Group Theory in Quantum Mechanics
Lecture 8 *(2.5.15)*
Spinor and vector representations of U(2) and R(3) Operators

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5)

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
- *Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$
- *Group product algebra*
  - *Jordan-Pauli identity and U(2) product* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$*- formula*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\sigma_\mu\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of spinor σ_μ-operators*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *of group-operators*
  - *Operator-on-Operator transformations*

Geometry of groups: Hamilton's turns and It's all done with mirrors!
- *Group product geometry*
  - *U(2) product* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$*- geometry*
  - *Transformation* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\boldsymbol{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ *replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ *and Hamilton's* $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor* σ_μ*-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Lecture 7 Review: The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \sigma_\omega$$

Notation for 2D Spinor space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1}=\sigma_0, \mathsf{S}_B=\mathsf{S}_X, \mathsf{S}_C=\mathsf{S}_Y, \mathsf{S}_A=\mathsf{S}_Z\}$

(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

"Crank" vector (2D-Spinor)

$$\vec{\varphi}=\begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t} = e^{-i\omega_0\cdot t}e^{-i\sigma_\omega \omega\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\sigma_\omega \sin\omega\cdot t\right)$$

$$= e^{-i(\Omega_0\mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t} = e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}} \qquad = e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2} - i\sigma_\omega \sin\frac{\Omega\cdot t}{2}\right)$$

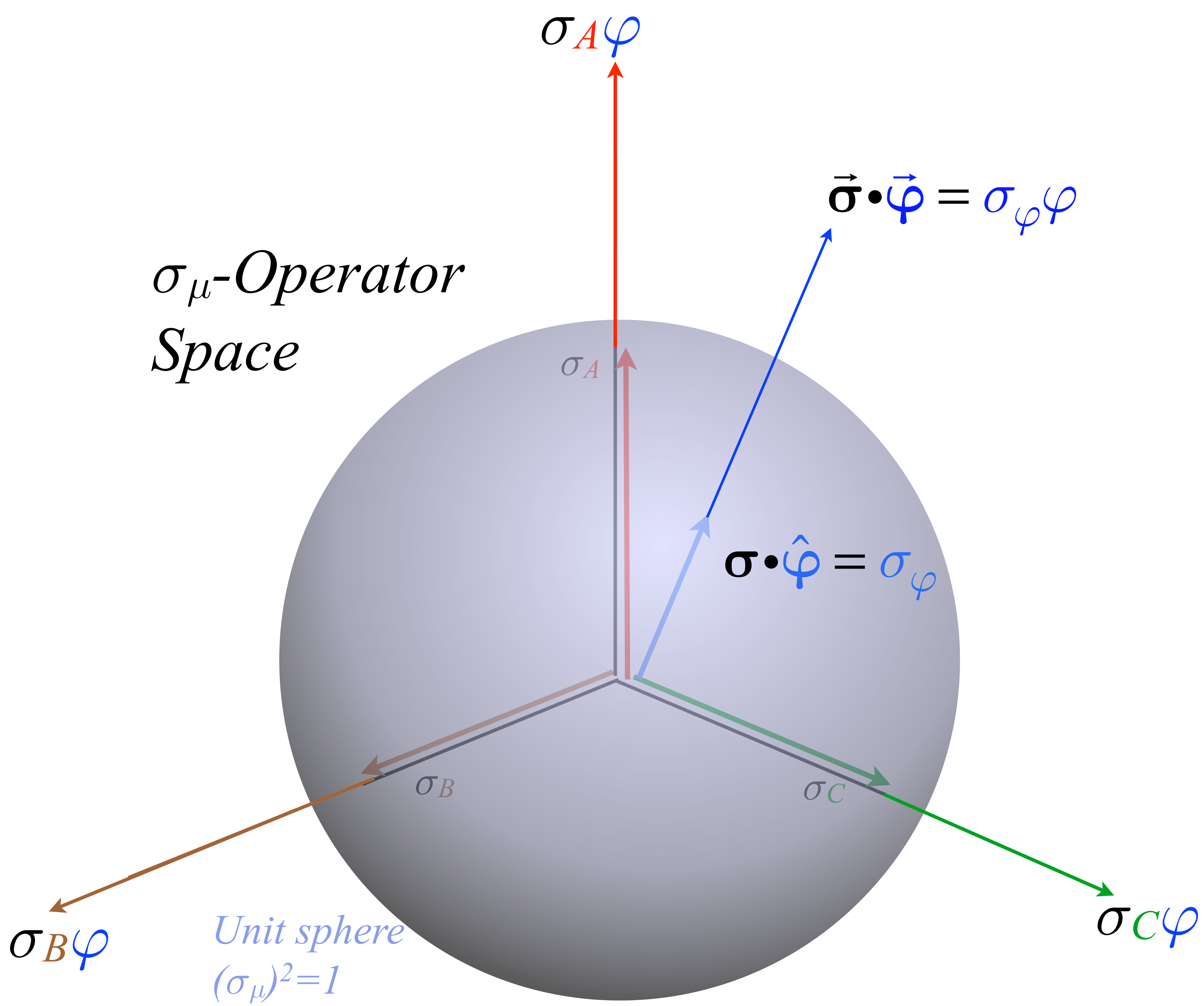
"Crank" vector (3D-Vector)

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix} = \vec{\Omega}\cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A \hat{\varphi}_A + \boldsymbol{\sigma}_B \hat{\varphi}_B + \boldsymbol{\sigma}_C \hat{\varphi}_C = \vec{\boldsymbol{\sigma}} \bullet \hat{\boldsymbol{\varphi}} = \vec{\boldsymbol{\sigma}} \bullet \vec{\boldsymbol{\varphi}} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$



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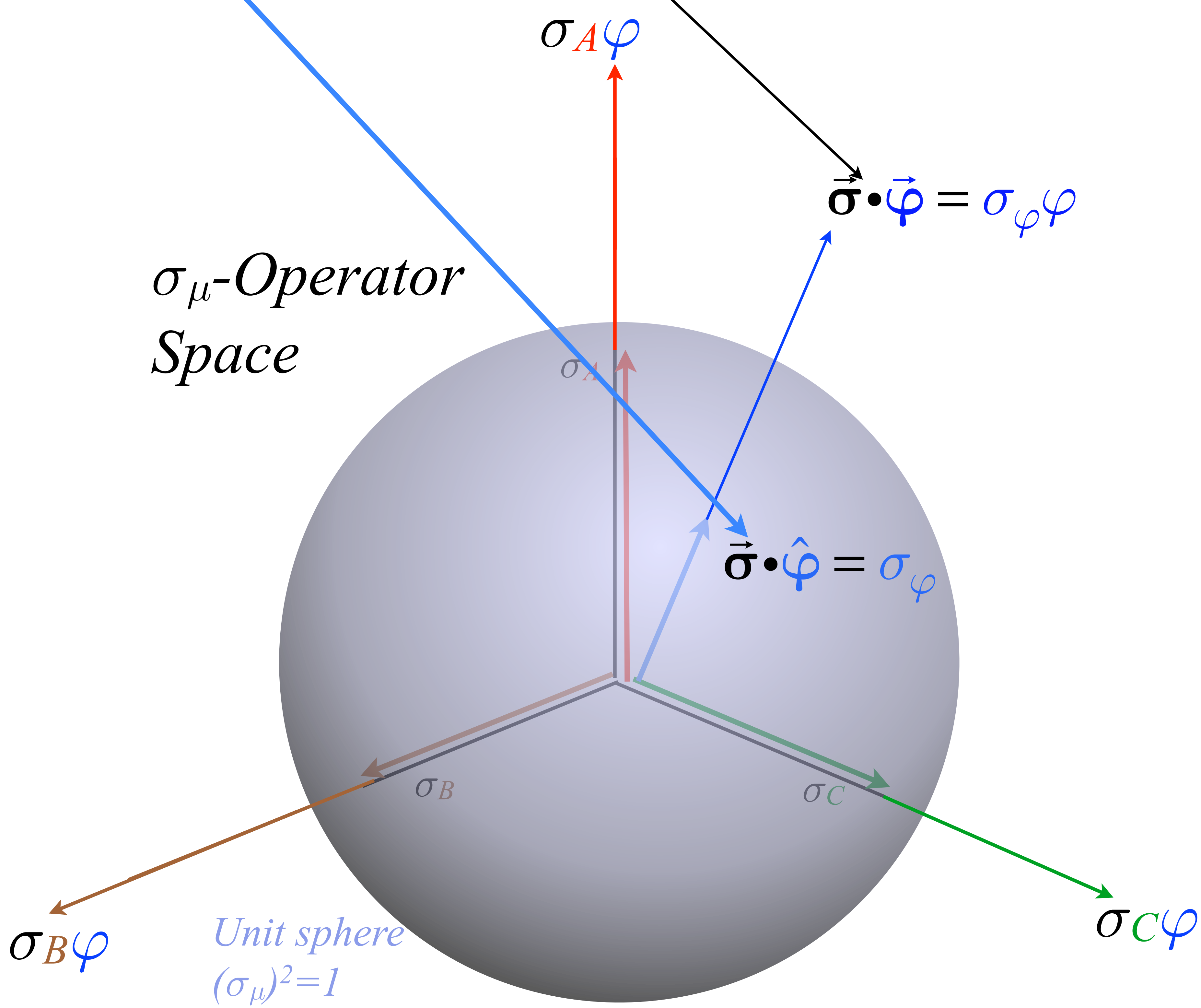
$\sigma_\varphi\varphi = \boldsymbol{\sigma}_A\varphi_A + \boldsymbol{\sigma}_B\varphi_B + \boldsymbol{\sigma}_C\varphi_C = \vec{\boldsymbol{\sigma}}\bullet\boldsymbol{\varphi} = \vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}$ where: $(\sigma_\varphi)^2=\mathbf{1}\varphi^2$

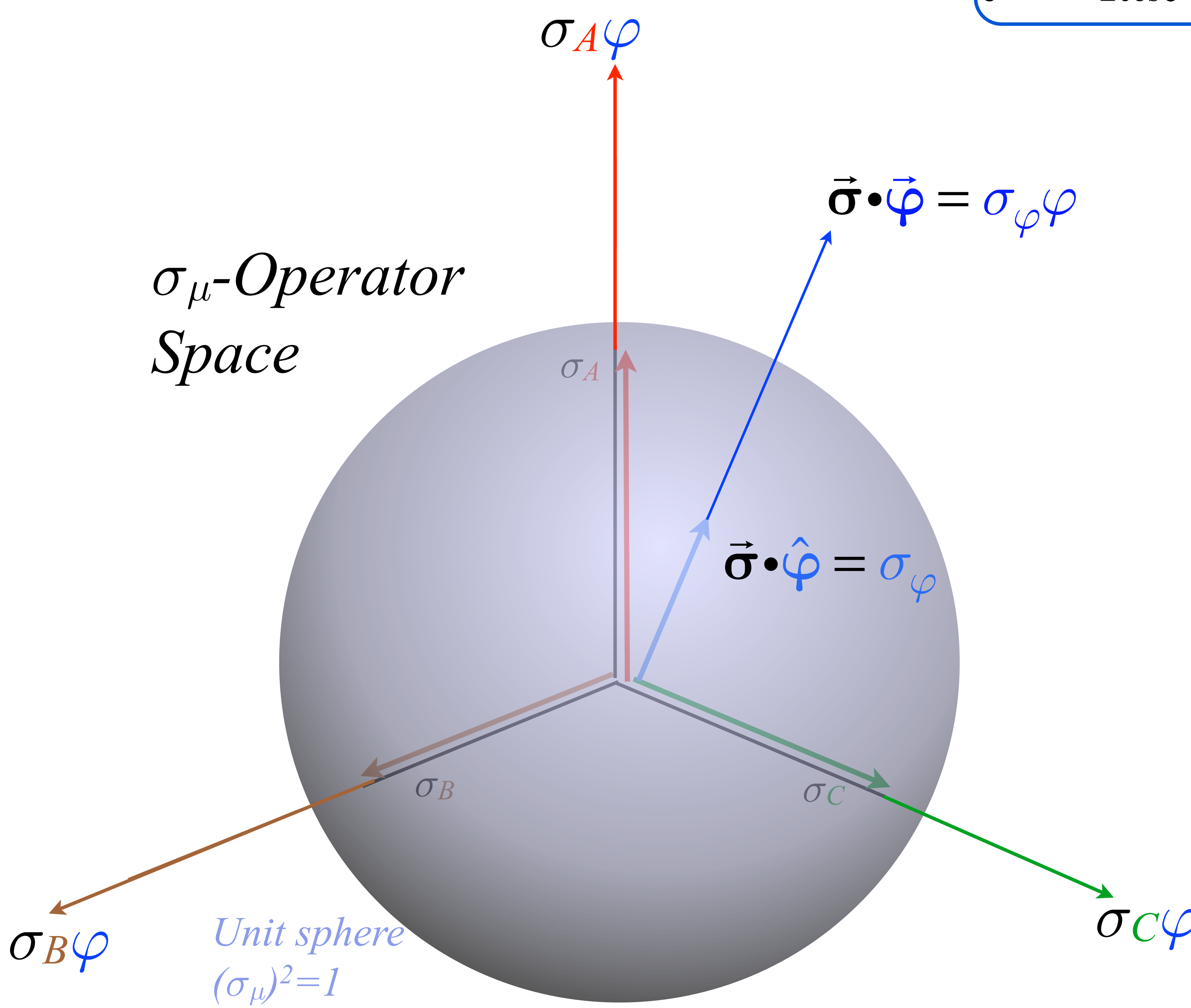


Not-so-Crazy Thing: $\sigma_{\varphi} = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \bullet \hat{\varphi} = \vec{\sigma} \bullet \vec{\varphi} / \varphi$ where: $(\sigma_{\varphi})^2 = \mathbf{1}$

Crazy Thing: (face) $= -i\sigma_{\varphi} = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \bullet \hat{\varphi} = -i\vec{\sigma} \bullet \vec{\varphi} / \varphi$

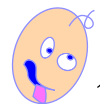
satisfies crazy requirement: ((face) $)^2 = (-i\sigma_{\varphi})^2 = \mathbf{-1}$

The Crazy Thing Theorem:
If ((face) $)^2 = \mathbf{-1}$
Then:
$e^{(\text{face})\theta} = \mathbf{1}\cos\theta + (\text{face})\sin\theta$



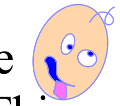
Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \vec{\sigma}\bullet\hat{\varphi} = \vec{\sigma}\bullet\vec{\varphi}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

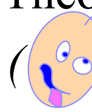
Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

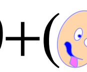
So:

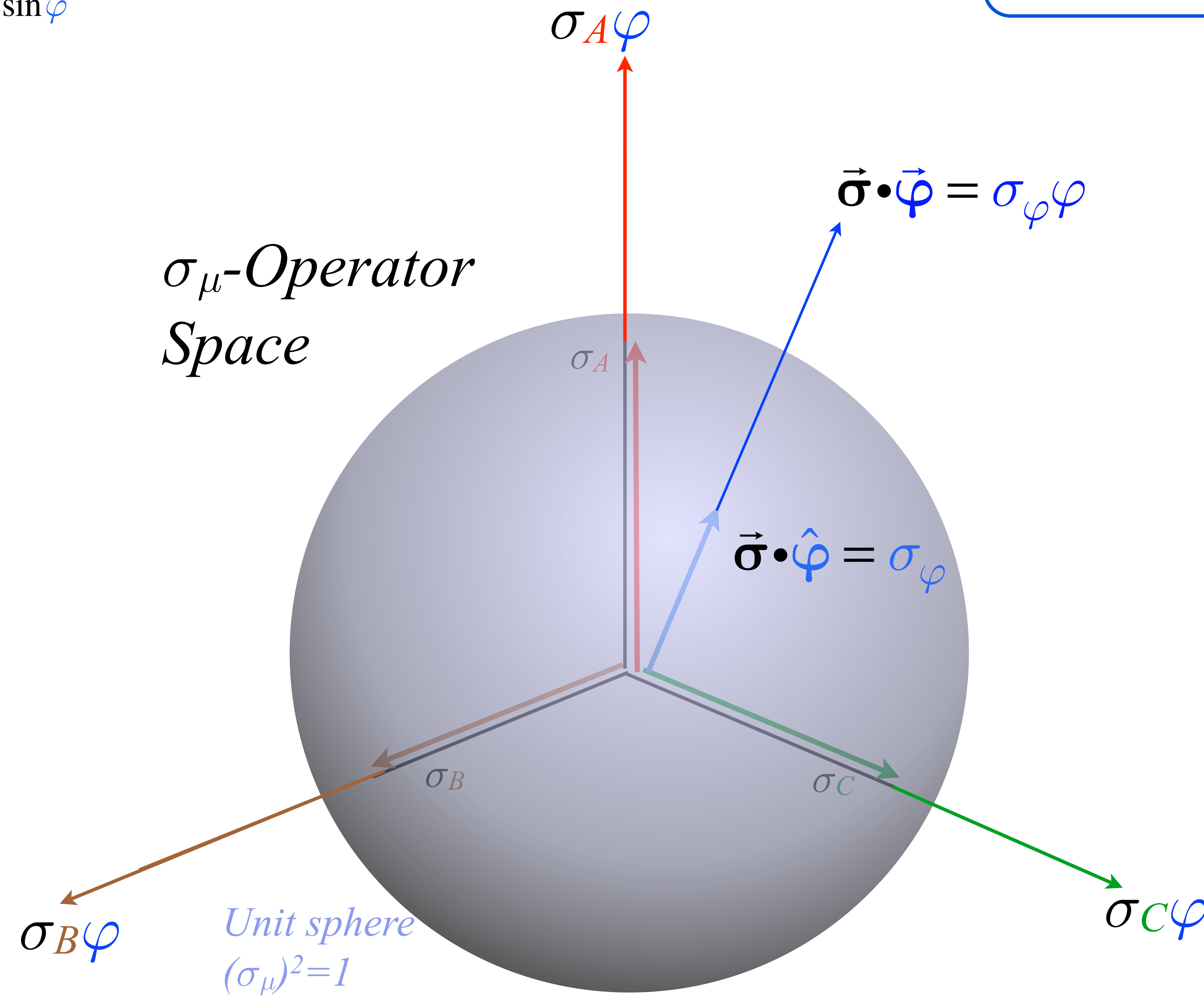
$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi\sin\varphi$

The  Crazy Thing Theorem:

If ( $)^2 = \mathbf{-1}$

Then:

e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$



Not-so-Crazy Thing: $\sigma_\varphi = \boldsymbol{\sigma}_A\hat{\varphi}_A + \boldsymbol{\sigma}_B\hat{\varphi}_B + \boldsymbol{\sigma}_C\hat{\varphi}_C = \vec{\boldsymbol{\sigma}} \bullet \hat{\boldsymbol{\varphi}} = \vec{\boldsymbol{\sigma}} \bullet \vec{\boldsymbol{\varphi}} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

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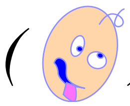
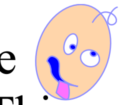
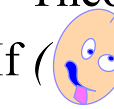
satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi \sin\varphi$$

$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A \sin\varphi - i\sigma_B\hat{\varphi}_B \sin\varphi - i\sigma_C\hat{\varphi}_C \sin\varphi$$

The 
Crazy Thing
Theorem:
If ( $)^2 = \mathbf{-1}$
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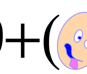
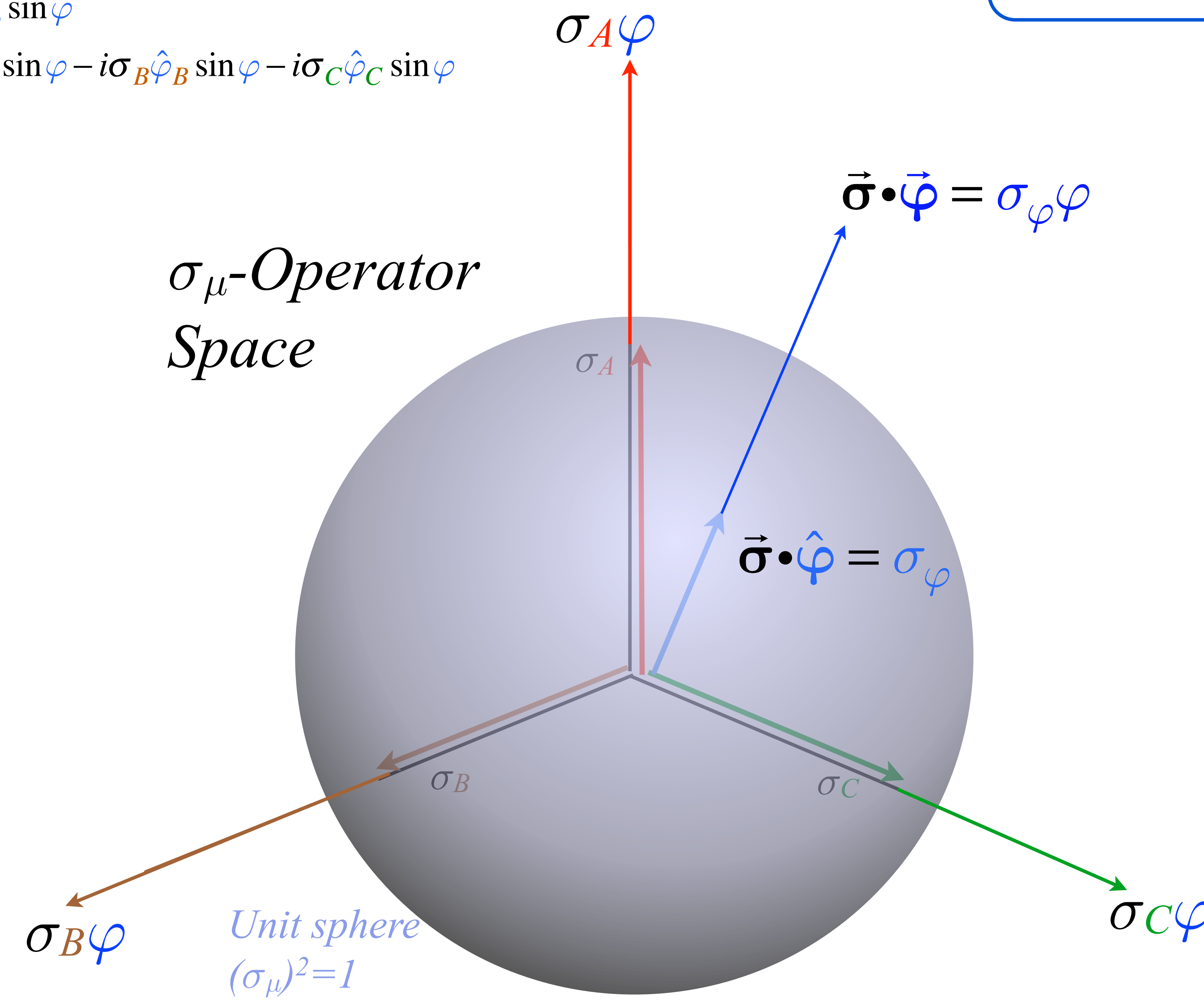
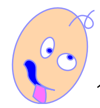
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \vec{\sigma}\bullet\hat{\varphi} = \vec{\sigma}\bullet\vec{\varphi}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

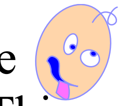
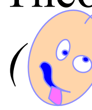
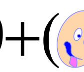
So:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi \sin\varphi$$

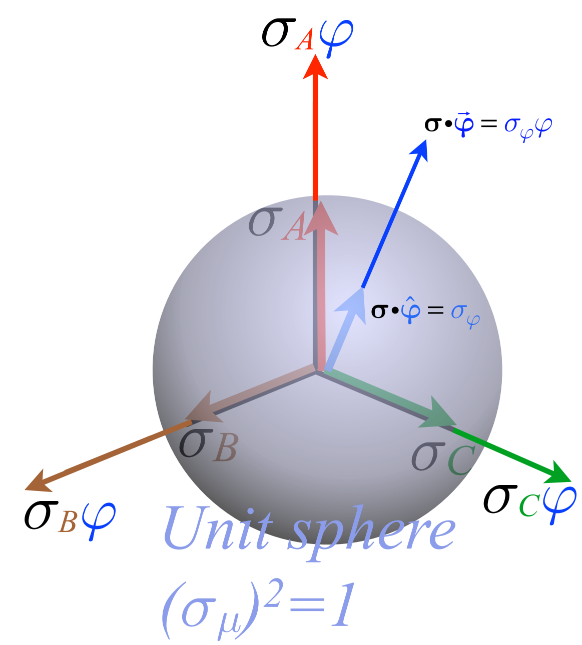
$$= \mathbf{1}\cos\varphi \quad - i\sigma_A\hat{\varphi}_A \sin\varphi \quad - i\sigma_B\hat{\varphi}_B \sin\varphi \quad - i\sigma_C\hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A \sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B \sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A \sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A \sin\varphi \end{pmatrix}$$

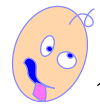
The 
Crazy Thing
Theorem:
If ( $)^2 = \mathbf{-1}$
Then:
e()θ $= \mathbf{1}\cos\theta + ($  $)\sin\theta$

σ_μ-Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \vec{\sigma}\bullet\hat{\varphi} = \vec{\sigma}\bullet\vec{\varphi}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi$$

$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A \sin\varphi - i\sigma_B\hat{\varphi}_B \sin\varphi - i\sigma_C\hat{\varphi}_C \sin\varphi$$

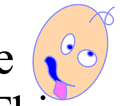
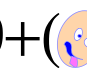
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A \sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B \sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A \sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A \sin\varphi \end{pmatrix}$$

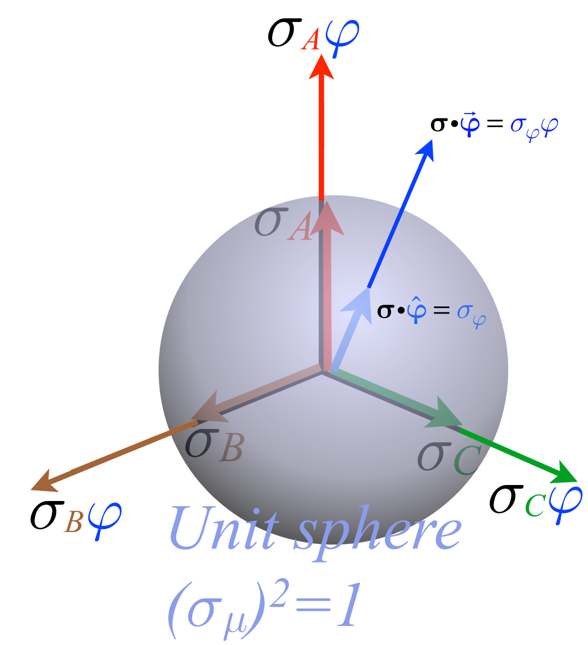
Case A

$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\varphi} = \boldsymbol{\sigma}_A$$

