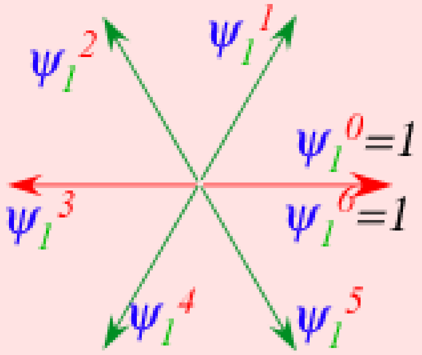
$$D^m(\mathbf{r}) = e^{-2\pi i m/6} = \chi_1^m = \psi_1^{m*}$$

$$D^m(\mathbf{r}^p) = e^{-2\pi i m\cdot p/6} = \chi_p^m = \psi_p^{m*}$$

p=power (exponent)
or position point

m = momentum
or wave-number

$\psi_1^0 = 1$
$\psi_1^1 = e^{2\pi i/6}$
$\psi_1^2 = \psi_2^1 = e^{4\pi i/6}$
$\psi_1^3 = \psi_3^1 = -1$
$\psi_1^4 = \psi_4^1 = \psi_1^{-2} = e^{-4\pi i/6}$
$\psi_1^5 = \psi_5^1 = \psi_1^{-1} = e^{-2\pi i/6}$



top-row flip
not needed...

$\mathbf{P}^{(m)} = \mathbf{P}^{(m)\dagger}$

6 *ring*	$\mathbf{P}^{(0)}$	$\mathbf{P}^{(1)}$	$\mathbf{P}^{(2)}$	$\mathbf{P}^{(3)}$	$\mathbf{P}^{(4)}$	$\mathbf{P}^{(5)}$
$^{(0)}$	$\mathbf{P}^{(0)}$	·	·	·	·	·
$\mathbf{P}^{(1)}$	·	$\mathbf{P}^{(1)}$	·	·	·	·
$\mathbf{P}^{(2)}$	·	·	$\mathbf{P}^{(2)}$	·	·	·
$\mathbf{P}^{(3)}$	·	·	·	$\mathbf{P}^{(3)}$	·	·
$\mathbf{P}^{(4)}$	·	·	·	·	$\mathbf{P}^{(4)}$	·
$\mathbf{P}^{(5)}$	·	·	·	·	·	$\mathbf{P}^{(5)}$

$$\mathbf{r}^p = \chi_p^0\,\mathbf{P}^{(0)} + \chi_p^1\,\mathbf{P}^{(1)} + \chi_p^2\,\mathbf{P}^{(2)} + \chi_p^3\,\mathbf{P}^{(3)} + \chi_p^4\,\mathbf{P}^{(4)} + \chi_p^5\,\mathbf{P}^{(5)}$$

$$\begin{pmatrix} \chi_p^0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \chi_p^1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \chi_p^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \chi_p^3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \chi_p^4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \chi_p^5 \end{pmatrix} = \chi_p^0 \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^1 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^3 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^4 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^5 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Projectors $\mathbf{P}^{(m)}$ *are eigenvalue "placeholders" having orthogonal-idempotent products, eigen-equations,*

$$\mathbf{P}^{(m)}\mathbf{P}^{(n)} = \delta^{mn}\mathbf{P}^{(m)} \qquad \mathbf{r}^p\,\mathbf{P}^{(n)} = \chi_p^n\,\mathbf{P}^{(n)}$$

and one completeness rule: $\mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \ldots + \mathbf{P}^{(5)} = \mathbf{1}$

2^{nd} Step (contd.)

H diagonalized by spectral resolution of $r, r^2,\ldots,r^6=1$

All $x=r^p$ satisfy $x^6=1$ and use **6^{th}-roots-of-1** *for eigenvalues*

$$D^m(\mathbf{r})=e^{-2\pi i m/6}=\chi_1^{\ m}=\psi_1^{\ m*}$$

$$D^m(\mathbf{r}^p)=e^{-2\pi i m\cdot p/6}=\chi_p^{\ m}=\psi_p^{\ m*}$$

p=power (exponent) or position point

m = momentum or wave-number

$\psi_1^{\ 0}=1$

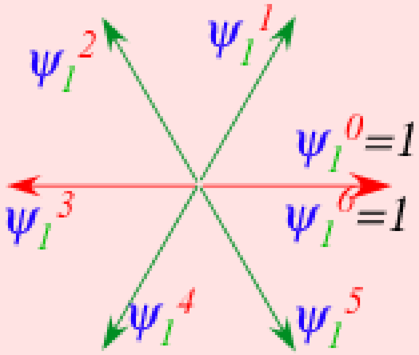
$\psi_1^{\ 1}=e^{2\pi i/6}$

$\psi_1^{\ 2}=\psi_2^{\ 1}=e^{4\pi i/6}$

$\psi_1^{\ 3}=\psi_3^{\ 1}=-1$

$\psi_1^{\ 4}=\psi_4^{\ 1}=\psi_1^{\ -2}=e^{-4\pi i/6}$

$\psi_1^{\ 5}=\psi_5^{\ 1}=\psi_1^{\ -1}=e^{-2\pi i/6}$



top-row flip not needed...

$$\mathbf{P}^{(m)}=\mathbf{P}^{(m)\dagger}$$

$$\begin{array}{c|cccccc}
6\ ring & \mathbf{P}^{(0)} & \mathbf{P}^{(1)} & \mathbf{P}^{(2)} & \mathbf{P}^{(3)} & \mathbf{P}^{(4)} & \mathbf{P}^{(5)} \\
\hline
\mathbf{P}^{(0)} & \mathbf{P}^{(0)} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\mathbf{P}^{(1)} & \cdot & \mathbf{P}^{(1)} & \cdot & \cdot & \cdot & \cdot \\
\mathbf{P}^{(2)} & \cdot & \cdot & \mathbf{P}^{(2)} & \cdot & \cdot & \cdot \\
\mathbf{P}^{(3)} & \cdot & \cdot & \cdot & \mathbf{P}^{(3)} & \cdot & \cdot \\
\mathbf{P}^{(4)} & \cdot & \cdot & \cdot & \cdot & \mathbf{P}^{(4)} & \cdot \\
\mathbf{P}^{(5)} & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{P}^{(5)}
\end{array}$$

$$\mathbf{r}^p=\chi_p^{\ 0}\,\mathbf{P}^{(0)}+\chi_p^{\ 1}\,\mathbf{P}^{(1)}+\chi_p^{\ 2}\,\mathbf{P}^{(2)}+\chi_p^{\ 3}\,\mathbf{P}^{(3)}+\chi_p^{\ 4}\,\mathbf{P}^{(4)}+\chi_p^{\ 5}\,\mathbf{P}^{(5)}$$

$$\begin{pmatrix}\chi_p^{\ 0} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \chi_p^{\ 1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \chi_p^{\ 2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \chi_p^{\ 3} & & \end{pmatrix}=\chi_p^{\ 0}\begin{pmatrix}1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}+\chi_p^{\ 1}\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}+\chi_p^{\ 2}\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}+\chi_p^{\ 3}\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & & \end{pmatrix}+\chi_p^{\ 4}\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}+\chi_p^{\ 5}\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & \cdot & 1\end{pmatrix}$$

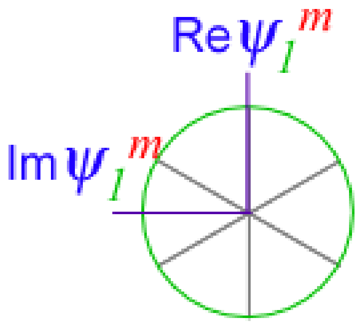
Inverse C_6 spectral resolution m-wave $\psi_p^{\ m}=D^{m*}(\mathbf{r}^p)=e^{+2\pi i m\cdot p/6}$:

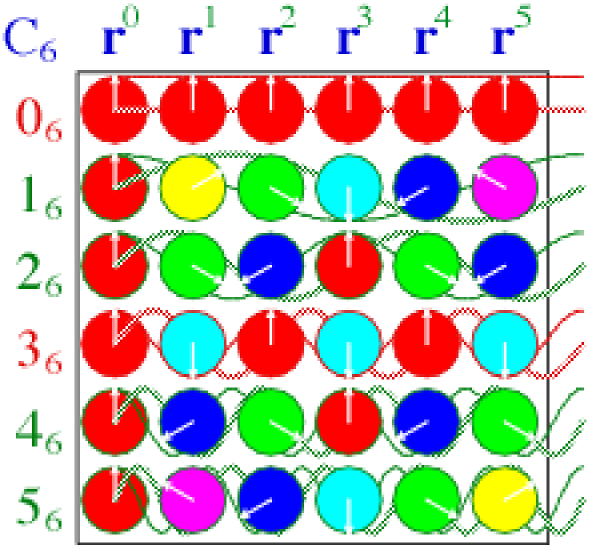
$$6\cdot\mathbf{P}^{(m)}=\psi_0^{\ m}\,\mathbf{r}^0+\psi_1^{\ m}\,\mathbf{r}^1+\psi_2^{\ m}\,\mathbf{r}^2+\psi_3^{\ m}\,\mathbf{r}^3+\psi_4^{\ m}\,\mathbf{r}^4+\psi_5^{\ m}\,\mathbf{r}^5$$

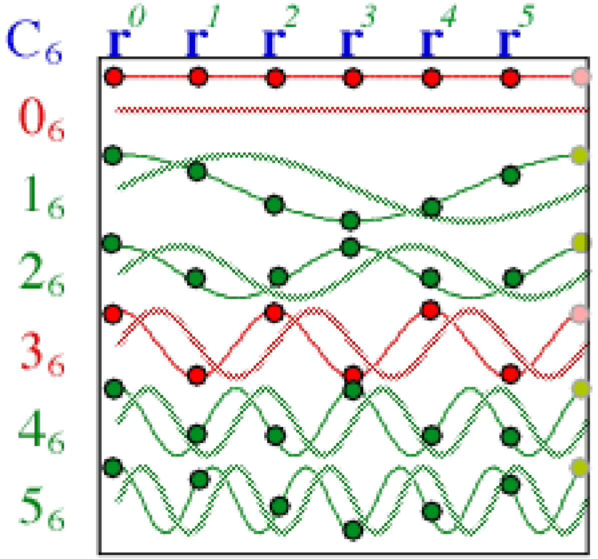
position p (or power of $\mathbf{r}^p$)

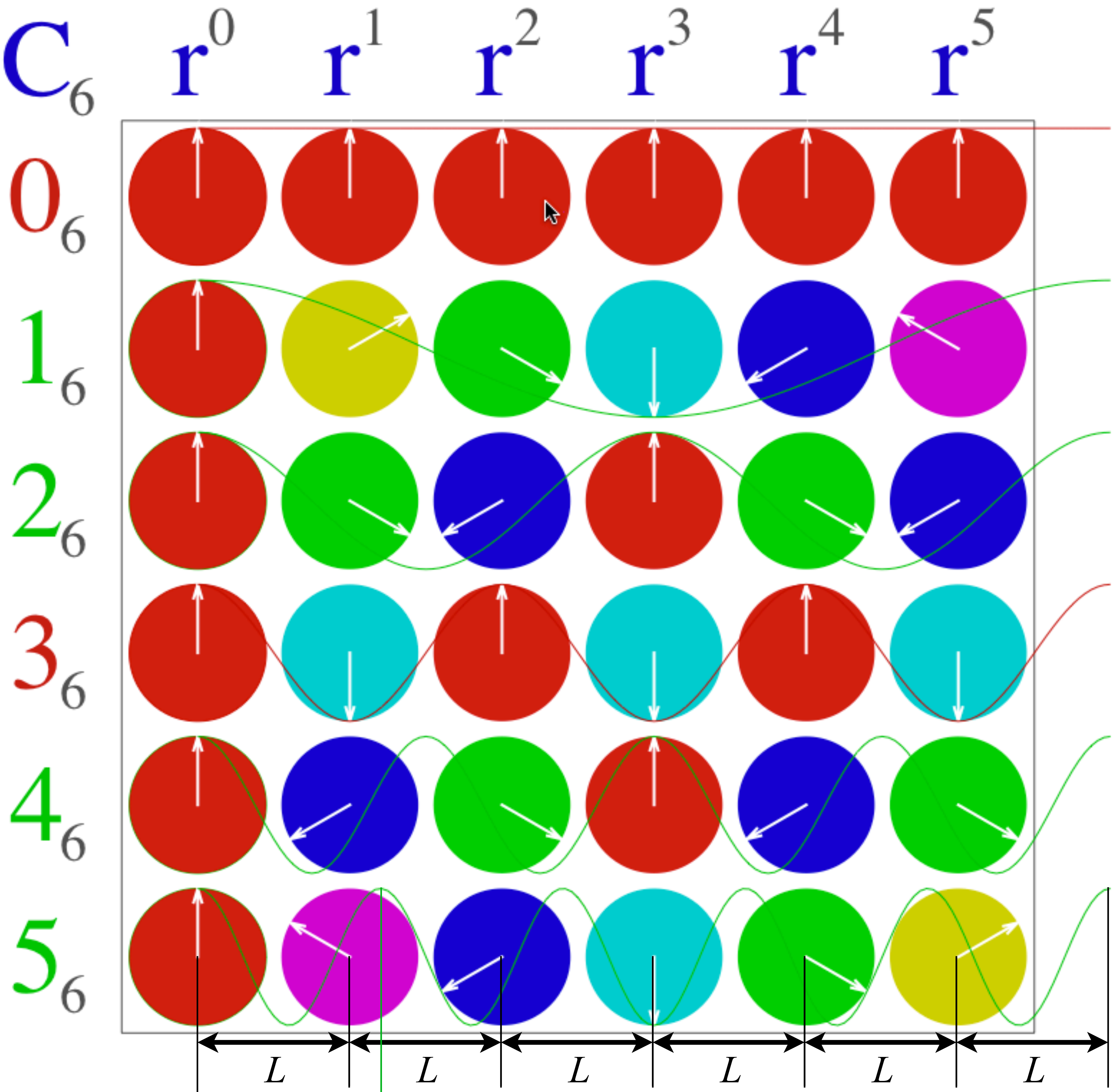
momentum m

	$p=0$	1	2	3	4	5
$m=0$	$\psi_0^{\ 0}$	$\psi_1^{\ 0}$	$\psi_2^{\ 0}$	$\psi_3^{\ 0}$	$\psi_4^{\ 0}$	$\psi_5^{\ 0}$
$m=1$	$\psi_0^{\ 1}$	$\psi_1^{\ 1}$	$\psi_2^{\ 1}$	$\psi_3^{\ 1}$	$\psi_4^{\ 1}$	$\psi_5^{\ 1}$
$m=2$	$\psi_0^{\ 2}$	$\psi_1^{\ 2}$	$\psi_2^{\ 2}$	$\psi_3^{\ 2}$	$\psi_4^{\ 2}$	$\psi_5^{\ 2}$
$m=3$	$\psi_0^{\ 3}$	$\psi_1^{\ 3}$	$\psi_2^{\ 3}$	$\psi_3^{\ 3}$	$\psi_4^{\ 3}$	$\psi_5^{\ 3}$
$m=4$	$\psi_0^{\ 4}$	$\psi_1^{\ 4}$	$\psi_2^{\ 4}$	$\psi_3^{\ 4}$	$\psi_4^{\ 4}$	$\psi_5^{\ 4}$
$m=5$	$\psi_0^{\ 5}$	$\psi_1^{\ 5}$	$\psi_2^{\ 5}$	$\psi_3^{\ 5}$	$\psi_4^{\ 5}$	$\psi_5^{\ 5}$









$\lambda_5 = 2\pi/k_5 = 6L/5$

C_6 character

$$\chi_{mp} = e^{-imp2\pi/6}$$

is wave function conjugate

$$\psi^*_m(r_p) = \frac{e^{-imp2\pi/6}}{\sqrt{6}}$$ (with norm √6)

C_6 Plane wave function

$$\psi_m(r_p) = \frac{e^{ik_m \cdot r_p}}{\sqrt{6}}$$

$$= \frac{e^{imp2\pi/6}}{\sqrt{6}}$$

C_6 Lattice position vector

$$r_p = L \cdot p$$

Wavevector

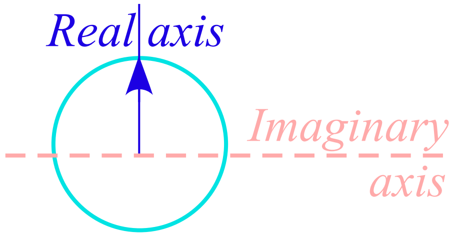
$$k_m = 2\pi m/6L = 2\pi/\lambda_m$$

Wavelength

$$\lambda_m = 2\pi/k_m = 6L/m$$

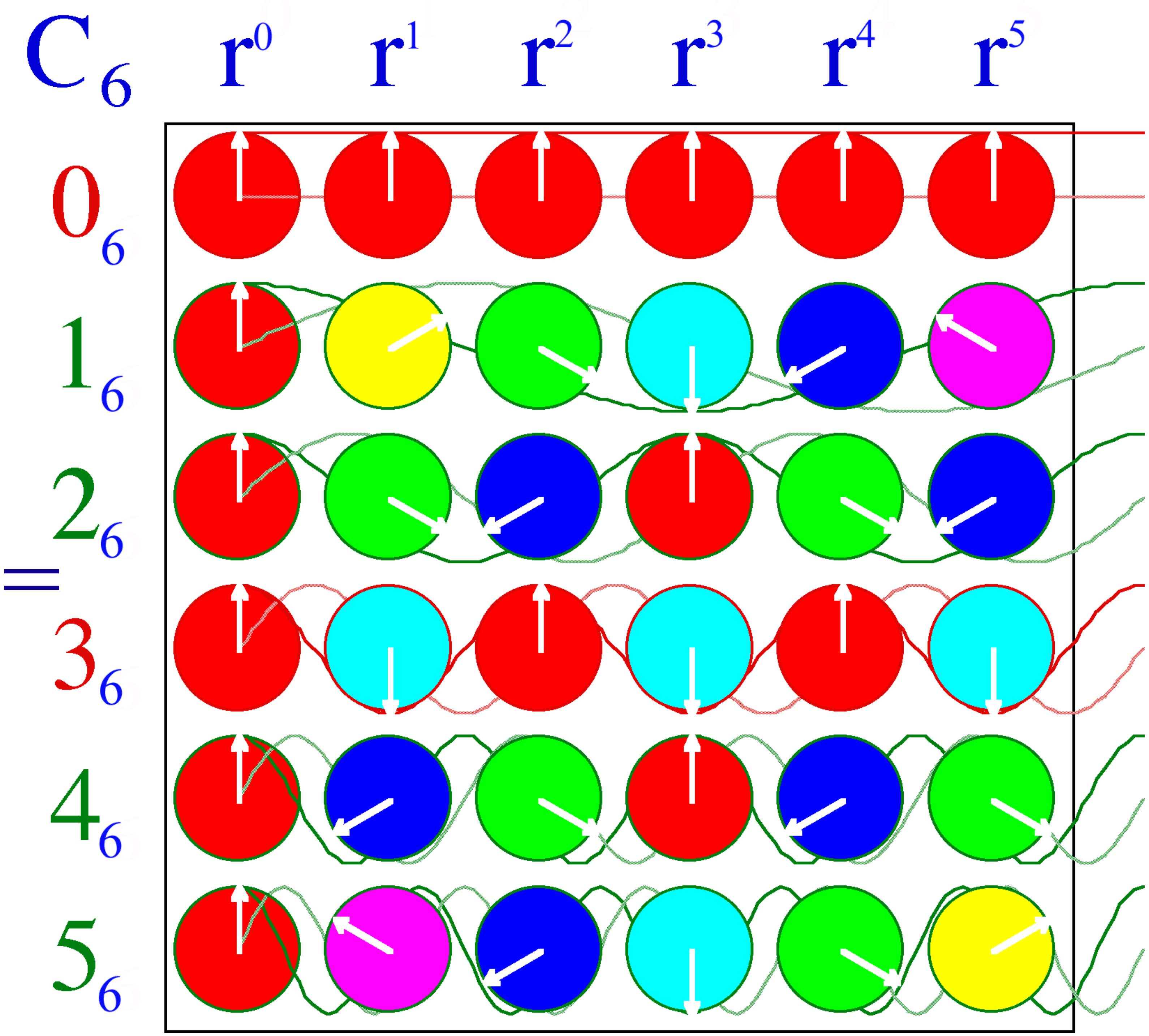
WaveIt App

Backwards phasors
for
conjugate waves
(turn counter-clockwise)



$\chi^{m}_{p}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε

$$\varepsilon=e^{i2\pi/6}$$

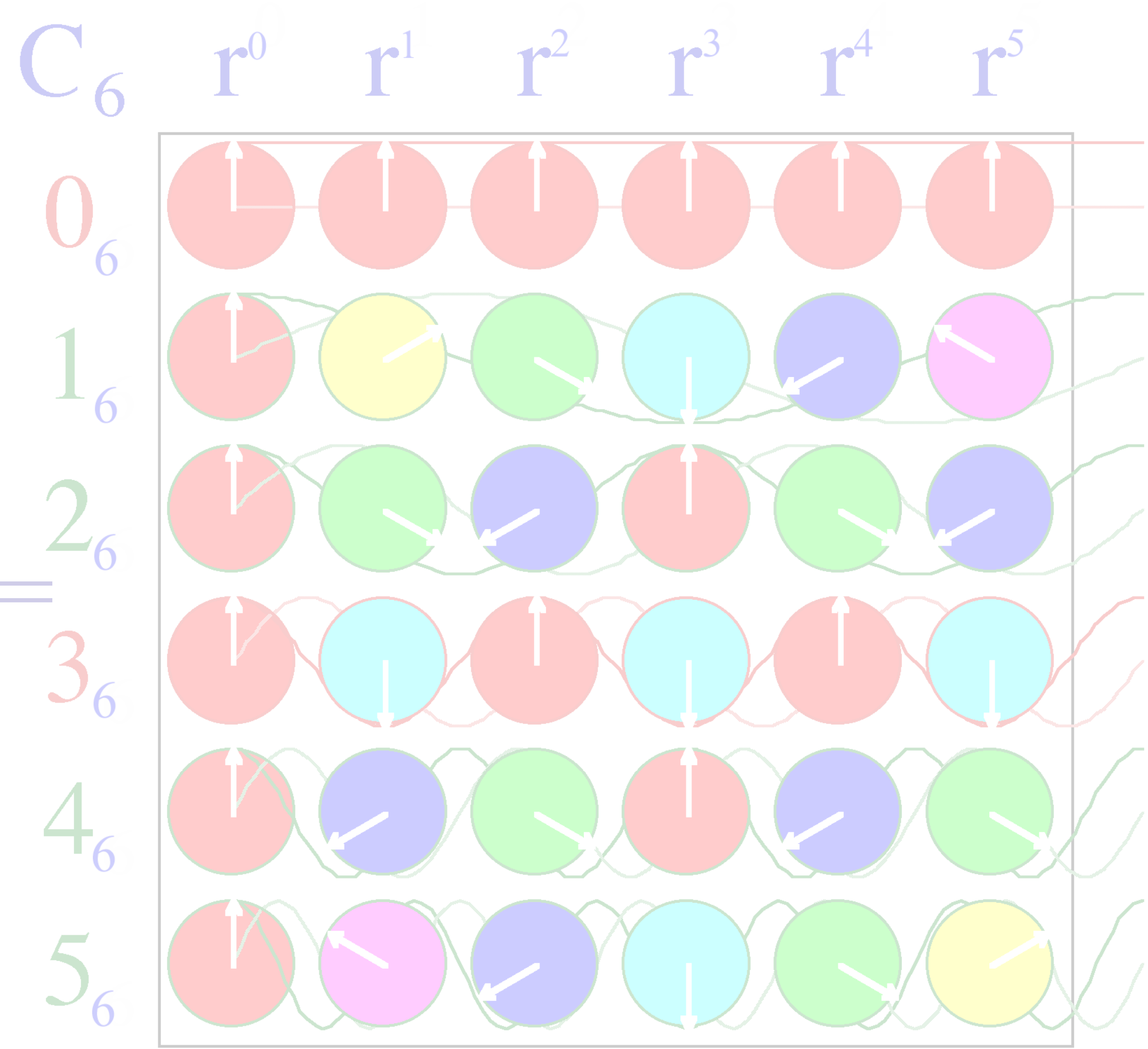


WaveIt App

	0°	*60°*	*120°*	*180°*	*-120°*	*-60°*
α	1	1	1	1	1	1
β	1	ε^*	ε^{2*}	-1	ε^2	ε
γ	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
δ	1	-1	1	-1	1	-1
γ^*	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
β^*	1	ε	ε^2	-1	ε^{2*}	ε

$$\varepsilon = e^{i2\pi/6}$$

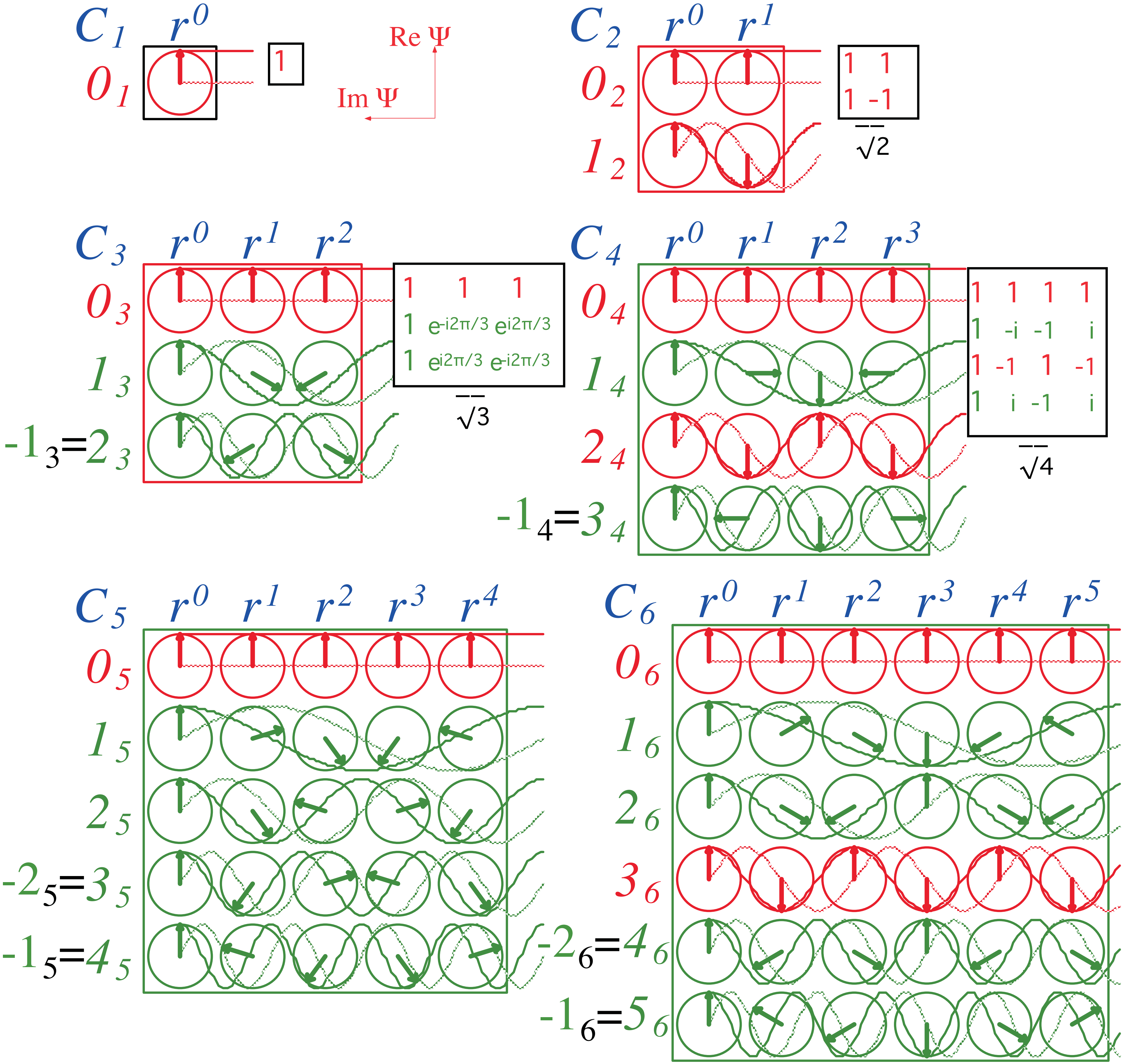
=



What you'll get
if you look up
C_6 characters in library

Wave phasor stuff? FUGggedd-aboudit!

WaveIt App



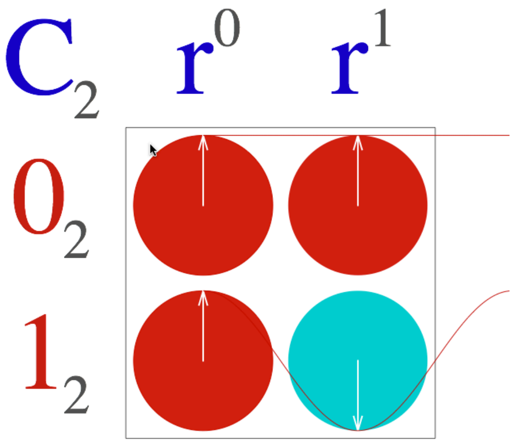
$$\begin{bmatrix}1\end{bmatrix}$$

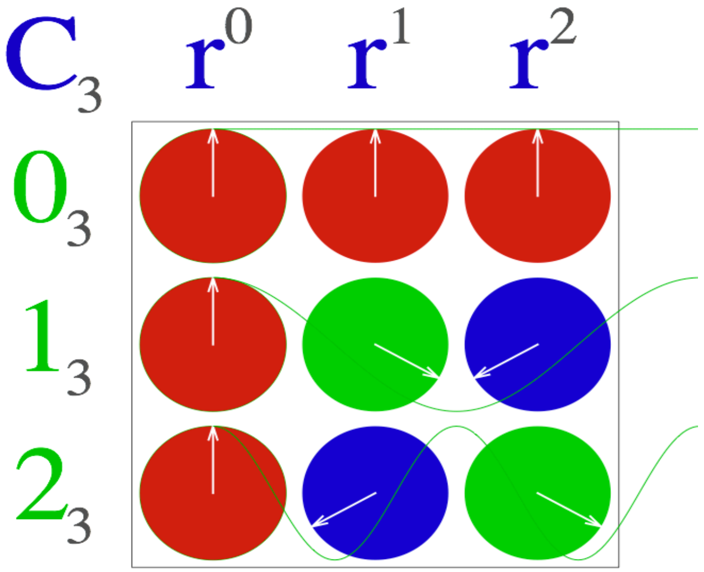
$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\Big/\sqrt{2}$$

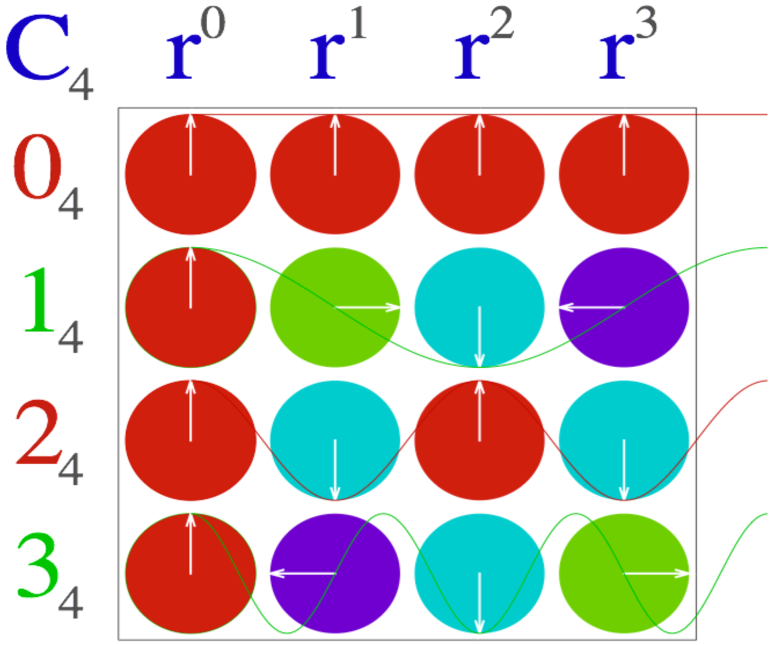
$$\begin{bmatrix}1 & 1 & 1\\ 1 & e^{-i2\pi/3} & e^{i2\pi/3}\\ 1 & e^{i2\pi/3} & e^{-i2\pi/3}\end{bmatrix}\Big/\sqrt{3}$$

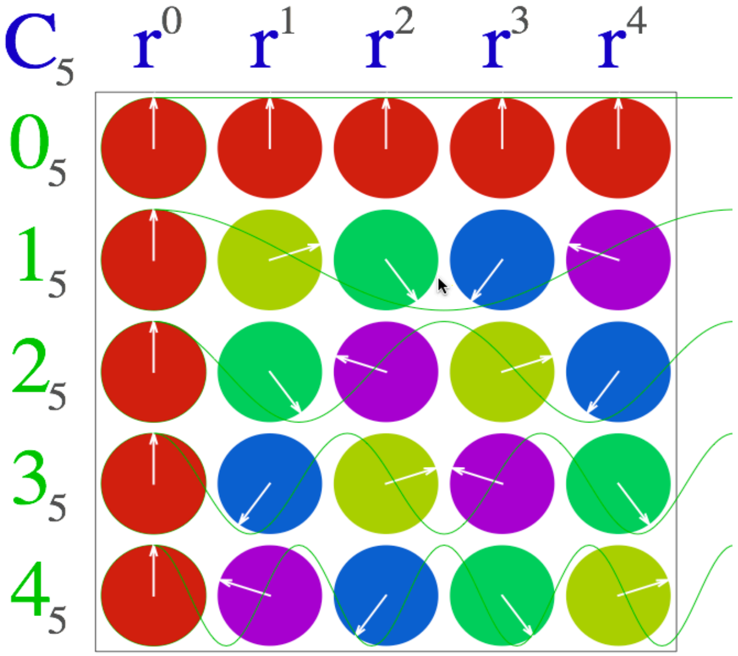
$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & i\end{bmatrix}\Big/\sqrt{4}$$

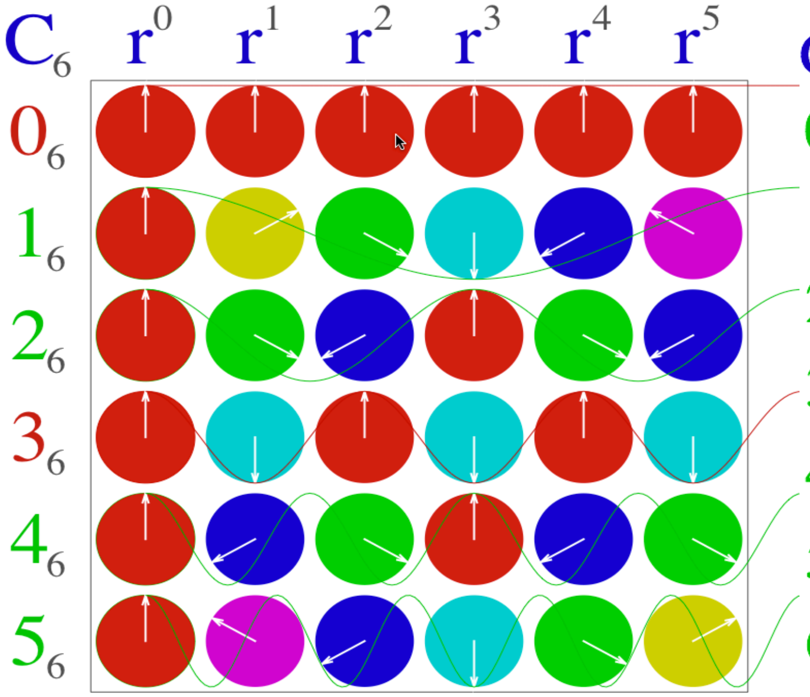
WaveIt App

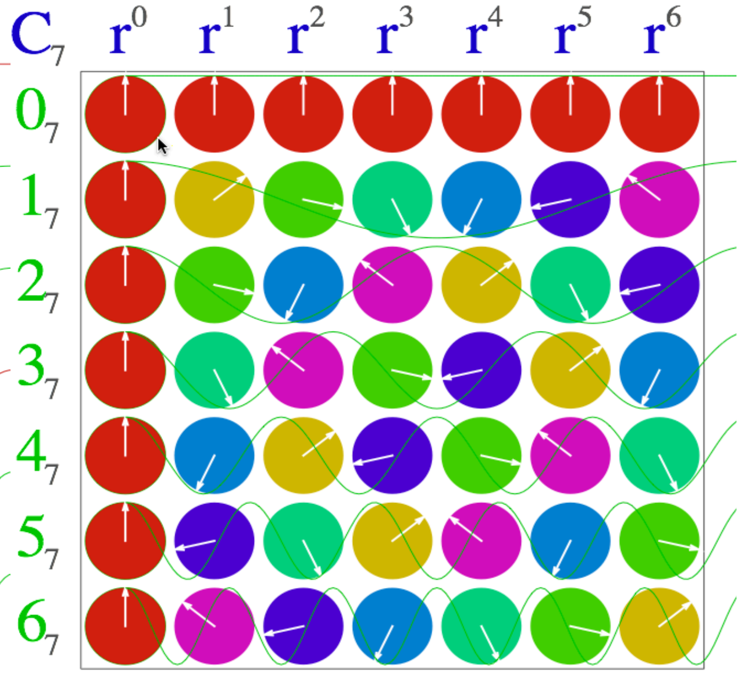

C_2
r^0
r^1
0_2
1_2


C_3
r^0
r^1
r^2
0_3
1_3
2_3

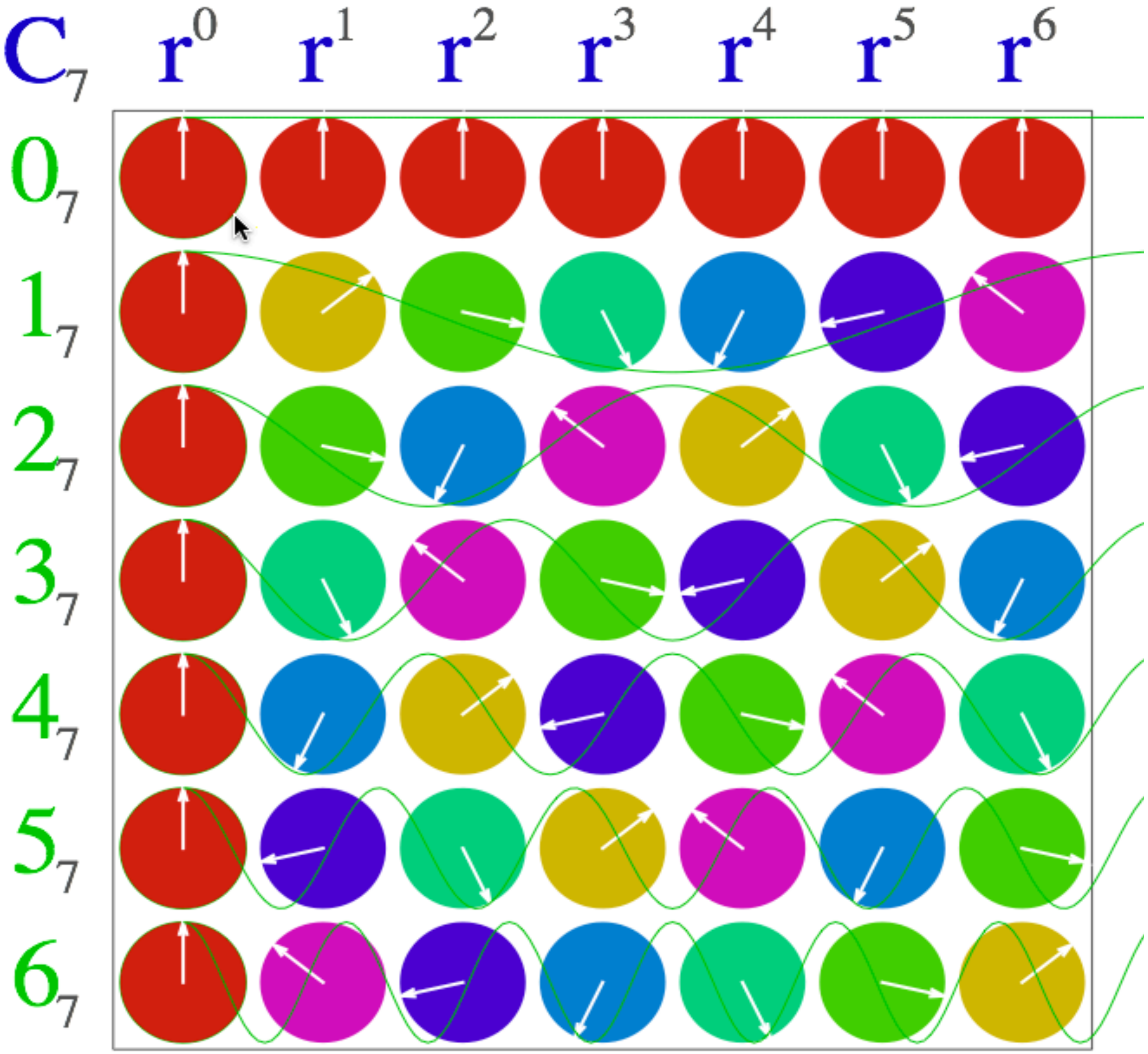

C_4
r^0
r^1
r^2
r^3
0_4
1_4
2_4
3_4


C_5
r^0
r^1
r^2
r^3
r^4
0_5
1_5
2_5
3_5
4_5


C_6
r^0
r^1
r^2
r^3
r^4
r^5
0_6
1_6
2_6
3_6
4_6
5_6


C_7
r^0
r^1
r^2
r^3
r^4
r^5
r^6
0_7
1_7
2_7
3_7
4_7
5_7
6_7

WaveIt App



WaveIt App

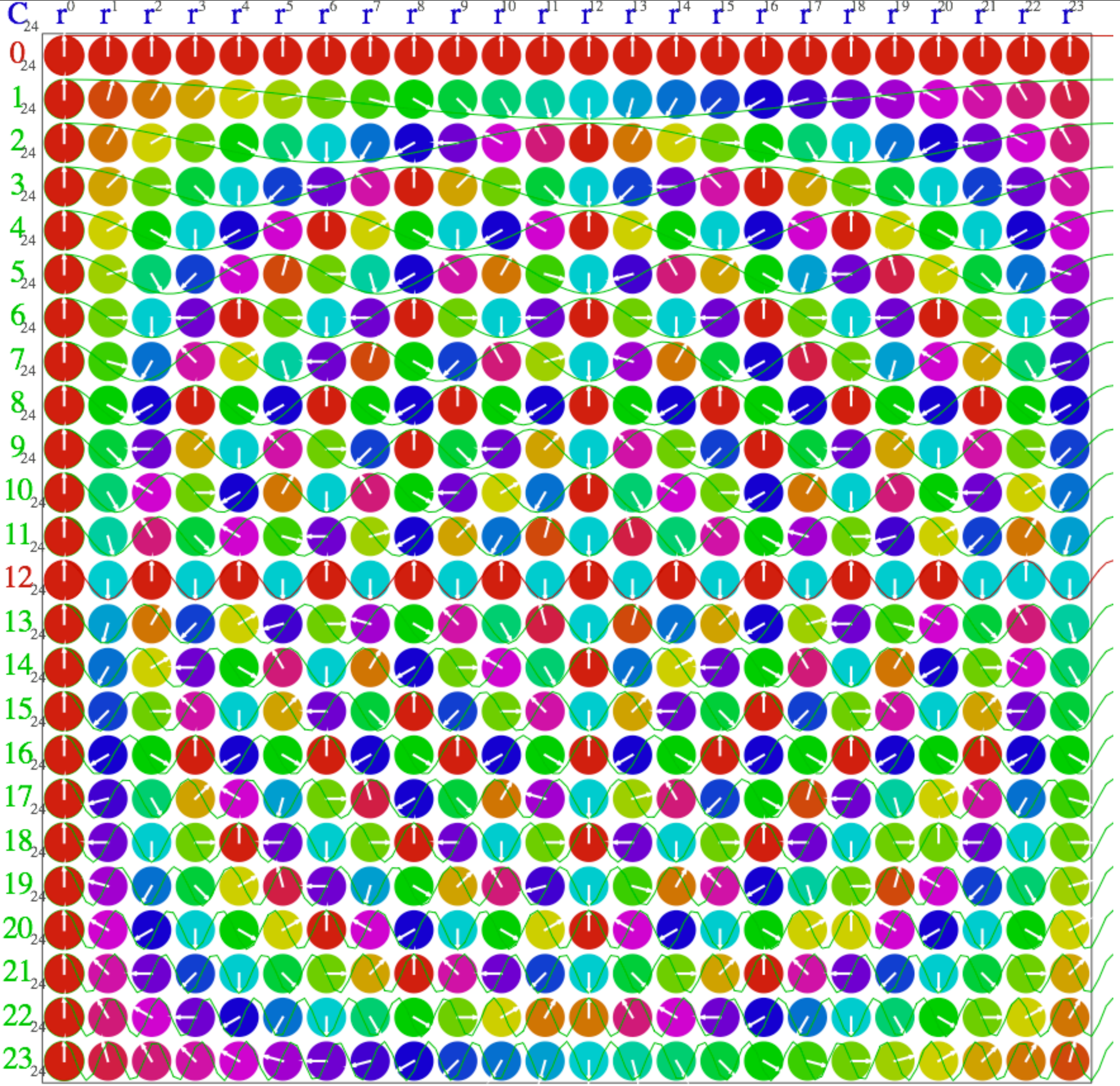
C_N Lattice position vector
$r_p = L \cdot p$

Wavevector
$k_m = 2\pi/\lambda_m$
$= 2\pi m/NL$

Wavelength
$\lambda_m = 2\pi/k_m$
$= NL/m$

C_N Plane wave function

$$\psi_m(x_p) = \frac{e^{ik_m \cdot x_p}}{\sqrt{N}} = \frac{e^{imp2\pi/N}}{\sqrt{N}}$$



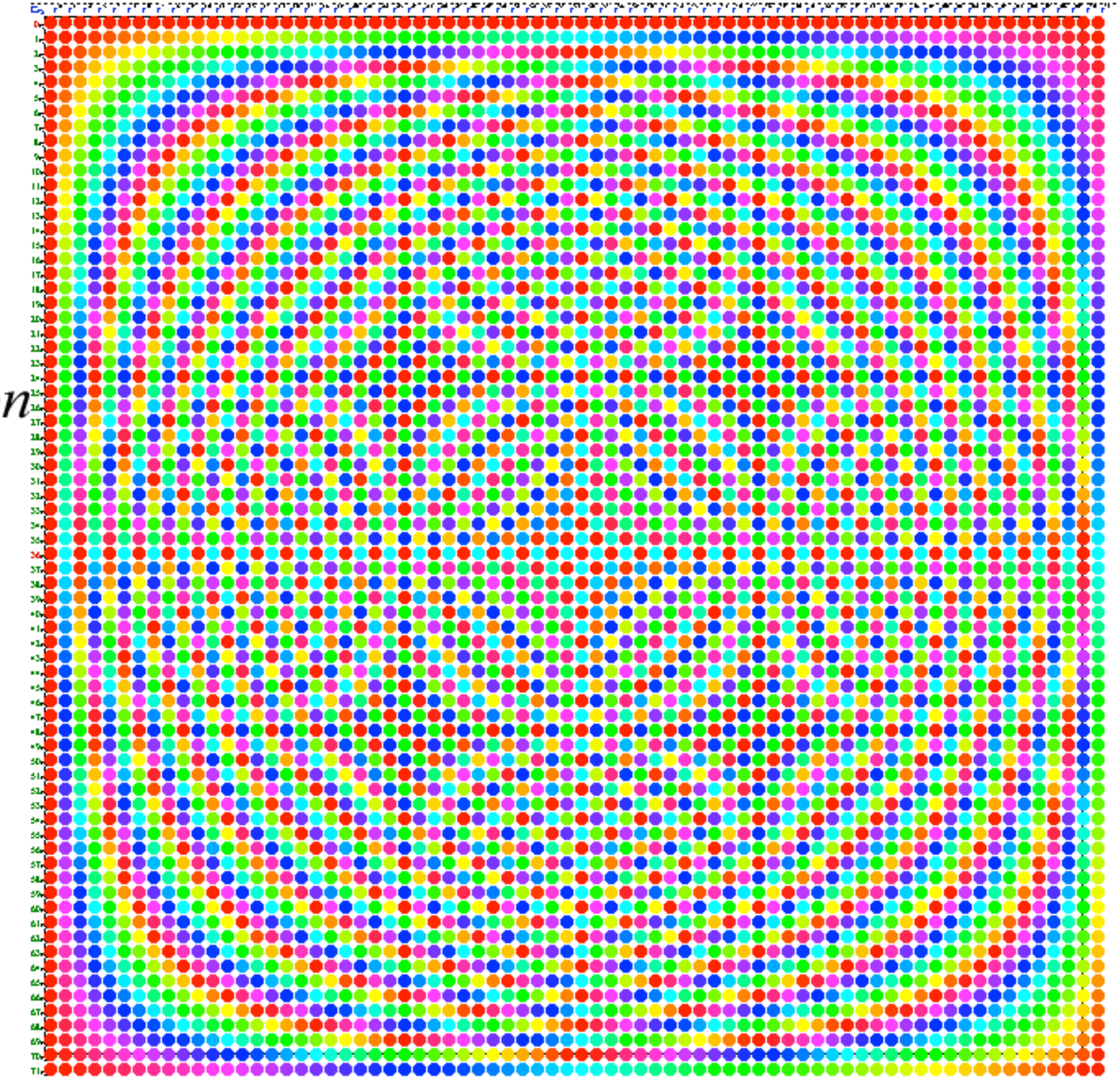
C_N Lattice position vector
$r_p = L \cdot p$

Wavevector
$k_m = 2\pi / \lambda_m$
$= 2\pi m / NL$

Wavelength
$\lambda_m = 2\pi / k_m$
$= NL / m$

$N=72$

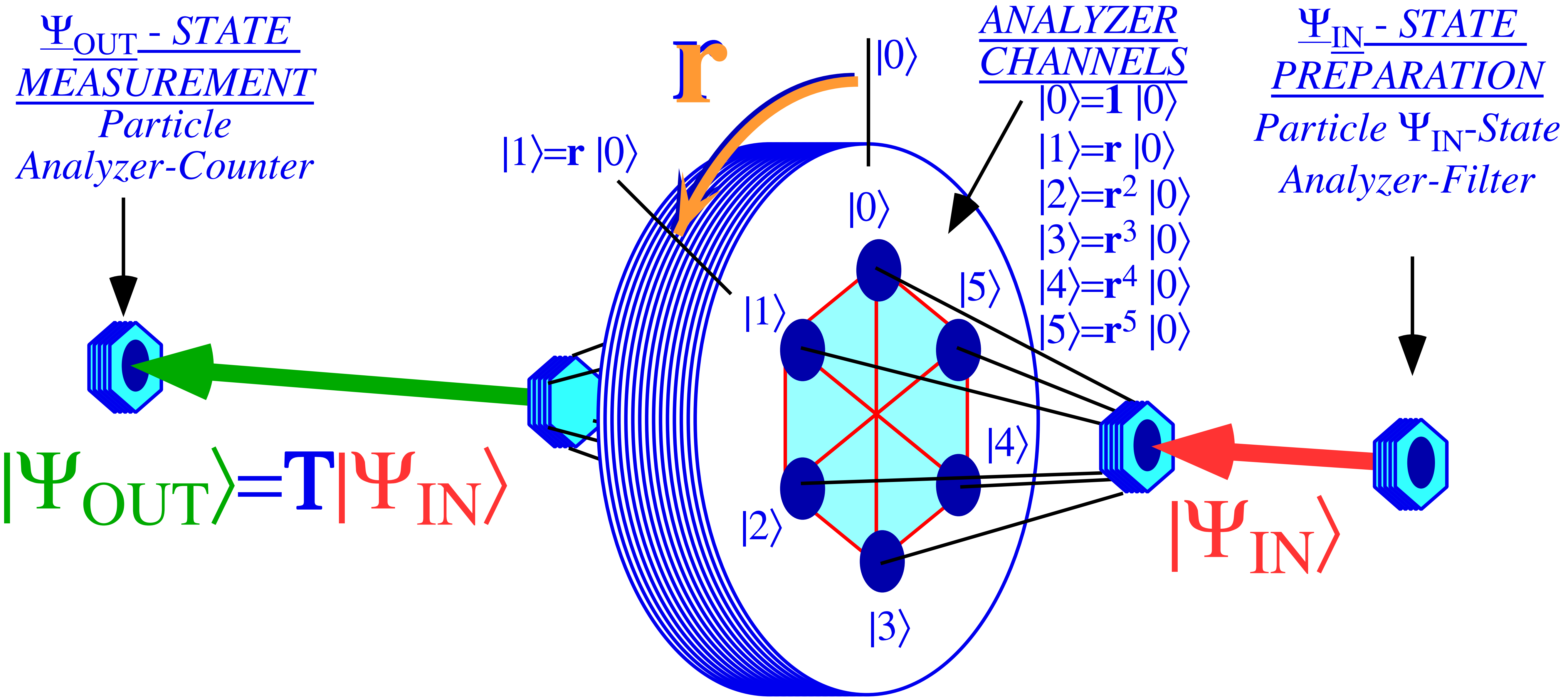
C_{72} Fourier transformation matrix



C_N Plane wave function

$$\psi_m(x_p) = \frac{e^{ik_m \cdot x_p}}{\sqrt{N}} = \frac{e^{imp2\pi/N}}{\sqrt{N}}$$

C_6 Beam analyzer used in Unit 3 Ch. 8 thru Ch. 9



Ψ_{OUT} - STATE MEASUREMENT
Particle
Analyzer-Counter

ANALYZER CHANNELS

$|0\rangle = \mathbf{1}\ |0\rangle$
$|1\rangle = \mathbf{r}\ |0\rangle$
$|2\rangle = \mathbf{r}^2\ |0\rangle$
$|3\rangle = \mathbf{r}^3\ |0\rangle$
$|4\rangle = \mathbf{r}^4\ |0\rangle$
$|5\rangle = \mathbf{r}^5\ |0\rangle$

Ψ_{IN} - STATE PREPARATION
Particle Ψ_{IN}-State
Analyzer-Filter

$$|\Psi_{OUT}\rangle = \mathbf{T}|\Psi_{IN}\rangle$$

$$|\Psi_{IN}\rangle$$

QTforCA Fig. 8.1.1

C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table and basic group representation theory*
    C_3 **H**-and-$\mathbf{r}^p$-*matrix representations and conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations
    C_3 *character table and modular labeling*

Ortho-completeness inversion for operators and states
    Modular quantum number arithmetic
    C_3-group jargon and structure of various tables

C_3 Eigenvalues and wave dispersion functions
    Standing waves vs Moving waves

C_6 Spectral resolution: 6th roots of unity and higher
→ *Complete sets of coupling parameters and Fourier dispersion*
    Gauge shifts due to complex coupling

3rd Step *Display all eigensolutions of all possible C_6 symmetric real H*

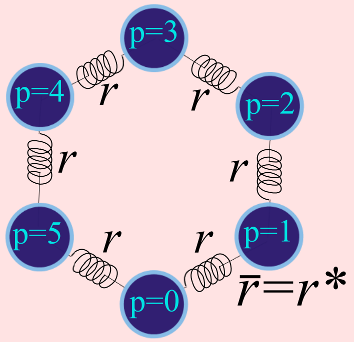
$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

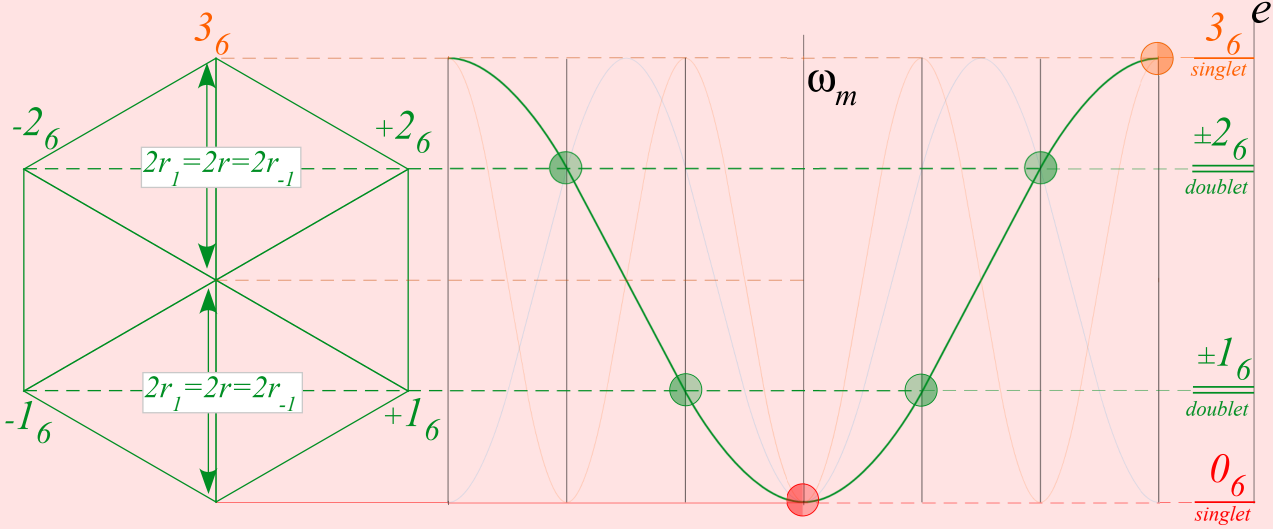
(Dispersion functions)

3rd Step *Display all eigensolutions of all possible C_6 symmetric real H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$ *(Dispersion functions)*

Elementary
Bloch Model
H= H_1**1** - r**r** - r**r**$^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & -r & \cdot & \cdot & \cdot & -r & 0 \\ -r & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -r & H_1 & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -r & H_1 & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -r & H_1 & -r & 4 \\ -r & \cdot & \cdot & \cdot & -r & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$=r_0\chi_0^m+r_1\chi_1^m+r_{-1}\chi_{-1}^m$$
$$=H_1-2r\cos(2\pi m/6)$$

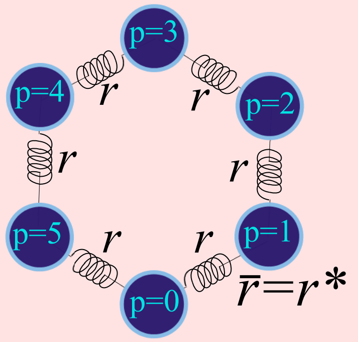
r_1 equals conjugate of r_5: ($r_1=r_5^*=-r$)

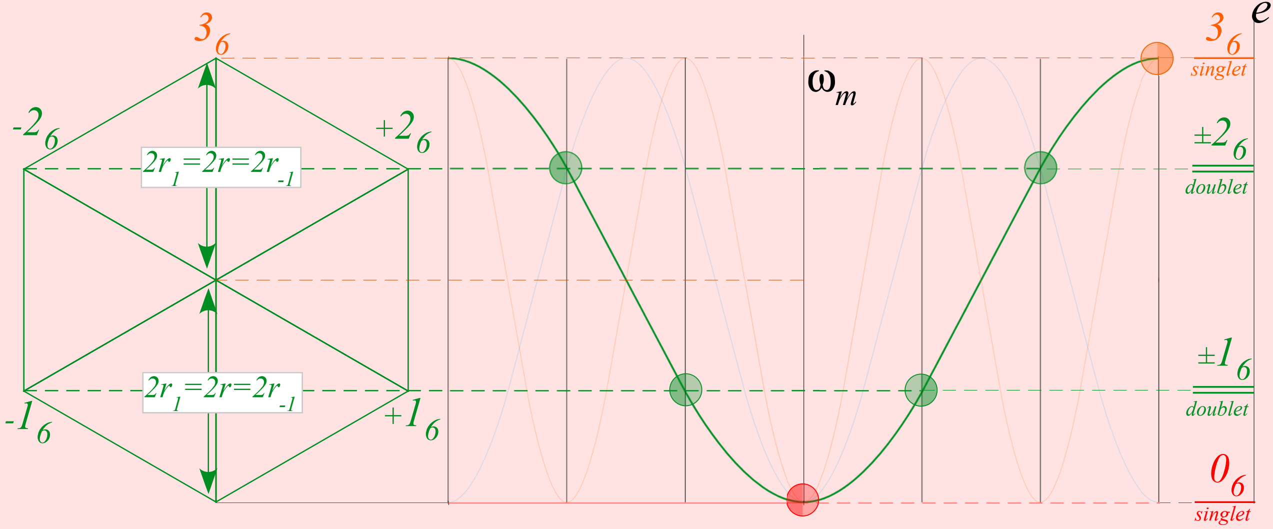
3^{rd} Step *Display all eigensolutions of all possible C_6 symmetric real H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$ *(Dispersion functions)*

Elementary
Bloch Model

H= H_1**1** - r**r** - r**r**$^{-1}$





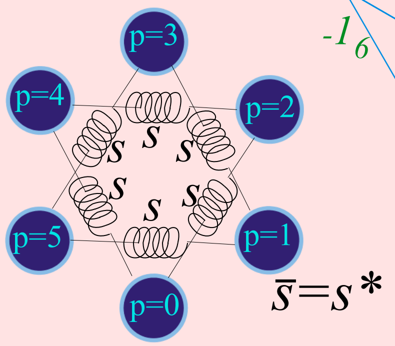
eigenvalues of $\mathbf{H}^{B1(6)}$

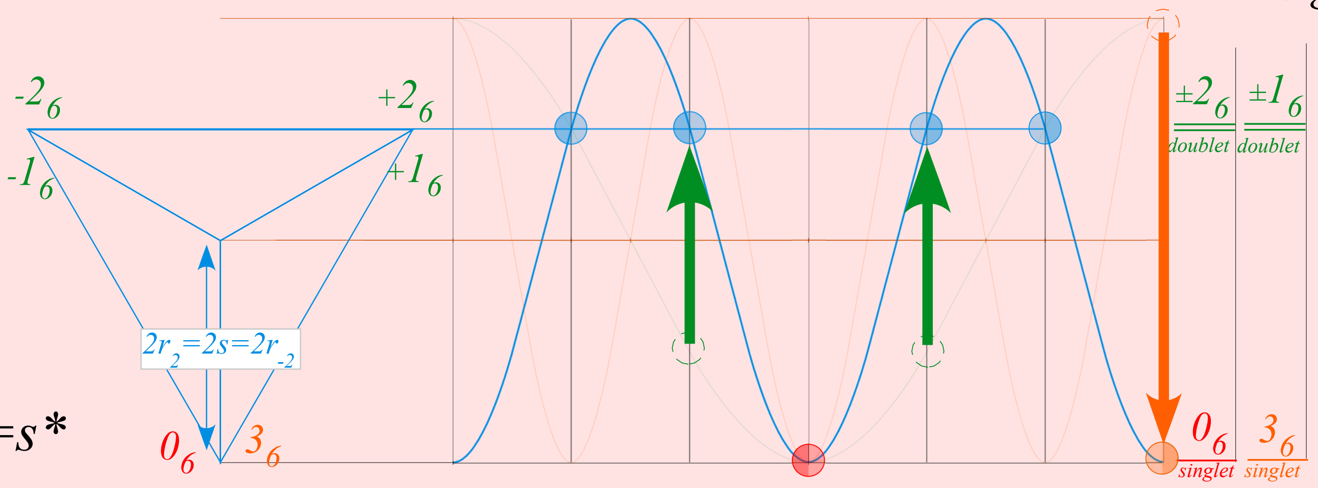
$$\begin{array}{c|cccccc|}
 & p=0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & H_1 & -r & \cdot & \cdot & \cdot & -r \\
1 & -r & H_1 & -r & \cdot & \cdot & \cdot \\
2 & \cdot & -r & H_1 & -r & \cdot & \cdot \\
3 & \cdot & \cdot & -r & H_1 & -r & \cdot \\
4 & \cdot & \cdot & \cdot & -r & H_1 & -r \\
5 & -r & \cdot & \cdot & \cdot & -r & H_1
\end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$=r_0\chi_0^m+r_1\chi_1^m+r_{-1}\chi_{-1}^m$$
$$=H_1-2r\,cos(2\pi m/6)$$

2^{nd} Neighbor
coupling

H= H_2**1** - s**r**^{2} - s**r**$^{-2}$





eigenvalues of $\mathbf{H}^{B2(6)}$

$$\begin{array}{c|cccccc|}
 & p=0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & H_2 & \cdot & -s & \cdot & -s & \cdot \\
1 & \cdot & H_2 & \cdot & -s & \cdot & -s \\
2 & -s & \cdot & H_2 & \cdot & -s & \cdot \\
3 & \cdot & -s & \cdot & H_2 & \cdot & -s \\
4 & -s & \cdot & -s & \cdot & H_2 & \cdot \\
5 & \cdot & -s & \cdot & -s & \cdot & H_2
\end{array}$$

$$\omega^{B2(n)}(k_m)$$
$$=r_0\chi_0^m+r_2\chi_2^m+r_{-2}\chi_{-2}^m$$
$$=H_2-2s\,cos(4\pi m/6)$$

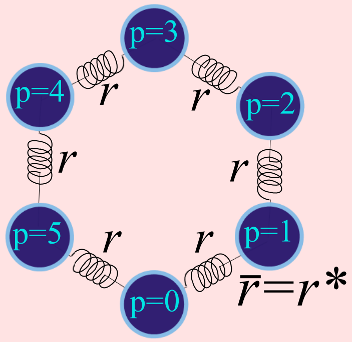
3rd Step *Display all eigensolutions of all possible C_6 symmetric real H*

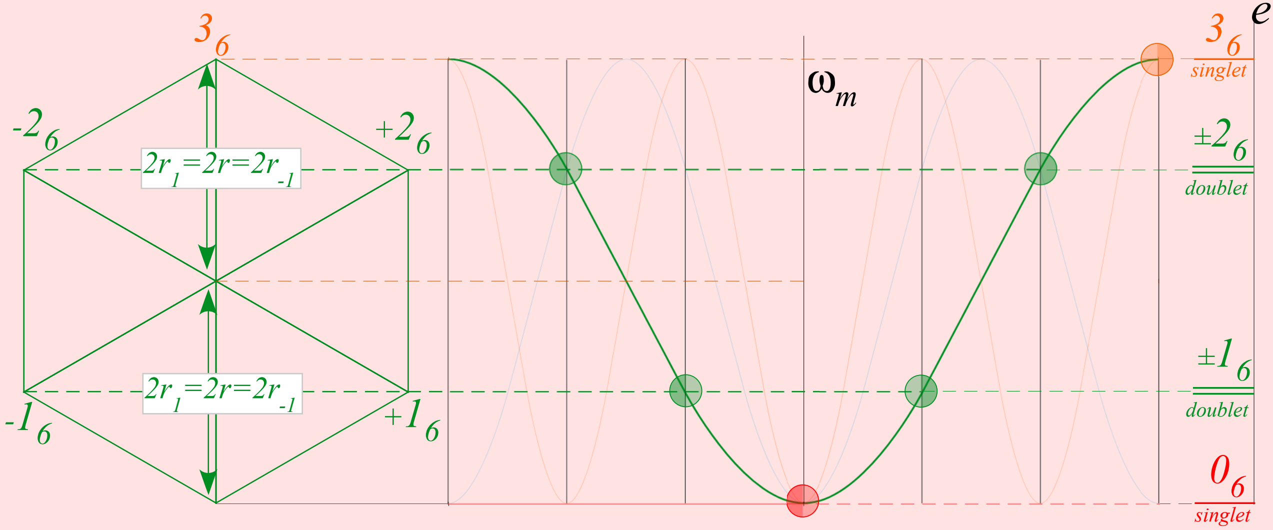
$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where:\ \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

(Dispersion functions)

Elementary Bloch Model

H= H_1**1** - r**r** - r**r**$^{-1}$





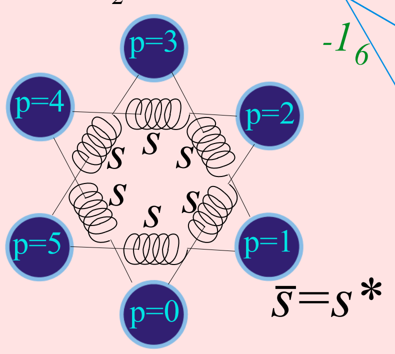
eigenvalues of $\mathbf{H}^{B1(6)}$

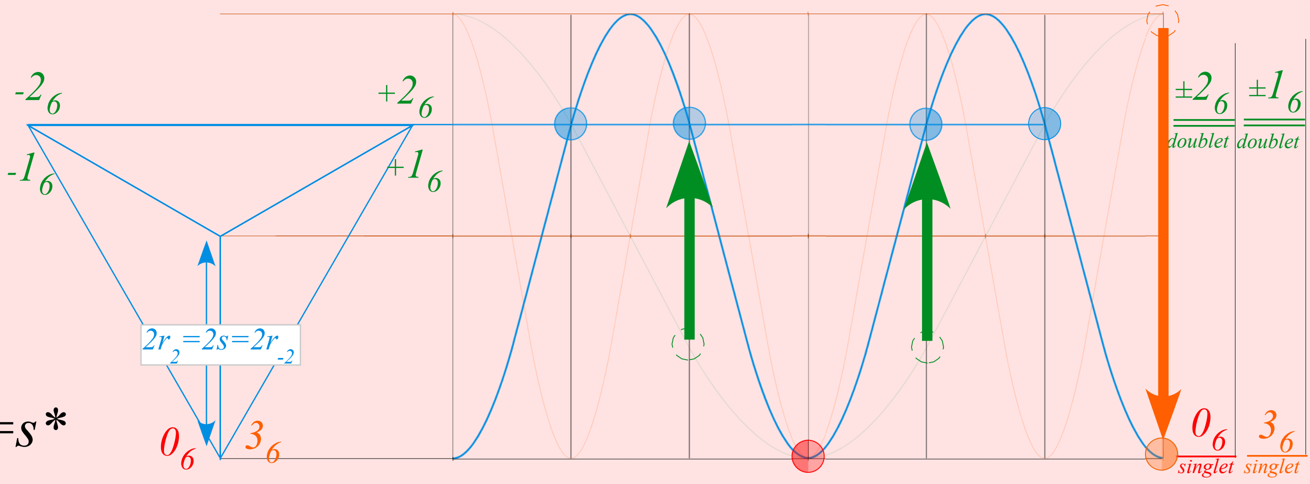
$$\begin{array}{c|cccccc} & p=0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & H_1 & -r & \cdot & \cdot & \cdot & -r \\ 1 & -r & H_1 & -r & \cdot & \cdot & \cdot \\ 2 & \cdot & -r & H_1 & -r & \cdot & \cdot \\ 3 & \cdot & \cdot & -r & H_1 & -r & \cdot \\ 4 & \cdot & \cdot & \cdot & -r & H_1 & -r \\ 5 & -r & \cdot & \cdot & \cdot & -r & H_1 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$=r_0\chi_0^m+r_1\chi_1^m+r_{-1}\chi_{-1}^m$$
$$=H_1-2r\,cos(2\pi m/6)$$

2nd Neighbor coupling

H= H_2**1** - s**r**^{2} - s**r**$^{-2}$





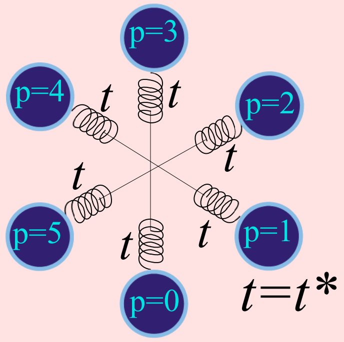
eigenvalues of $\mathbf{H}^{B2(6)}$

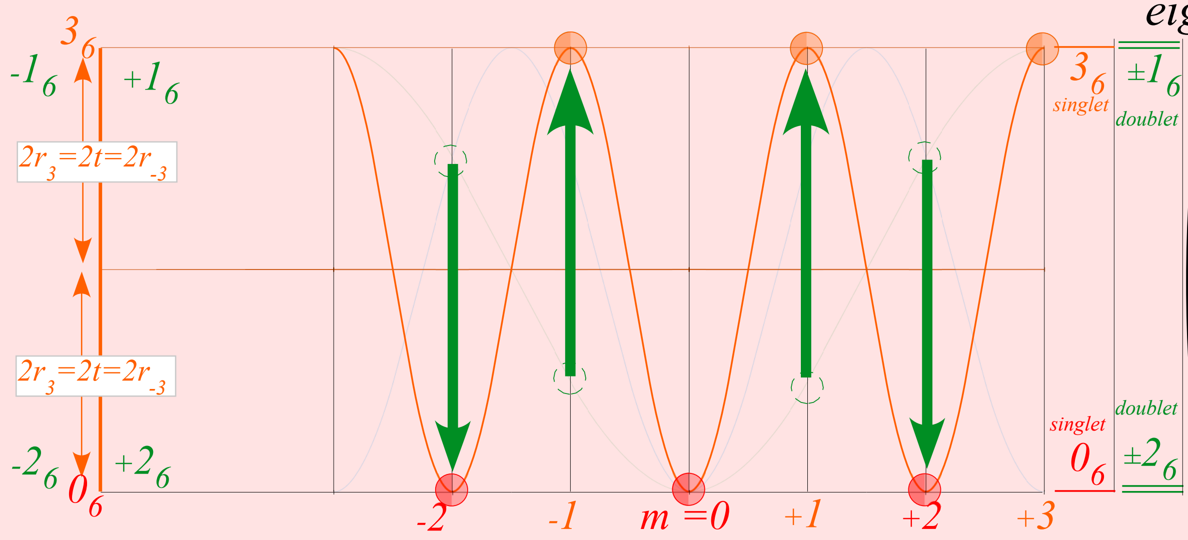
$$\begin{array}{c|cccccc} & p=0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & H_2 & \cdot & -s & \cdot & -s & \cdot \\ 1 & \cdot & H_2 & \cdot & -s & \cdot & -s \\ 2 & -s & \cdot & H_2 & \cdot & -s & \cdot \\ 3 & \cdot & -s & \cdot & H_2 & \cdot & -s \\ 4 & -s & \cdot & -s & \cdot & H_2 & \cdot \\ 5 & \cdot & -s & \cdot & -s & \cdot & H_2 \end{array}$$

$$\omega^{B2(n)}(k_m)$$
$$=r_0\chi_0^m+r_2\chi_2^m+r_{-2}\chi_{-2}^m$$
$$=H_2-2s\,cos(4\pi m/6)$$

3rd Neighbor coupling

H= H_3**1** - t**r**^{3} - t**r**$^{-3}$





eigenvalues of $\mathbf{H}^{B3(6)}$

$$\begin{array}{c|cccccc} & p=0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & H_3 & \cdot & \cdot & -t & \cdot & \cdot \\ 1 & \cdot & H_3 & \cdot & \cdot & -t & \cdot \\ 2 & \cdot & \cdot & H_3 & \cdot & \cdot & -t \\ 3 & -t & \cdot & \cdot & H_3 & \cdot & \cdot \\ 4 & \cdot & -t & \cdot & \cdot & H_3 & \cdot \\ 5 & \cdot & \cdot & -t & \cdot & \cdot & H_3 \end{array}$$

$$\omega^{B3(n)}(k_m)$$
$$=r_0\chi_0^m+r_3\chi_3^m+r_{-3}\chi_{-3}^m$$
$$=H_3-2t\,(-1)^m)$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N} - \phi_p)}$$

C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero ($\phi_p = 0$)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \ , r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m | \sum_{p=0} r_p \mathbf{r}^p | m \rangle = \sum_{p=0} r_p \langle m | \mathbf{r}^p | m \rangle = \sum_{p=0} r_p e^{-i2\pi \frac{m \cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi \frac{m \cdot p}{N} - \phi_p)}$$

Setting gauge to zero ($\phi_p = 0$)

Real C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \;, r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}_{real}{}^{GB(6)}) = r_0 + r_1 (e^{i\pi \frac{m \cdot 1}{3}} + e^{-i\pi \frac{m \cdot 1}{3}}) \quad + r_2 (e^{i\pi \frac{m \cdot 2}{3}} + e^{-i\pi \frac{m \cdot 2}{3}}) \quad + r_3 (e^{i\pi \frac{m \cdot 3}{3}}) \qquad \text{(for real: } r_p = r_{-p} = r_p^*)$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero ($\phi_p=0$)

Real C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \ , r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \qquad \text{(for real: } r_p = r_{-p} = r_p^*\text{)}$$

$$= H \ + \ 2r\cos\pi\frac{m\cdot 1}{3} \quad + \quad 2s\cos\pi\frac{m\cdot 2}{3} \quad + \ t(-1)^m$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m | \sum_{p=0} r_p \mathbf{r}^p | m \rangle = \sum_{p=0} r_p \langle m | \mathbf{r}^p | m \rangle = \sum_{p=0} r_p e^{-i2\pi \frac{m \cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi \frac{m \cdot p}{N} - \phi_p)}$$

Setting gauge to zero ($\phi_p = 0$)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \ , r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi \frac{m \cdot 1}{3}} + e^{-i\pi \frac{m \cdot 1}{3}}) \quad + r_2(e^{i\pi \frac{m \cdot 2}{3}} + e^{-i\pi \frac{m \cdot 2}{3}}) \quad + r_3(e^{i\pi \frac{m \cdot 3}{3}}) \qquad \text{(for real: } r_p = r_{-p} = r_p^*\text{)}$$

$$= H \ + \ 2r \cos \pi \frac{m \cdot 1}{3} \quad + \quad 2s \cos \pi \frac{m \cdot 2}{3} \quad + \ t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 & = H + 2r + 2s + t \\ \omega_{\pm 1} & = H + r - s - t \\ \omega_{\pm 2} & = H - r - s + t \\ \omega_3 & = H - 2r + 2s - t \end{cases}$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero ($\phi_p=0$)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \;, r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \qquad (\text{for real: } r_p = r_{-p} = r_p^*)$$

$$= H \;+\; 2r\cos\pi\frac{m\cdot 1}{3} \quad + \quad 2s\cos\pi\frac{m\cdot 2}{3} \quad + \; t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 & = H + 2r + 2s + t \\ \omega_{\pm 1} & = H + r - s - t \\ \omega_{\pm 2} & = H - r - s + t \\ \omega_3 & = H - 2r + 2s - t \end{cases}$$

...in terms of 4 solvable r_p-parameters:

$$r_p = \begin{cases} H = \frac{1}{4}(\omega_0 + \omega_1 + \omega_2 + \omega_3) \\ r = \frac{1}{6}(\omega_0 + \omega_1 - \omega_2 - \omega_3) \\ s = \frac{1}{6}(\omega_0 - \omega_1 - \omega_2 + \omega_3) \\ t = \frac{1}{6}(\omega_0 - 2\omega_1 + 2\omega_2 - \omega_3) \end{cases}$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m|\sum_{p=0} r_p \mathbf{r}^p|m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} \left|r_p\right| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero (ϕ_p =0)

Real C_6 Bloch **H** $^{GB(N)}$ eigenvalues are Fourier series with 4 (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = r = r_{-1}\ ,\ r_2 = s = r_{-2},\quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}})$$

(for real: $r_p = r_{-p} = r_p^*$)

$$= H + 2r\cos\pi\frac{m\cdot 1}{3} + 2s\cos\pi\frac{m\cdot 2}{3} + t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 = H + 2r + 2s + t \\ \omega_{\pm 1} = H + r - s - t \\ \omega_{\pm 2} = H - r - s + t \\ \omega_3 = H - 2r + 2s - t \end{cases}$$

...in terms of 4 solvable r_p-parameters:

$$r_p = \begin{cases} H = \frac{1}{4}(\omega_0 + \omega_1 + \omega_2 + \omega_3) \\ r = \frac{1}{6}(\omega_0 + \omega_1 - \omega_2 - \omega_3) \\ s = \frac{1}{6}(\omega_0 - \omega_1 - \omega_2 + \omega_3) \\ t = \frac{1}{6}(\omega_0 - 2\omega_1 + 2\omega_2 - \omega_3) \end{cases}$$

General Bloch **H** $^{GB(N)}$ eigenvalues are Fourier series with six (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = re^{i\phi_1},\quad r_{-1} = re^{-i\phi_1},\quad r_2 = se^{i\phi_2},\quad r_{-2} = se^{-i\phi_2},\quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p ,

or complex: $r_{-p} = r_p^*$

$$\omega_m(\mathbf{H}_{complex}{}^{GZB(6)}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

C_3 $\mathbf{g}^{\dagger}\mathbf{g}$-*product-table and basic group representation theory*

C_3 **H**-and-$\mathbf{r}^p$-*matrix representations and conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations

C_3 *character table and modular labeling*

Ortho-completeness inversion for operators and states

Modular quantum number arithmetic

C_3-group jargon and structure of various tables

C_3 Eigenvalues and wave dispersion functions

Standing waves vs Moving waves

C_6 Spectral resolution: 6th roots of unity and higher

Complete sets of coupling parameters and Fourier dispersion

→ *Gauge shifts due to complex coupling*

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 6 (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = re^{i\phi_1},\quad r_{-1} = re^{-i\phi_1},\quad r_2 = se^{i\phi_2},\quad r_{-2} = se^{-i\phi_2},\quad r_3 = t = r_{-3} \}$ Nonzero gauge ϕ_p ,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m | \sum_{p=0} r_p \mathbf{r}^p | m \rangle = \sum_{p=0} r_p \langle m | \mathbf{r}^p | m \rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} \left| r_p \right| e^{-i(2\pi\frac{m\cdot p}{N} - \phi_p)}$$

Complex Bloch matrix **H** $^{GB(N)}$ eigenvalues are Fourier series with 6 (for N=6) Fourier parameters

$\{ r_0 = H, \quad r_1 = re^{i\phi_1}, \quad r_{-1} = re^{-i\phi_1}, \quad r_2 = se^{i\phi_2}, \quad r_{-2} = se^{-i\phi_2}, \quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r_p^*$

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 6 (for $N=6$) Fourier parameters

$\{ r_0 = H,\quad r_1 = re^{i\phi_1},\quad r_{-1} = re^{-i\phi_1},\quad r_2 = se^{i\phi_2},\quad r_{-2} = se^{-i\phi_2},\quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p ,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r_p^*$

giving 6 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 = r_0 + r_1 + r_{-1} + r_2 + r_{-2} + r_3 \\ \omega_{+1} = r_0 + r_1 e^{\frac{i\pi}{3}} + r_{-1} e^{\frac{-i\pi}{3}} + r_2 e^{\frac{i2\pi}{3}} + r_{-2} e^{\frac{-i2\pi}{3}} - r_3 \\ \omega_{-1} = r_0 + r_1 e^{\frac{-i\pi}{3}} + r_{-1} e^{\frac{i\pi}{3}} + r_2 e^{\frac{-i2\pi}{3}} + r_{-2} e^{\frac{i2\pi}{3}} - r_3 \\ \omega_{+2} = r_0 + r_1 e^{\frac{i2\pi}{3}} + r_{-1} e^{\frac{-i2\pi}{3}} - r_2 e^{\frac{i\pi}{3}} - r_{-2} e^{\frac{-i\pi}{3}} + r_3 \\ \omega_{-2} = r_0 + r_1 e^{\frac{-i2\pi}{3}} + r_{-1} e^{\frac{i2\pi}{3}} - r_2 e^{\frac{-i\pi}{3}} - r_{-2} e^{\frac{i\pi}{3}} + r_3 \\ \omega_3 = r_0 - r_1 - r_{-1} + r_2 + r_{-2} - r_3 \end{cases}$$

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m|\sum_{p=0} r_p \mathbf{r}^p|m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 6 (for N=6) Fourier parameters

$\{ r_0 = H, \quad r_1 = re^{i\phi_1}, \quad r_{-1} = re^{-i\phi_1}, \quad r_2 = se^{i\phi_2}, \quad r_{-2} = se^{-i\phi_2}, \quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p ,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r_p^*$

giving 6 ω_m-levels:

...in terms of 6 solvable r_p-parameters:

$$\omega_m = \begin{cases} \omega_0 = r_0 + r_1 + r_{-1} + r_2 + r_{-2} + r_3 \\ \omega_{+1} = r_0 + r_1 e^{\frac{i\pi}{3}} + r_{-1} e^{\frac{-i\pi}{3}} + r_2 e^{\frac{i2\pi}{3}} + r_{-2} e^{\frac{-i2\pi}{3}} - r_3 \\ \omega_{-1} = r_0 + r_1 e^{\frac{-i\pi}{3}} + r_{-1} e^{\frac{i\pi}{3}} + r_2 e^{\frac{-i2\pi}{3}} + r_{-2} e^{\frac{i2\pi}{3}} - r_3 \\ \omega_{+2} = r_0 + r_1 e^{\frac{i2\pi}{3}} + r_{-1} e^{\frac{-i2\pi}{3}} - r_2 e^{\frac{i\pi}{3}} - r_{-2} e^{\frac{-i\pi}{3}} + r_3 \\ \omega_{-2} = r_0 + r_1 e^{\frac{-i2\pi}{3}} + r_{-1} e^{\frac{i2\pi}{3}} - r_2 e^{\frac{-i\pi}{3}} - r_{-2} e^{\frac{i\pi}{3}} + r_3 \\ \omega_3 = r_0 - r_1 - r_{-1} + r_2 + r_{-2} - r_3 \end{cases}$$

$$r_p = \begin{cases} r_0 = ? \\ r_1 = ? \\ r_{-1} = ? \\ r_2 = ? \\ r_{-2} = ? \\ r_3 = ? \end{cases}$$

Left as an exercise...