$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$

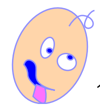
The 
Crazy Thing
Theorem:
If ( $)^2 = \mathbf{-1}$
Then:
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

σ_μ-Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \vec{\sigma}\bullet\hat{\varphi} = \vec{\sigma}\bullet\vec{\varphi}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \quad - i\sigma_\varphi\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\sigma_A\hat{\varphi}_A\sin\varphi \quad - i\sigma_B\hat{\varphi}_B\sin\varphi \quad - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

Case A

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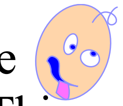
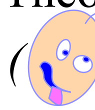
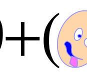
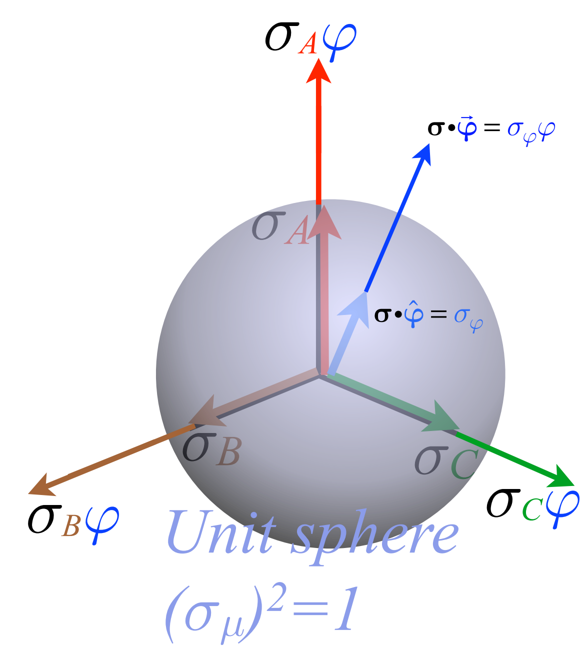
$$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$$

Case B

$$= \mathbf{R}_B = e^{-i\sigma_B\varphi} = \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\varphi} = \sigma_B$$

The  Crazy Thing Theorem:
If ( $)^2 = \mathbf{-1}$
Then:
e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

σ_μ-Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C = \vec{\sigma}\bullet\hat{\varphi} = \vec{\sigma}\bullet\vec{\varphi}/\varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

Crazy Thing: (face) $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

The Crazy Thing Theorem:
If (face)$^2 = -\mathbf{1}$
Then:
$e^{(\text{face})\theta} = \mathbf{1}\cos\theta + (\text{face})\sin\theta$

satisfies crazy requirement: (face)$^2 = (-i\sigma_\varphi)^2 = -\mathbf{1}$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$$

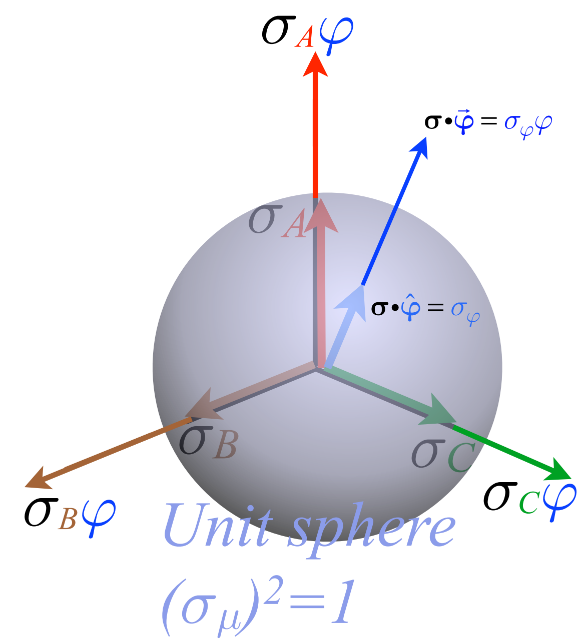
$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A\sin\varphi - i\sigma_B\hat{\varphi}_B\sin\varphi - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

σ

σ_μ-Operator Space



Case A

$$= \begin{pmatrix} \cos\varphi - i\sin\varphi & 0 \\ 0 & \cos\varphi + i\sin\varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\varphi} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A\varphi}$$

Case B

$$= \mathbf{R}_B = e^{-i\sigma_B\varphi} = \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \boldsymbol{\sigma}\bullet\hat{\varphi} = \sigma_B$$

Case C

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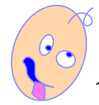
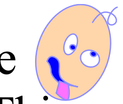
Crazy Thing:  $= -i\sigma_\varphi = -i\sigma_A\hat{\varphi}_A - i\sigma_B\hat{\varphi}_B - i\sigma_C\hat{\varphi}_C = -i\vec{\sigma}\bullet\hat{\varphi} = -i\vec{\sigma}\bullet\vec{\varphi}/\varphi$

satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

The Crazy Thing Theorem: 

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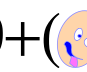
Then:

e()θ $= \mathbf{1}\cos\theta +$ ()$\sin\theta$

So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$$

$$= \mathbf{1}\cos\varphi - i\sigma_A\hat{\varphi}_A\sin\varphi - i\sigma_B\hat{\varphi}_B\sin\varphi - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\varphi - i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\hat{\varphi}_C\sin\varphi$$

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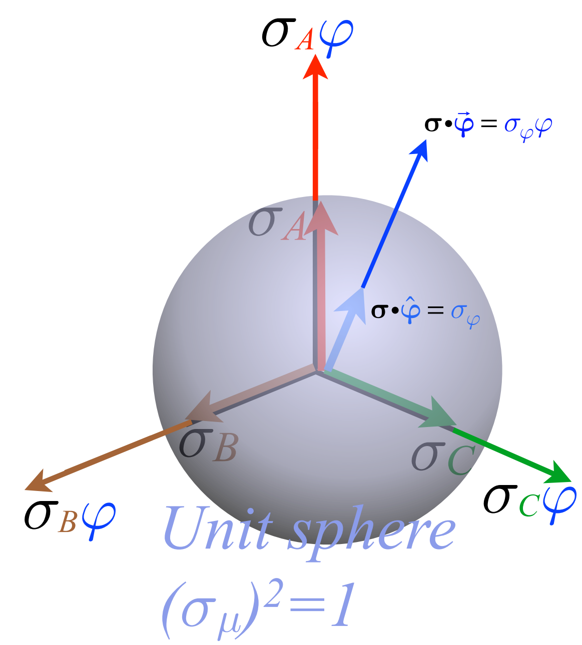
σ_μ-Operator Space



Case A

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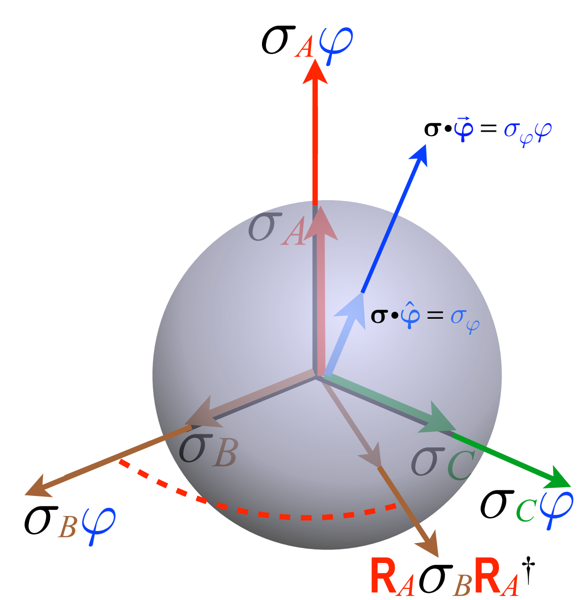

$\mathbf{R}_A\sigma_B\mathbf{R}_A^\dagger$ *example:* σ_B *rotated by* $\mathbf{R}_A = e^{-i\sigma_A\varphi}$ *shows 3D-space double angle* $\Theta = 2\varphi$

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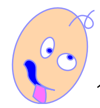
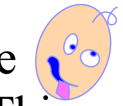
satisfies crazy requirement: ( $)^2 = (-i\sigma_\varphi)^2 = \mathbf{-1}$

The Crazy Thing Theorem: 

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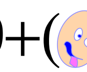
e()θ $= \mathbf{1}\cos\theta +$ ()sinθ

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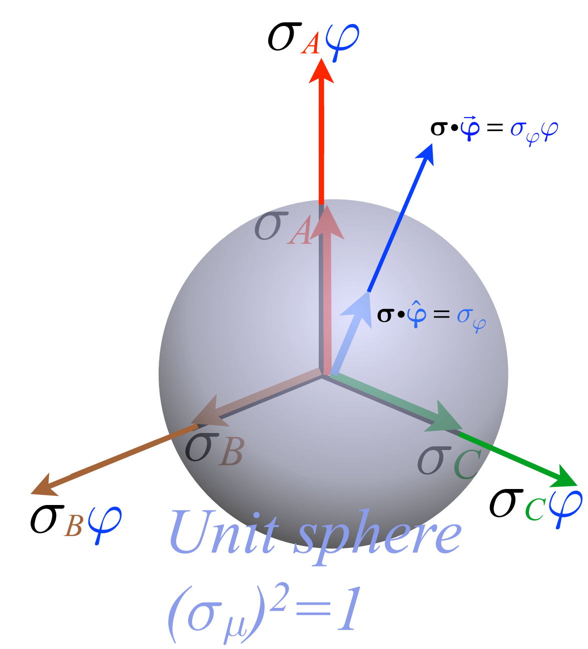
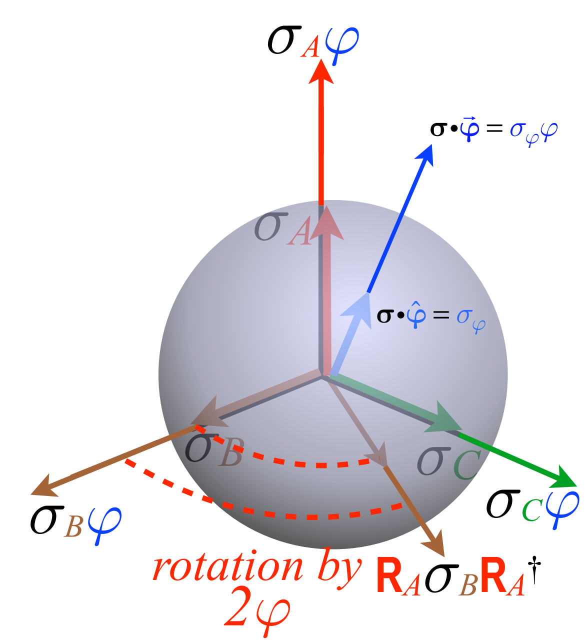
σ_μ-Operator Space





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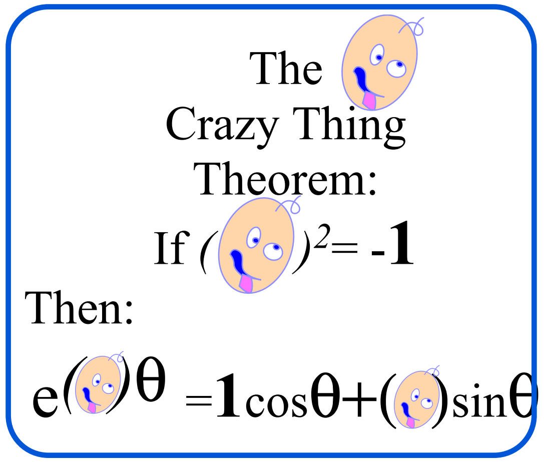
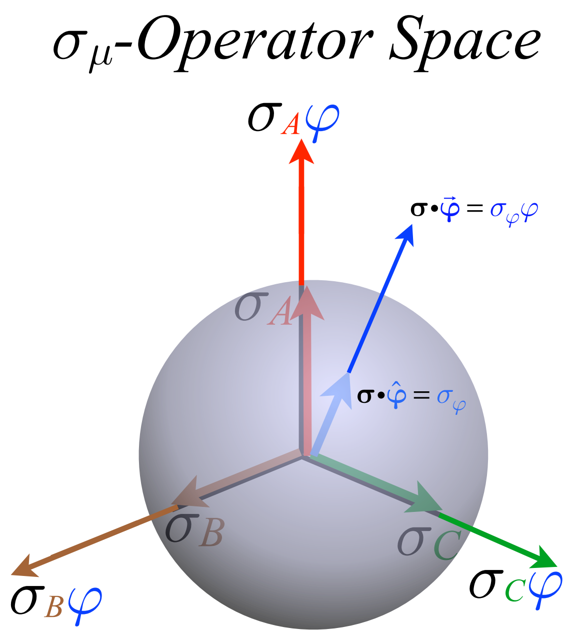
Replace spinor angle φ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\varphi}}\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2} = \varphi$





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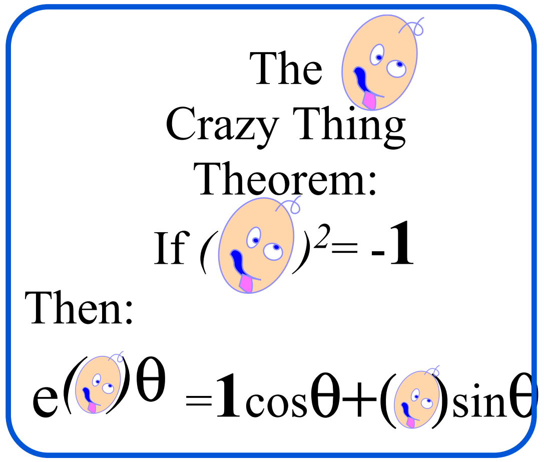
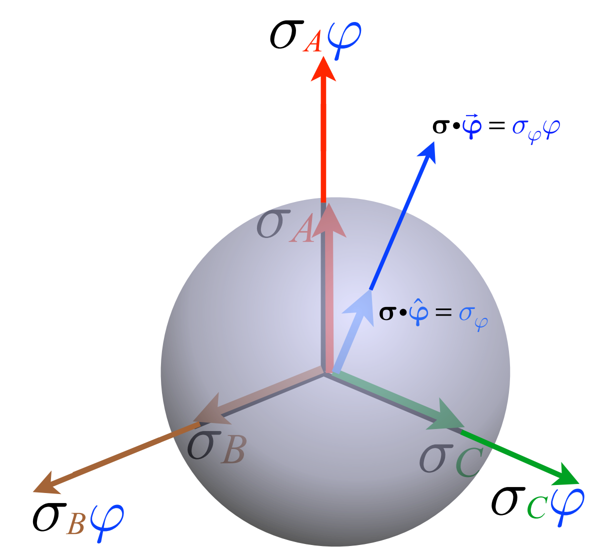


σ_μ-Operator Space



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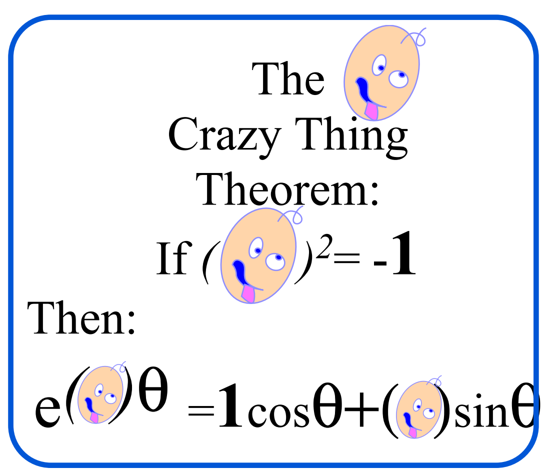
Unit rotation axis vector $\hat{\Theta} = \left(\hat{\Theta}_X, \;\; \hat{\Theta}_Y, \;\; \hat{\Theta}_Z \right) = \left(\cos\varphi\sin\vartheta \quad \sin\varphi\sin\vartheta \quad \cos\vartheta \right)$ is the same as the unit $\hat{\varphi}$.



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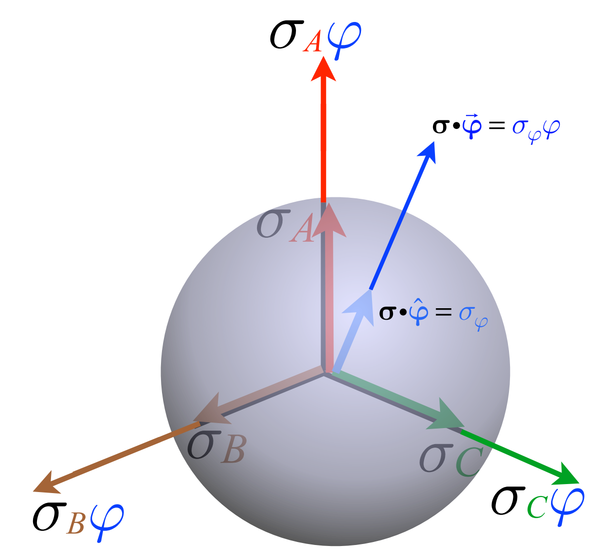

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with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1}\cos\frac{\Theta}{2} - i\vec{\boldsymbol{\sigma}}\bullet\hat{\Theta}\sin\frac{\Theta}{2} = e^{-i\frac{\vec{\boldsymbol{\sigma}}}{2}\bullet\vec{\Theta}} = e^{-i\mathbf{S}\bullet\vec{\Theta}}$

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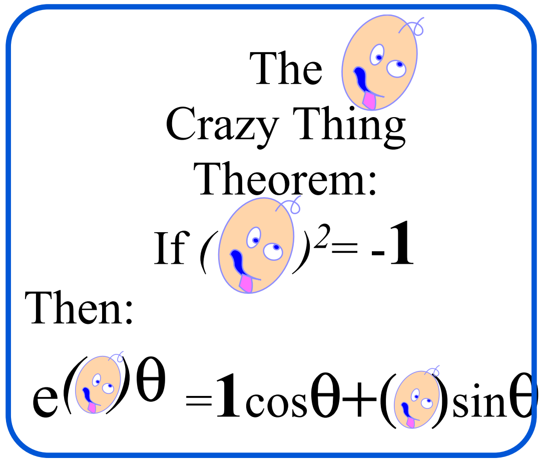
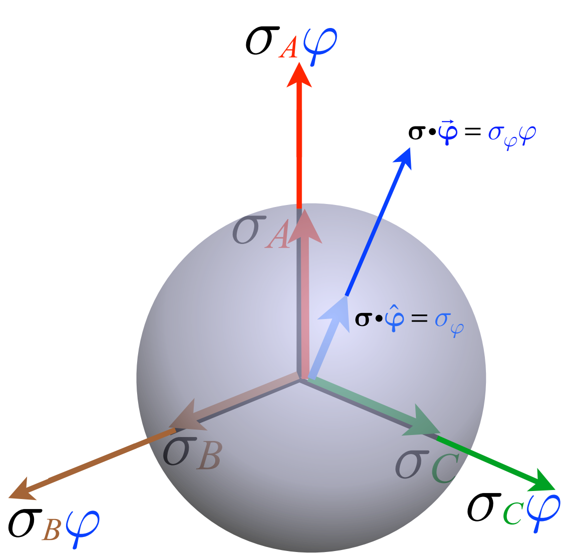




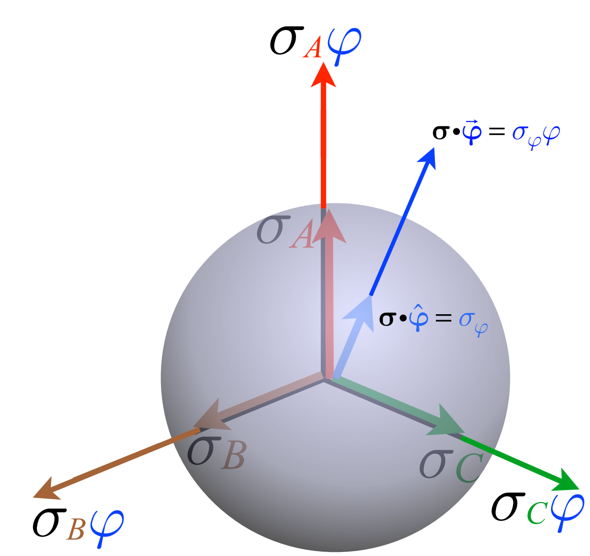
Half-angle $\Theta/2 = \varphi$ *replacement in rotation* $\mathbf{R}_\varphi = e^{-i\sigma_\varphi \varphi} = e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\varphi}}$ *where:* $\sigma_\varphi = \vec{\boldsymbol{\sigma}} \bullet \vec{\boldsymbol{\varphi}} = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\boldsymbol{\sigma}} \bullet \hat{\boldsymbol{\varphi}} \varphi$

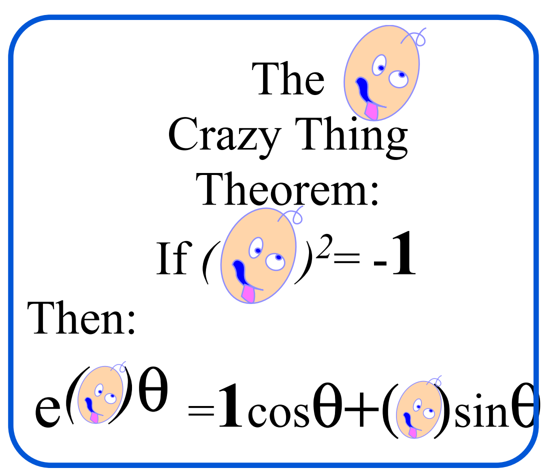
Replace spinor angle φ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\vec{\boldsymbol{\sigma}} \bullet \hat{\boldsymbol{\varphi}} \sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\frac{\vec{\boldsymbol{\Theta}}}{2}} = \mathbf{R}[\vec{\boldsymbol{\Theta}}] = \mathbf{1}\cos\frac{\Theta}{2} - i\vec{\boldsymbol{\sigma}} \bullet \hat{\boldsymbol{\Theta}} \sin\frac{\Theta}{2} = e^{-i\frac{\vec{\boldsymbol{\sigma}}}{2}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\mathbf{S}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\sigma_\varphi \varphi} = e^{-i\vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}}$

Unit rotation axis vector $\hat{\boldsymbol{\Theta}} = \left(\hat{\boldsymbol{\Theta}}_X, \hat{\boldsymbol{\Theta}}_Y, \hat{\boldsymbol{\Theta}}_Z \right) = \left(\cos\varphi\sin\vartheta \quad \sin\varphi\sin\vartheta \quad \cos\vartheta \right)$ is the same as the unit $\hat{\boldsymbol{\varphi}}$.

σ_μ-Operator Space





Half-angle $\Theta/2=\varphi$ *replacement in rotation* $\mathbf{R}_\varphi = e^{-i\sigma_\varphi \varphi} = e^{-i\vec{\sigma}\bullet\vec{\varphi}}$ *where:* $\sigma_\varphi = \vec{\sigma}\bullet\vec{\varphi} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \vec{\sigma}\bullet\hat{\varphi}\varphi$

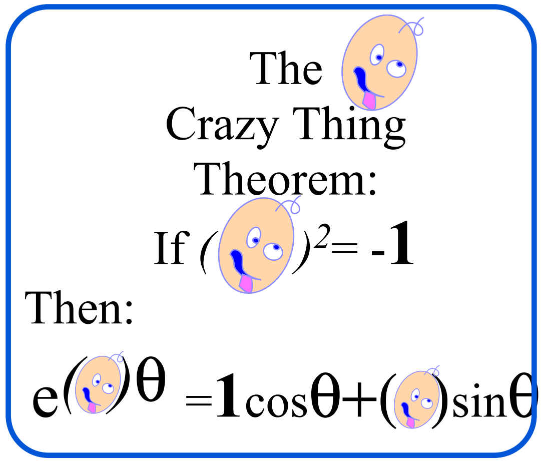
Replace spinor angle φ in: $e^{-i\vec{\sigma}\bullet\vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\vec{\sigma}\bullet\hat{\varphi}\,\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2}=\varphi$ in: $e^{-i\vec{\sigma}\bullet\frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1}\cos\frac{\Theta}{2} - i\vec{\sigma}\bullet\hat{\Theta}\sin\frac{\Theta}{2} = e^{-i\frac{\vec{\sigma}}{2}\bullet\vec{\Theta}} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\sigma}\bullet\vec{\varphi}}$

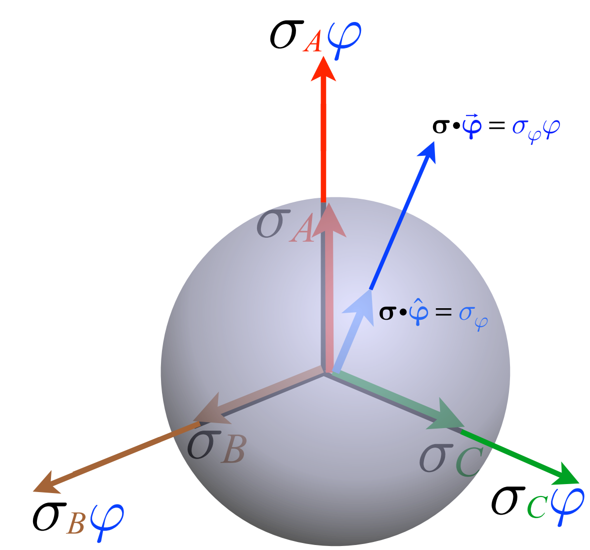
Unit rotation axis vector $\hat{\Theta} = \begin{pmatrix} \hat{\Theta}_X, & \hat{\Theta}_Y, & \hat{\Theta}_Z \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta \end{pmatrix}$ is the same as the unit $\hat{\varphi}$.

$$\mathbf{R}[\vec{\Theta}] = \cos\frac{\Theta}{2}\,\mathbf{1} - i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2} - i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$



σ_μ-Operator Space



Half-angle $\Theta/2=\varphi$ replacement in rotation $\mathbf{R}_\varphi=e^{-i\sigma_\varphi\varphi}=e^{-i\vec{\sigma}\bullet\vec{\varphi}}$ *where:* $\sigma_\varphi=\vec{\sigma}\bullet\vec{\varphi}=\sigma_A\varphi_A+\sigma_B\varphi_B+\sigma_C\varphi_C=\vec{\sigma}\bullet\hat{\varphi}\varphi$

Replace spinor angle φ in: $e^{-i\vec{\sigma}\bullet\vec{\varphi}}=\mathbf{R}_\varphi=\mathbf{1}\cos\varphi-i\vec{\sigma}\bullet\hat{\varphi}\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2}=\varphi$ in: $e^{-i\vec{\sigma}\bullet\frac{\vec{\Theta}}{2}}=\mathbf{R}[\vec{\Theta}]=\mathbf{1}\cos\frac{\Theta}{2}-i\vec{\sigma}\bullet\hat{\Theta}\sin\frac{\Theta}{2}=e^{-i\frac{\vec{\sigma}}{2}\bullet\vec{\Theta}}=e^{-i\mathbf{S}\bullet\vec{\Theta}}=e^{-i\sigma_\varphi\varphi}=e^{-i\vec{\sigma}\bullet\vec{\varphi}}$