Geometric solution shown next...

3rd Step (contd.)

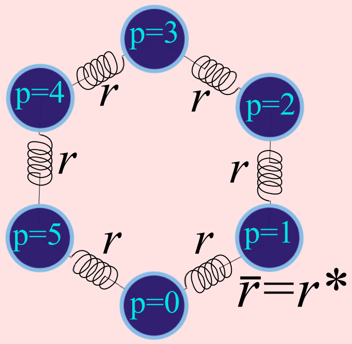
...*eigensolutions for all possible C_6 symmetric complex H*

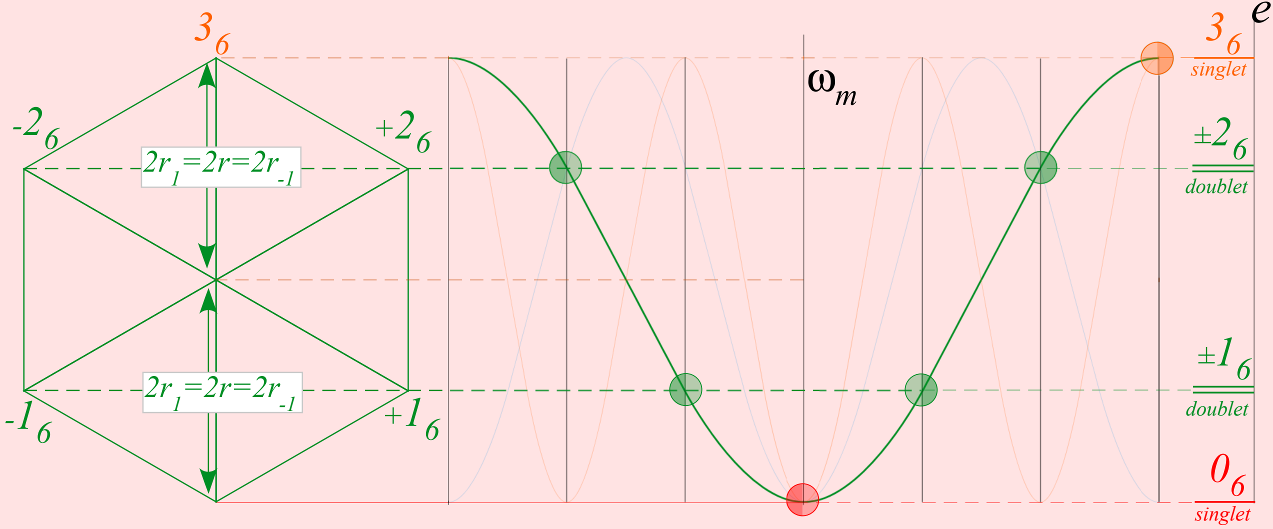
$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

(Dispersion function)

Elementary
Bloch Model

$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$= r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m$$
$$= H_1 - 2r\,cos(2\pi m/6)$$

Nearest neighbor coupling

$$\begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

3rd Step (contd.)

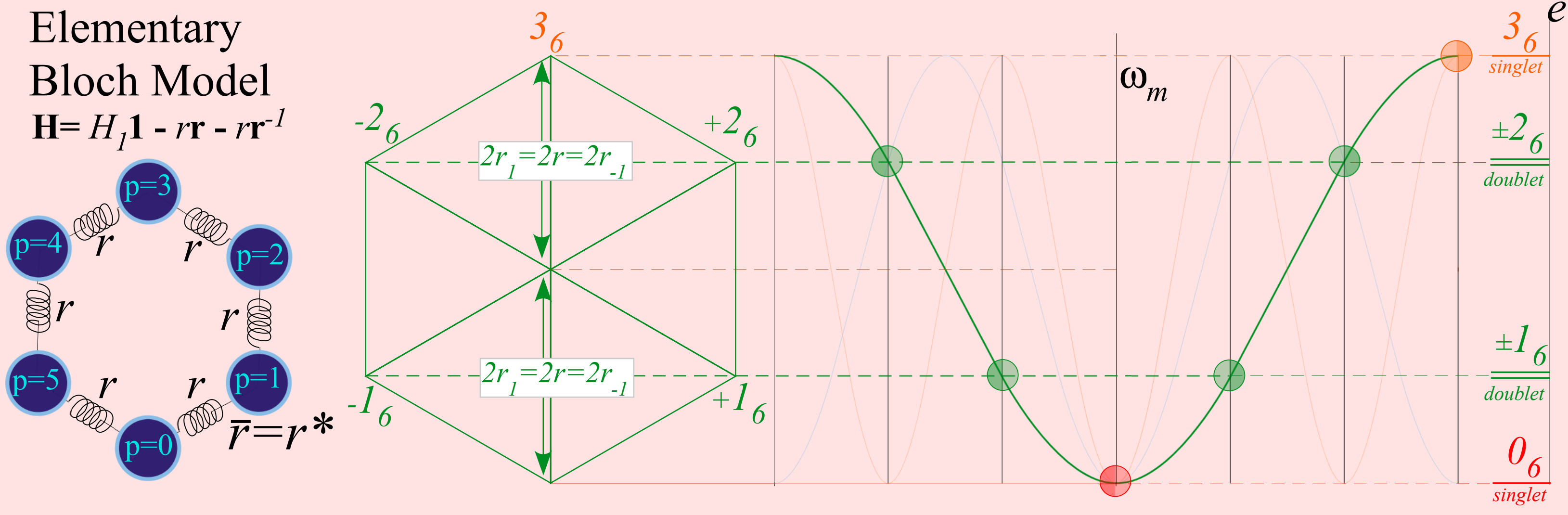
...*eigensolutions for all possible C_6 symmetric complex H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

(Dispersion function)

Elementary Bloch Model

$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$



eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c}
p=0 & 1 & 2 & 3 & 4 & 5 & \\
H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0\\
r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1\\
\cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2\\
\cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3\\
\cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4\\
r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5
\end{array}$$

$$\omega^{B1(n)}(k_m) = r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m = H_1 - 2r\cos(2\pi m/6)$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)}=\begin{pmatrix}
r_0 & r_{-1} & & & & r_1\\
r_1 & r_0 & r_{-1} & & & \\
 & r_1 & r_0 & r_{-1} & & \\
 & & r_1 & r_0 & r_{-1} & \\
 & & & r_1 & r_0 & r_{-1}\\
r_{-1} & & & & r_1 & r_0
\end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)}=(\mathbf{H}^{B1(6)})^\dagger$

complex components

$r_1=-re^{i\phi}$ *imply*

conjugate components

$r^*_1=r_{-1}=-re^{-i\phi}$

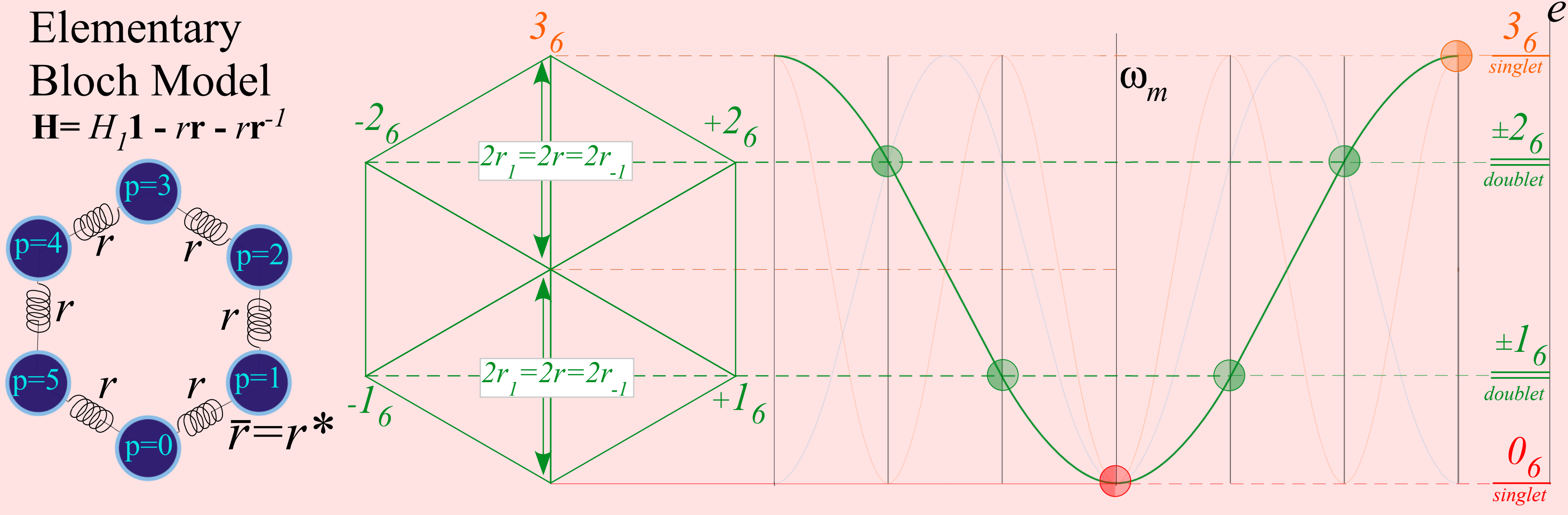
3rd Step (contd.)

...eigensolutions for all possible C_6 symmetric complex H

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

(Dispersion function)

Elementary
Bloch Model
$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$



eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc|c} & p=0 & 1 & 2 & 3 & 4 & 5 & \\ \hline & H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ & r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ & \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ & \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ & \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ & r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$= r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m$$
$$= H_1 - 2r\,cos(2\pi m/6)$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)}=\begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)}=(\mathbf{H}^{B1(6)})^\dagger$
complex components
$r_1=-re^{i\phi}$ *imply*
conjugate components
$r^*_1=r_{-1}=-re^{-i\phi}$

$$\omega^{B1(6)}(k_m) = r_0\chi^m{}_0 + r_1\chi^m{}_1 + r_{-1}\chi^m{}_{-1}$$
$$= r_0 - r\,e^{i\phi}e^{i2\pi m/6} - r\,e^{-i\phi}e^{-i2\pi m/6}$$
$$= r_0 - 2r\,cos(2\pi m/6 + \phi)$$

3^{rd} Step (contd.)

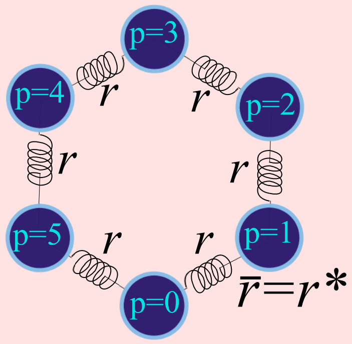
...eigensolutions for all possible C_6 symmetric complex H

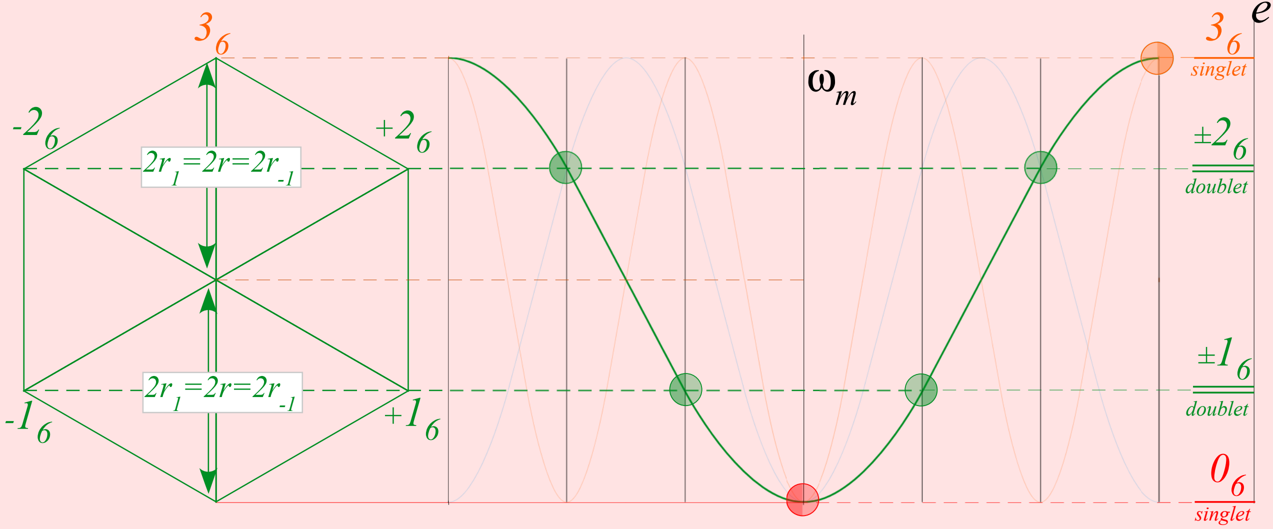
$$\mathbf{H} = \sum_{p=0}^{n-1} r_p \mathbf{r}^p = \sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)} = \sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)} = \sum_{p=0}^{n-1} r_p \chi_p^m = \omega(k_m)$$

(Dispersion function)

Elementary Bloch Model

$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccccc} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$= r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m$$
$$= H_1 - 2r\,cos(2\pi m/6)$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)}=\begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)}=(\mathbf{H}^{B1(6)})^\dagger$

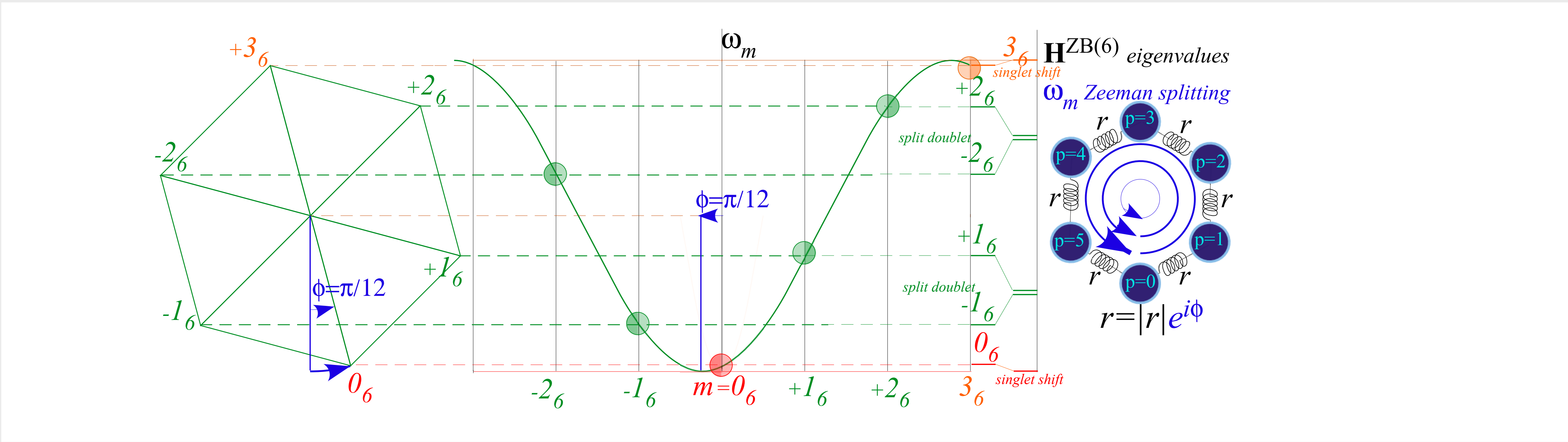
complex components

$r_1=-re^{i\phi}$ *imply*

conjugate components

$r^*_1=r_{-1}=-re^{-i\phi}$

$$\omega^{B1(6)}(k_m) = r_0\chi^m{}_0 + r_1\chi^m{}_1 + r_{-1}\chi^m{}_{-1}$$
$$= r_0 - r\,e^{i\phi}e^{i2\pi m/6} - r\,e^{-i\phi}e^{-i2\pi m/6}$$
$$= r_0 - 2r\,cos(2\pi m/6 + \phi)$$



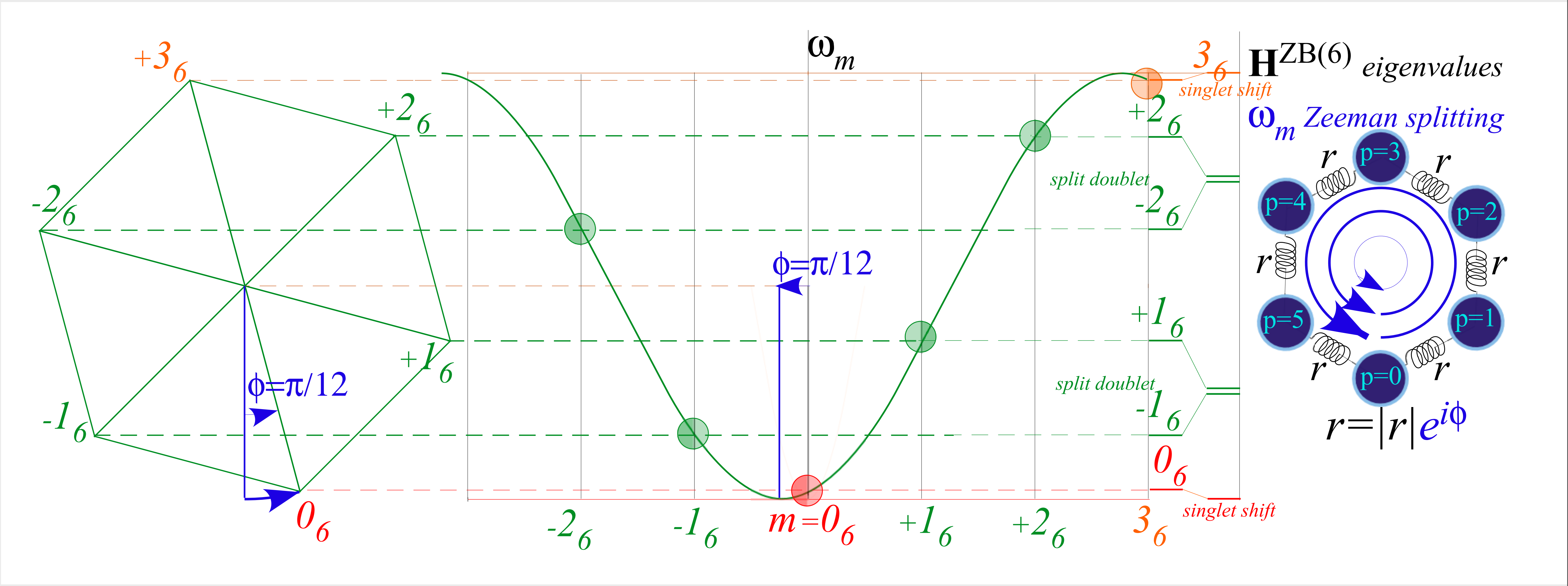
3rd Step (contd.)

...eigensolutions for all possible C_6 symmetric complex H

$$\mathbf{H} = \sum_{p=0}^{n-1} r_p \mathbf{r}^p = \sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)} = \sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)} = \sum_{p=0}^{n-1} r_p \chi_p^m = \omega(k_m)$$

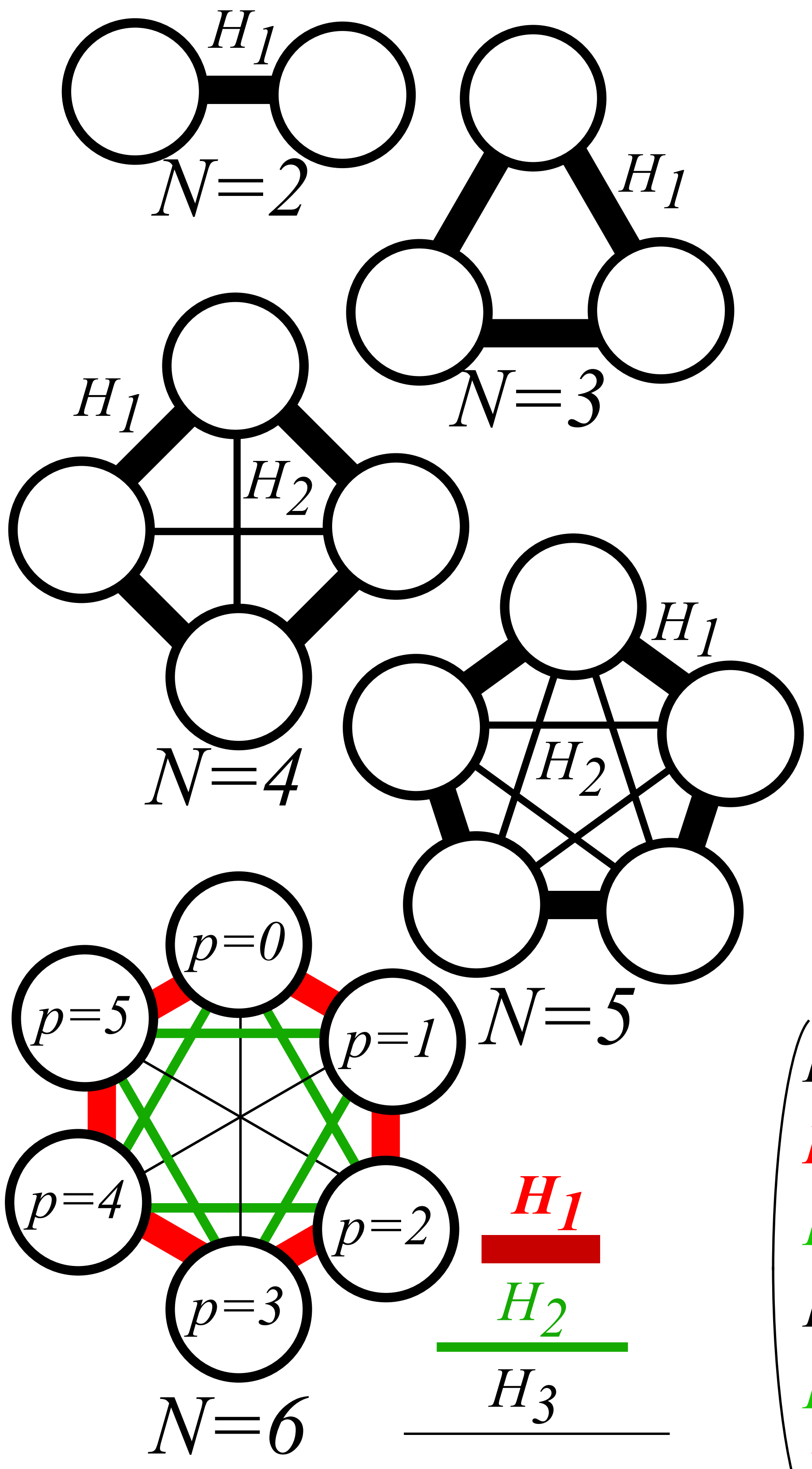
(Dispersion function)

In this C-Type situation m-eigenstates
are required to be moving waves $e^{ik_m \cdot x_p}$



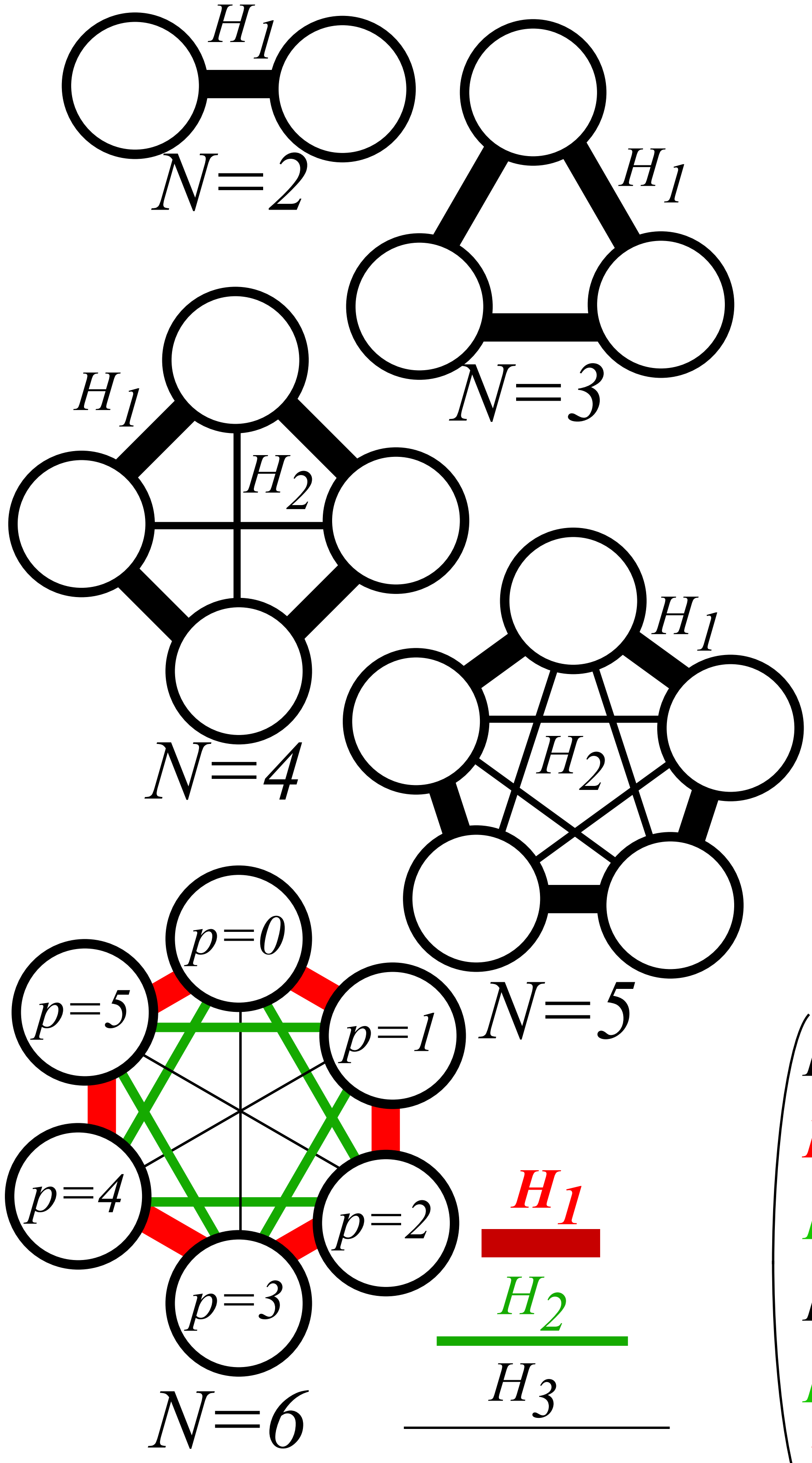
Simulating Complex Systems With Simpler Ones

Discrete Rotor Waves
Bohr-Rotors Made of Quantum Dots

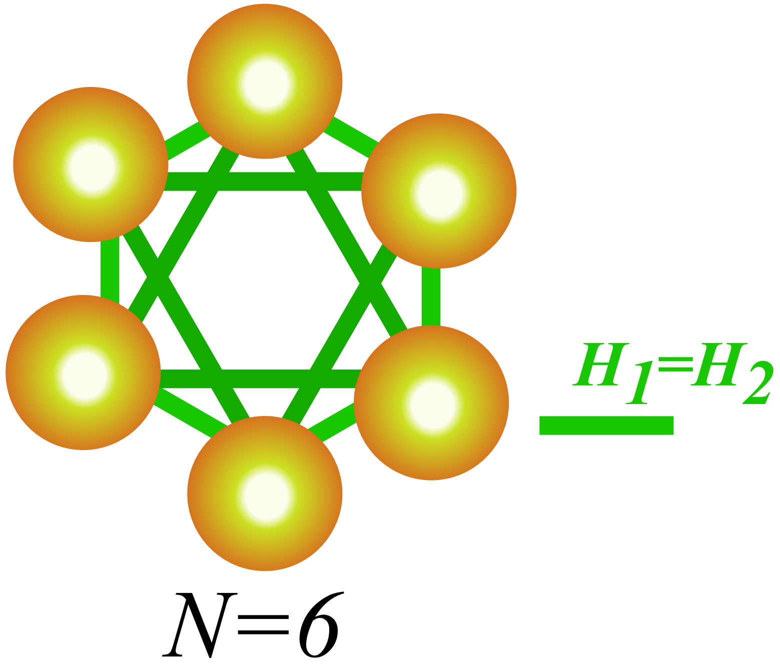


$$\begin{pmatrix}
H_0 & \boldsymbol{H_1} & H_2 & H_3 & H_2 & \boldsymbol{H_1} \\
\boldsymbol{H1} & H_0 & \boldsymbol{H_1} & H_2 & H_3 & H_2 \\
H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} & H_2 & H_3 \\
H_3 & H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} & H_2 \\
H_2 & H_3 & H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} \\
\boldsymbol{H_1} & H_2 & H_3 & H_2 & \boldsymbol{H_1} & H_0
\end{pmatrix}$$

Simulating Complex Systems With Simpler Ones



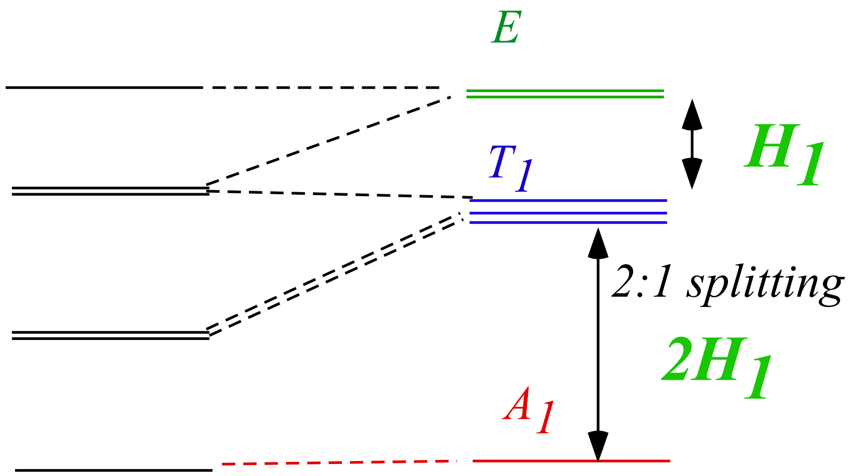
Discrete Rotor Waves
Bohr-Rotors Made of Quantum Dots



$$\begin{pmatrix}
H_0 & \mathbf{H_1} & \mathbf{H_1} & 0 & \mathbf{H_1} & \mathbf{H_1}\\
\mathbf{H_1} & H_0 & \mathbf{H_1} & \mathbf{H_1} & 0 & \mathbf{H_1}\\
\mathbf{H_1} & \mathbf{H_1} & H_0 & \mathbf{H_1} & \mathbf{H_1} & 0\\
0 & \mathbf{H_1} & \mathbf{H_1} & H_0 & \mathbf{H_1} & \mathbf{H_1}\\
\mathbf{H_1} & 0 & \mathbf{H_1} & \mathbf{H_1} & H_0 & \mathbf{H_1}\\
\mathbf{H_1} & \mathbf{H_1} & 0 & \mathbf{H_1} & \mathbf{H_1} & H_0
\end{pmatrix}$$

Hexagonal becomes Octahedral

$$\begin{pmatrix}
H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1}\\
\mathbf{H1} & H_0 & \mathbf{H_1} & H_2 & H_3 & H_2\\
H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 & H_3\\
H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2\\
H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1}\\
\mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} & H_0
\end{pmatrix}$$



Polygonal geometry of U(2)⊃C_N character spectral function
Algebra
Geometry

➡ *Introduction to wave dynamics of phase, mean phase, and group velocity*
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals"
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = \underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}} \underbrace{\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)}_{2\cos\left(\frac{a-b}{2}\right)}$$

OUTSIDE Group

Anatomy of a 2-State Wavefunction

$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos\left(\frac{a-b}{2}\right)$

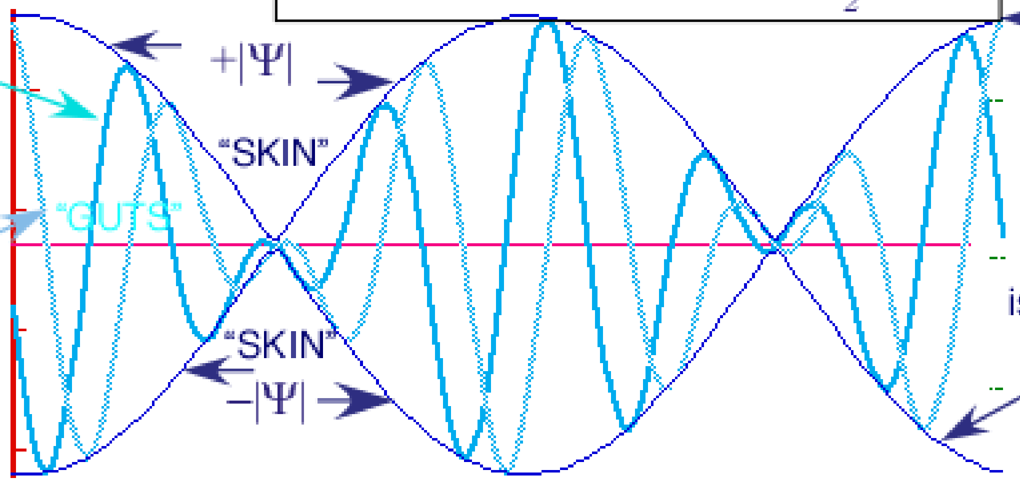
Real Part

$\mathrm{Re}\Psi = |\Psi|\cos\left(\frac{a+b}{2}\right)$

and

Imaginary Part

$\mathrm{Im}\Psi = |\Psi|\sin\left(\frac{a+b}{2}\right)$



Envelope or Modulus

Wave "SKIN" $\pm|\Psi| = \pm 2\cos\left(\frac{a-b}{2}\right)$

is PROBABILITY wave for classical "stuff" $|\Psi| = \sqrt{\Psi^*\Psi}$

Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia} + e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 = e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad$ $b=k_2x-\omega_2t$

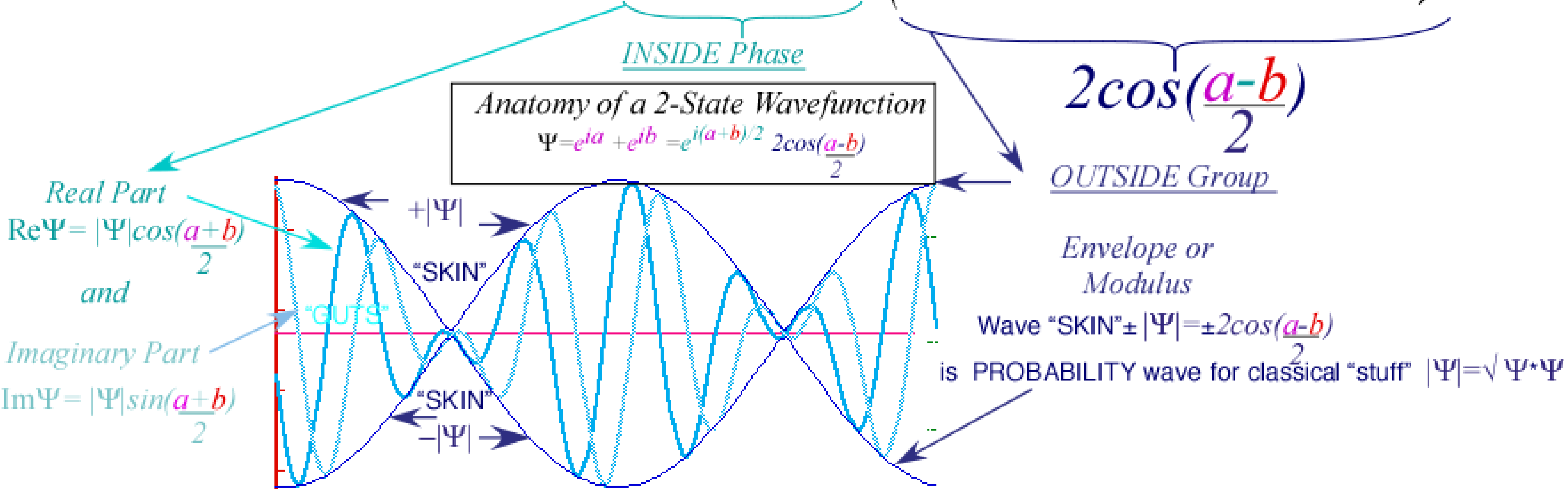
Balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi=e^{ia}+e^{ib}=\underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}}\underbrace{(e^{i(a-b)/2}+e^{-i(a-b)/2})}_{2\cos\left(\frac{a-b}{2}\right)}$$



Wave "SKIN"± $|\Psi|=\pm 2\cos\left(\frac{a-b}{2}\right)$

is PROBABILITY wave for classical "stuff" $|\Psi|=\sqrt{\Psi^*\Psi}$

Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad b=k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$ $\quad \omega_g=(\omega_1-\omega_2)/2$

$k_p=(k_1+k_2)/2$ $\quad k_g=(k_1-k_2)/2$

Overall or Mean phase

Relative or Group phase

Balanced (50-50) plane wave combination:

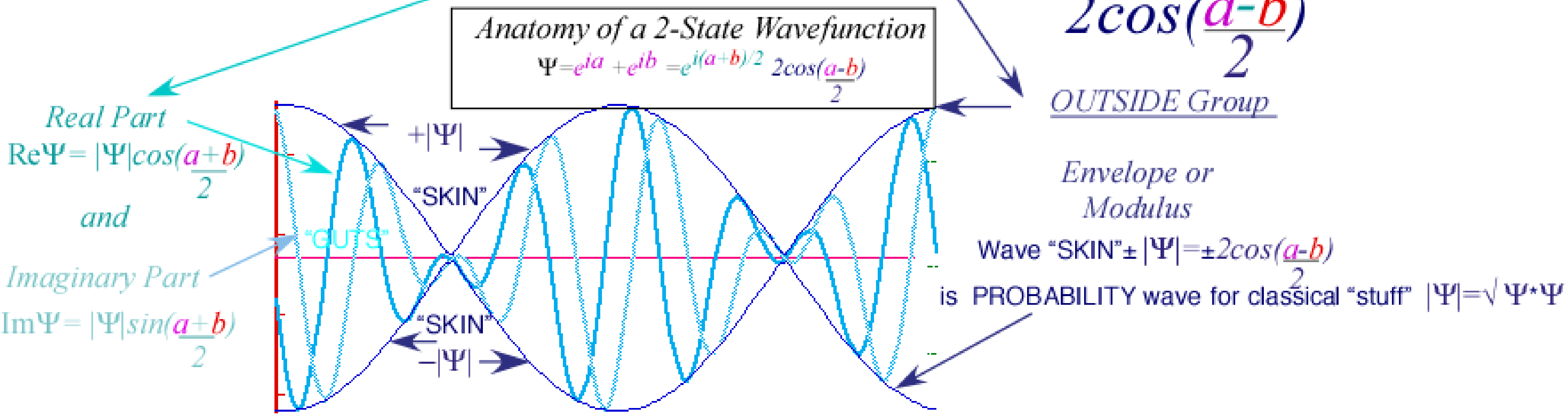
$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi=e^{ia}+e^{ib}=\underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}}\underbrace{(e^{i(a-b)/2}+e^{-i(a-b)/2})}_{2\cos(\frac{a-b}{2})}$$

OUTSIDE Group



Wave "SKIN"± $|\Psi|=\pm 2\cos(\frac{a-b}{2})$

is PROBABILITY wave for classical "stuff" $|\Psi|=\sqrt{\Psi^*\Psi}$

Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; 2e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\qquad$ $b=k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$ $\qquad$ $\omega_g=(\omega_1-\omega_2)/2$

$k_p=(k_1+k_2)/2$ $\qquad$ $k_g=(k_1-k_2)/2$

Overall or Mean phase ↓

Relative or Group phase ↓

Balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) \;+\; (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

1st plane phase velocity

2nd plane phase velocity

Phase or Carrier velocity

Group or Envelope velocity

$$V_1=\frac{\omega_1}{k_1} \qquad V_2=\frac{\omega_2}{k_2} \qquad V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2} \qquad V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$$

Define **K***-vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1, k_1)$ $\qquad$ $\mathbf{K}_2=(\omega_2, k_2)$ $\qquad$ $\mathbf{K}_{phase}=(\omega_p, k_p)=(\mathbf{K}_1+\mathbf{K}_2)/2$ $\qquad$ $\mathbf{K}_{group}=(\omega_g, k_g)=(\mathbf{K}_1-\mathbf{K}_2)/2$

Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; 2e^{i(a+b)/2} \;\bullet\; \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad b= k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$

$k_p=(k_1+k_2)/2$

Overall or Mean phase

$\omega_g=(\omega_1-\omega_2)/2$

$k_g=(k_1-k_2)/2$

Relative or Group phase

Balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \;\bullet\; \cos(k_gx-\omega_gt)$$

1st plane phase velocity

2nd plane phase velocity

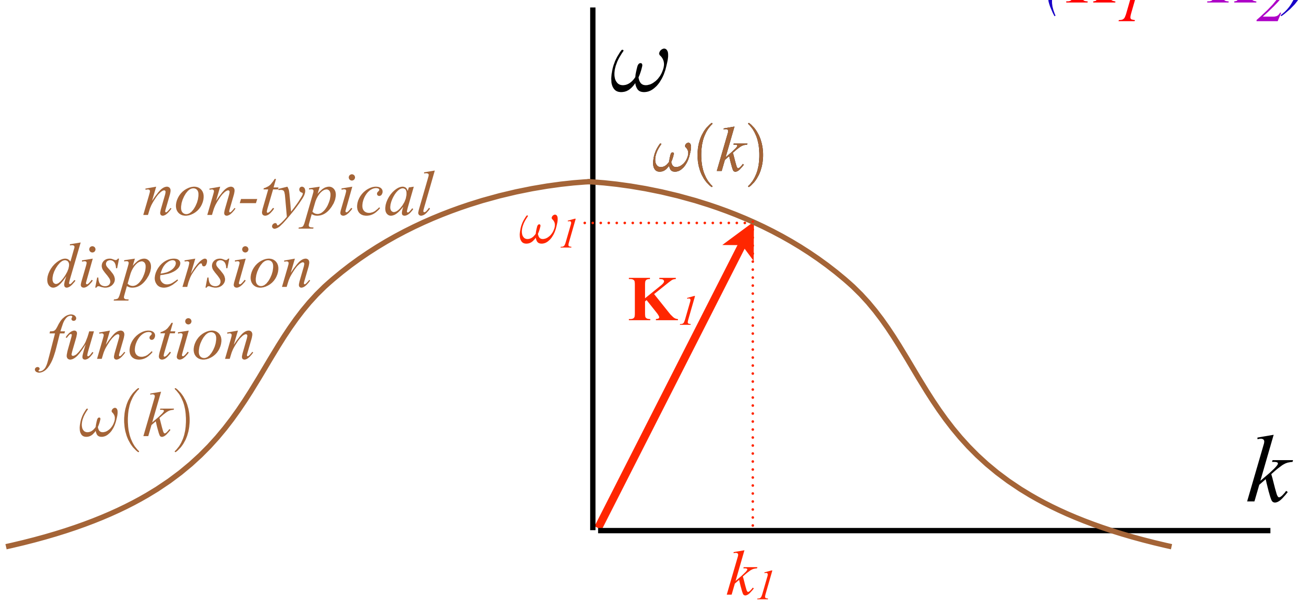
Phase or Carrier velocity

Group or Envelope velocity

$$V_1=\frac{\omega_1}{k_1} \qquad V_2=\frac{\omega_2}{k_2} \qquad V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2} \qquad V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$$

Define **K**-*vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1,k_1)$ $\quad \mathbf{K}_2=(\omega_2,k_2)$ $\quad \mathbf{K}_{phase}=(\omega_p,k_p)=(\mathbf{K}_1+\mathbf{K}_2)/2$ $\quad \mathbf{K}_{group}=(\omega_g,k_g)=(\mathbf{K}_1-\mathbf{K}_2)/2$



Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 = 2e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad$ $b=k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$	$\omega_g=(\omega_1-\omega_2)/2$
$k_p=(k_1+k_2)/2$	$k_g=(k_1-k_2)/2$
Overall or Mean phase	*Relative or Group phase*

Balanced (50-50) plane wave combination:

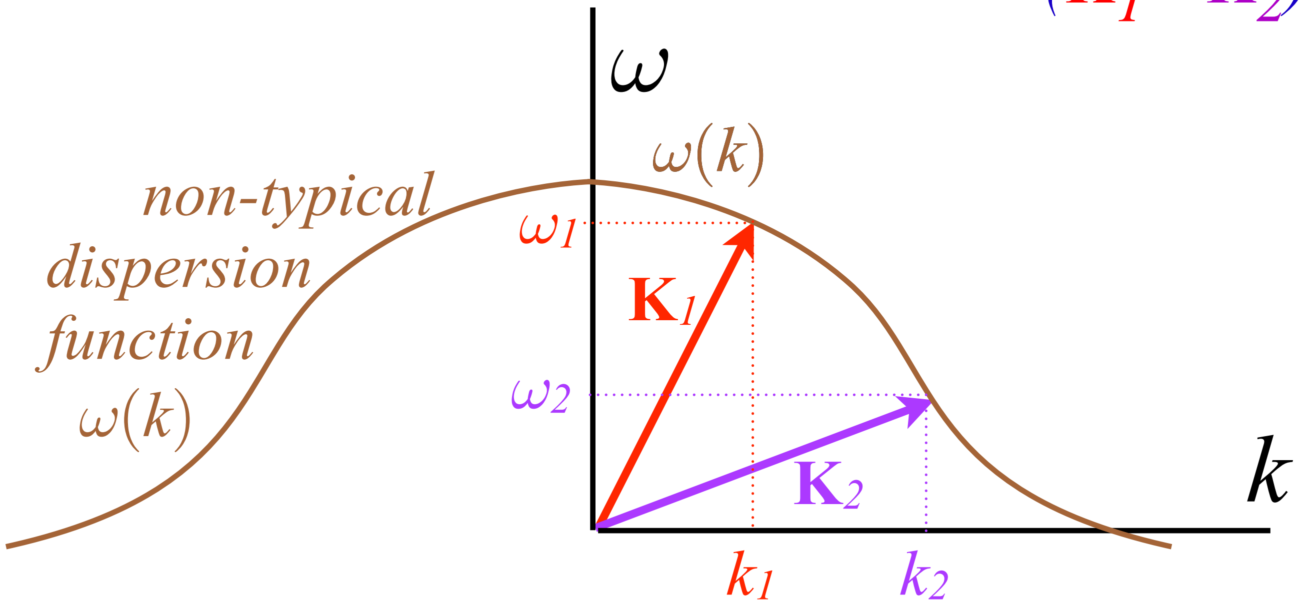
$$\Psi_{50_1-50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

1st plane phase velocity	*2nd plane phase velocity*	*Phase or Carrier velocity*	*Group or Envelope velocity*
$V_1=\frac{\omega_1}{k_1}$	$V_2=\frac{\omega_2}{k_2}$	$V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2}$	$V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$

Define **K***-vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1,k_1)$	$\mathbf{K}_2=(\omega_2,k_2)$	$\mathbf{K}_{phase}=(\omega_p,k_p)$ $=(\mathbf{K}_1+\mathbf{K}_2)/2$	$\mathbf{K}_{group}=(\omega_g,k_g)$ $=(\mathbf{K}_1-\mathbf{K}_2)/2$



Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; 2e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad b=k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$

$k_p=(k_1+k_2)/2$

Overall or
Mean phase

$\omega_g=(\omega_1-\omega_2)/2$

$k_g=(k_1-k_2)/2$

Relative or
Group phase

Balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) \;+\; (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

1st plane
phase
velocity

$$V_1=\frac{\omega_1}{k_1}$$

2nd plane
phase
velocity

$$V_2=\frac{\omega_2}{k_2}$$

Phase or
Carrier
velocity

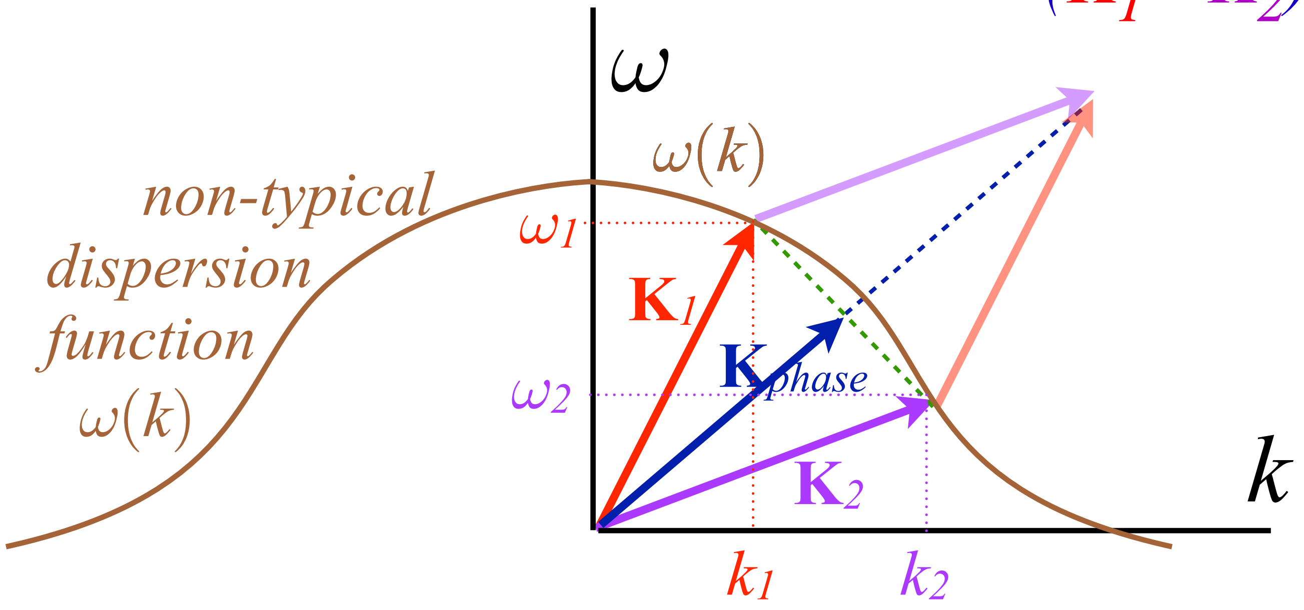
$$V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2}$$

Group or
Envelope
velocity

$$V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$$

Define **K***-vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1,k_1)$ $\quad \mathbf{K}_2=(\omega_2,k_2)$ $\quad \mathbf{K}_{phase}=(\omega_p,k_p)=(\mathbf{K}_1+\mathbf{K}_2)/2$ $\quad \mathbf{K}_{group}=(\omega_g,k_g)=(\mathbf{K}_1-\mathbf{K}_2)/2$



Fundamental wave dynamics based on Euler Expo-cosine Identity

$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 = e^{i(a+b)/2}\bullet\cos(a-b)/2$$

$a=k_1x-\omega_1t$ $\quad b=k_2x-\omega_2t$

$\omega_p=(\omega_1+\omega_2)/2$

$k_p=(k_1+k_2)/2$

Overall or
Mean phase

$\omega_g=(\omega_1-\omega_2)/2$

$k_g=(k_1-k_2)/2$

Relative or
Group phase

Balanced (50-50) plane wave combination:

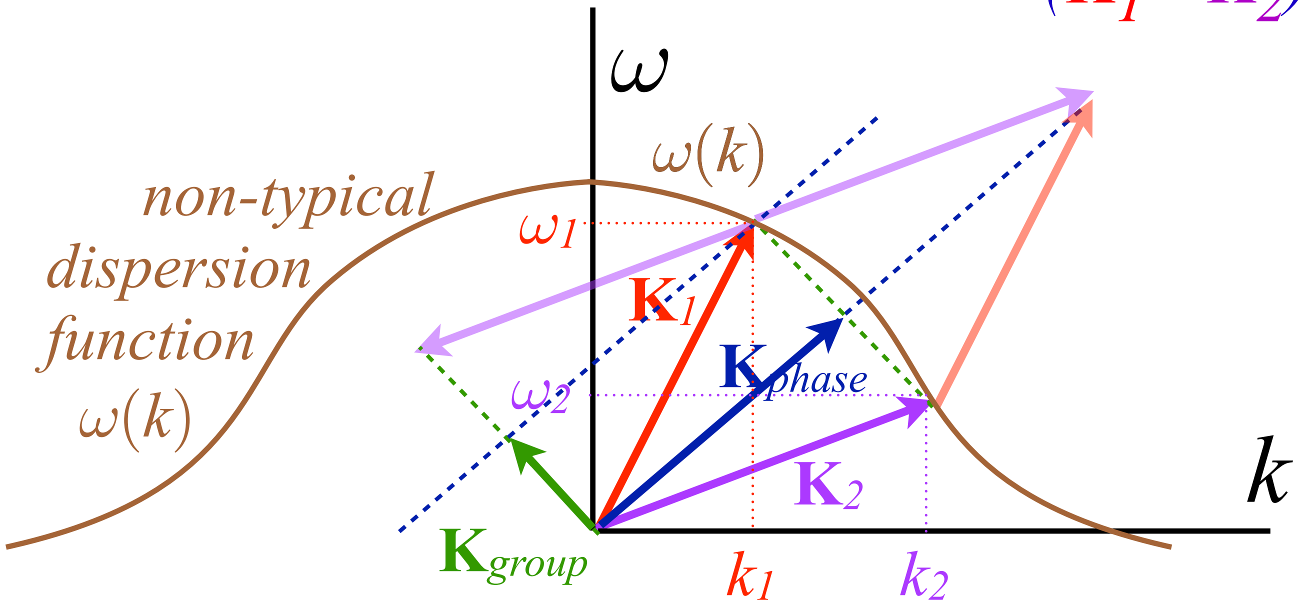
$$\Psi_{50_1\text{-}50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

1st plane phase velocity	2nd plane phase velocity	Phase or Carrier velocity	Group or Envelope velocity
$V_1=\frac{\omega_1}{k_1}$	$V_2=\frac{\omega_2}{k_2}$	$V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2}$	$V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$

Define **K**-*vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1,k_1)$ $\quad\mathbf{K}_2=(\omega_2,k_2)$ $\quad\mathbf{K}_{phase}=(\omega_p,k_p)=(\mathbf{K}_1+\mathbf{K}_2)/2$ $\quad\mathbf{K}_{group}=(\omega_g,k_g)=(\mathbf{K}_1-\mathbf{K}_2)/2$



Fundamental wave dynamics based on Euler Expo-cosine Identity

$(e^{ia} + e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 \;=\; 2e^{i(a+b)/2} \bullet \cos(a-b)/2$

$a=k_1x-\omega_1t$ $\quad b= k_2x-\omega_2t$

$\omega_p=(\omega_1+ \omega_2)/2$ $\quad \omega_g=(\omega_1- \omega_2)/2$

$k_p=(k_1+ k_2)/2$ $\quad k_g=(k_1- k_2)/2$

Overall or Mean phase — *Relative or Group phase*

Balanced (50-50) plane wave combination:

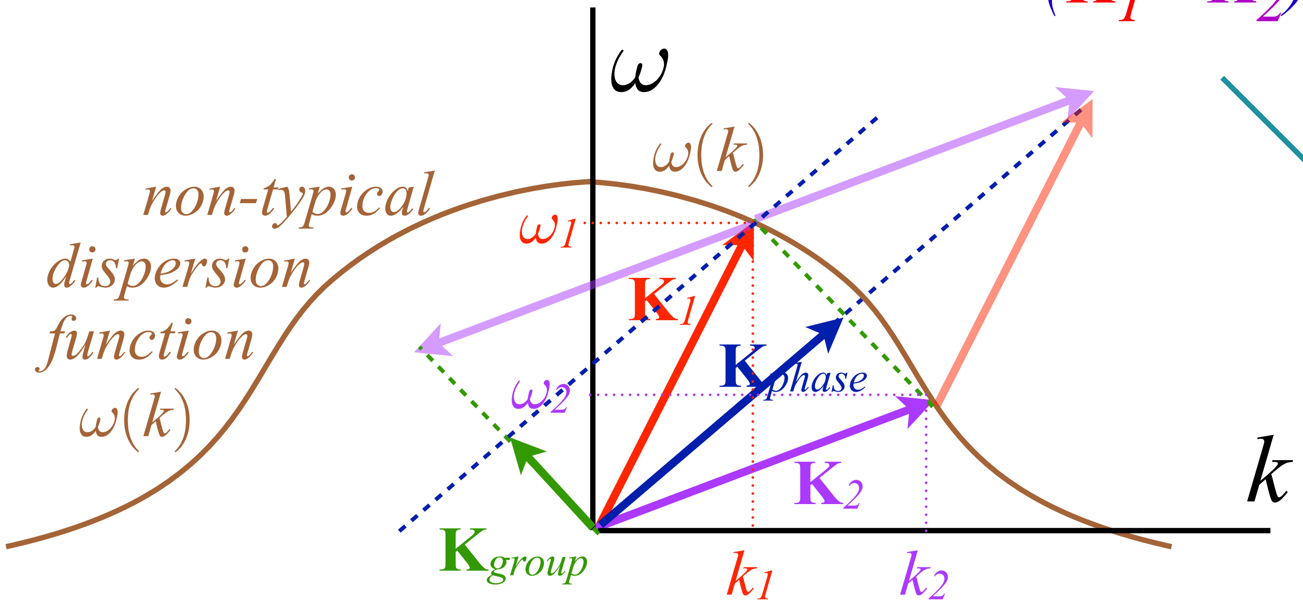
$\Psi_{50_1-50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$

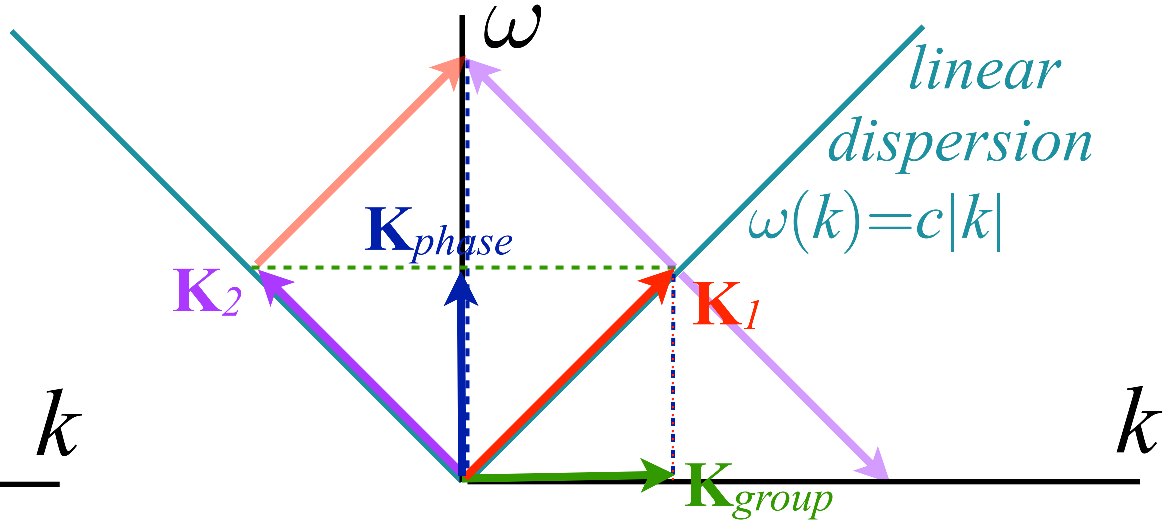
$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

1st plane phase velocity	*2nd plane phase velocity*	*Phase or Carrier velocity*	*Group or Envelope velocity*
$V_1 = \frac{\omega_1}{k_1}$	$V_2 = \frac{\omega_2}{k_2}$	$V_p = \frac{\omega_p}{k_p} = \frac{\omega_1+ \omega_2}{k_1+ k_2}$	$V_g = \frac{\omega_g}{k_g} = \frac{\omega_1- \omega_2}{k_1- k_2}$

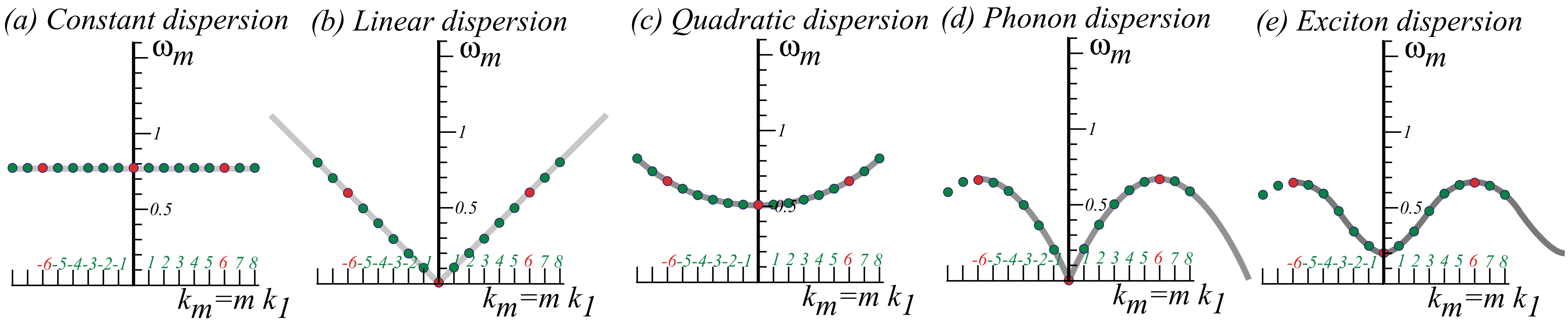
*Define **K**-vectors in per-spacetime*

$\mathbf{K}_1 =(\omega_1, k_1)$ $\quad \mathbf{K}_2 =(\omega_2, k_2)$ $\quad \mathbf{K}_{phase} =(\omega_p, k_p) =(\mathbf{K}_1 +\mathbf{K}_2)/2$ $\quad \mathbf{K}_{group} =(\omega_g, k_g) =(\mathbf{K}_1 -\mathbf{K}_2)/2$



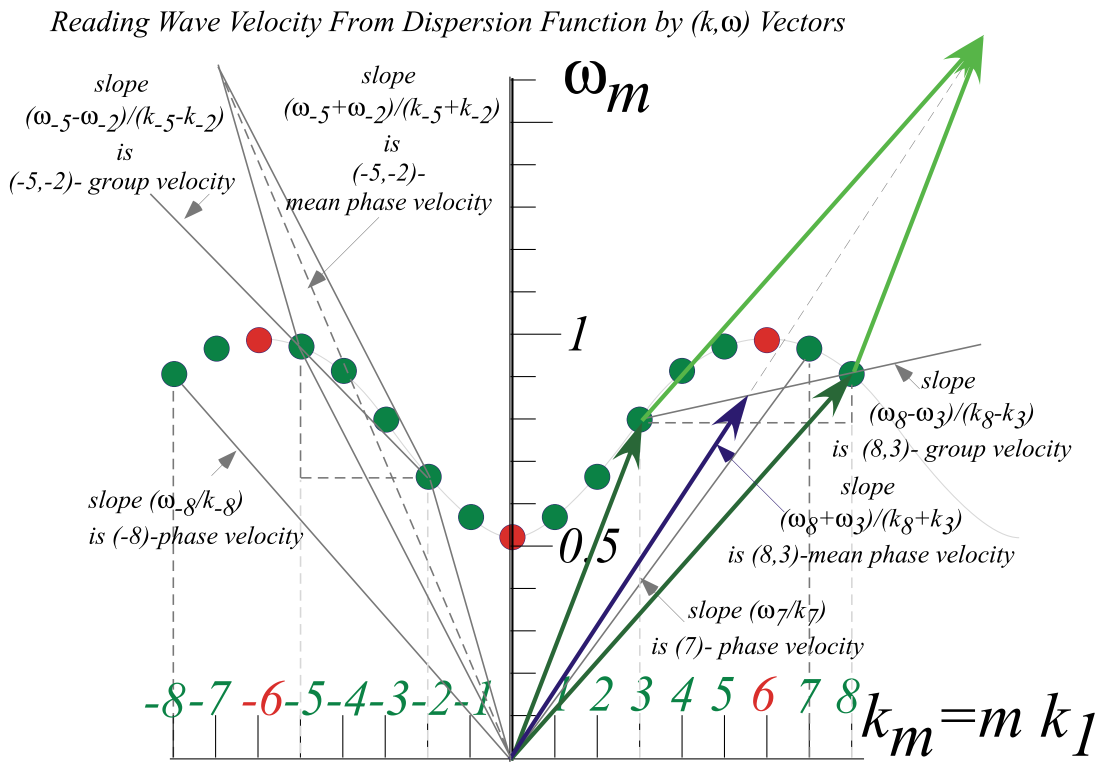


Archetypical Examples of Dispersion Functions


(a) Constant dispersion
(b) Linear dispersion
(c) Quadratic dispersion
(d) Phonon dispersion
(e) Exciton dispersion

Applications:

Uncoupled pendulums	Weakly coupled pendulums (No gravity)	Weakly coupled pendulums (With gravity)	Strongly coupled pendulums (No gravity)	Strongly coupled pendulums (With gravity)
Movie marquis Xmas lights	Light in vacuum (Exactly) Sound (Approximately)	Light in fiber (Approx) Non-relativistic Schrodinger matter wave	Acoustic mode in solids	Optical mode in solids Relativistic matter (If exact hyperbola)


Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

slope $(\omega_{-5}-\omega_{-2})/(k_{-5}-k_{-2})$ is (-5,-2)- group velocity

slope $(\omega_{-5}+\omega_{-2})/(k_{-5}+k_{-2})$ is (-5,-2)- mean phase velocity

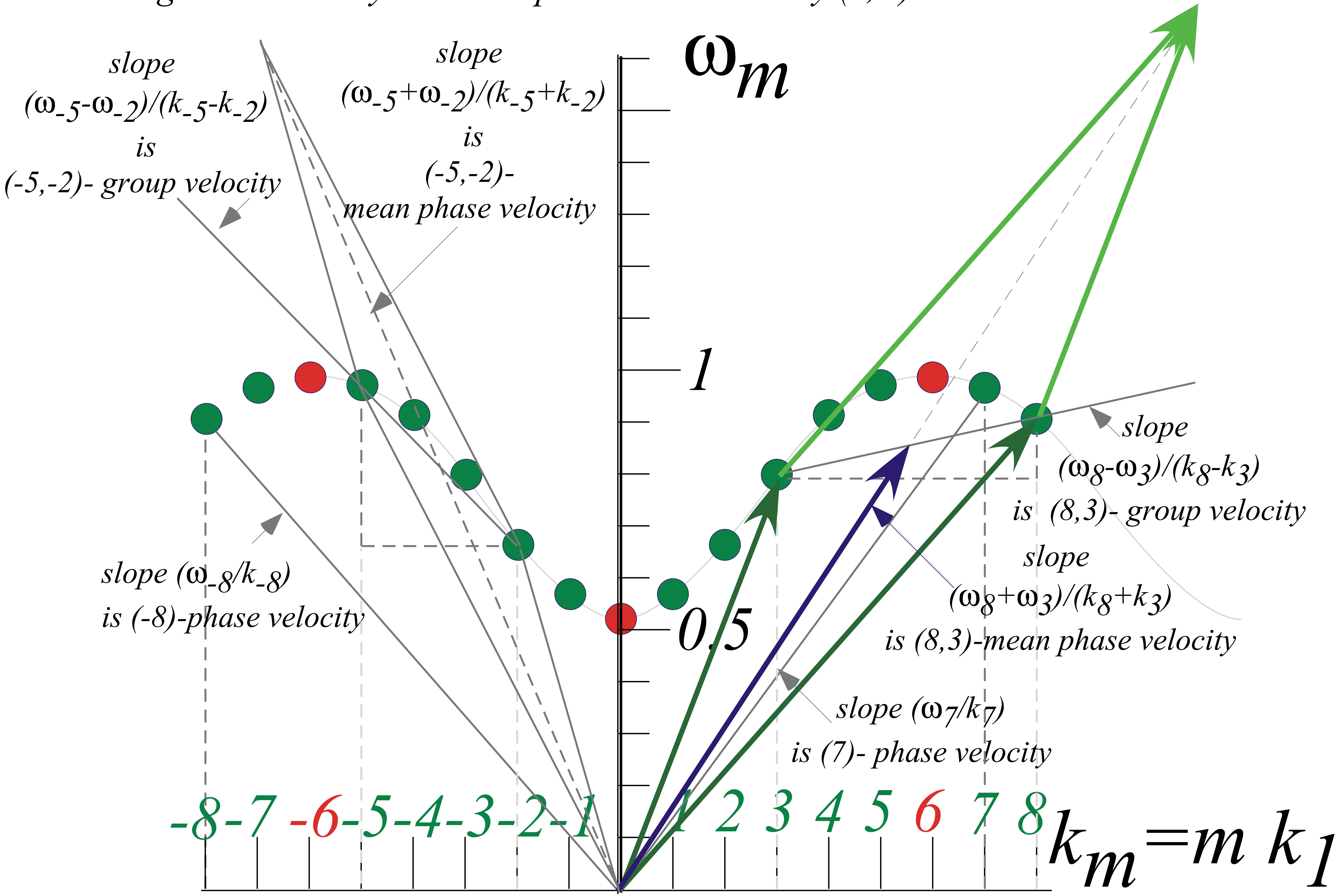
slope (ω_{-8}/k_{-8}) is (-8)-phase velocity

slope $(\omega_8-\omega_3)/(k_8-k_3)$ is (8,3)- group velocity

slope $(\omega_8+\omega_3)/(k_8+k_3)$ is (8,3)-mean phase velocity

slope (ω_7/k_7) is (7)- phase velocity

Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

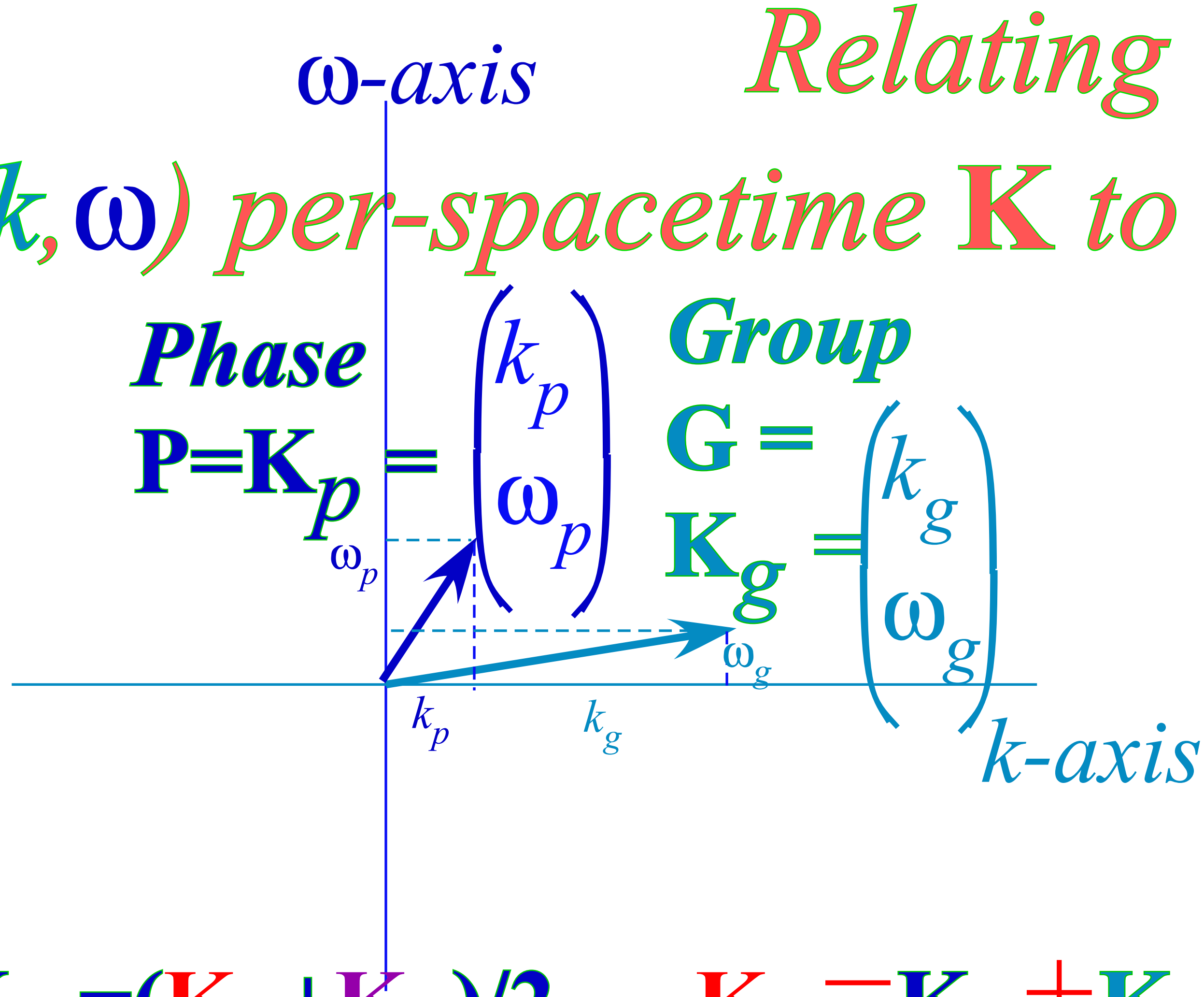


Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
→ *Relating space-time and per-space-time*
  - *Wave coordinates*
  - *Pulse-waves (PW) vs Continuous -waves (CW)*
  - *Wave coordinates for Linear Dispersion*
  - *Wave coordinates for Bohr-Schrodinger Dispersion*
  - *Einstein-Lorentz-Minkowski laser coordinates*

Introduction to C_N beat dynamics and "Revivals"
  - *Farey-Sums and Ford-products*
  - *Phase dynamics*

Relating
(k,ω) per-spacetime $\mathbf{K}$ to (x,t) spacetime



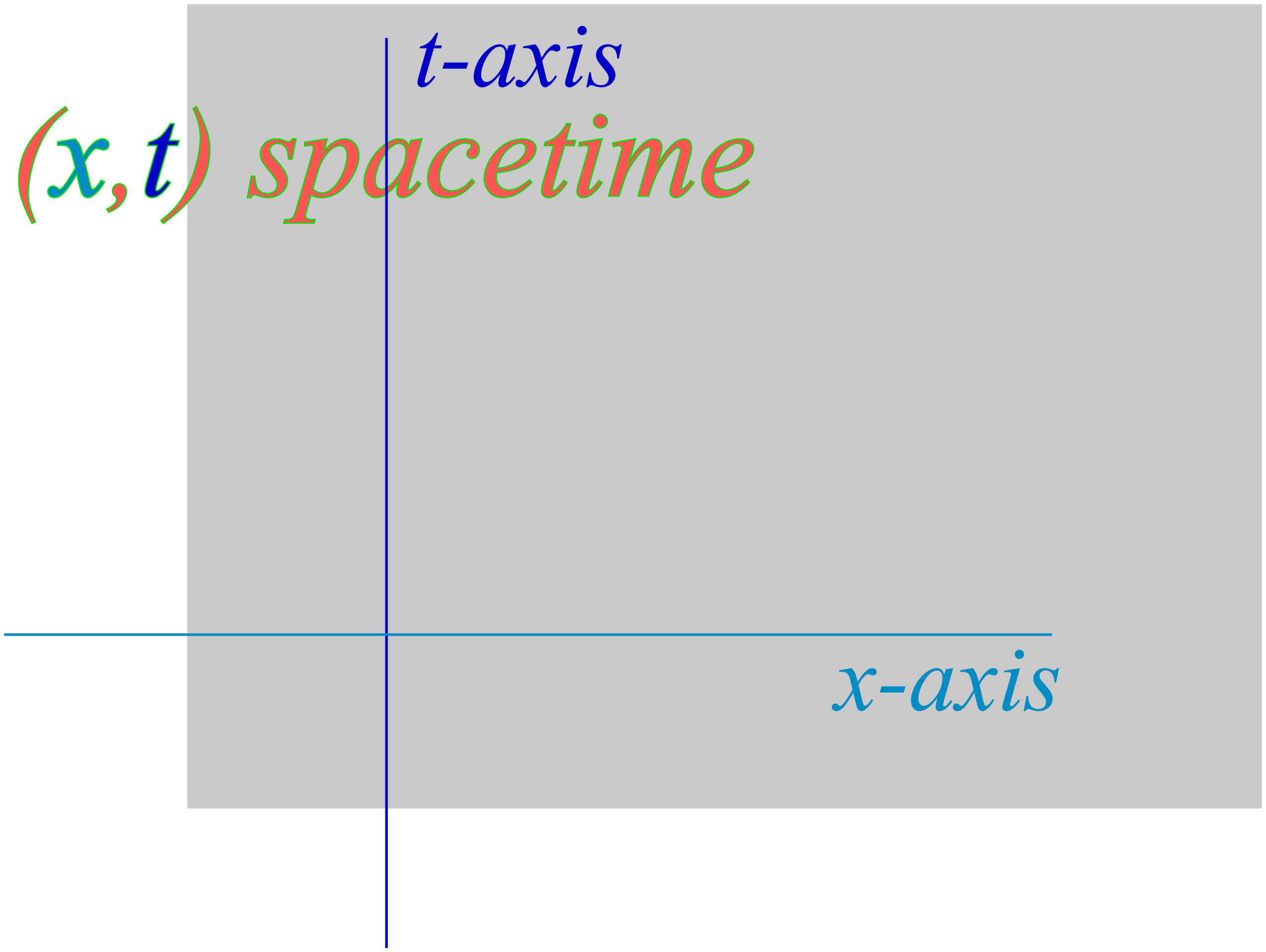
Phase

$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$

Group

$\mathbf{G}=$

$\mathbf{K}_g=\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$



$\mathbf{K}_p=(\mathbf{K}_1+\mathbf{K}_2)/2$ $\qquad$ $\mathbf{K}_1=\mathbf{K}_p+\mathbf{K}_g$