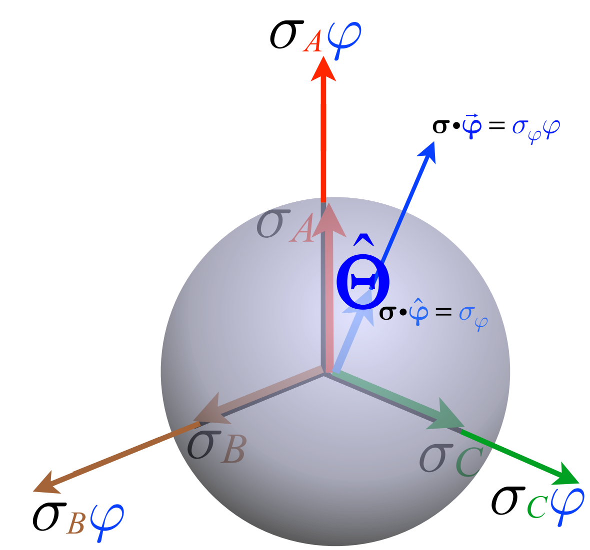
Unit rotation axis vector $\hat{\Theta}=\left(\hat{\Theta}_X,\ \hat{\Theta}_Y,\ \hat{\Theta}_Z\right)=\left(\cos\varphi\sin\vartheta\quad\sin\varphi\sin\vartheta\quad\cos\vartheta\right)$ is the same as the unit $\hat{\varphi}$.

$$\mathbf{R}[\vec{\Theta}]=\cos\frac{\Theta}{2}\,\mathbf{1}-i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2}-i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2}-i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$\begin{pmatrix}\langle 1|\mathbf{R}[\vec{\Theta}]|1\rangle & \langle 1|\mathbf{R}[\vec{\Theta}]|2\rangle\\ \langle 2|\mathbf{R}[\vec{\Theta}]|1\rangle & \langle 2|\mathbf{R}[\vec{\Theta}]|2\rangle\end{pmatrix}=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}$$

σ_μ-Operator Space



The Crazy Thing Theorem:
If $(\ )^2=-\mathbf{1}$
Then:
$e^{(\ )\theta}=\mathbf{1}\cos\theta+(\ )\sin\theta$

Polar coordinates for unit axis vector $\hat{\Theta}$

$\hat{\Theta}_X=\cos\varphi\ \sin\vartheta$
$\hat{\Theta}_Y=\sin\varphi\ \sin\vartheta$
$\hat{\Theta}_Z=\qquad\cos\vartheta$

Half-angle $\Theta/2=\varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi \varphi} = e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\varphi}}$ *where:* $\sigma_\varphi = \vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\varphi}}\varphi$

Replace spinor angle φ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\varphi}}\,\sin\varphi$

with 3D $\frac{1}{2}$-angle $\frac{\Theta}{2}=\varphi$ in: $e^{-i\vec{\boldsymbol{\sigma}}\bullet\frac{\vec{\boldsymbol{\Theta}}}{2}} = \mathbf{R}[\vec{\boldsymbol{\Theta}}] = \mathbf{1}\cos\frac{\Theta}{2} - i\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}\sin\frac{\Theta}{2} = e^{-i\frac{\vec{\boldsymbol{\sigma}}}{2}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\mathbf{S}\bullet\vec{\boldsymbol{\Theta}}} = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\boldsymbol{\sigma}}\bullet\vec{\boldsymbol{\varphi}}}$

Unit rotation axis vector $\hat{\boldsymbol{\Theta}} = \left(\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z\right) = \left(\cos\varphi\sin\vartheta \quad \sin\varphi\sin\vartheta \quad \cos\vartheta\right)$ is the same as the unit $\hat{\boldsymbol{\varphi}}$.

$$\mathbf{R}[\vec{\boldsymbol{\Theta}}] = \cos\frac{\Theta}{2}\;\mathbf{1} - i\,\sigma_X\,\hat{\Theta}_X\sin\frac{\Theta}{2} - i\,\sigma_Y\,\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\,\sigma_Z\,\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

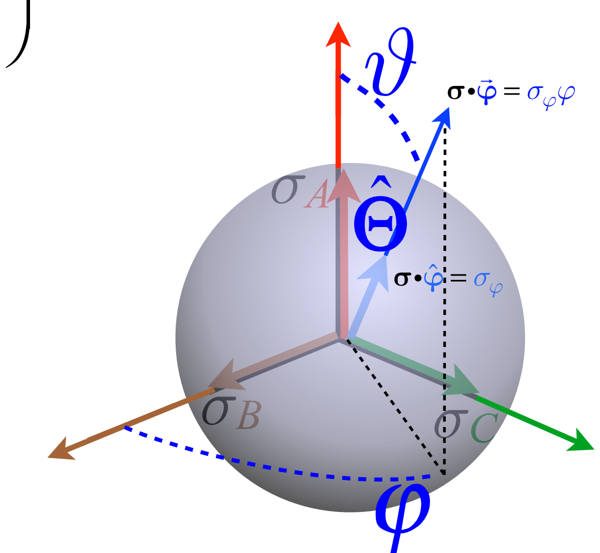
$$= \cos\frac{\Theta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$$\begin{pmatrix}\langle 1|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|1\rangle & \langle 1|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|2\rangle\\ \langle 2|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|1\rangle & \langle 2|\mathbf{R}[\vec{\boldsymbol{\Theta}}]|2\rangle\end{pmatrix} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\cos\varphi\sin\vartheta - i\sin\varphi\sin\vartheta)\\ -i\sin\frac{\Theta}{2}(\cos\varphi\sin\vartheta + i\sin\varphi\sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

$$= \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\vec{\boldsymbol{\Theta}}\bullet\mathbf{S}} = e^{-i\mathbf{H}t}$$

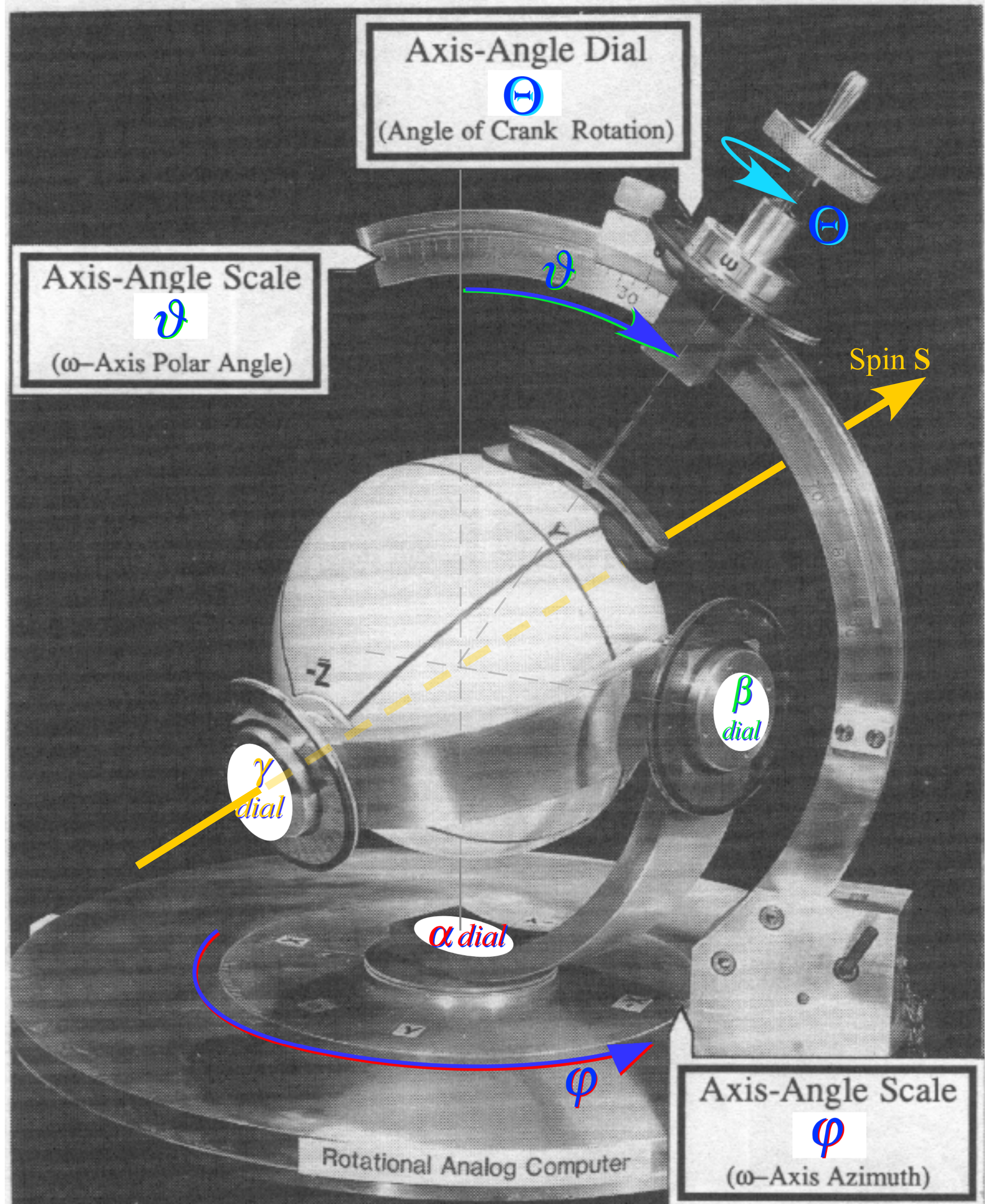
***Θ*-crank polar angle**



***Θ*-crank azimuth angle**

Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

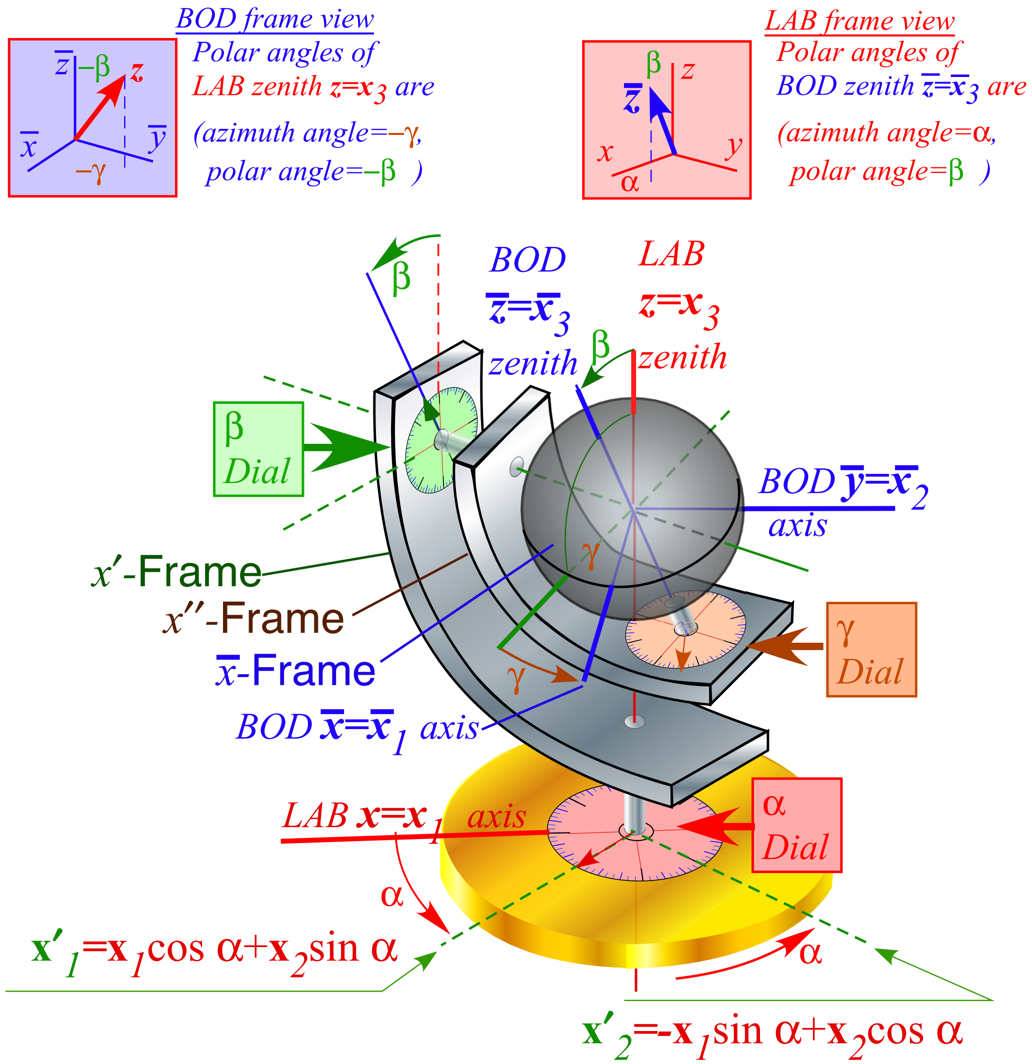
$\hat{\Theta}_X = \cos\varphi\ \sin\vartheta$
$\hat{\Theta}_Y = \sin\varphi\ \sin\vartheta$
$\hat{\Theta}_Z = \cos\vartheta$

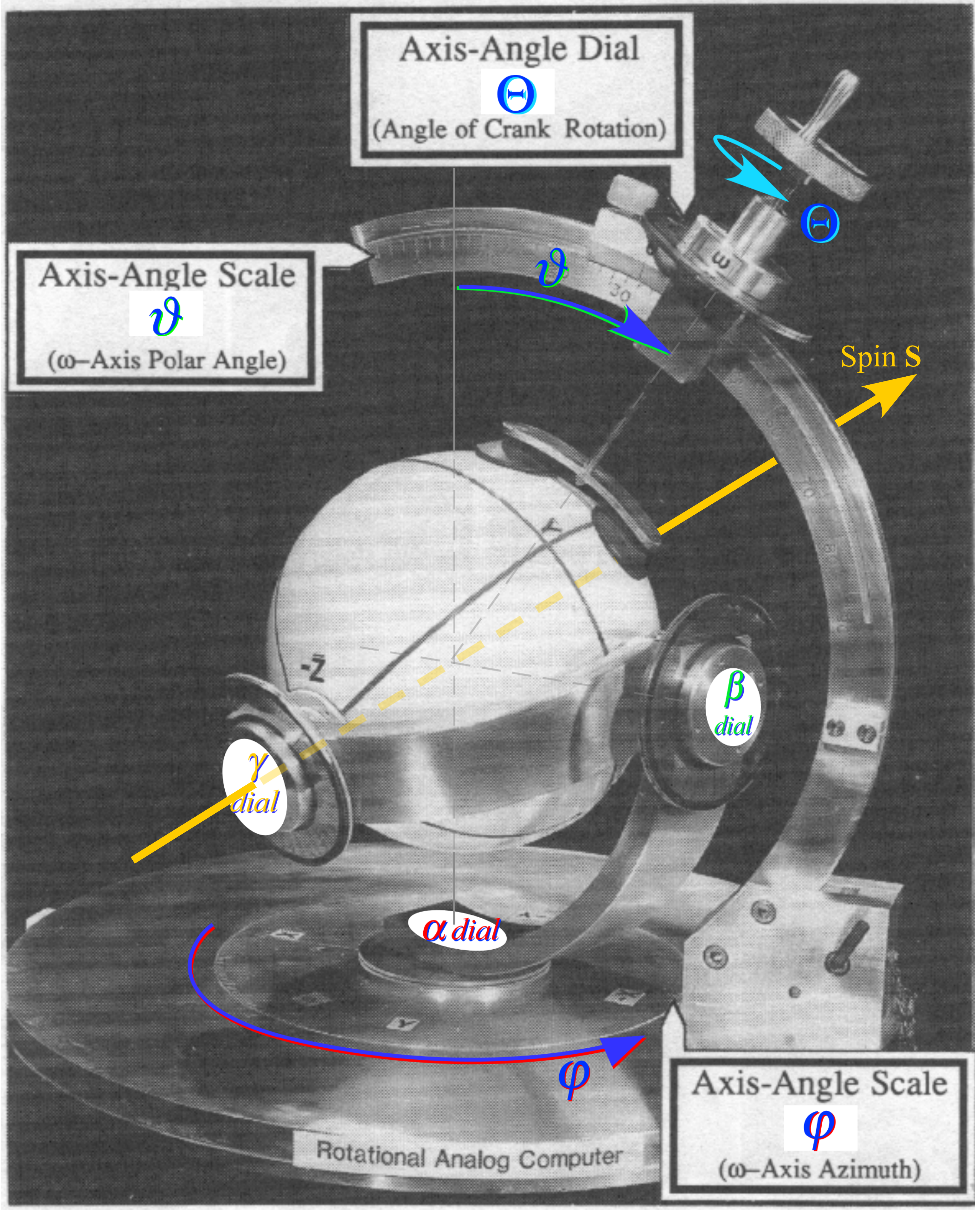


Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

Here spin-rotor S-polar coordinates are Euler angles





Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor* σ_μ*-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	**1**	σ_X	σ_Y	σ_Z
1	**1**	σ_X	σ_Y	σ_Z
σ_X	σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\, \mathbf{1} + \varepsilon_{\mu\nu\lambda}\, \mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\, \mathbf{1} + i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

Operator-on-Operator transformations

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•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\, \mathbf{1} + \varepsilon_{\mu\nu\lambda}\, \mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\, \mathbf{1} + i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

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•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \, \mathbf{q}_\nu = -\delta_{\mu\nu} \, \mathbf{1} + \varepsilon_{\mu\nu\lambda} \, \mathbf{q}_\lambda$$

$$\sigma_\mu \, \sigma_\nu = \delta_{\mu\nu} \, \mathbf{1} + i \, \varepsilon_{\mu\nu\lambda} \, \sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i \, \varepsilon_{\mu\nu\lambda} \, \sigma_\lambda$$

Jordan's spin-ops: $\mathbf{J}_\mu = \mathbf{S}_\mu = \sigma_\mu/2$.

$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i \, \varepsilon_{\mu\nu\lambda} \, \mathbf{S}_\lambda$$

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Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\, \mathbf{1} + \varepsilon_{\mu\nu\lambda}\, \mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\, \mathbf{1} + i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
$$\sigma_\mu\,\sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

This NOT just $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\, \mathbf{1} + \varepsilon_{\mu\nu\lambda}\, \mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\, \mathbf{1} + i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

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$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\, \varepsilon_{\mu\nu\lambda}\, \sigma_\lambda$$

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$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
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Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
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$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	q_X	q_Y	q_Z
1	1	q_X	q_Y	q_Z
q_X	q_X	-1	q_Z	$-q_Y$
q_Y	q_Y	$-q_Z$	-1	q_X
q_Z	q_Z	q_Y	$-q_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
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$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

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$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
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Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$= \left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$= \left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient ( ) of **unit 1**
derives ***angle of rotation***: Θ_{ab}

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

$\bullet$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

$\bullet$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\mathbf{1} + \varepsilon_{\mu\nu\lambda}\mathbf{q}_\lambda$$
$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This <u>NOT</u> just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient ( ) of **unit 1** derives ***angle of rotation***: Θ_{ab}

2nd Step: Coefficient { } of $-i\bullet\vec{\sigma}$ derives ***unit-vector*** $\hat{\Theta}_{ab}$ ***of rotation***:

Operator-on-Operator transformations

Product algebra $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

This NOT just $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

(...except when $\hat{\Theta}_a=\hat{\Theta}_b$ *!)*

$$= \left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$= \left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient ( ) of **unit 1** derives ***angle of rotation:*** Θ_{ab}

2nd Step: Coefficient { } of $-i\bullet\vec{\sigma}$ derives ***unit-vector*** $\hat{\Theta}_{ab}$ ***of rotation:***

Now easy to find the *product angle* Θ_{ab} and *crank unit vector* $\hat{\Theta}_{ab}$.

U(2) and R(3) Group Product Formulae

$$\frac{\Theta_{ab}}{2} = \cos^{-1}\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\hat{\Theta}_a\bullet\hat{\Theta}_b\right)$$

$$\vec{\Theta}_{ab} = \left[\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\hat{\Theta}_a + \cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\hat{\Theta}_b + \sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}_a\times\hat{\Theta}_b\right]/\sin\frac{\Theta_{ab}}{2}$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$- *formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor* σ_μ*-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$- *geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta = 0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

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➡ *Transformation* $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

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Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\vec{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\vec{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

Sum over repeated indices is implied:

$$a_M b_M = \sum_{M=1}^{3} a_M b_M$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)\boldsymbol{\sigma}_L\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\boldsymbol{\Theta}}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\boldsymbol{\sigma}_L\left(\cos\frac{\Theta}{2}\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

Sum over repeated indices is implied:

$$a_M b_M = \sum_{M=1}^{3} a_M b_M$$

(Left as an exercise)

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

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Geometry of groups: Hamilton's turns and It's all done with mirrors!

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

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Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

General transformation of rotational **R[Θ′]***-operators*

$$\mathbf{R}[\vec{\Theta}]\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$\mathbf{R}\left[\mathbf{R}[\Theta]\cdot\vec{\Theta}'\right] = \left(\cos\frac{\Theta}{2}\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

Transformation of spinor σ_μ-operators

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$= \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

$$= \sigma'_L \qquad = \sigma_L\cos\Theta - \varepsilon_{LKN}\hat{\Theta}_K\sigma_N\sin\Theta + (1-\cos\Theta)\hat{\Theta}_L\left(\hat{\Theta}_M\sigma_M\right)$$

General transformation of rotational $\mathbf{R}[\Theta']$*-operators*

$$\mathbf{R}[\vec{\Theta}]\mathbf{R}[\vec{\Theta}']\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\,\hat{\Theta}\bullet\boldsymbol{\sigma}\right)^\dagger$$

$$\mathbf{R}\left[\mathbf{R}[\Theta]\cdot\vec{\Theta}'\right] = \left(\cos\frac{\Theta}{2}\,\mathbf{1} - i\sin\frac{\Theta}{2}\hat{\Theta}_K\sigma_K\right)\mathbf{R}[\vec{\Theta}']\left(\cos\frac{\Theta}{2}\,\mathbf{1} + i\sin\frac{\Theta}{2}\hat{\Theta}_M\sigma_M\right)$$

This one is better seen geometrically. Algebra not so quick.

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

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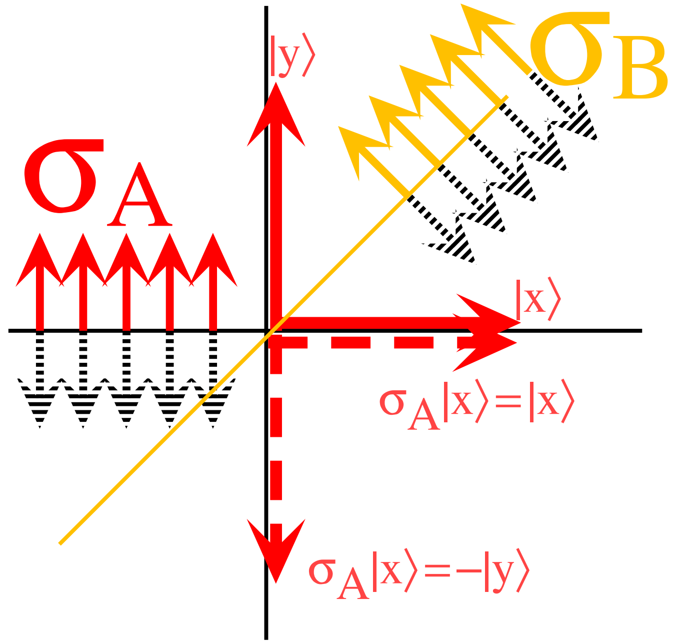
Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_A$ is an x-plane mirror



Geometry of U(2) transformations. It's all done with mirrors!

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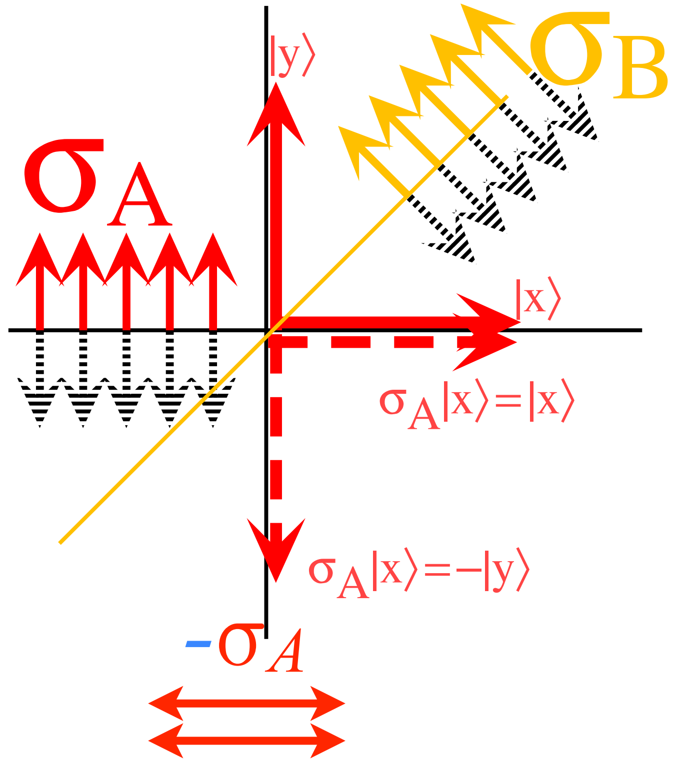
$+\sigma_A$ is an x-plane mirror



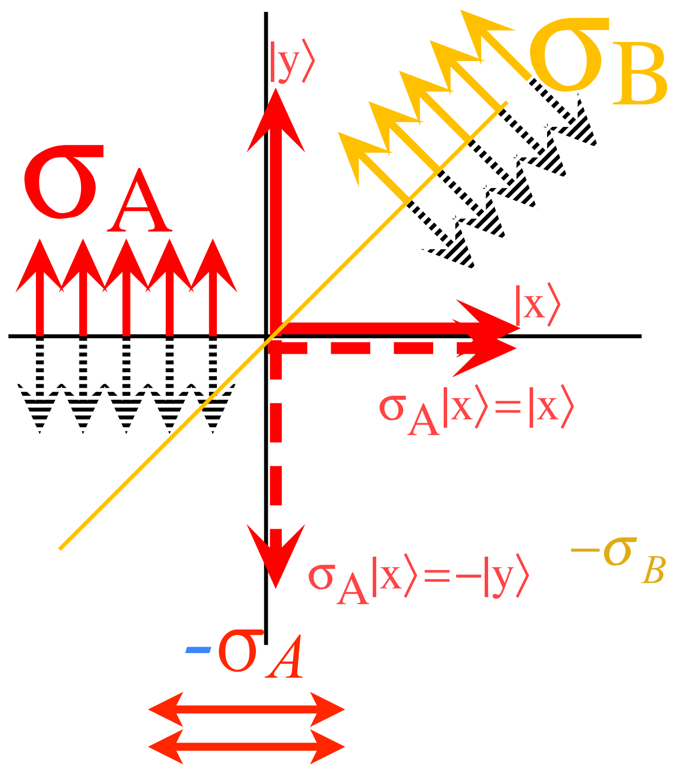
Note that $-\sigma_A$ is a y-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_A$ is an x-plane mirror



$$-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = ?$$

Note that $-\sigma_A$ is a y-plane mirror

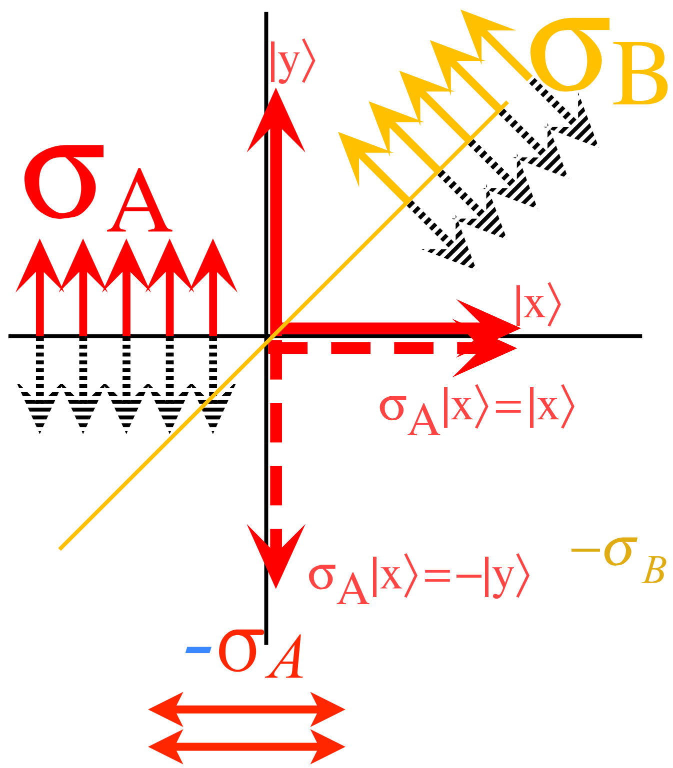
$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_B$ is an 45°-plane mirror

$+\sigma_A$ is an x-plane mirror



$$-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = ?$$

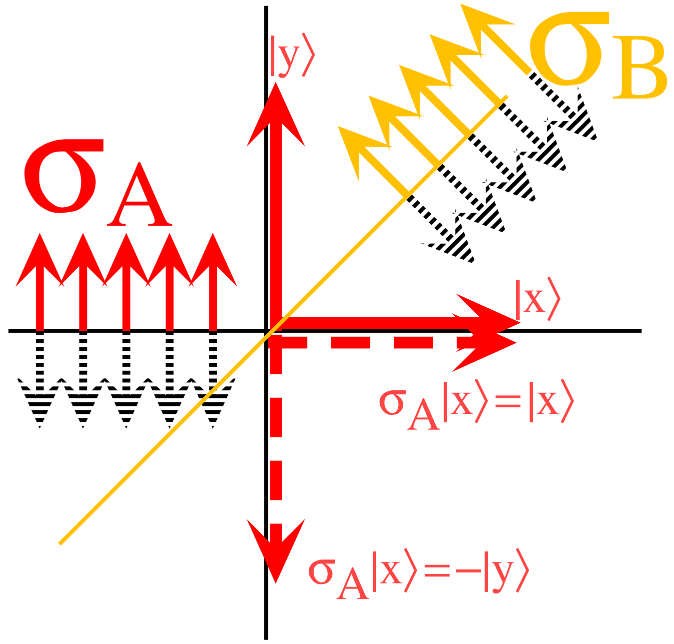
$-\sigma_B$ is an -45°-plane mirror

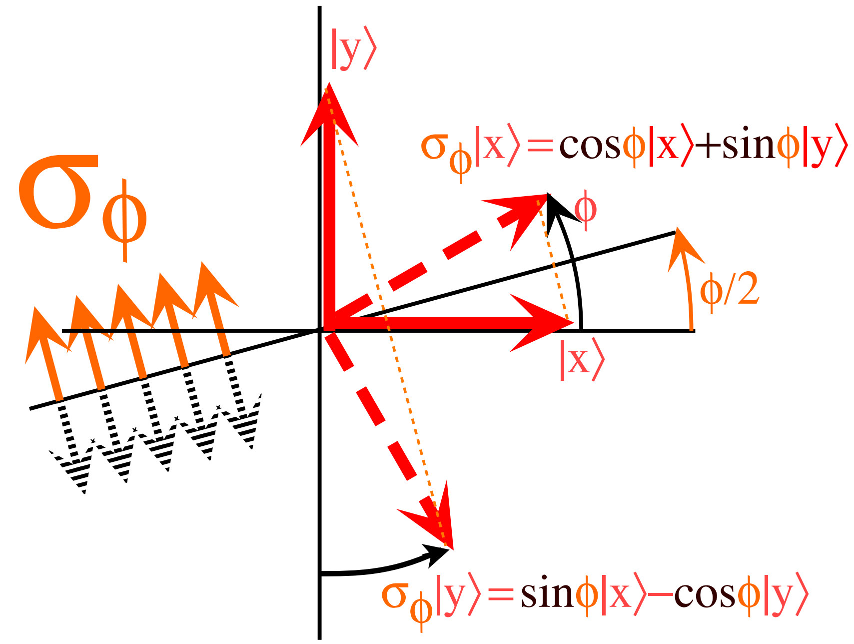
Note that $-\sigma_A$ is a y-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

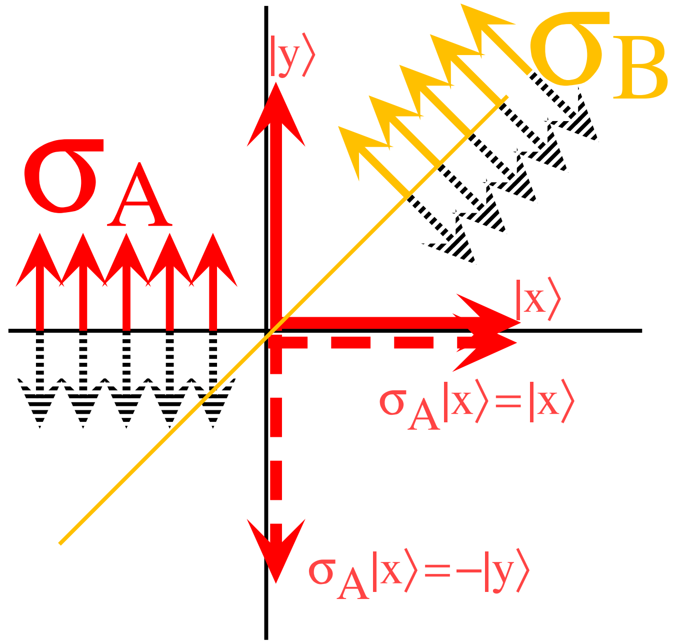
$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$

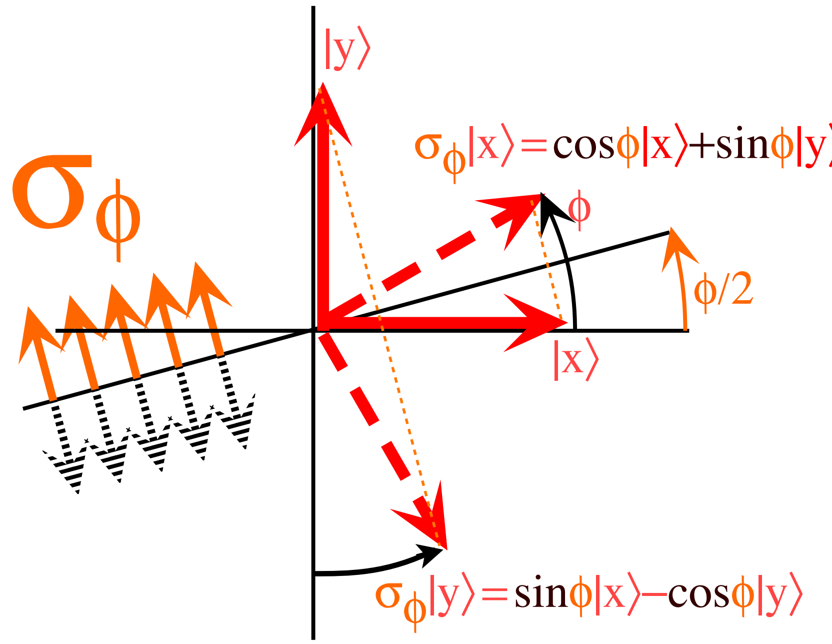




Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



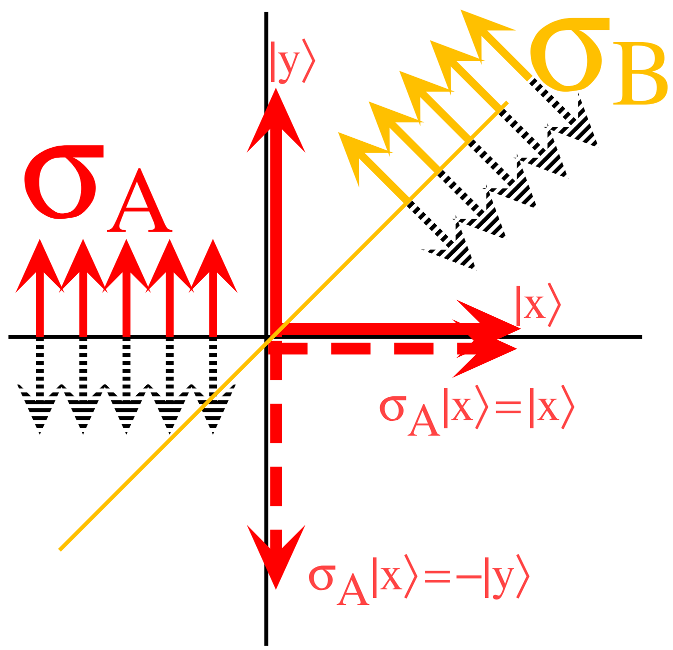


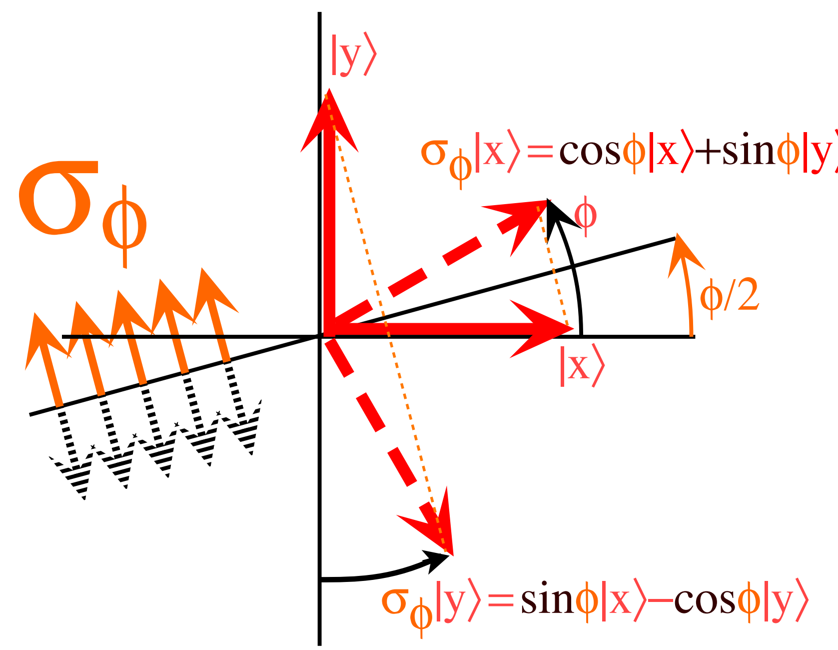
implies : $\langle x|\sigma_\phi|x\rangle = \cos\phi$

and : $\langle y|\sigma_\phi|x\rangle = \sin\phi$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





implies: $\langle x|\sigma_\phi|x\rangle = \cos\phi$

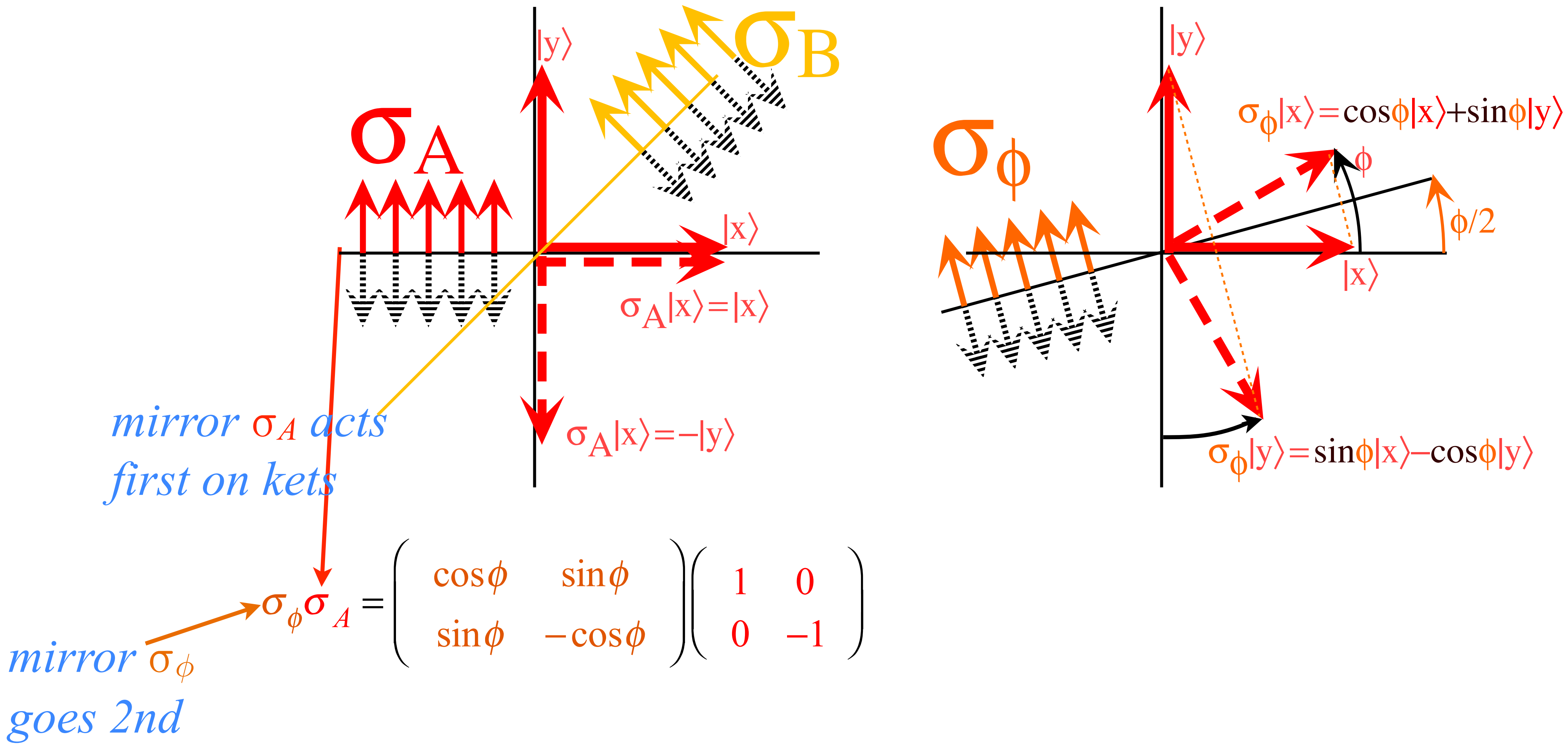
and: $\langle y|\sigma_\phi|x\rangle = \sin\phi$

implies: $\langle x|\sigma_\phi|y\rangle = \sin\phi$

and: $\langle y|\sigma_\phi|y\rangle = -\cos\phi$

Geometry of U(2) transformations. It's all done with mirrors!

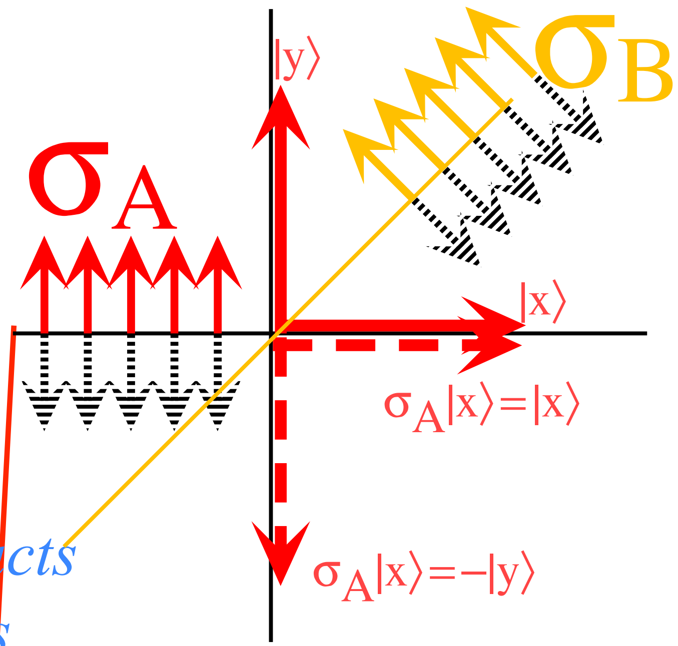
$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$

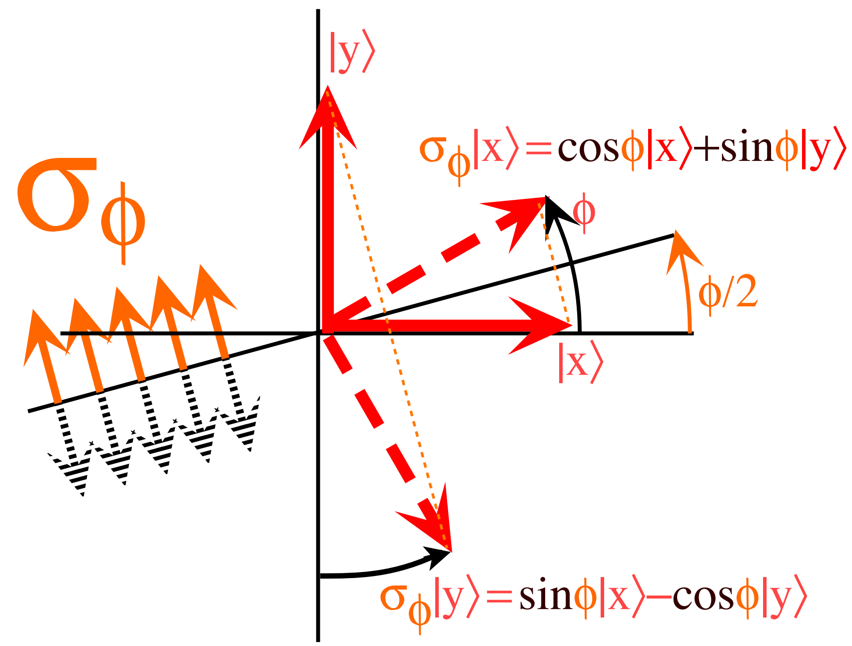


$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A *acts first on kets*

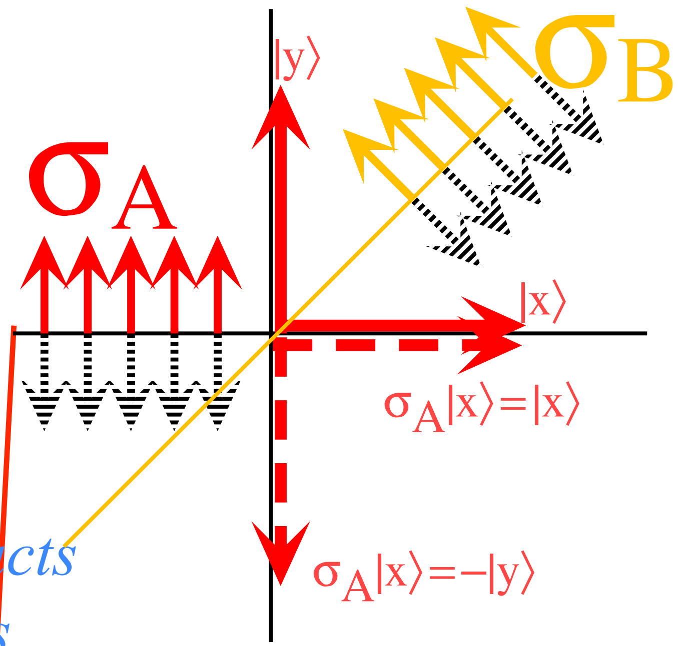
mirror σ_ϕ *goes 2nd*

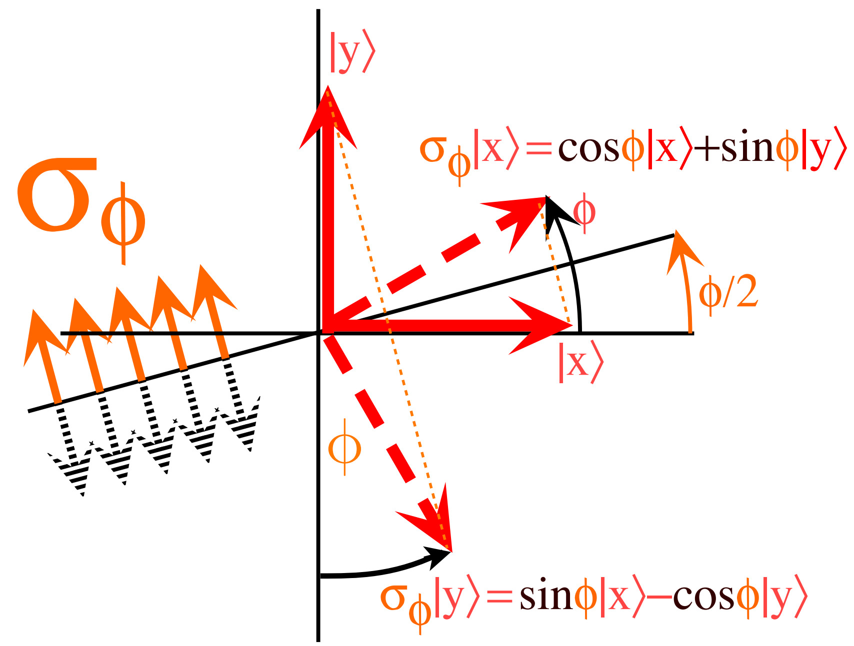
$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \;,$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts first on kets

mirror σ_ϕ goes 2nd

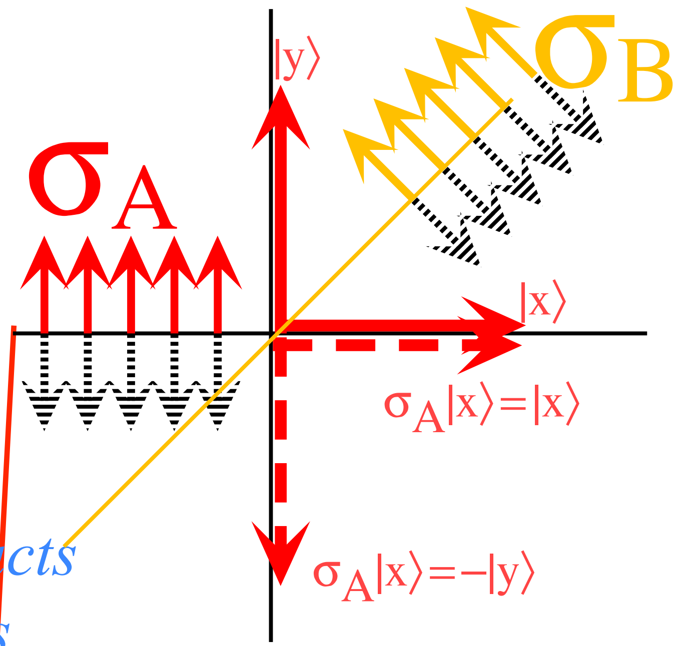
$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

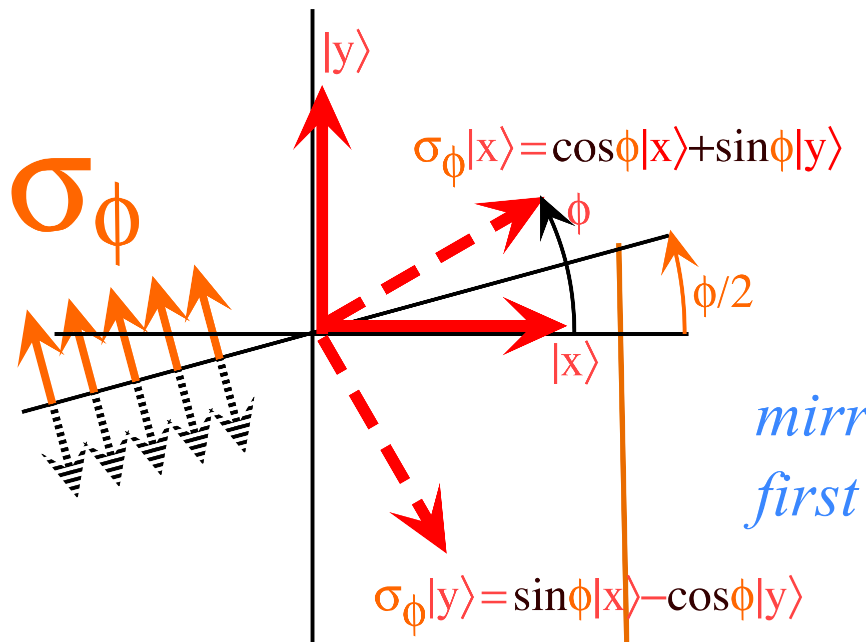
$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi],$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts first on kets

mirror σ_ϕ acts first on kets

mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \, ,$$

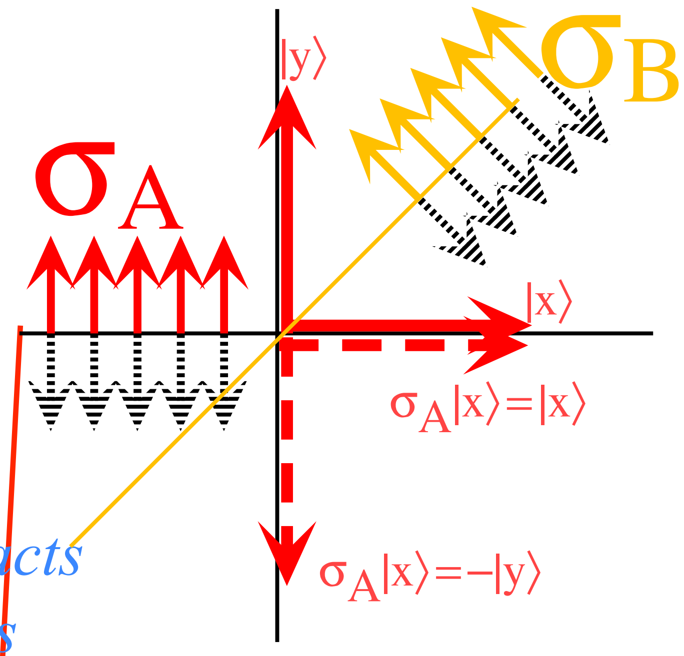
mirror σ_A goes 2nd

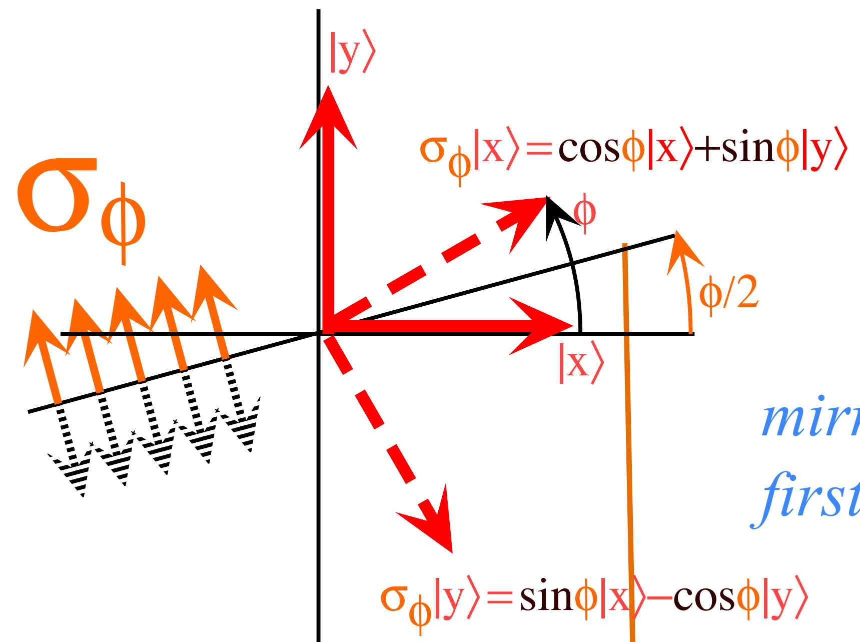
$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts first on kets

mirror σ_ϕ acts first on kets

mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] ,$$

mirror σ_A goes 2nd

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

determinant det = -1 · *determinant det = -1* · *determinant det = +1* · *determinant det = -1*

$$\sigma_A \sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_C$$

xy-rotation by -90° · *imaginary reflection?*

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\boldsymbol{q}_\mu = -i\sigma_\mu$

Group product algebra

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Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

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Geometry of groups: Hamilton's turns and It's all done with mirrors!