$\mathbf{K}_g=(\mathbf{K}_1-\mathbf{K}_2)/2$ $\qquad$ $\mathbf{K}_2=\mathbf{K}_p-\mathbf{K}_g$

$\omega_p=(\omega_1+\omega_2)/2$ $\qquad$ $\omega_g=(\omega_1-\omega_2)/2$

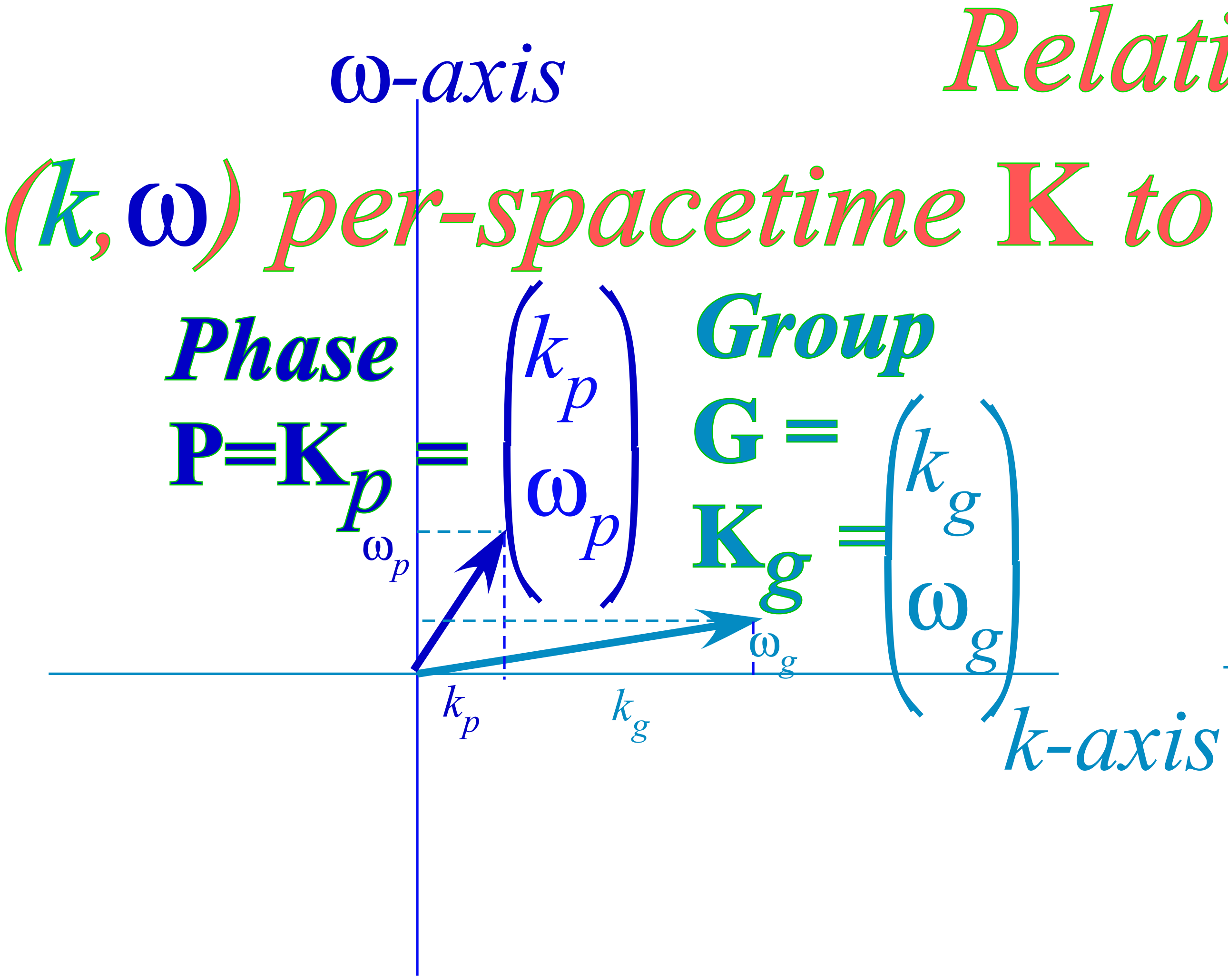
$k_p=(k_1+k_2)/2$ $\qquad$ $k_g=(k_1-k_2)/2$

Overall or Mean phase $\qquad$ *Relative or Group phase*

Find tracks in space-time of a balanced (50-50) plane wave combination:

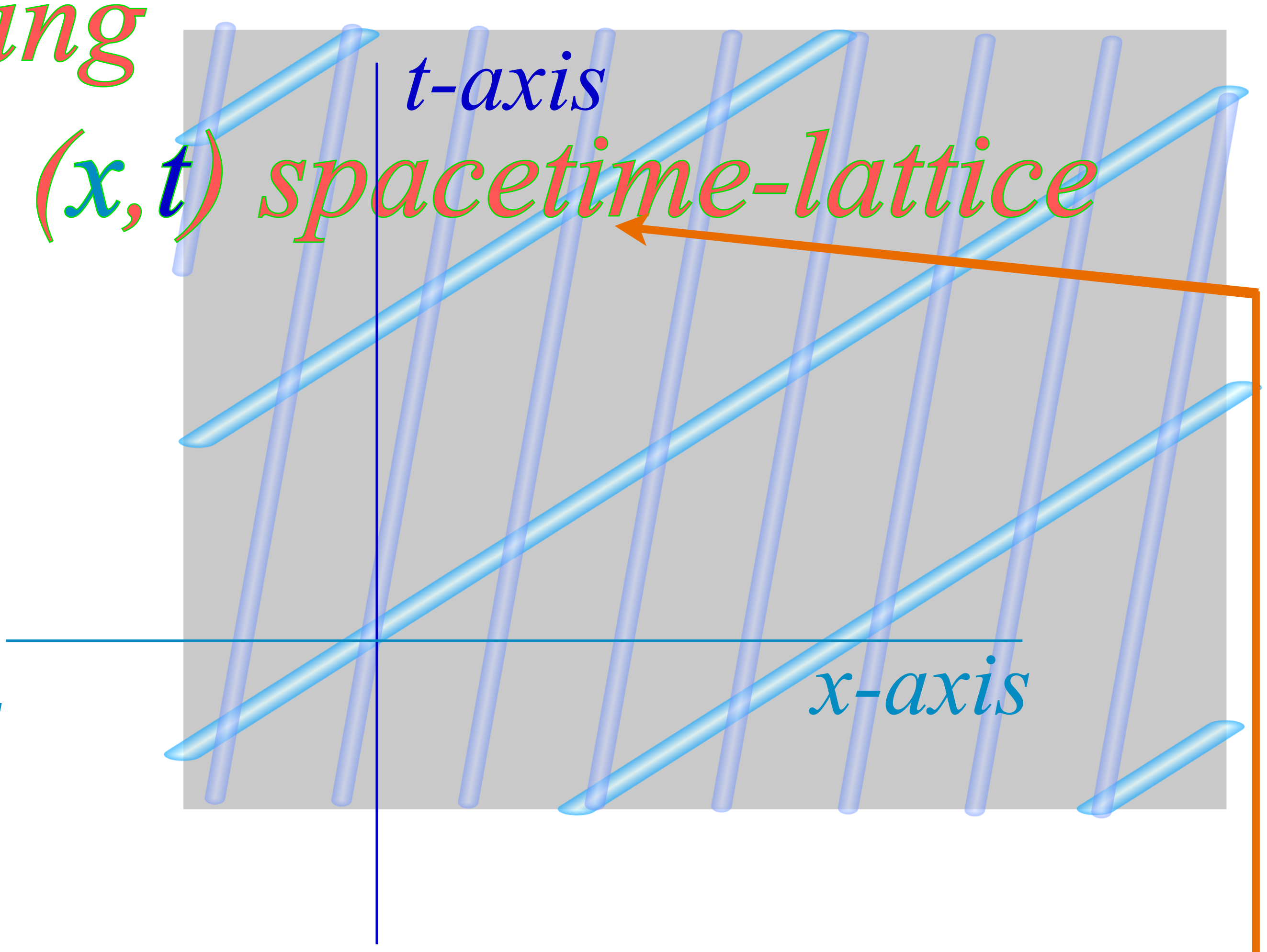
$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

Relating (k,ω) per-spacetime **K** to (x,t) spacetime-lattice



Phase $\mathbf{P}=\mathbf{K}_p=\begin{pmatrix}k_p\\ \omega_p\end{pmatrix}$

Group $\mathbf{G}=\mathbf{K_g}=\begin{pmatrix}k_g\\ \omega_g\end{pmatrix}$



$\omega_p=(\omega_1+\omega_2)/2$ $\omega_g=(\omega_1-\omega_2)/2$

$k_p=(k_1+k_2)/2$ $k_g=(k_1-k_2)/2$

Overall or Mean phase   *Relative or Group phase*

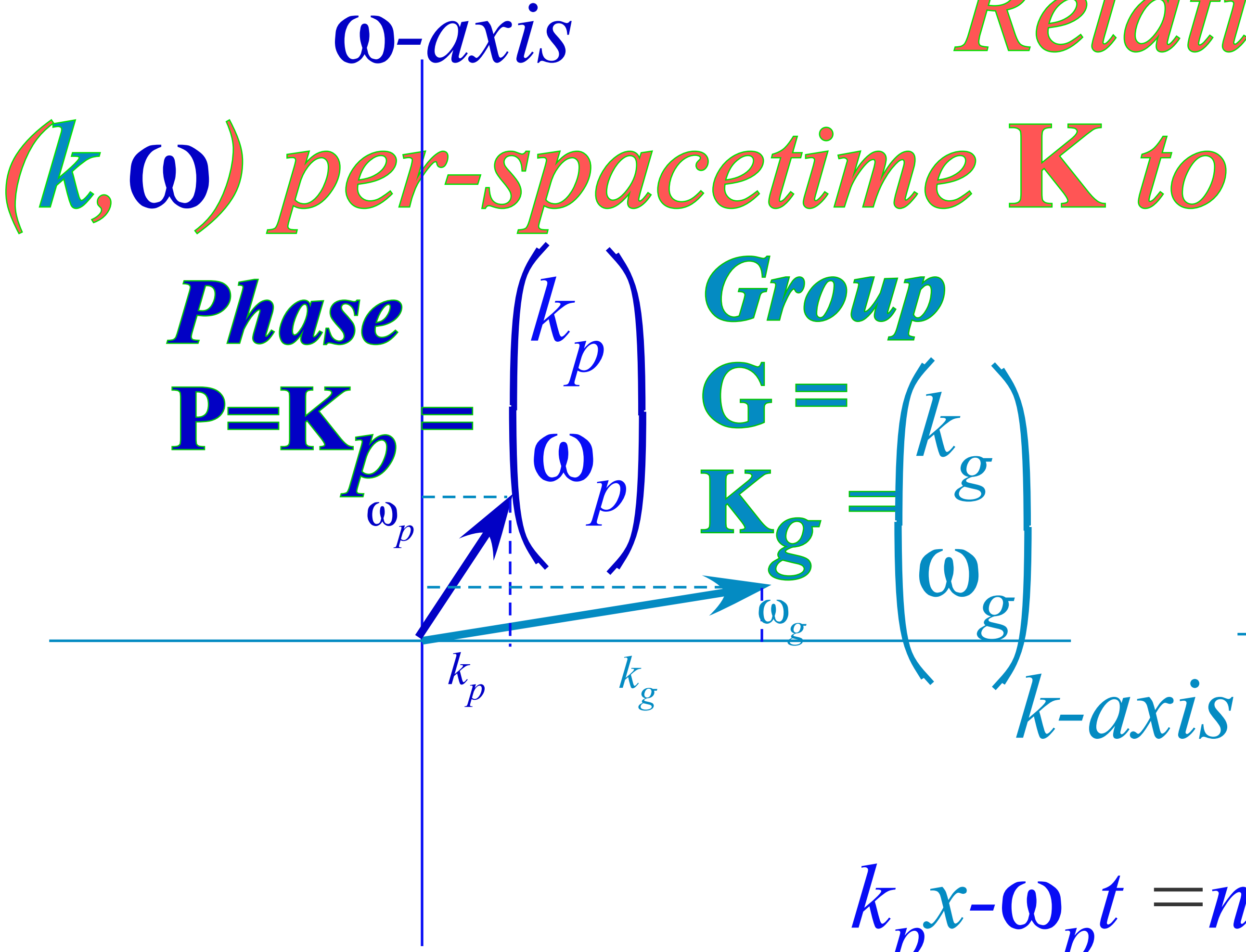
Find tracks in space-time of a balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

$$Re[\Psi_{50_1\text{-}50_2}(x,t)]= \quad =\cos(k_px-\omega_pt)\bullet\cos(k_gx-\omega_gt)=0$$

Real part has ZEROS that make: (x,t) spacetime-lattice

Relating
(k,ω) per-spacetime **K** *to (x,t) spacetime-lattice*



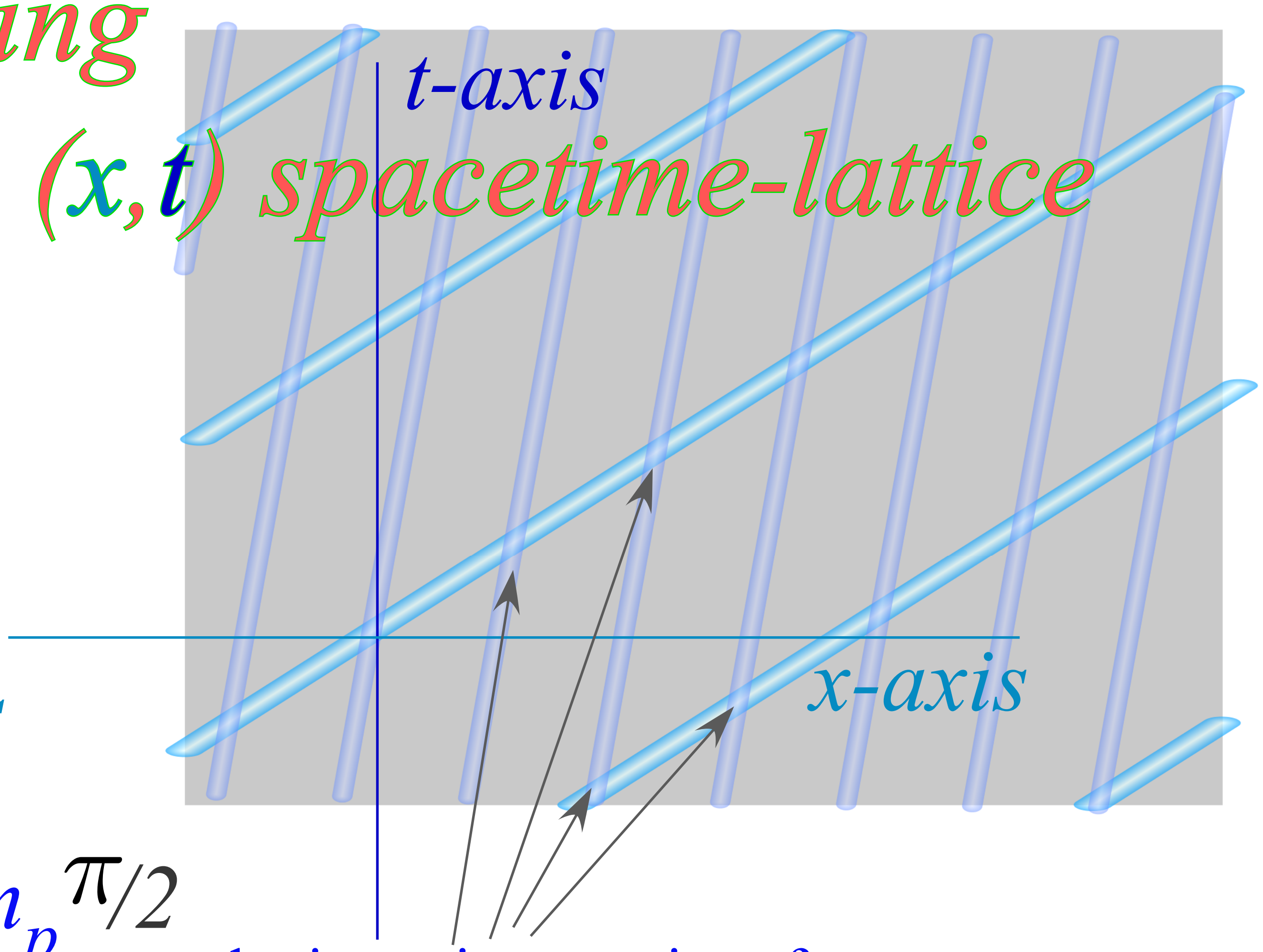
Phase

P=K$_p$ = $\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$

Group

G =

K$_g$ = $\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$



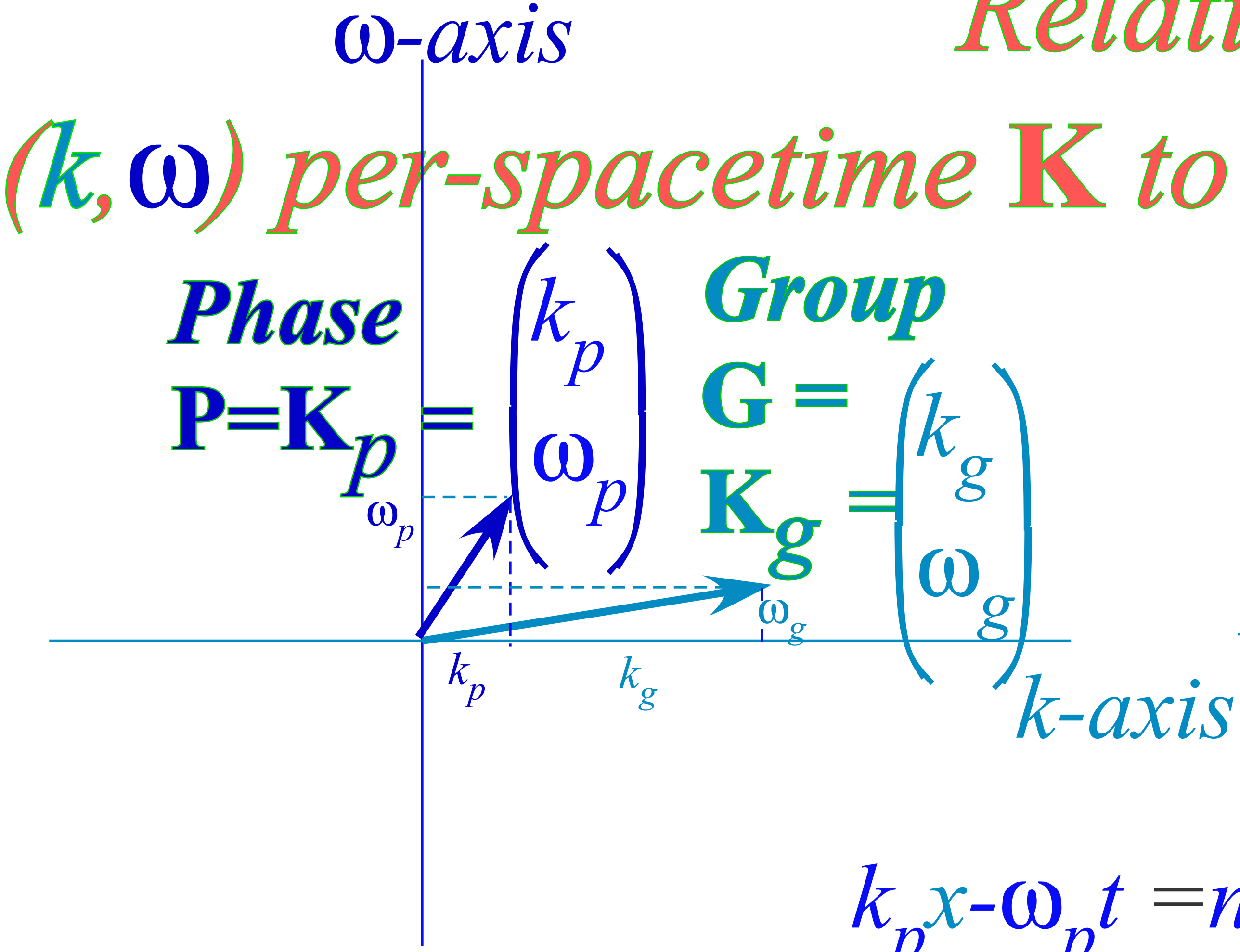
$$k_p x - \omega_p t = n_p \pi/2$$
$$k_g x - \omega_g t = n_g \pi/2$$

lattice point equations for:
$n_p = \pm 1, \pm 2...$ and $n_g = \pm 1, \pm 2...$

Re$[\Psi_{50_1-50_2}(x,t)]=$ $= \cos(k_p x - \omega_p t) \bullet \cos(k_g x - \omega_g t) = 0$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating
(k,ω) per-spacetime **K** to (x,t) spacetime-lattice

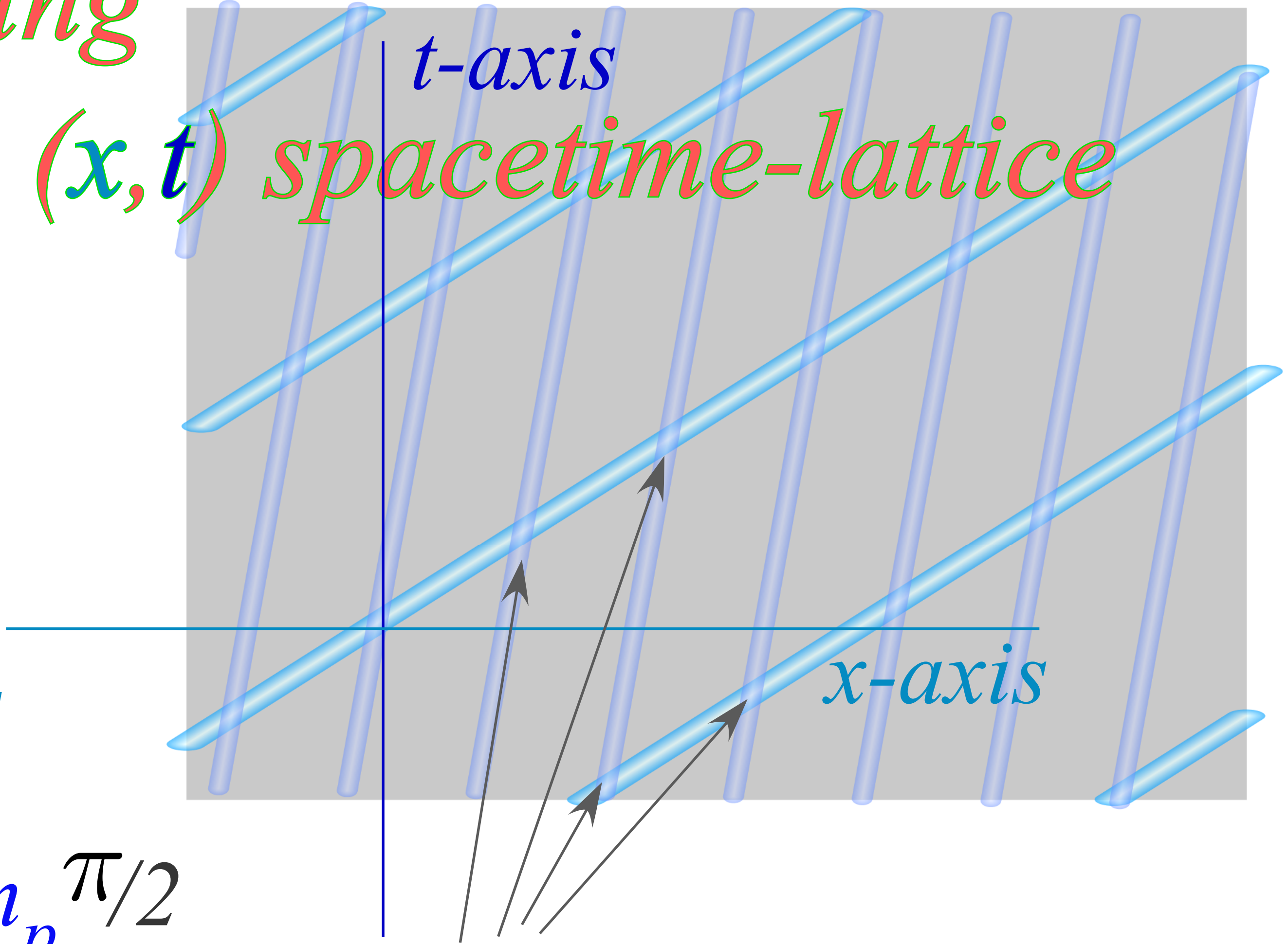


Phase

$$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix}k_p\\ \omega_p\end{pmatrix}$$

Group

$$\mathbf{G}=\mathbf{K}_g=\begin{pmatrix}k_g\\ \omega_g\end{pmatrix}$$



$$k_p x-\omega_p t = n_p {}^{\pi}/2$$

$$k_g x-\omega_g t = n_g {}^{\pi}/2$$

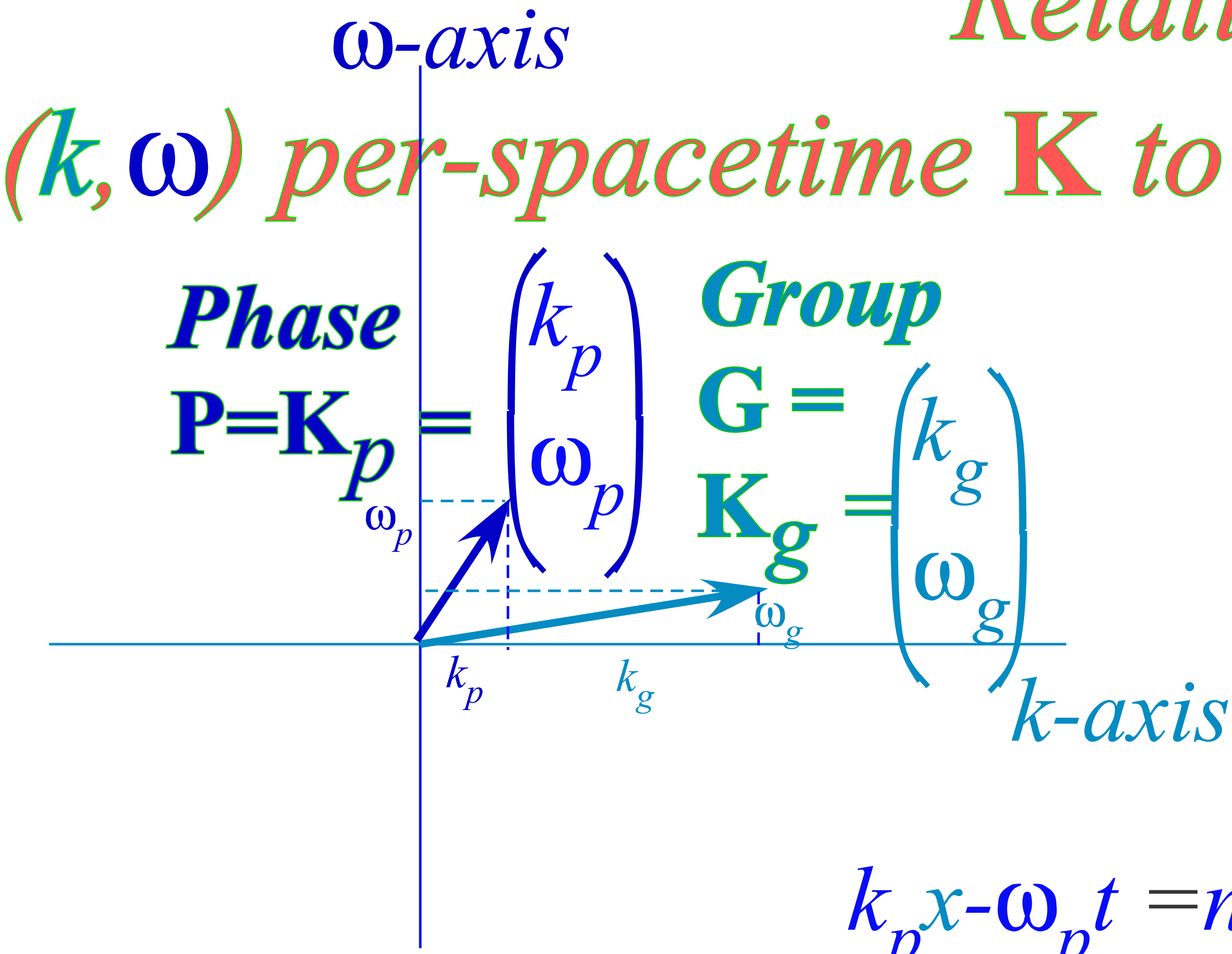
lattice point equations for:
$n_p=\pm1,\pm2...$ and $n_g=\pm1,\pm2...$

$$\begin{pmatrix}k_p & -\omega_p\\ k_g & -\omega_g\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}=\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2$$

$Re[\Psi_{50_1\text{-}50_2}(x,t)]=$ $= \cos(k_p x-\omega_p t) \bullet \cos(k_g x-\omega_g t)=0$

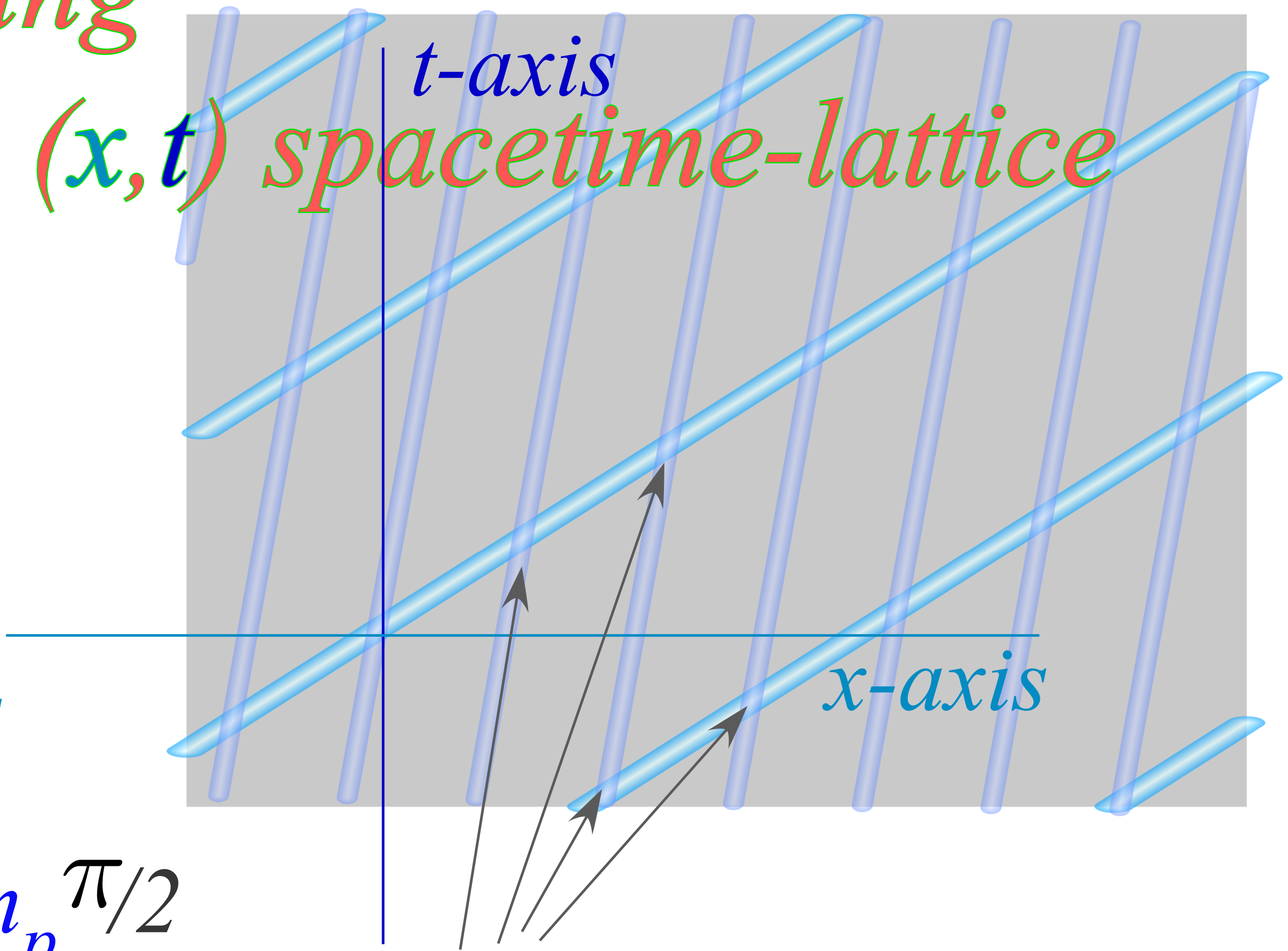
Real part has ZEROS that make: (x,t) CW spacetime-lattice

Relating
(k,ω) per-spacetime **K** to (x,t) spacetime-lattice



Phase $\mathbf{P}=\mathbf{K}_p=\begin{pmatrix}k_p\\ \omega_p\end{pmatrix}$

Group $\mathbf{G}=\mathbf{K}_g=\begin{pmatrix}k_g\\ \omega_g\end{pmatrix}$



$$k_p x-\omega_p t=n_p\pi/2$$

$$k_g x-\omega_g t=n_g\pi/2$$

lattice point equations for:
$n_p=\pm 1,\pm 2...$ and $n_g=\pm 1,\pm 2...$

$$\begin{pmatrix}k_p & -\omega_p\\ k_g & -\omega_g\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}=\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2$$

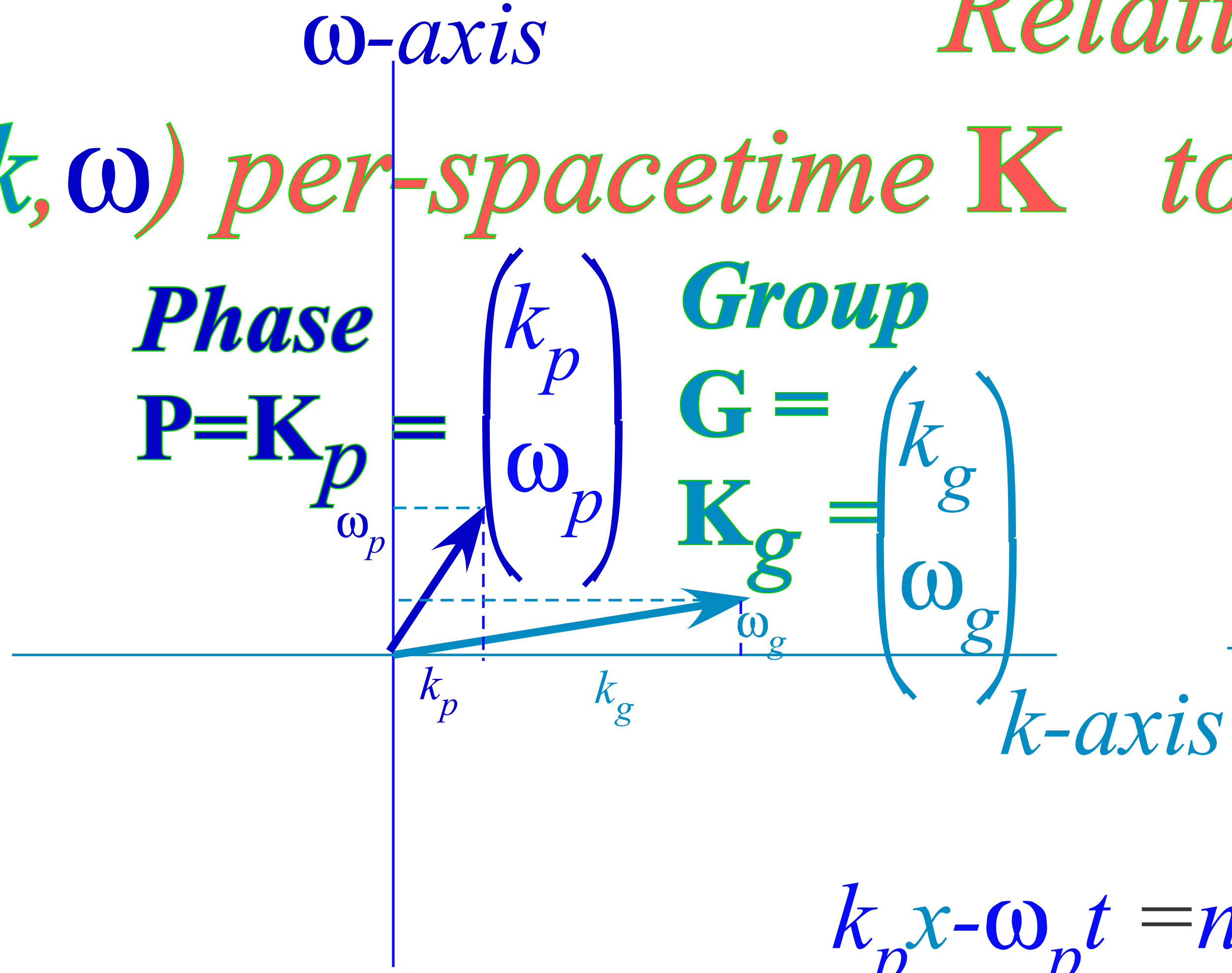
inverted

$$\begin{pmatrix}x\\ t\end{pmatrix}=\frac{\begin{pmatrix}-\omega_g & \omega_p\\ -k_g & k_p\end{pmatrix}}{\det\left|\mathbf{K}_g\times\mathbf{K}_p\right|}\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2=-n_p\frac{\begin{pmatrix}\omega_g\\ k_g\end{pmatrix}+n_g\begin{pmatrix}\omega_p\\ k_p\end{pmatrix}}{\det\left|\mathbf{K}_g\times\mathbf{K}_p\right|2/\pi}$$

$Re[\Psi_{50_1-50_2}(x,t)]=$ $=\cos(k_p x-\omega_p t)\bullet\cos(k_g x-\omega_g t)=0$

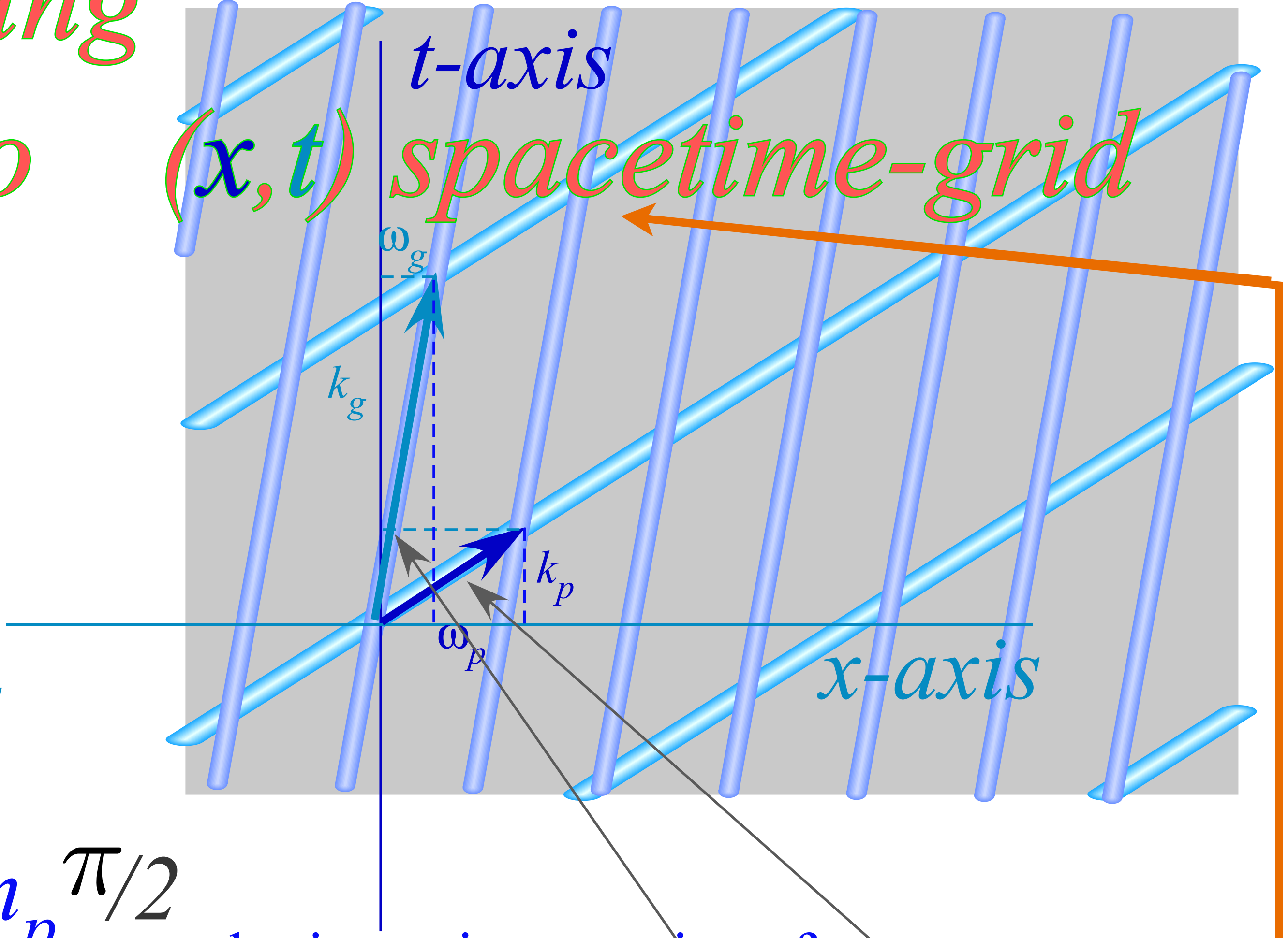
Real part has ZEROS that make: (x,t) CW spacetime-lattice

Relating (k,ω) per-spacetime K to (x,t) spacetime-grid



Phase $\mathbf{P}=\mathbf{K}_p=\begin{pmatrix}k_p\\ \omega_p\end{pmatrix}$

Group $\mathbf{G}=\mathbf{K}_g=\begin{pmatrix}k_g\\ \omega_g\end{pmatrix}$



$$k_p x-\omega_p t = n_p \pi/2$$

$$k_g x-\omega_g t = n_g \pi/2$$

lattice point equations for: $n_p=\pm 1,\pm 2...$ and $n_g=\pm 1,\pm 2...$

$$\begin{pmatrix}k_p & -\omega_p\\ k_g & -\omega_g\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}=\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2 \xrightarrow[\text{inverted}]{} \begin{pmatrix}x\\ t\end{pmatrix}=\frac{\begin{pmatrix}-\omega_g & \omega_p\\ -k_g & k_p\end{pmatrix}}{\det\left|\mathbf{K}_g \mathrm{x} \mathbf{K}_p\right|}\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2=-n_p\frac{\begin{pmatrix}\omega_g\\ k_g\end{pmatrix}+n_g\begin{pmatrix}\omega_p\\ k_p\end{pmatrix}}{\det\left|\mathbf{K}_g \mathrm{x} \mathbf{K}_p\right| 2/\pi}$$

$Re[\Psi_{50_1-50_2}(x,t)]=$ $\quad = \cos(k_p x-\omega_p t) \bullet \cos(k_g x-\omega_g t)=0$

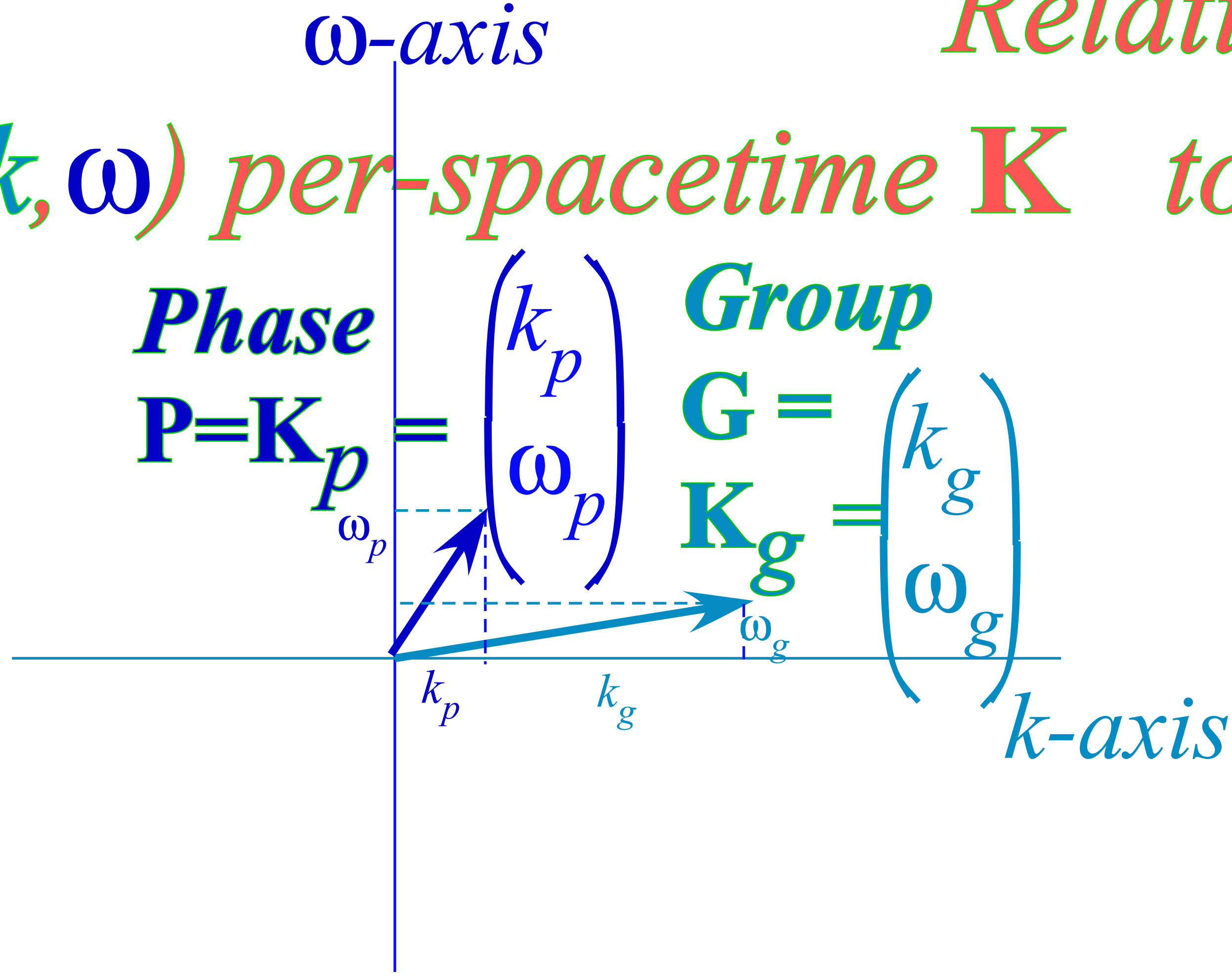
Real part has ZEROS that make: (x,t) CW spacetime-lattice

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- → *Pulse-waves (PW) vs Continuous -waves (CW)*
- *Wave coordinates for Linear Dispersion*
- *Wave coordinates for Bohr-Schrodinger Dispersion*
- *Einstein-Lorentz-Minkowski laser coordinates*

Introduction to C_N beat dynamics and "Revivals"
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Relating (k,ω) per-spacetime $\mathbf{K}$ to (x,t) spacetime-grid



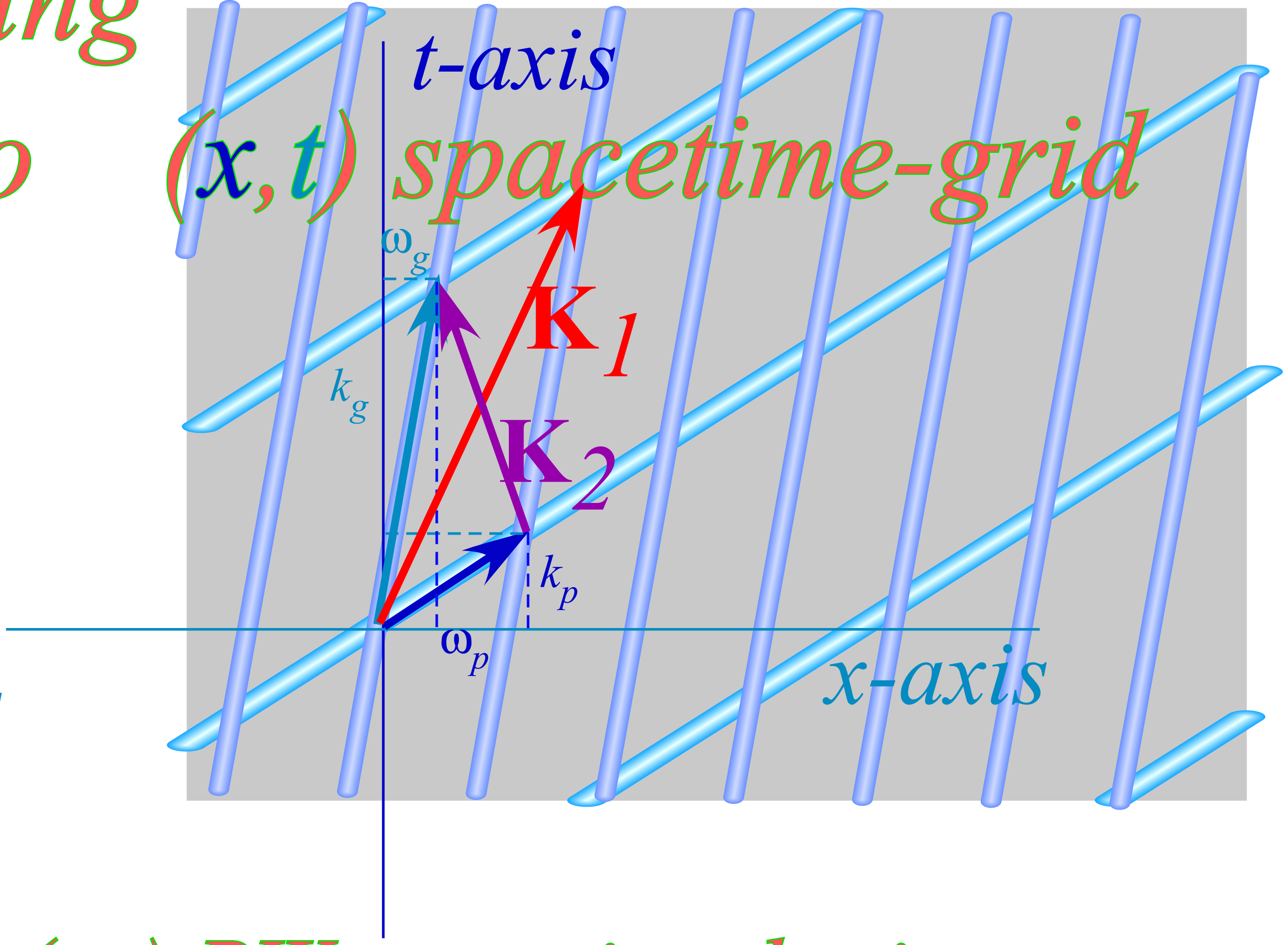
Phase

$\mathbf{P}=\mathbf{K}_p = \begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$

Group

$\mathbf{G} =$

$\mathbf{K}_g = \begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$



while primitive $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *make:* *(x,t) PW spacetime-lattice*

$\mathbf{K}_p = (\mathbf{K}_1+\mathbf{K}_2)/2 \qquad \mathbf{K}_1 = \mathbf{K}_p + \mathbf{K}_g$

$\mathbf{K}_g = (\mathbf{K}_1-\mathbf{K}_2)/2 \qquad \mathbf{K}_2 = \mathbf{K}_p - \mathbf{K}_g$

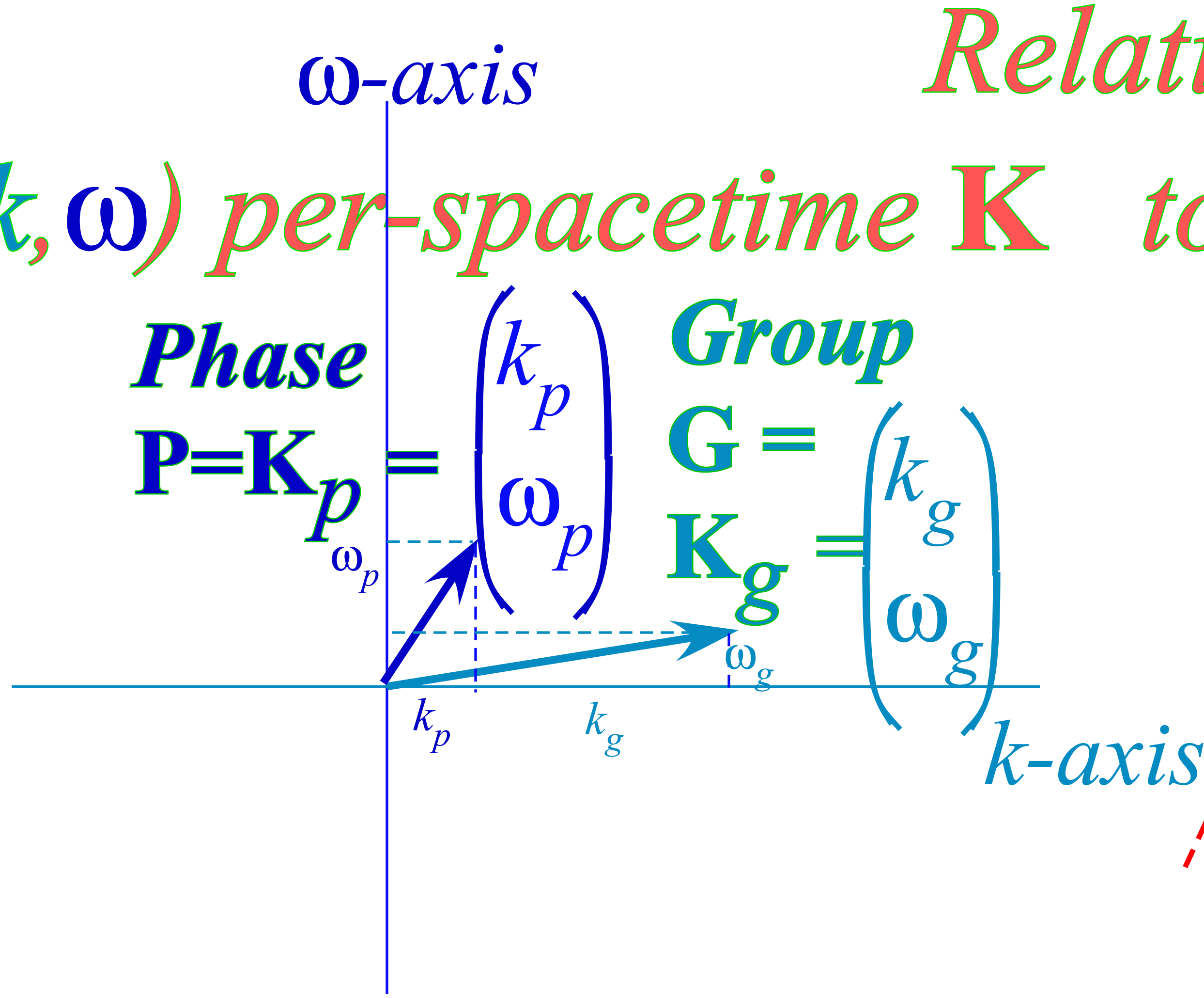
Find tracks in space-time of a
balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t) = 1/2e^{i(k_1x-\omega_1t)} + 1/2e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$

$$Re[\Psi_{50_1\text{-}50_2}(x,t)] = \qquad = \cos(k_px-\omega_pt) \bullet \cos(k_gx-\omega_gt)=0$$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating (k,ω) per-spacetime $\mathbf{K}$ to (x,t) spacetime-grid

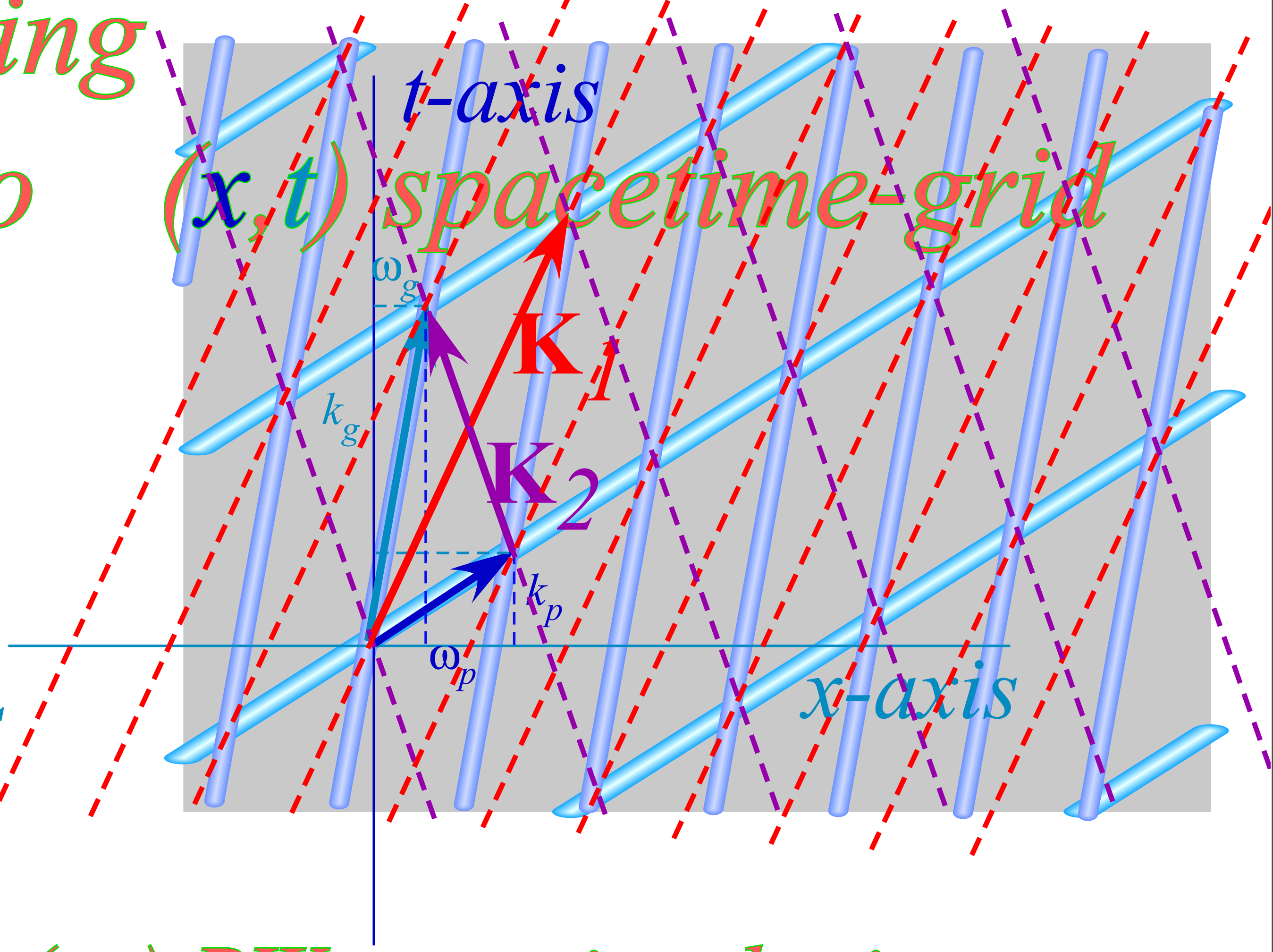


Phase

$$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$$

Group

$$\mathbf{G}=\mathbf{K}_g=\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$$



while primitive $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *make:* *(x,t) PW spacetime-lattice*

$$\mathbf{K}_p=(\mathbf{K}_1+\mathbf{K}_2)/2 \qquad \mathbf{K}_1=\mathbf{K}_p+\mathbf{K}_g$$

$$\mathbf{K}_g=(\mathbf{K}_1-\mathbf{K}_2)/2 \qquad \mathbf{K}_2=\mathbf{K}_p-\mathbf{K}_g$$

Find tracks in space-time of a
balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

$$Re[\Psi_{50_1\text{-}50_2}(x,t)]= \quad =\cos(k_px-\omega_pt)\bullet\cos(k_gx-\omega_gt)=0$$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- Wave coordinates
- Pulse-waves (PW) vs Continuous -waves (CW)
- → Wave coordinates for Linear Dispersion
- Wave coordinates for Bohr-Schrodinger Dispersion
- Einstein-Lorentz-Minkowski laser coordinates

Introduction to C_N beat dynamics and "Revivals"
- Farey-Sums and Ford-products
- Phase dynamics

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = \underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}} \underbrace{\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)}_{2\cos\left(\frac{a-b}{2}\right)}$$

OUTSIDE Group

Anatomy of a 2-State Wavefunction

$$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos\left(\frac{a-b}{2}\right)$$

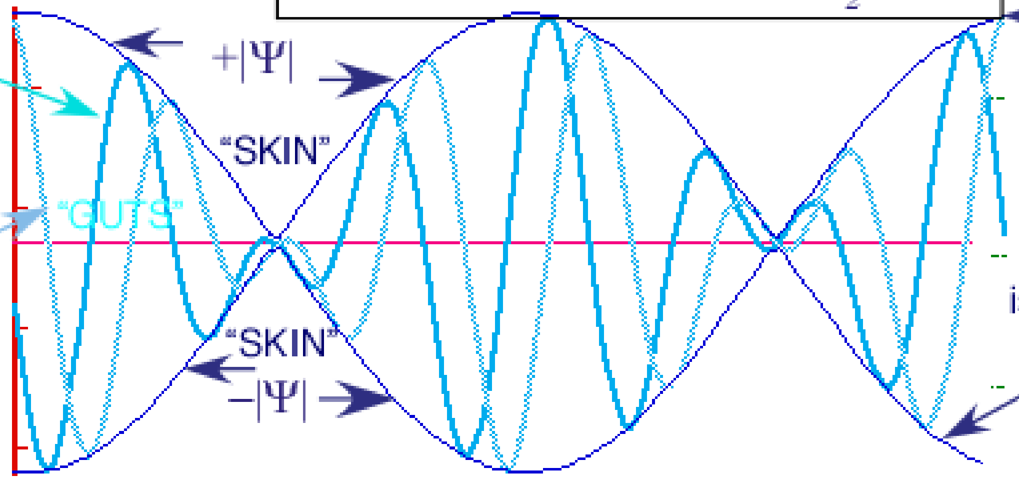
Real Part

$\mathrm{Re}\Psi = |\Psi|\cos\left(\frac{a+b}{2}\right)$

and

Imaginary Part

$\mathrm{Im}\Psi = |\Psi|\sin\left(\frac{a+b}{2}\right)$



Linear Dispersion

Envelope or Modulus

Wave "SKIN" $\pm|\Psi| = \pm 2\cos\left(\frac{a-b}{2}\right)$

is PROBABILITY wave for classical "stuff" $|\Psi| = \sqrt{\Psi^*\Psi}$

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi=e^{ia}+e^{ib}=\underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}}\underbrace{\left(e^{i(a-b)/2}+e^{-i(a-b)/2}\right)}_{2\cos\left(\frac{a-b}{2}\right)}$$

OUTSIDE Group

Anatomy of a 2-State Wavefunction

$\Psi=e^{ia}+e^{ib}=e^{i(a+b)/2}\,2\cos(\frac{a-b}{2})$

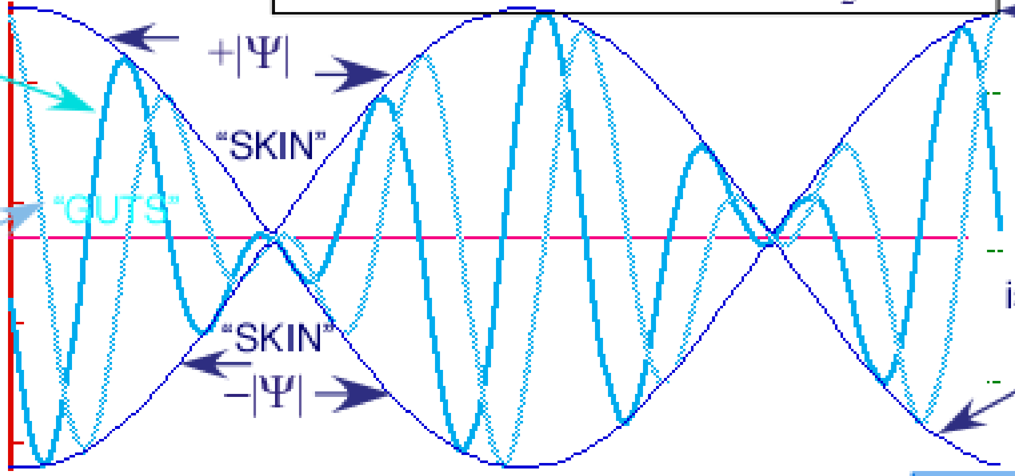
Real Part

$\mathrm{Re}\Psi=|\Psi|\cos(\frac{a+b}{2})$

and

Imaginary Part

$\mathrm{Im}\Psi=|\Psi|\sin(\frac{a+b}{2})$



Envelope or Modulus

Wave "SKIN" $\pm|\Psi|=\pm 2\cos(\frac{a-b}{2})$

is PROBABILITY wave for classical "stuff" $|\Psi|=\sqrt{\Psi^*\Psi}$

Input phases

$a=k_a x-\omega_a t$

1st base vector

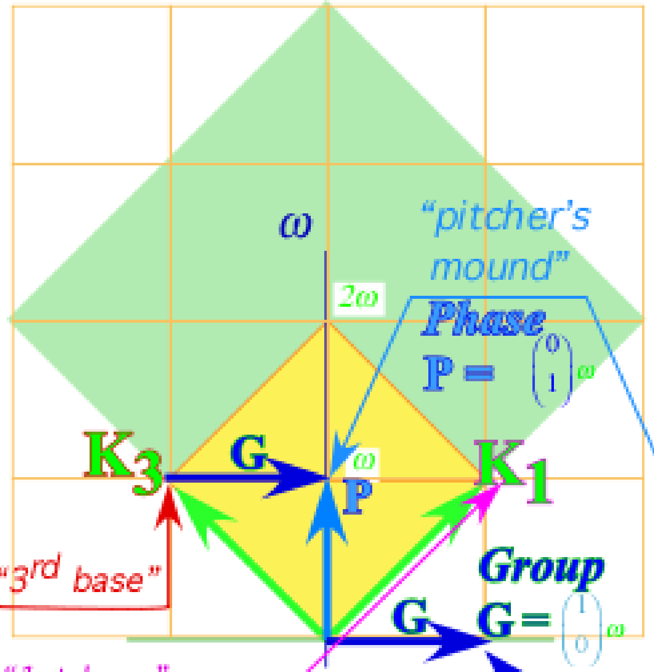
$$\mathbf{K}_1=\begin{pmatrix}ck_a\\ \omega_a\end{pmatrix}=\omega_a\begin{pmatrix}1\\1\end{pmatrix}$$

$b=k_b x-\omega_b t$

3rd base vector

$$\mathbf{K}_3=\begin{pmatrix}ck_b\\ \omega_b\end{pmatrix}=\omega_b\begin{pmatrix}-1\\1\end{pmatrix}$$

Linear Dispersion



(Here: $\omega_a=\omega=\omega_b$)

$\frac{1}{2}$-Sum

Phase vector

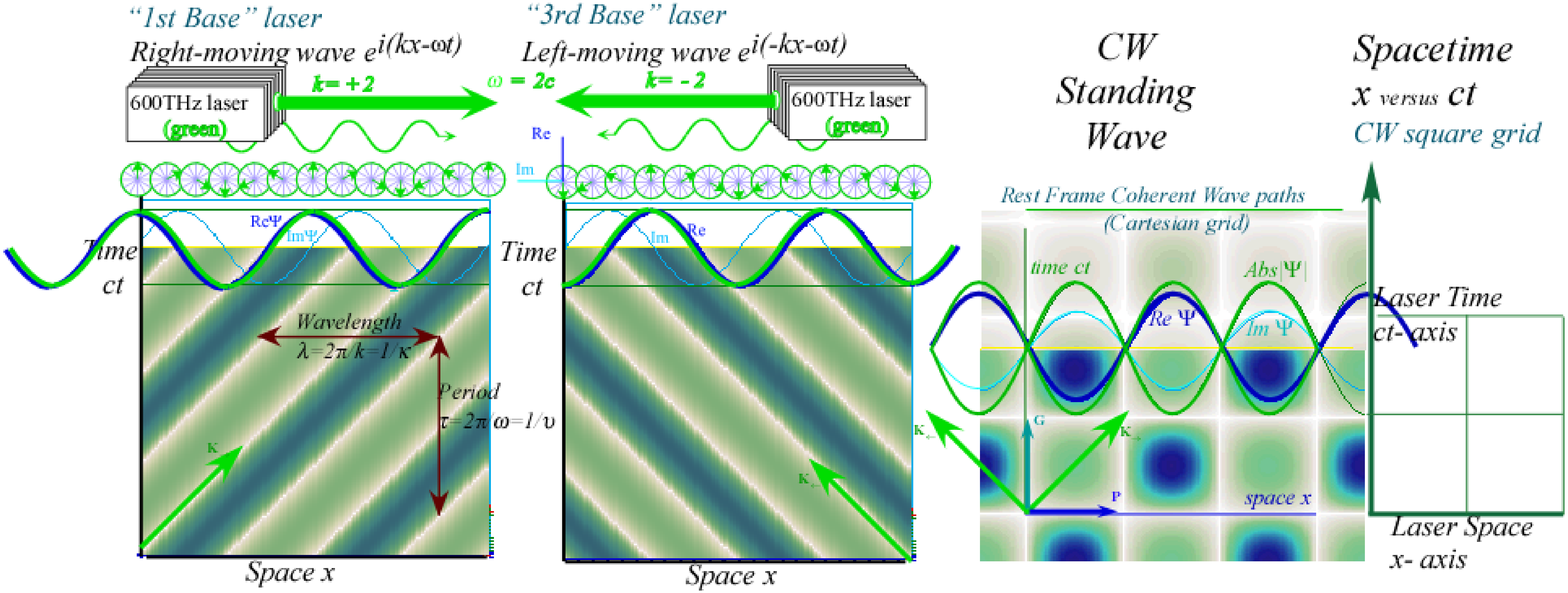
$$\mathbf{P}=\frac{1}{2}\begin{pmatrix}ck_a+ck_b\\ \omega_a+\omega_b\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\omega_a-\omega_b\\ \omega_a+\omega_b\end{pmatrix}=\omega\begin{pmatrix}0\\1\end{pmatrix}$$

$\frac{1}{2}$-Difference

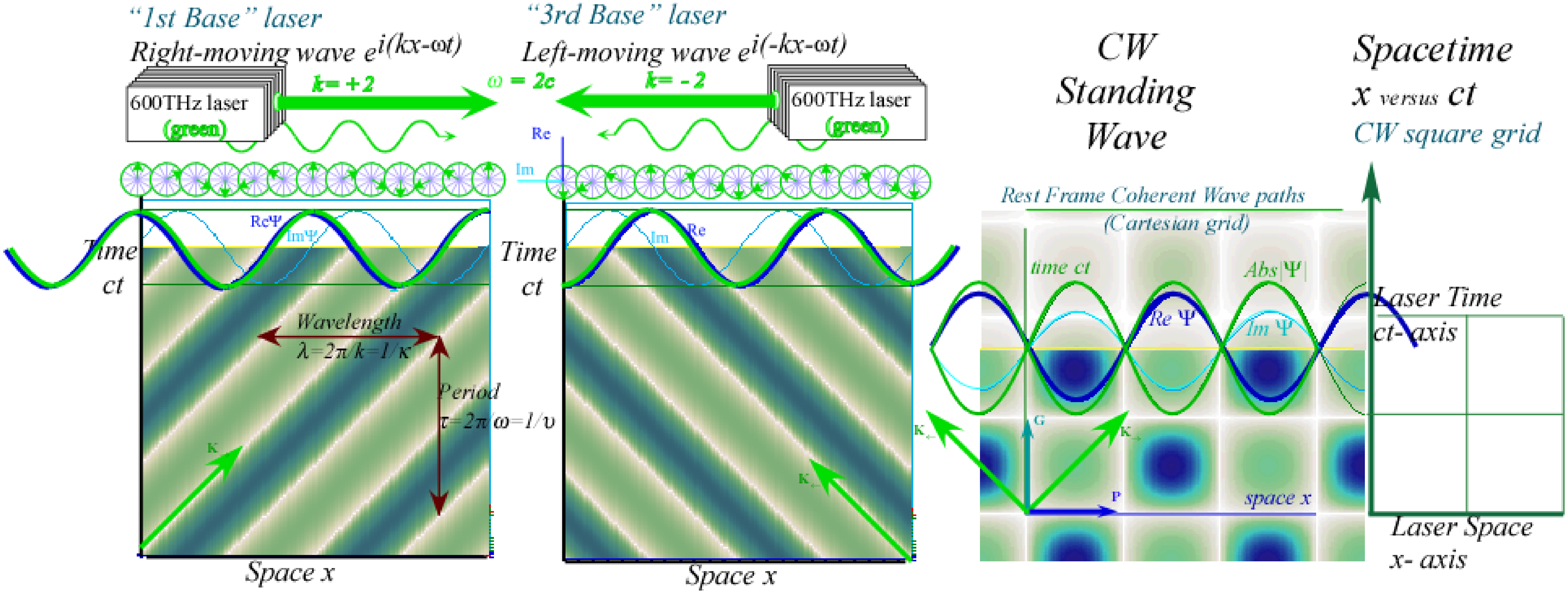
Group vector

$$\mathbf{G}=\frac{1}{2}\begin{pmatrix}ck_a-ck_b\\ \omega_a-\omega_b\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\omega_a+\omega_b\\ \omega_a-\omega_b\end{pmatrix}=\omega\begin{pmatrix}1\\0\end{pmatrix}$$

Zeros of head-on CW sum gives (x,ct)-grid



Zeros of head-on CW sum gives (x,ct)-grid

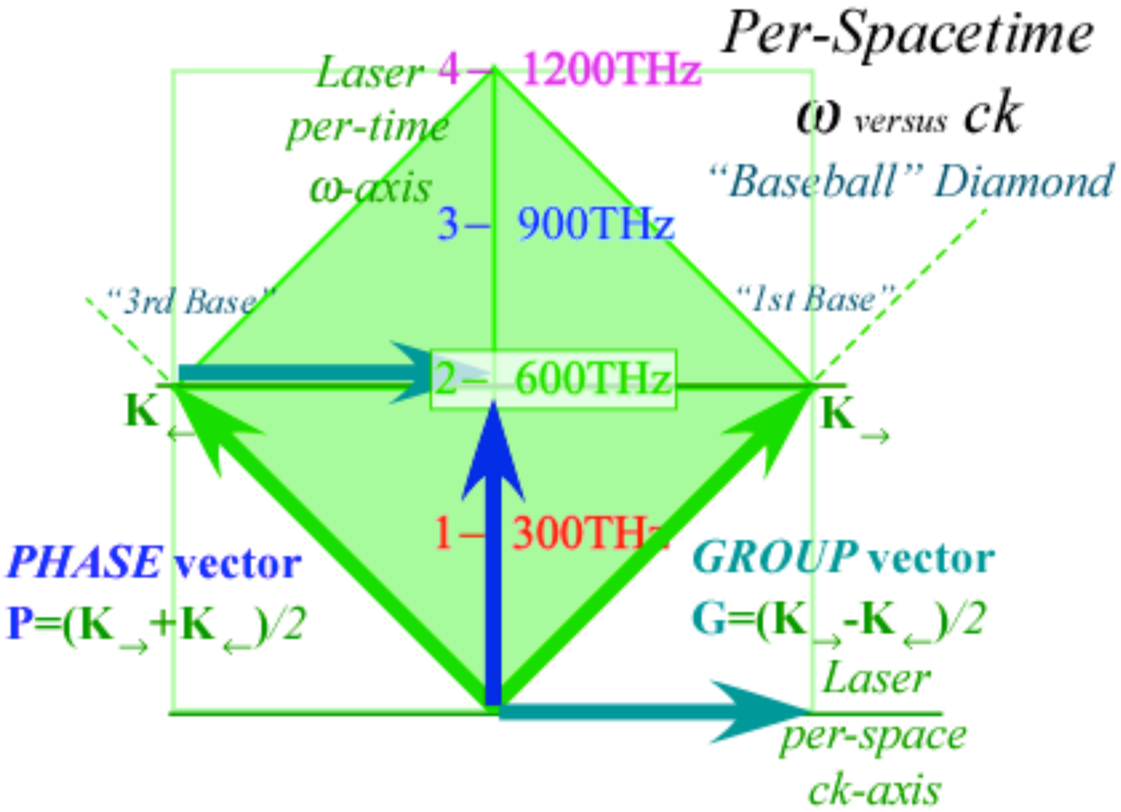


Find zeros by factoring sum:

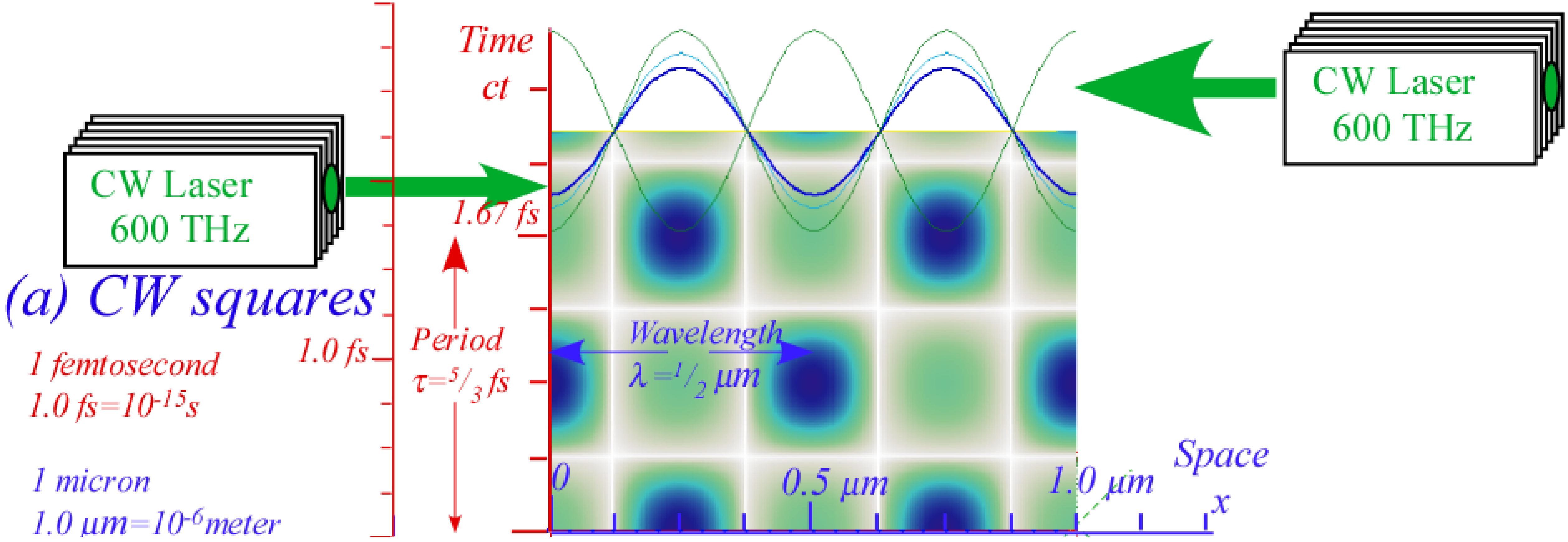
$$\Psi = \overbrace{e^{ia}}^{(kx-\omega t)} + \overbrace{e^{ib}}^{(-kx-\omega t)}$$

$$= \underbrace{e^{i(a+b)/2}}_{\text{Phase factor:}} \underbrace{\left(e^{i(a-b)/2}+e^{-i(a-b)/2}\right)}_{\text{Group factor:}}$$

$\exp i\left(\frac{a+b}{2}\right)=e^{-i\omega t}$ $\quad$ $2\cos\left(\frac{a-b}{2}\right) = 2\cos(kx)$