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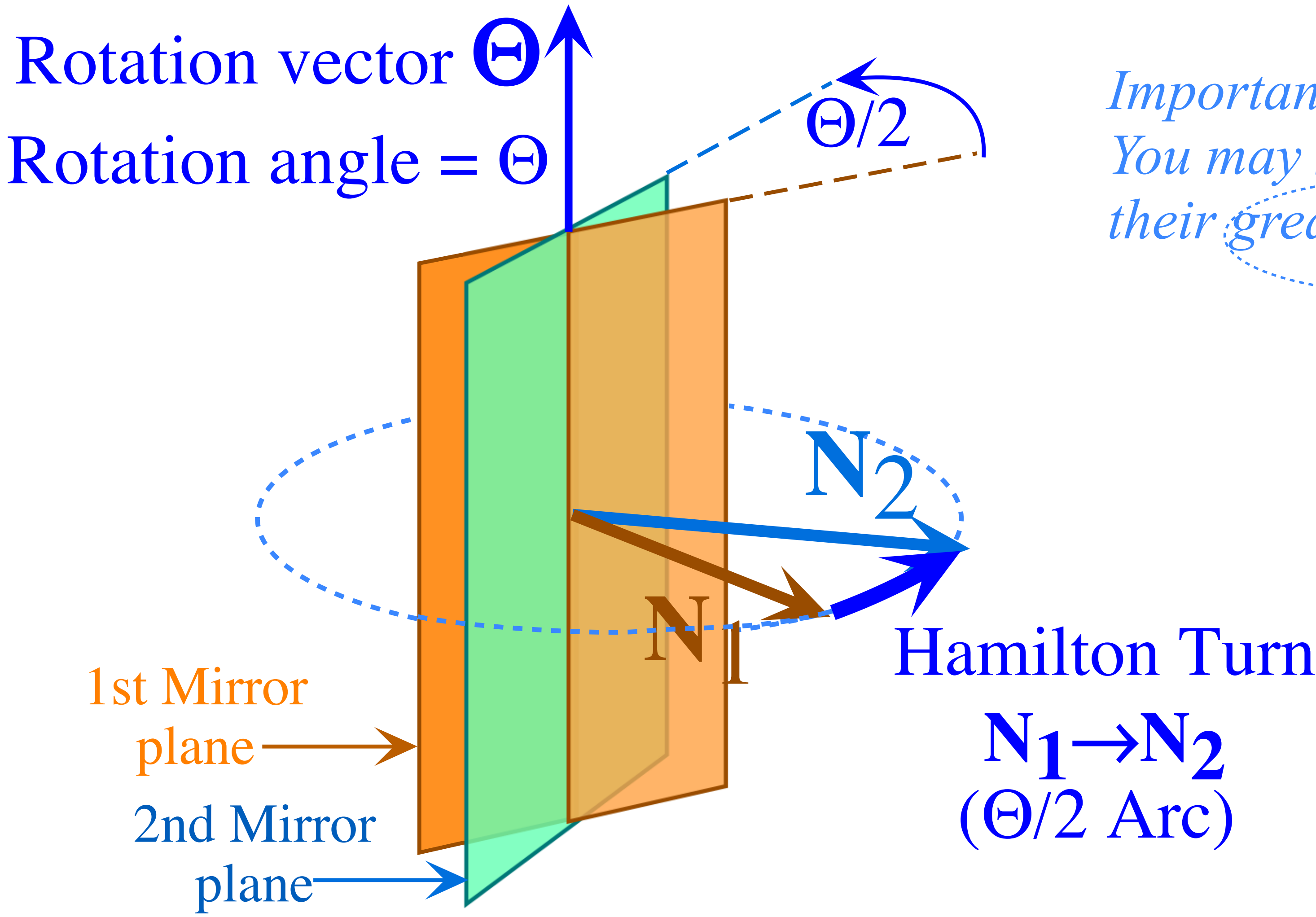
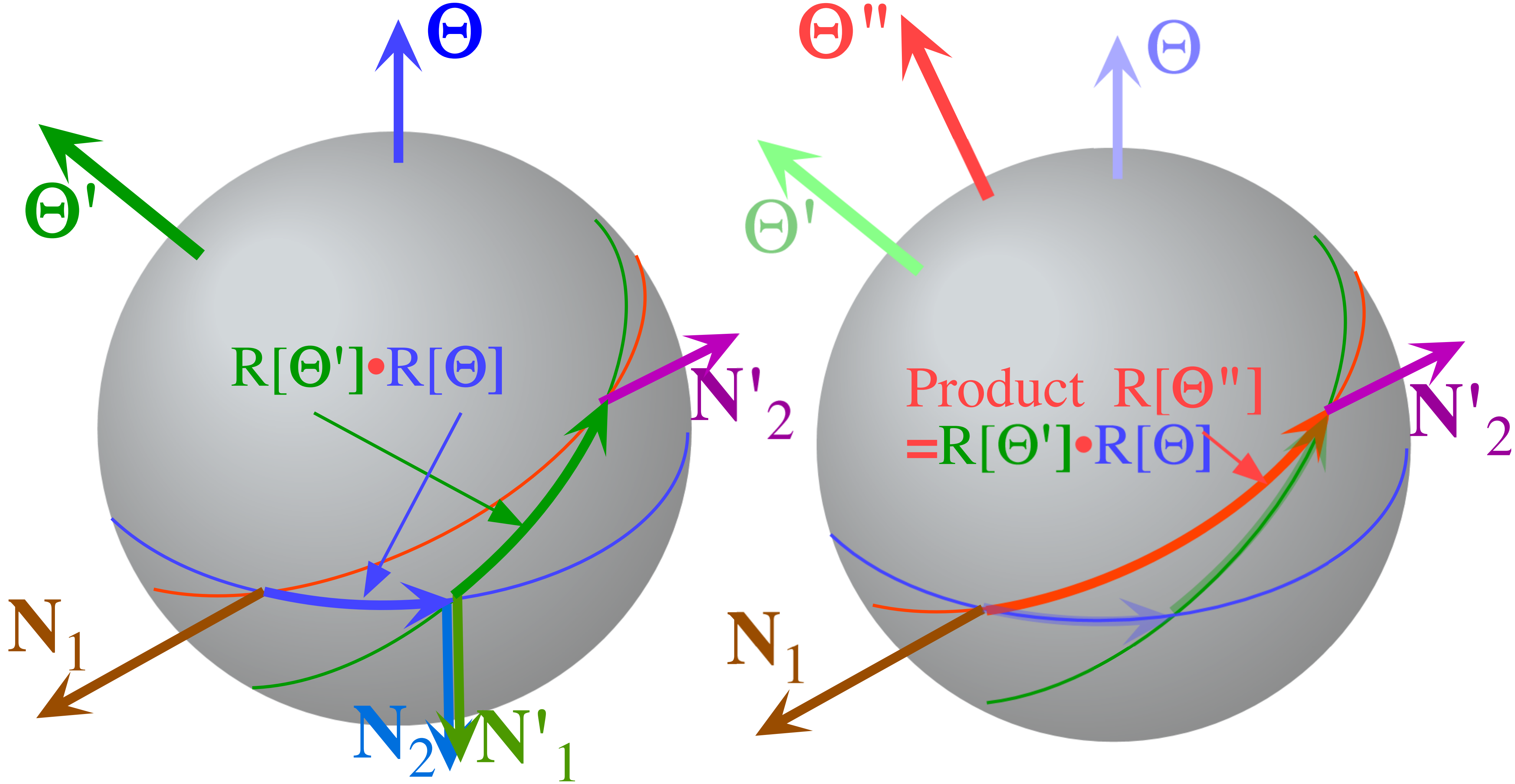
Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*



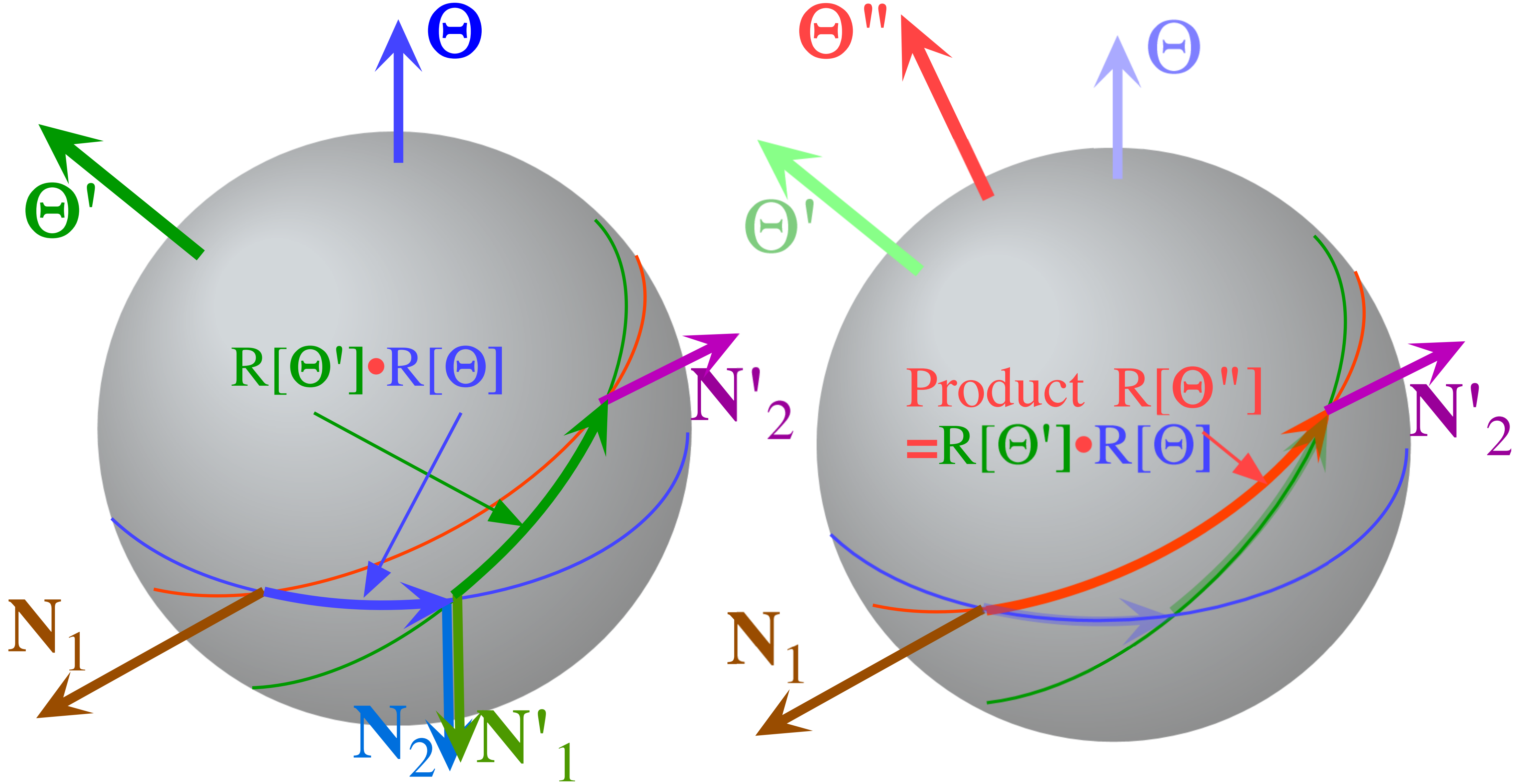
Important point about ***Turn arcs****:*
You may slide them anywhere on their great-circle arc

Fig. 10.A.7 Mirror reflection planes, normals, and Hamilton-turn arc vector.



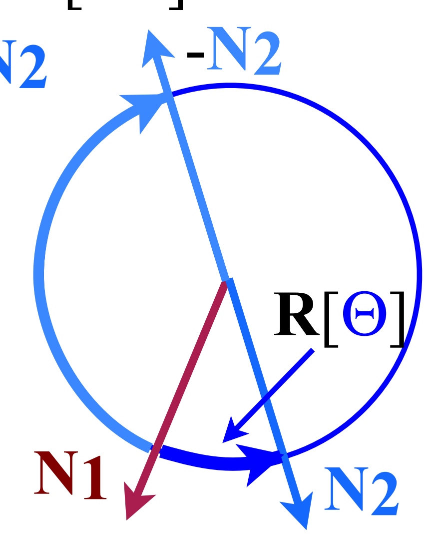
QTforCA Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R**[Θ"]=**R**[Θ']**R**[Θ].

Each arc Θ/2, Θ'/2, or Θ"/2 is 1/2 actual angle Θ, Θ', or Θ" of rotation **R**[Θ], **R**[Θ'], or **R**[Θ"].

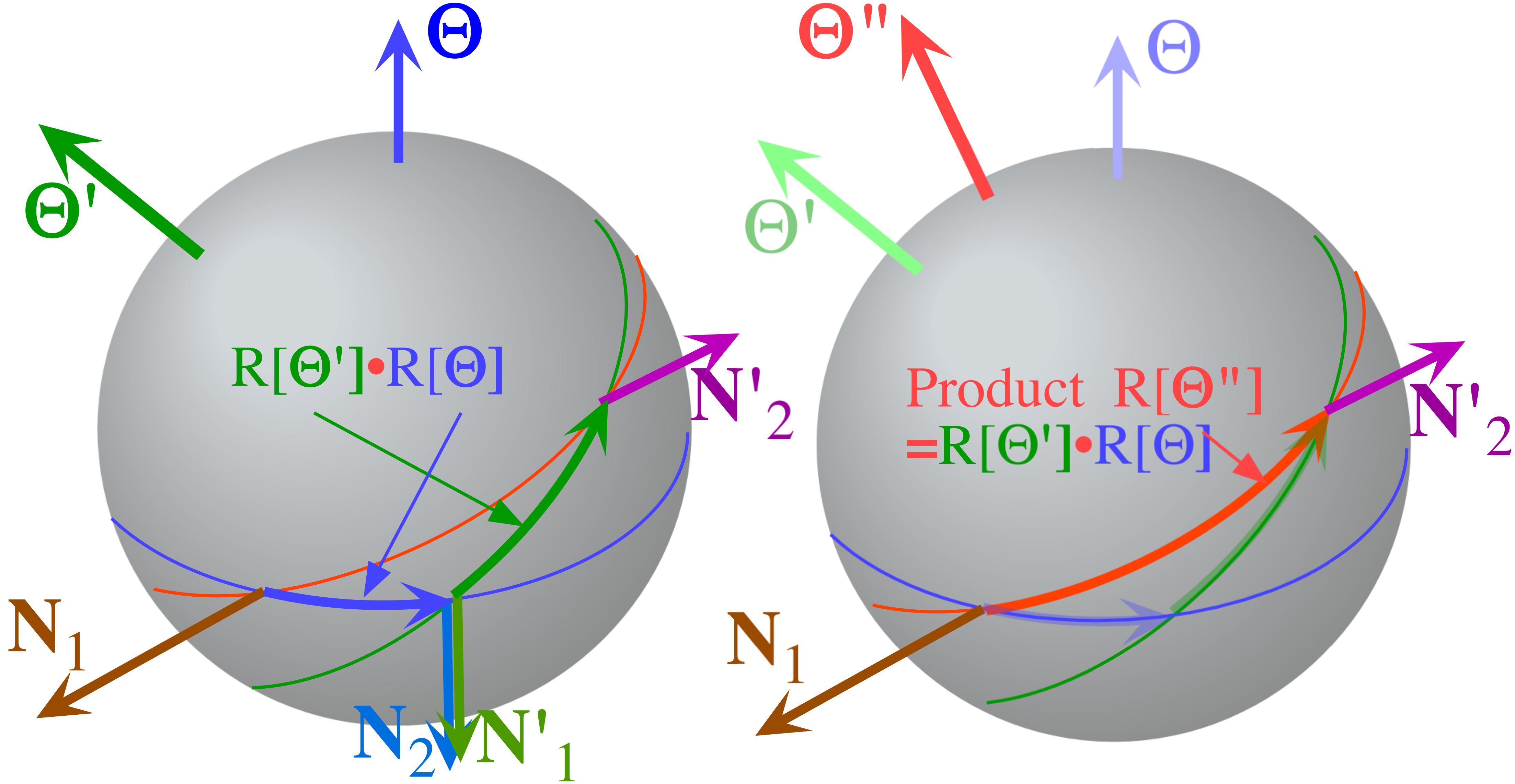


QTforCA Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R**[Θ"]=**R**[Θ']**R**[Θ].

Each arc Θ/2, Θ'/2, or Θ"/2 is 1/2 actual angle Θ, Θ', or Θ" of rotation **R**[Θ], **R**[Θ'], or **R**[Θ"].
Arc Θ/2 between **N1** and **N2** and its supplement (Θ±2π)/2=Θ/2±π between **N1** and -**N2** represent the same <u>classical</u> rotation by Θ.



Geometry of U(2) group products: Hamilton's Turns



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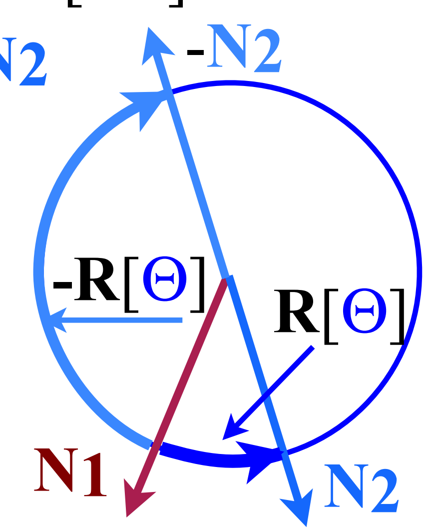
Arc Θ/2 between **N1** and **N2** and its supplement $(\Theta \pm 2\pi)/2 = \Theta/2 \pm \pi$ between **N1** and -**N2** represent the same classical rotation by Θ.



For quantum spin-1/2 object, the arc pointing from **N1** to the antipodal normal -**N2** represents a Θ-rotation with an extra π-phase factor $e^{\pm i\pi} = -1$, that is, -**R**[Θ] .

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

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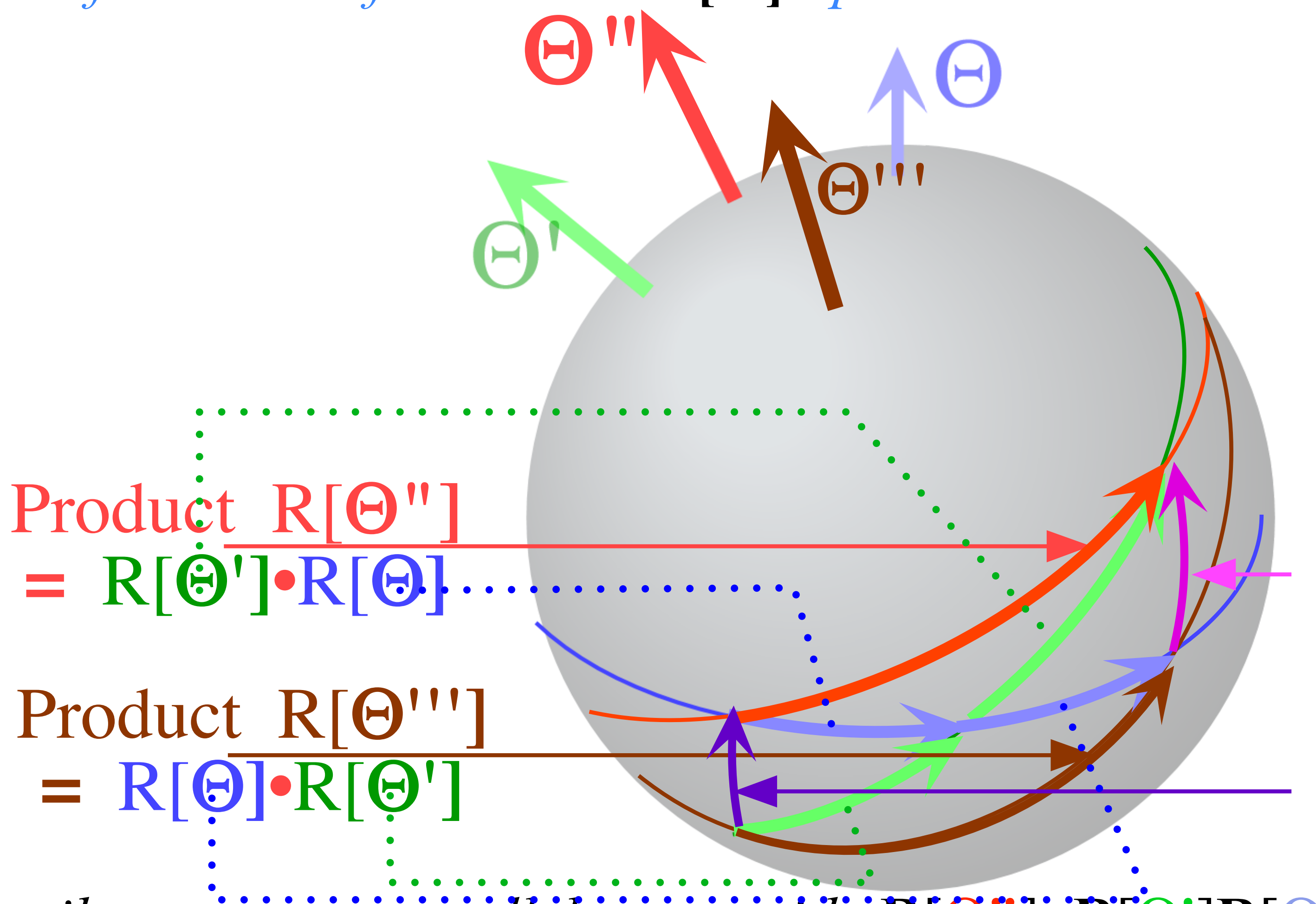

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Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

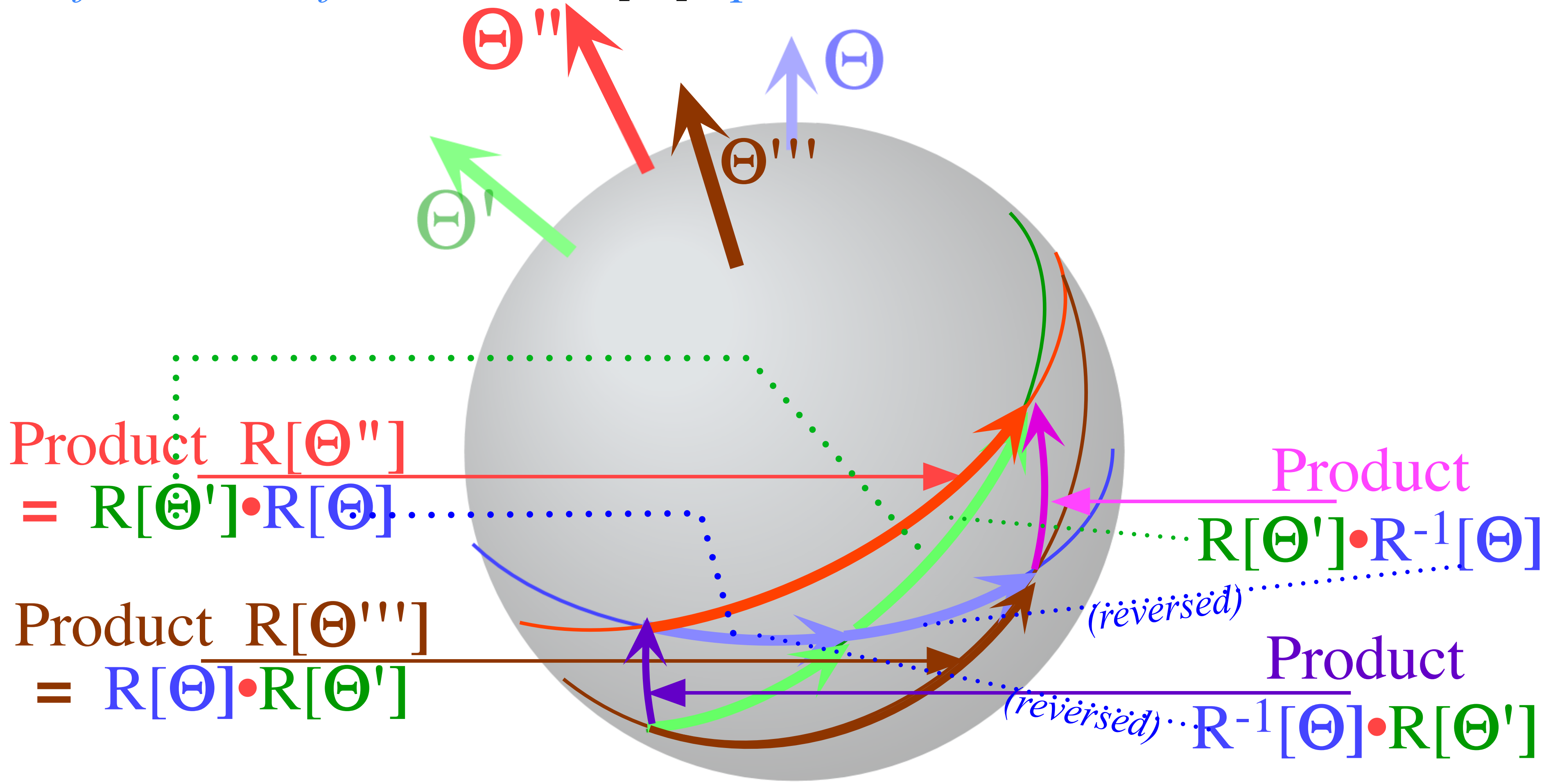

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Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Geometry of transformation of rotational **R[Θ]**-*operators*