Wave coordinates for Linear Dispersion

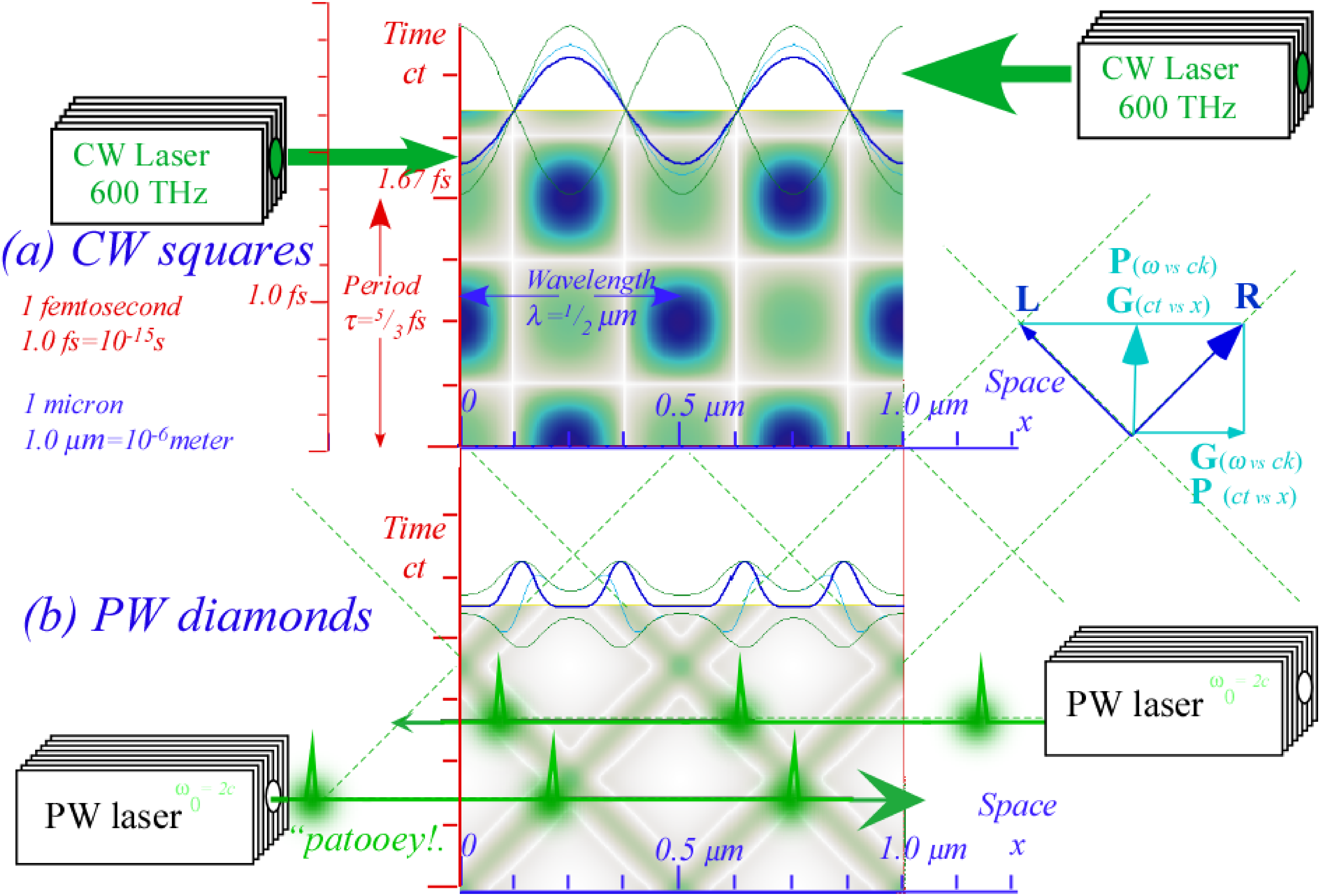


(a) CW squares

1 femtosecond
1.0 fs$=10^{-15}$*s*

1 micron
1.0 μm$=10^{-6}$*meter*

Wave coordinates for Linear Dispersion

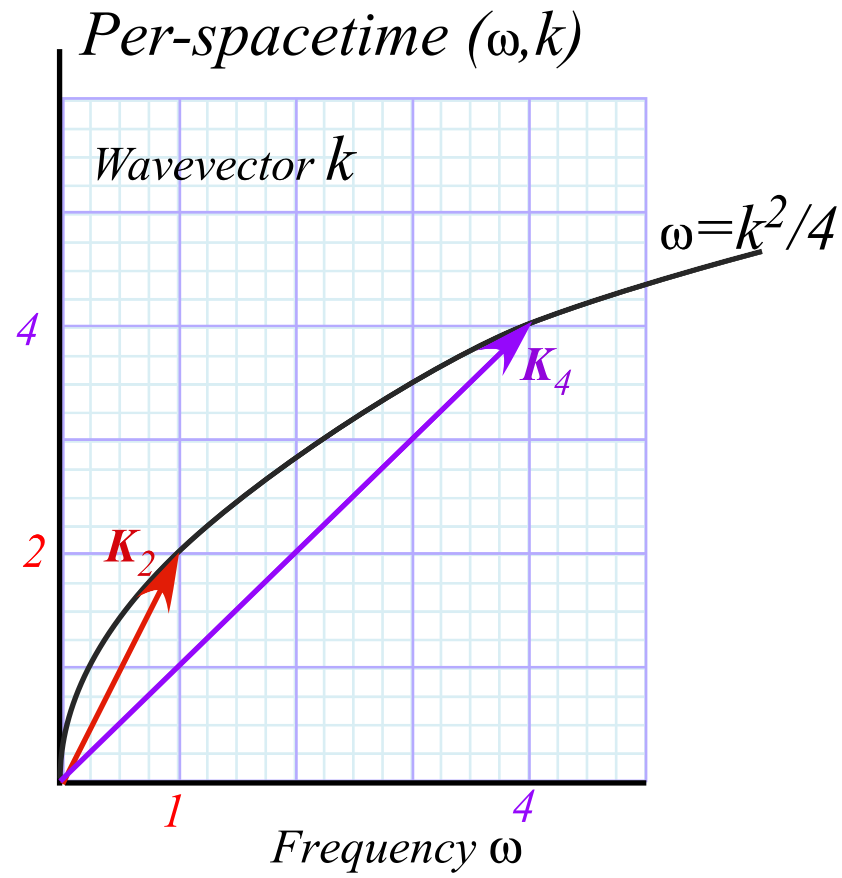


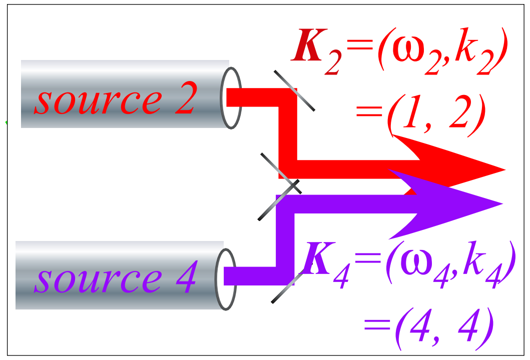
Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*
- *Wave coordinates for Linear Dispersion*
- → *Wave coordinates for Bohr-Schrodinger Dispersion*
- *Einstein-Lorentz-Minkowski laser coordinates*

Introduction to C_N beat dynamics and "Revivals"
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

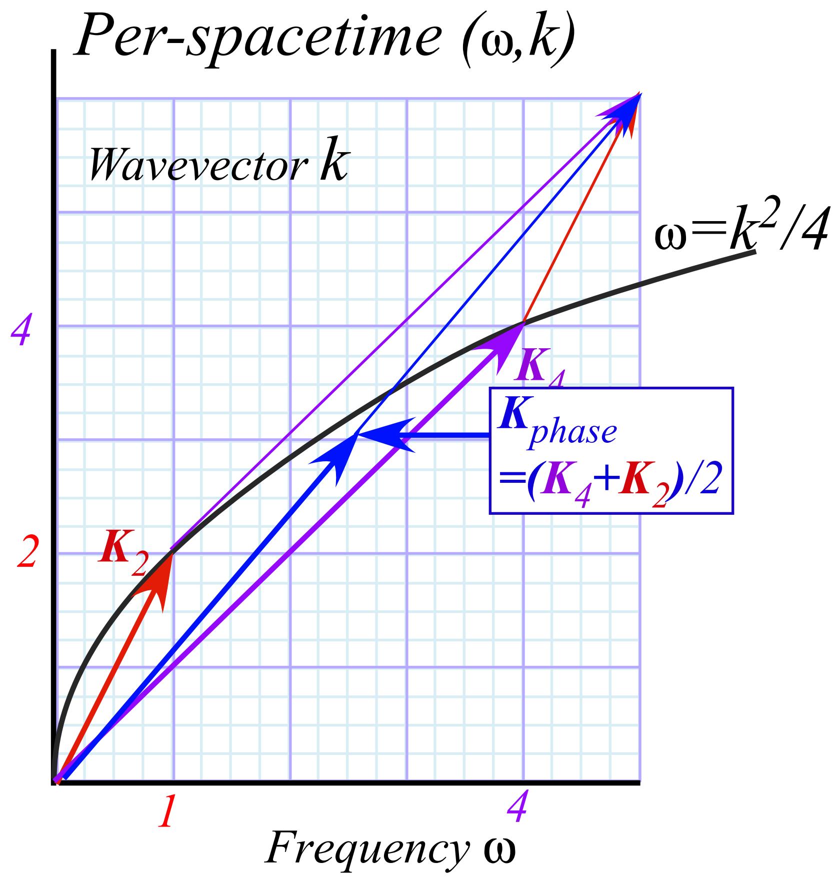


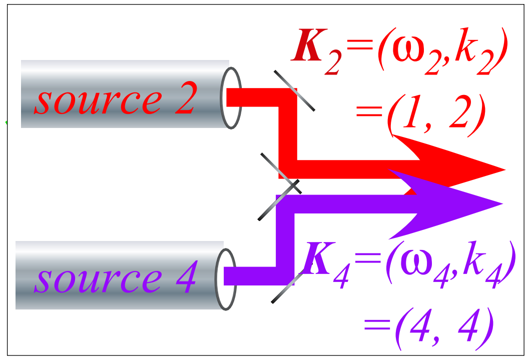


Suppose we are given two "mystery† sources"

† Bohr-Schrodinger "matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

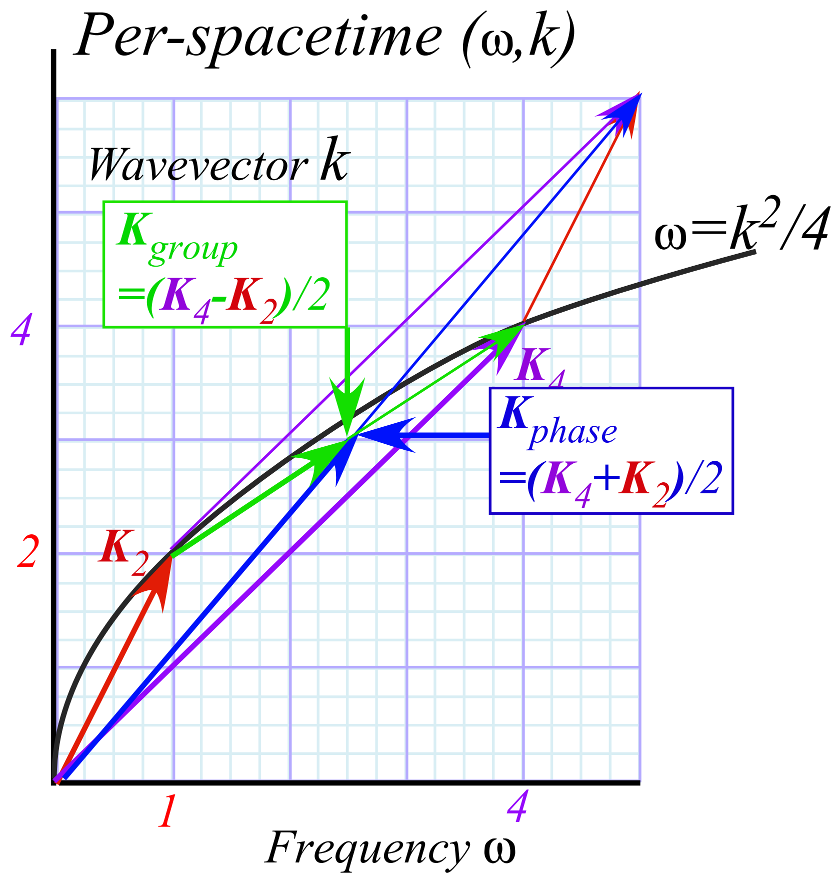


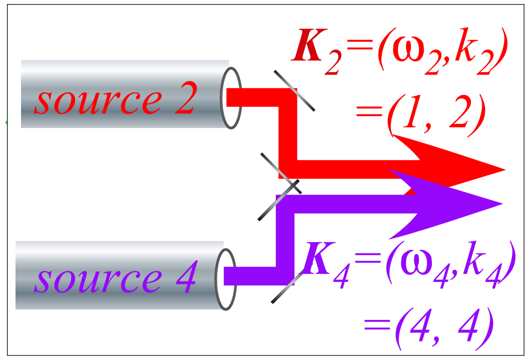


Suppose we are
given two
“mystery† sources”

† Bohr-Schrodinger
“matter-waves”

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

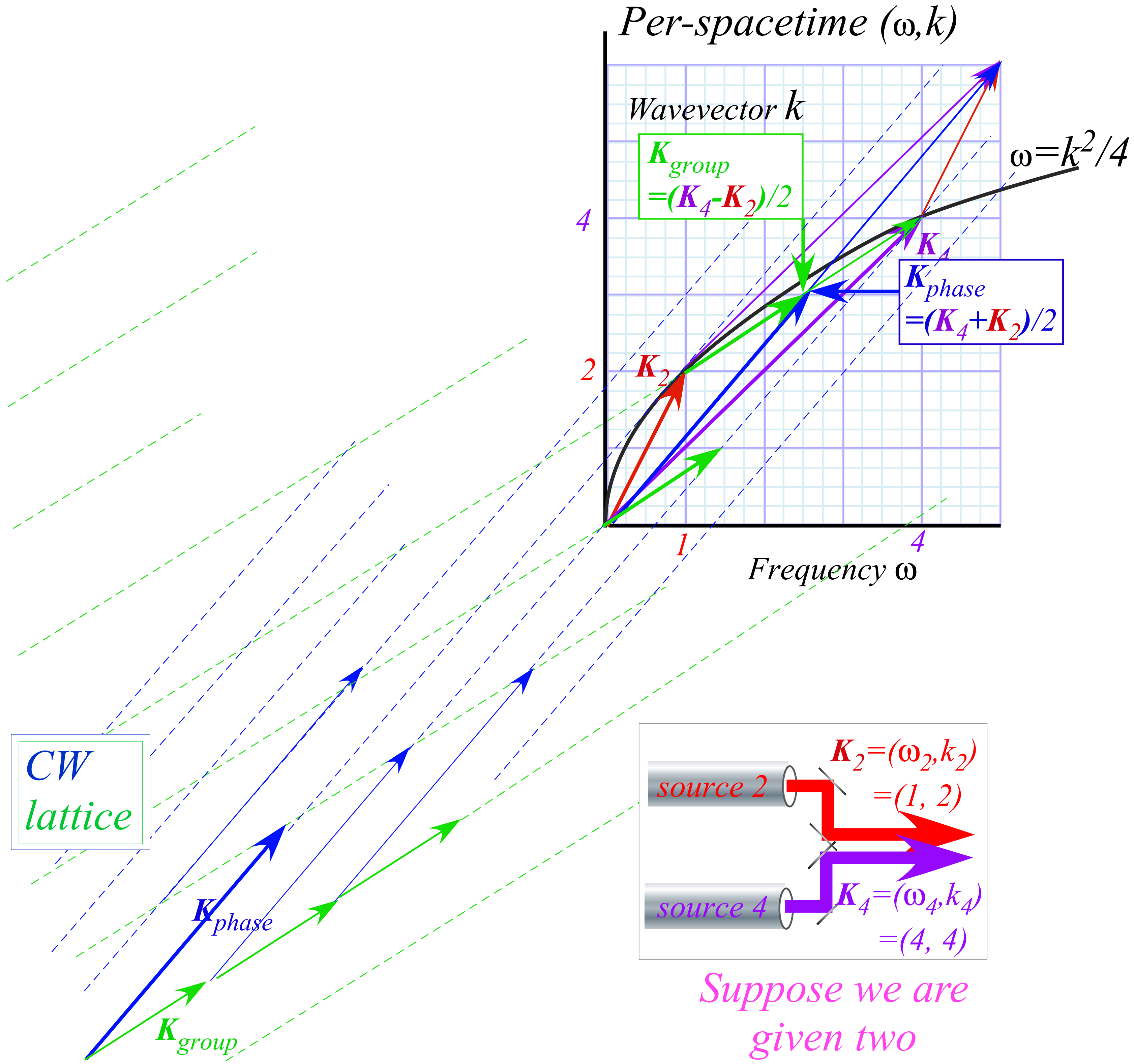




Suppose we are given two "mystery† sources"

†Bohr-Schrodinger "matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

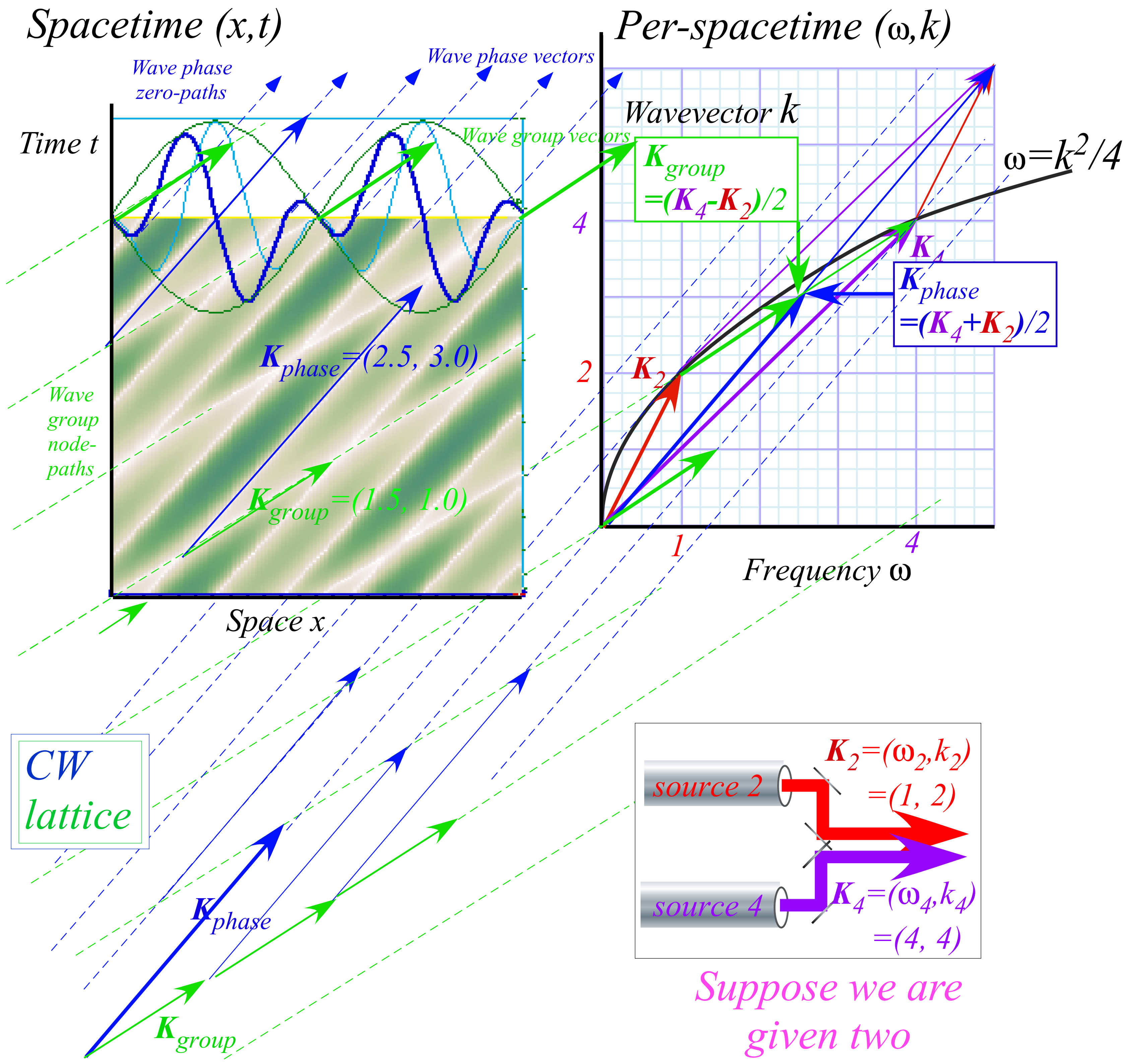


Continuous Wave
or
Coherent Wave
lattice

Suppose we are
given two
"mystery† sources"

†Bohr-Schrodinger
"matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



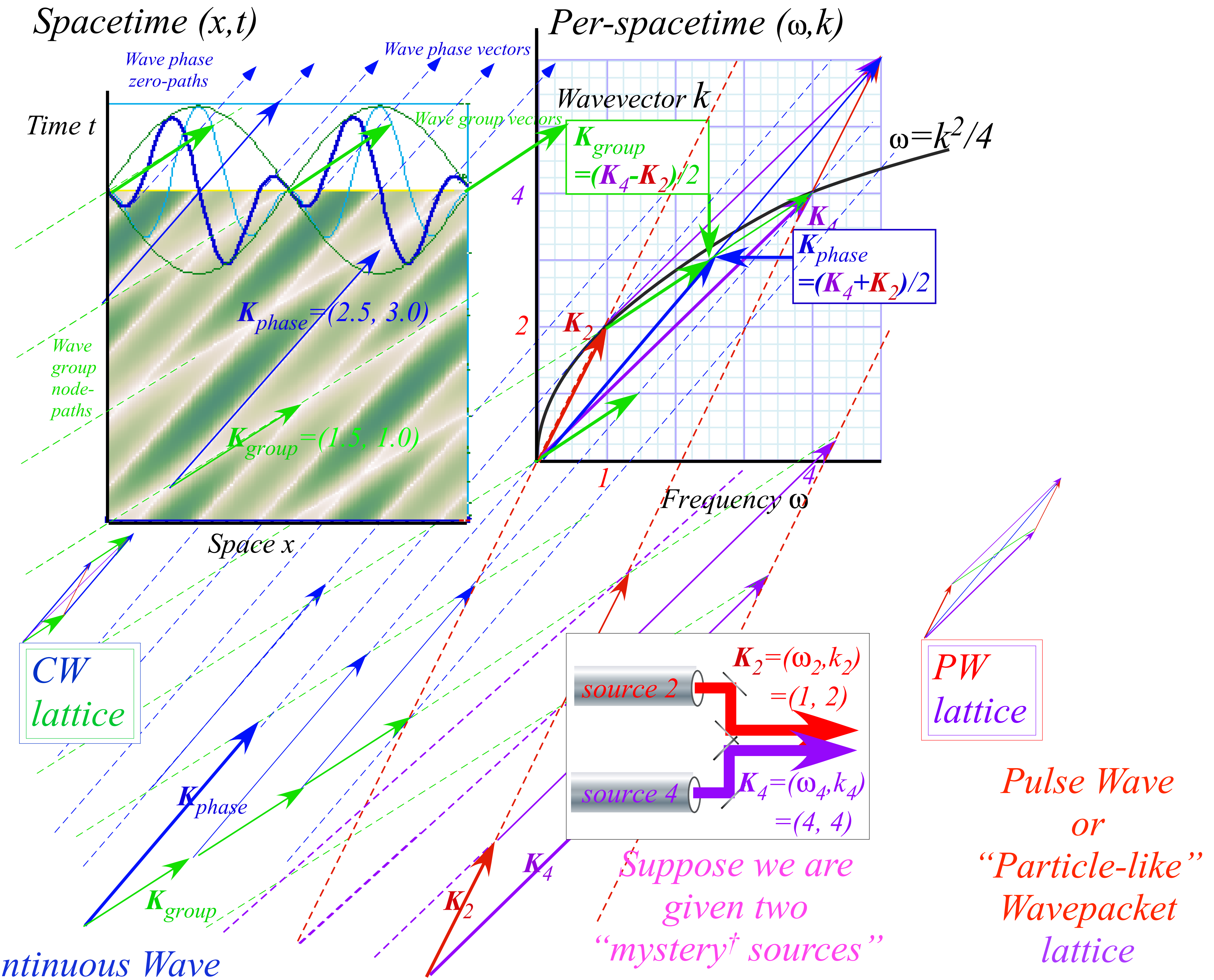
CW lattice

Continuous Wave
or
Coherent Wave
lattice

Suppose we are
given two
"mystery† sources"

† Bohr-Schrodinger
"matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



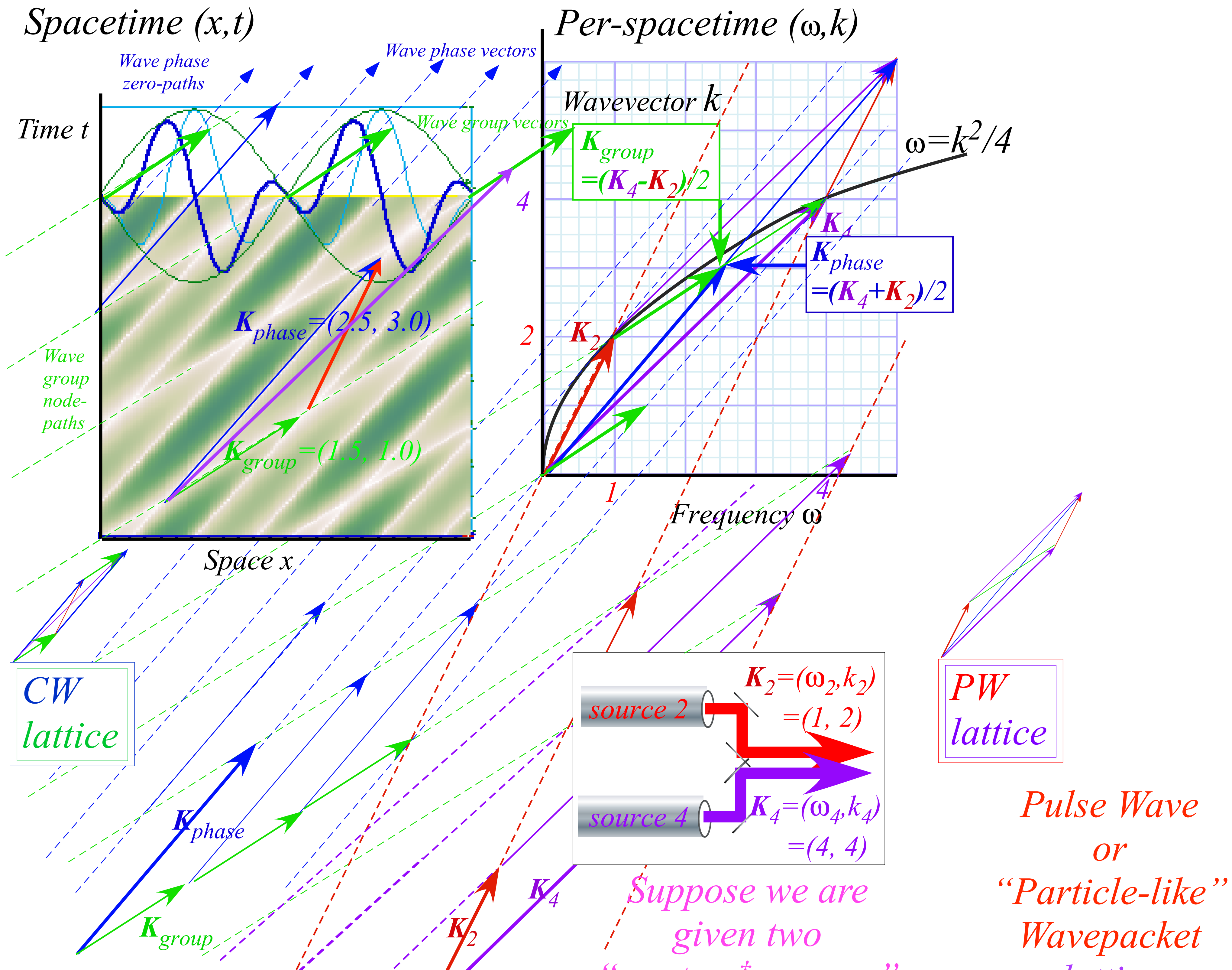
Suppose we are given two "mystery† sources"

Pulse Wave or "Particle-like" Wavepacket lattice

Continuous Wave or Coherent Wave lattice

†Bohr-Schrodinger "matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



Continuous Wave
or
Coherent Wave
lattice

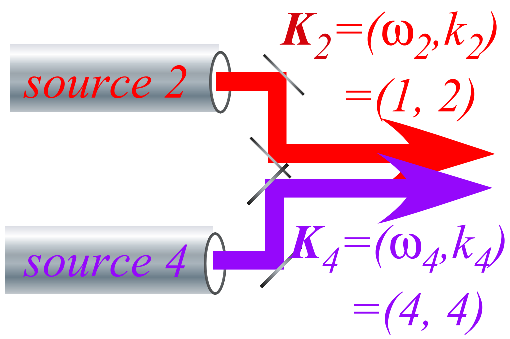
Suppose we are
given two
"mystery† sources"

† Bohr-Schrodinger
"matter-waves"

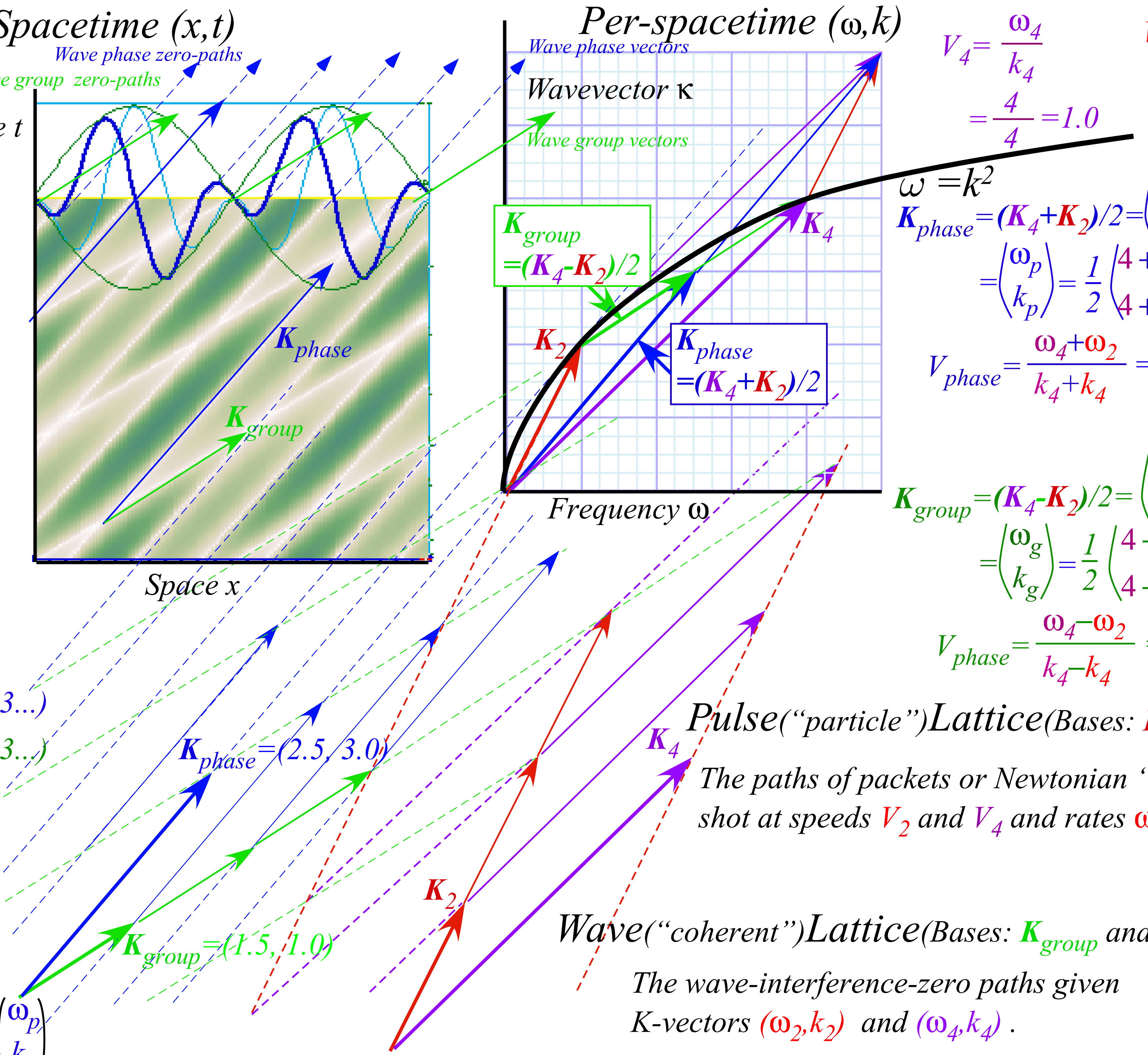
Pulse Wave
or
"Particle-like"
Wavepacket
lattice

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

Suppose we are given two “mystery† sources”



†Shrodinger matter waves



$$V_4 = \frac{\omega_4}{k_4} = \frac{4}{4} = 1.0 \qquad V_2 = \frac{\omega_2}{k_2} = \frac{1}{2} = 0.5$$

$$K_{phase} = (K_4 + K_2)/2 = \begin{pmatrix} \omega_4+\omega_2 \\ k_4+k_4 \end{pmatrix}/2 = \begin{pmatrix} \omega_p \\ k_p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4+1 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.0 \end{pmatrix}$$

$$V_{phase} = \frac{\omega_4+\omega_2}{k_4+k_4} = \frac{2.5}{3.0} = 0.83$$

$$K_{group} = (K_4 - K_2)/2 = \begin{pmatrix} \omega_4-\omega_2 \\ k_4-k_4 \end{pmatrix}/2 = \begin{pmatrix} \omega_g \\ k_g \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$$

$$V_{phase} = \frac{\omega_4-\omega_2}{k_4-k_4} = \frac{1.5}{1.0} = 1.5$$

Wave(“coherent”)Lattice

Bases: K_{group} and K_{phase}

$k_p x - \omega_p t = n_p = N_p\ /2 \quad (N_p = \pm1, \pm3 ...)$

$k_g x - \omega_g t = n_g = N_g\ /2 \quad (N_g = \pm1, \pm3 ...)$

$$\begin{pmatrix} k_p & -\omega_p \\ k_g & -\omega_g \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} n_p \\ n_g \end{pmatrix}$$

Pulse(“particle”)Lattice(Bases: K_2 and K_4)

The paths of packets or Newtonian “corpuscles” shot at speeds V_2 and V_4 and rates ω_2 and ω_2

Wave(“coherent”)Lattice(Bases: K_{group} and K_{phase})

The wave-interference-zero paths given K-vectors (ω_2,k_2) and (ω_4,k_4) .

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{\begin{pmatrix} -\omega_g & \omega_p \\ -k_g & k_p \end{pmatrix}\begin{pmatrix} n_p \\ n_g \end{pmatrix}}{\omega_p k_g - \omega_g k_p} = \frac{-n_p\begin{pmatrix} \omega_g \\ k_g \end{pmatrix} + n_g\begin{pmatrix} \omega_p \\ k_p \end{pmatrix}}{\omega_p k_g - \omega_g k_p} = \frac{n_p}{D} K_{group} + \frac{n_g}{D} K_{phase}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

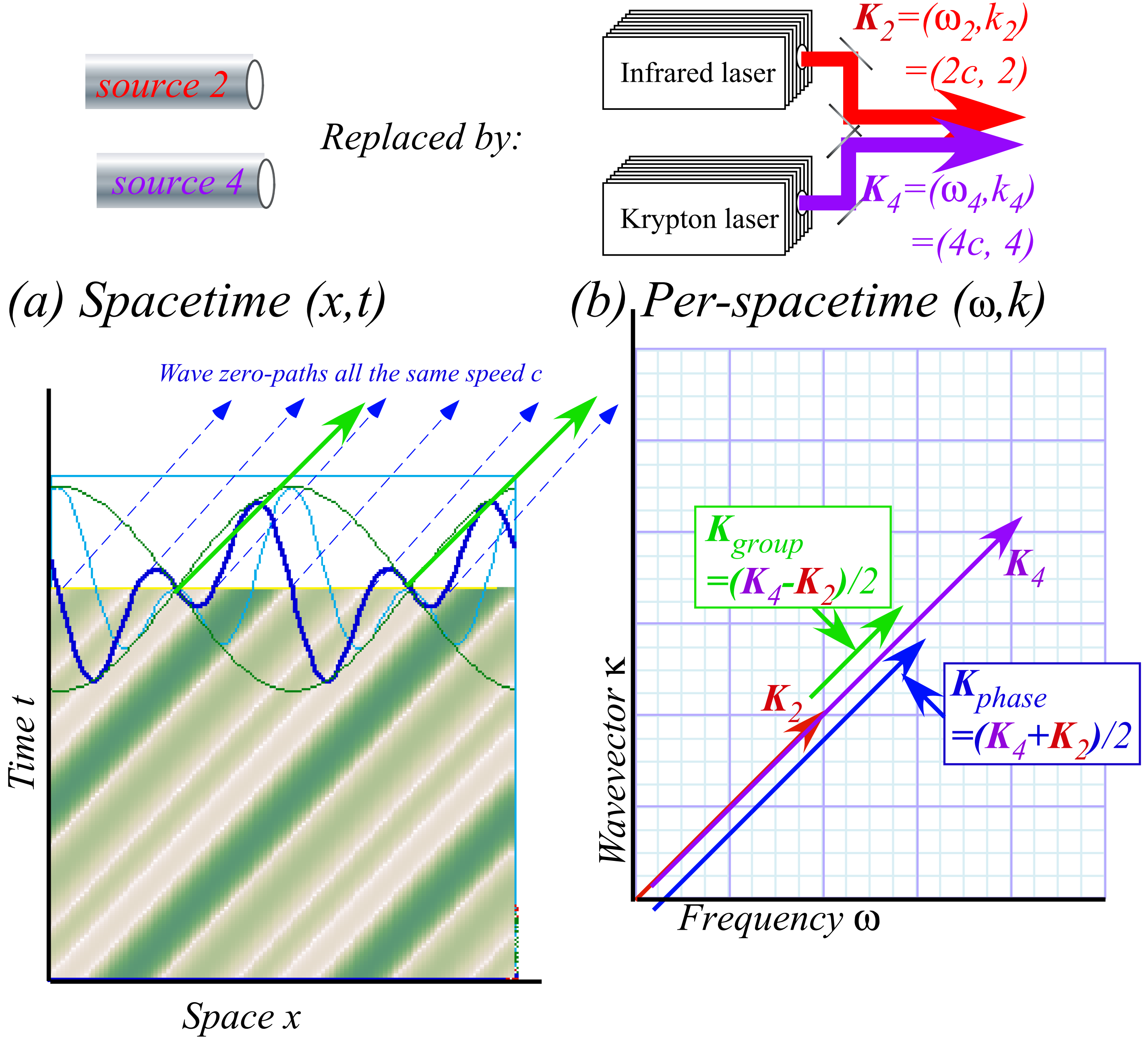
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
  Wave coordinates
  Pulse-waves (PW) vs Continuous -waves (CW)
→ *(Back to)Wave coordinates for Linear Dispersion*
  Wave coordinates for Bohr-Schrodinger Dispersion
  Einstein-Lorentz-Minkowski laser coordinates

Introduction to C_N beat dynamics and "Revivals"
  Farey-Sums and Ford-products
  Phase dynamics

(Back to) Wave coordinates for Linear Dispersion

*For co-propagating laser * sources...*
...the wave-coordinate lattice collapses to lines..

**simple linear*
ω=ck dispersion



(a) Spacetime (x,t)   *(b) Per-spacetime (ω,k)*

But, for counter-propagating laser sources...
...the wave coordinate lattice is the Lorentz-Einstein-Minkowski frame!!

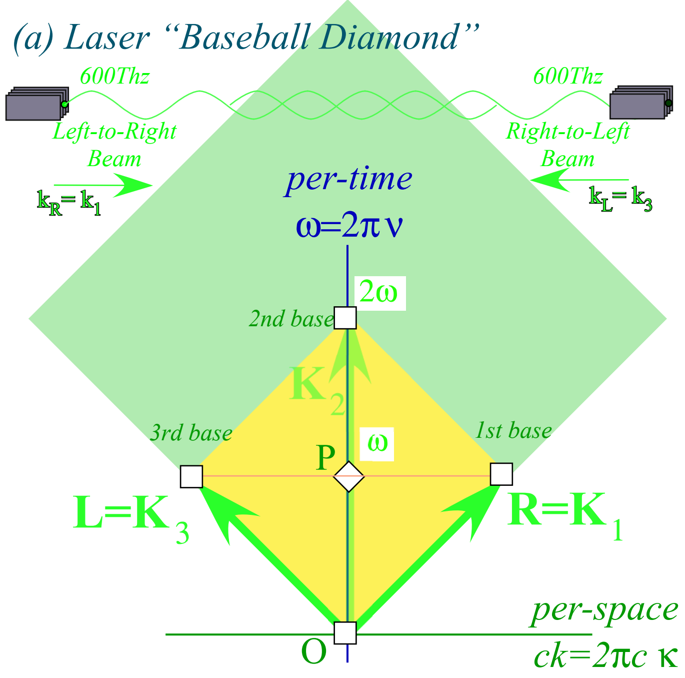
Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
  Wave coordinates
  Pulse-waves (PW) vs Continuous -waves (CW)
  Wave coordinates for Linear Dispersion
  Wave coordinates for Bohr-Schrodinger Dispersion
  Einstein-Lorentz-Minkowski laser coordinates

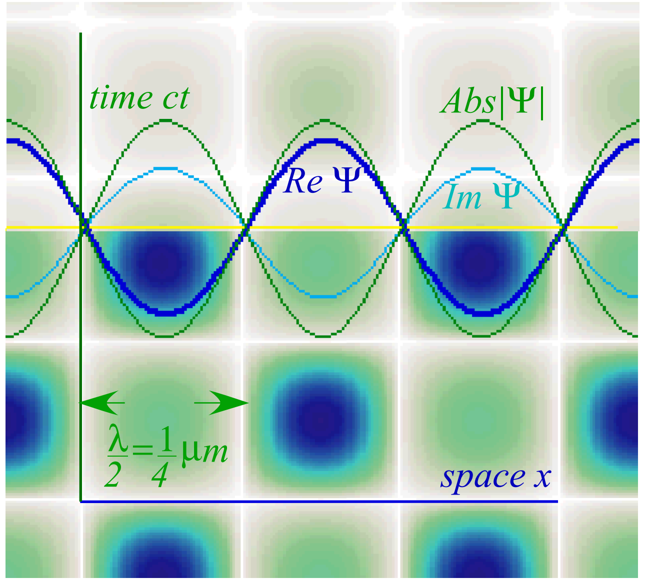
Introduction to C_N beat dynamics and "Revivals"
  Farey-Sums and Ford-products
  Phase dynamics

(Back to) Wave coordinates for Linear Dispersion
u=0 space-time coordinates

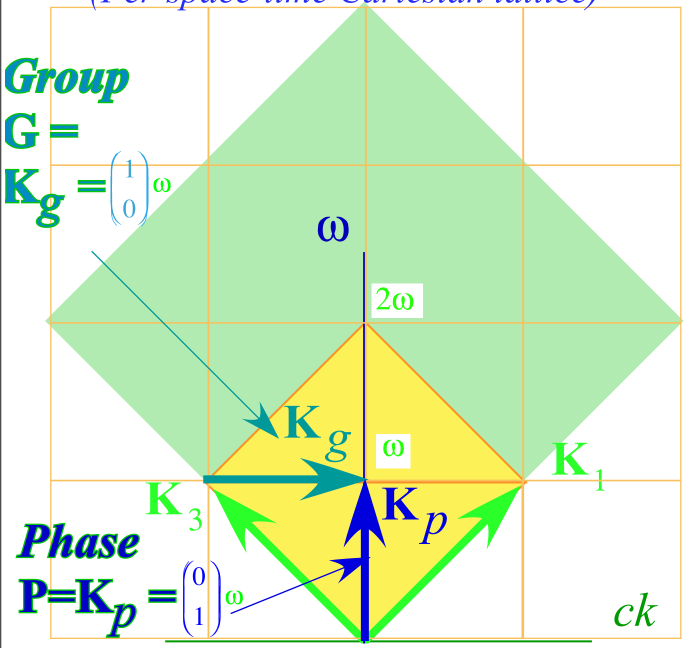
(a) Laser "Baseball Diamond"



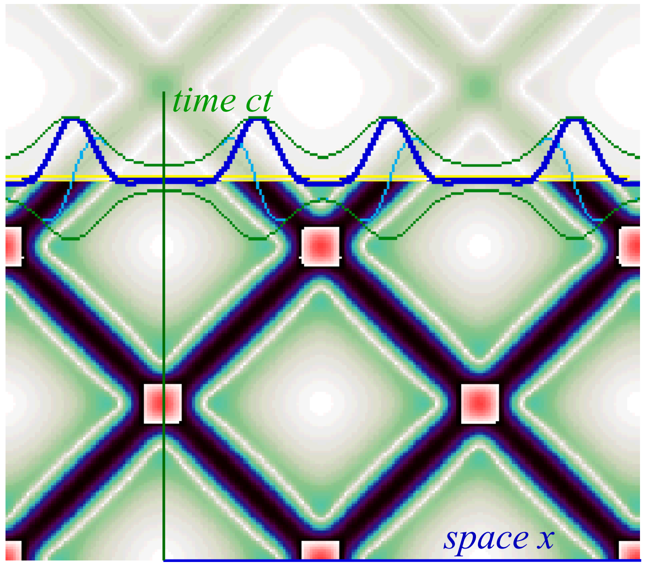
(c) Laser Coherent Wave (CW) paths
(Space-time Cartesian grid)



(b) Laser group and phase wavevectors
(Per-space-time Cartesian lattice)



(d) Laser Pulse Wave (PW) Paths
(Space-time Diamond grid)

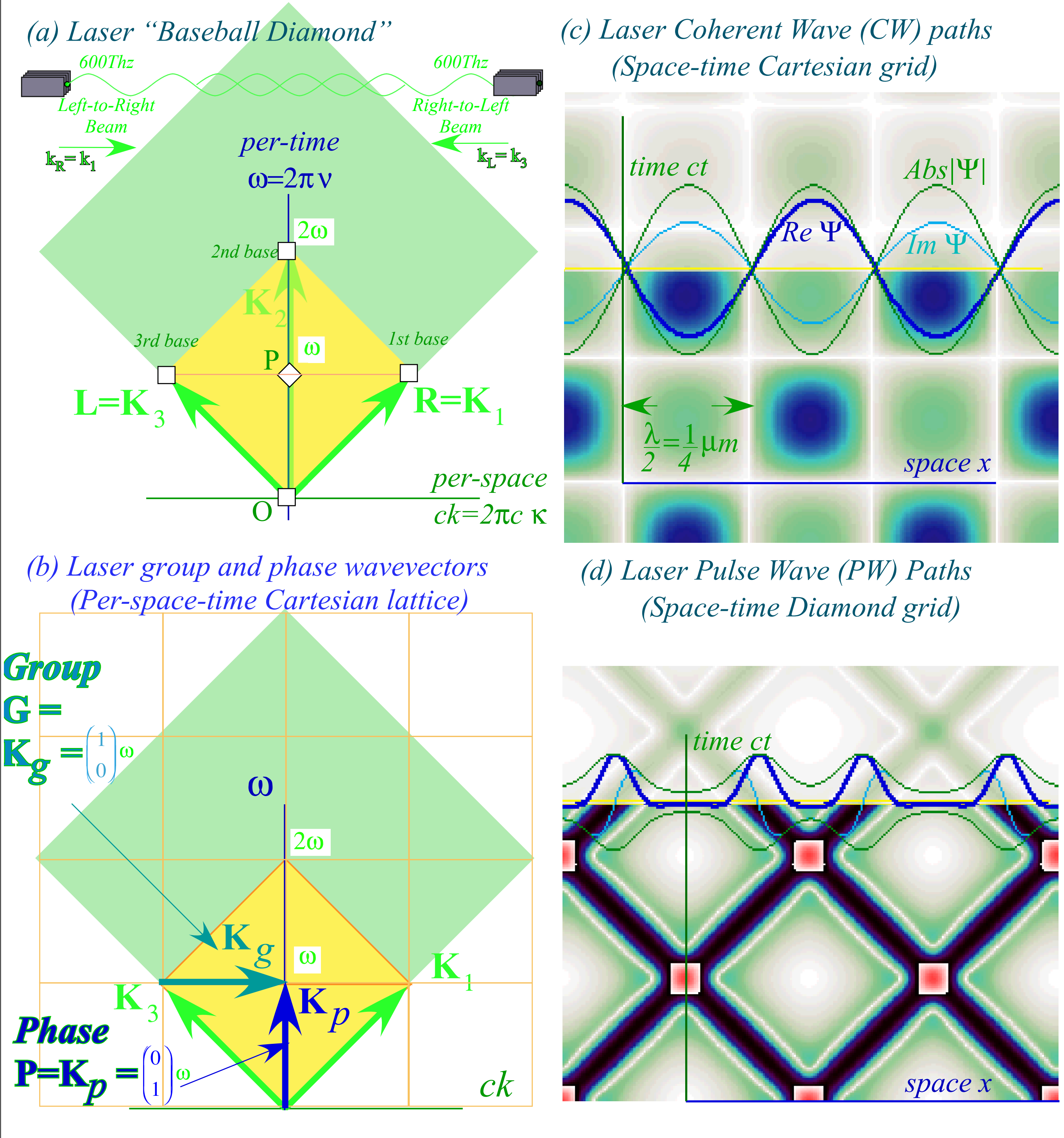


u=0 space-time pulse waves

CMwith a BANG! Fig. 8.2.1

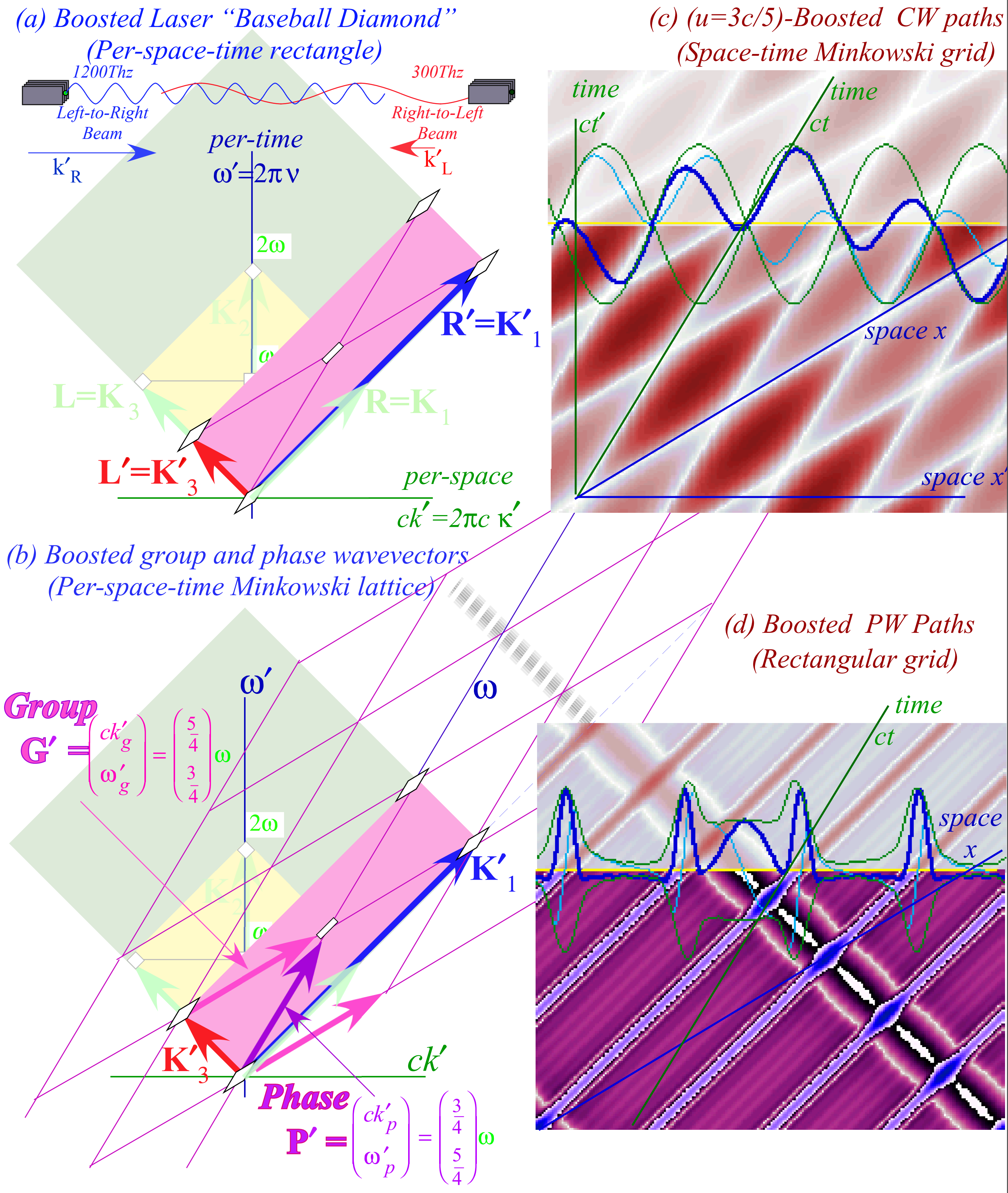
(Back to) Wave coordinates for Linear Dispersion
u=0 space-time coordinates

Doppler shifted by factor of 2
Gives u=3c/5 Einstein-Lorentz-Minkowski coordinates



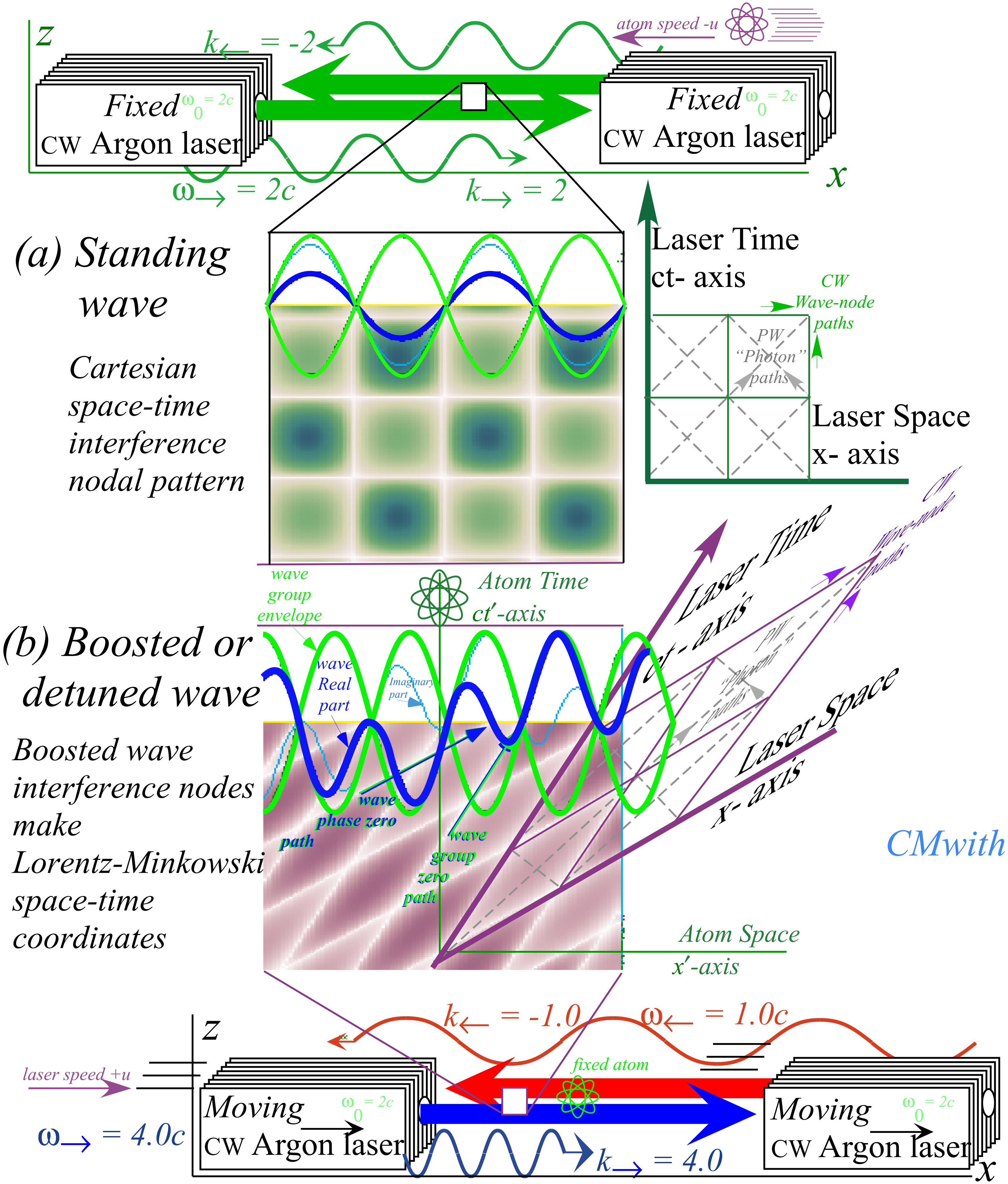
u=0 space-time pulse waves

CMwith a BANG! Fig. 8.2.1



u=3c/5 space-time pulse waves

CMwith a BANG! Fig. 8.2.2



(a) Standing wave

Cartesian space-time interference nodal pattern

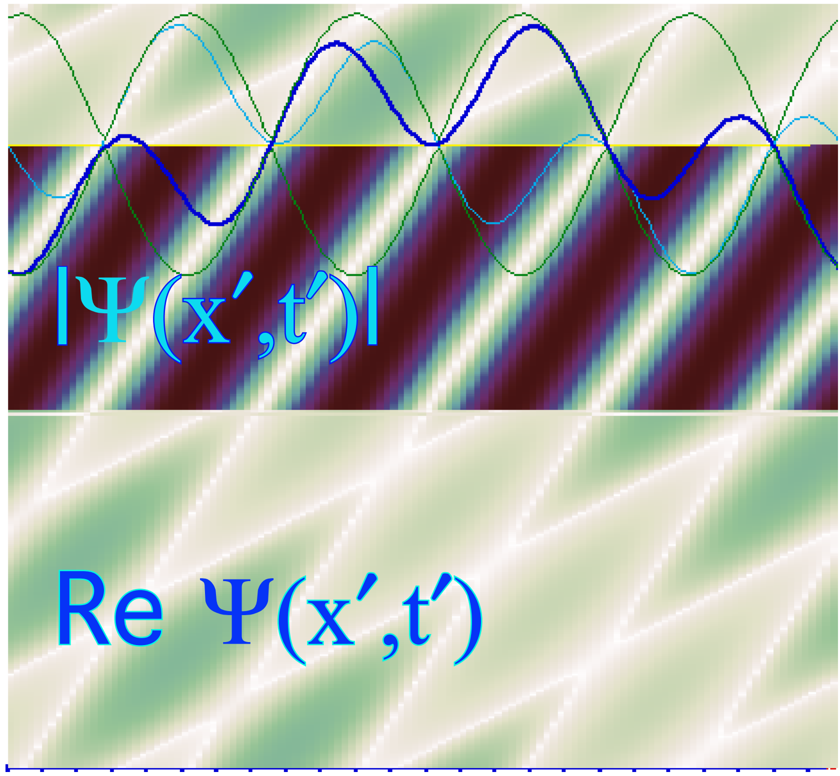
(b) Boosted or detuned wave

Boosted wave interference nodes make Lorentz-Minkowski space-time coordinates

CMwith a BANG! Fig. 8.2.3

Phase lines may not show up in Magnitude ($|\Psi(x',t')|$) *orProbability* ($\Psi(x',t')^*\Psi(x',t')$) *plots.*

Unbiased $\Psi=\psi_{-1}+\psi_{+4}$

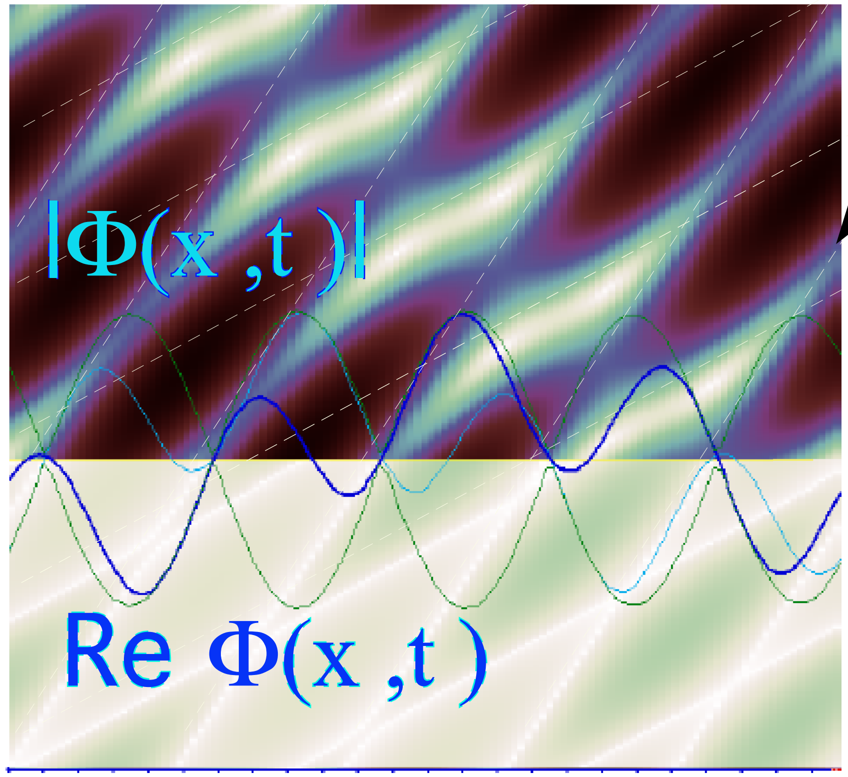


Only the group wave paths appear

The “inside phase” $e^{i[]}$ *gets killed in* ($\Psi(x',t')^*\Psi(x',t')$)
because $(e^{i[]})^*= e^{-i[]}$
and $(e^{i[]})^*\bullet e^{i[]}= 1$

Phase structure begins to show up if ground-state (k=0) component is added.

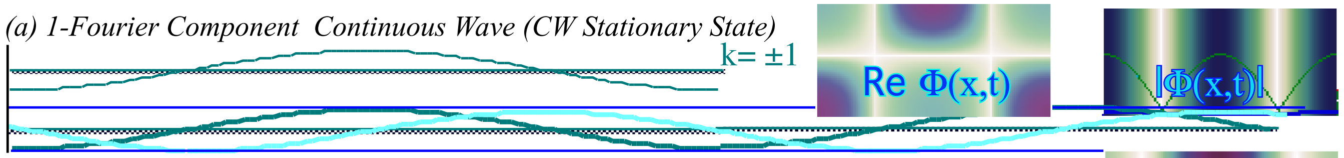
DC biased $\Phi=\psi_0+\Psi$

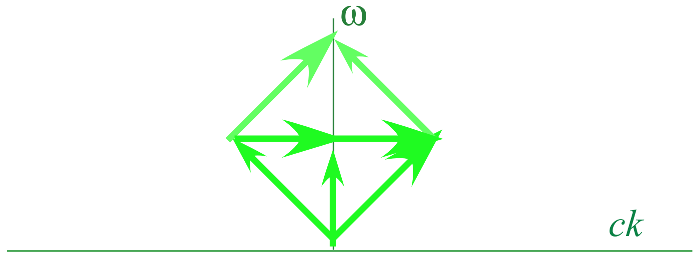


Group and phase paths begin to appear

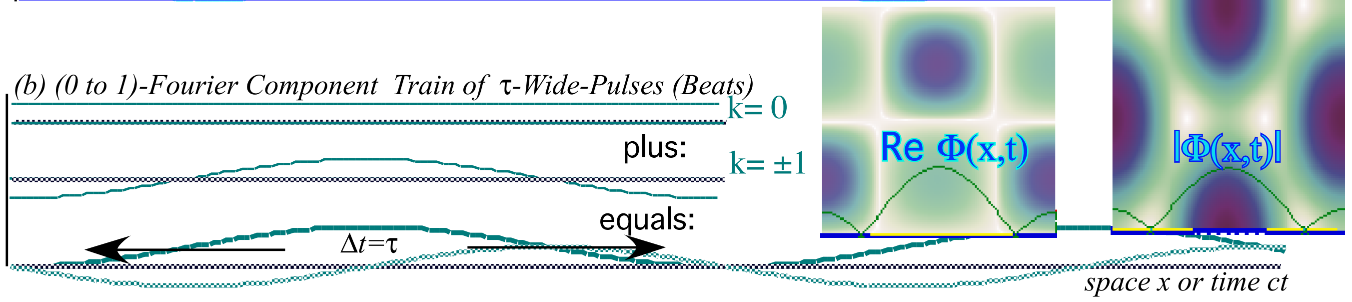
Each counter-propagating pair of beams makes a wave-interference-lattice.
"Packets" or pulses made by adding more pairs. Finally, pulse lattice apppears.

(a) 1-Fourier Component Continuous Wave (CW Stationary State)

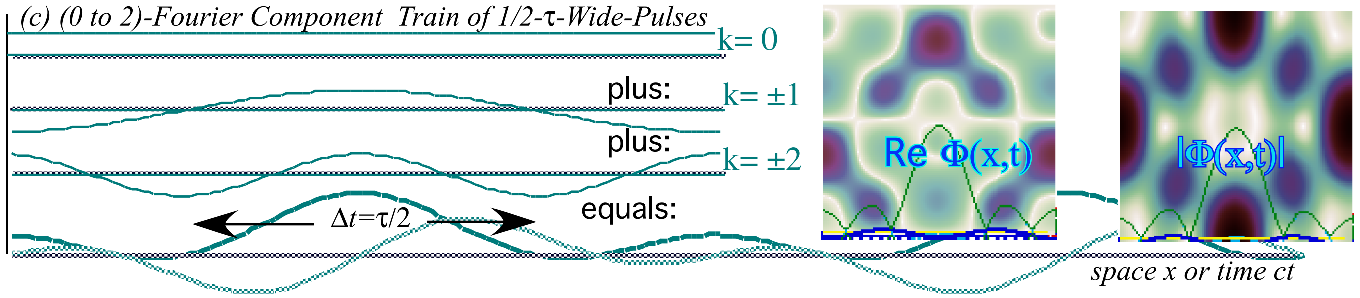


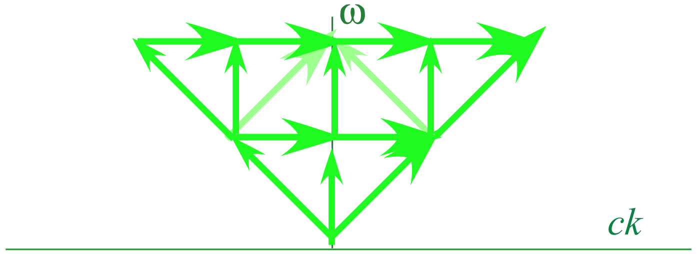


(b) (0 to 1)-Fourier Component Train of τ-Wide-Pulses (Beats)

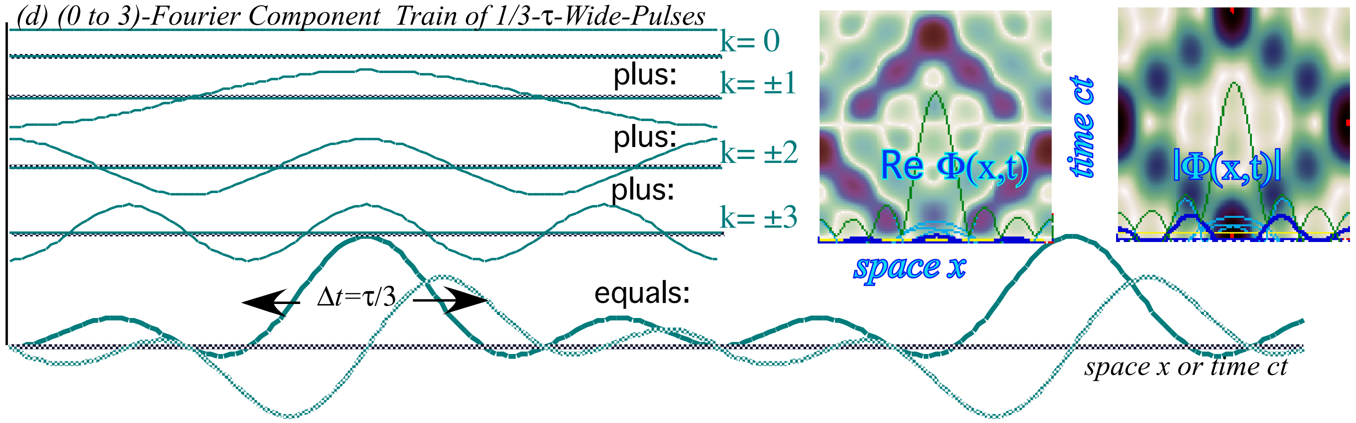


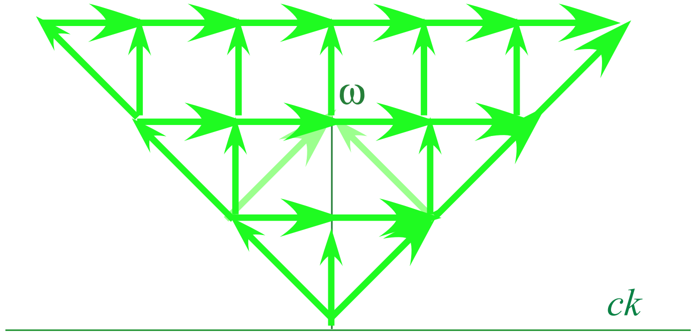
(c) (0 to 2)-Fourier Component Train of 1/2-τ-Wide-Pulses



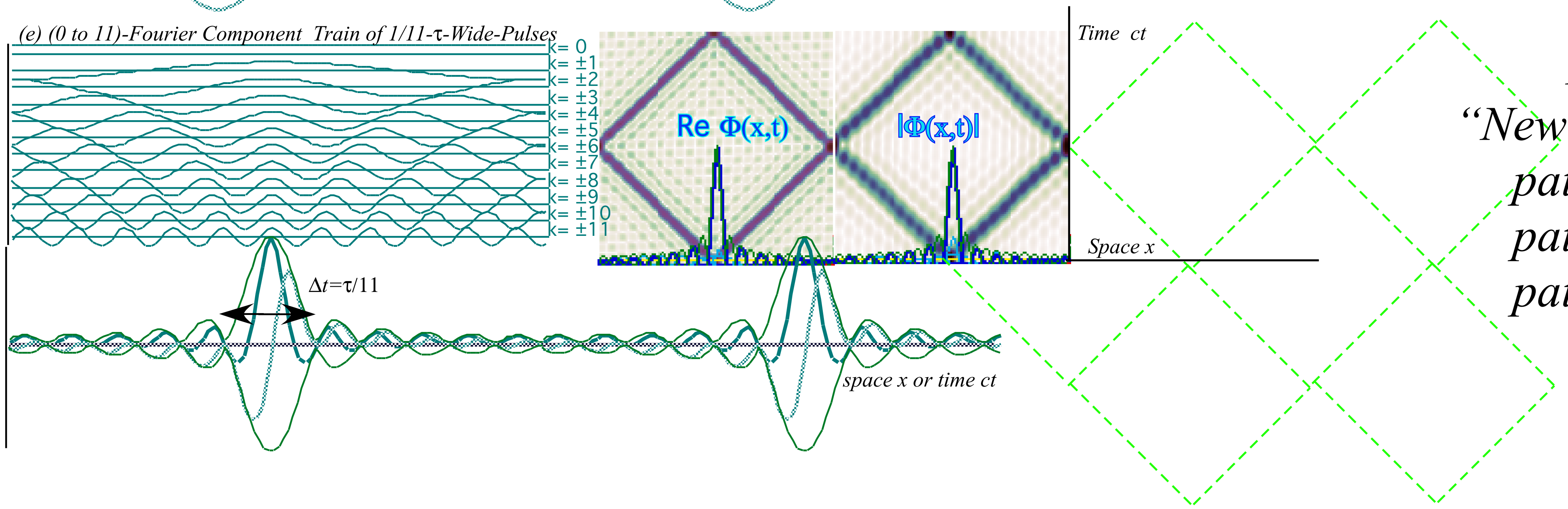


(d) (0 to 3)-Fourier Component Train of 1/3-τ-Wide-Pulses





(e) (0 to 11)-Fourier Component Train of 1/11-τ-Wide-Pulses



It's
"Newton-like"
patooey!
patooey!
patooey!
...