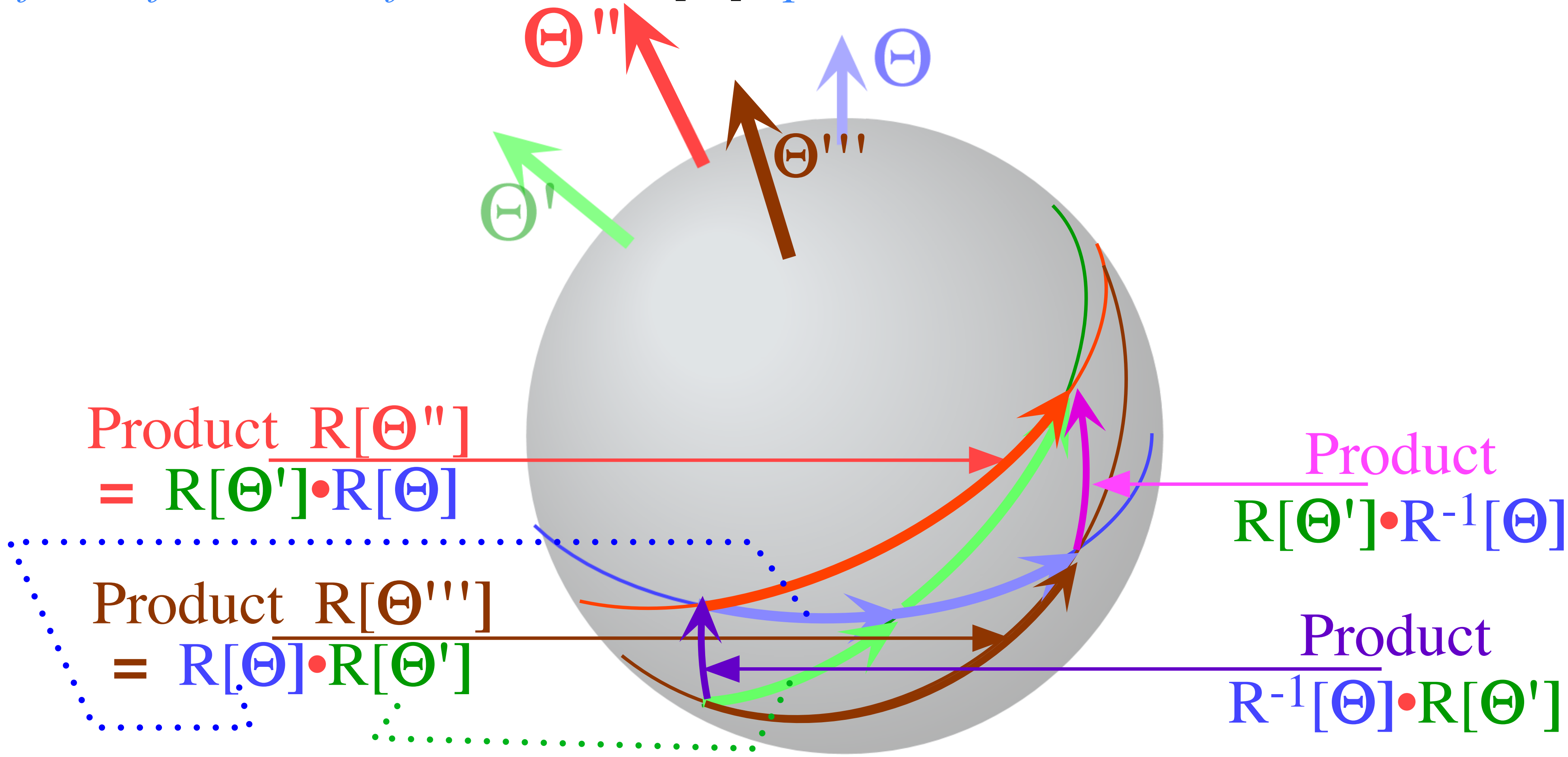

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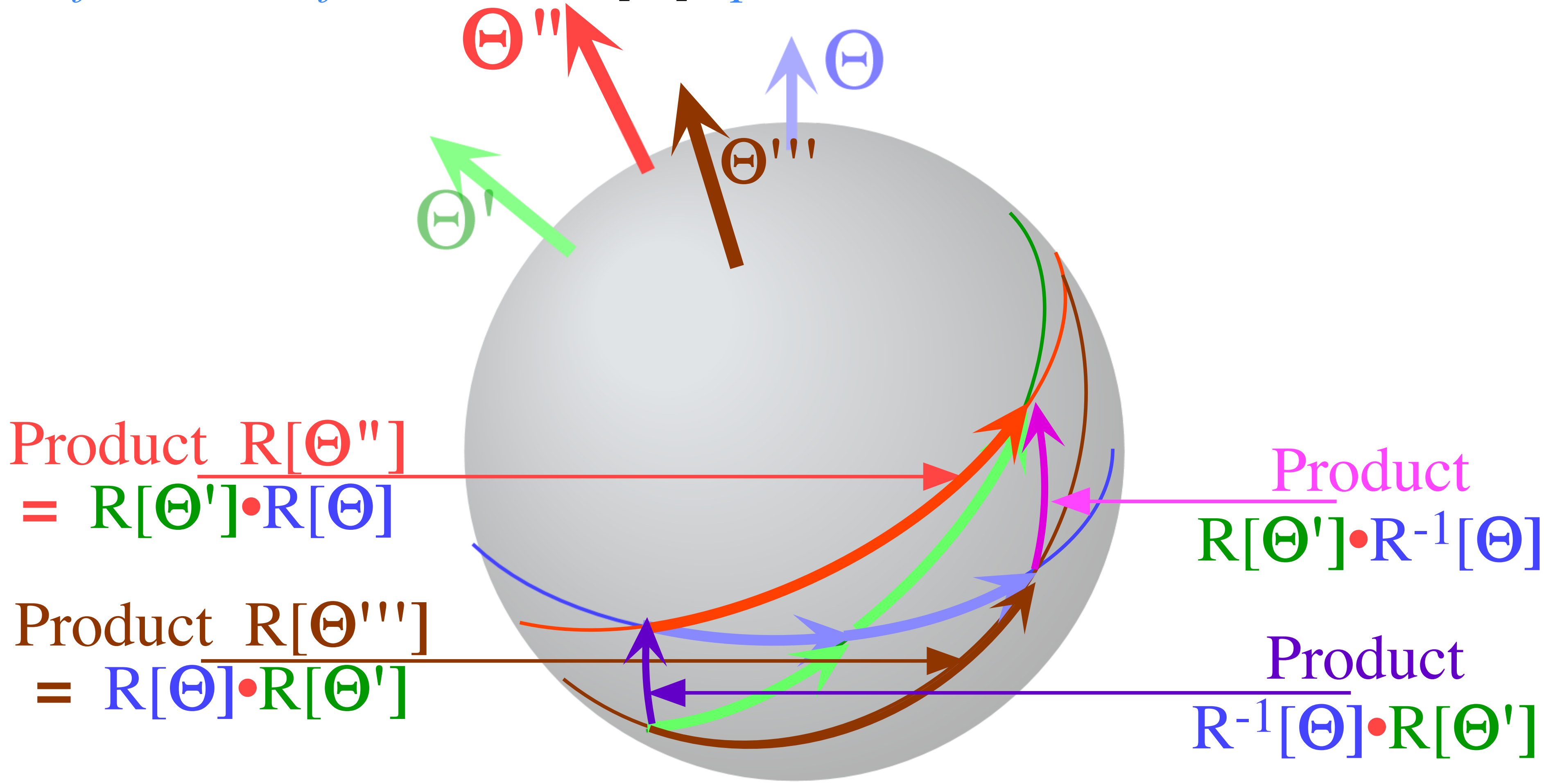
Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].



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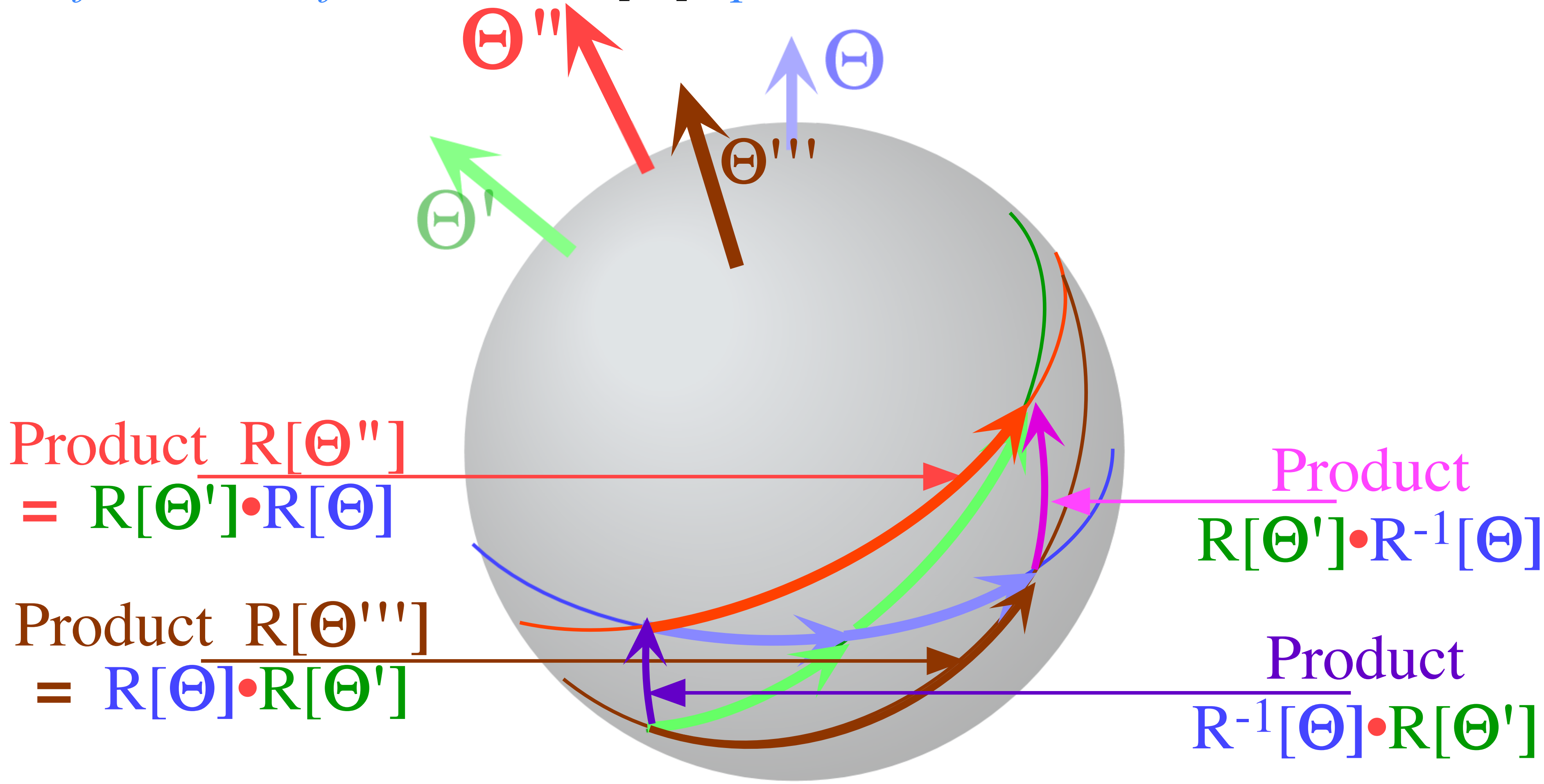
Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ"]=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ"] by rotation **R**[Θ] gives rotation **R**[Θ''']

$$\mathbf{R}[\Theta]\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta''']$$



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Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

$$\mathbf{R}[\Theta]\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

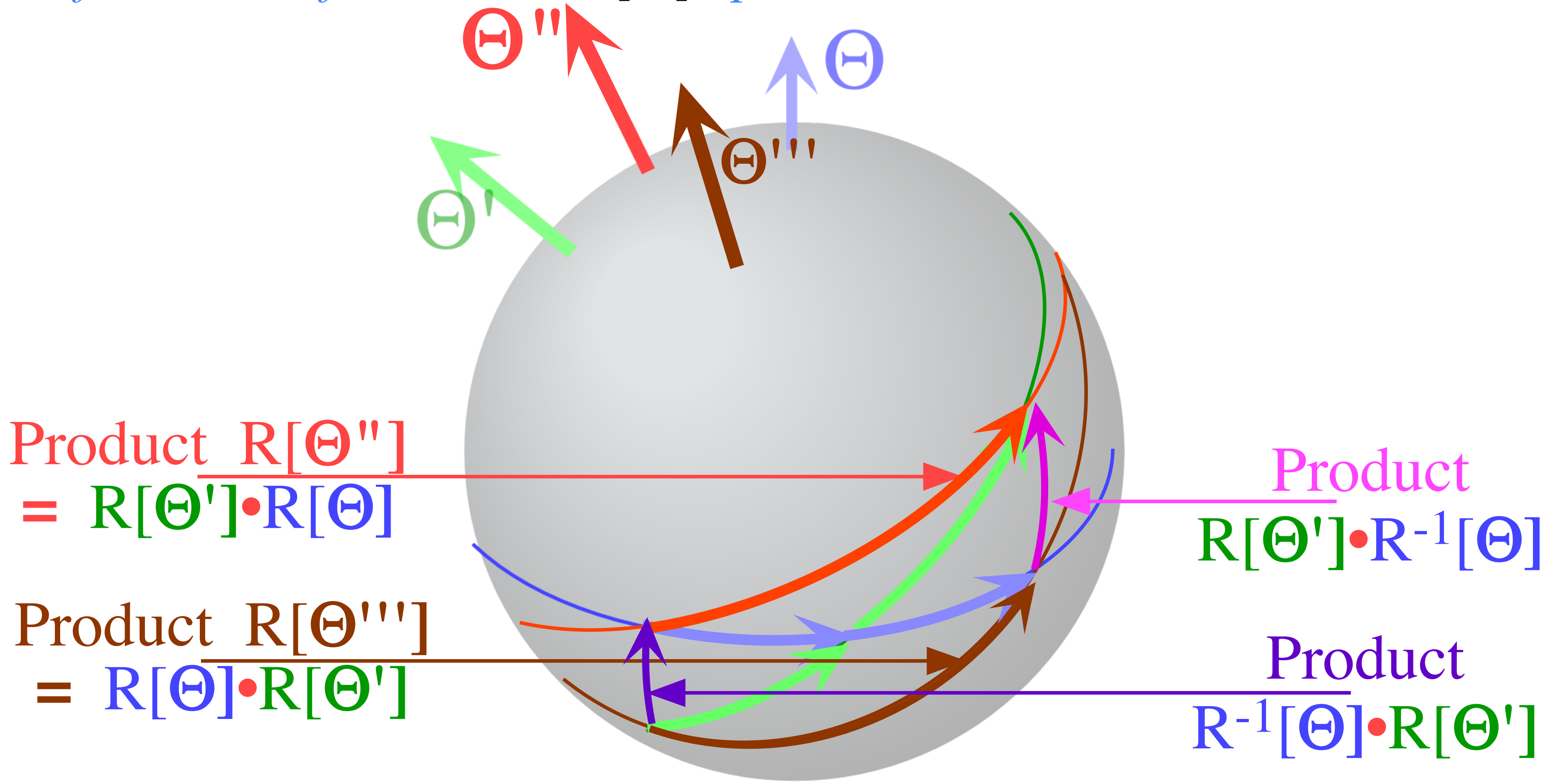

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Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

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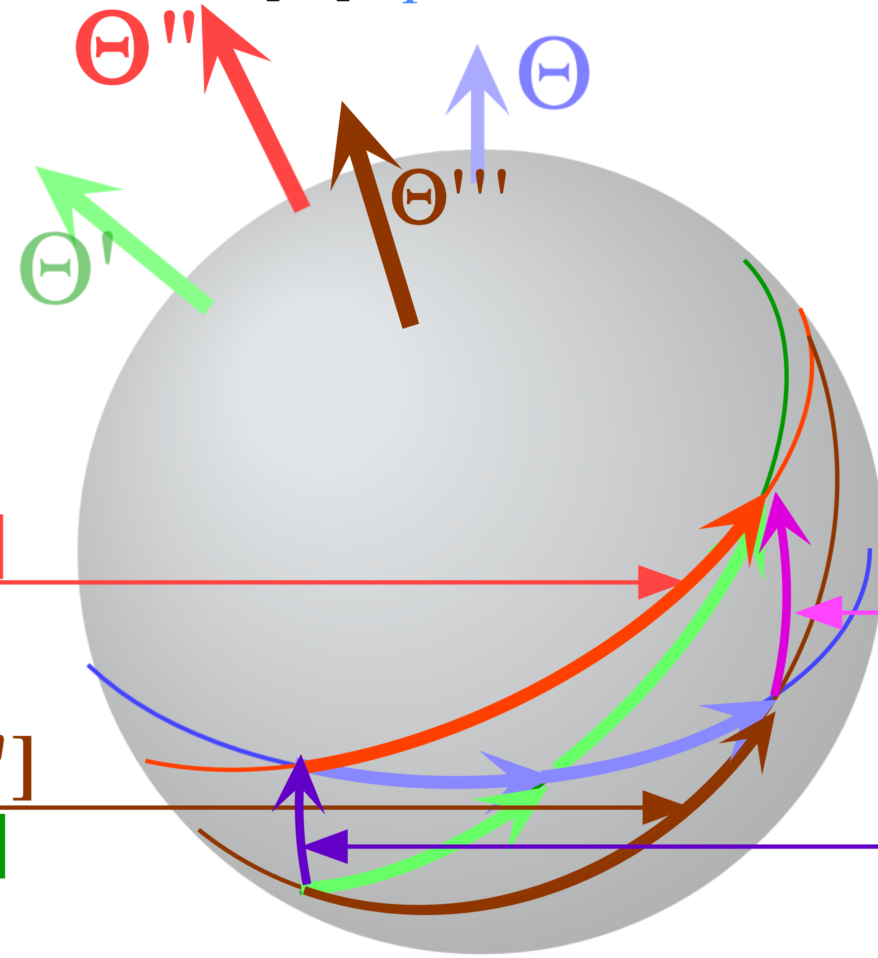
$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.



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$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

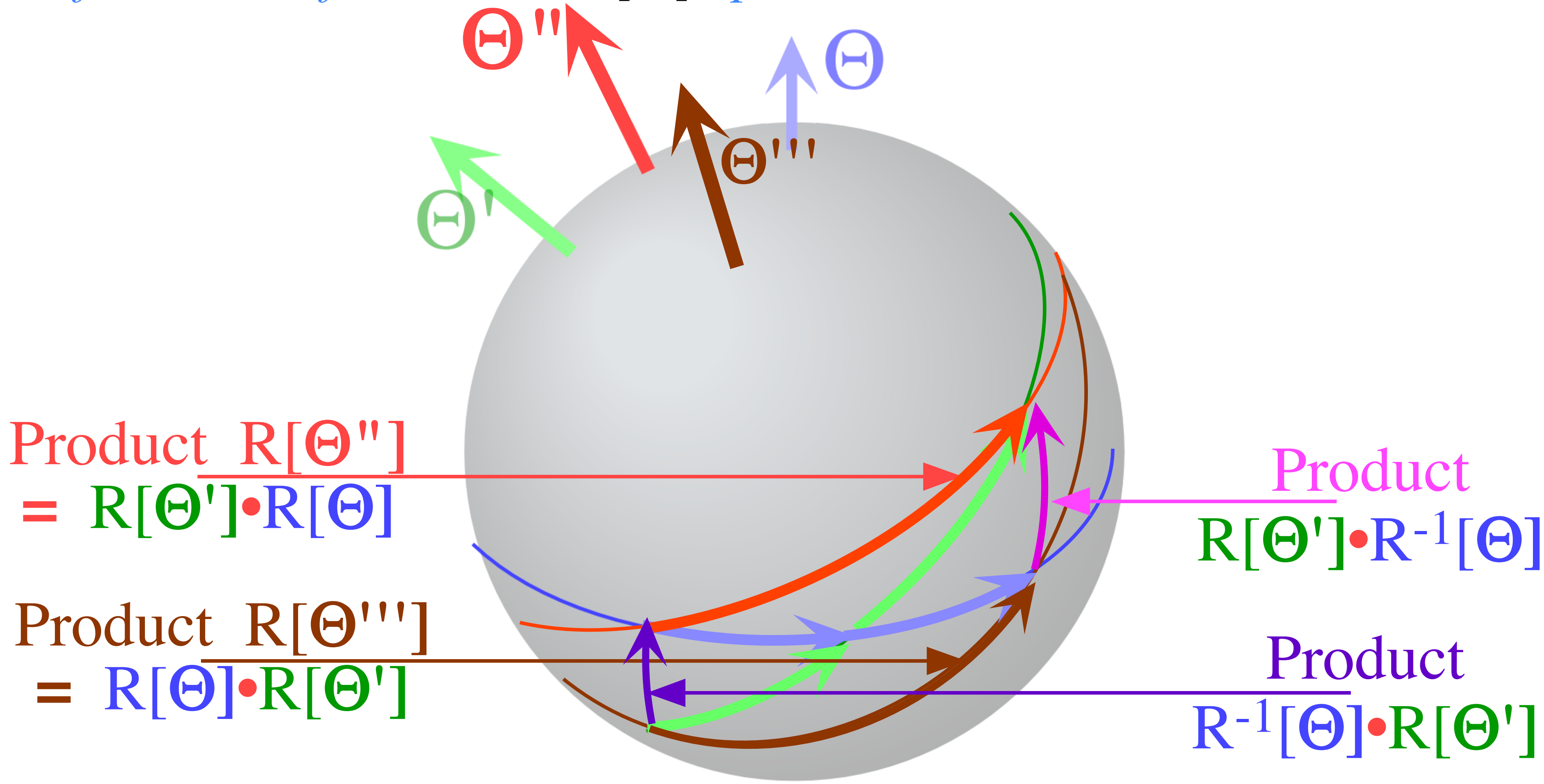

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Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$ *and* $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

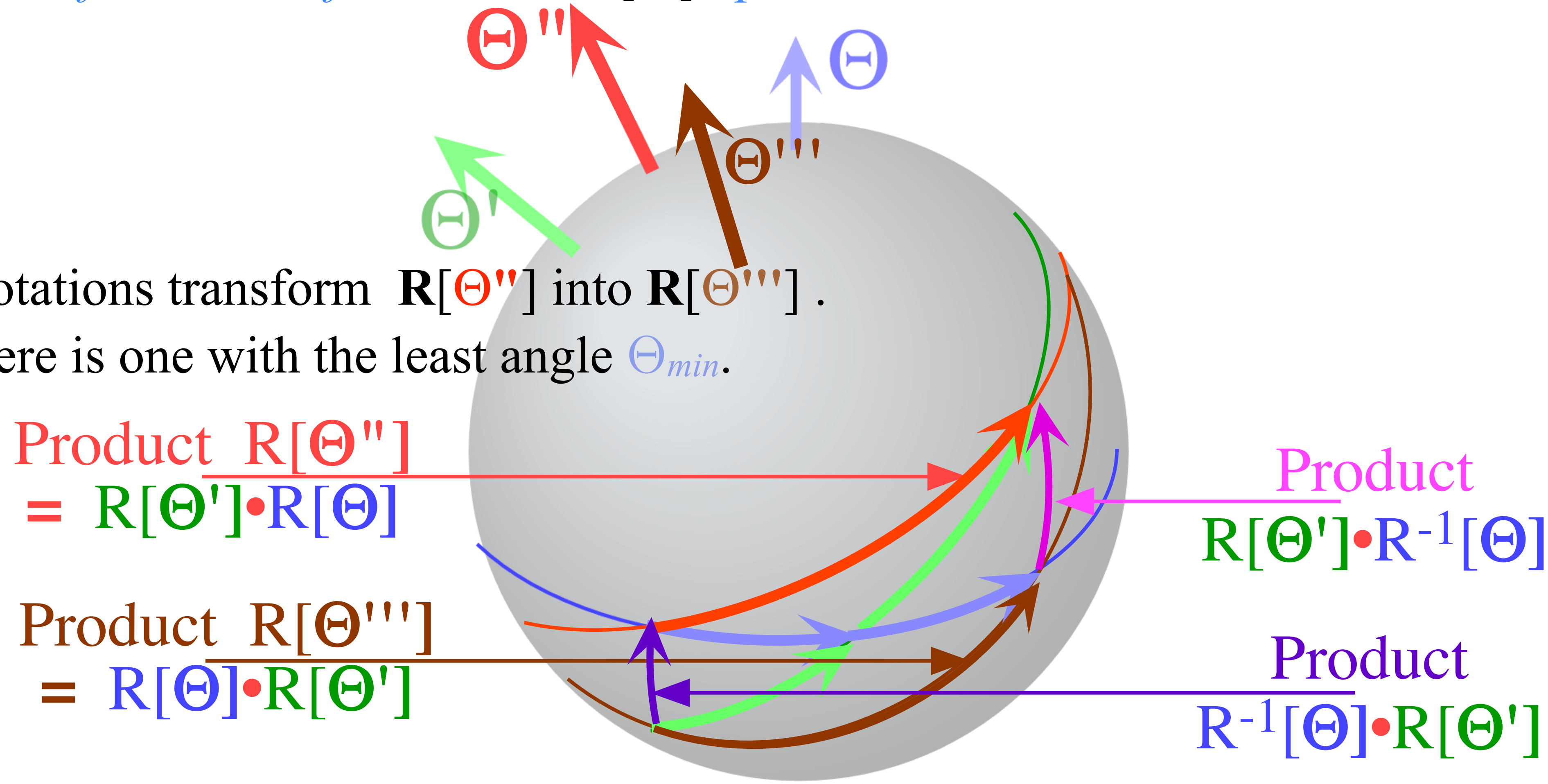
Everything associated with rotation $\mathbf{R}[\Theta'']$ is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> $\mathbf{R}[\Theta]$ turn arcs into turn arc of $\mathbf{R}[\Theta''']$ below it.