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
    Wave coordinates
    Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
→ ∞-Square well PE versus Bohr rotor
    SinNx/x wavepackets bandwidth and uncertainty
        SinNx/x revivals
    Gaussian wave-packet bandwidth and uncertainty
        Gaussian revivals
    Farey-Sums and Ford-products
    Phase dynamics

∞-Square well PE versus Bohr rotor

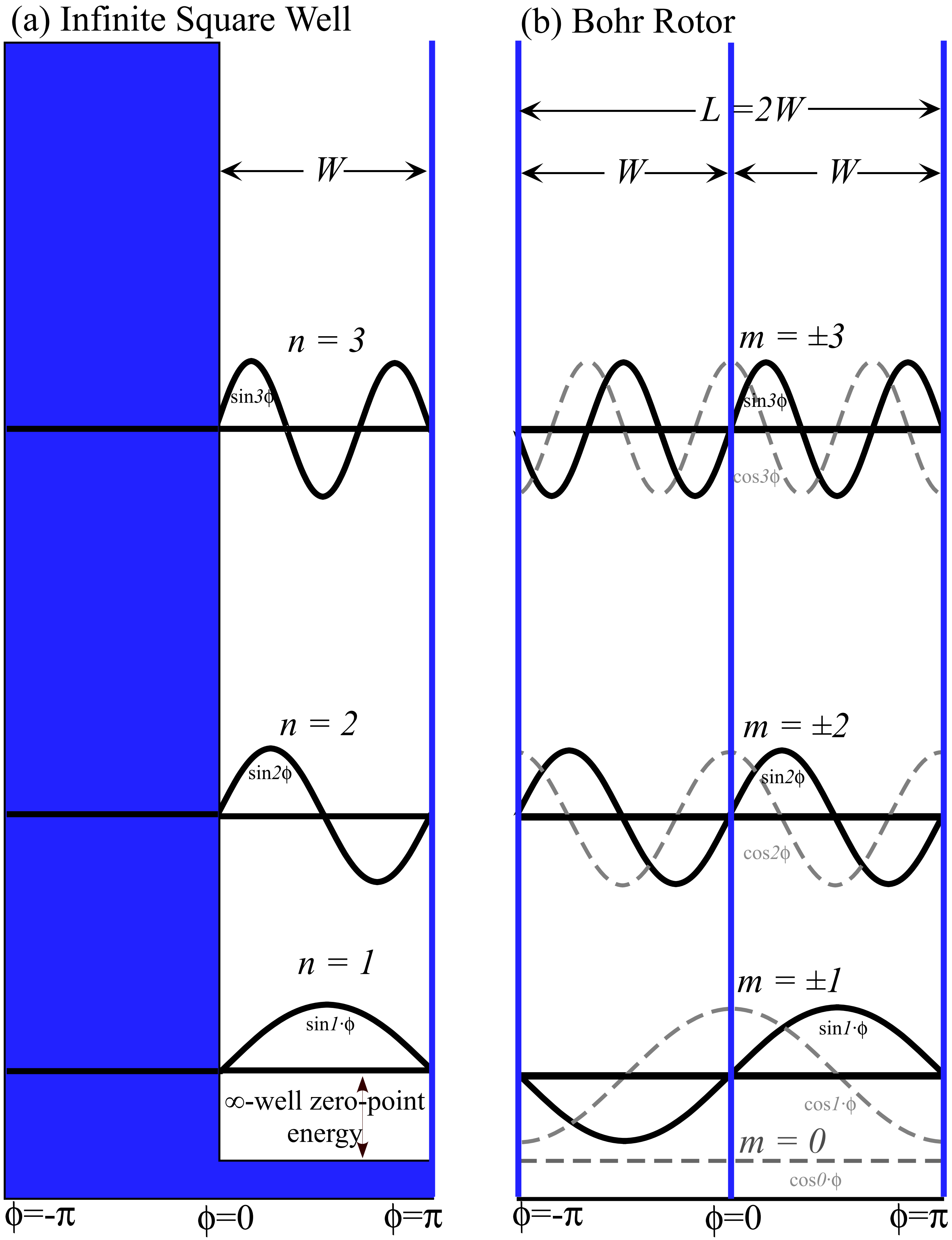
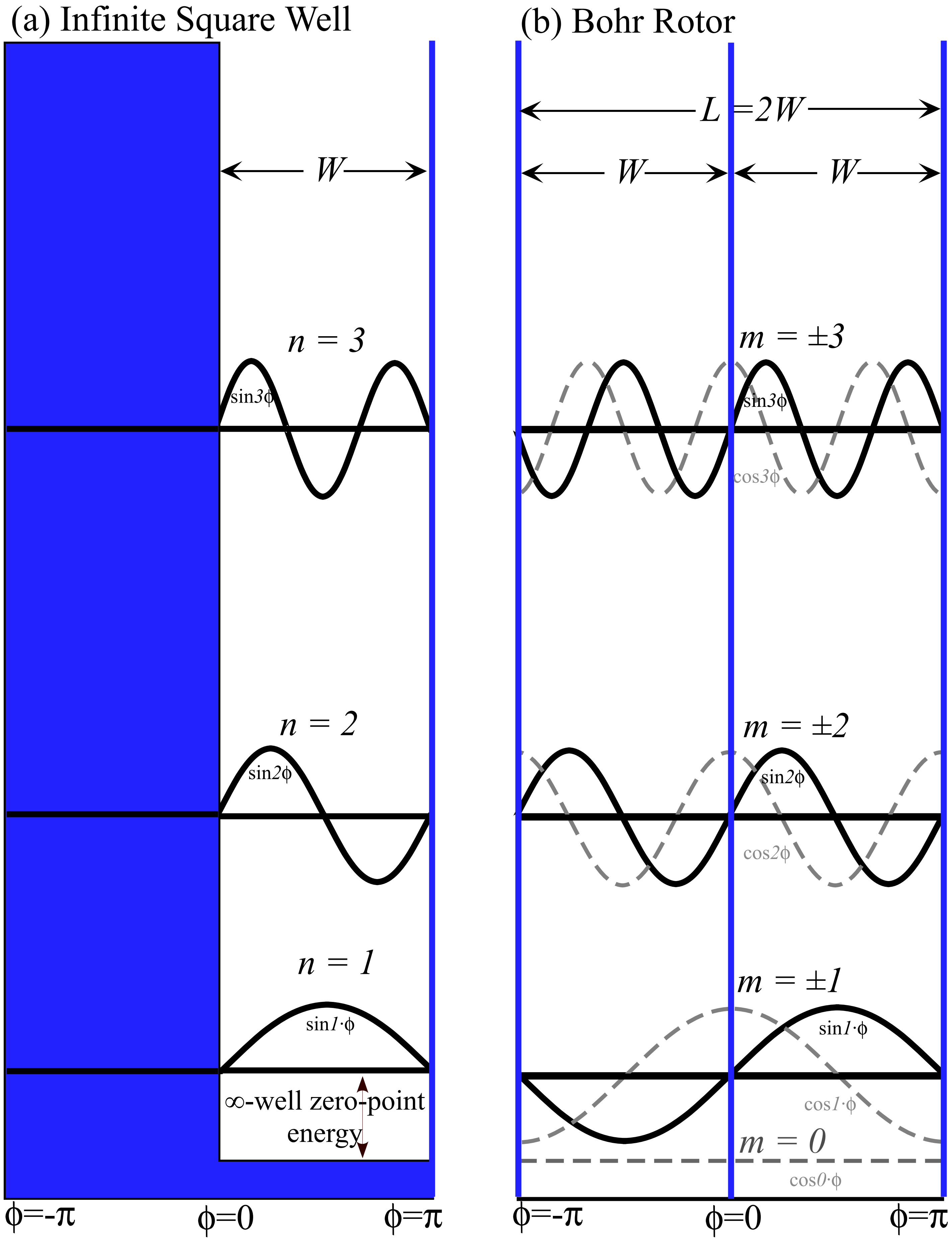


Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m = 2\pi m/L$

$(k_m = m \quad if: L = 2\pi)$

∞-Square well PE versus Bohr rotor


(a) Infinite Square Well

(b) Bohr Rotor

Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

$m=0, \pm1, \pm2, \pm3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

($k_m=m$ if: $L=2\pi$)

$$E_m= (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$

$$= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

∞-Square well PE versus Bohr rotor

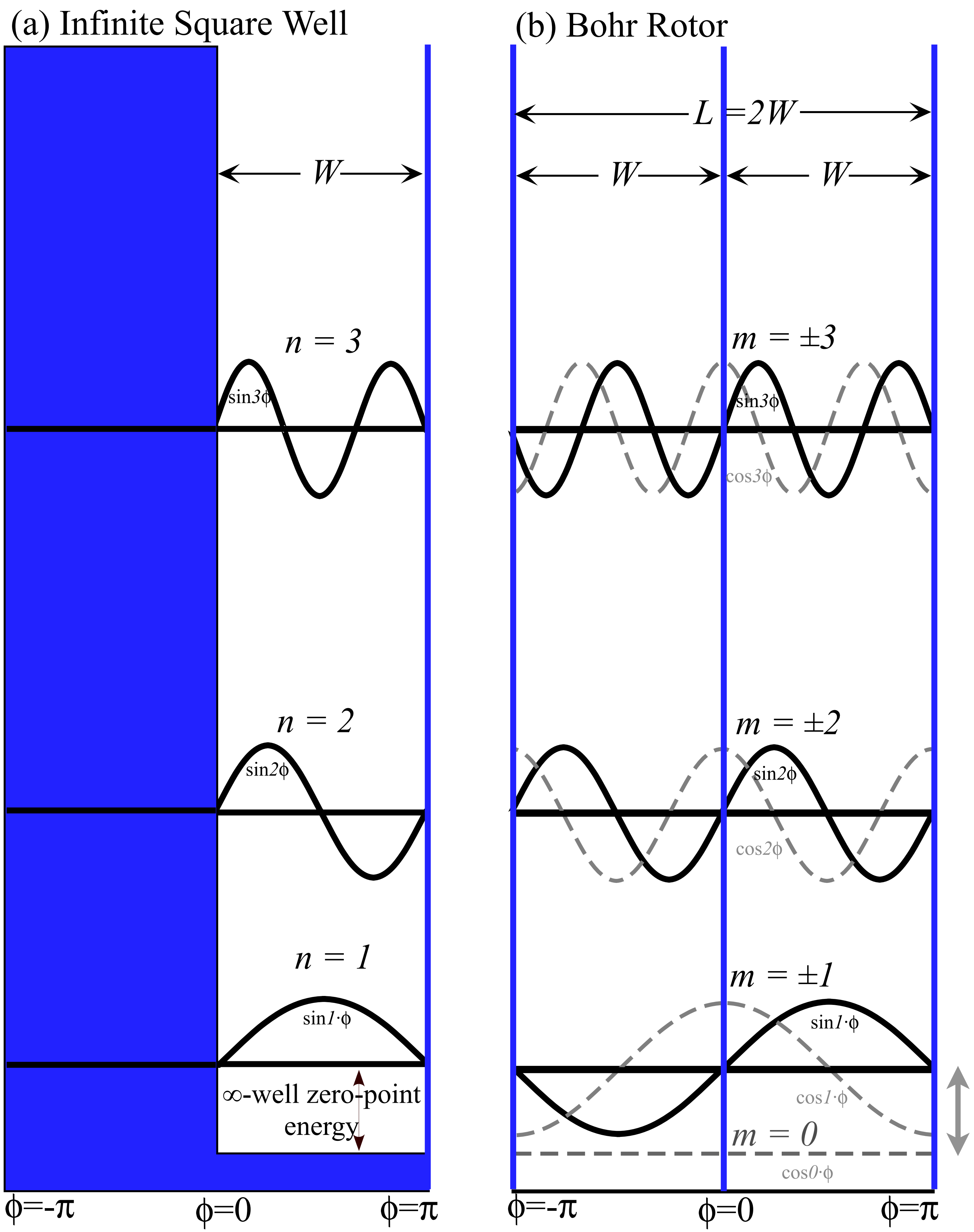


Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

$(k_m=m \quad if: L=2\pi)$

$$E_m= (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$

$$= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency

$\omega_1=2\pi\upsilon_1$

lowest *transition (beat) frequency*

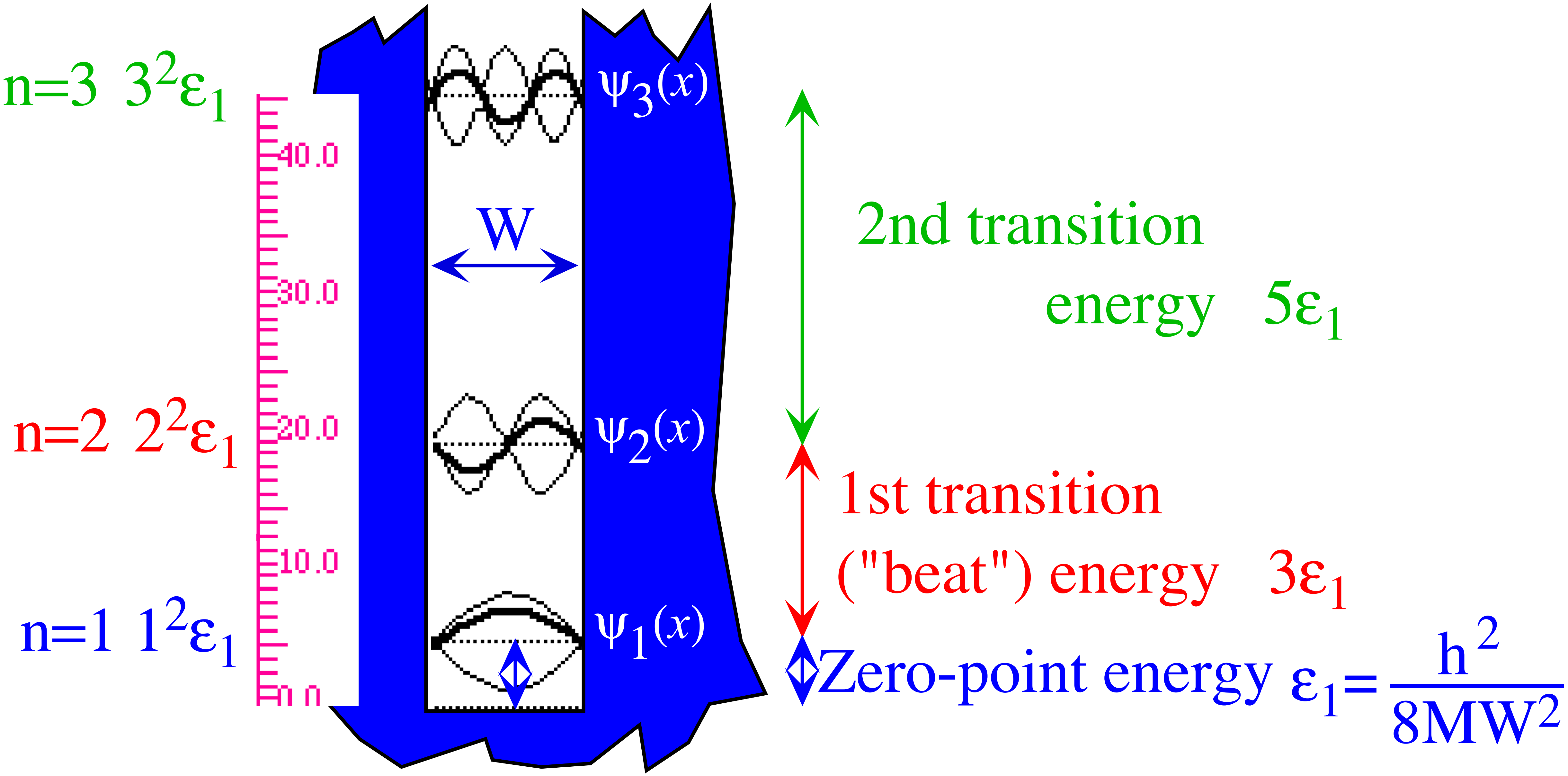
$\upsilon_1 =(E_1-E_0)/h$ *(E_0 is defined as zero)*

Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
  Wave coordinates
  Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
  ∞-Square well PE versus Bohr rotor
  → *SinNx/x wavepackets bandwidth and uncertainty*
    SinNx/x explosion and revivals
  Gaussian wave-packet bandwidth and uncertainty
    Gaussian revivals
  Farey-Sums and Ford-products
  Phase dynamics

∞-*Square well PE versus Bohr rotor*



n=3 $3^2\varepsilon_1$

n=2 $2^2\varepsilon_1$

n=1 $1^2\varepsilon_1$

2nd transition energy $5\varepsilon_1$

1st transition ("beat") energy $3\varepsilon_1$

Zero-point energy $\varepsilon_1 = \frac{h^2}{8MW^2}$

∞-Square well PE versus Bohr rotor

$kW = n\pi \quad or: \; k = n\pi/W$

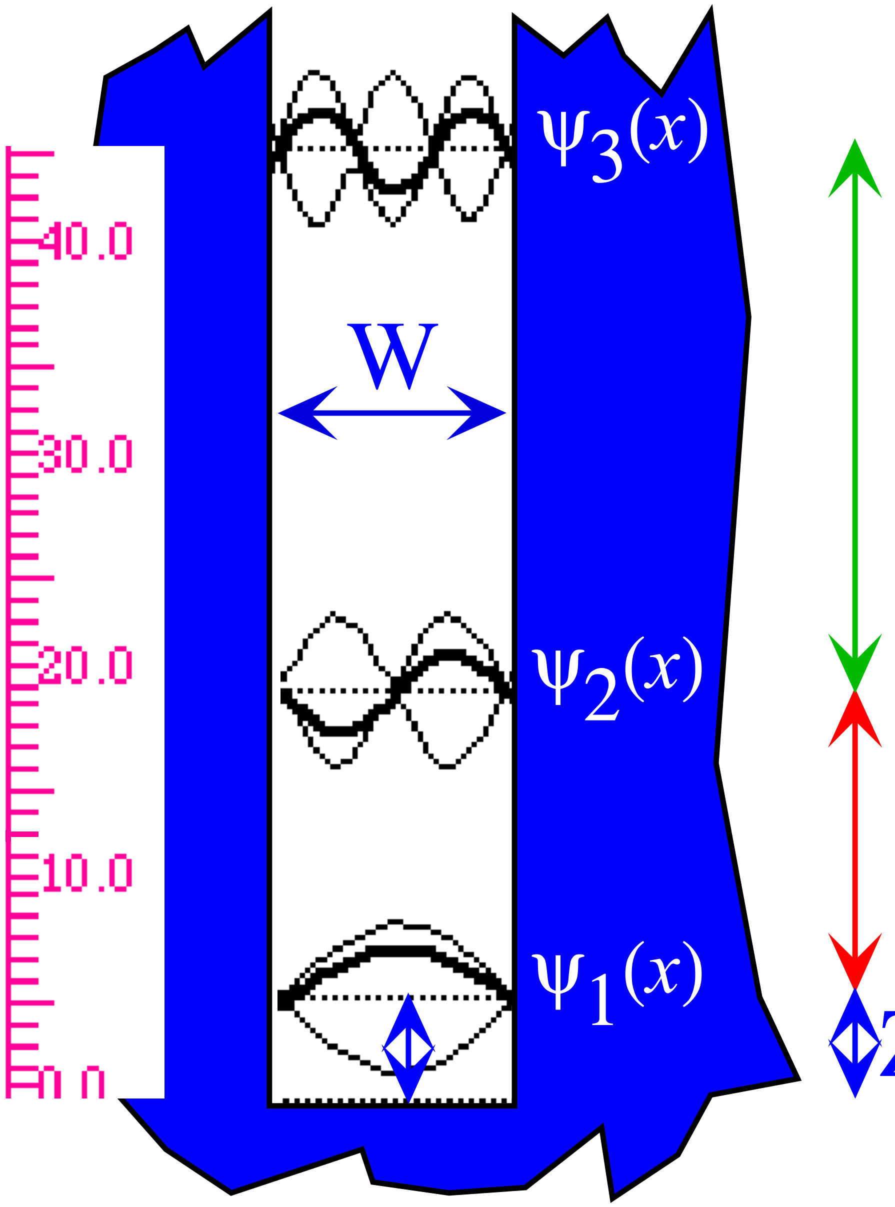
$$\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi x}{W}\right) \quad (n=1,2,3,\ldots\infty)$$

$$\varepsilon_n = \frac{\hbar^2}{2M}k^2 = \frac{\hbar^2 n^2\pi^2}{2MW^2} = \left(1^2, 2^2, 3^2, \ldots \text{or } n^2\right)\frac{h^2}{8MW^2}$$

n=3 $3^2\varepsilon_1$

n=2 $2^2\varepsilon_1$

n=1 $1^2\varepsilon_1$



2nd transition

energy $5\varepsilon_1$

1st transition

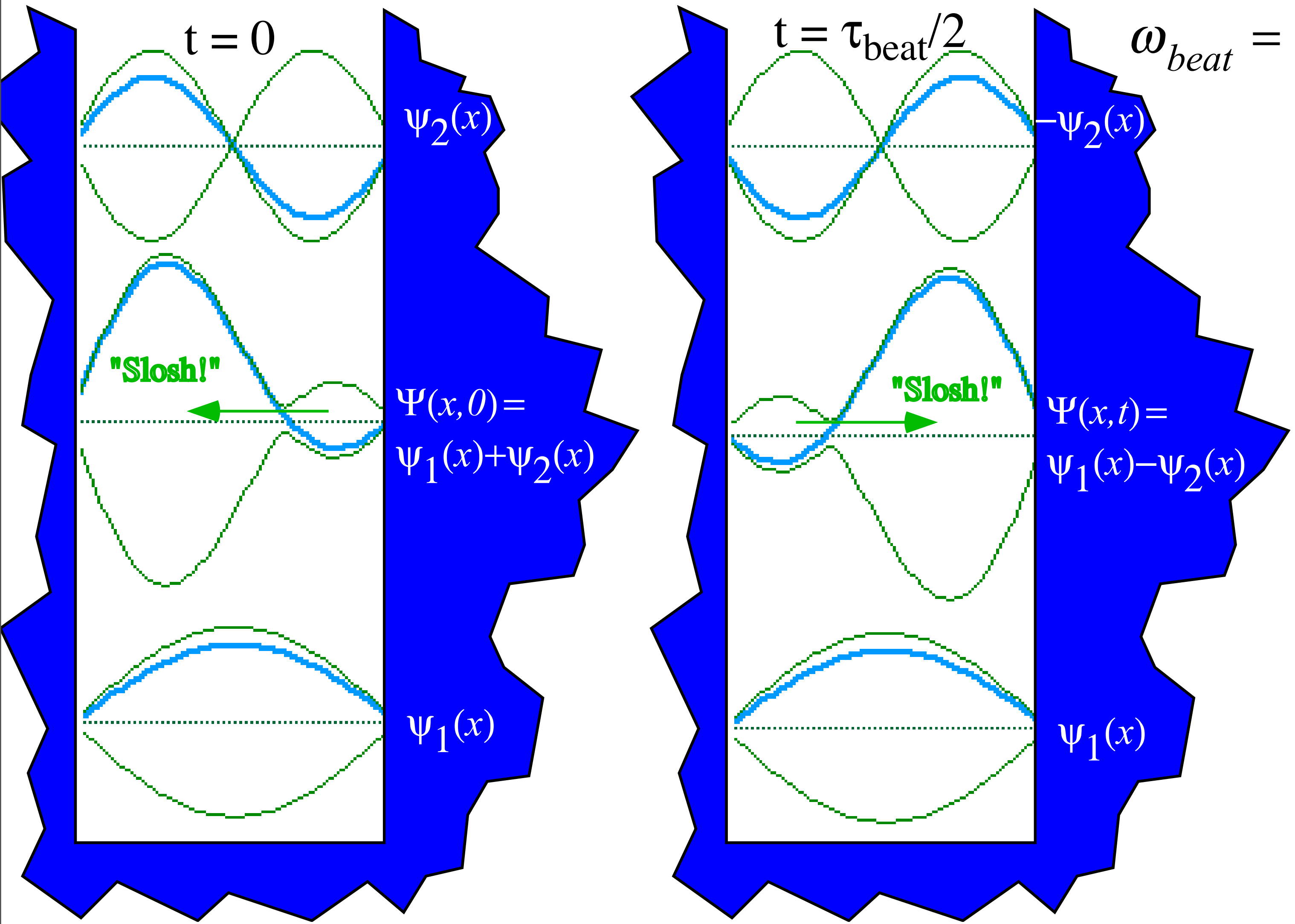
("beat") energy $3\varepsilon_1$

Zero-point energy $\varepsilon_1 = \frac{h^2}{8MW^2}$

∞-Square well PE versus Bohr rotor

$$kW = n\pi \quad or: \; k = n\pi/W$$

$$\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi\, x}{W}\right) \quad (n=1,2,3,\ldots\infty)$$



$$\omega_{beat} = \omega_2\text{-}\omega_1 = \frac{\varepsilon_2 - \varepsilon_1}{\hbar} = \frac{2^2 - 1^2}{\hbar}\,\frac{h^2}{8MW^2} = 3\,\frac{2\pi\, h}{8MW^2} = 3\,\omega_1$$

Fig. 12.1.2 Exercise in prison. Infinite square well eigensolution combination "sloshes" back and forth.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\Sigma_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle=\Sigma_{n=1}^{\infty}a_n\sin k_n x$$

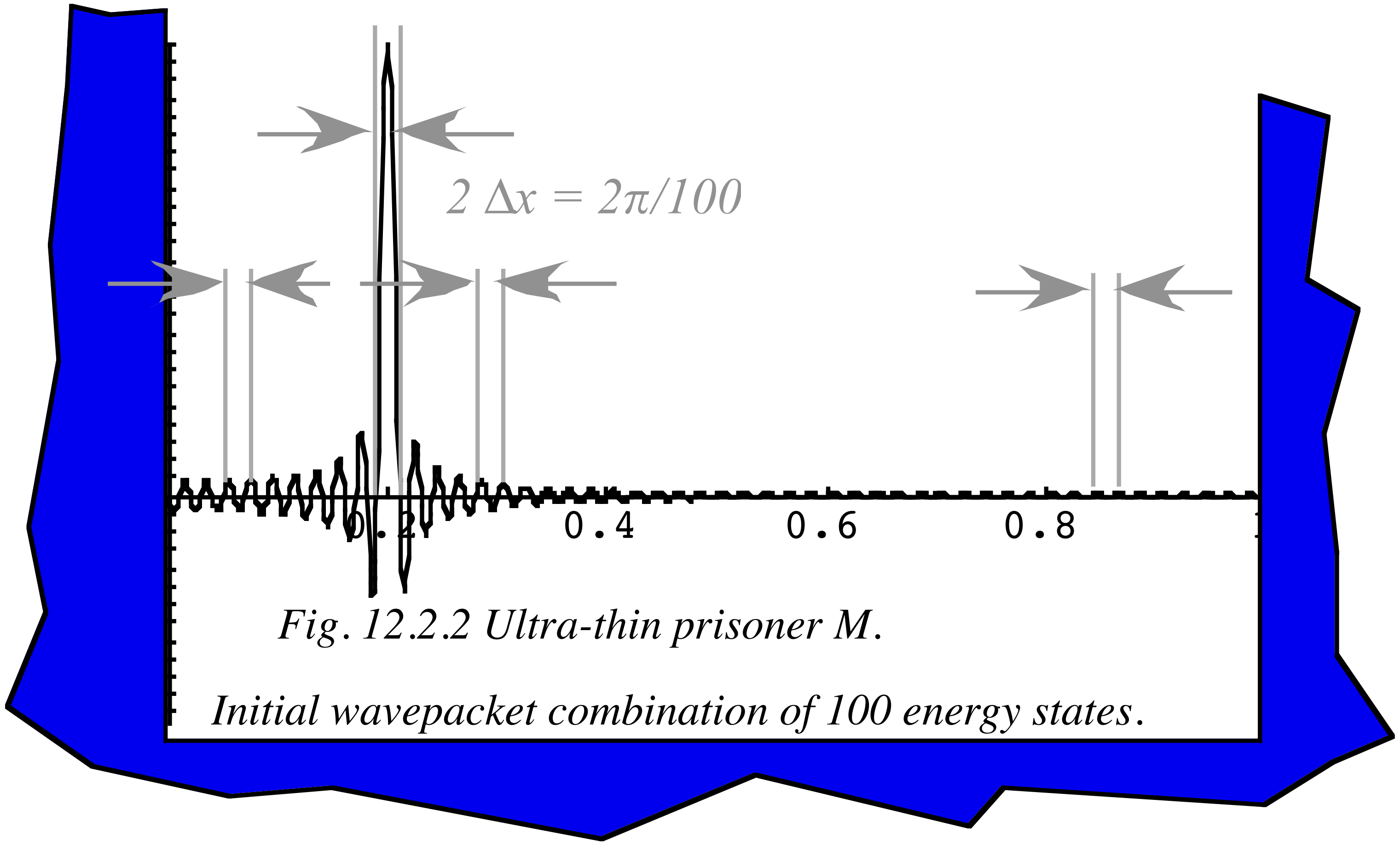


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

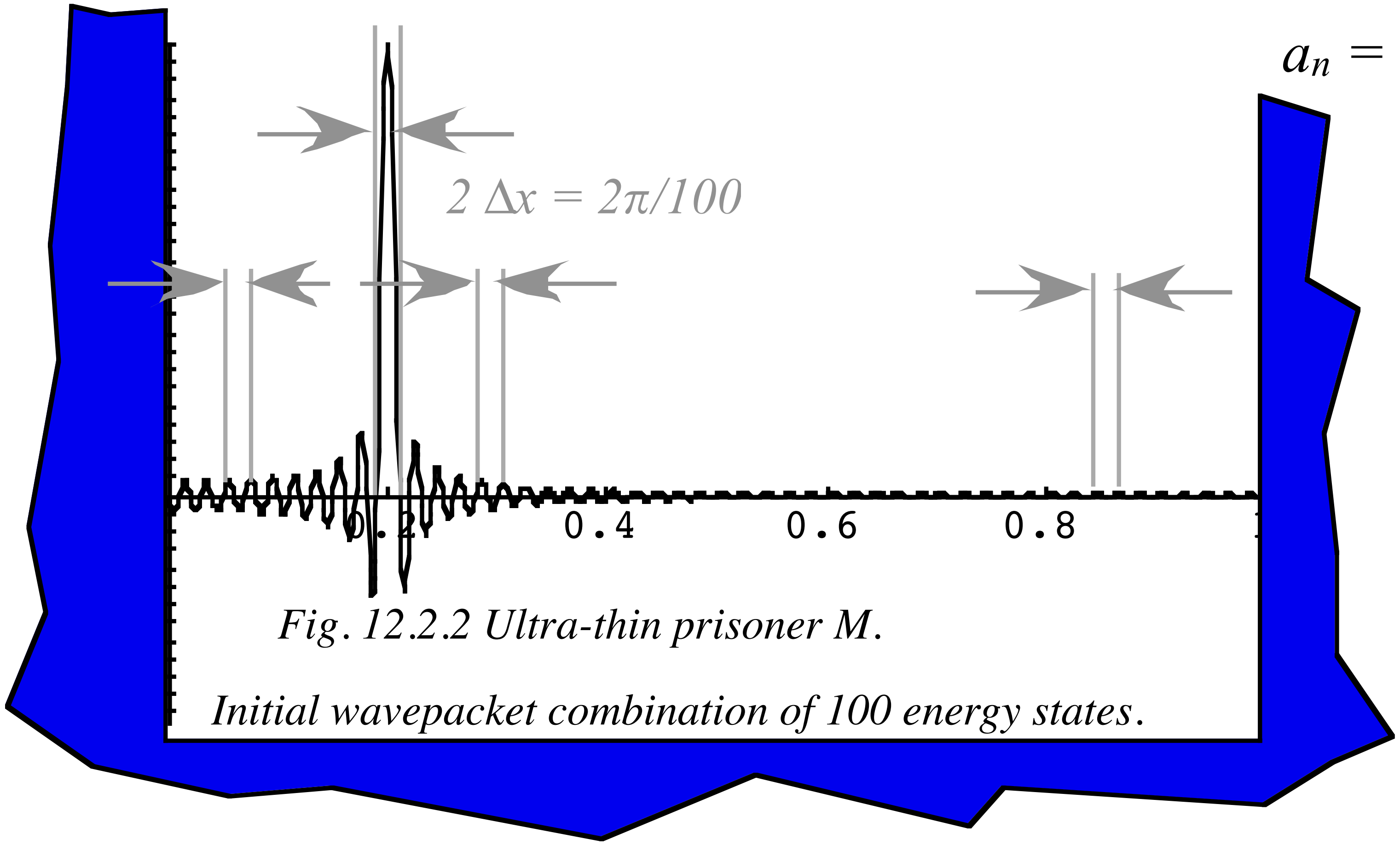


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

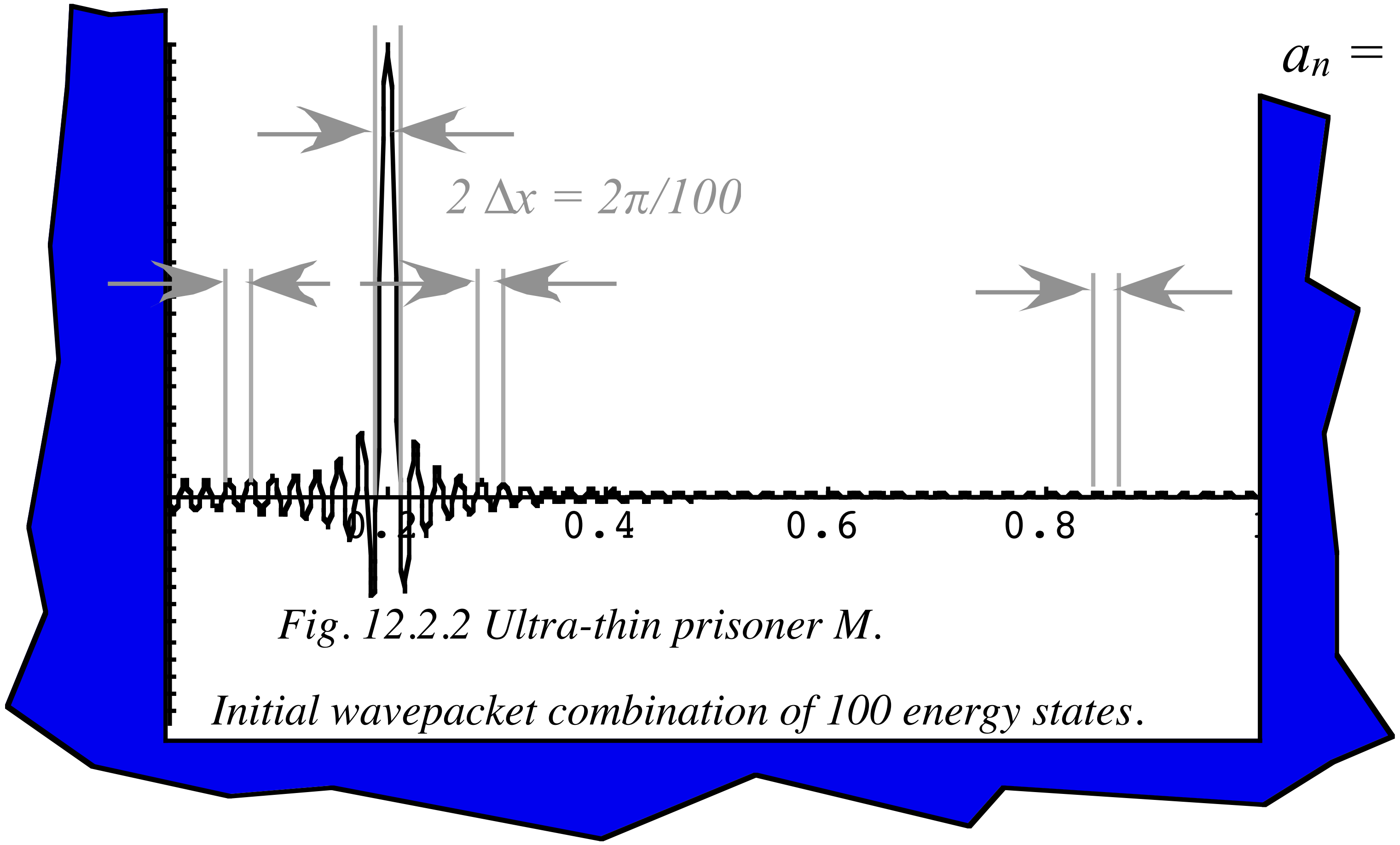


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$$

$$\Psi\left(x\right)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a \sin k_n x$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\textstyle\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

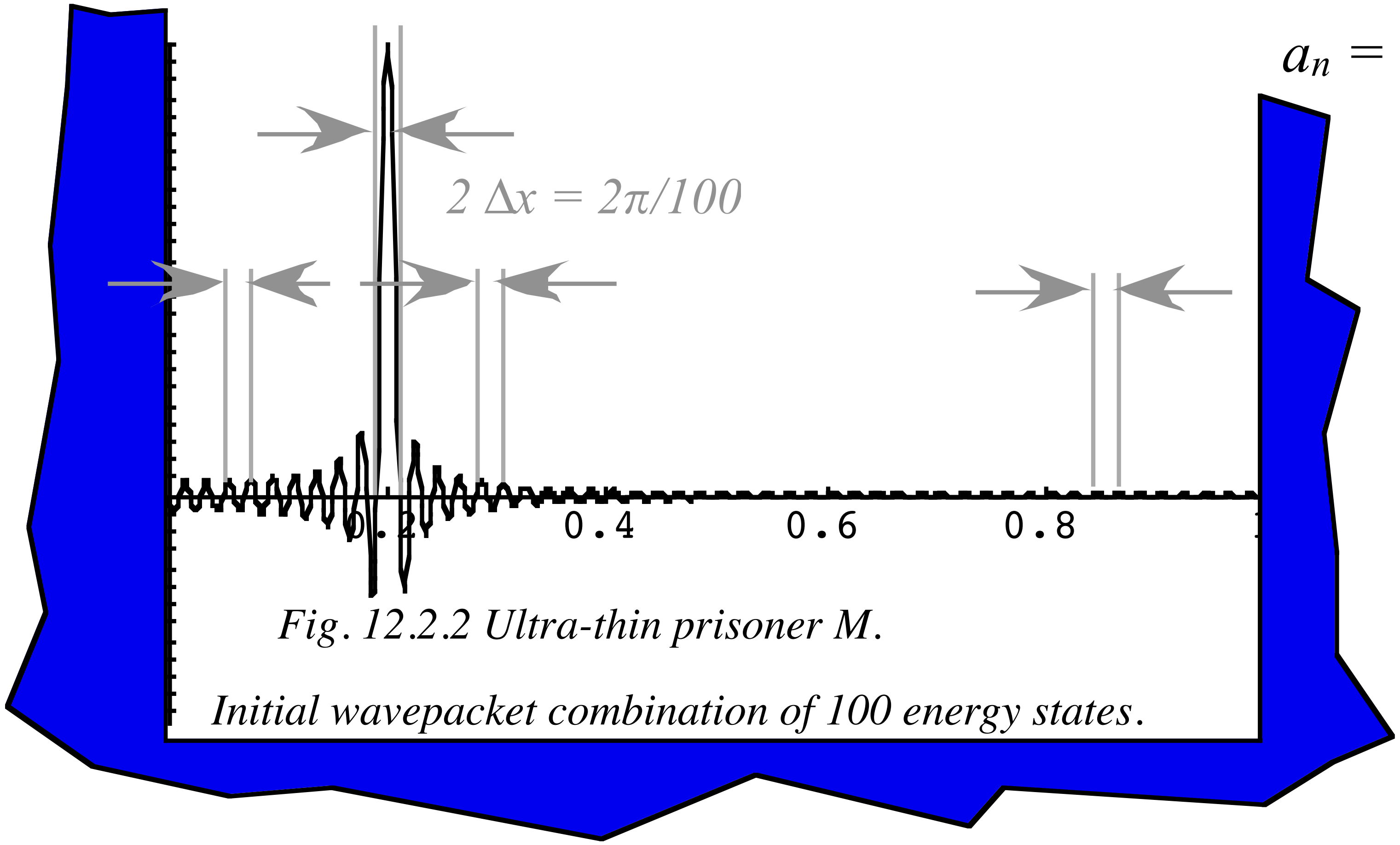


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_{0}^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\sum_{n=1}^{\infty}\left\langle x\middle|\varepsilon_n\right\rangle\left\langle\varepsilon_n\middle|a\right\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

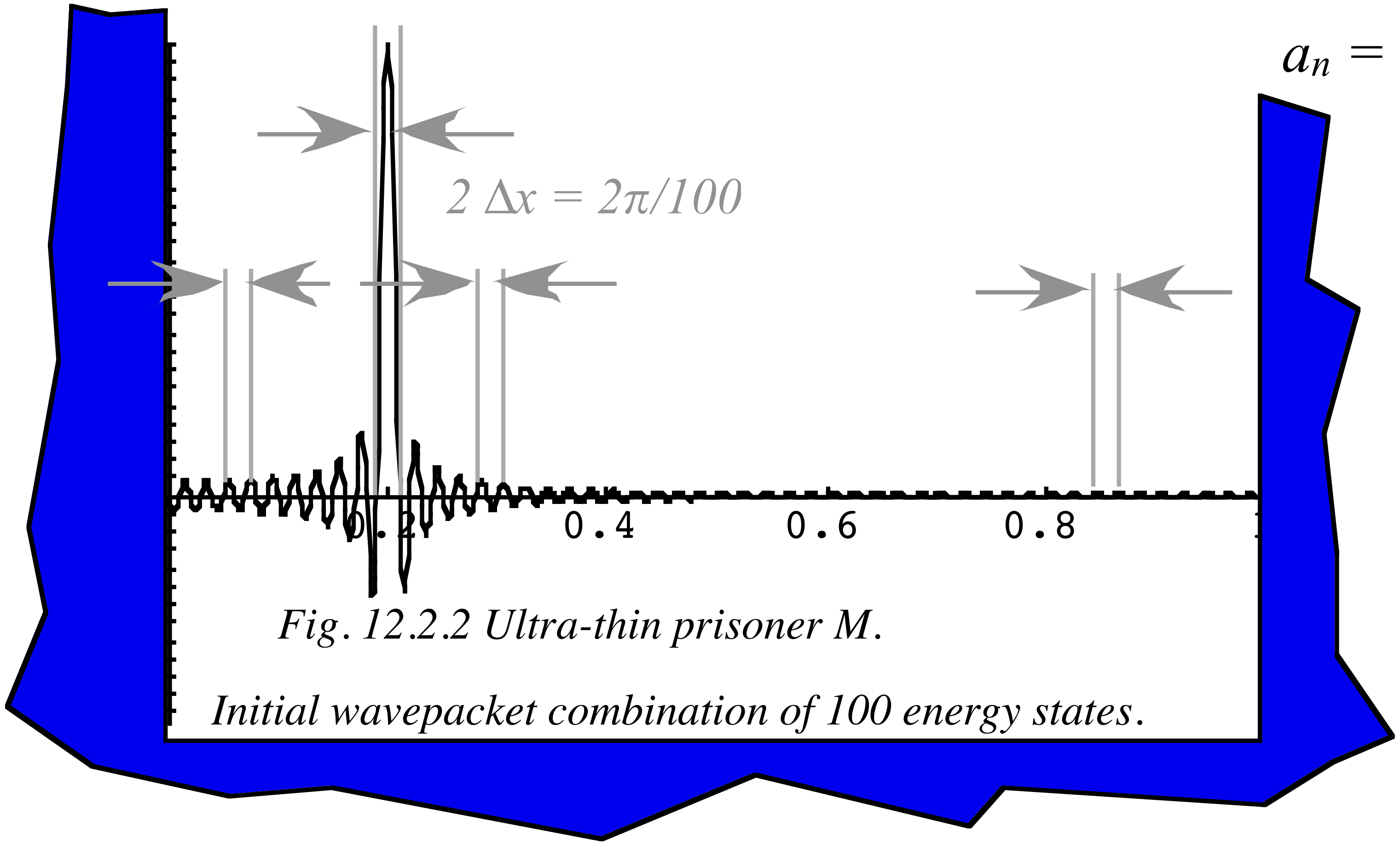


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi\left(x\right)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_{0}^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_{0}^{K_{\max}}dk\sin ka\sin kx$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\Sigma_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\Sigma_{n=1}^{\infty}a_n\sin k_n x$$

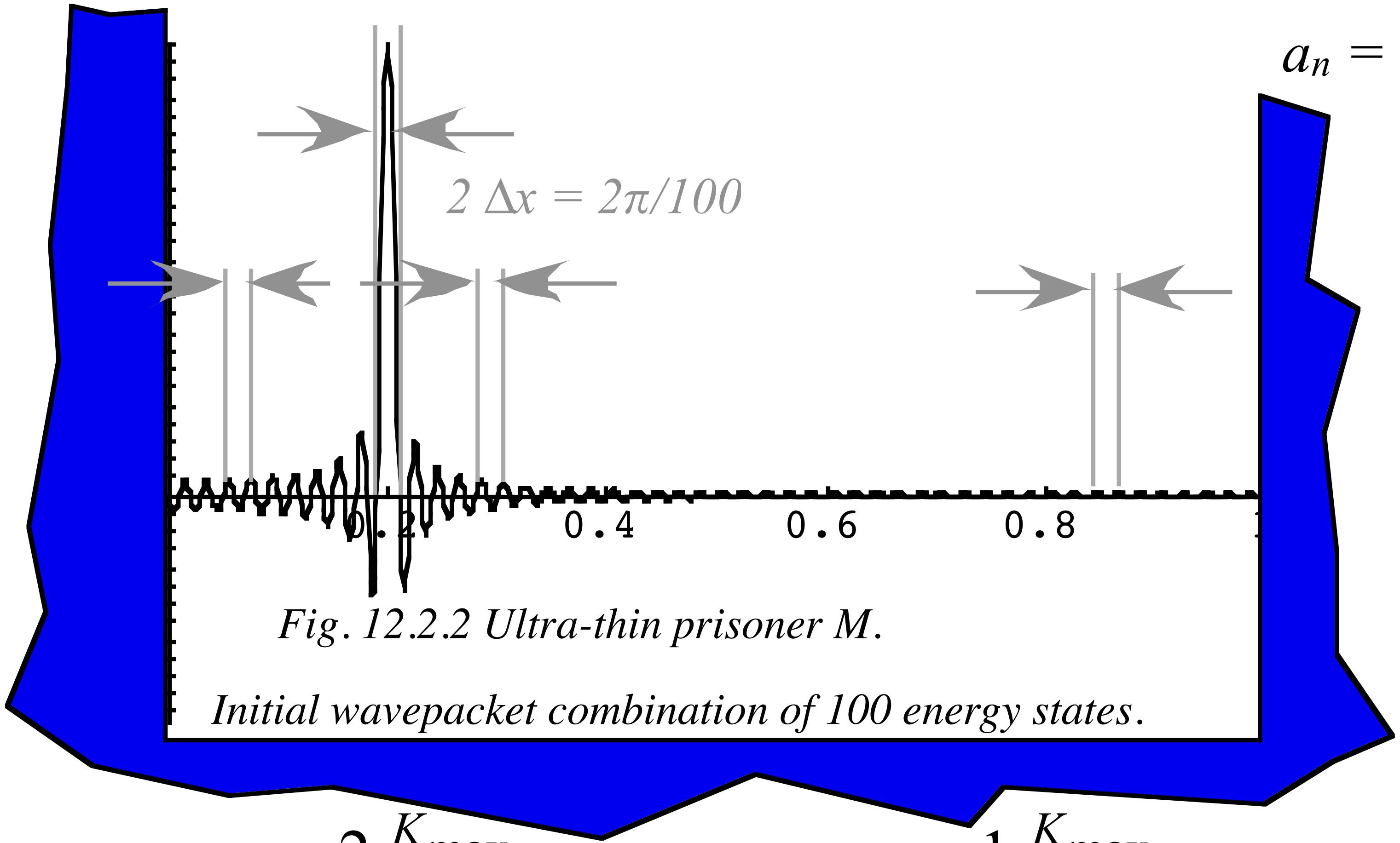


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{2}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx=\frac{1}{\pi}\int_0^{K_{\max}}dk\left(\cos k(x-a)-\cos k(x+a)\right)$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

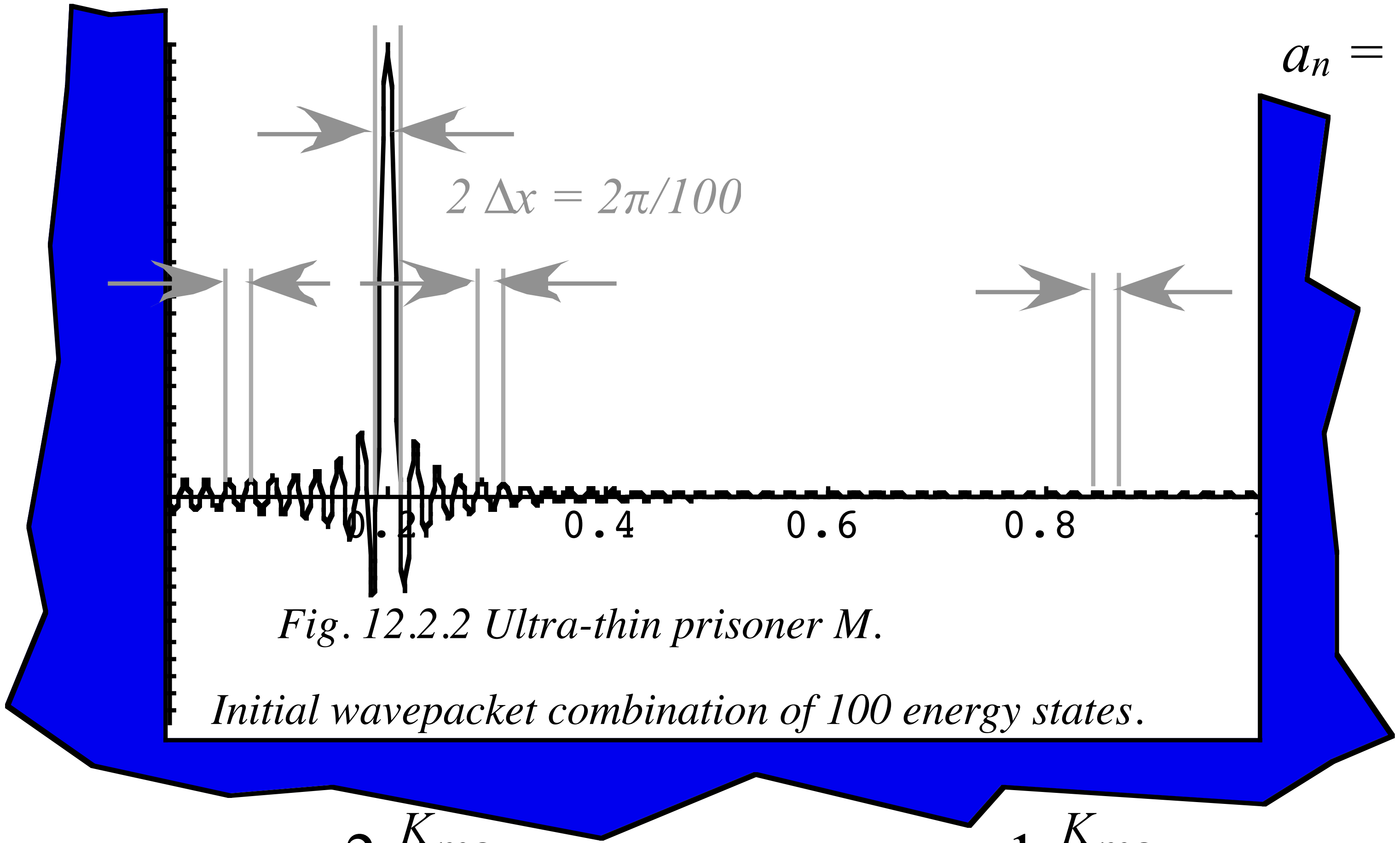


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\,\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\,\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\,\sin ka\,\sin kx$$

$$\Psi(x)\cong\frac{2}{\pi}\int_0^{K_{\max}}dk\,\sin ka\,\sin kx=\frac{1}{\pi}\int_0^{K_{\max}}dk\big(\cos k(x-a)-\cos k(x+a)\big)$$

$$\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}-\frac{\sin K_{\max}(x+a)}{\pi(x+a)}\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}\quad\text{for: } x\approx a$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a) = \langle x|a\rangle = \sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$

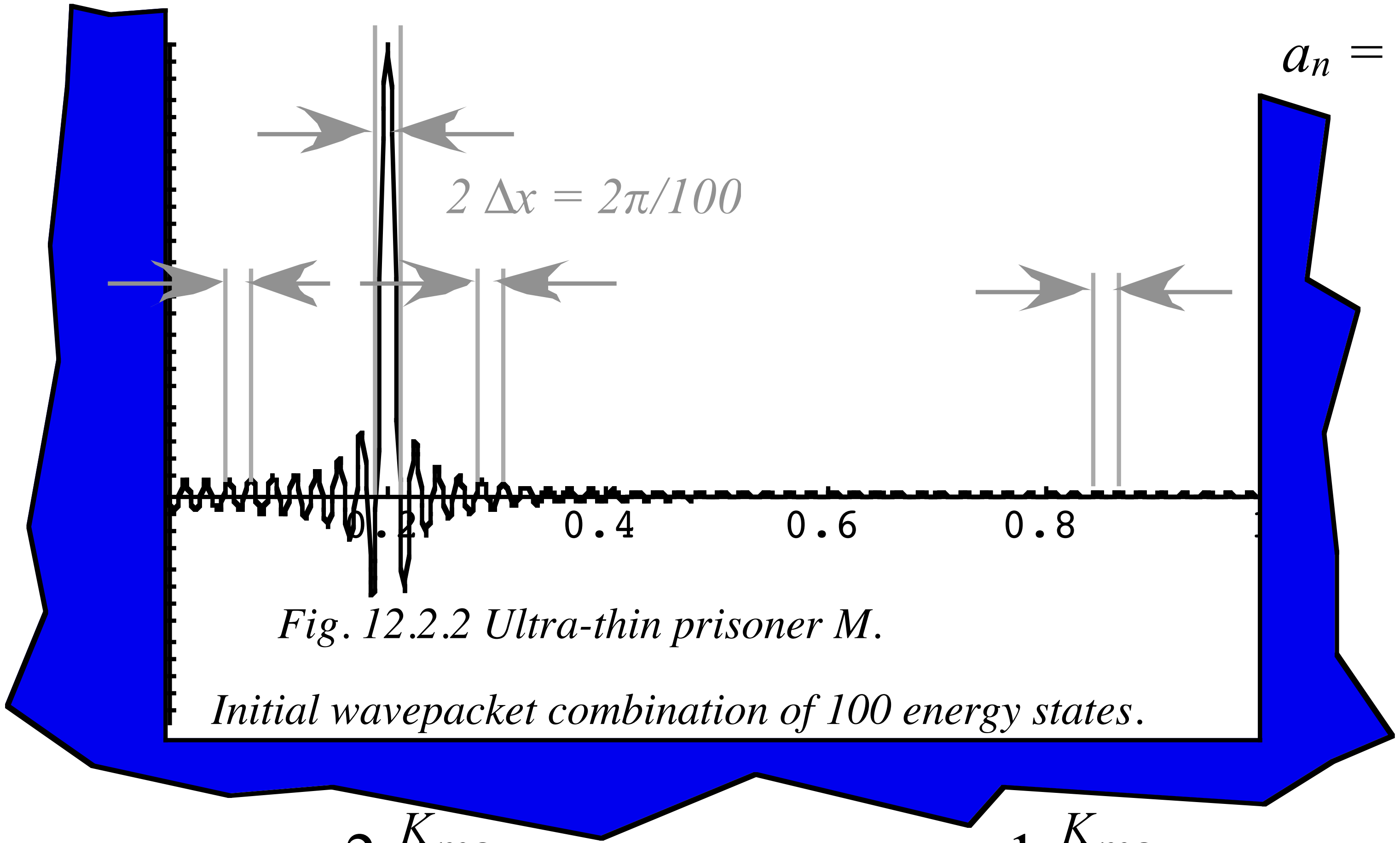


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x) = \frac{2}{W}\sum_{n}^{N_{\max}} \sin k_n a \sin k_n x$$

$$\rightarrow \frac{2}{W}\int_0^{K_{\max}} dk \frac{\Delta n}{\Delta k} \sin ka \sin kx$$

$$= \frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}} dk \sin ka \sin kx$$

$$\Psi(x) \cong \frac{2}{\pi}\int_0^{K_{\max}} dk \sin ka \sin kx = \frac{1}{\pi}\int_0^{K_{\max}} dk\left(\cos k(x-a) - \cos k(x+a)\right)$$

$$\cong \frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)} - \frac{\sin K_{\max}(x+a)}{\pi(x+a)} \cong \frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)} \quad \text{for: } x \approx a$$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a) = \langle x|a\rangle = \sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$

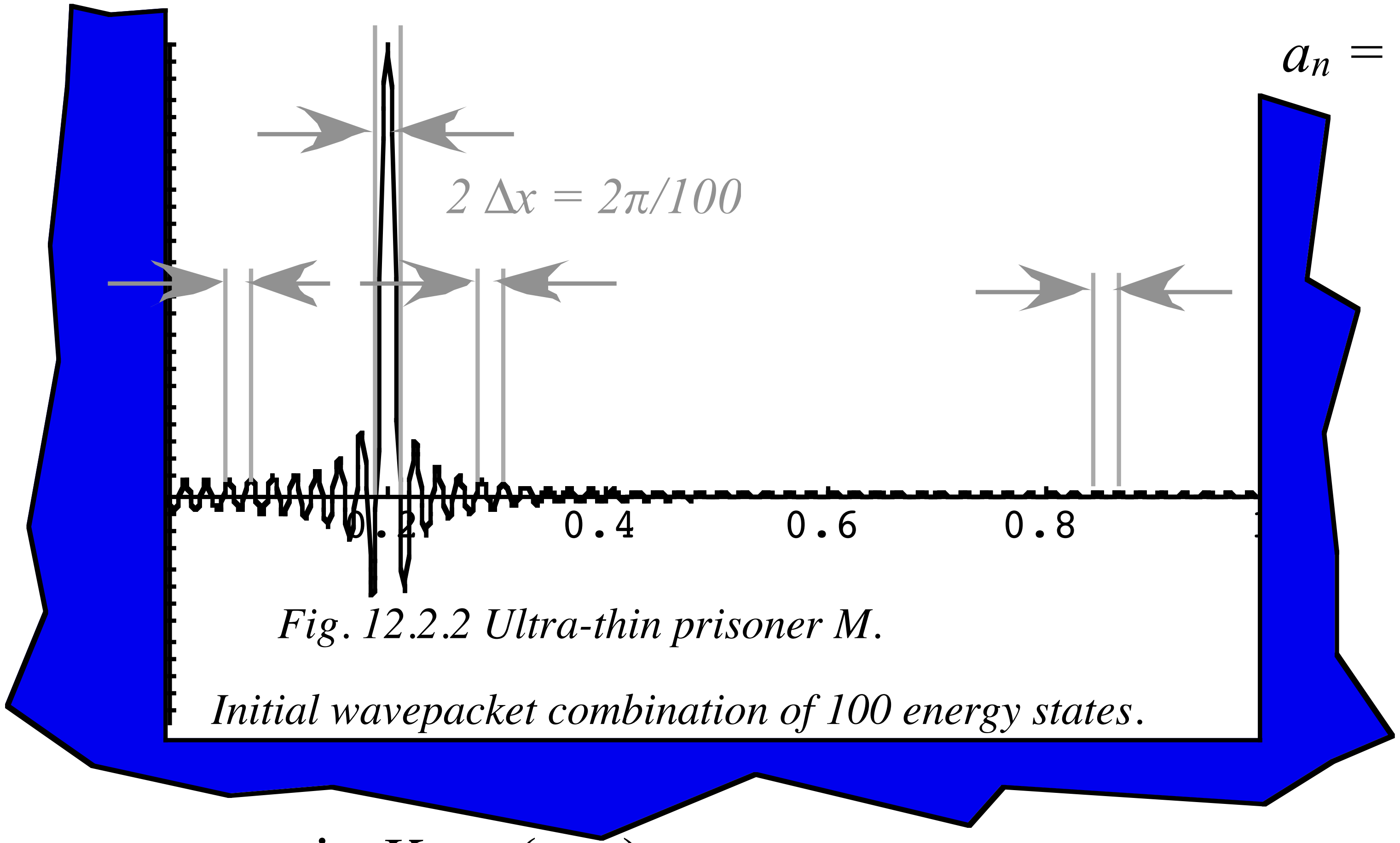


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$$

$$\Psi(x) = \frac{2}{W}\sum_{n}^{N_{\max}} \sin k_n a \sin k_n x$$

$$\rightarrow \frac{2}{W}\int_{0}^{K_{\max}} dk \frac{\Delta n}{\Delta k} \sin ka \sin kx$$

$$= \frac{2}{W}\frac{W}{\pi}\int_{0}^{K_{\max}} dk \sin ka \sin kx$$

$$\Psi(x) \cong \frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)} \quad \text{for: } x \approx a$$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and “inside” K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

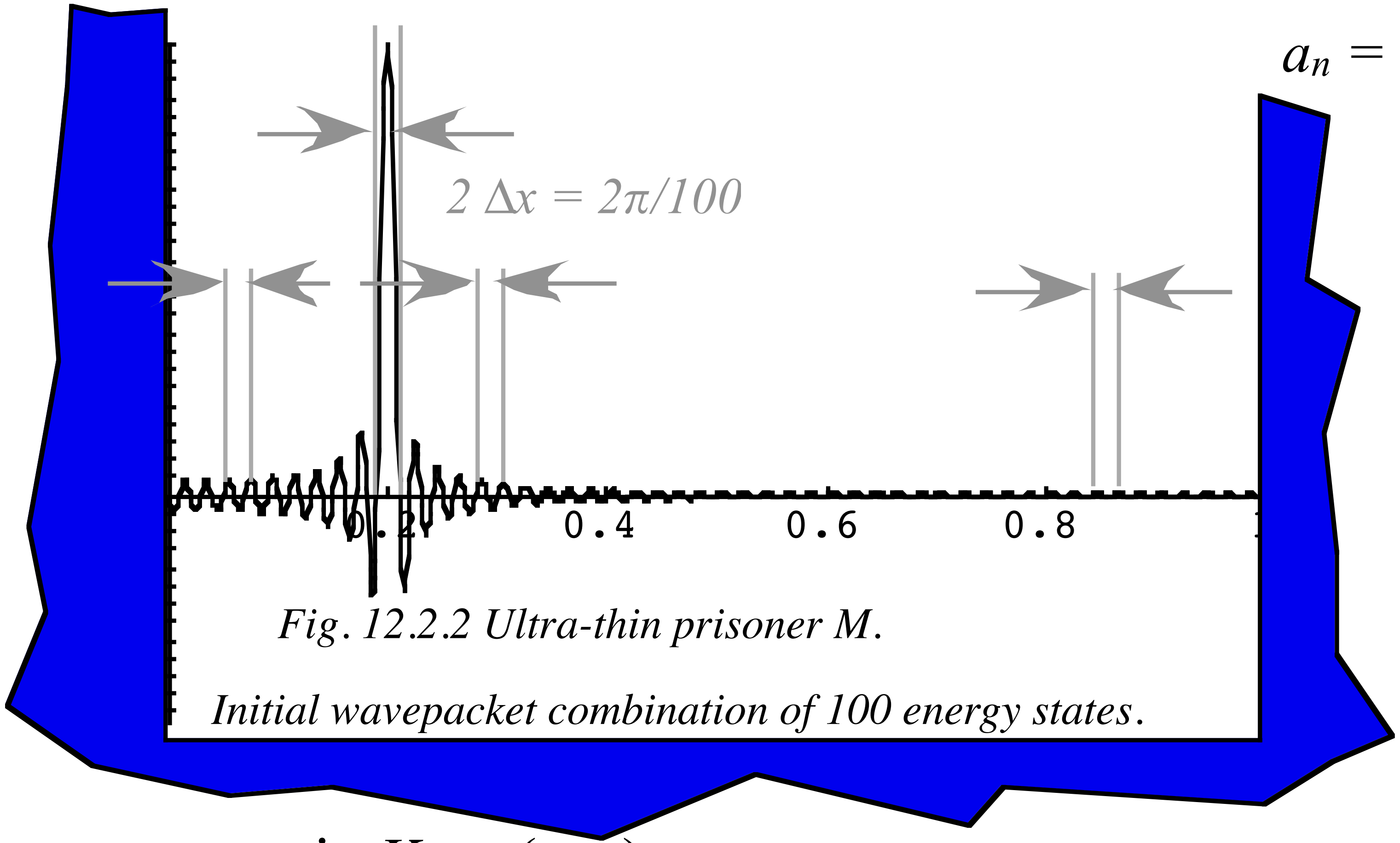


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)} \quad \text{for: } x\approx a$$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width* Δx

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

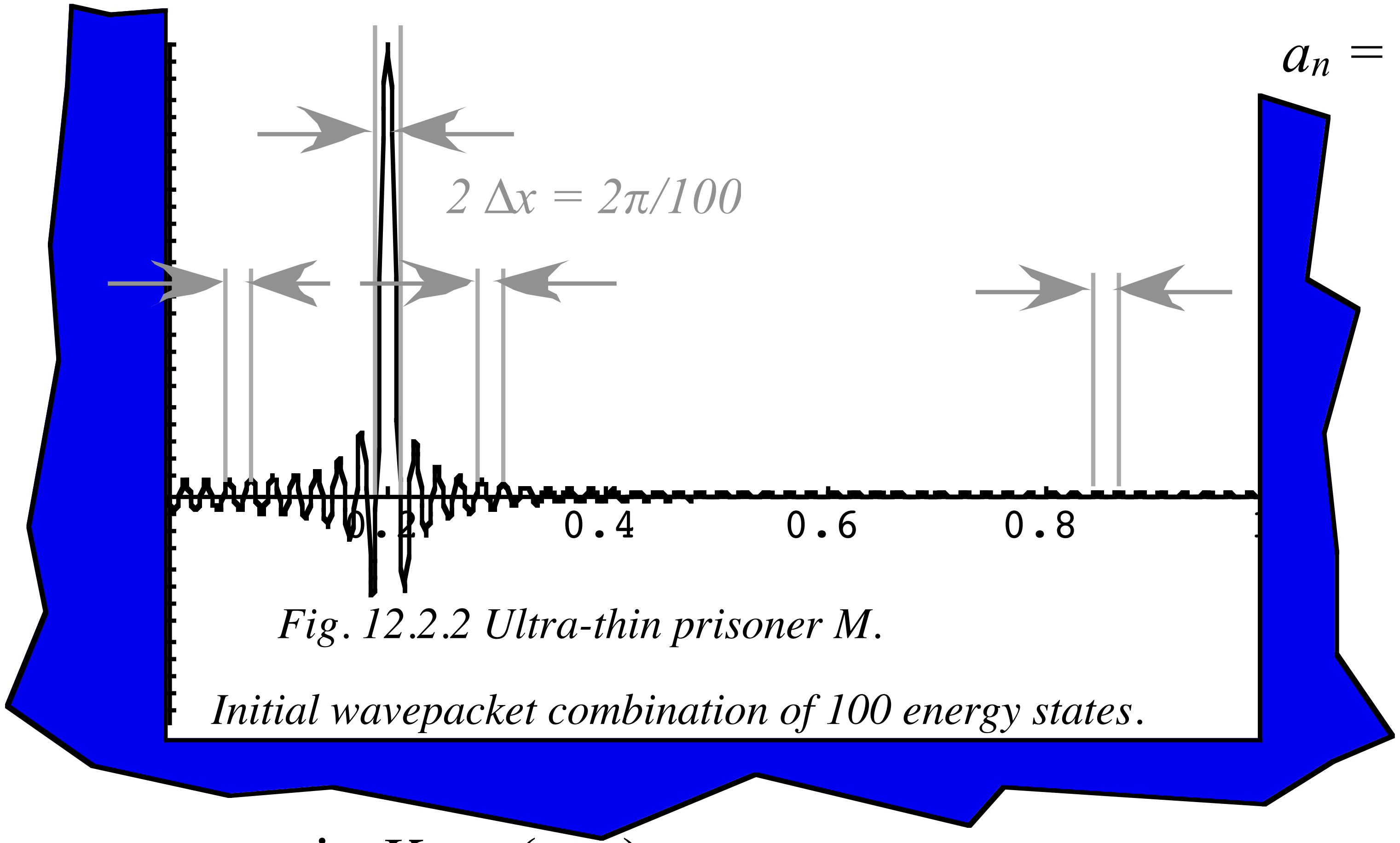


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}\quad\text{for: } x\approx a$$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width Δx*

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and “inside” K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\sum_{n=1}^{\infty}a_n\sin k_n x$$

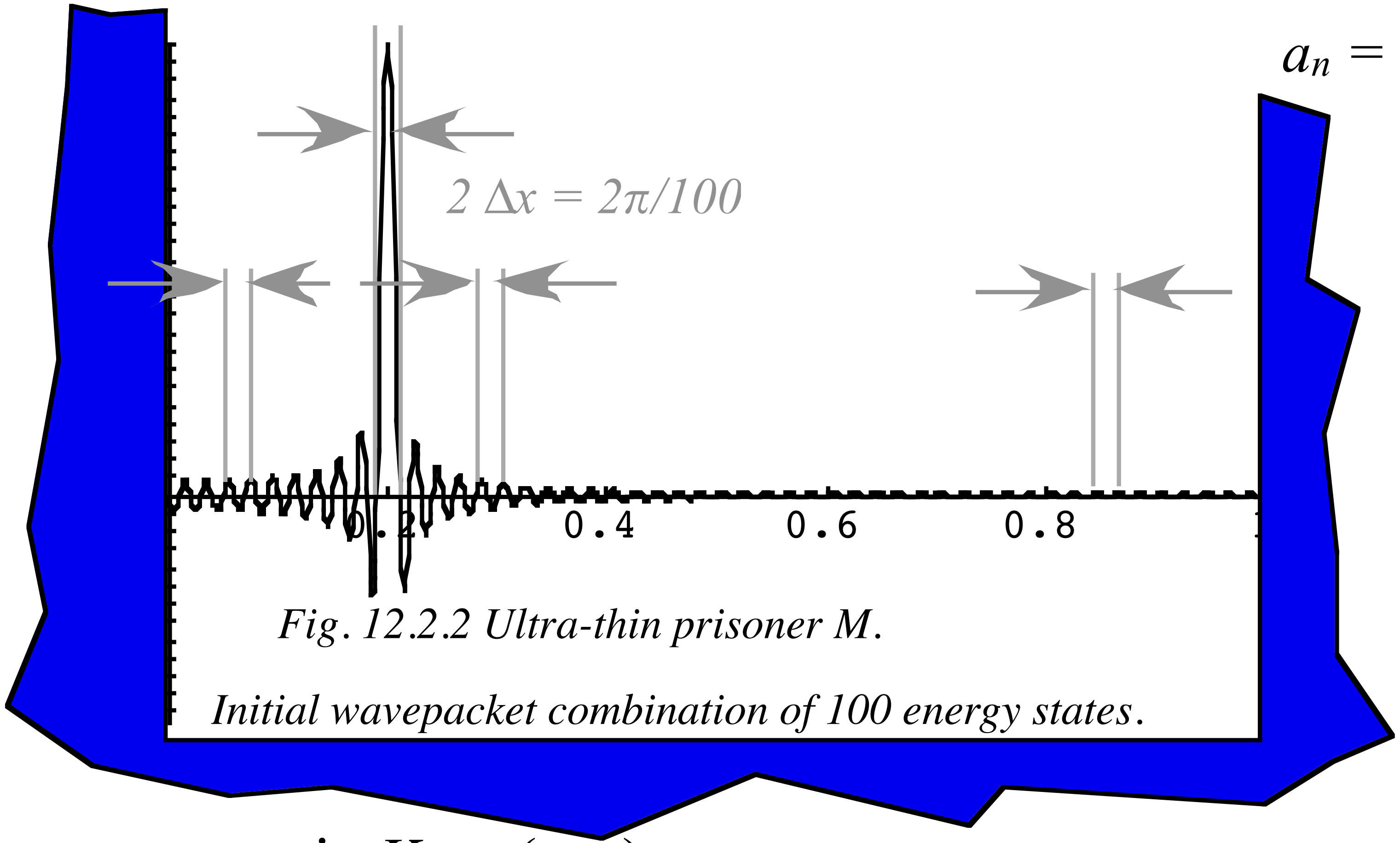


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}\quad\text{for: } x\approx a$$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width* Δx

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$, or: $\Delta x=\pm\pi/K_{\max}$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\Sigma_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\Sigma_{n=1}^{\infty}a_n\sin k_n x$$

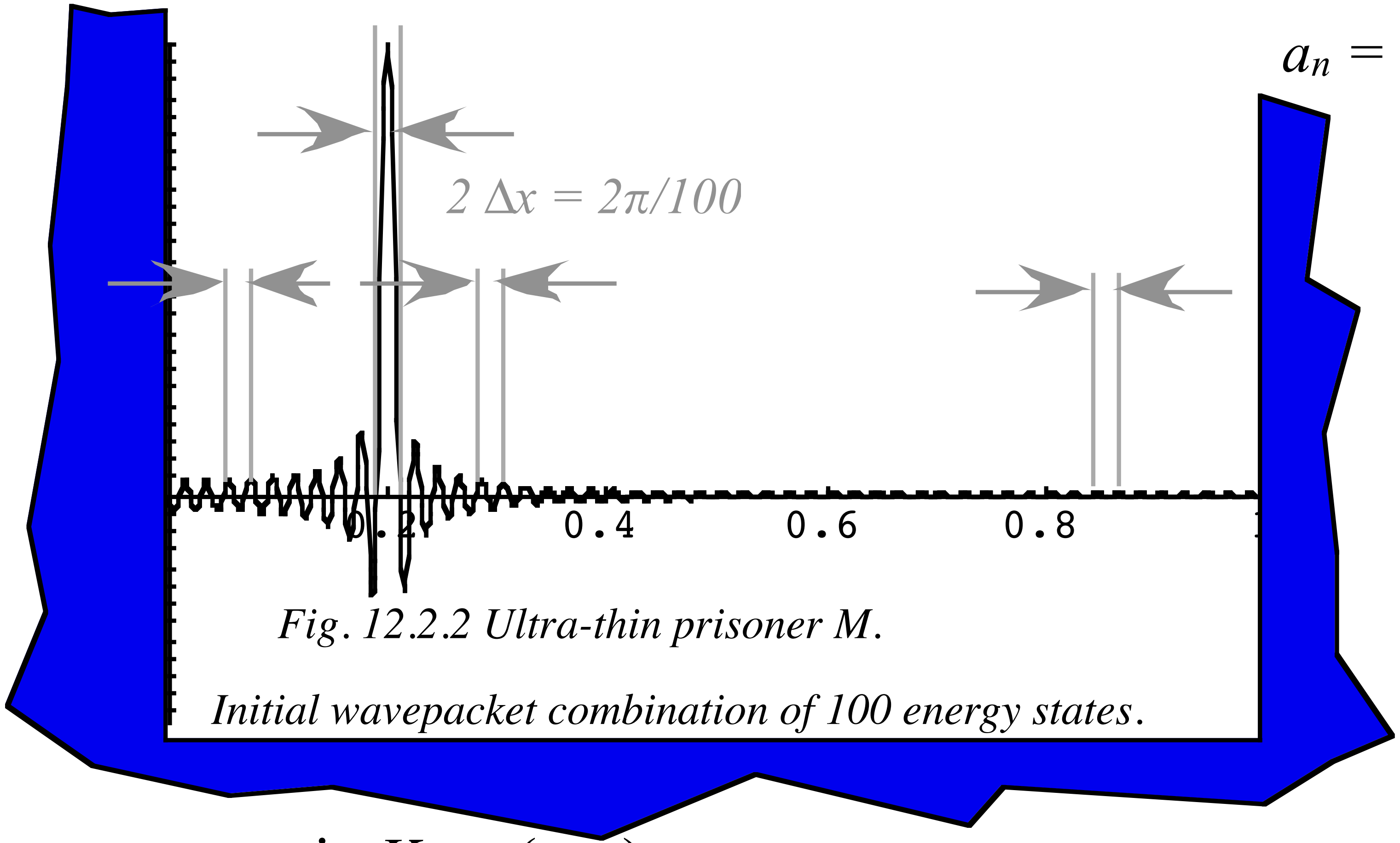


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}\quad\text{for: } x\approx a$$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width Δx*

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$, or: $\Delta x=\pm\pi/K_{\max}$

$$\Delta x\cdot|K_{\max}|=\Delta x\cdot\Delta k=\pi$$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\Sigma_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\Sigma_{n=1}^{\infty}a_n\sin k_n x$$

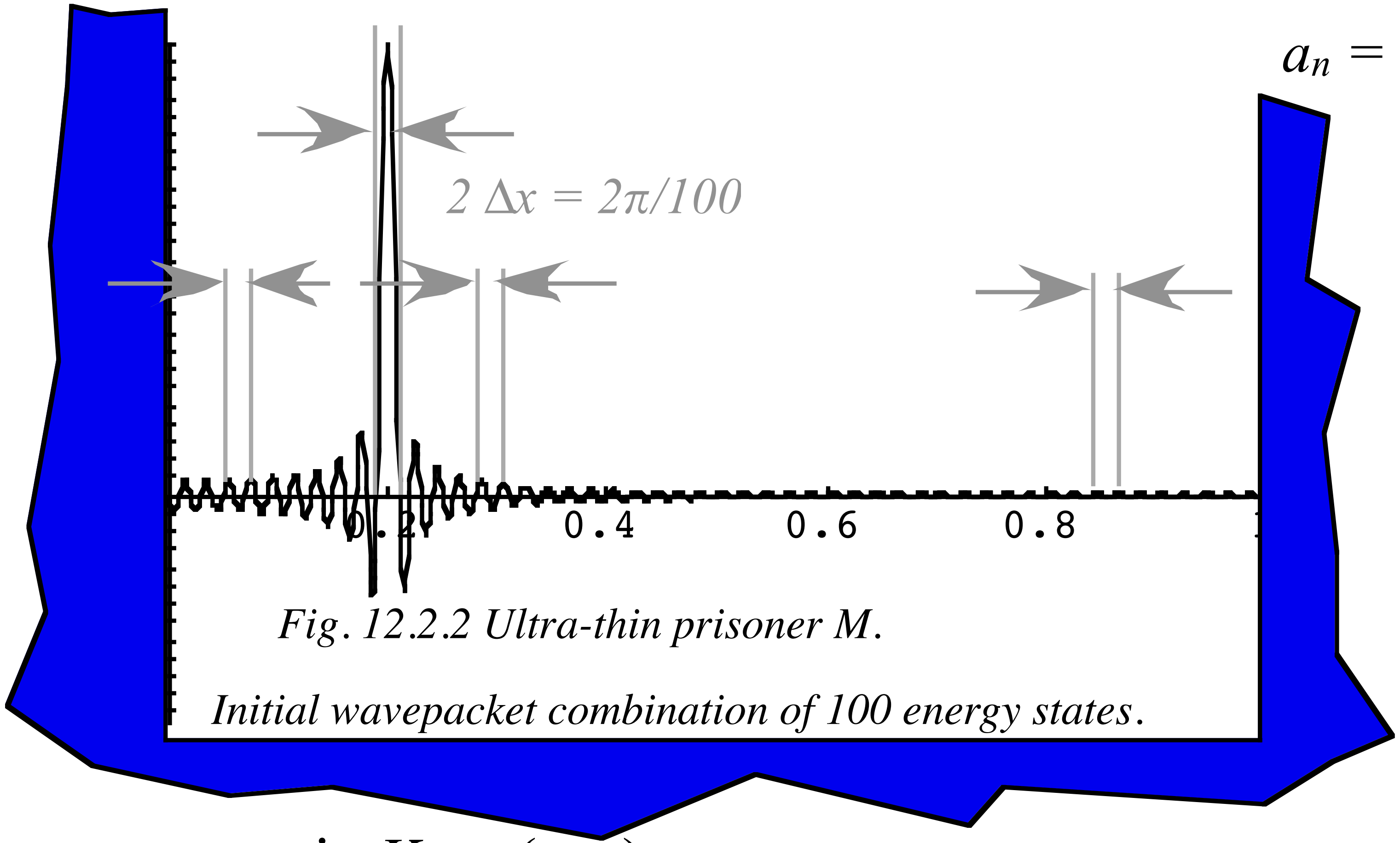


Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle= (2/W) \sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_0^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_0^{K_{\max}}dk\sin ka\sin kx$$

$\Psi(x)\cong\dfrac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}$ for: $x\approx a$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width* Δx

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$, or: $\Delta x=\pm\pi/K_{\max}$

$\Delta x\cdot|K_{\max}|=\Delta x\cdot\Delta k=\pi$ or: $\boxed{\Delta x\cdot\Delta p=\pi\hbar=h/2 \quad \infty\text{-Well uncertainty relation}}$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
Geometry

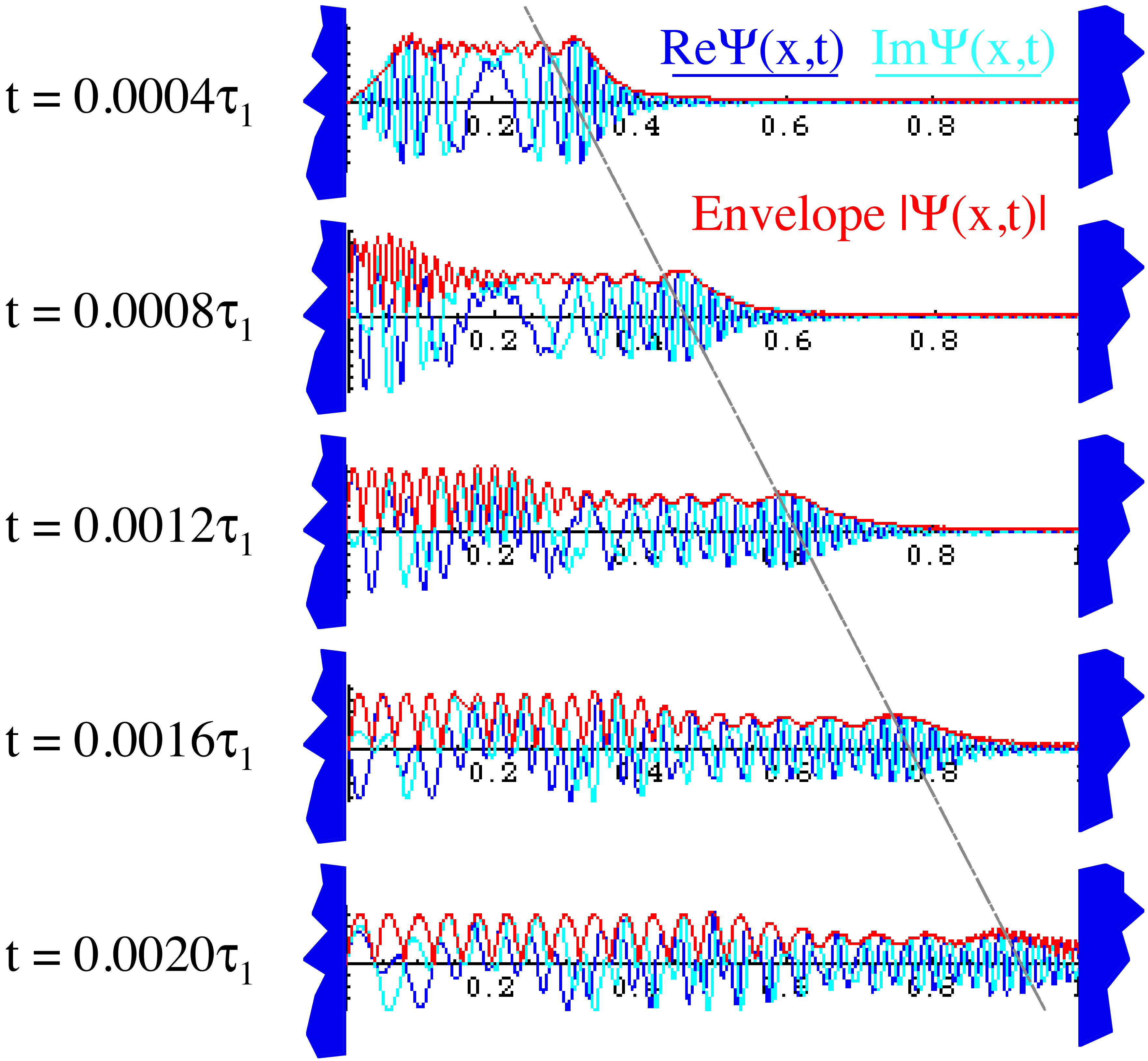
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
    Wave coordinates
    Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
    ∞-Square well PE versus Bohr rotor
    SinNx/x wavepackets bandwidth and uncertainty
→   *SinNx/x explosion and revivals*
    Gaussian wave-packet bandwidth and uncertainty
        Gaussian revivals
    Farey-Sums and Ford-products
    Phase dynamics

Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$



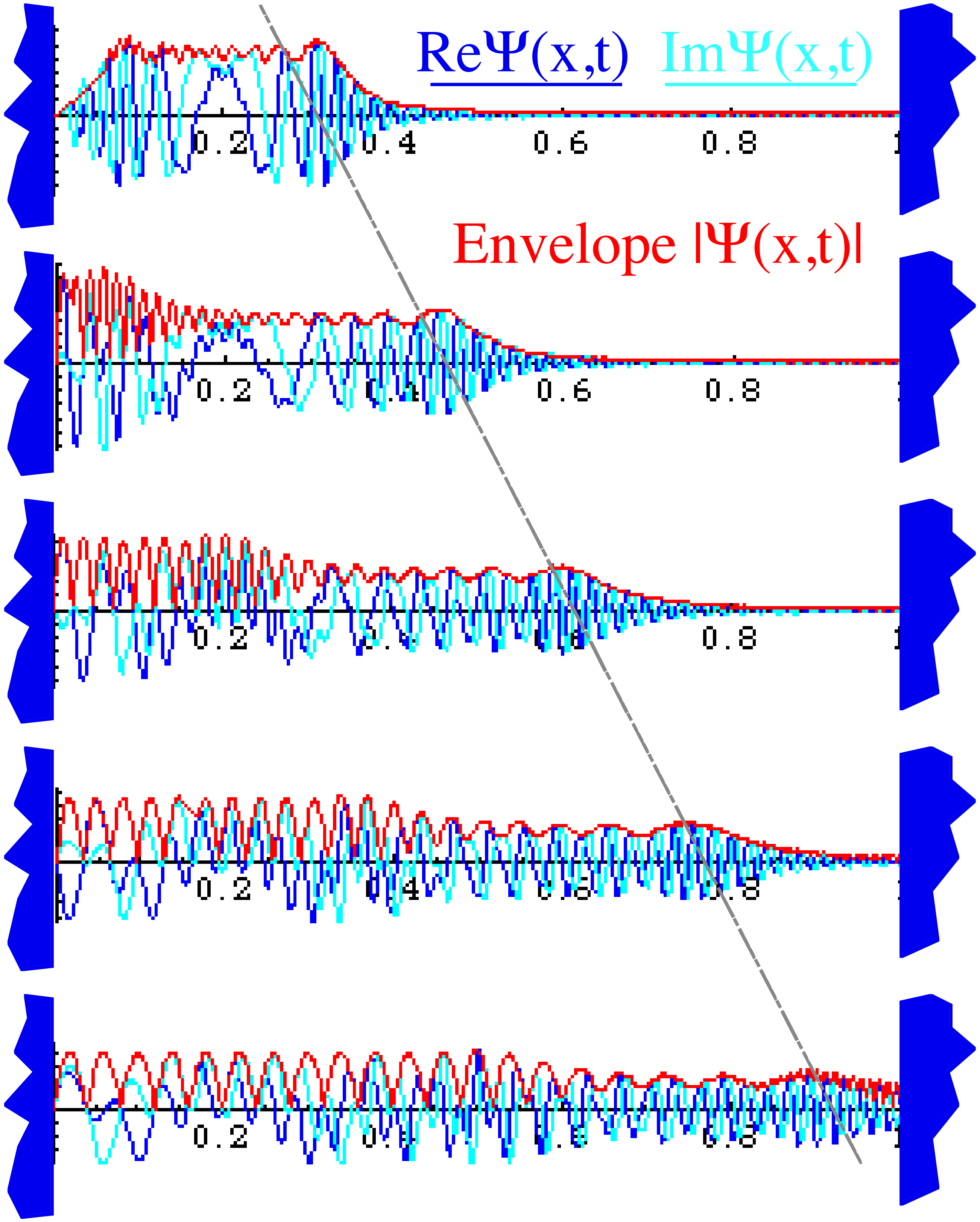
Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



$t = 0.0004\tau_1$

$t = 0.0008\tau_1$

$t = 0.0012\tau_1$

$t = 0.0016\tau_1$

$t = 0.0020\tau_1$

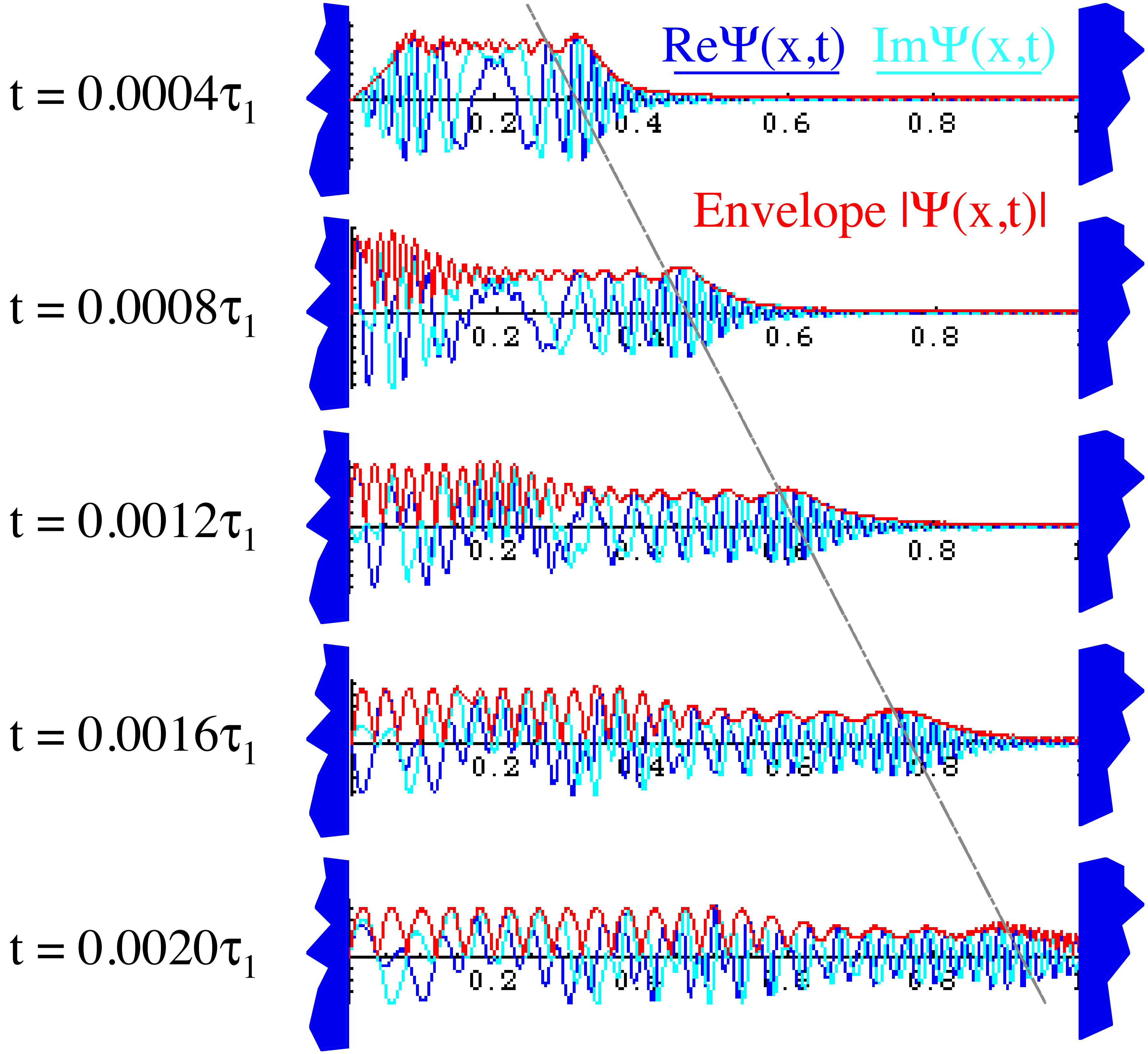
Wavepacket explodes!