Another similarity transformation of rotation $\mathbf{R}[\Theta''']$ by rotation $\mathbf{R}[\Theta']$ to $\mathbf{R}[\Theta'']$

$$\mathbf{R}[\Theta']\,\underbrace{\mathbf{R}[\Theta''']\,\mathbf{R}[-\Theta']}_{\mathbf{R}[\Theta]} = \mathbf{R}[\Theta'']$$

Many (∞) rotations transform **R**[Θ''] into **R**[Θ'''] .
Of these, there is one with the least angle Θ_{min}.



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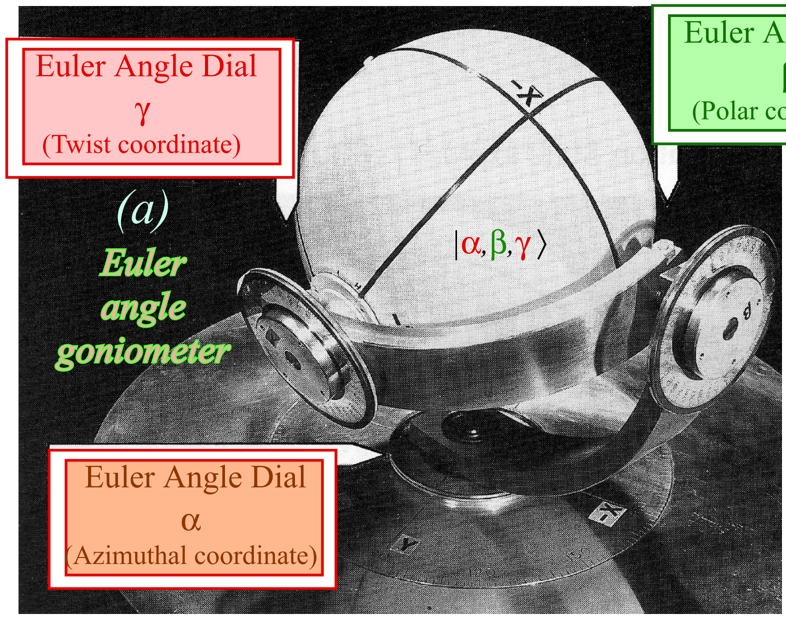
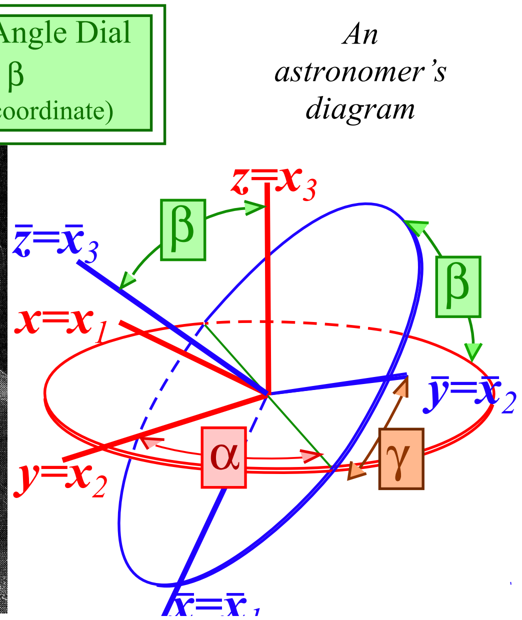
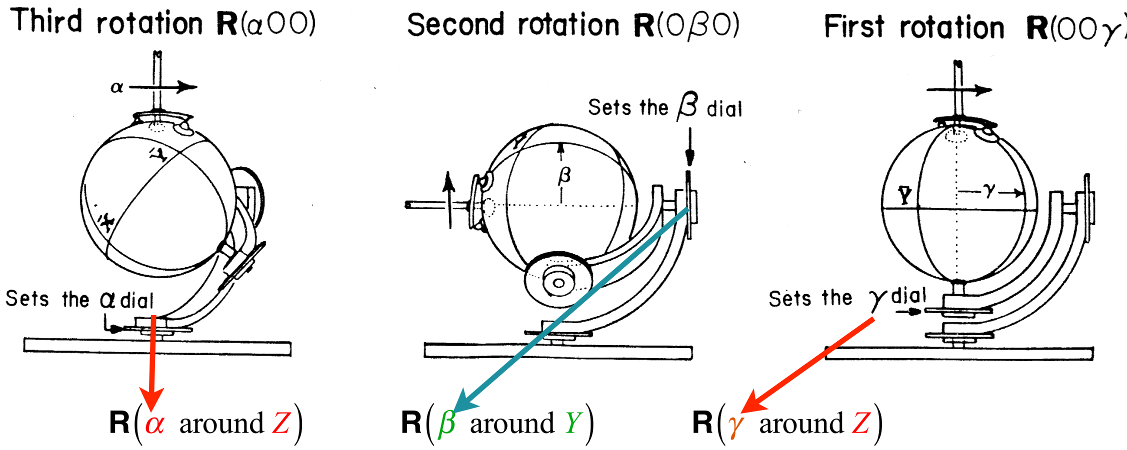




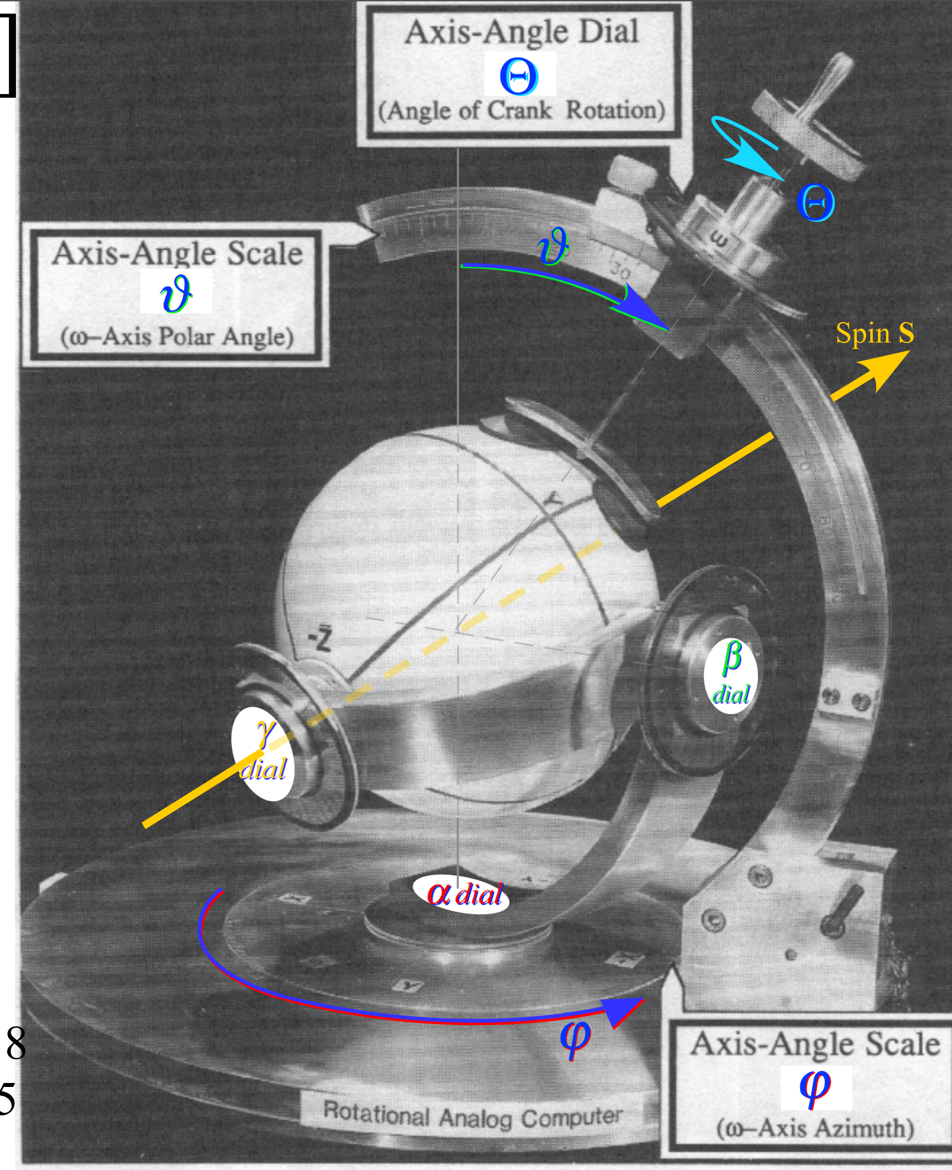


From Lecture 7
page 80 to 89

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.



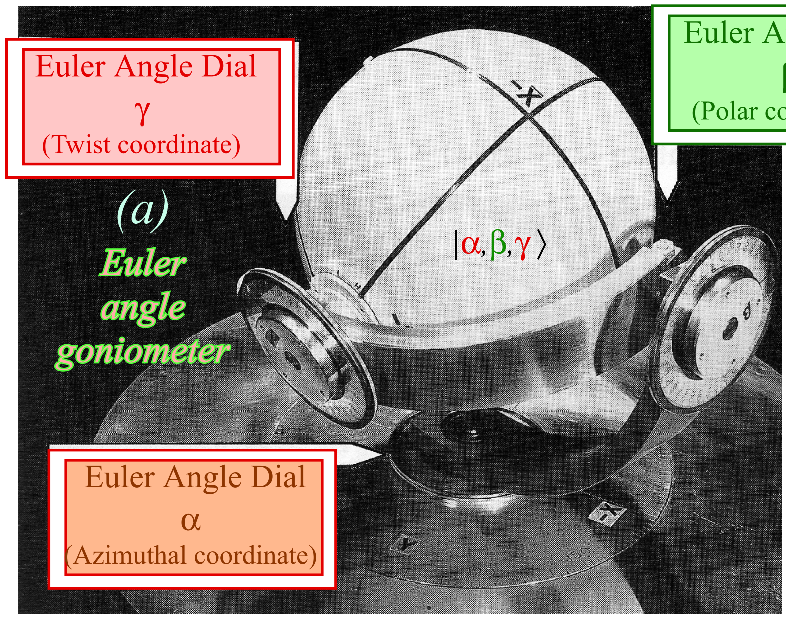
This Lecture 8
page 21 to 25

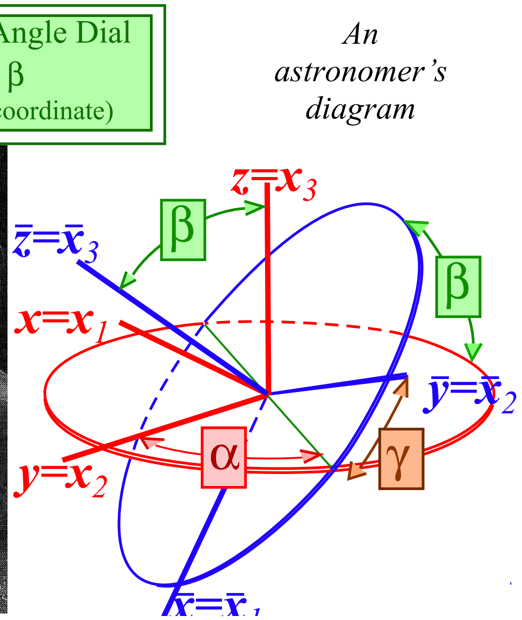
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

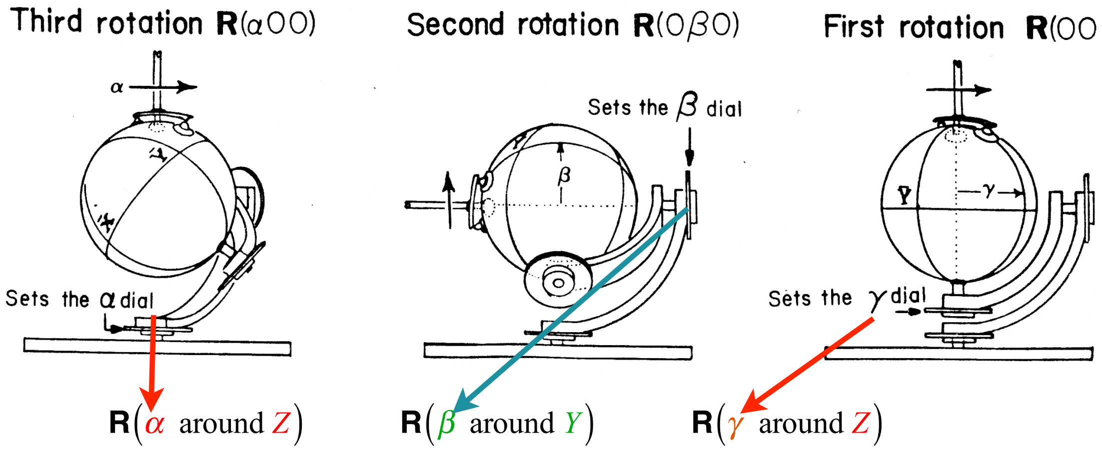
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







From Lecture 7 page 80 to 89

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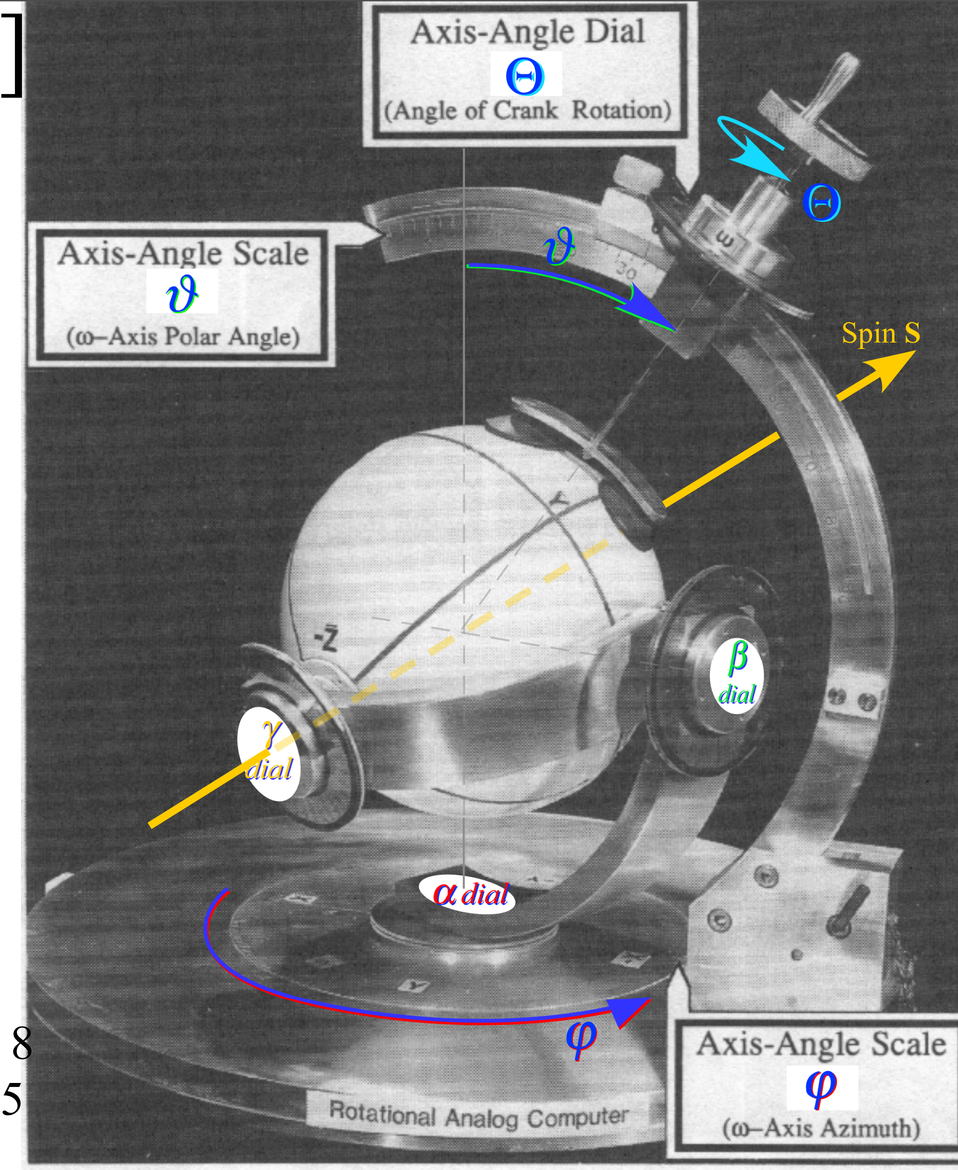


$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition*:

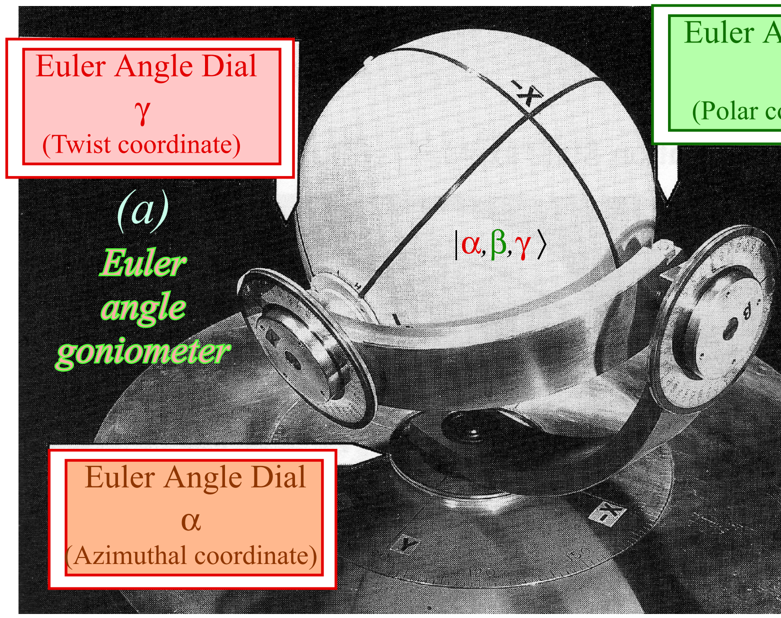
$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ ($\alpha\beta\gamma$ *make better coordinates)*

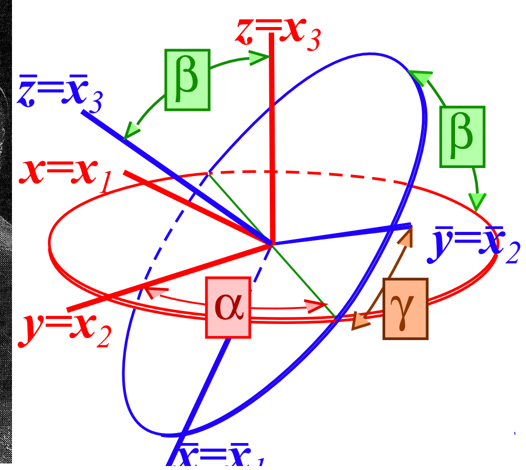
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

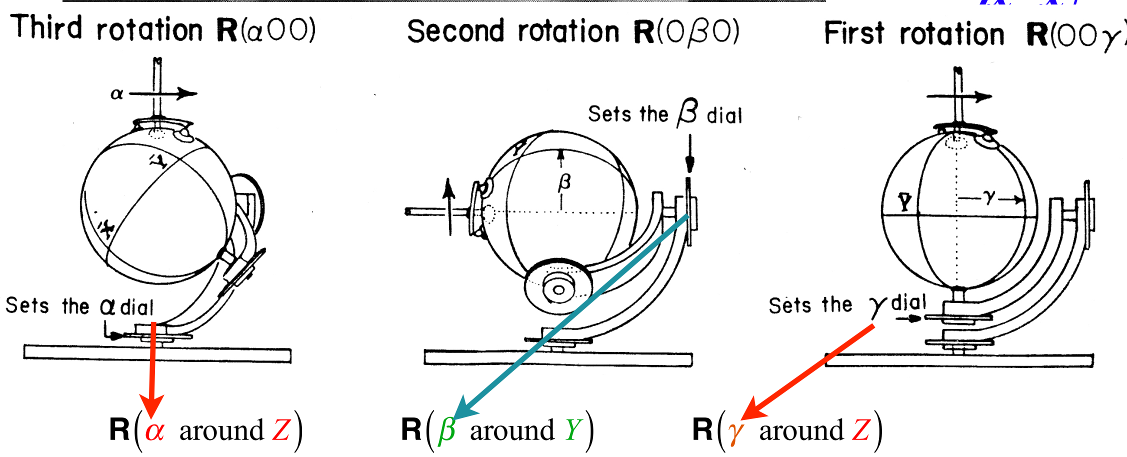
$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]

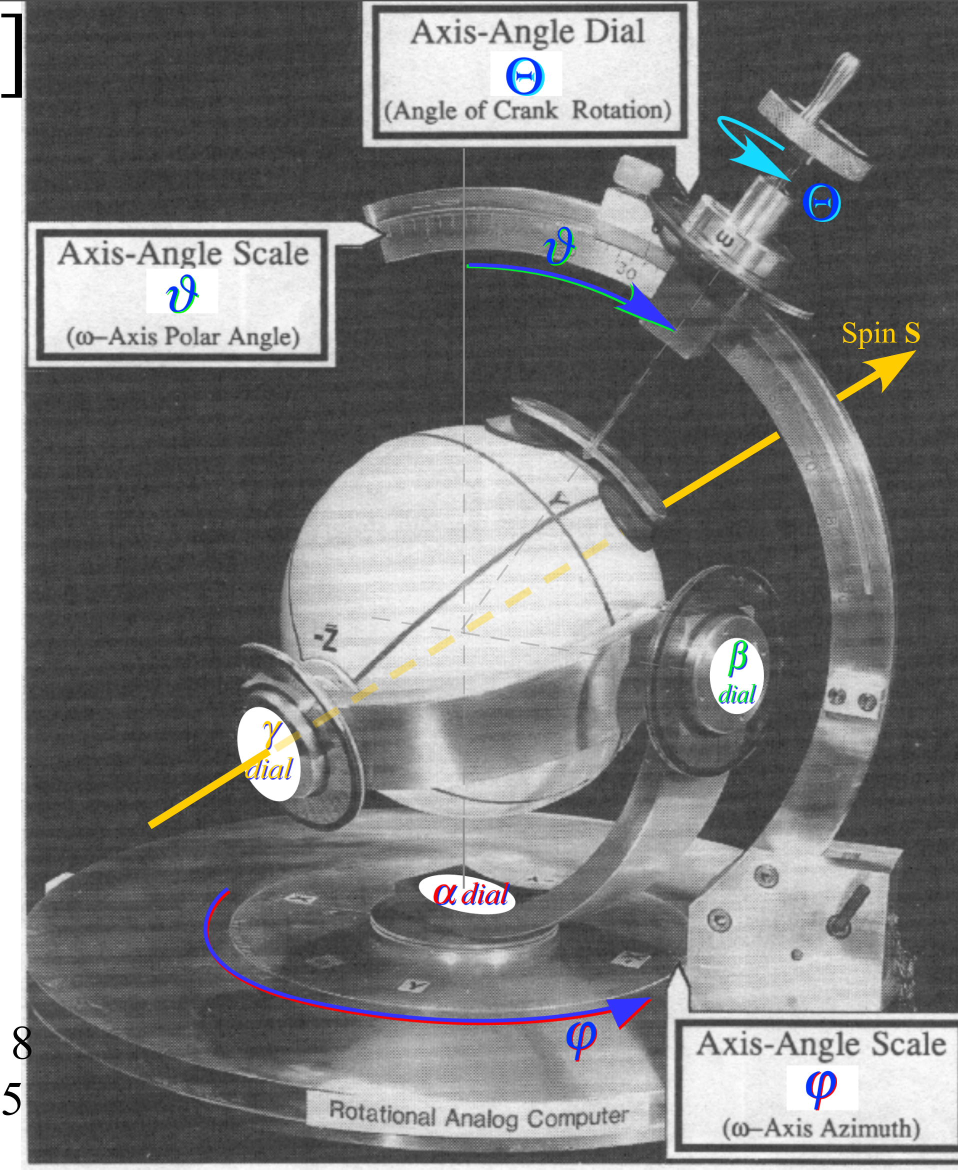






From Lecture 7
page 80 to 89

This Lecture 8
page 21 to 25



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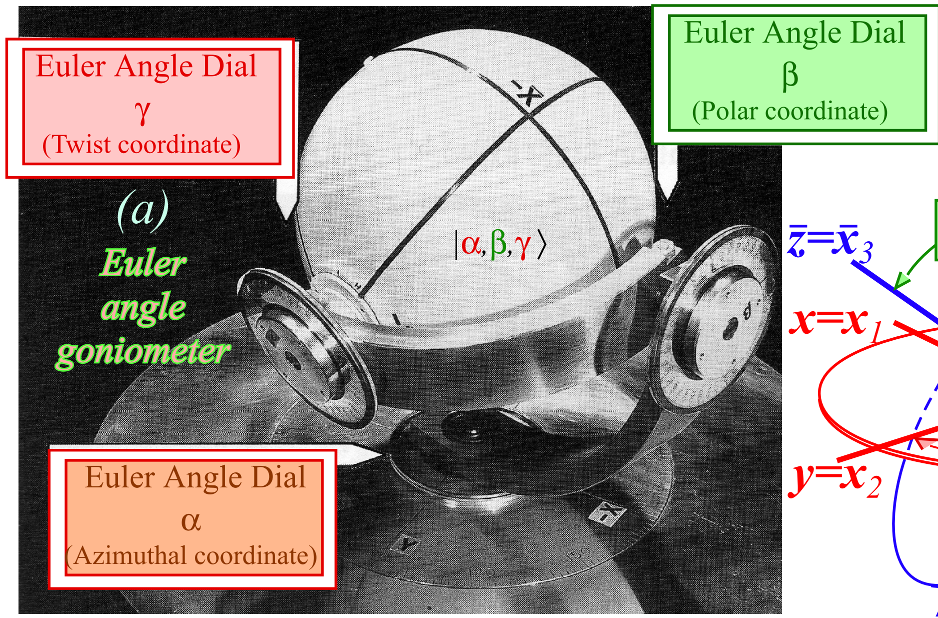
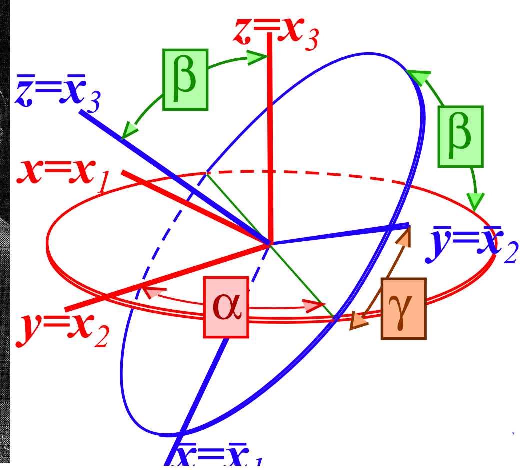
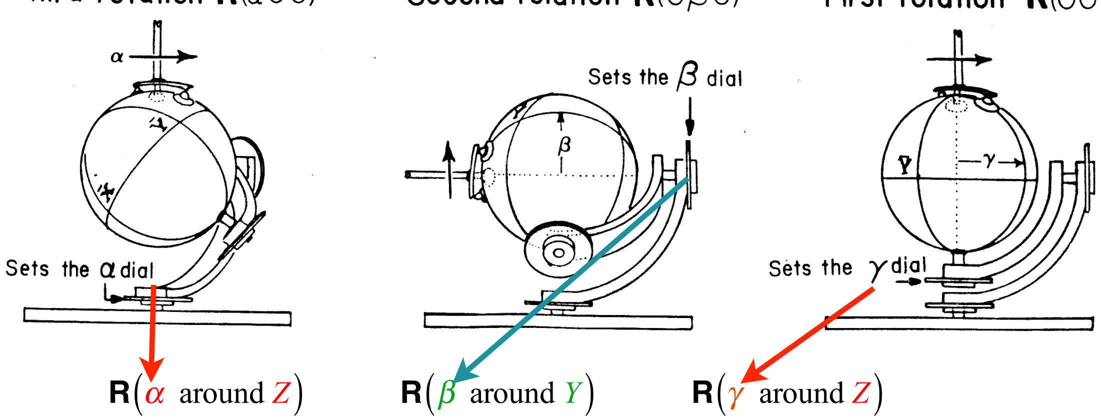