Time given in units of period τ1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2 / 8MW^2} = \frac{8MW^2}{h}$$



ε_n-*level classical velocity*:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

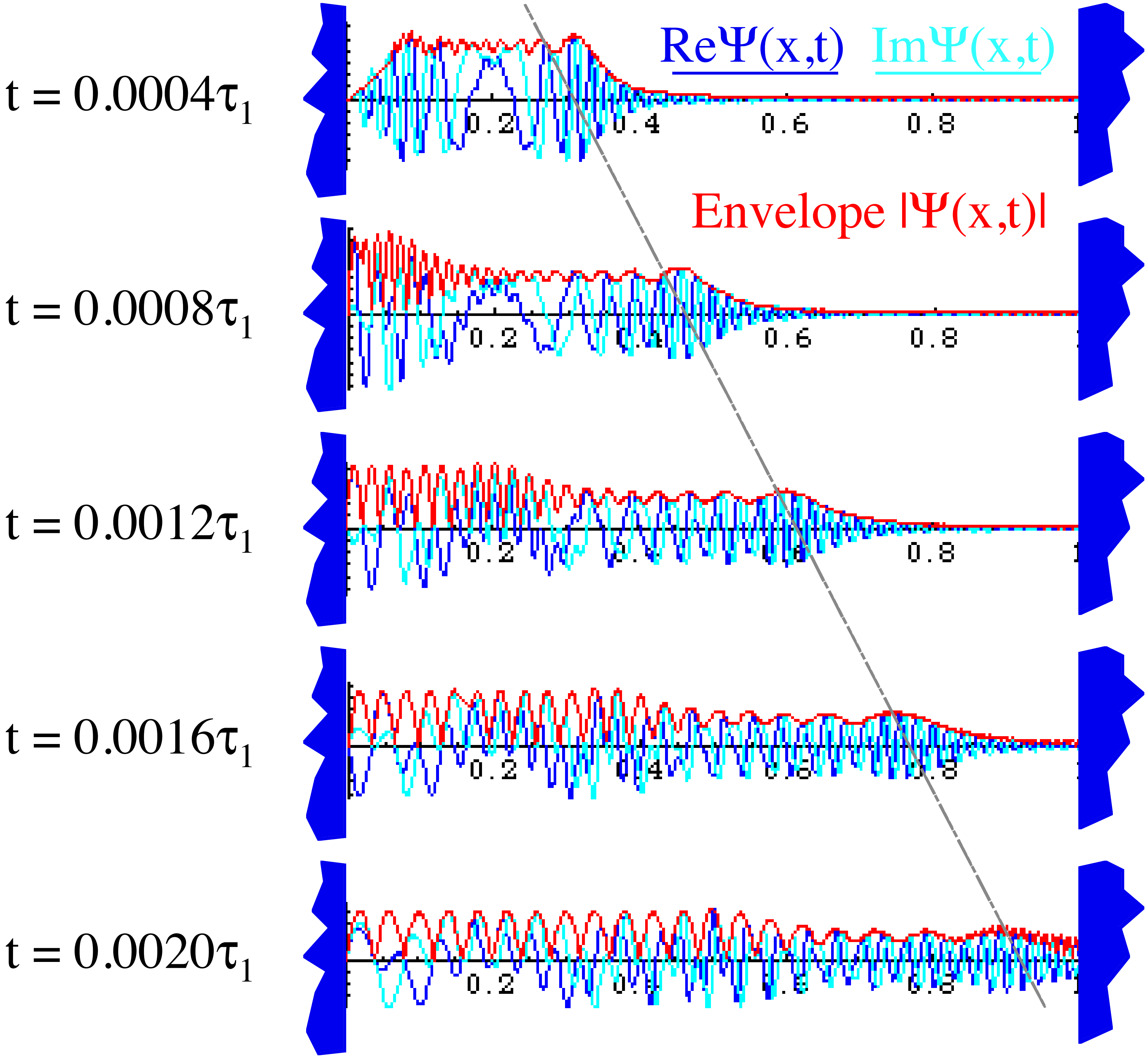
Wavepacket explodes!

Time given in units of period τ₁ (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



εn-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

εn-level classical round trip time Tn(2W)

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

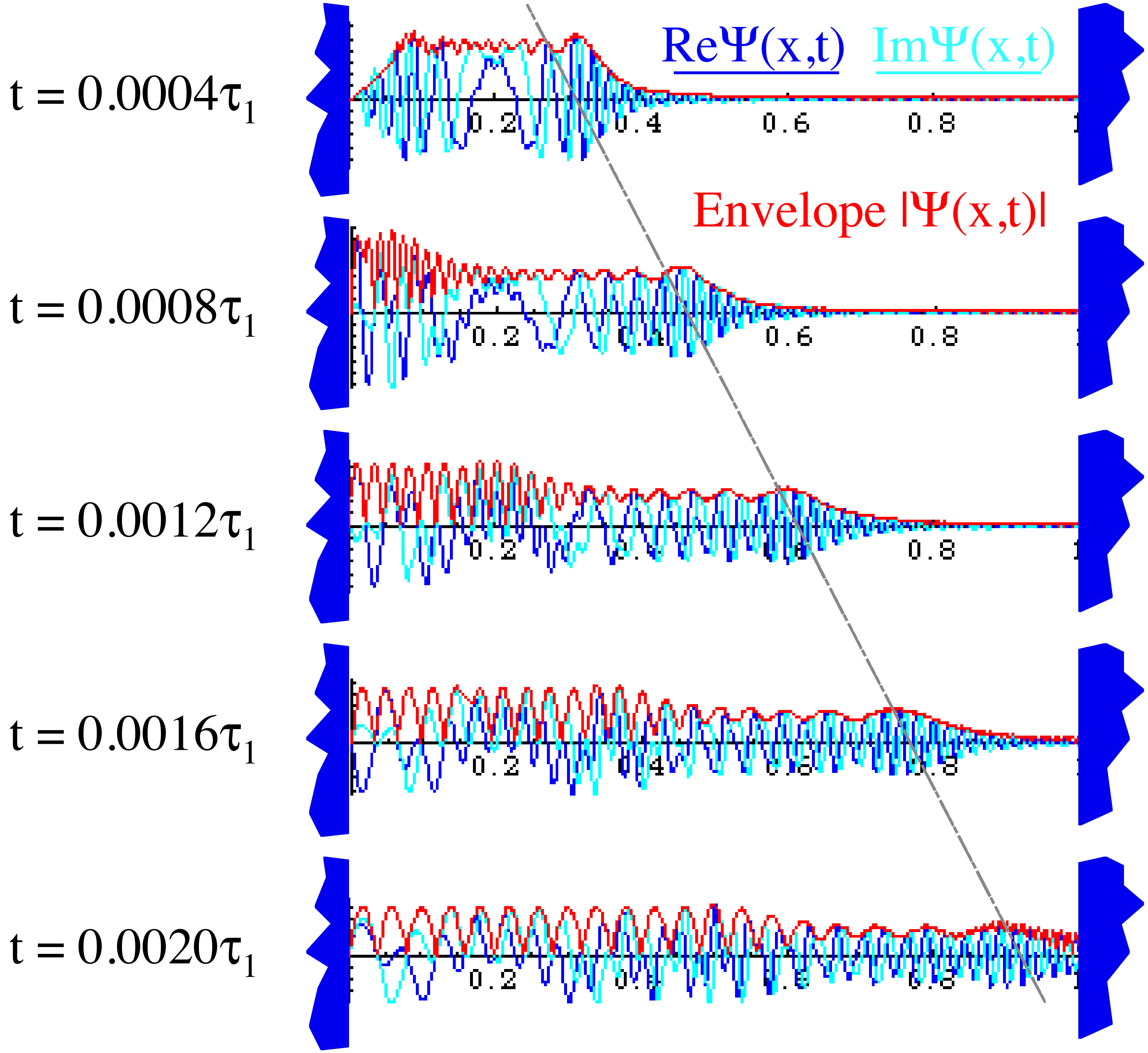
Wavepacket explodes!

Time given in units of period τ₁ (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n=100\right)$$

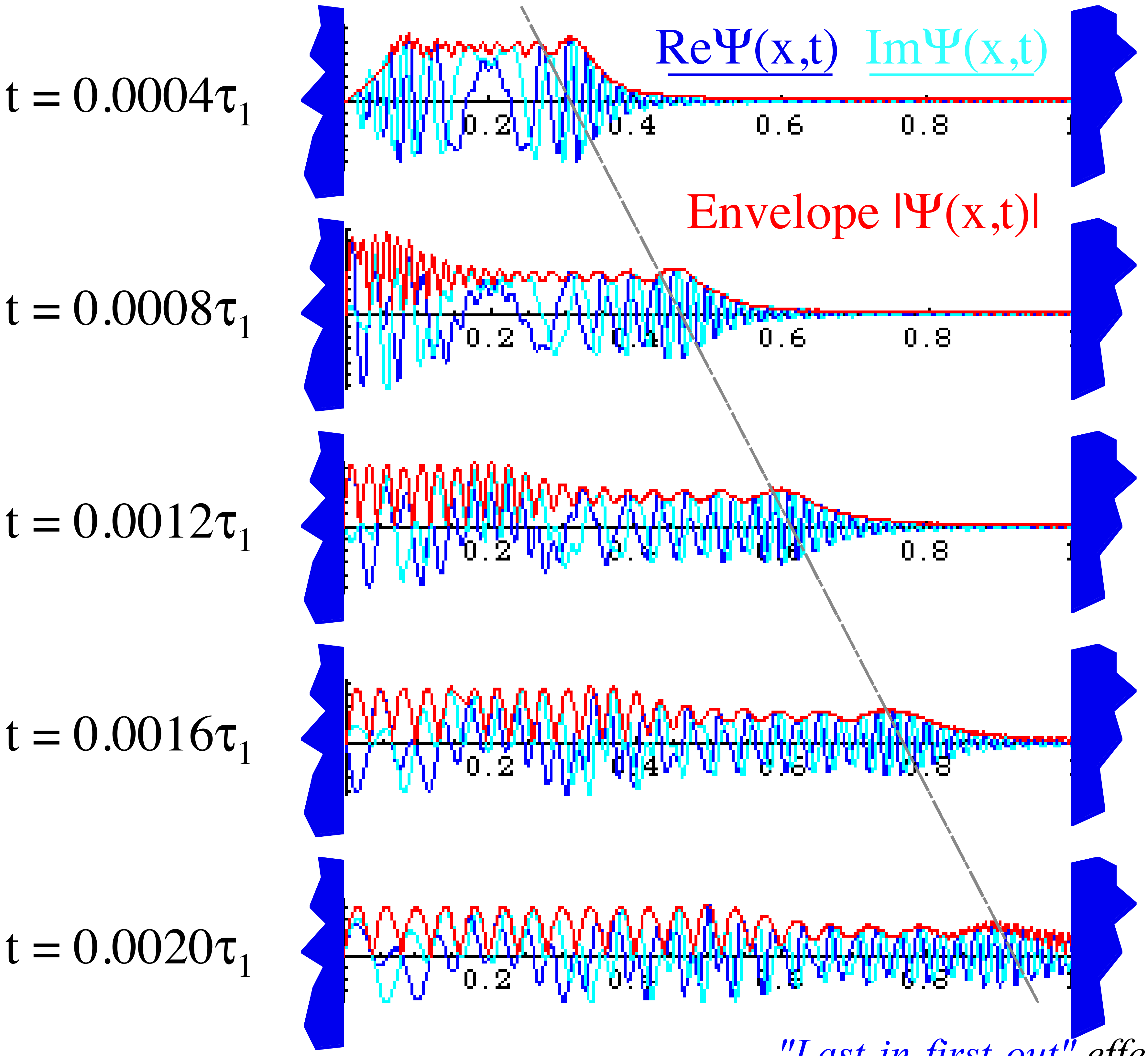
Wavepacket explodes!

Time given in units of period τ₁ (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



"Last-in-first-out" effect

ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-level 1-way time $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n=100\right)$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
    Wave coordinates
    Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
    ∞-Square well PE versus Bohr rotor
    SinNx/x wavepackets bandwidth and uncertainty
        → SinNx/x explosion and revivals ←
    Gaussian wave-packet bandwidth and uncertainty
        Gaussian revivals
    Farey-Sums and Ford-products
    Phase dynamics

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\,T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, M undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

Wavepacket explodes! (Then revives)

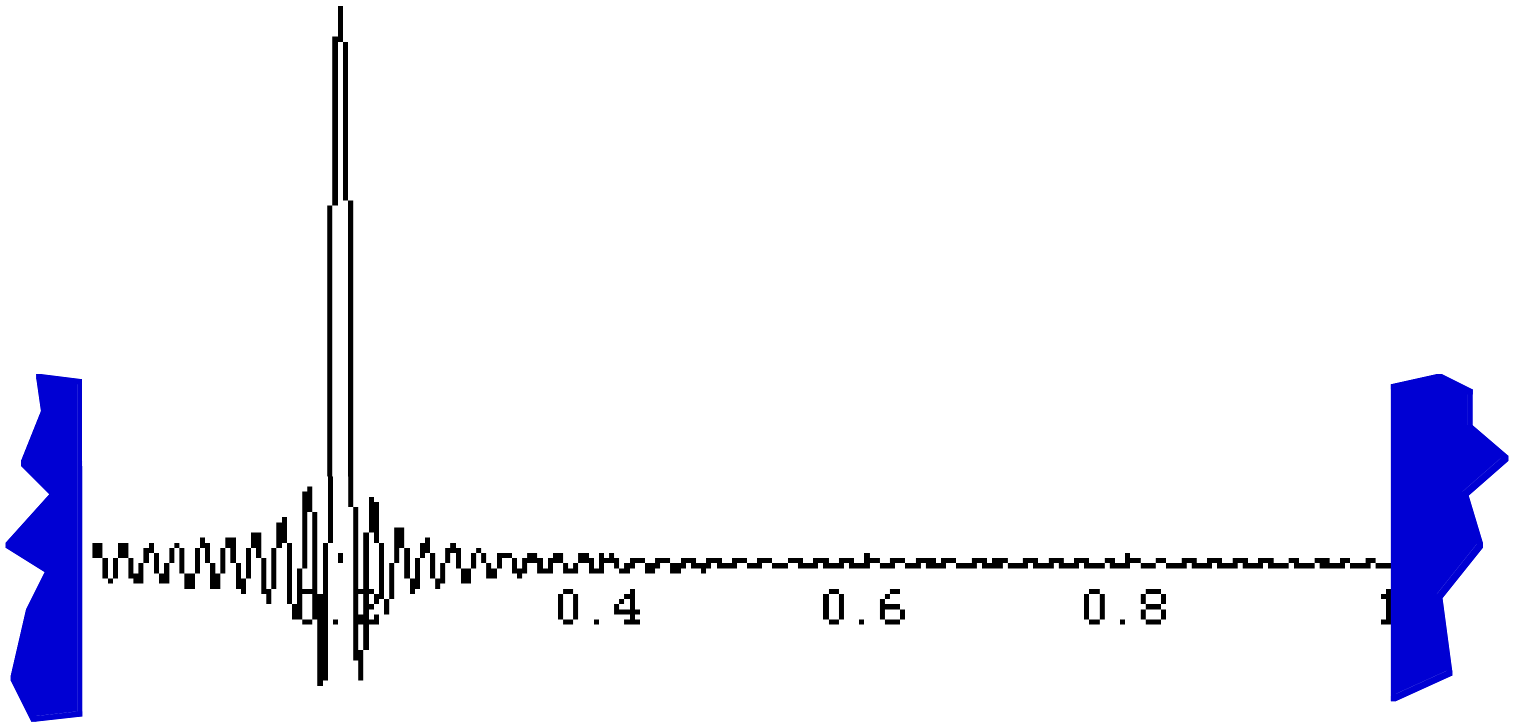
Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\,T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,
ε_2-level particle makes 4 round trips,
ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

$$t = 1.0000\tau_1$$
$$= 3.0\tau_{\text{beat}}$$



Wavepacket explodes! (Then revives)

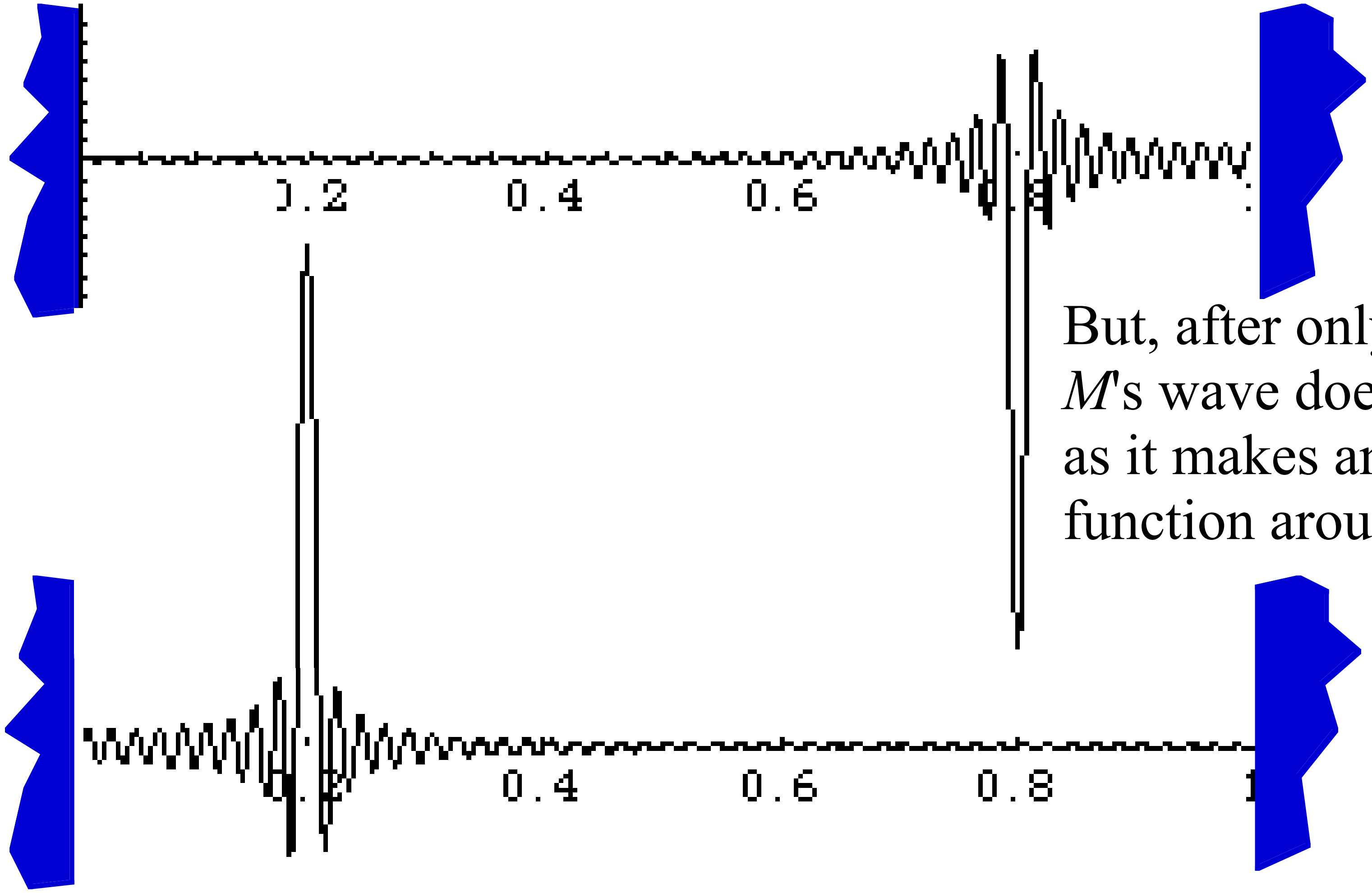
Zero-point period τ₁ is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,
ε_2-level particle makes 4 round trips,
ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at *x=0.2W*,

$t = 0.5000\tau_1$
$= 1.5\tau_{beat}$



But, after only 50 round-trips *M*'s wave does a *partial revival* as it makes an upside down-delta function around *x=0.8W*.

$t = 1.0000\tau_1$
$= 3.0\tau_{beat}$

At fractional times τ_1/n M undergoes a number of *fractional revivals*

$t = \tau_1/3$

$t = \tau_1/5$

$t = \tau_1/7$

$t = \tau_1/9$

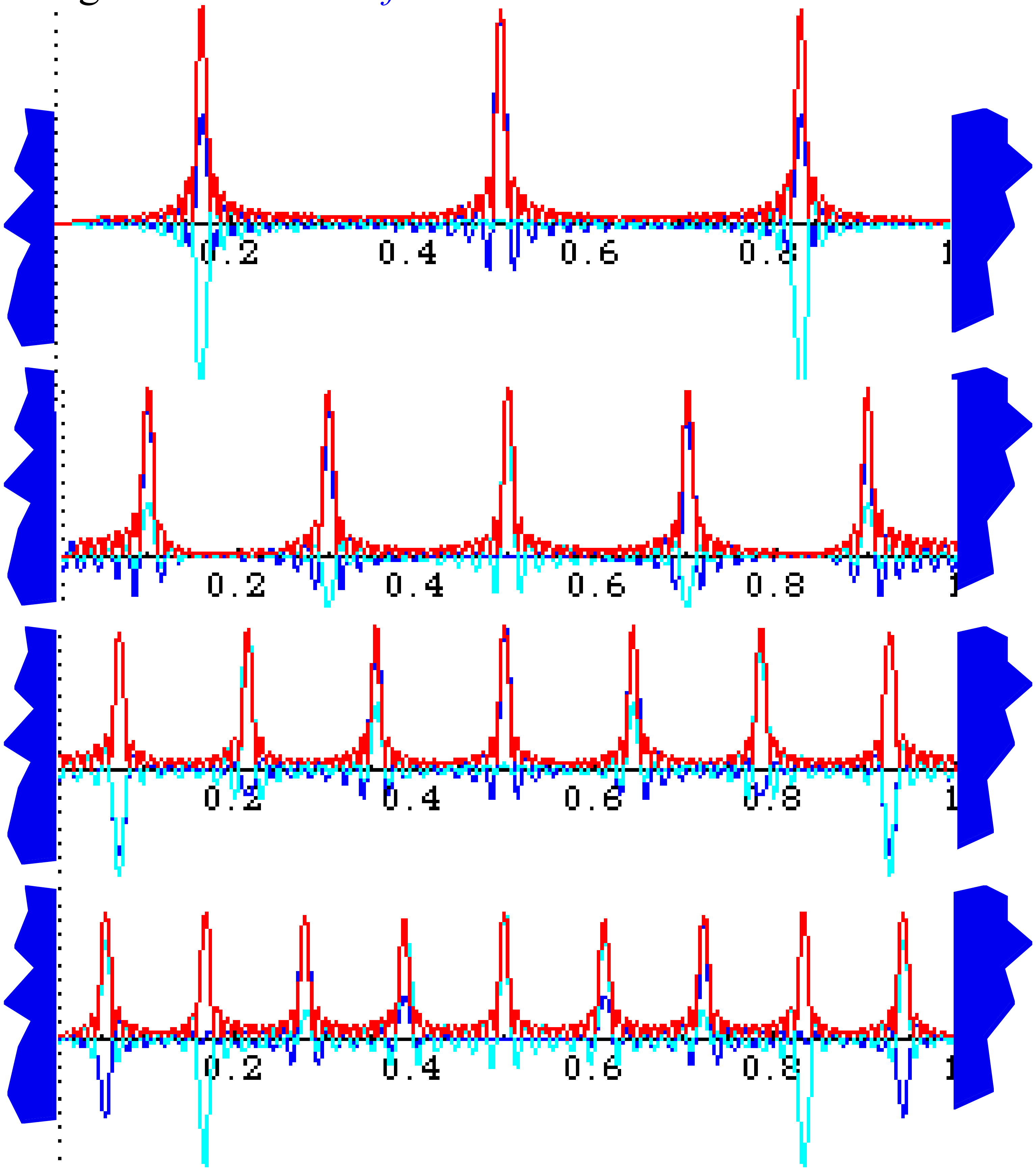


Fig. 12.2.5 The "Dance of the deltas." Mini-Revivals for prisoner M's wavepacket envelope function.

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

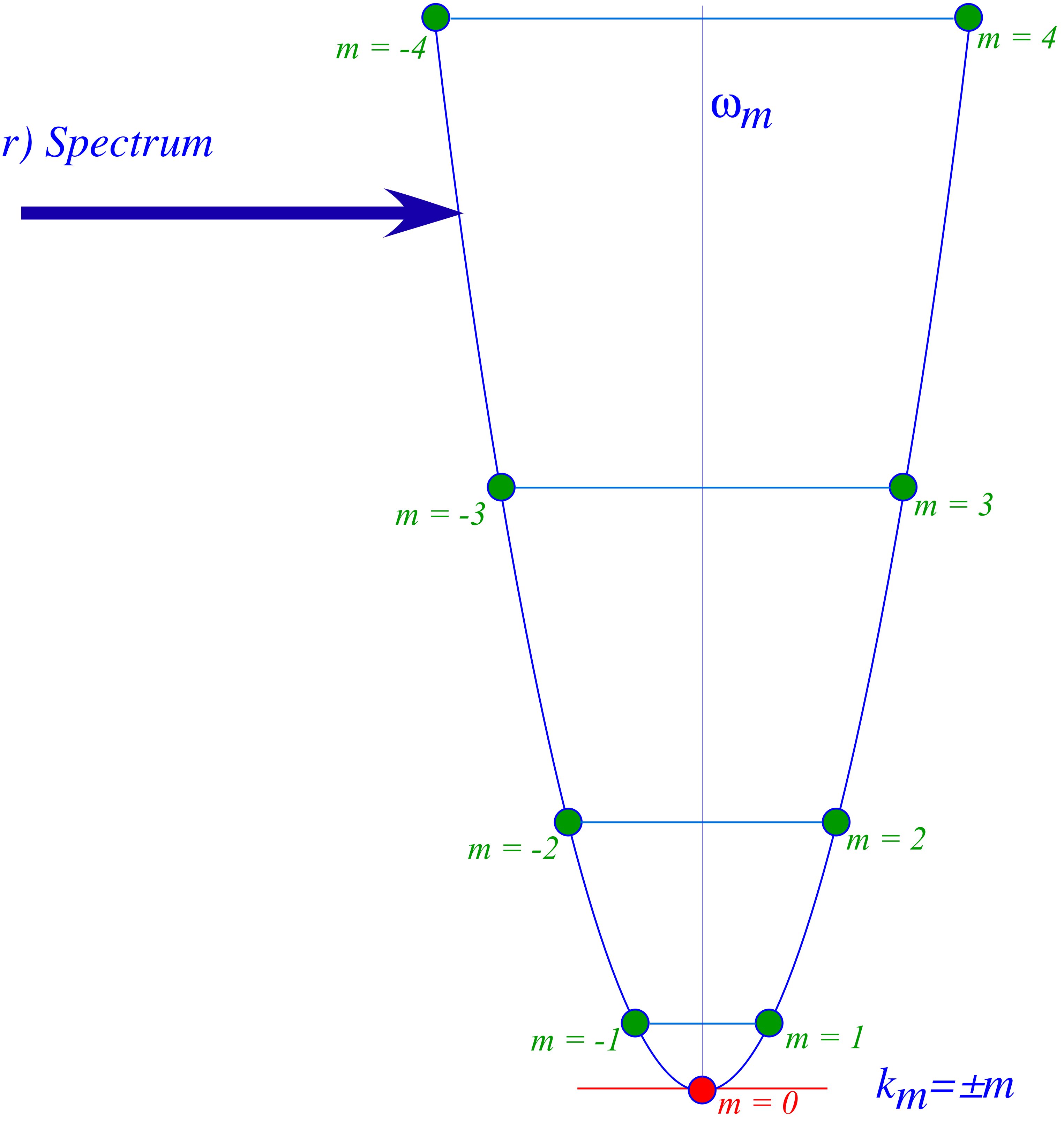
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

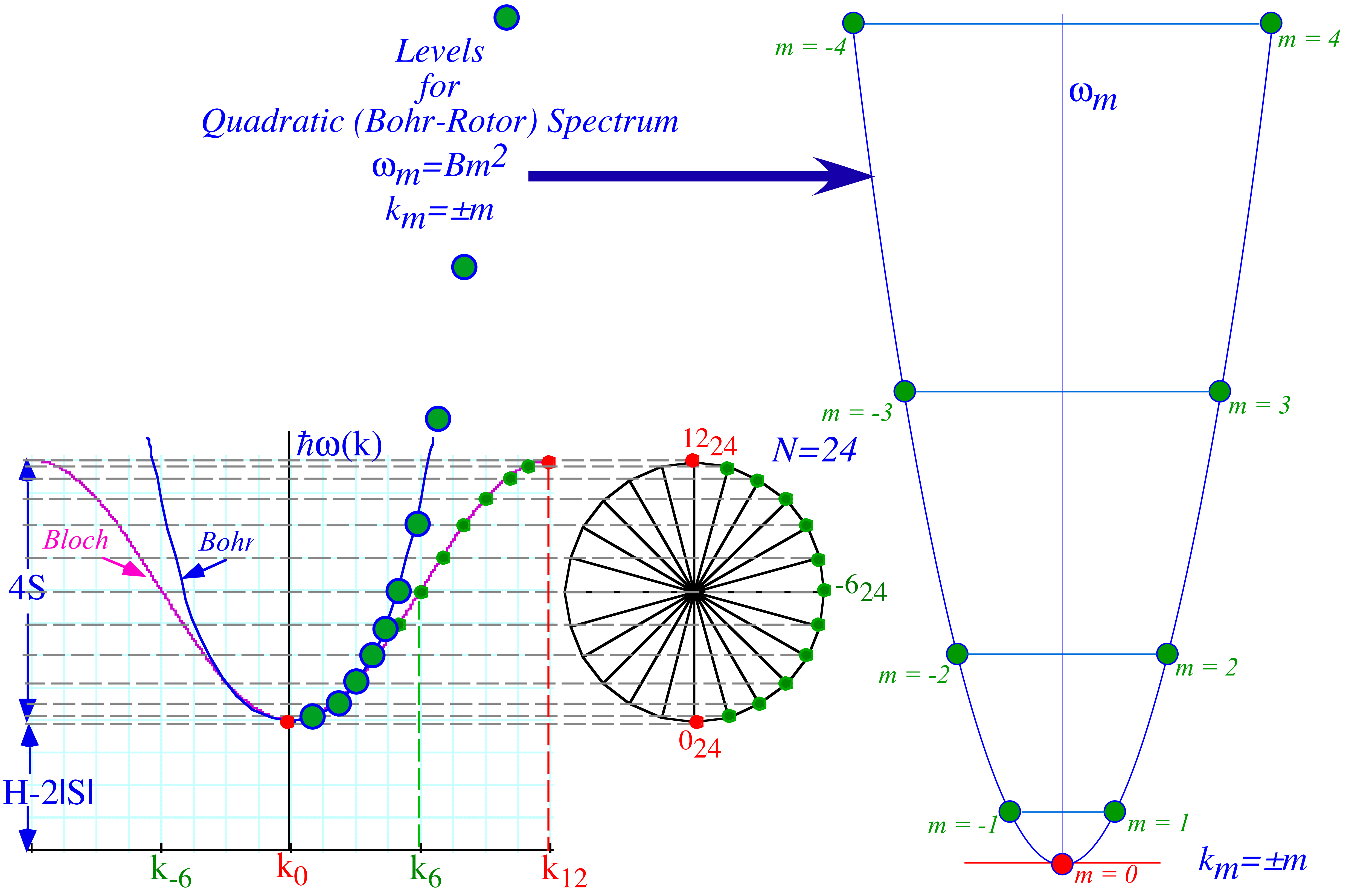
Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
    - *SinNx/x explosion and revivals*
- *Bohr-rotor dynamics*
    - *Gaussian wave-packet bandwidth and uncertainty*
    - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$





Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

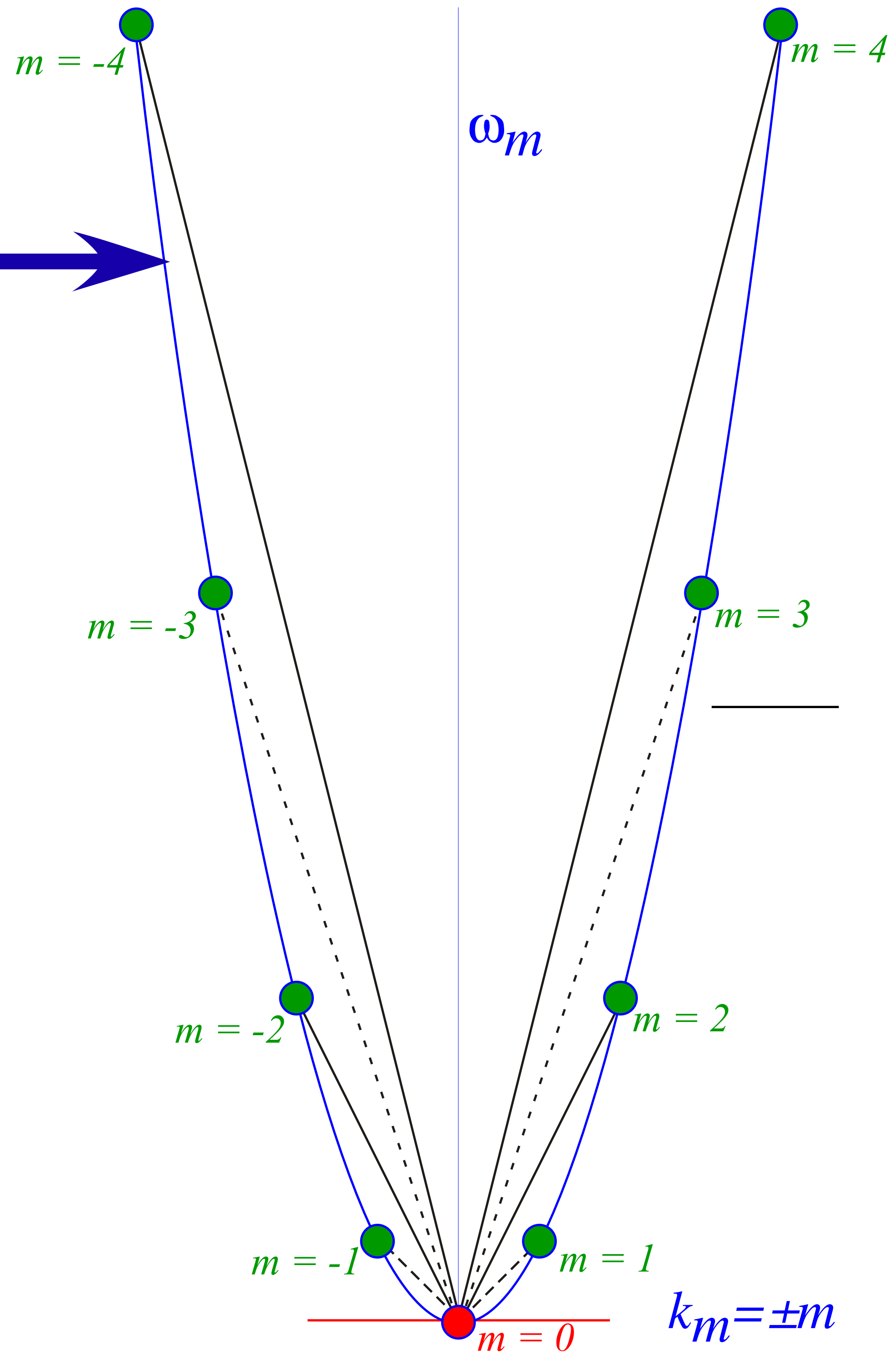
$k_m = \pm m$

Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$\boxed{V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB}$$

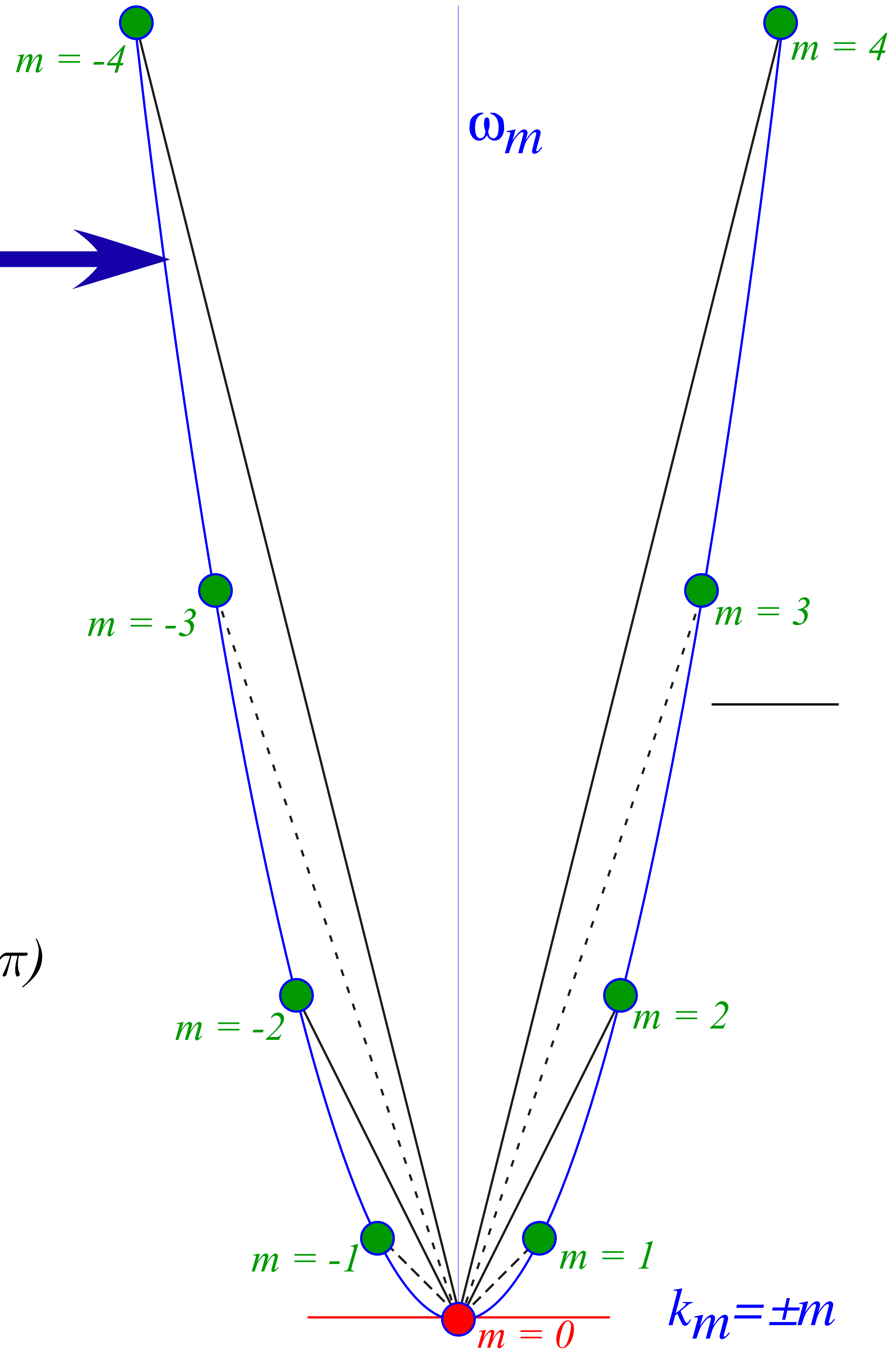


Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$\boxed{V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB}$$



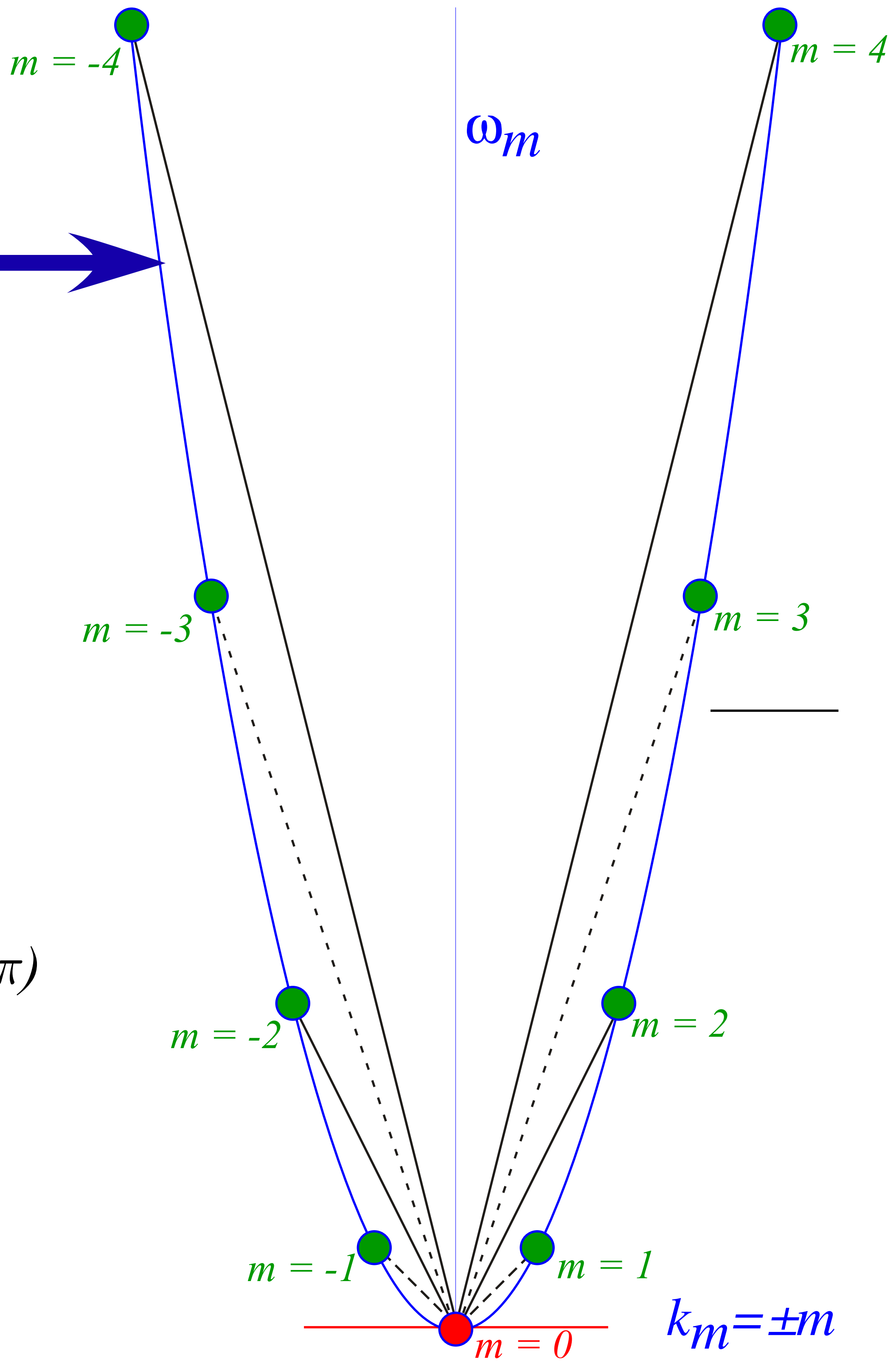
$m = 0, \pm 1, \pm 2, \pm 3$,...are momentum quanta
in wavevector formula: $k_m = 2\pi m/L$ $(k_m = m$ if: $L = 2\pi)$

Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$



$m = 0, \pm 1, \pm 2, \pm 3, \ldots$ are momentum quanta in wavevector formula: $k_m = 2\pi m/L$ $(k_m = m$ *if:* $L = 2\pi)$

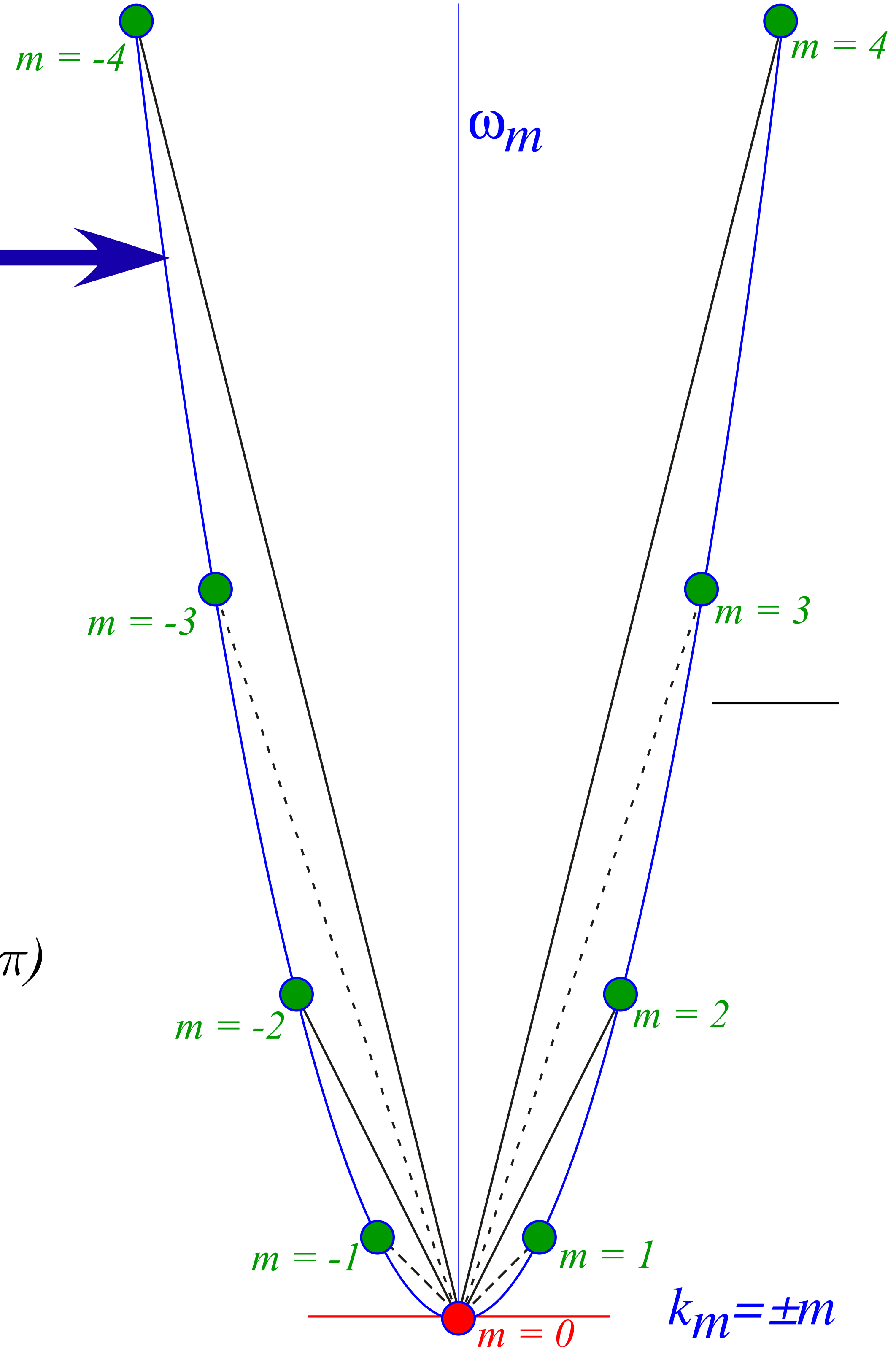
$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$\boxed{V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB}$$



$m=0, \pm 1, \pm 2, \pm 3,$...are momentum quanta
in wavevector formula: $k_m=2\pi m/L$ $(k_m=m$ *if:* $L=2\pi)$

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1=2\pi\upsilon_1$
and lowest *transition (beat) frequency* $\upsilon_1 = (E_1-E_0)/h$

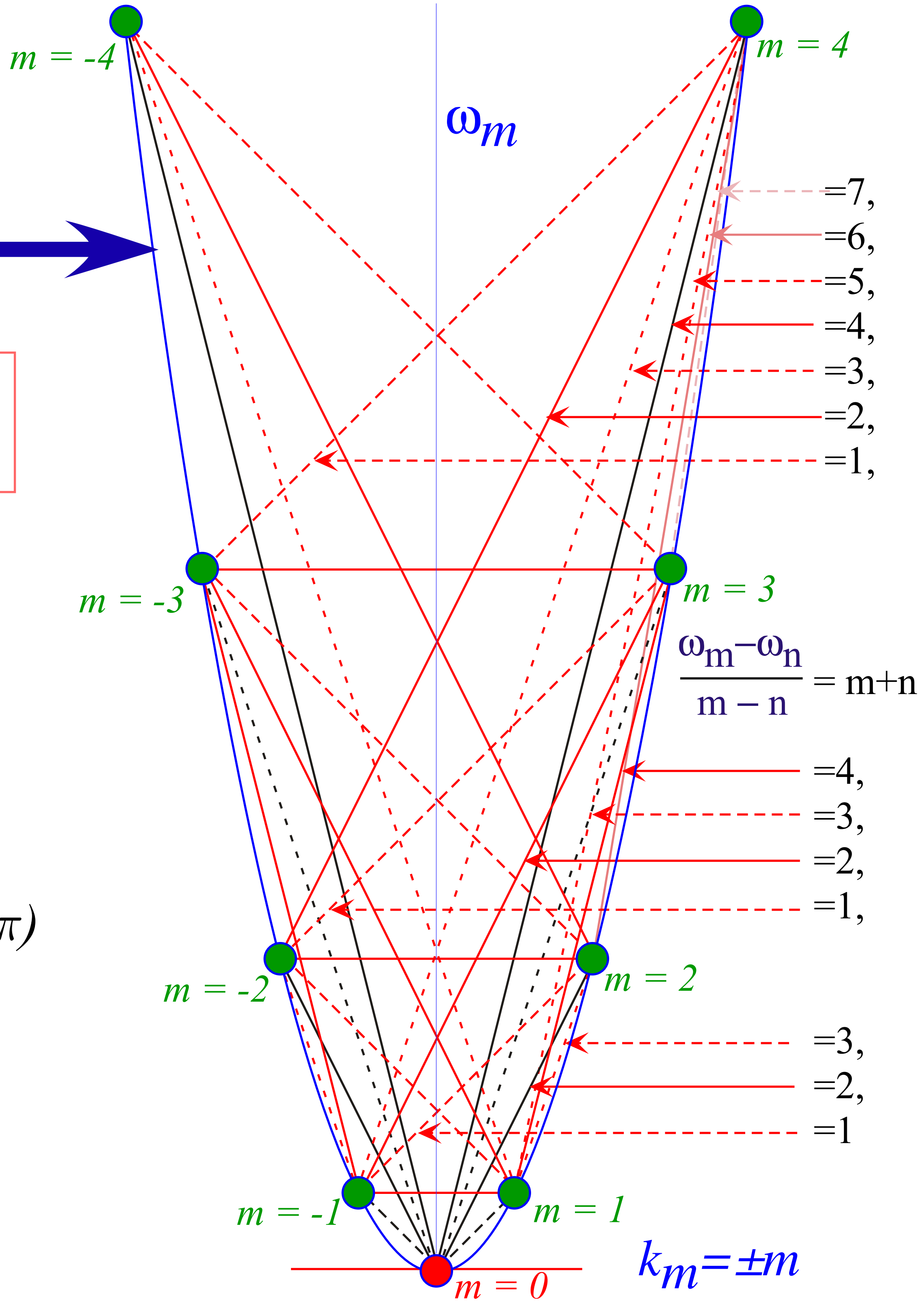
Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$\omega_m=Bm^2$

$k_m=\pm m$

$$V_{phase}=\frac{\quad}{k_m}=\frac{Bm^2}{m}=mB$$

$$V_{group}=\frac{\omega_m-\omega_n}{k_m-k_n}=\frac{m^2-n^2}{m\pm n}B=(m\pm n)B$$



$m=0, \pm1, \pm2, \pm3,$...are momentum quanta
in wavevector formula: $k_m=2\pi m/L$ $(k_m=m$ if: $L=2\pi)$

$$E_m= (\hbar k_m)^2/2M = m^2[h^2/2ML^2]= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1=2\pi\upsilon_1$

and lowest *transition (beat) frequency* $\upsilon_1 =(E_1-E_0)/h$

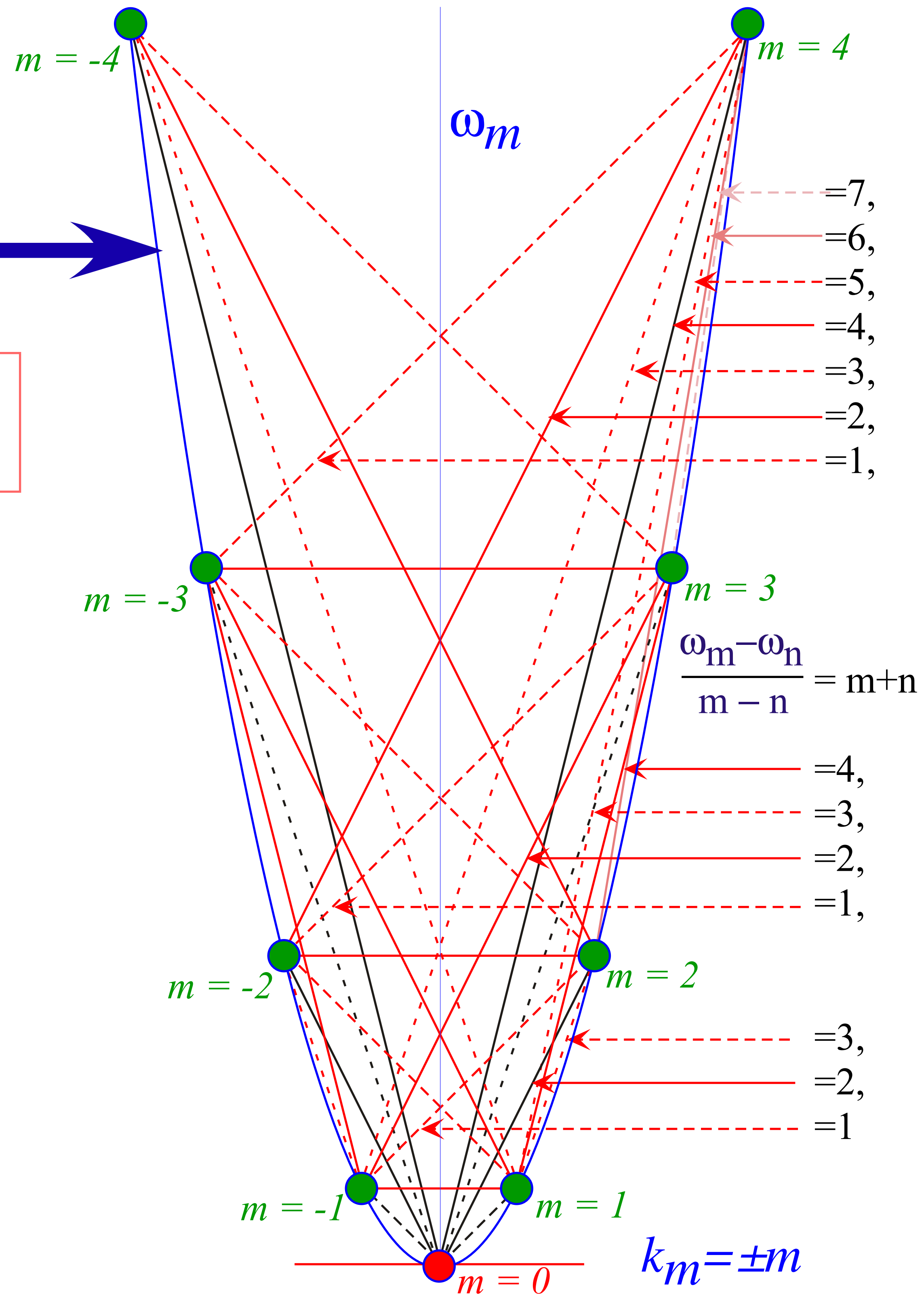
Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$

$$k_m = \pm m$$

$$V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB$$

$$V_{group} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{m^2 - n^2}{m \pm n} B = (m \pm n)B$$

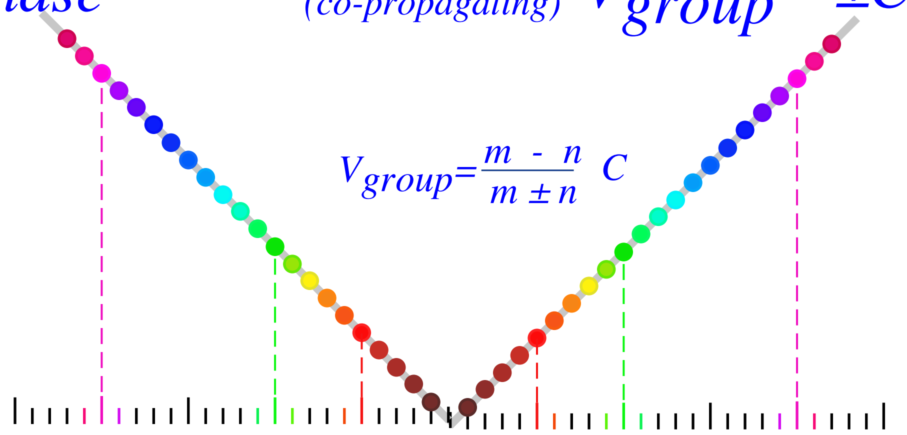


Possible wave velocities for Linear (Optical) Spectrum

$$\omega_m = C|m|^1$$

$$k_m = m$$

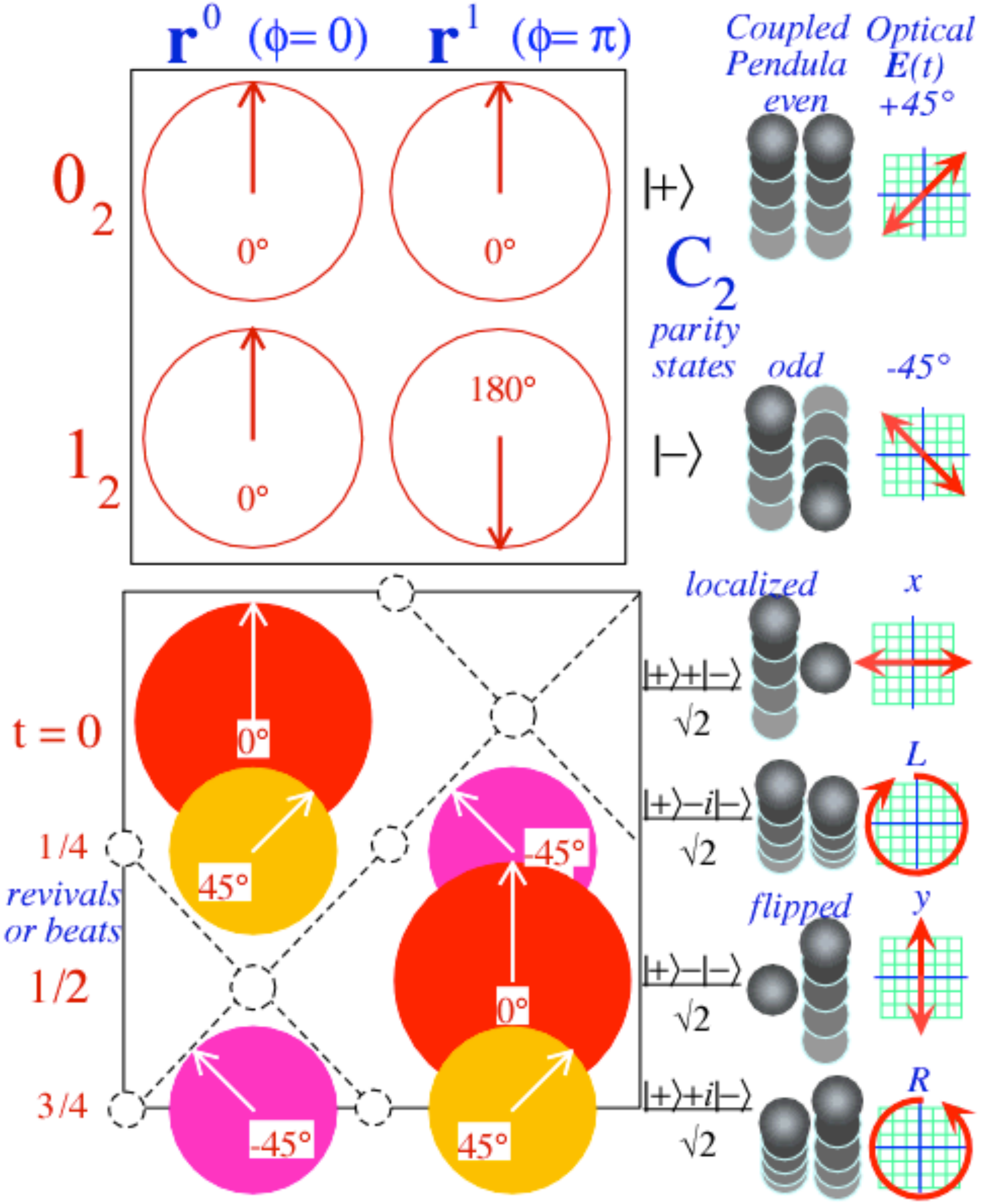
$V_{phase} = \pm C$ (co-propagating) $V_{group} = \pm C$



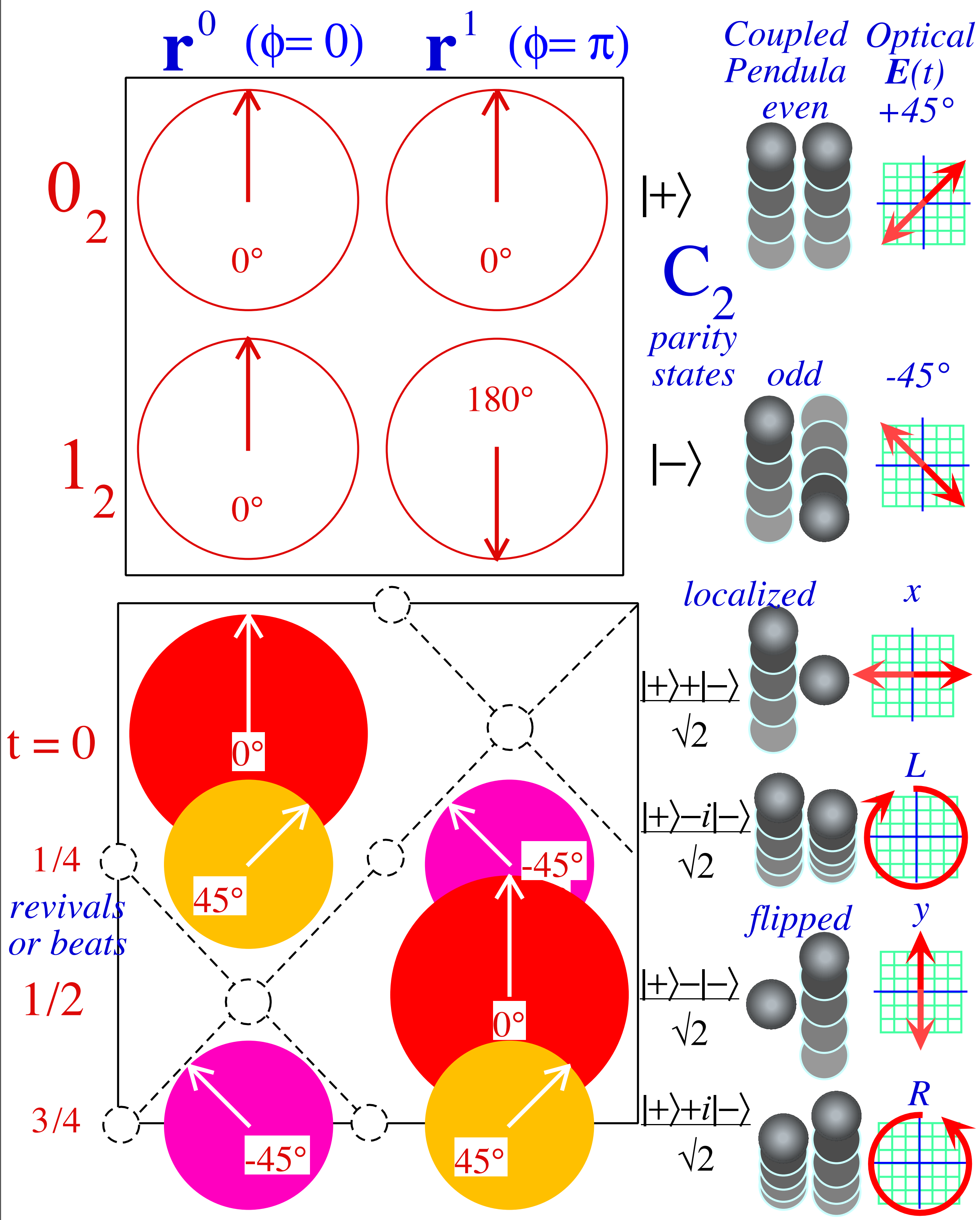
C_2
Fourier transformation matrix

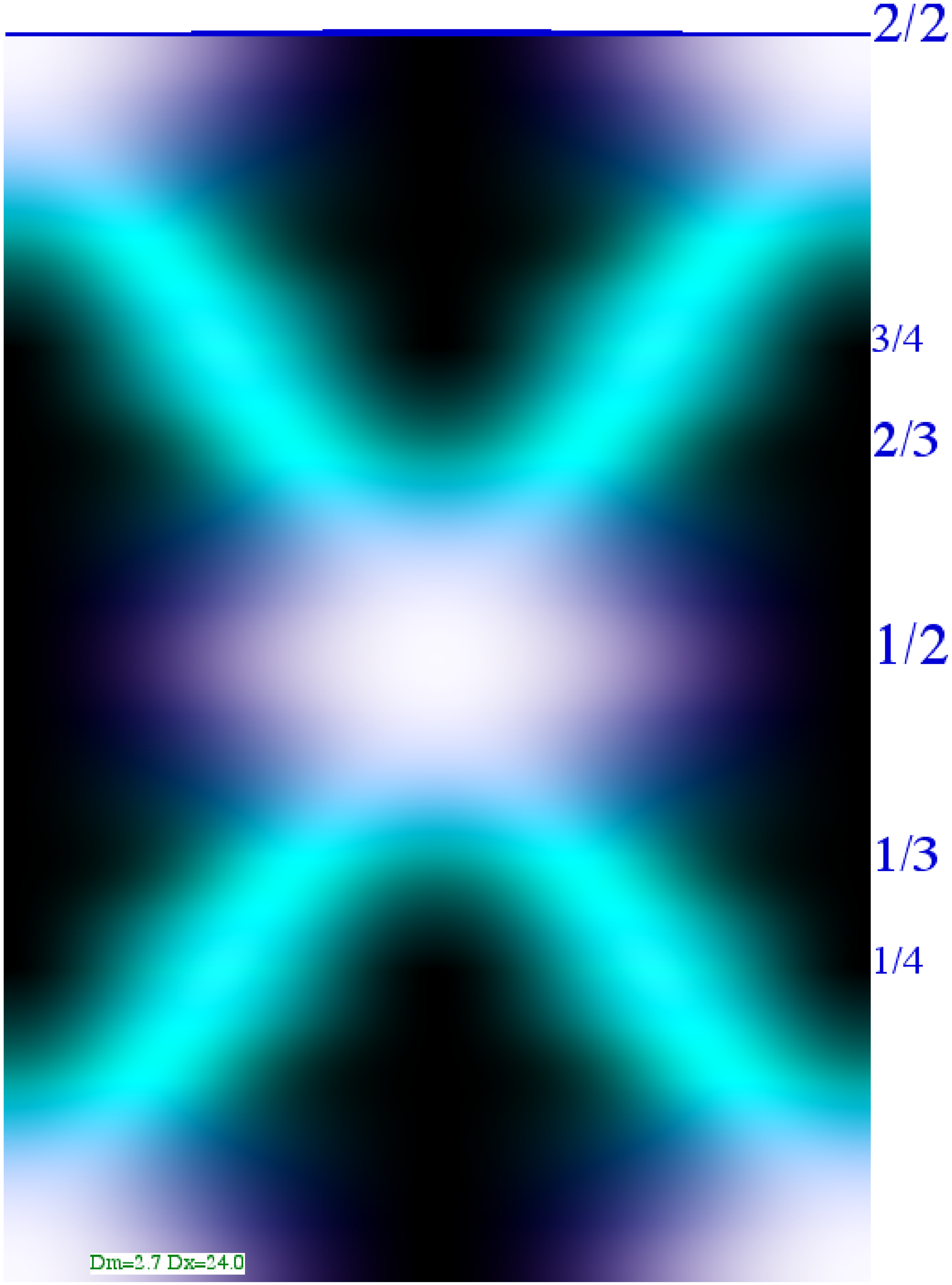
and

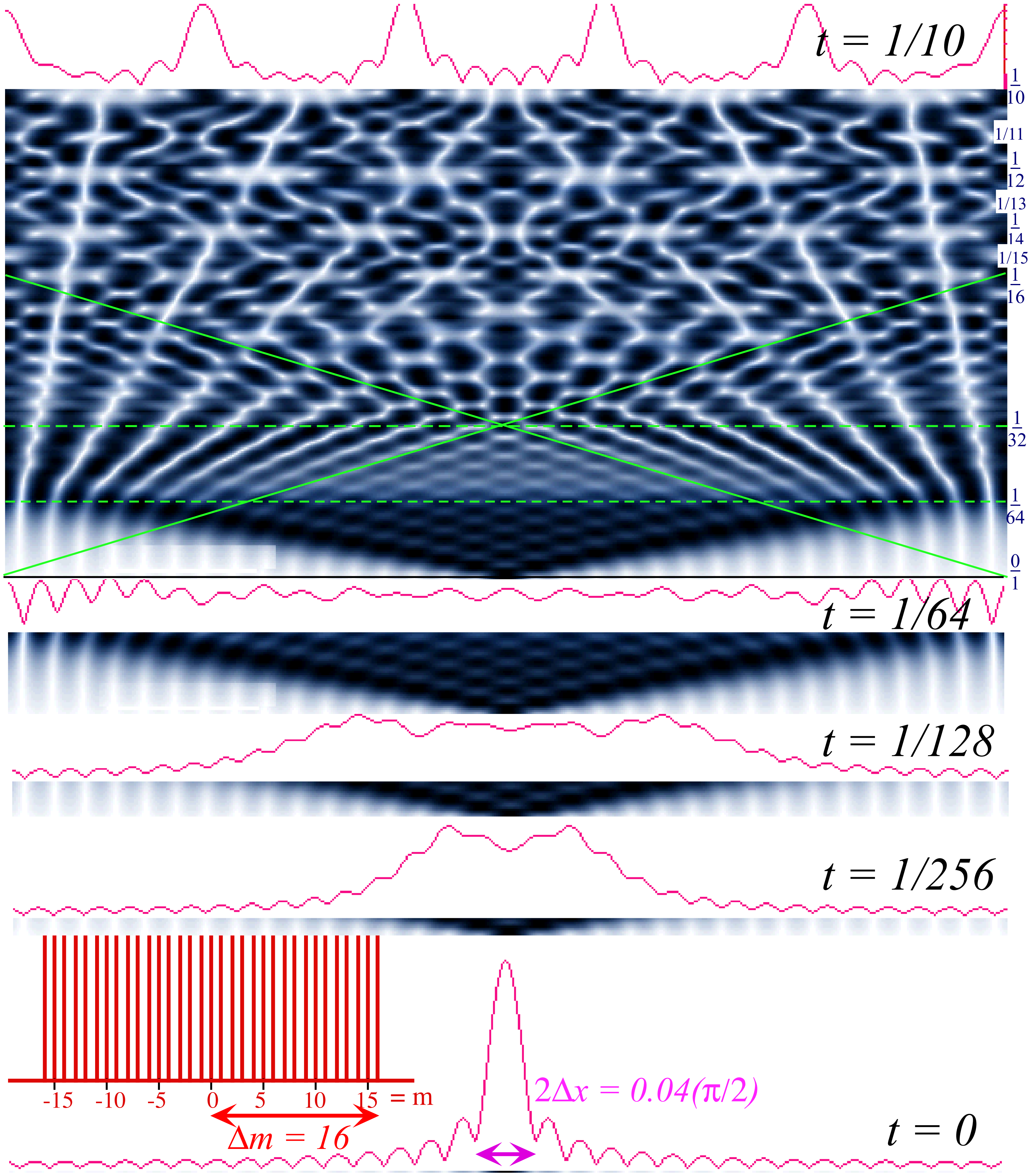
dynamics

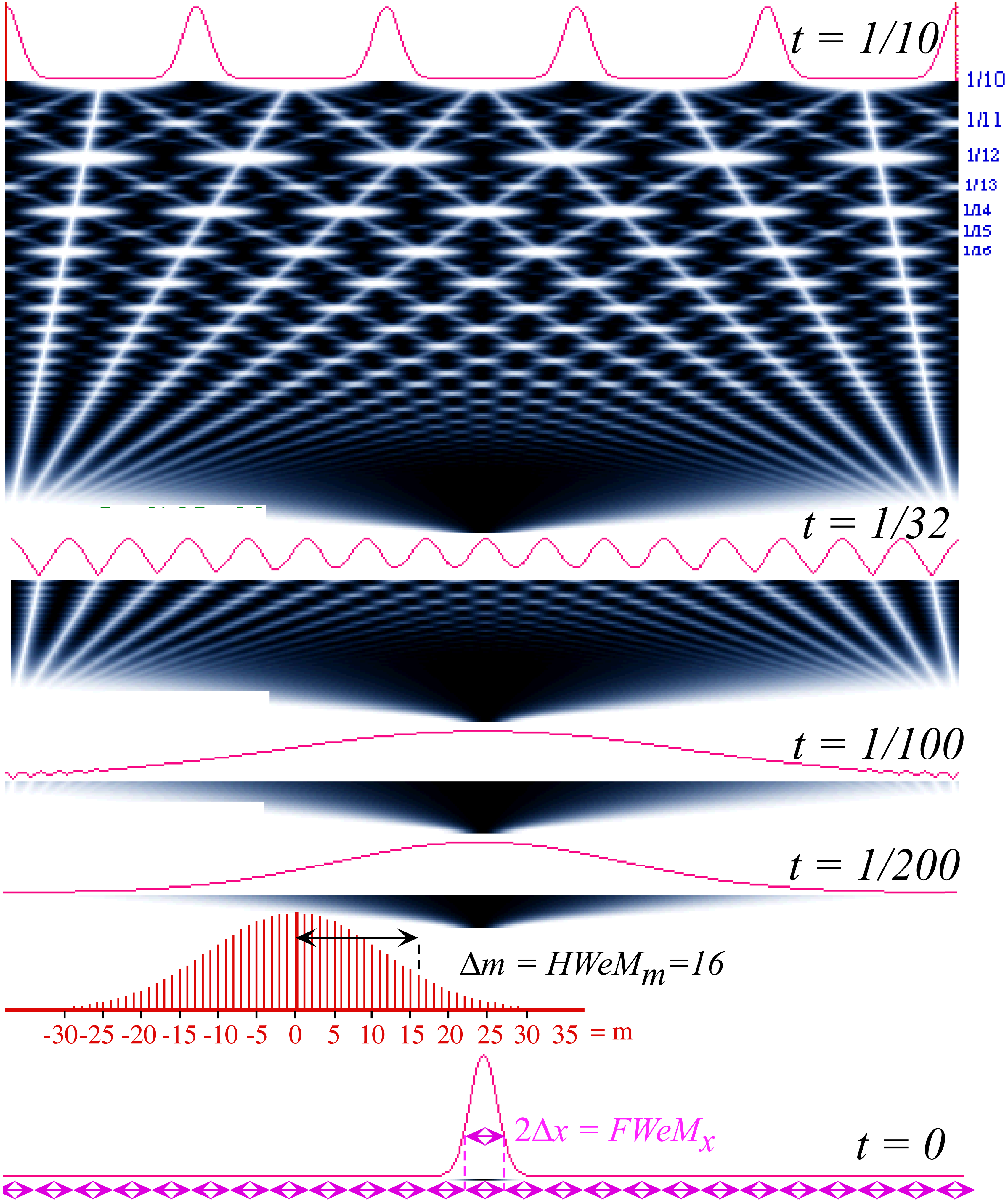


Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2






t = 1/10
1/10
1/11
1/12
1/13
1/14
1/15
1/16
1/32
1/64
0/1
t = 1/64
t = 1/128
t = 1/256
-15 -10 -5 0 5 10 15 = m
Δm = 16
2Δx = 0.04(π/2)
t = 0


t = 1/10
1/10
1/11
1/12
1/13
1/14
1/15
1/16
t = 1/32
t = 1/100
t = 1/200
Δm = HWeM_m=16
-30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 = m
2Δx = FWeM_x
t = 0

Polygonal geometry of U(2)⊃C$_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
    Wave coordinates
    Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C$_N$ beat dynamics and "Revivals" due to Bohr-dispersion
    ∞-Square well PE versus Bohr rotor
    SinNx/x wavepackets bandwidth and uncertainty
        SinNx/x explosion and revivals
    Bohr-rotor dynamics
        → *Gaussian wave-packet bandwidth and uncertainty* ←
        Gaussian Bohr-rotor revivals
    Farey-Sums and Ford-products
    Phase dynamics

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{i m\phi}$$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t=0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{i\,m\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + i\,m\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ*-angle wavefunction.*

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \quad e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \qquad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{A\left(\Delta_m,\phi\right)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A\left(\Delta_m,\phi\right)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{-\left(\frac{m}{\Delta_m}\right)^2}e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\quad e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}\qquad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\qquad\frac{A(\Delta_m,\phi)}{2\pi}\qquad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A(\Delta_m,\phi)=\sum_{m=-\infty}^{\infty}e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}\xrightarrow[\Delta_m>>1]{}\int_{-\infty}^{\infty}dk\ e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad (k_m=m \quad$ *if:* $L=2\pi)$

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \quad e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \quad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \qquad \frac{A\left(\Delta_m,\phi\right)}{2\pi} \qquad e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A\left(\Delta_m,\phi\right)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\; e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \quad \text{so: } dk=\Delta_m dK\right]$$

m=0, ±1, ±2, ±3,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t=0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{A\left(\Delta_m,\phi\right)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Gaussian integral:

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx}\sqrt{\int_{-\infty}^{\infty} e^{-y^2}dy}=\sqrt{\iint e^{-(x^2+y^2)}\,dx\,dy}$$

$$=\sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta}=\sqrt{2\pi\int_0^{\infty} e^{-r^2}\frac{dr^2}{2}}=\sqrt{\pi}$$

$$A\left(\Delta_m,\phi\right)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}\xrightarrow[\Delta_m>>1]{}\int_{-\infty}^{\infty}dk\; e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \quad \text{so: } dk=\Delta_m dK\right] \quad \text{then: } A\left(\Delta_m,\phi\right)\simeq\Delta_m\int_{-\infty}^{\infty}dK\; e^{-(K)^2}=\Delta_m\sqrt{\pi}$$

$m=0, \pm 1, \pm 2, \pm 3,\ldots$ are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$A(\Delta_m,\phi)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\; e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \;\; \text{so: } dk=\Delta_m dK\right] \text{ then: } A(\Delta_m,\phi)\simeq \Delta_m\int_{-\infty}^{\infty} dK\; e^{-(K)^2}=\Delta_m\sqrt{\pi}$$

$m=0, \pm1, \pm2, \pm3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ $\quad$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-m plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2+im\phi+\left(\frac{\Delta_m}{2}\phi\right)^2-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$=\frac{A(\Delta_m,\phi)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A(\Delta_m,\phi)=\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\; e^{-\left(\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K=\frac{k}{\Delta_m}-i\frac{\Delta_m}{2}\phi \quad \text{so: } dk=\Delta_m dK\right] \quad \text{then: } A(\Delta_m,\phi)\simeq\Delta_m\int_{-\infty}^{\infty} dK\; e^{-(K)^2}=\Delta_m\sqrt{\pi}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$\Psi(\phi,t{=}0)=\frac{\Delta_m}{2\sqrt{\pi}}\, e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$\Psi(\phi,t{=}0)\simeq\frac{\Delta_m}{2\sqrt{\pi}}\, e^{-\left(\frac{\phi}{\Delta_\phi}\right)^2}$$

where: $\Delta_\phi=\frac{2}{\Delta_m}$ or: $\Delta_\phi\Delta_m=2$

$m=0, \pm1, \pm2, \pm3,$...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ ($k_m=m$ *if:* $L=2\pi$)

Gaussian wave-packet bandwidth and uncertainty

Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2} e^{im\phi}$$

Complete the square in exponent to simplify ϕ-angle wavefunction.

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m}\right)^2 + im\phi + \left(\frac{\Delta_m}{2}\phi\right)^2 - \left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$= \frac{A\left(\Delta_m,\phi\right)}{2\pi} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

$$A\left(\Delta_m,\phi\right) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2} \xrightarrow[\Delta_m>>1]{} \int_{-\infty}^{\infty} dk\, e^{-\left(\frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi\right)^2}$$

$$\left[\text{let: } K = \frac{k}{\Delta_m} - i\frac{\Delta_m}{2}\phi \quad \text{so: } dk = \Delta_m dK\right] \quad \text{then: } A\left(\Delta_m,\phi\right) \simeq \Delta_m \int_{-\infty}^{\infty} dK\, e^{-(K)^2} = \Delta_m\sqrt{\pi}$$

$$\Psi(\phi,t{=}0) = \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_m}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

$$\Psi(\phi,t{=}0) \simeq \frac{\Delta_m}{2\sqrt{\pi}} e^{-\left(\frac{\phi}{\Delta_\phi}\right)^2}$$

where: $\Delta_\phi = \frac{2}{\Delta_m}$ or: $\Delta_\phi\Delta_m = 2$

Gaussian uncertainty relation

(Compare to $\Delta x \cdot \Delta k = \pi$ *for ∞-Well)*

m=0, ±1, ±2, ±3,...are momentum quanta in wavevector formula: $k_m = 2\pi m/L$ $\quad (k_m = m \quad \text{if: } L = 2\pi)$

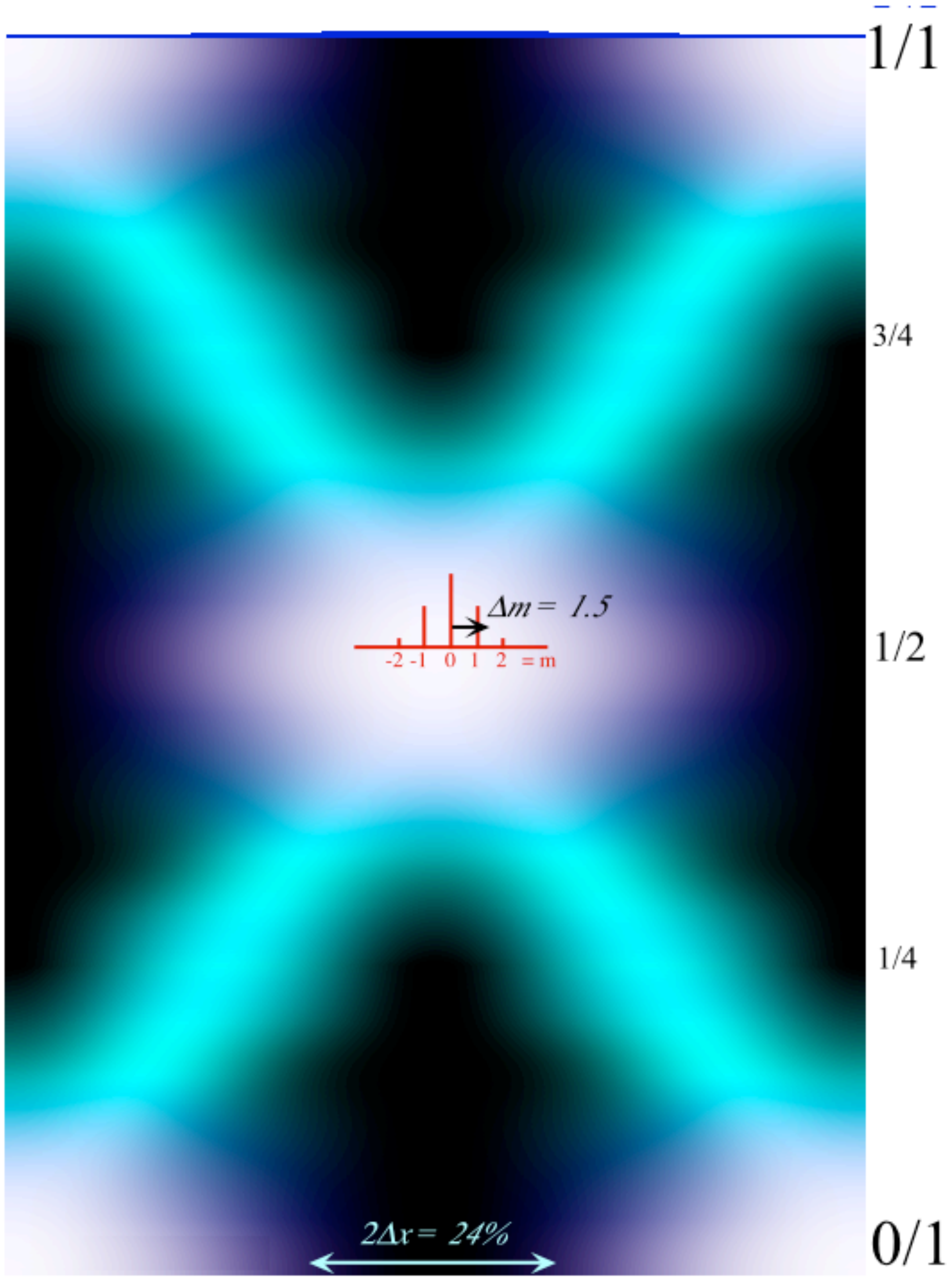
$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

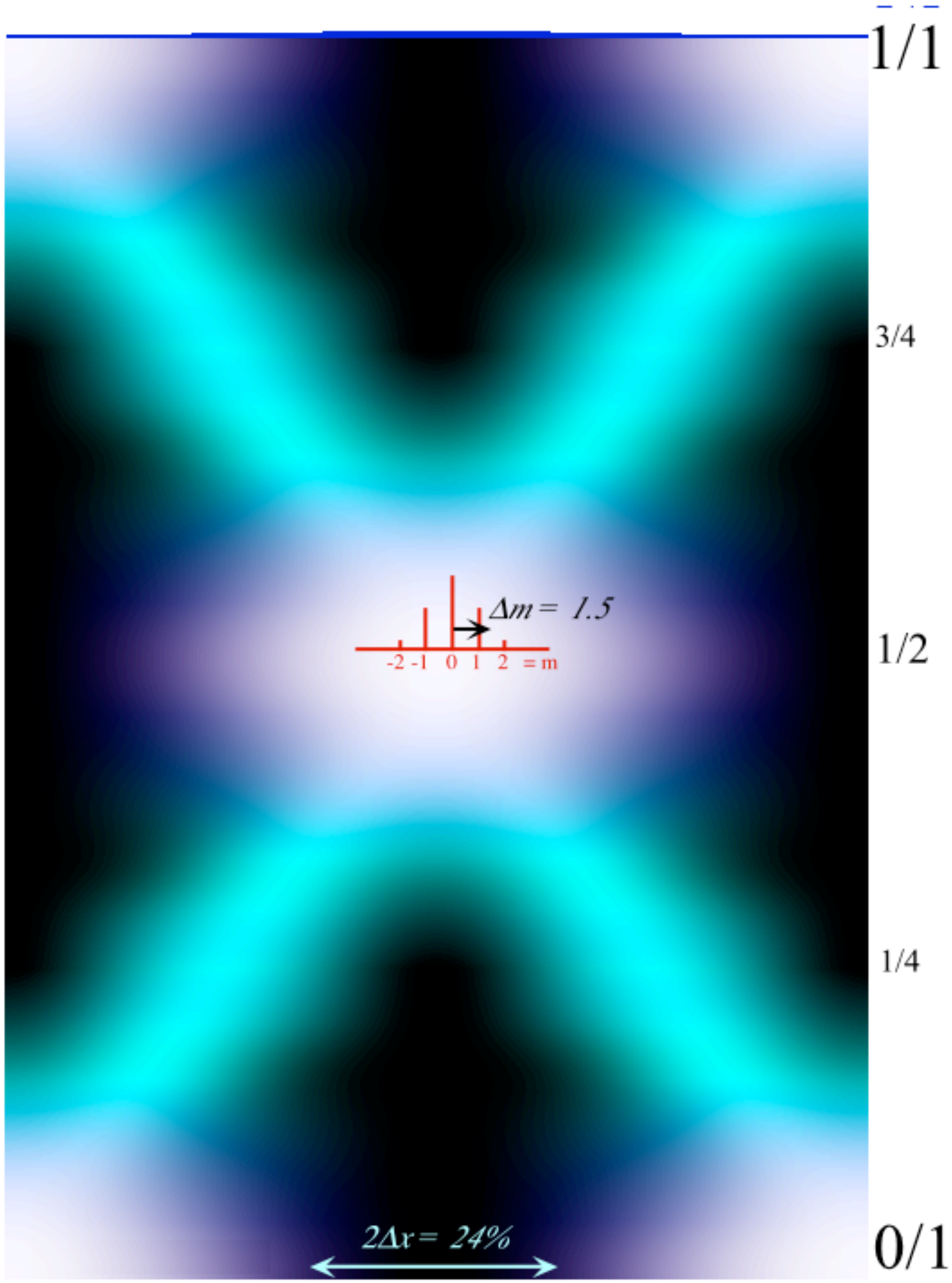
fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$ and lowest *transition (beat) frequency* $\upsilon_1 = (E_1 - E_0)/h$

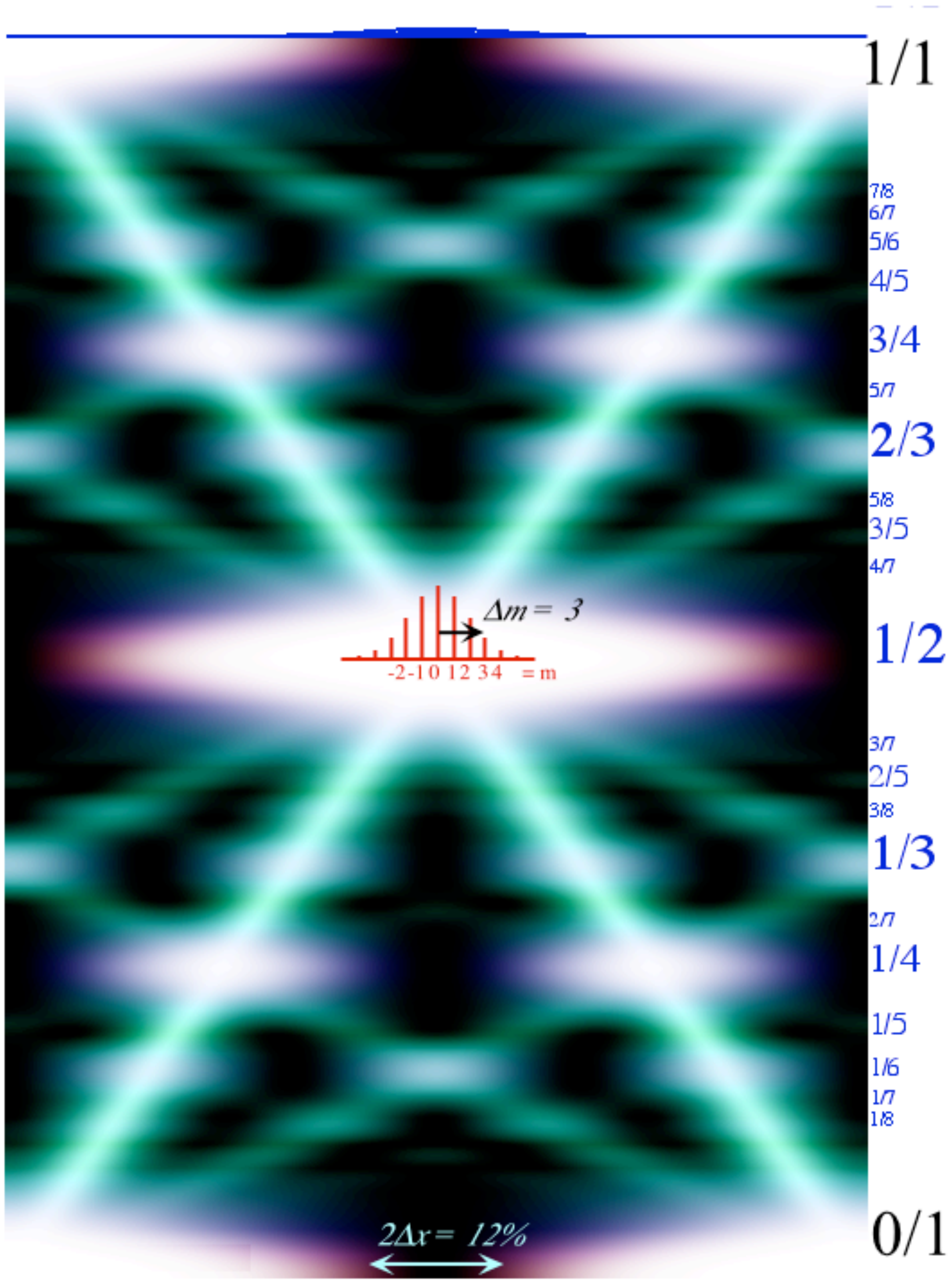
Polygonal geometry of U(2)⊃C_N character spectral function
Algebra
Geometry

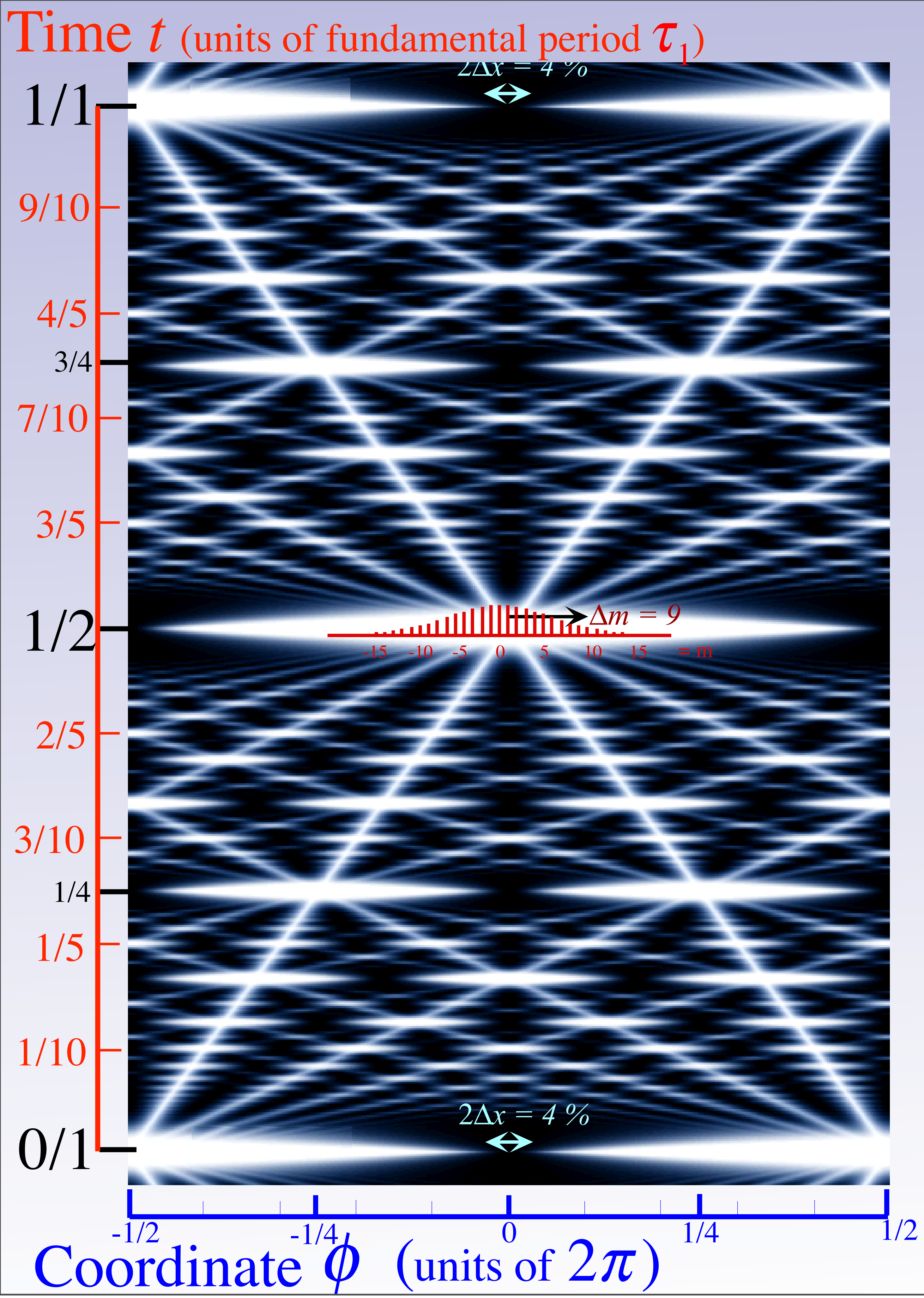
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x explosion and revivals*
- *Bohr-rotor dynamics*
  - *Gaussian wave-packet bandwidth and uncertainty*
  - → *Gaussian Bohr-rotor revivals* ←
- *Farey-Sums and Ford-products*
- *Phase dynamics*

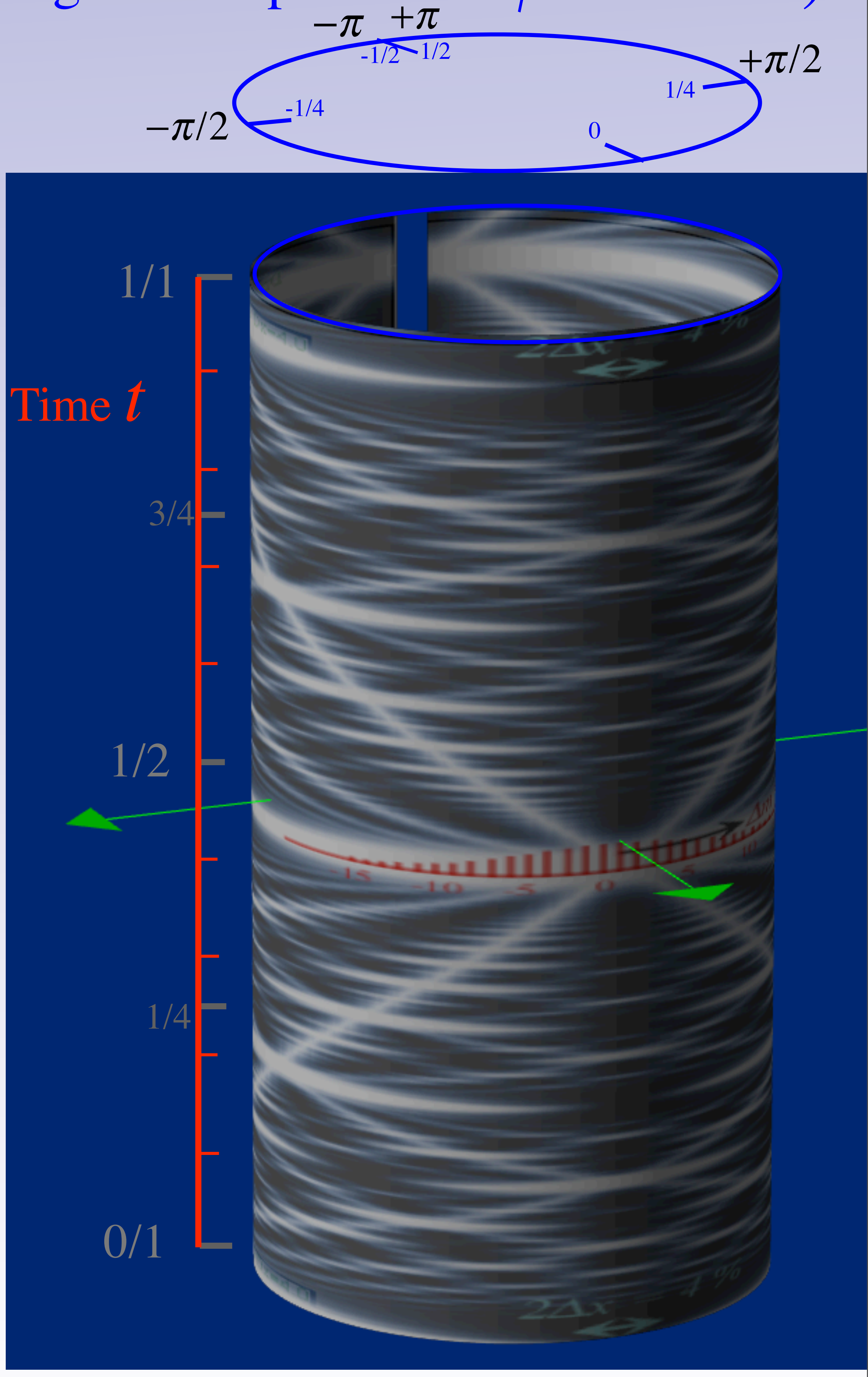

1/1
3/4
Δm = 1.5
-2 -1 0 1 2 = m
1/2
1/4
2Δx = 24%
0/1


1/1
3/4
Δm = 1.5
-2 -1 0 1 2 = m
1/2
1/4
2Δx = 24%
0/1


1/1
7/8
6/7
5/6
4/5
3/4
5/7
2/3
5/8
3/5
4/7
Δm = 3
-2-1 0 1 2 3 4 = m
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
2Δx = 12%
0/1



(Imagine "wrap-around" ϕ-coordinate)

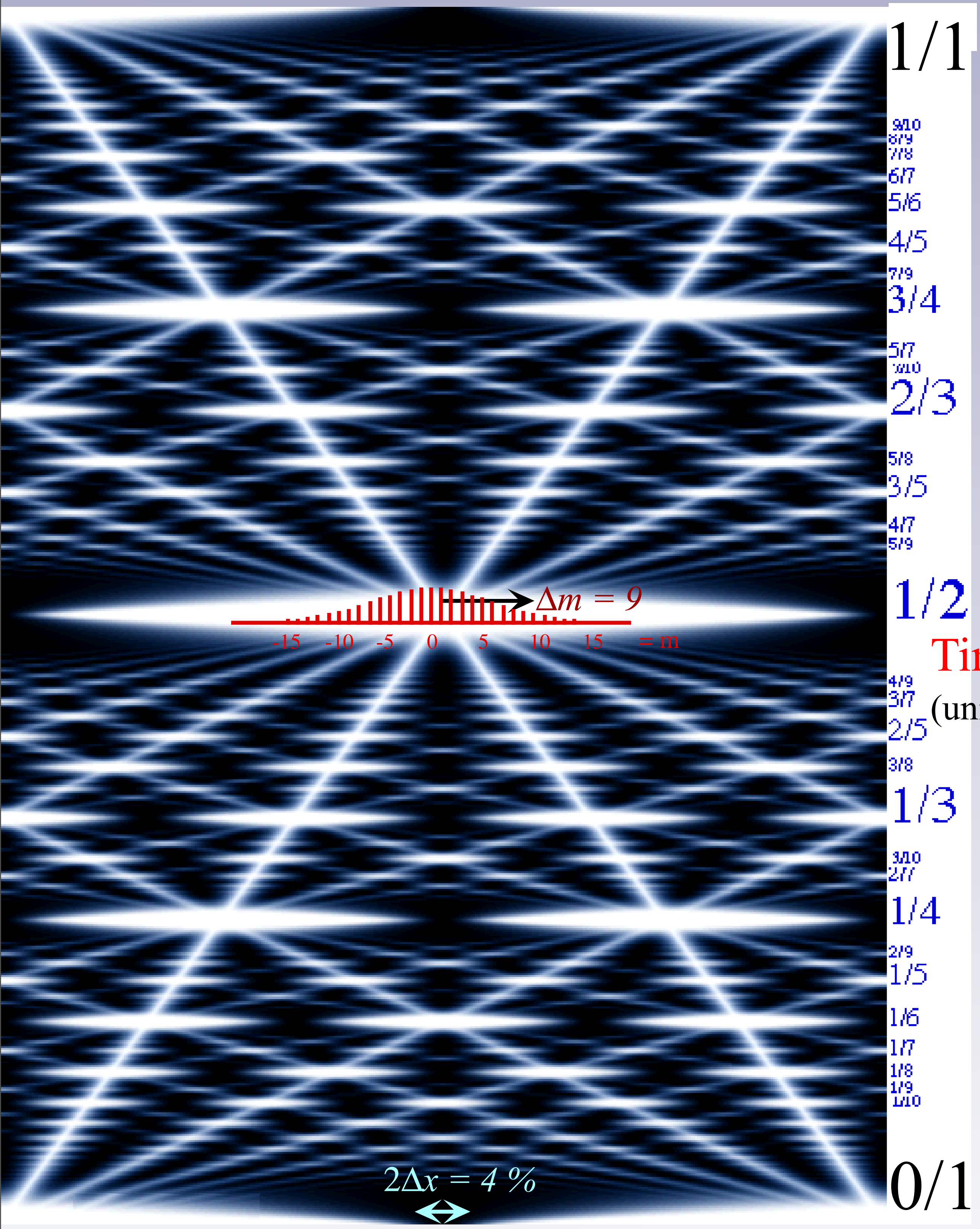


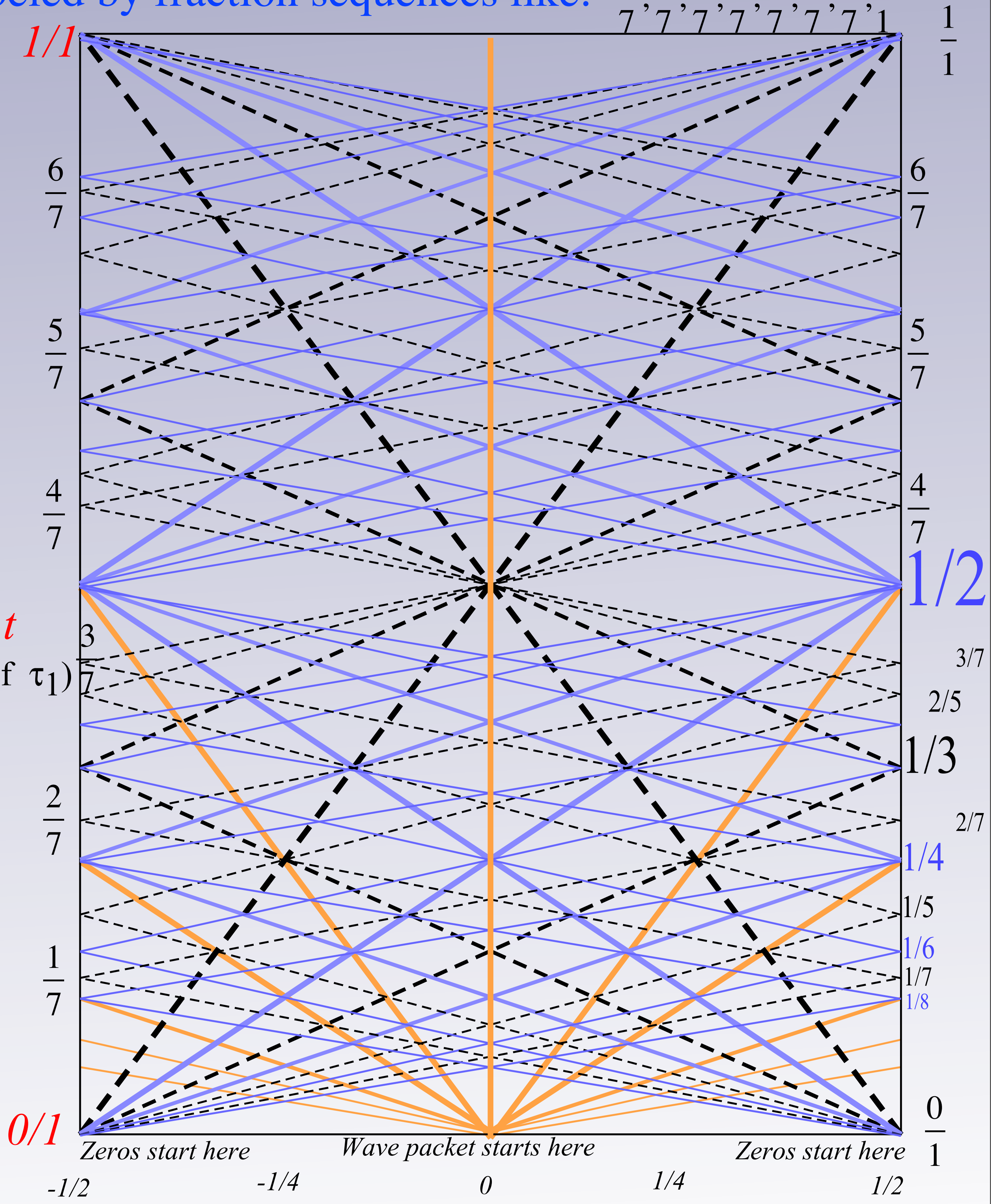
[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

N-level-system and revival-beat wave dynamics

(9 or10-levels $(0, \pm1, \pm2, \pm3, \pm4, ..., \pm9, \pm10, \pm11...)$ excited)

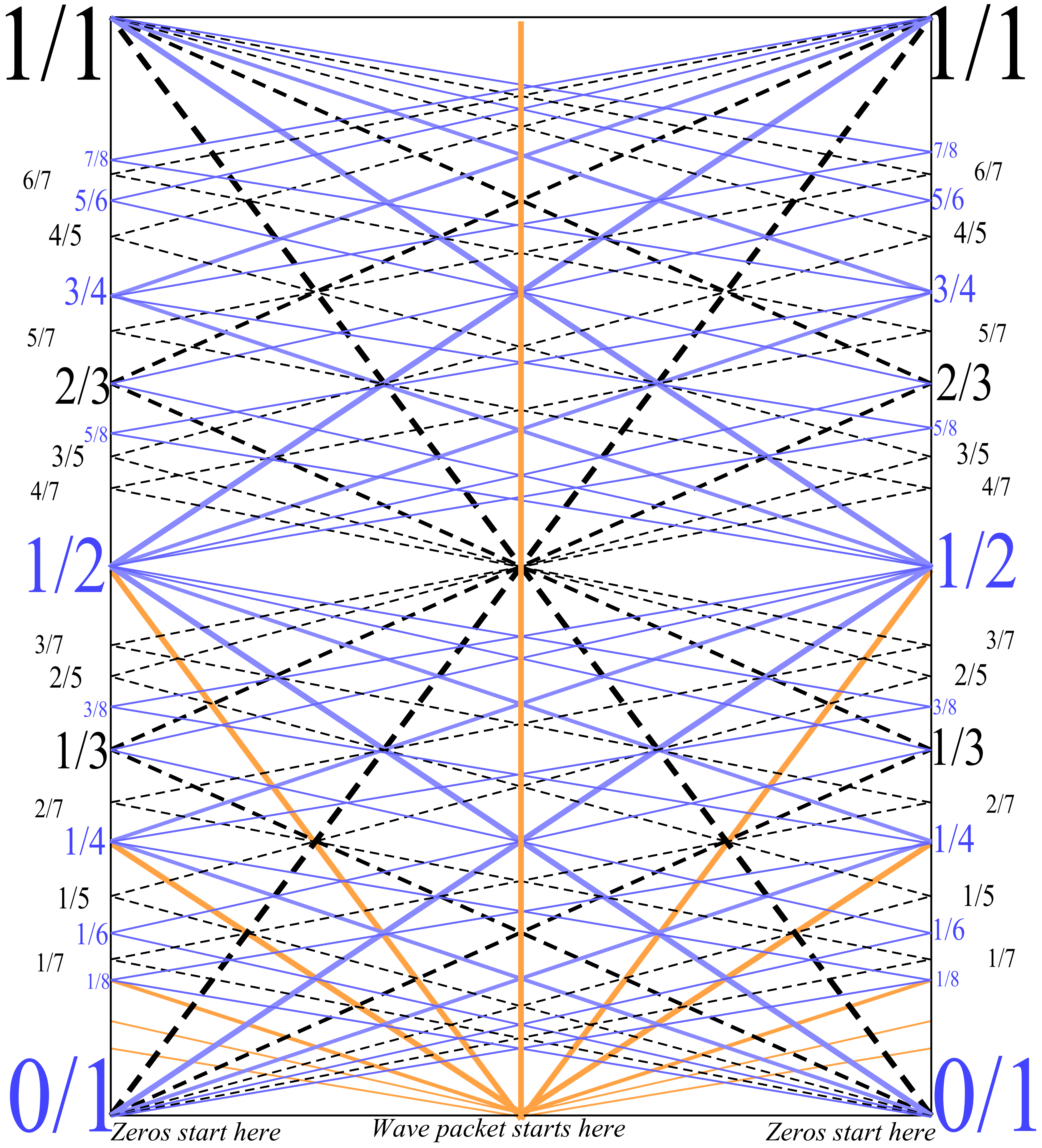
Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$





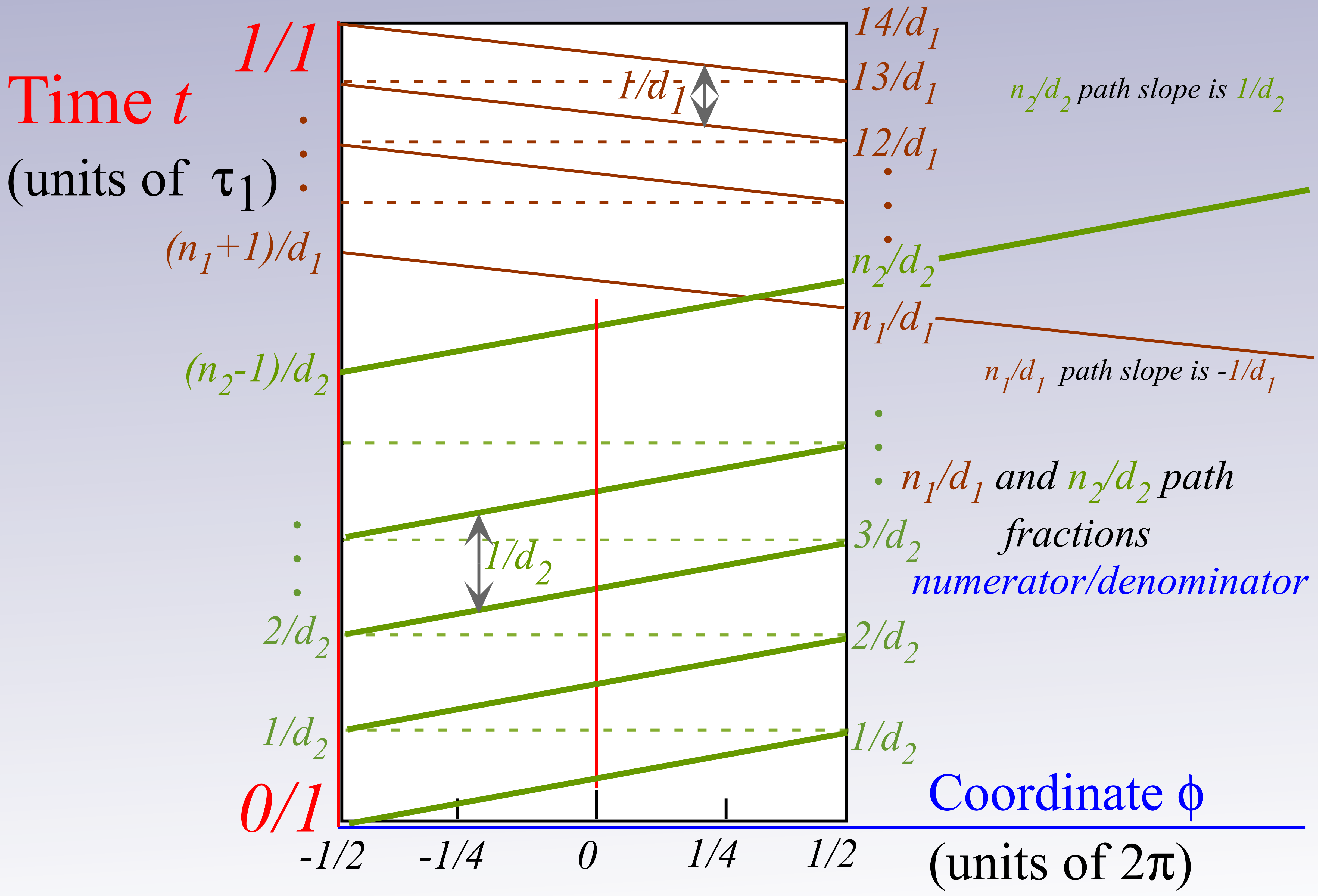
Coordinate ϕ
(units of 2π)

[Harter, J. Mol. Spec. 210, 166-182 (2001)]


1/1
1/1
7/8
7/8
6/7
6/7
5/6
5/6
4/5
4/5
3/4
3/4
5/7
5/7
2/3
2/3
5/8
5/8
3/5
3/5
4/7
4/7
1/2
1/2
3/7
3/7
2/5
2/5
3/8
3/8
1/3
1/3
2/7
2/7
1/4
1/4
1/5
1/5
1/6
1/6
1/7
1/7
1/8
1/8
0/1
0/1
Zeros start here
Wave packet starts here
Zeros start here

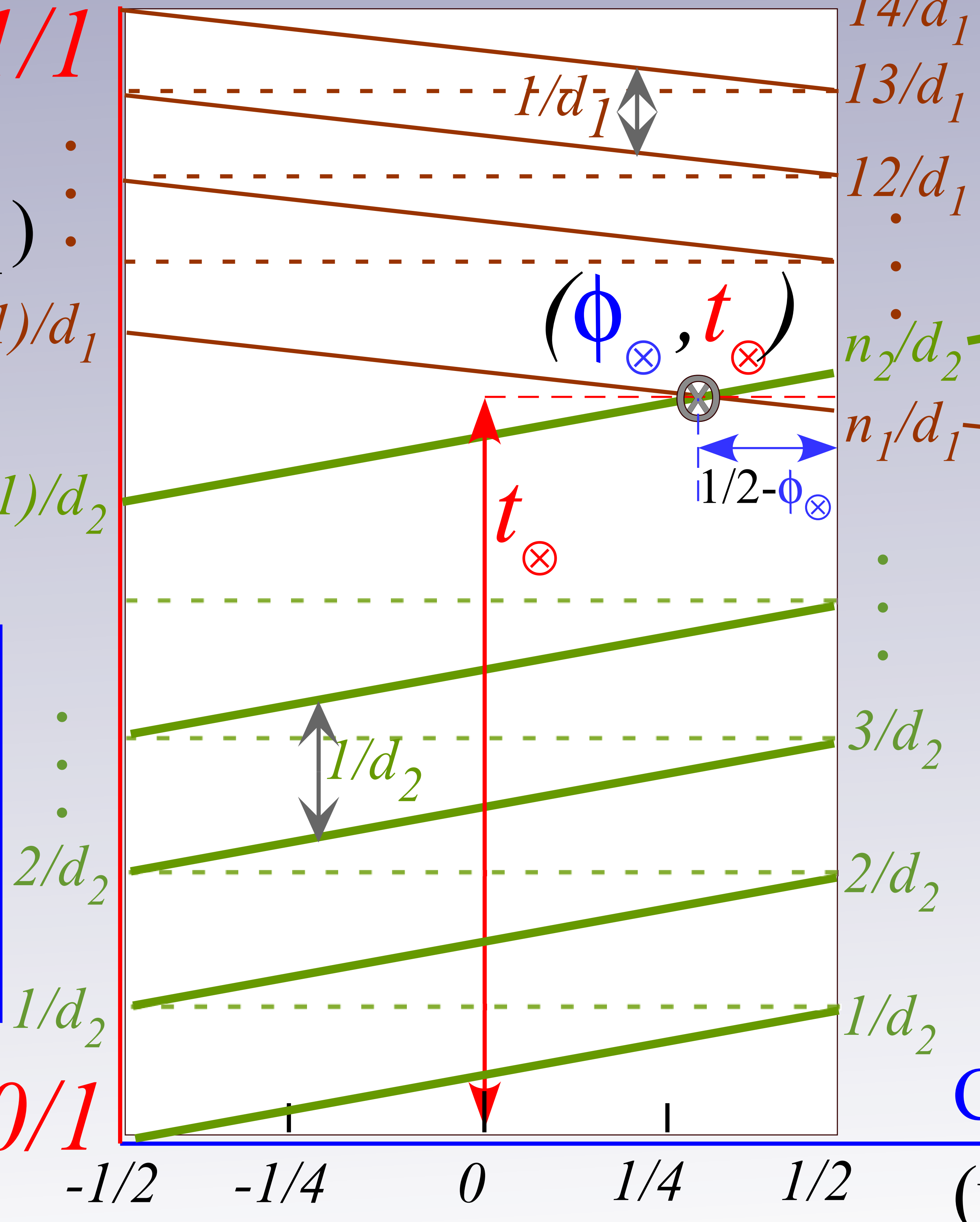
Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_2/d_2 *path slope is* $1/d_2$

$$\frac{n_2/d_2 - t_\otimes}{1/2 - \phi_\otimes} = 1/d_2$$

$$\frac{n_1/d_1 - t_\otimes}{1/2 - \phi_\otimes} = -1/d_1$$

n_1/d_1 *path slope is* $-1/d_1$

n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

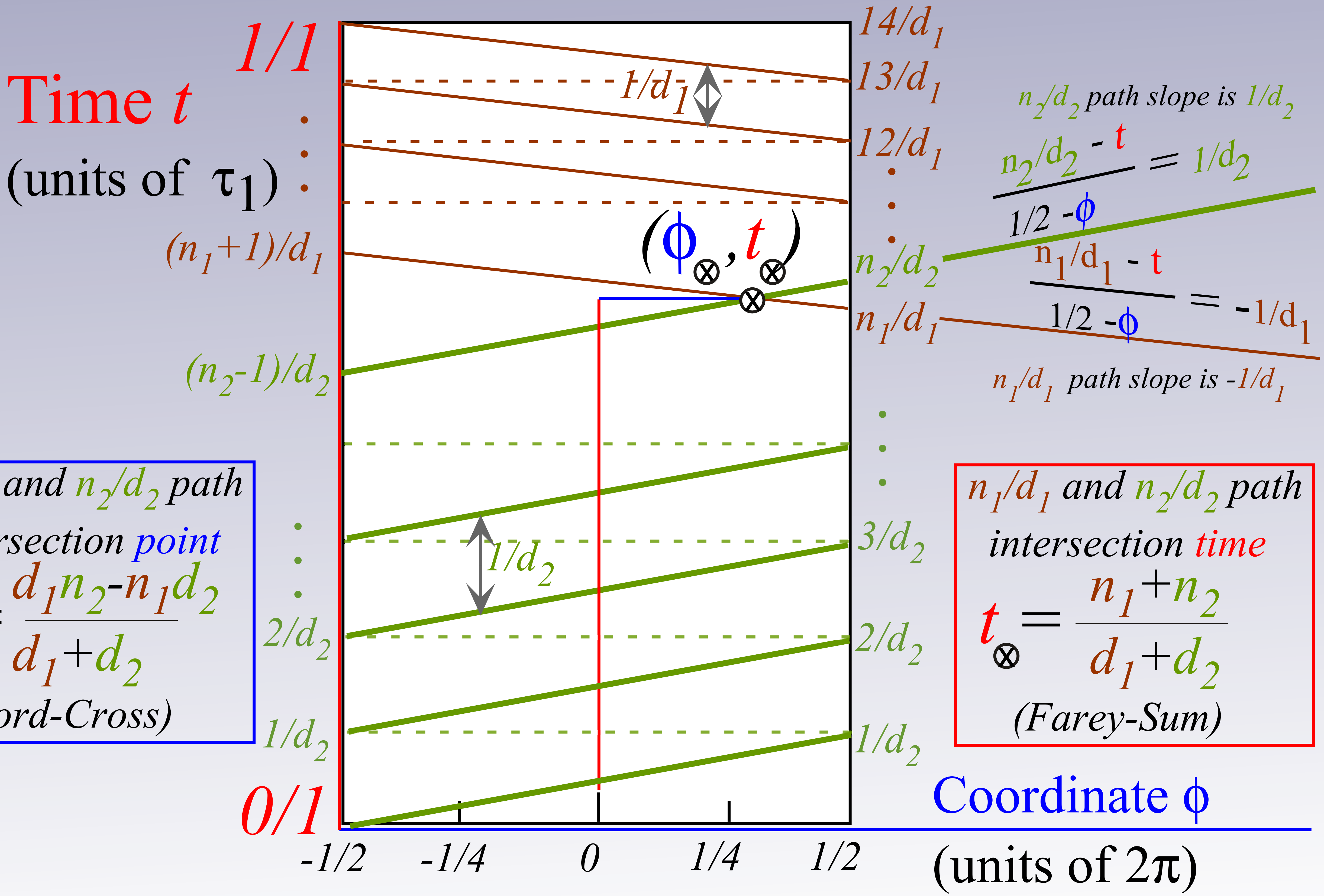
(Farey-Sum)

[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection **point**

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

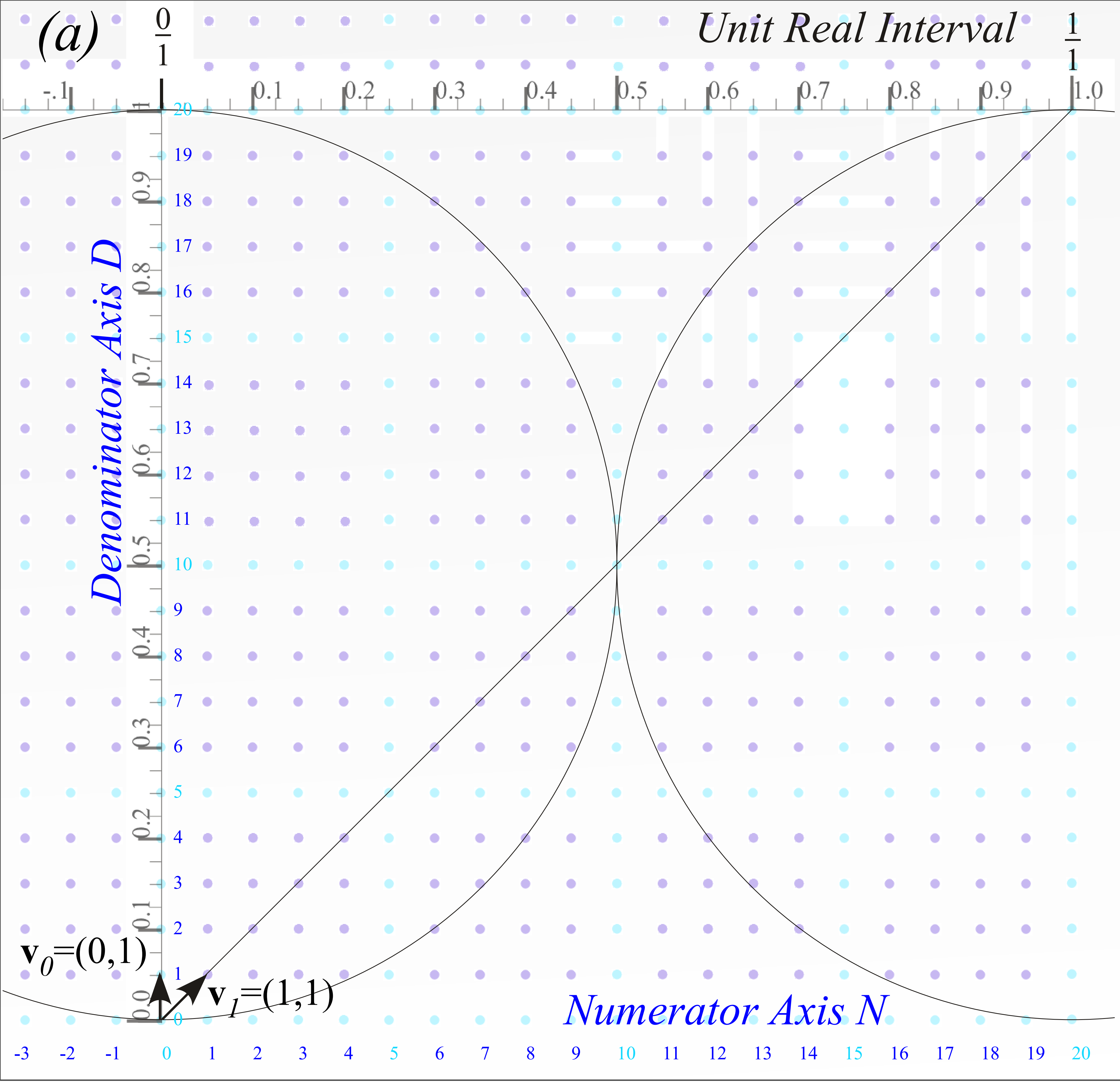
n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

(Farey-Sum)

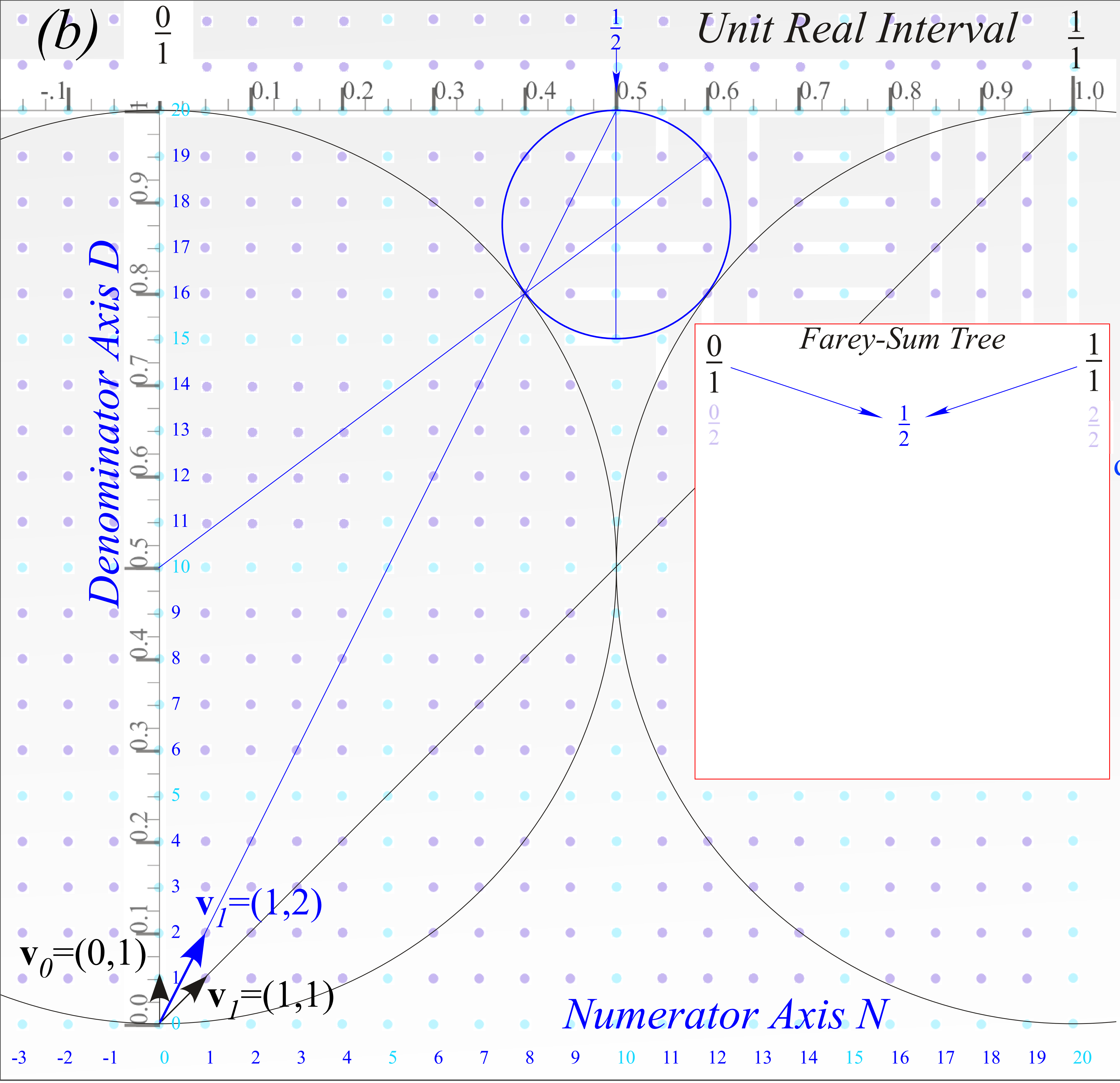
[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

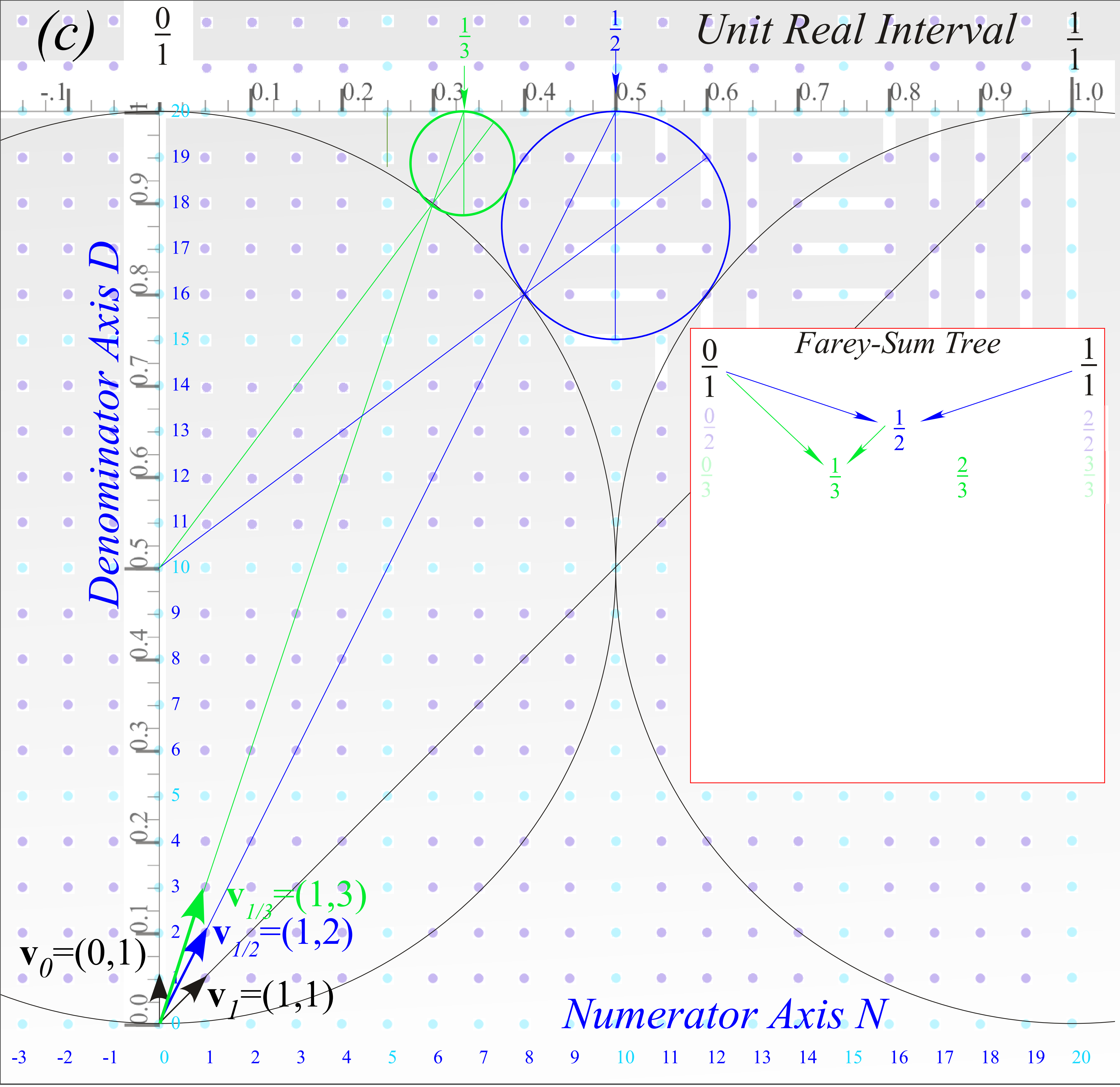


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

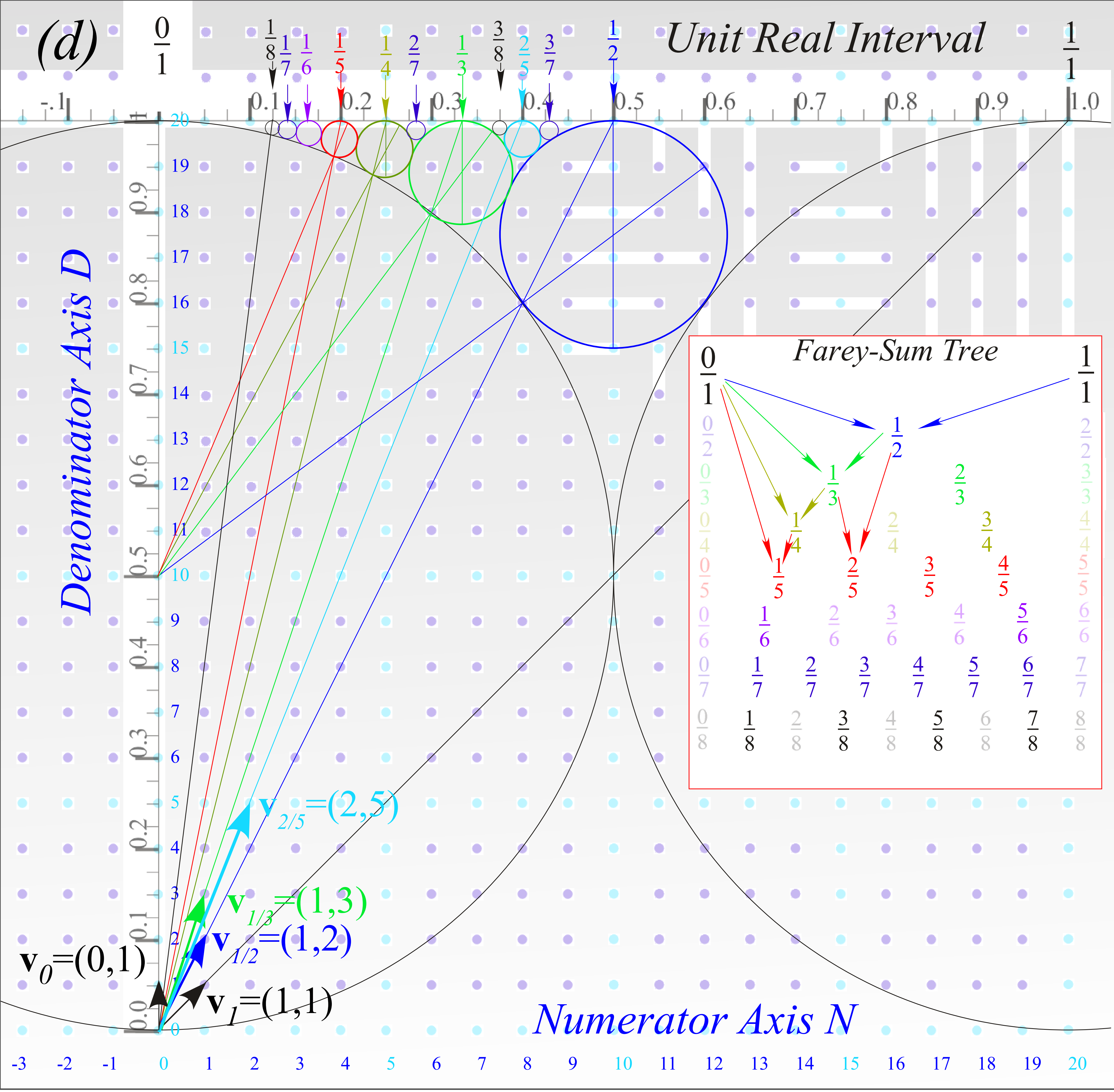
1/2-circle has diameter $1/2^2=1/4$

Farey Sum related to vector sum and *Ford Circles*



1/2-circle has diameter *$1/2^2=1/4$*

1/3-circles have diameter *$1/3^2=1/9$*



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

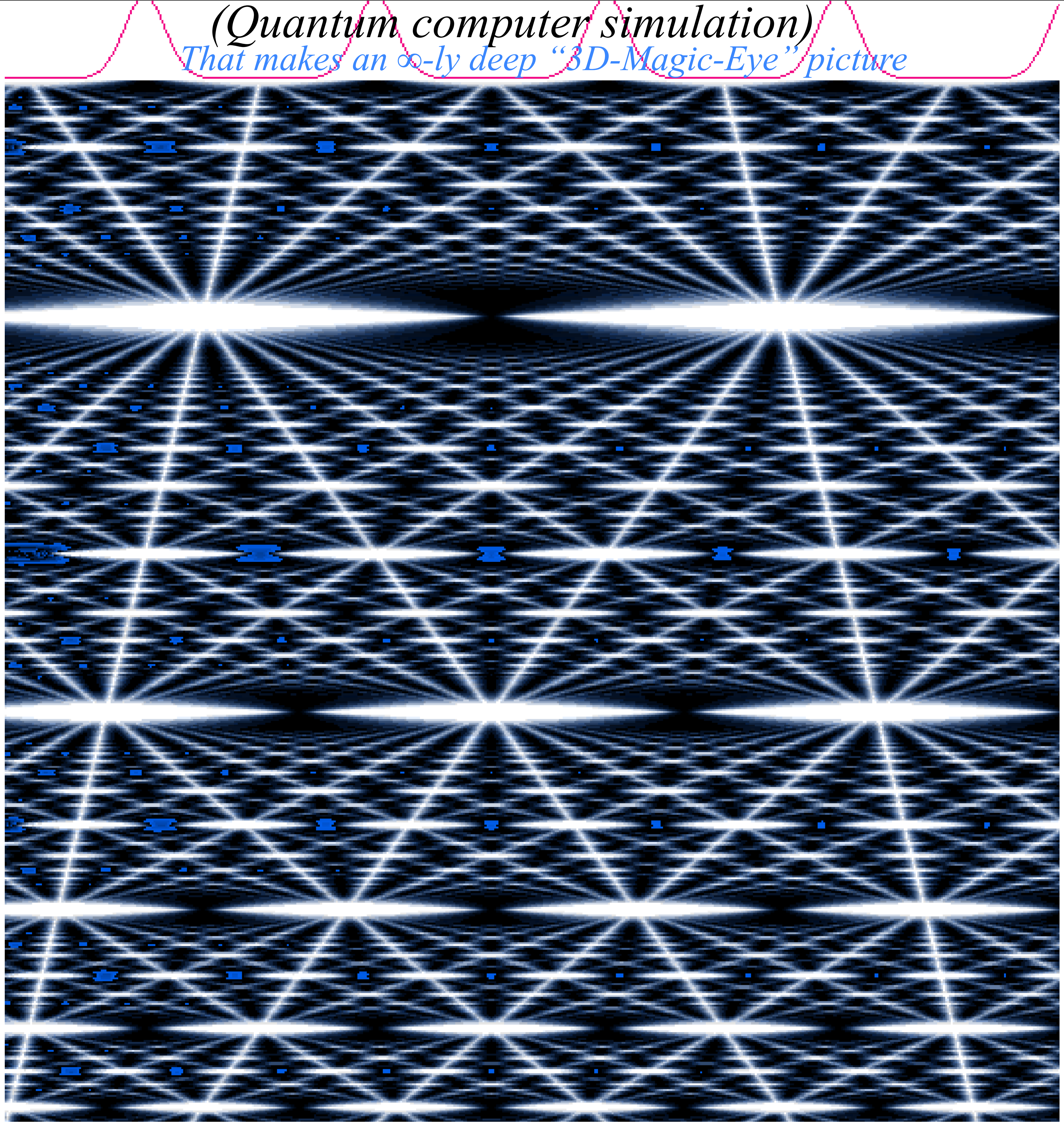
1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

$D \leq 1$	$\boldsymbol{\frac{0}{1}}$																						$\boldsymbol{\frac{1}{1}}$
$D \leq 2$	$\frac{0}{1}$											$\boldsymbol{\frac{1}{2}}$											$\frac{1}{1}$
$D \leq 3$	$\frac{0}{1}$							$\boldsymbol{\frac{1}{3}}$				$\frac{1}{2}$				$\boldsymbol{\frac{2}{3}}$							$\frac{1}{1}$
$D \leq 4$	$\frac{0}{1}$					$\boldsymbol{\frac{1}{4}}$		$\frac{1}{3}$				$\frac{1}{2}$				$\frac{2}{3}$		$\boldsymbol{\frac{3}{4}}$					$\frac{1}{1}$
$D \leq 5$	$\frac{0}{1}$				$\boldsymbol{\frac{1}{5}}$	$\frac{1}{4}$		$\frac{1}{3}$		$\boldsymbol{\frac{2}{5}}$		$\frac{1}{2}$		$\boldsymbol{\frac{3}{5}}$		$\frac{2}{3}$		$\frac{3}{4}$	$\boldsymbol{\frac{4}{5}}$				$\frac{1}{1}$
$D \leq 6$	$\frac{0}{1}$			$\boldsymbol{\frac{1}{6}}$	$\frac{1}{5}$	$\frac{1}{4}$		$\frac{1}{3}$		$\frac{2}{5}$		$\frac{1}{2}$		$\frac{3}{5}$		$\frac{2}{3}$		$\frac{3}{4}$	$\frac{4}{5}$	$\boldsymbol{\frac{5}{6}}$			$\frac{1}{1}$
$D \leq 7$	$\frac{0}{1}$		$\boldsymbol{\frac{1}{7}}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\boldsymbol{\frac{2}{7}}$	$\frac{1}{3}$		$\frac{2}{5}$	$\boldsymbol{\frac{3}{7}}$	$\frac{1}{2}$	$\boldsymbol{\frac{4}{7}}$	$\frac{3}{5}$		$\frac{2}{3}$	$\boldsymbol{\frac{5}{7}}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\boldsymbol{\frac{6}{7}}$		$\frac{1}{1}$
$D \leq 8$	$\frac{0}{1}$	$\boldsymbol{\frac{1}{8}}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\boldsymbol{\frac{3}{8}}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\boldsymbol{\frac{5}{8}}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\boldsymbol{\frac{7}{8}}$	$\frac{1}{1}$

(Quantum computer simulation)

That makes an ∞-ly deep "3D-Magic-Eye" picture

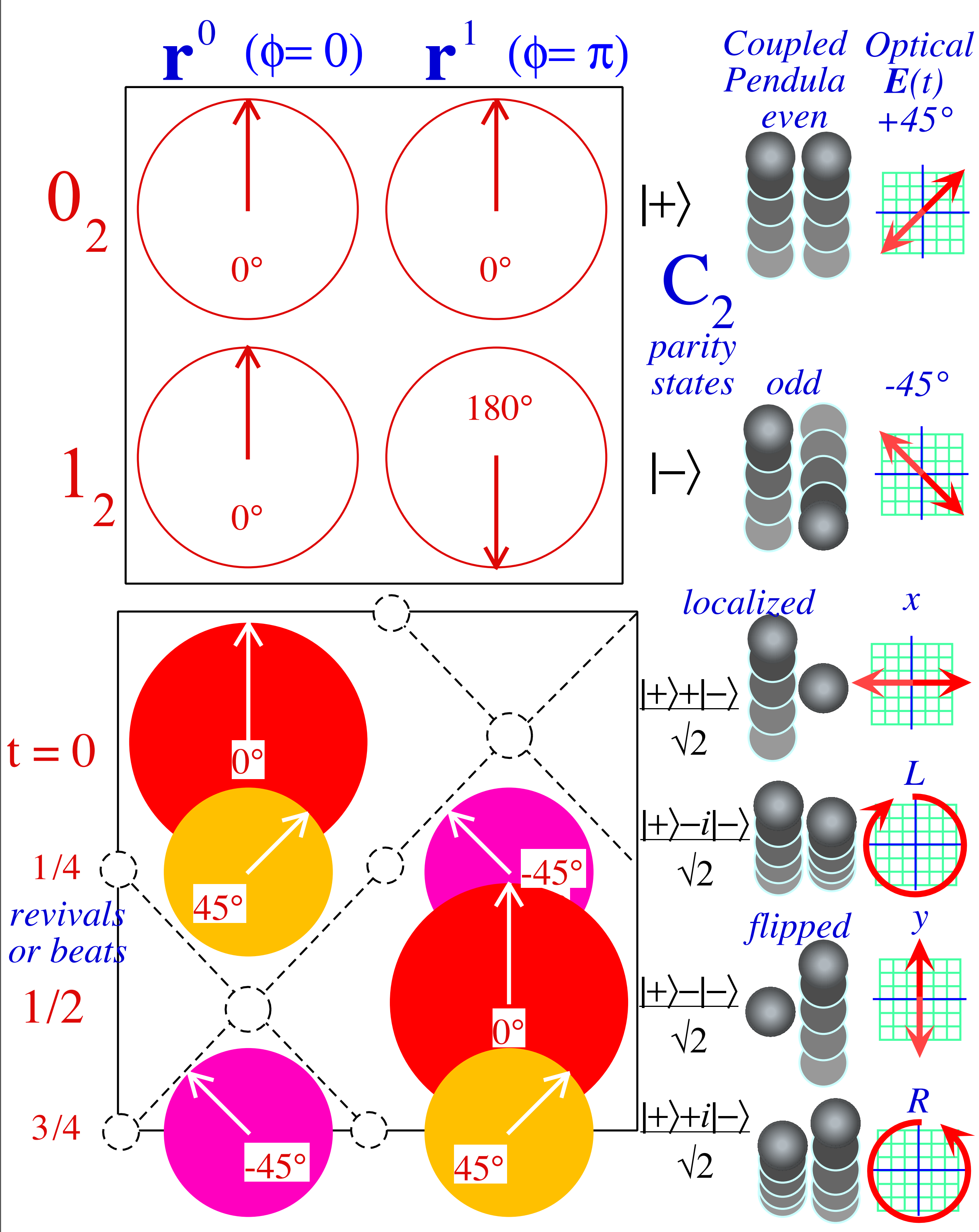


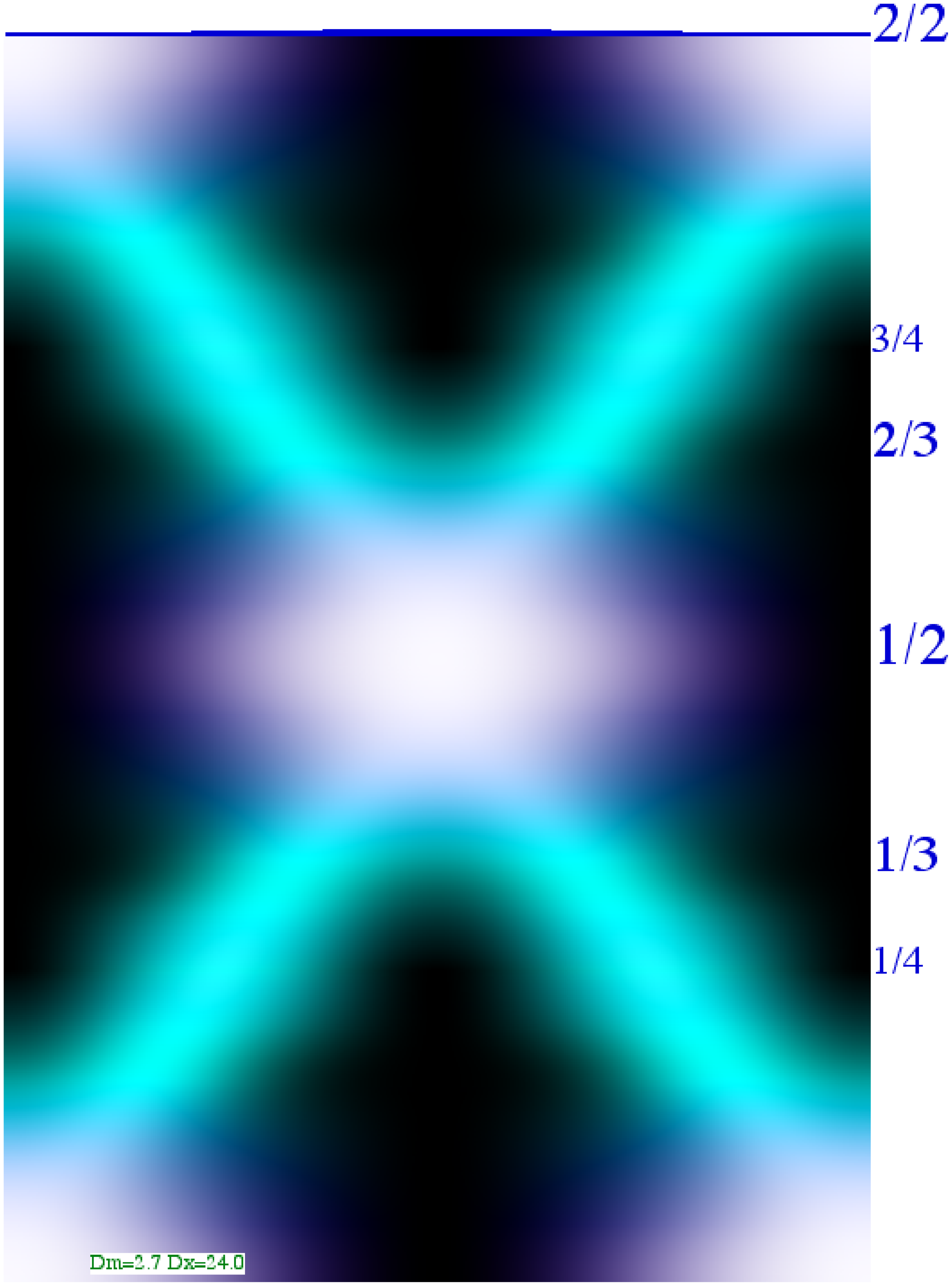
Polygonal geometry of U(2)⊃C_N character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

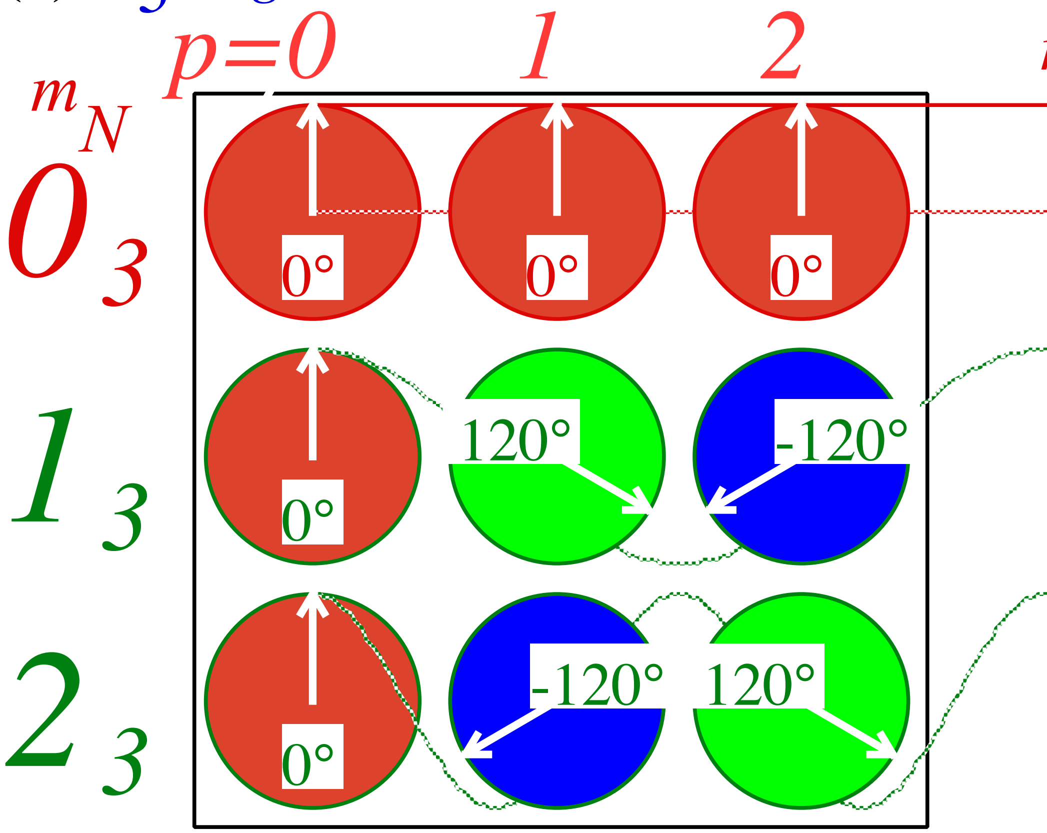
Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x explosion and revivals*
- *Bohr-rotor dynamics*
  - *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- → *Phase dynamics* ←

Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2

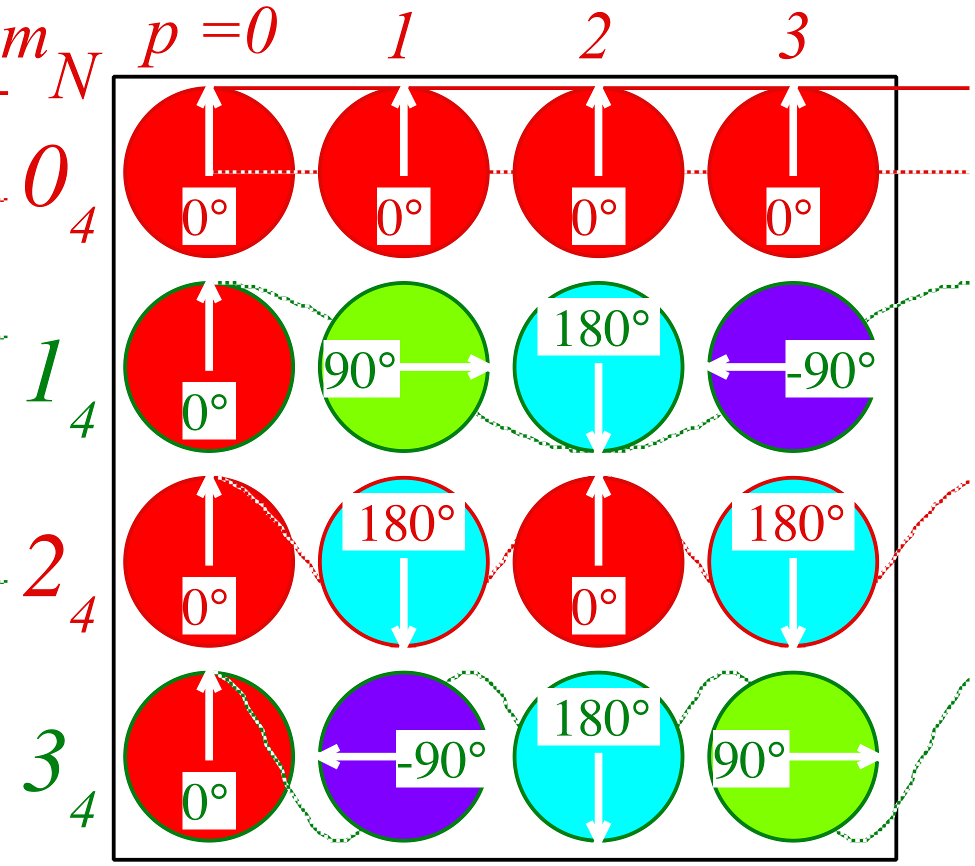




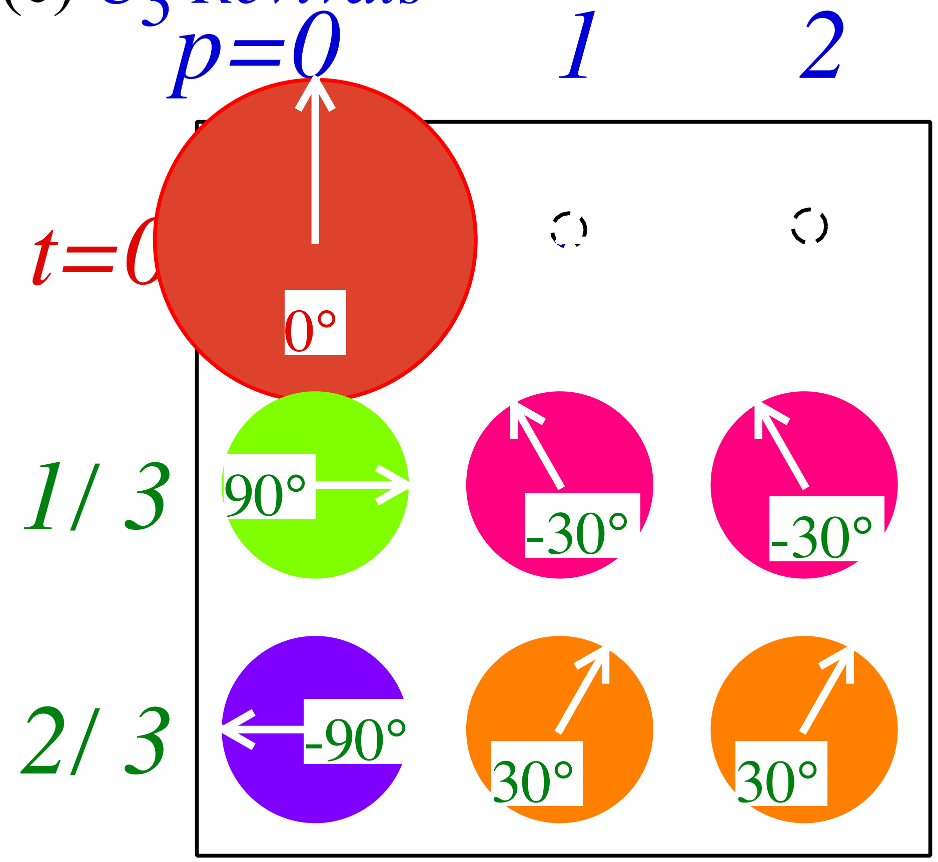
(a) *C_3 Eigenstate Characters*



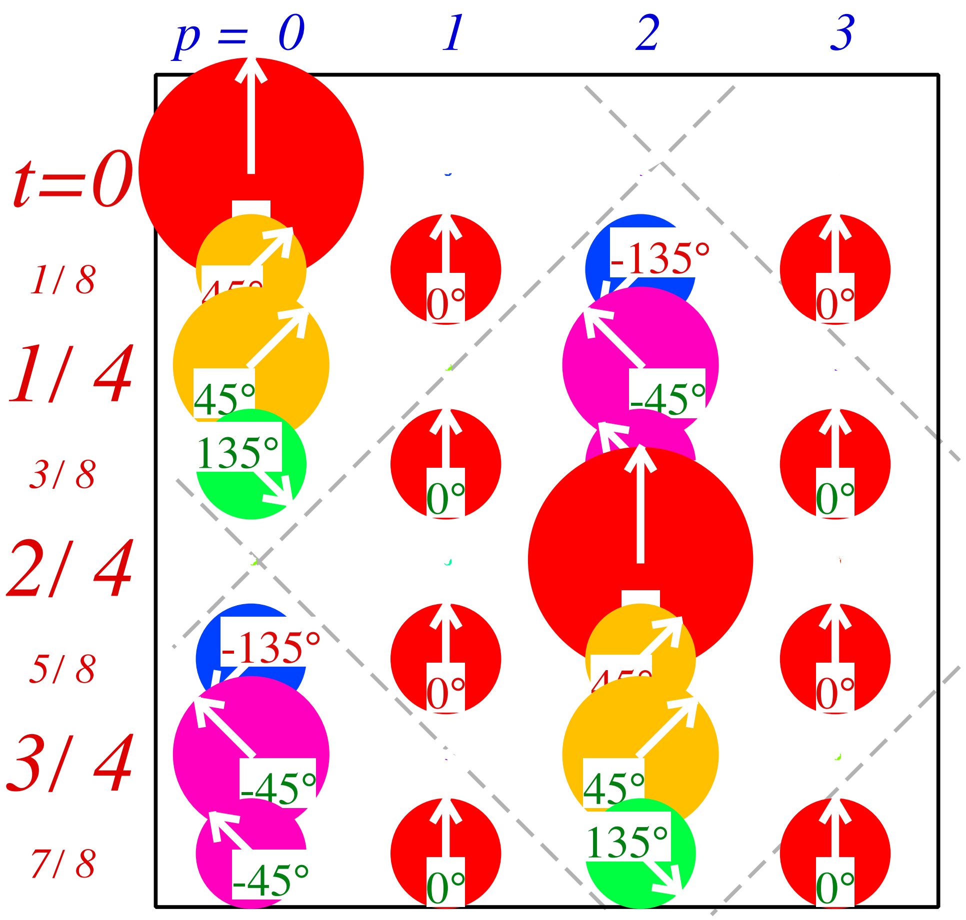
(b) *C_4 Eigenstate Characters*



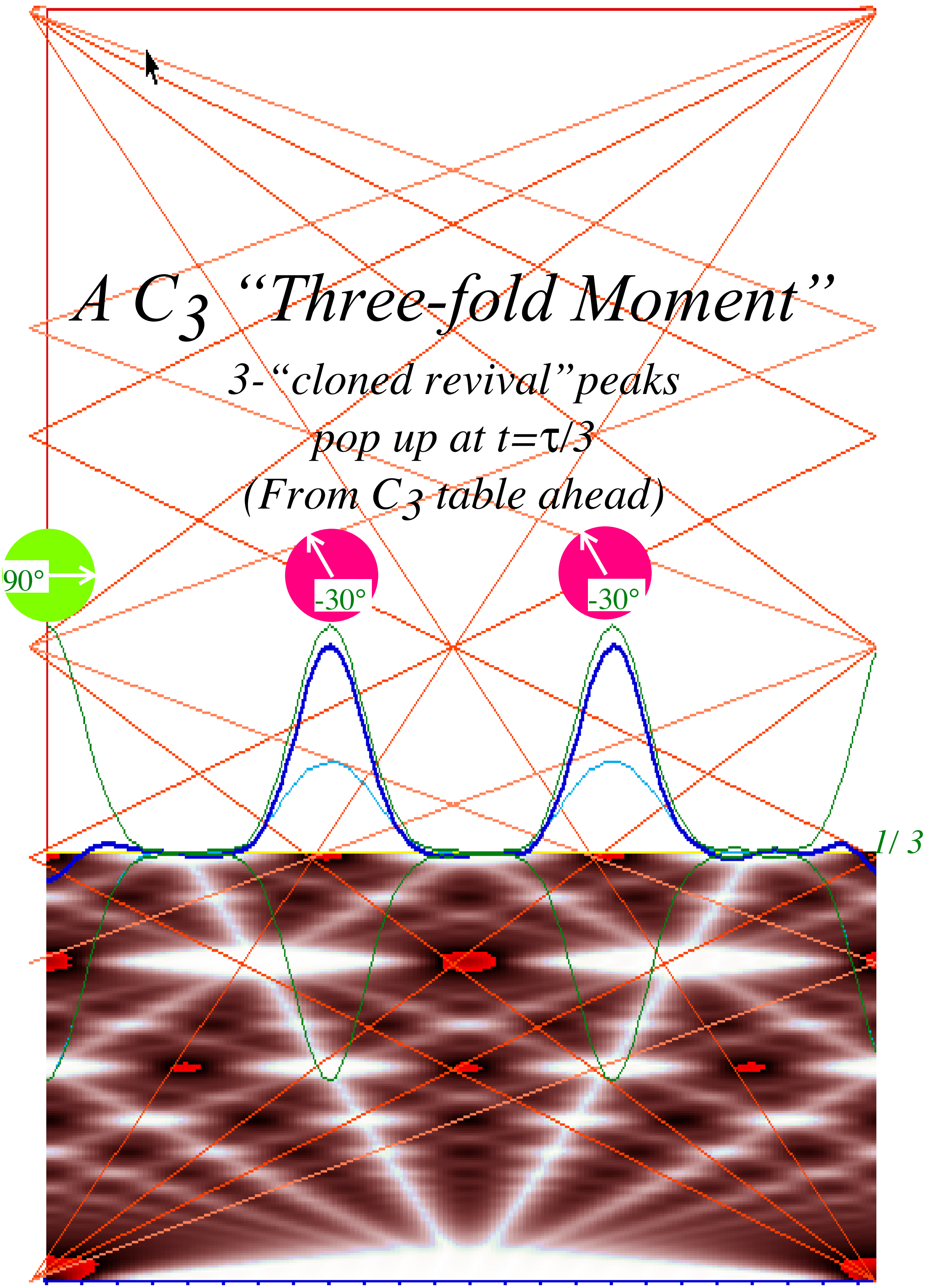
(c) *C_3 Revivals*



(d) *C_4 Revivals*

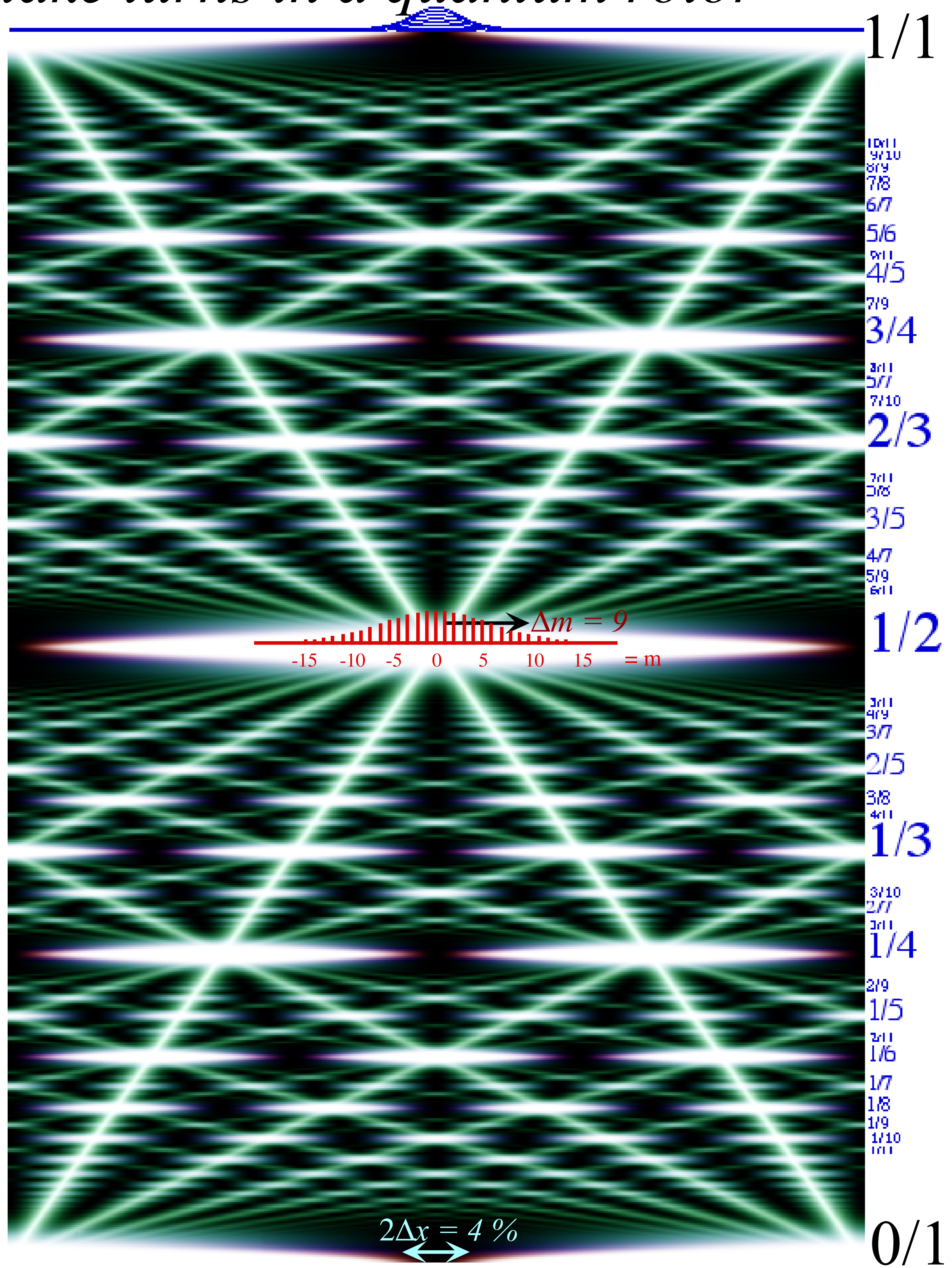


Revivals: All excited transitions take turns in a quantum rotor

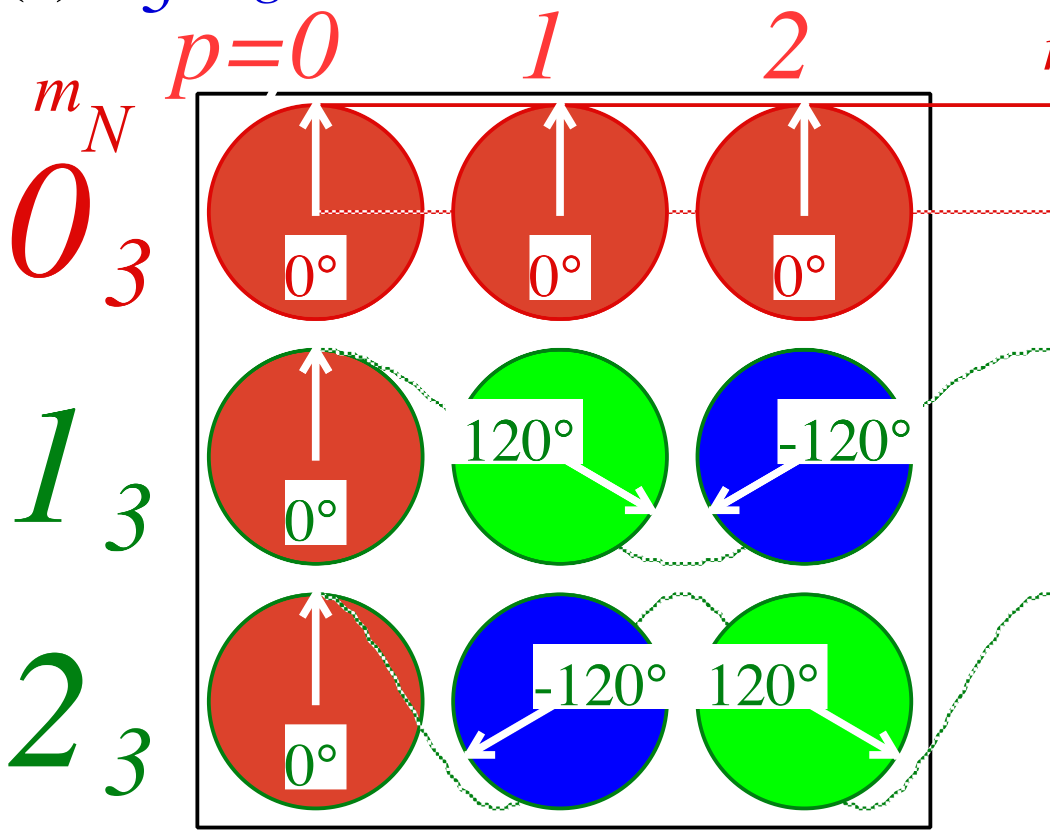


A C_3 "Three-fold Moment"

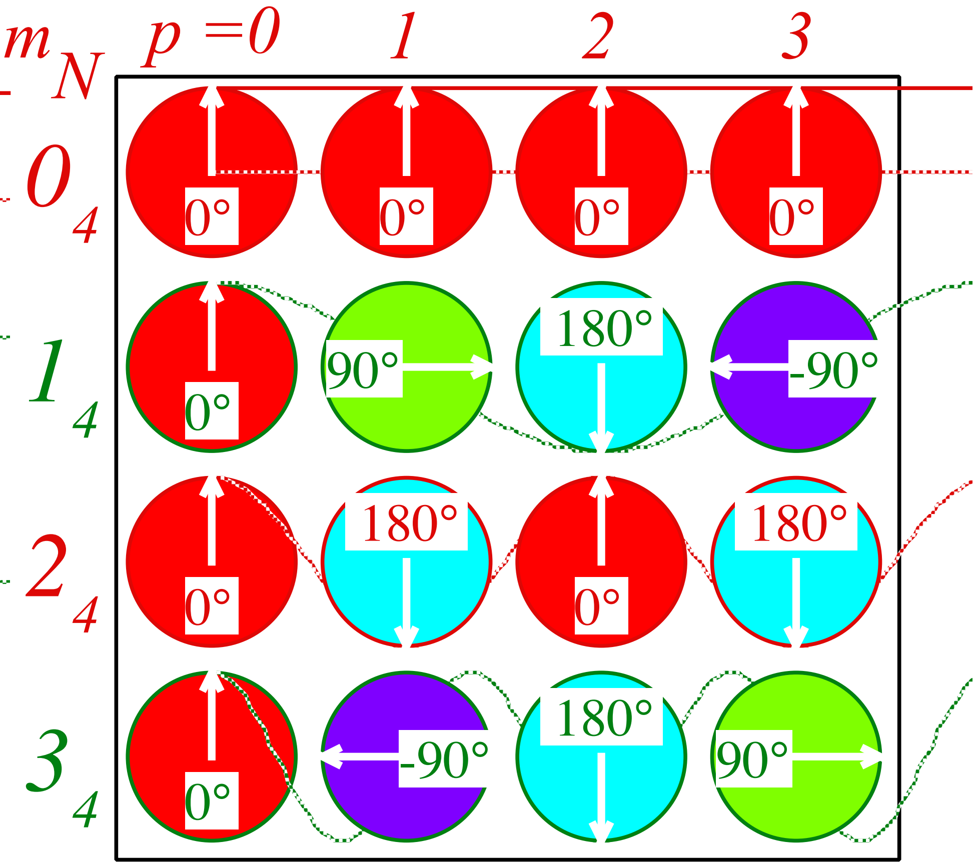
3-"cloned revival" peaks pop up at t=τ/3 (From C_3 table ahead)



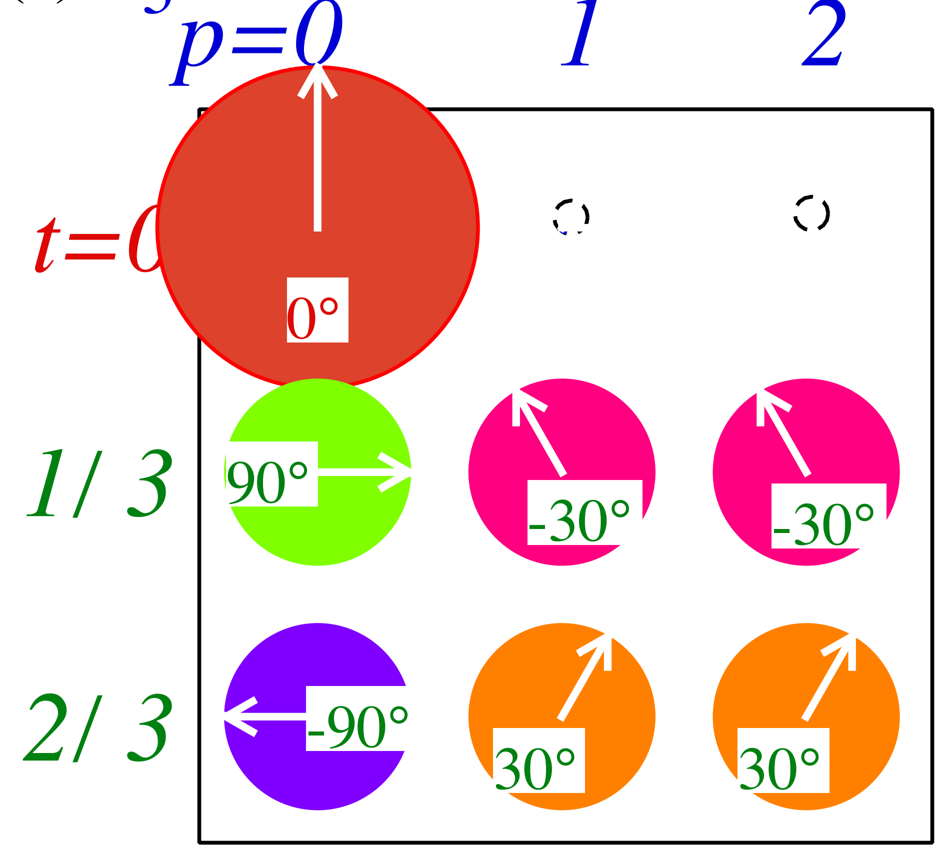
(a) C_3 *Eigenstate Characters*



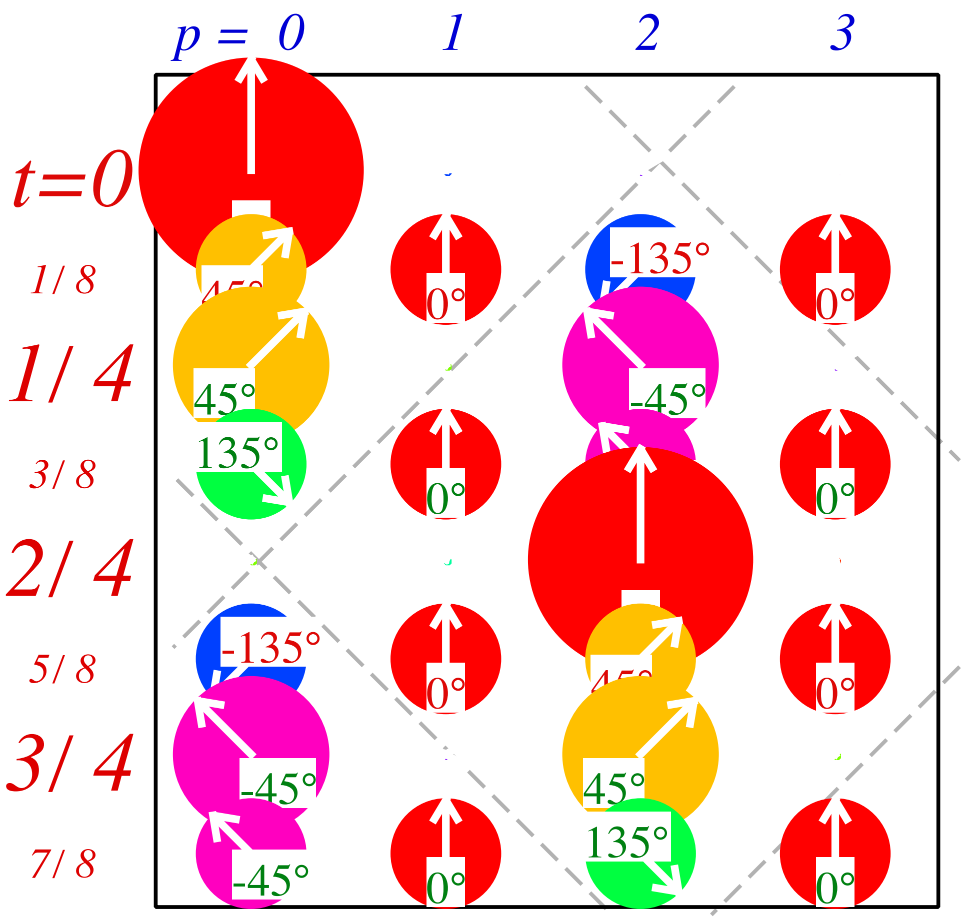
(b) C_4 *Eigenstate Characters*



(c) C_3 *Revivals*



(d) C_4 *Revivals*



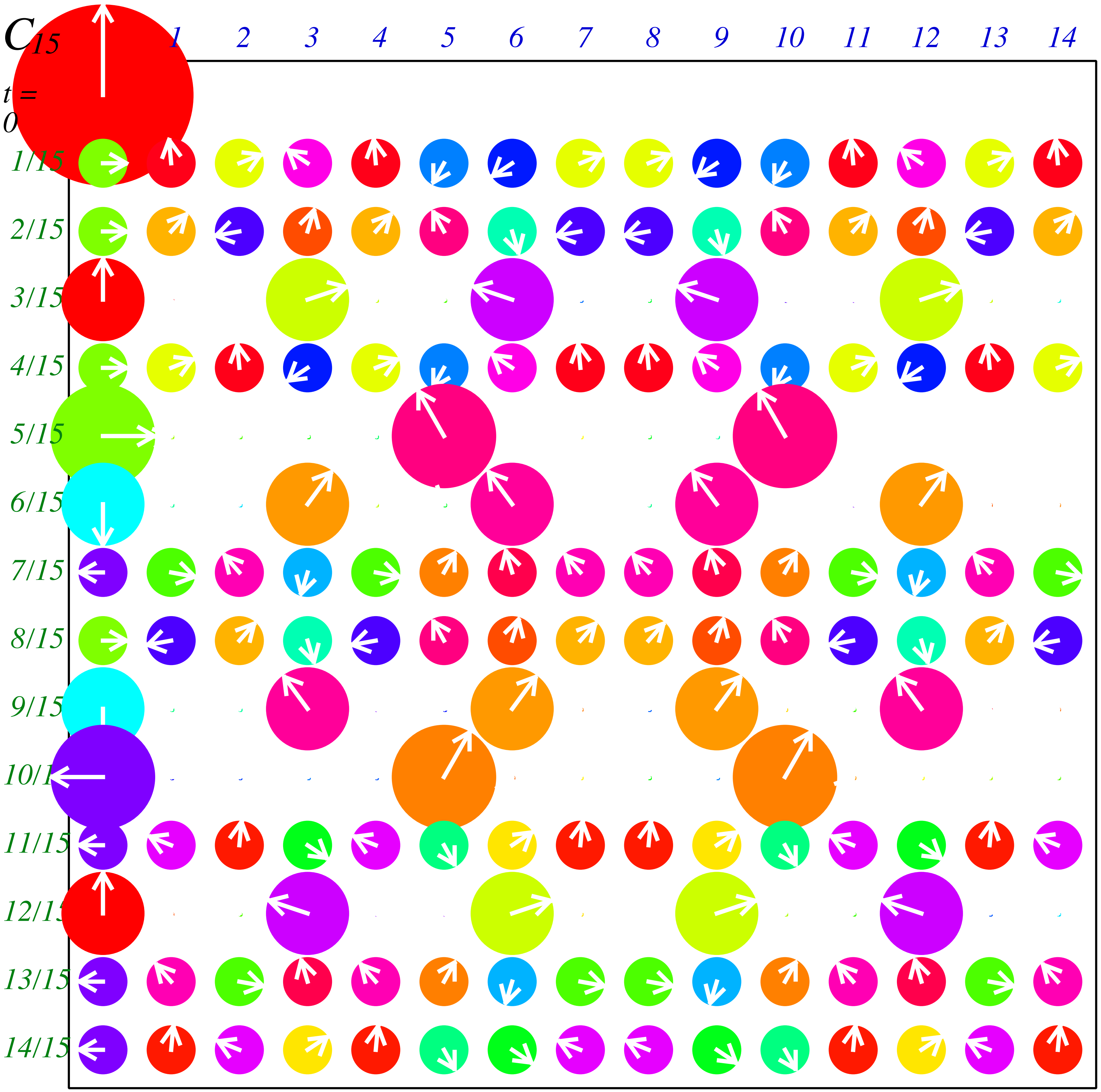
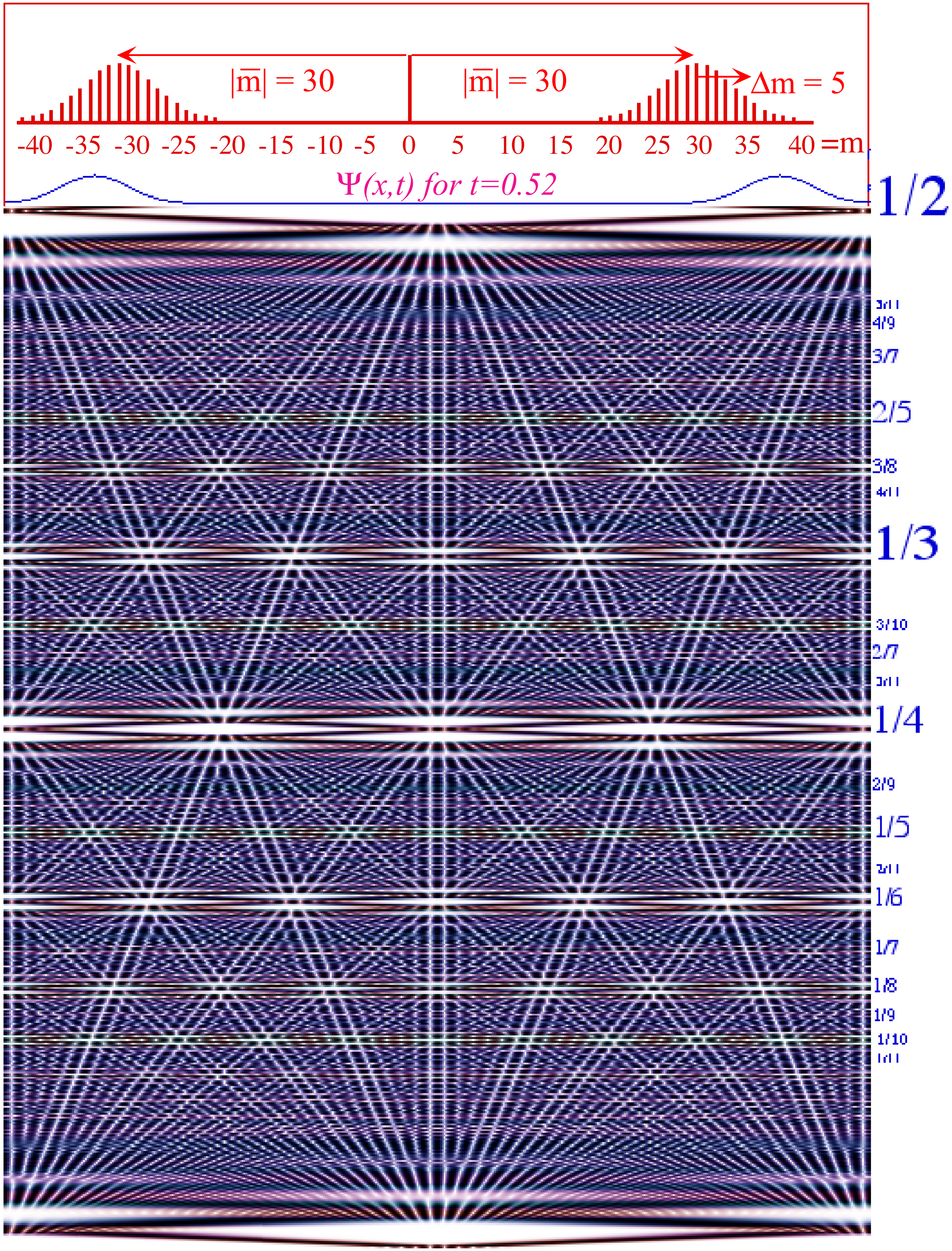
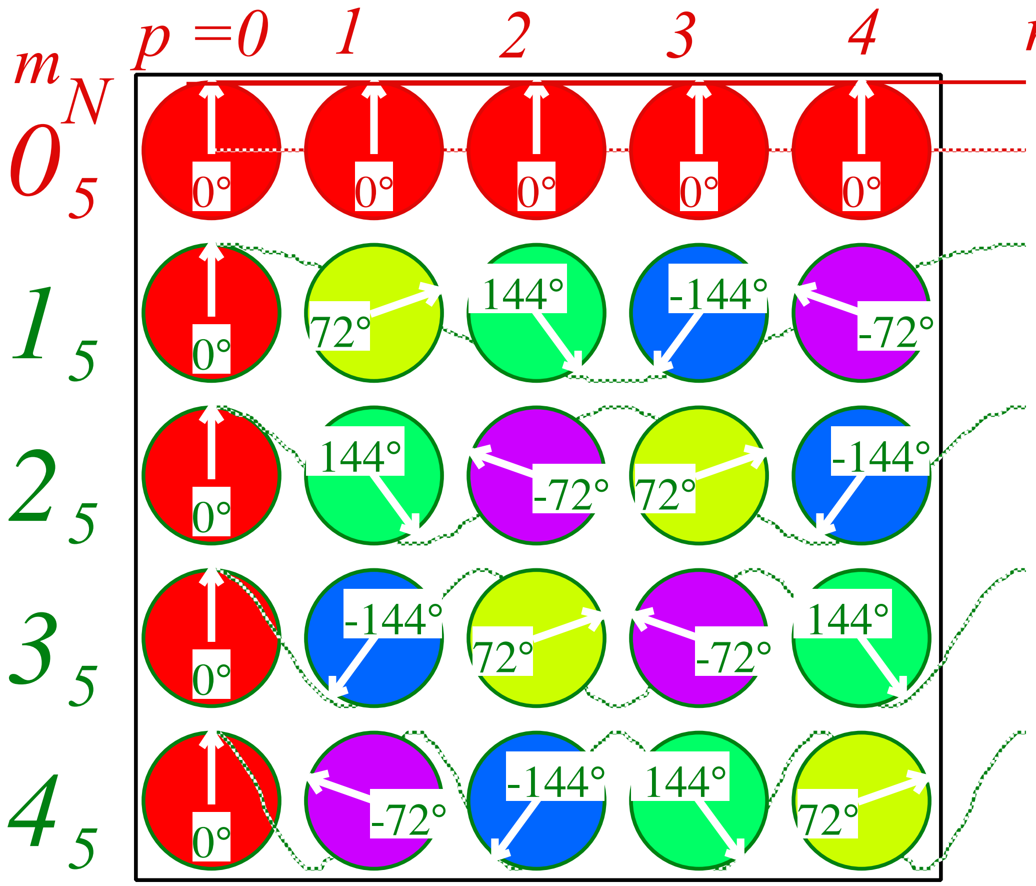


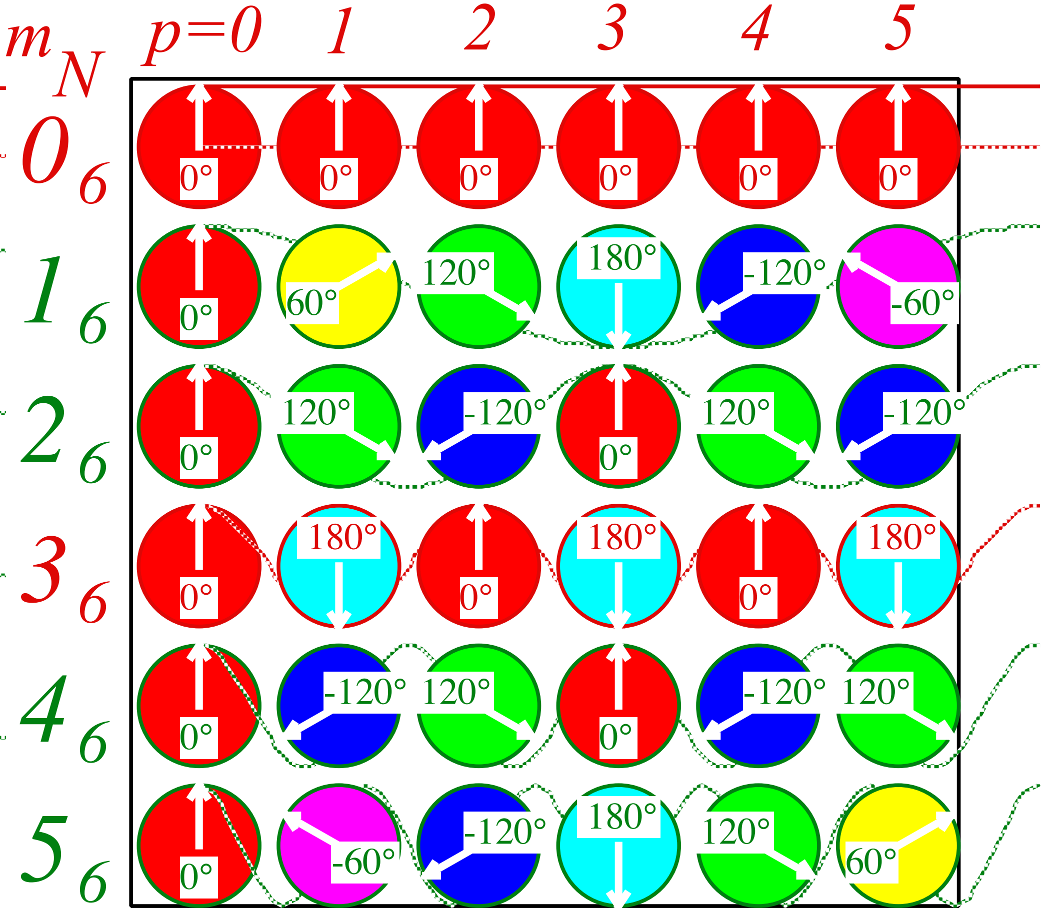
Fig. 9.4.4 Bohr space-time revival pattern for C_{15} Bohr system.


|m̄| = 30
|m̄| = 30
Δm = 5
-40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40 =m
Ψ(x,t) for t=0.52
1/2
5/11
4/9
3/7
2/5
3/8
4/11
1/3
3/10
2/7
3/11
1/4
2/9
1/5
2/11
1/6
1/7
1/8
1/9
1/10
1/11

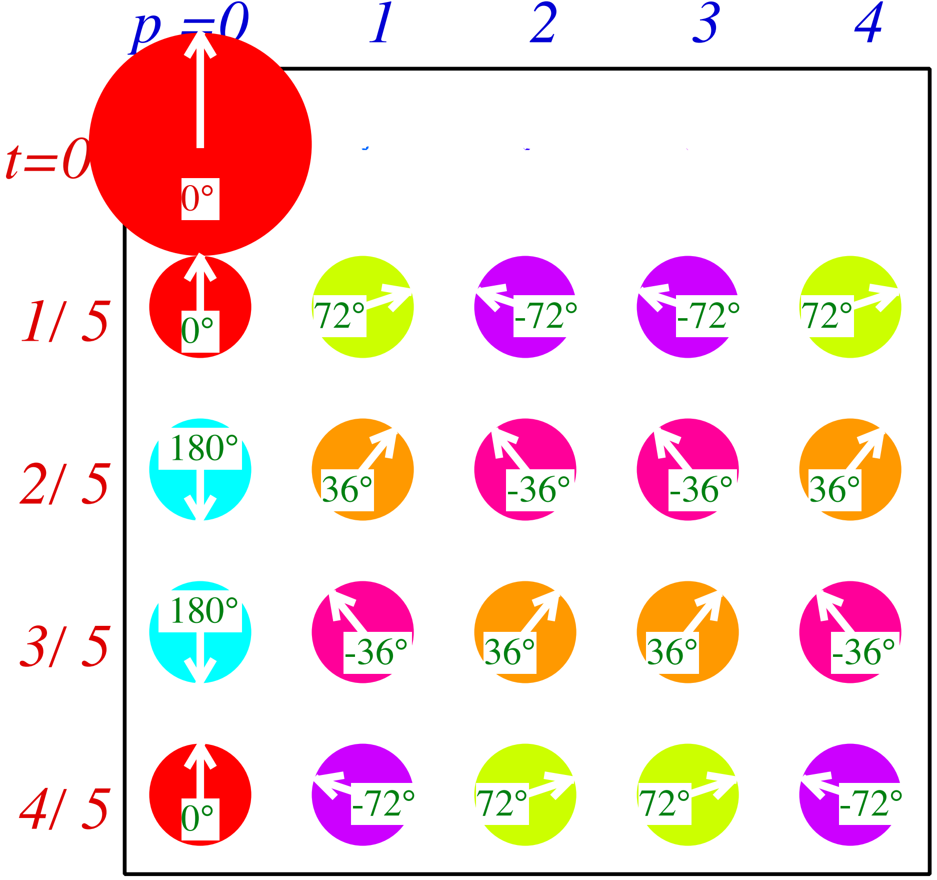
(a) C_5 *Eigenstate Characters*



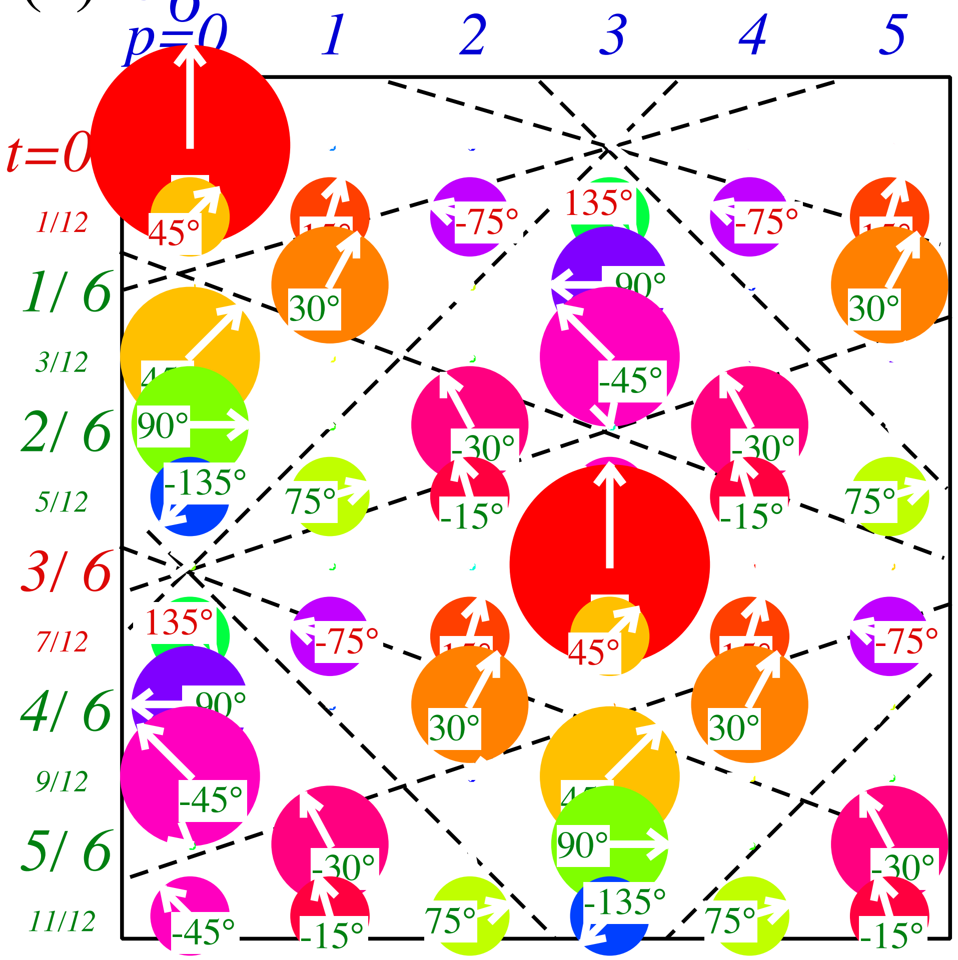
(b) C_6 *Eigenstate Characters*



(c) C_5 *Revivals*



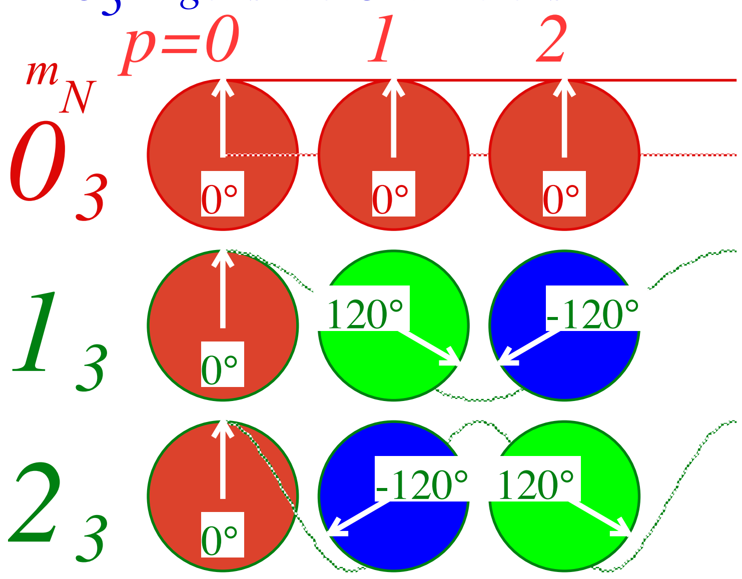
(d) C_6 *Revivals*



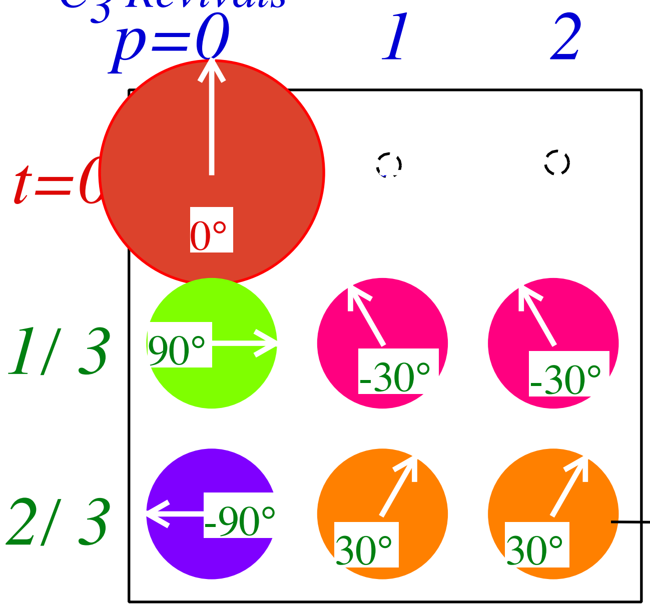
Simulating Complex Systems With Simpler Ones

Discrete 3-State or Trigonal System
(Tesla's 3-Phase AC)

C_3 Eigenstate Characters



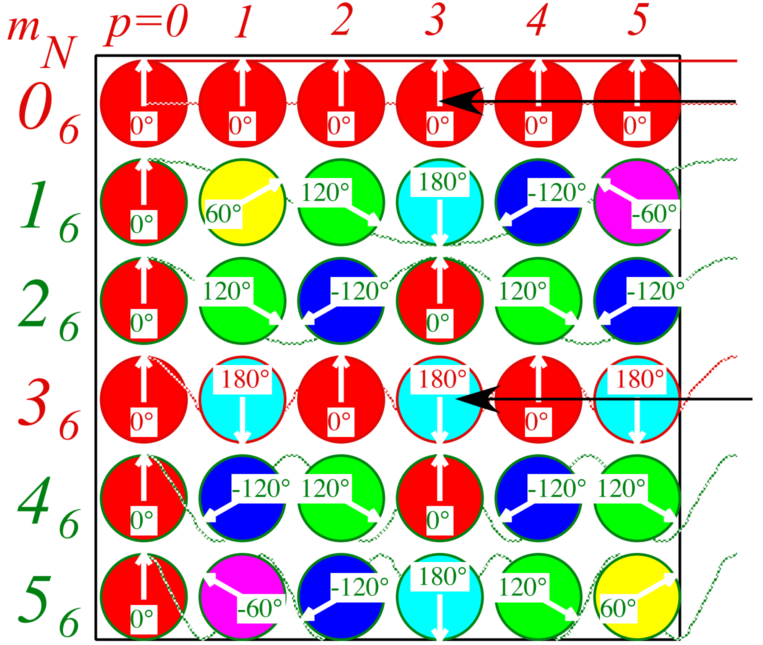
C_3 Revivals



Note 3-phase sub-symmetry

Discrete 6-State or Hexagonal System
(6-Phase AC)

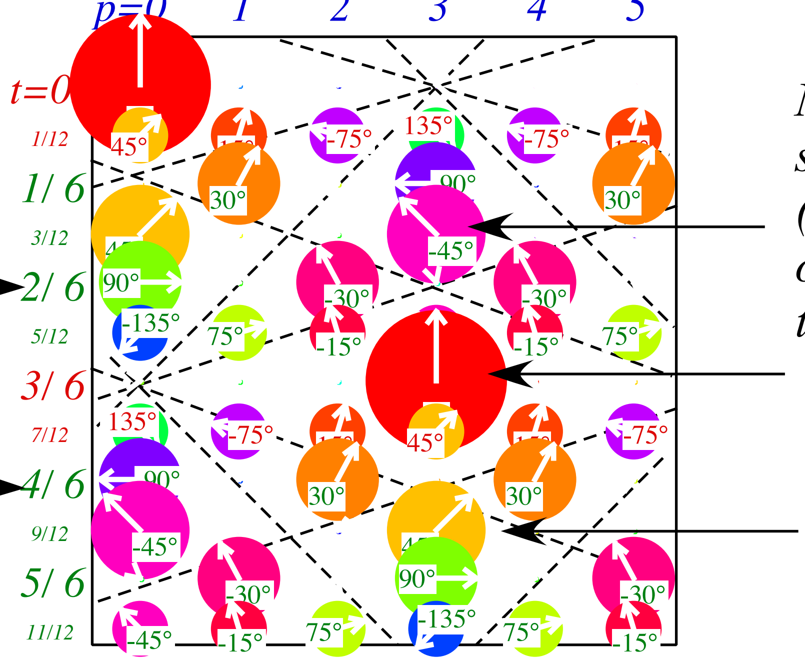
C_6 Eigenstate Characters



Note 2-phase AC

C_2

C_6 Revivals



Note 2-phase sub-symmetry (The "Mother of all symmetry" is C_2)

→ *Polygonal geometry of $U(2)\supset C_N$ character spectral function*

Algebra

Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity

Expo-Cosine identity

Relating space-time and per-space-time

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals"

- *Farey-Sums and Ford-products*
- *Phase dynamics*

Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=trace D^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=trace D^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos2\Theta$$

(tensor-j=2)

Polygonal geometry of $U(2)\supset C_N$ character spectral function

→ *Algebra*

Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity

Expo-Cosine identity

Relating space-time and per-space-time

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals"

- *Farey-Sums and Ford-products*
- *Phase dynamics*

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

Polygonal geometry of U(2)⊃C_N character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \dots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}}=2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0}=1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\dots+e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\dots+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\dots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\dots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\dots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Polygonal geometry of U(2)⊃C_N character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=trace D^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=trace D^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^{j}(\Theta)(1-e^{-i\Theta})=-e^{-i\Theta(j+1)} \qquad + \qquad e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an $(\ell^j=2j+1)$-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+\cancel{e^{+i\Theta(j-1)}}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+\cancel{e^{+i\Theta(j-1)}}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+\cancel{e^{+i\Theta(j-1)}}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+\cancel{e^{+i\Theta(j-1)}}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

$$\chi^{j}(\frac{2\pi}{n})=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^{j}}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Polygonal geometry of U(2)⊃C_N character spectral function

Algebra

→ *Geometry*

Introduction to wave dynamics of phase, mean phase, and group velocity

Expo-Cosine identity

Relating space-time and per-space-time

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

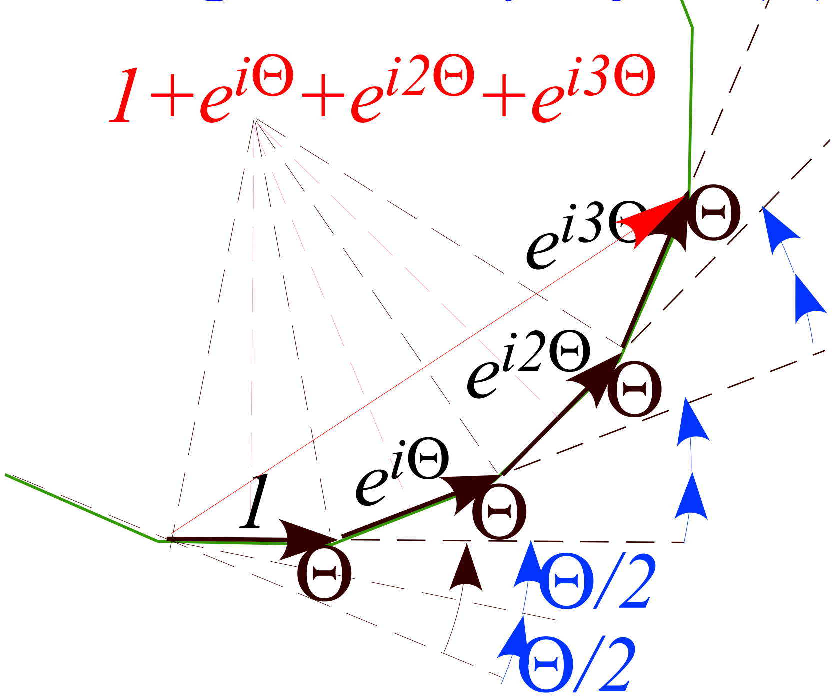
Introduction to C_N beat dynamics and “Revivals”

- *Farey-Sums and Ford-products*
- *Phase dynamics*

Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

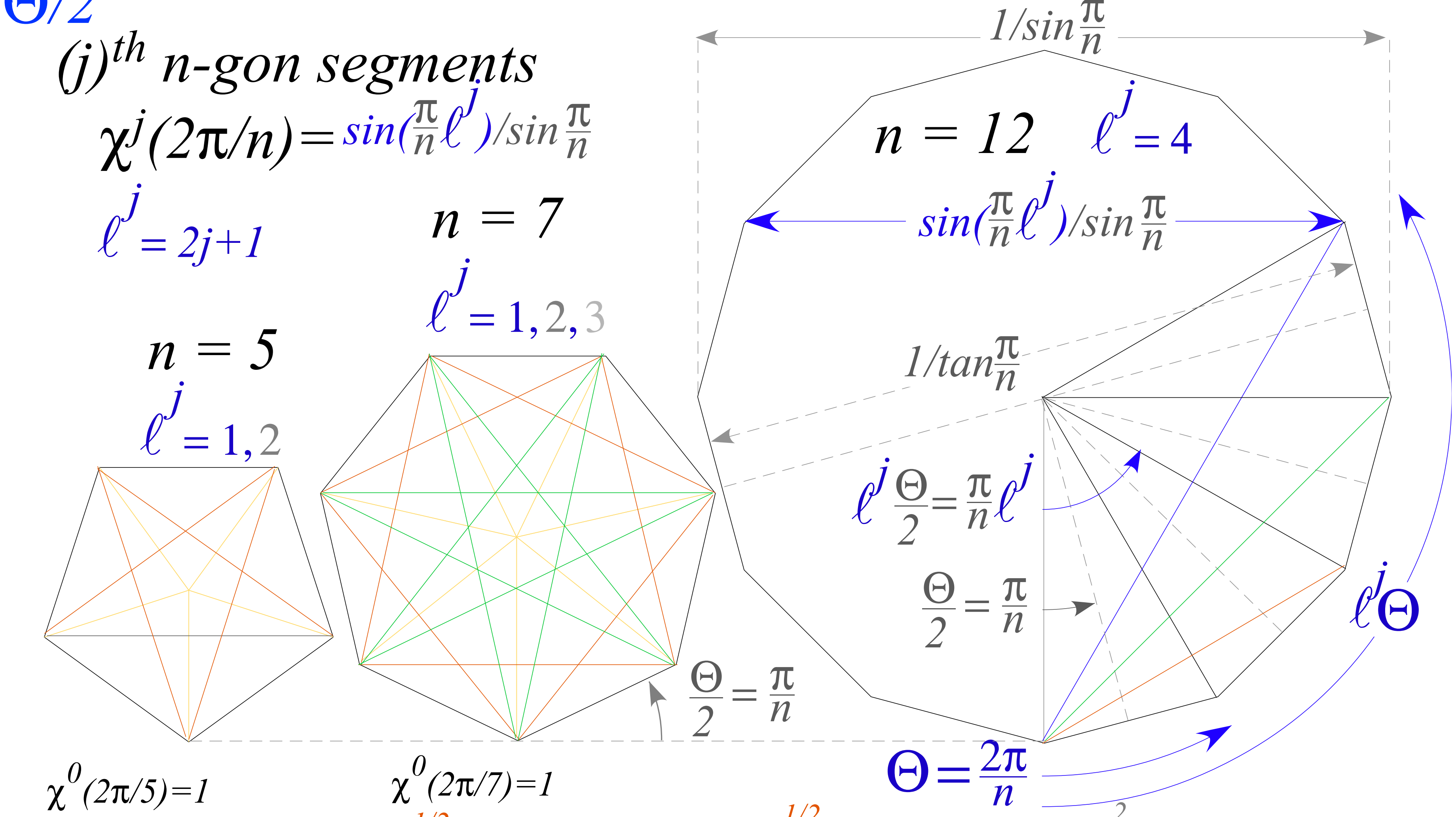
Character Spectral Function
where: $\ell^j=2j+1$
is $U(2)$ irrep dimension



$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin(\tfrac{\pi}{n}\ell^j)/\sin\tfrac{\pi}{n}$$

$\ell^j = 2j+1$



$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

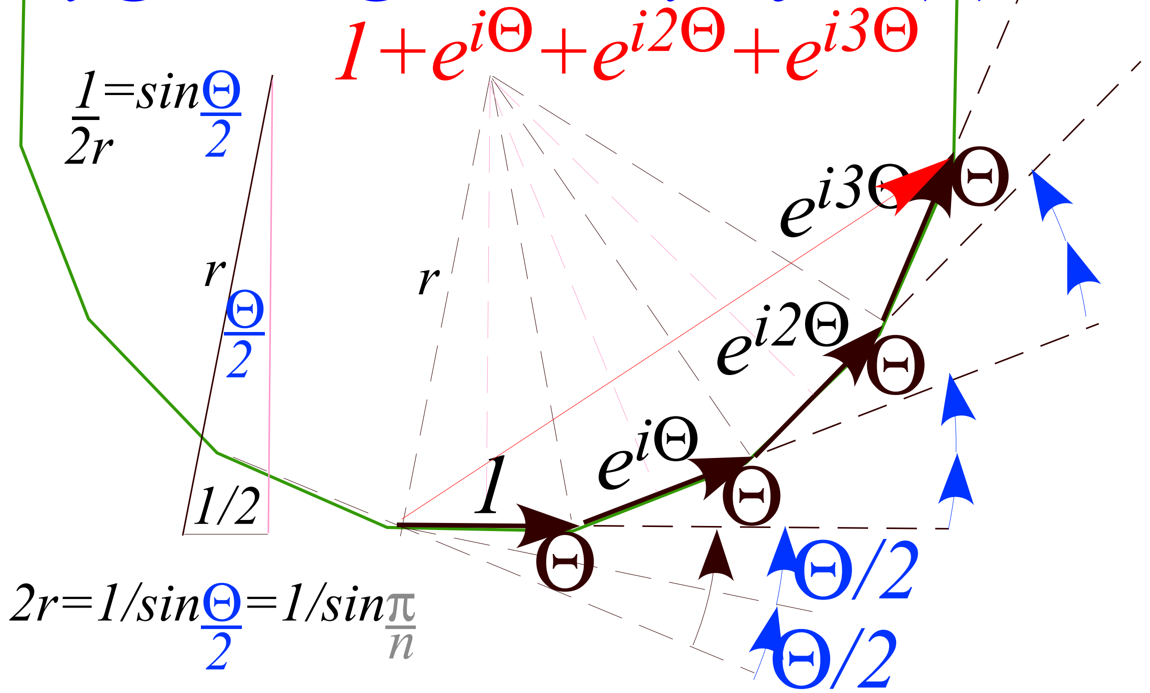
Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function

where: $\ell^j=2j+1$

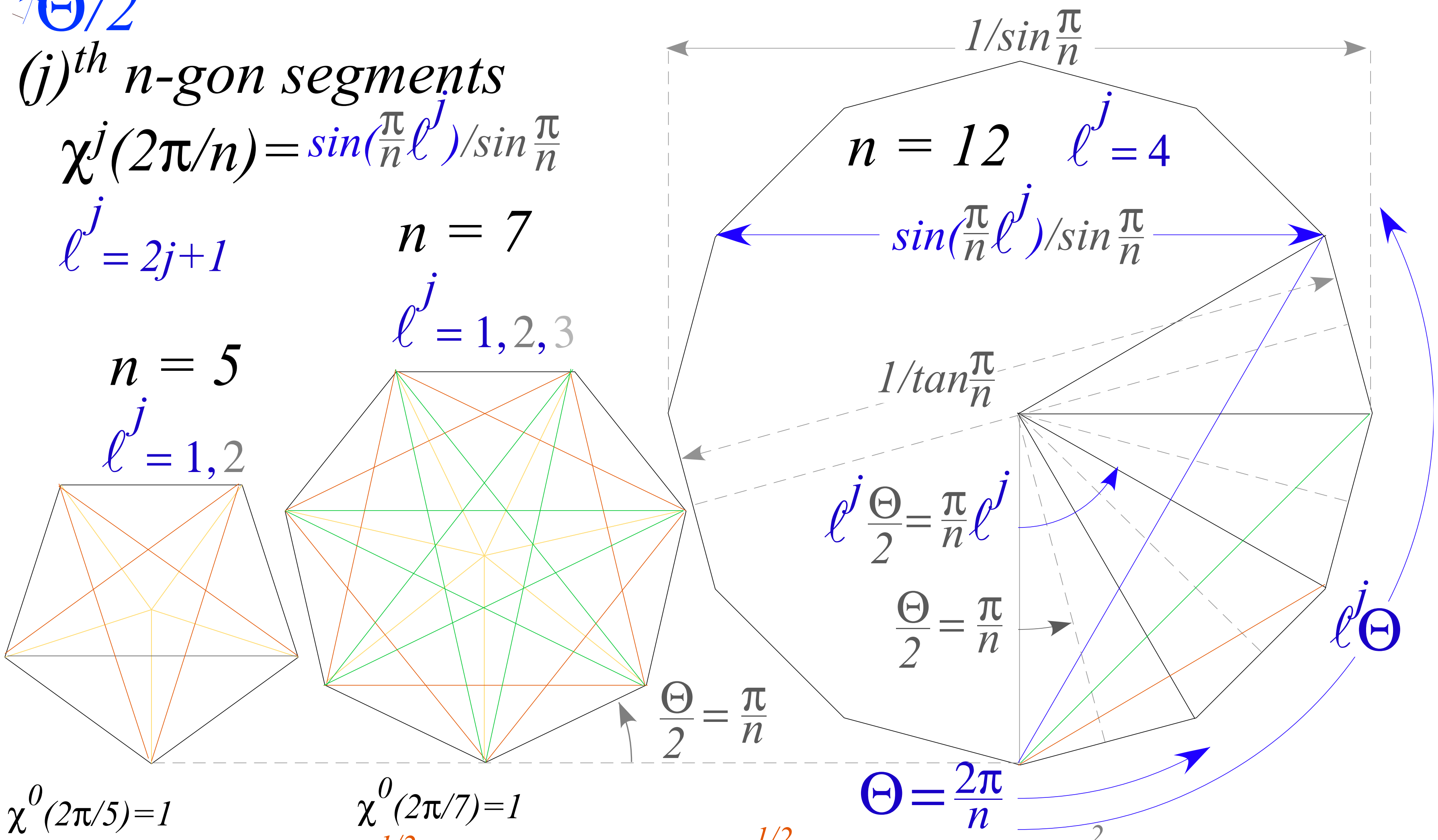
is U(2) irrep dimension



$(j)^{th}$ n-gon segments

$\chi^j(2\pi/n)=\sin(\frac{\pi}{n}\ell^j)/\sin\frac{\pi}{n}$

$\ell^j=2j+1$



$\chi^0(2\pi/5)=1$

$\chi^{1/2}(2\pi/5)=1.618...$

$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$

$\chi^{1/2}(2\pi/7)=1.802...$

$\chi^1(2\pi/7)=2.247...$

$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$

$\chi^1(2\pi/12)=2.732...$

$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$

$\chi^{5/2}(2\pi/12)=3.864...$

$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

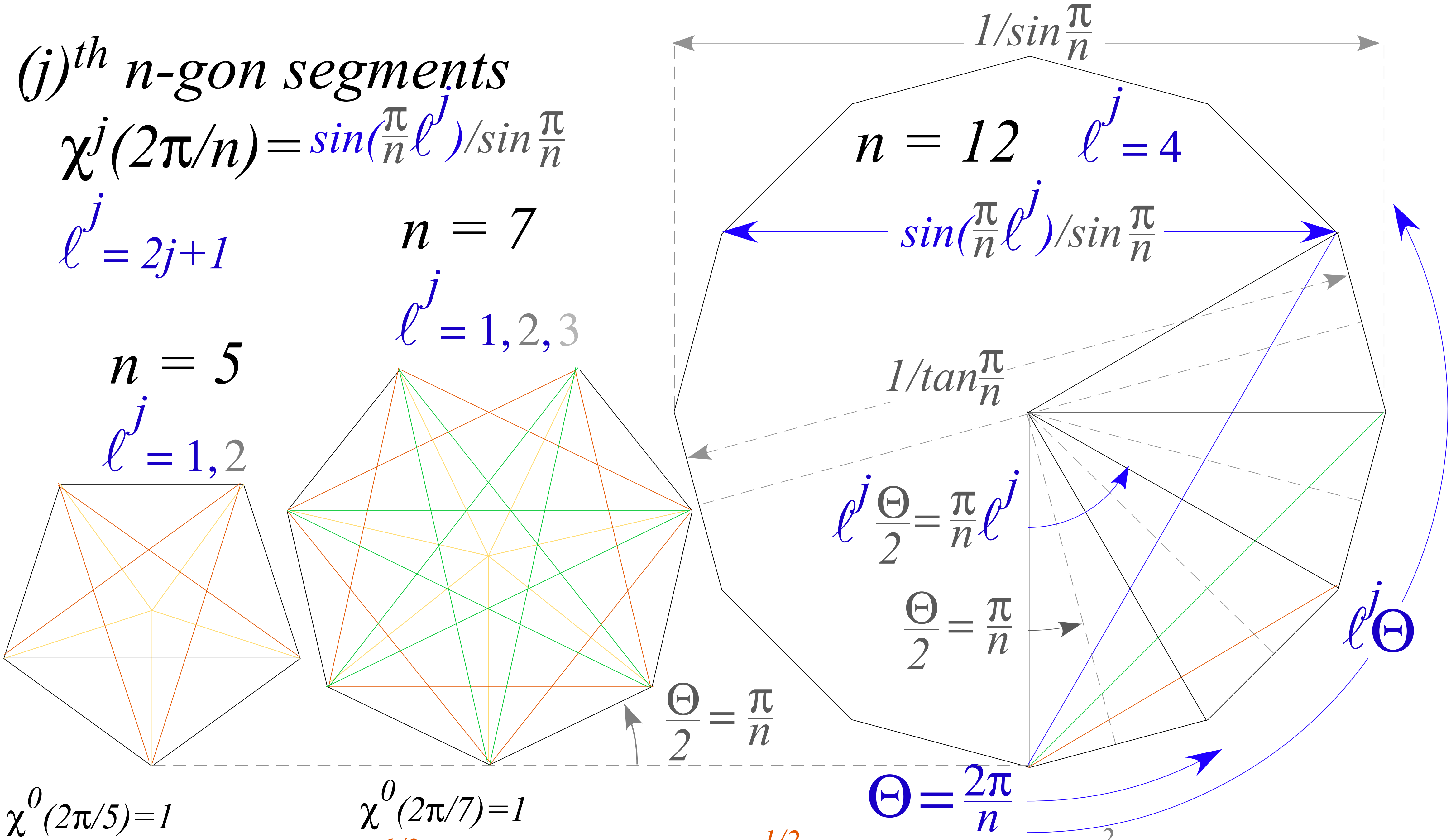
$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin\left(\frac{\pi}{n}\ell^j\right)/\sin\frac{\pi}{n}$$

$$\ell^j=2j+1$$



$\chi^0(2\pi/5)=1$

$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$

$\chi^{1/2}(2\pi/7)=1.802...$

$\chi^1(2\pi/7)=2.247...$

$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$

$\chi^1(2\pi/12)=2.732...$

$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$

$\chi^{5/2}(2\pi/12)=3.864...$

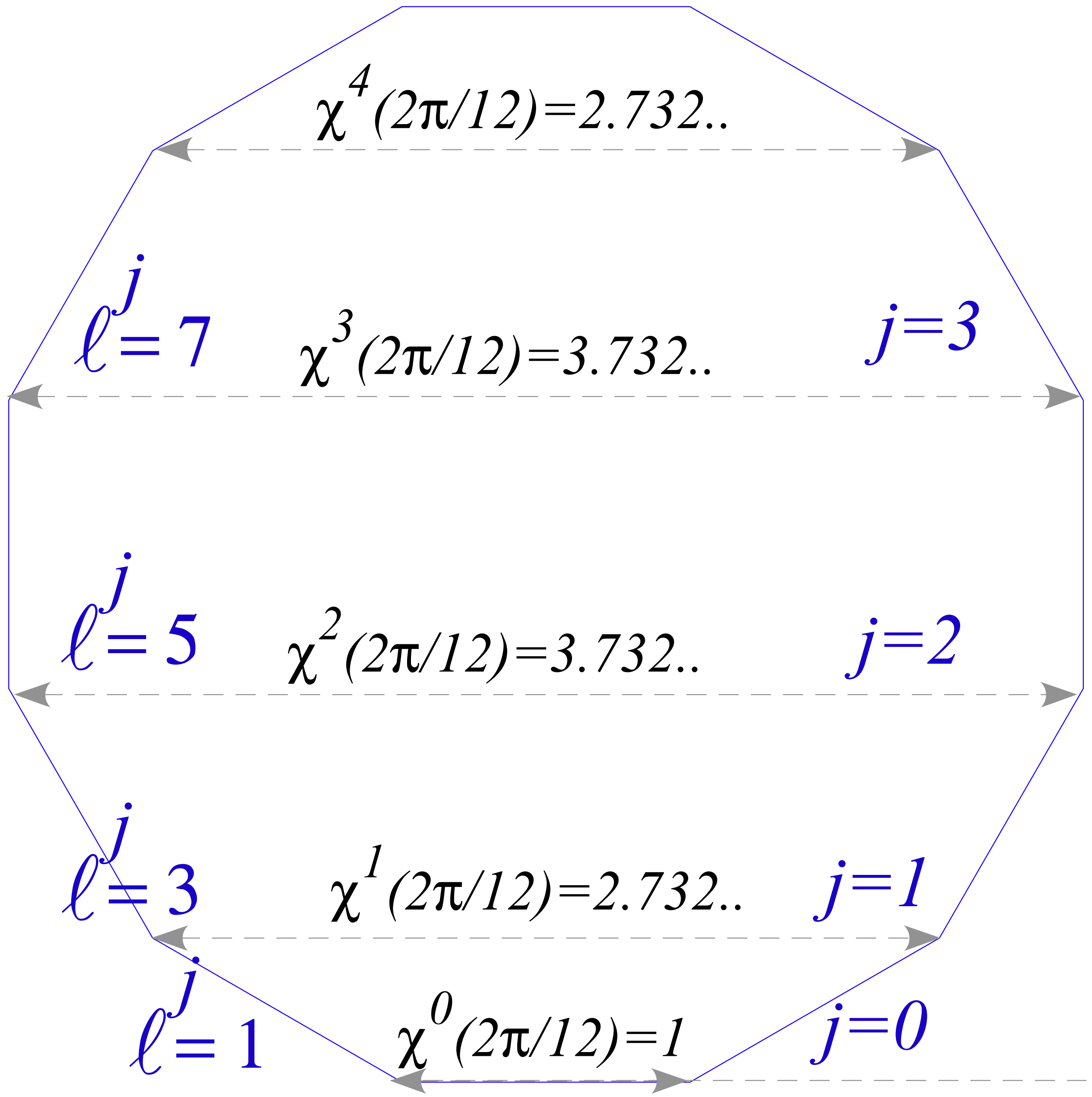
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Integer j for n=12

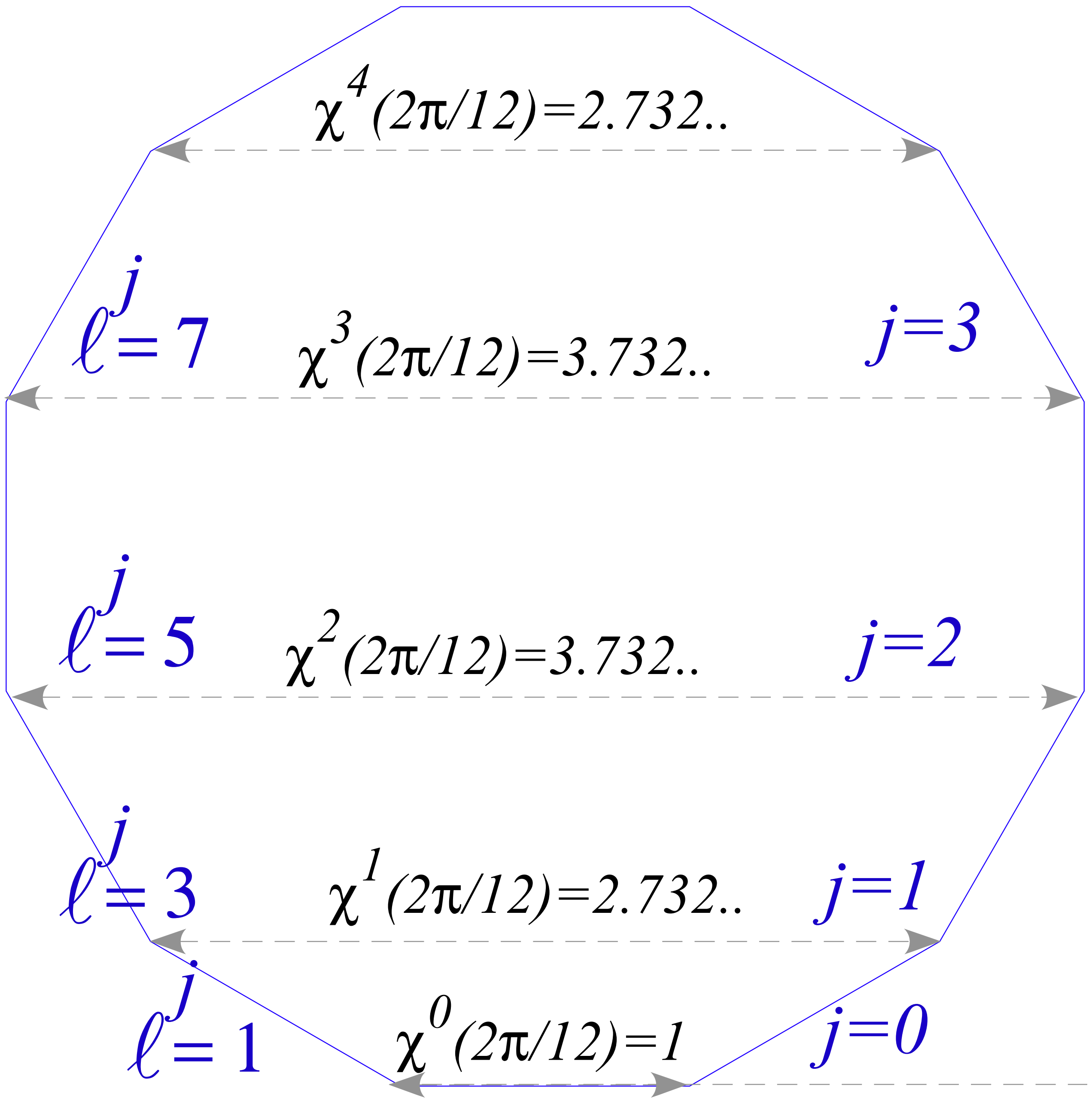


Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Integer j for n=12



$\chi^4(2\pi/12)=2.732..$

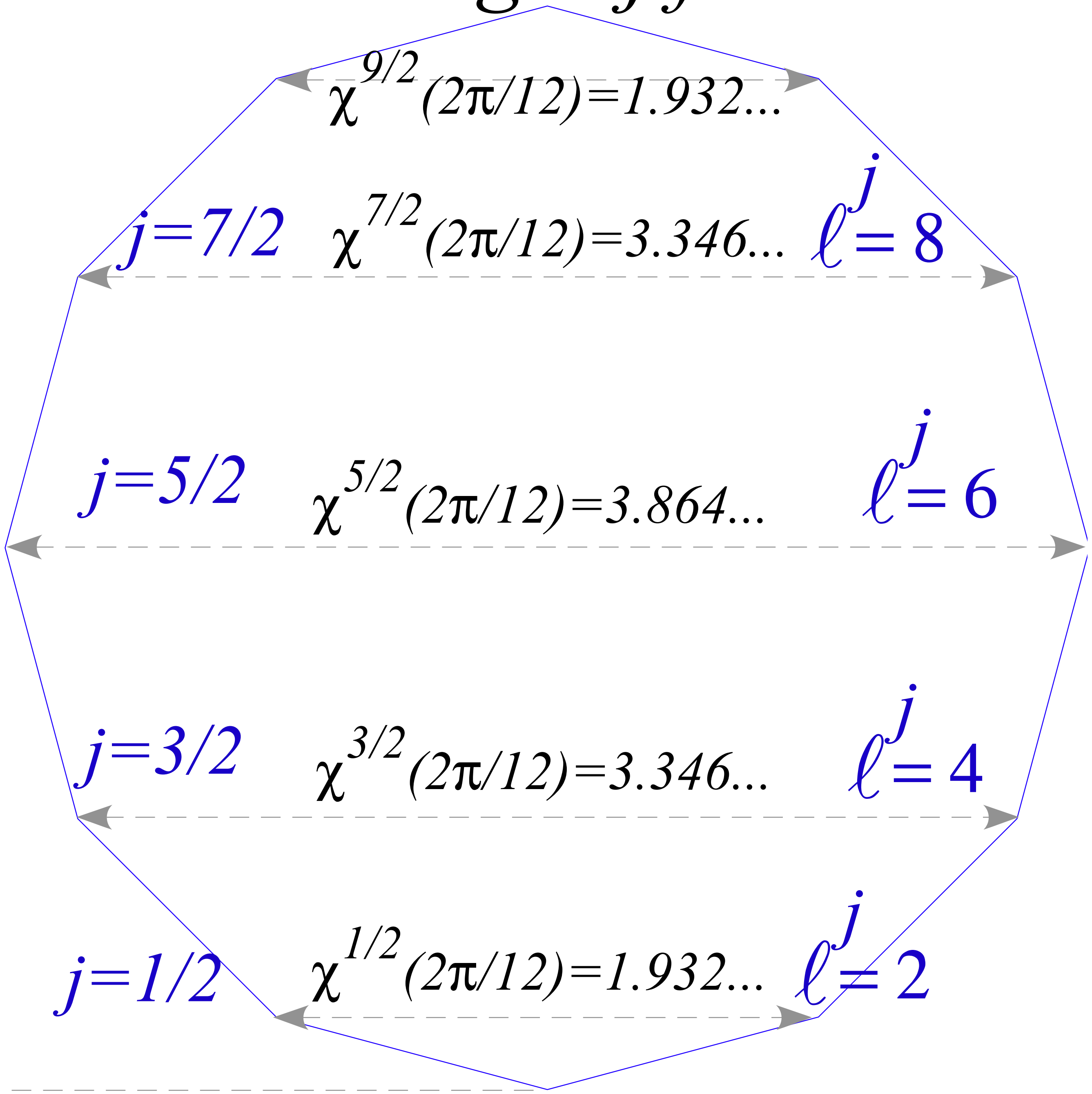
$\ell^j=7$ $\chi^3(2\pi/12)=3.732..$ $j=3$

$\ell^j=5$ $\chi^2(2\pi/12)=3.732..$ $j=2$

$\ell^j=3$ $\chi^1(2\pi/12)=2.732..$ $j=1$

$\ell^j=1$ $\chi^0(2\pi/12)=1$ $j=0$

1/2-Integer j for n=12



$\chi^{9/2}(2\pi/12)=1.932...$

$j=7/2$ $\chi^{7/2}(2\pi/12)=3.346...$ $\ell^j=8$

$j=5/2$ $\chi^{5/2}(2\pi/12)=3.864...$ $\ell^j=6$

$j=3/2$ $\chi^{3/2}(2\pi/12)=3.346...$ $\ell^j=4$

$j=1/2$ $\chi^{1/2}(2\pi/12)=1.932...$ $\ell^j=2$