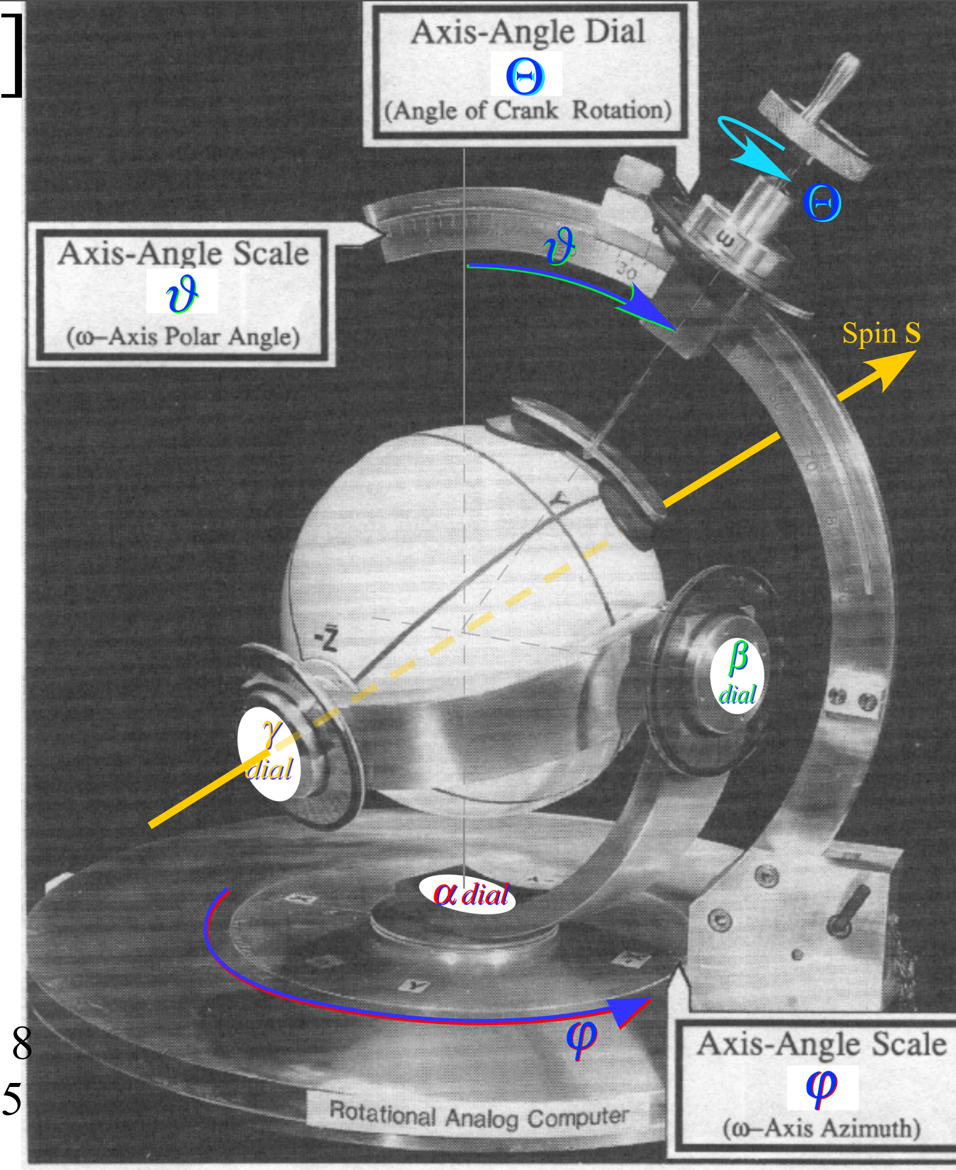
An astronomer's diagram





From Lecture 7 page 80 to 89

This Lecture 8 page 21 to 25



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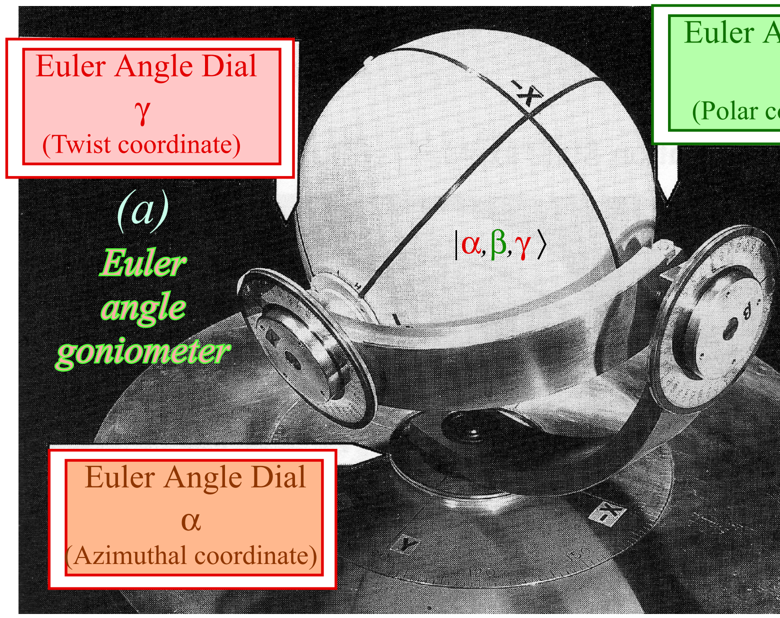
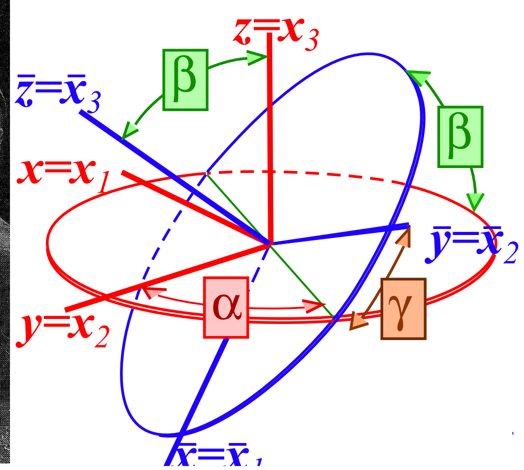
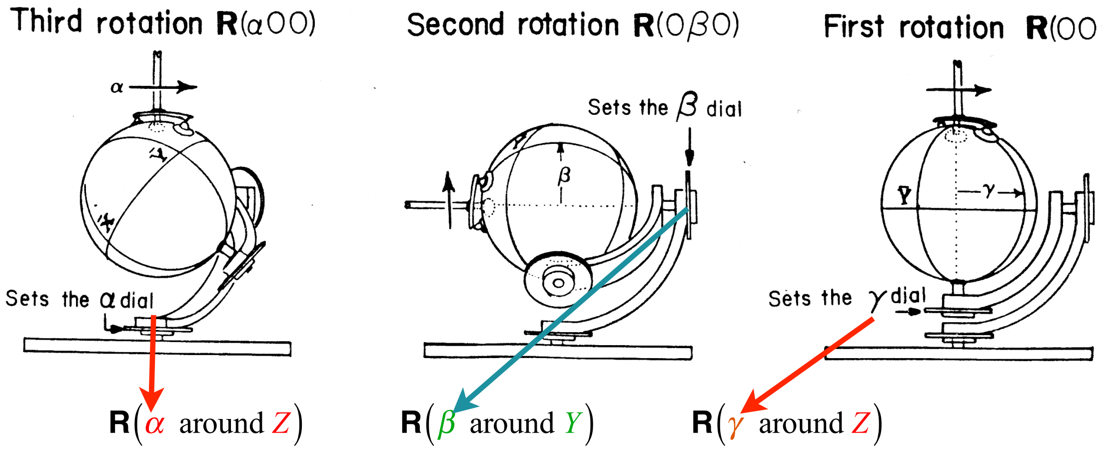
Euler R(αβγ) versus *Darboux* R[φϑΘ]

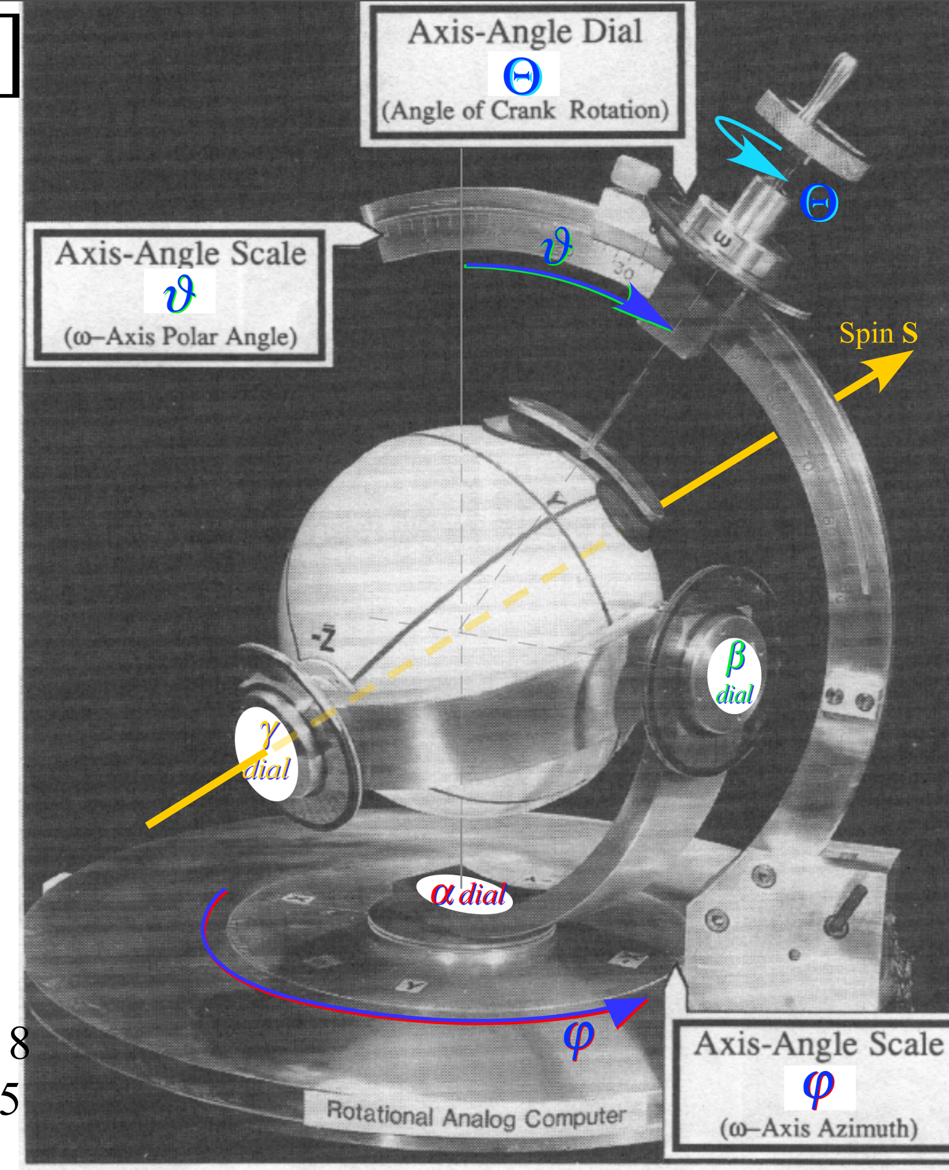






From Lecture 7 page 80 to 89

This Lecture 8 page 21 to 25



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|αβγ⟩= R(αβγ)|000⟩ *(αβγ make better coordinates)*

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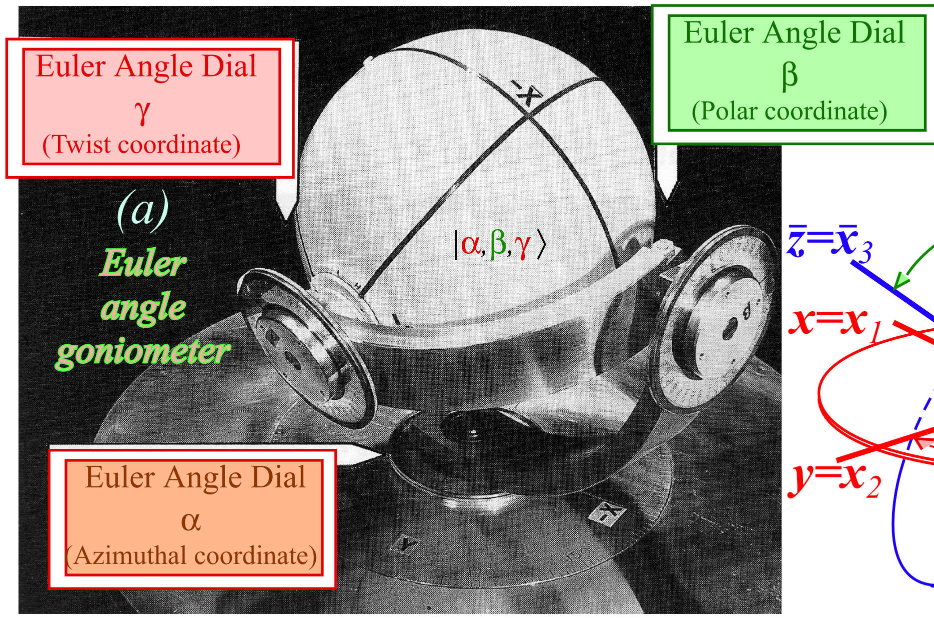
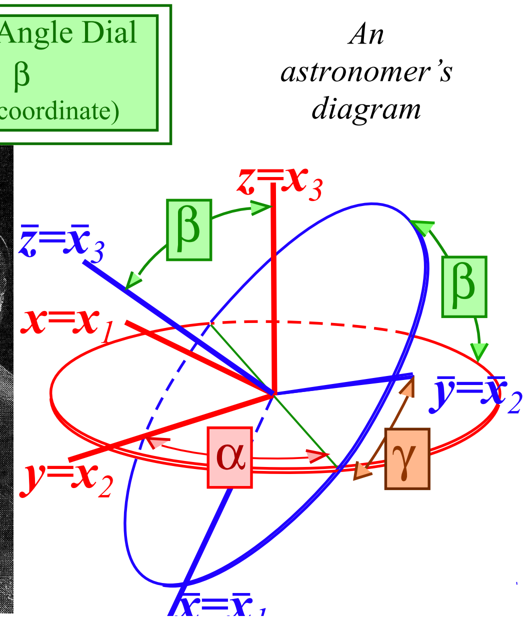
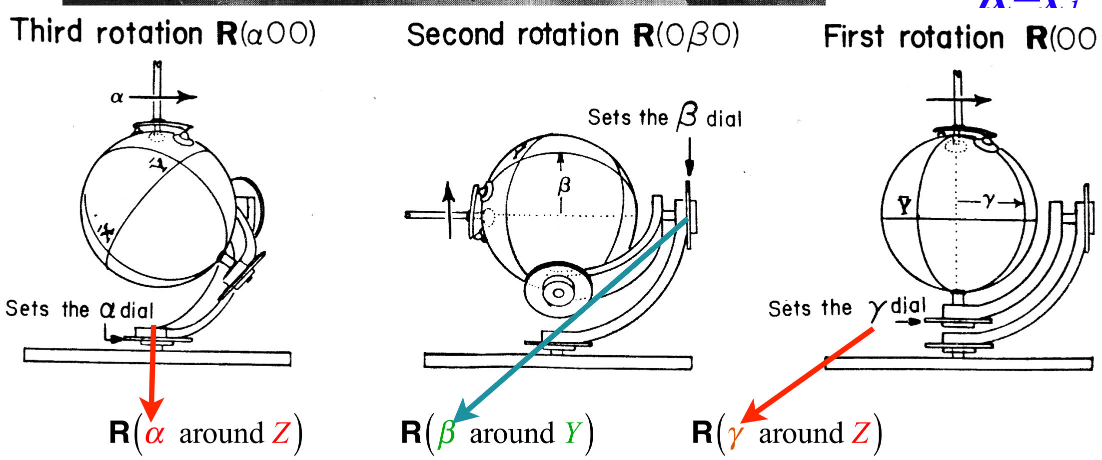
x_1 = cos[(γ+α)/2] cosβ/2 = cosΘ/2

Euler R(αβγ) versus *Darboux* R[φϑΘ]

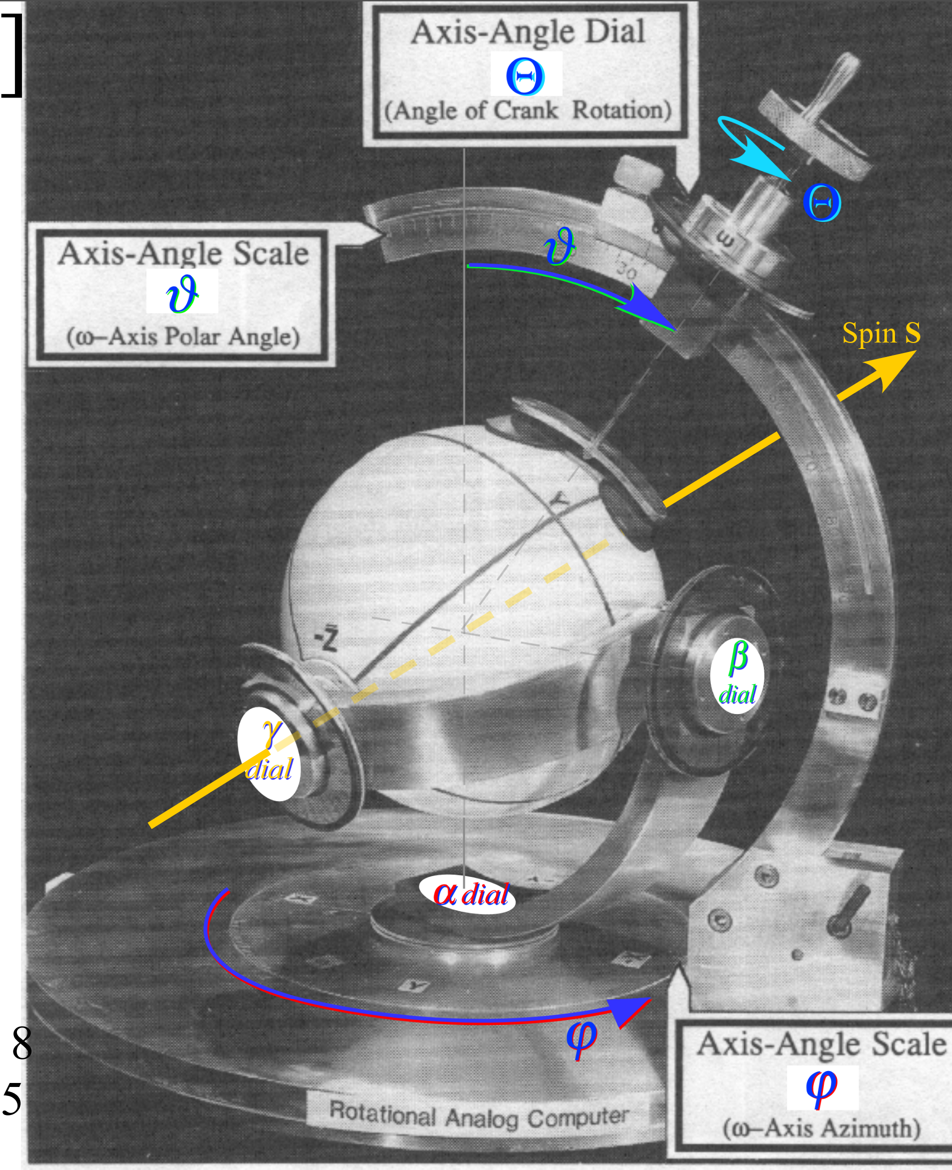






From Lecture 7
page 80 to 89

This Lecture 8
page 21 to 25



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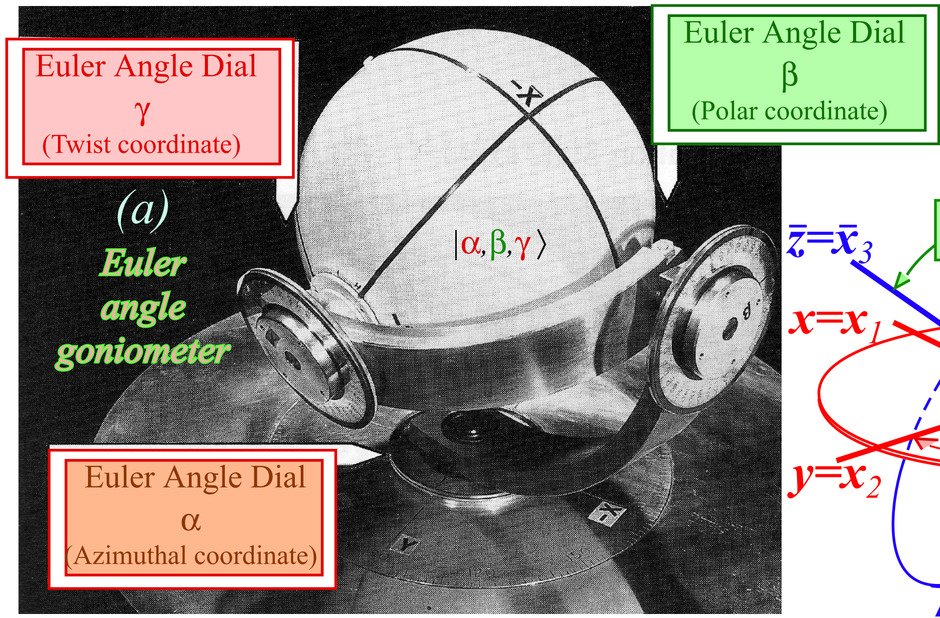
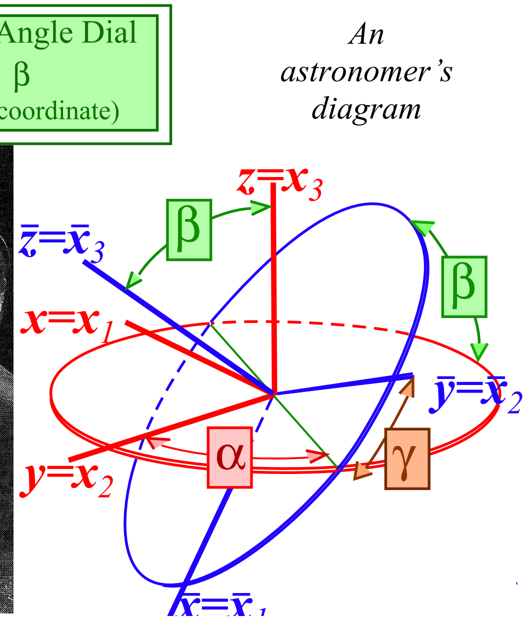
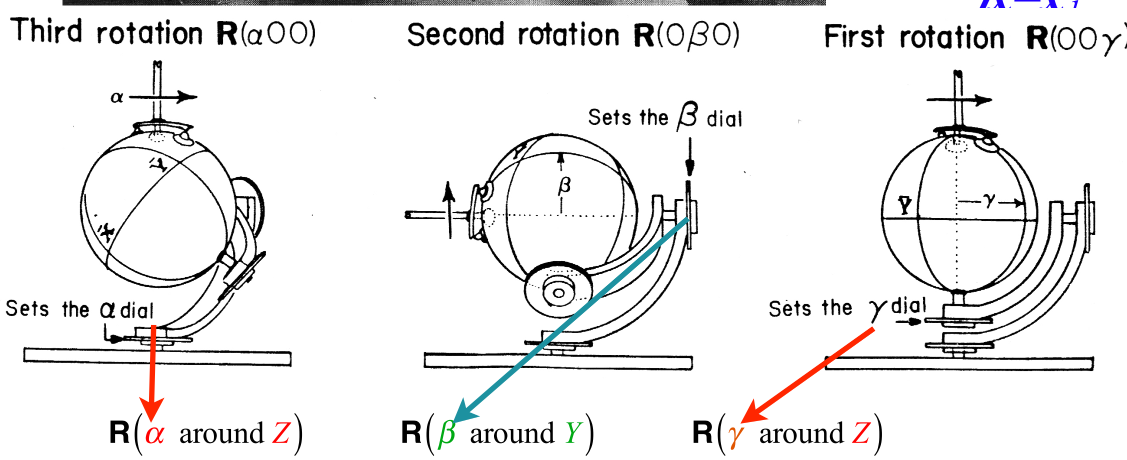
$x_1 = \cos[(\gamma+\alpha)/2]\ \cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

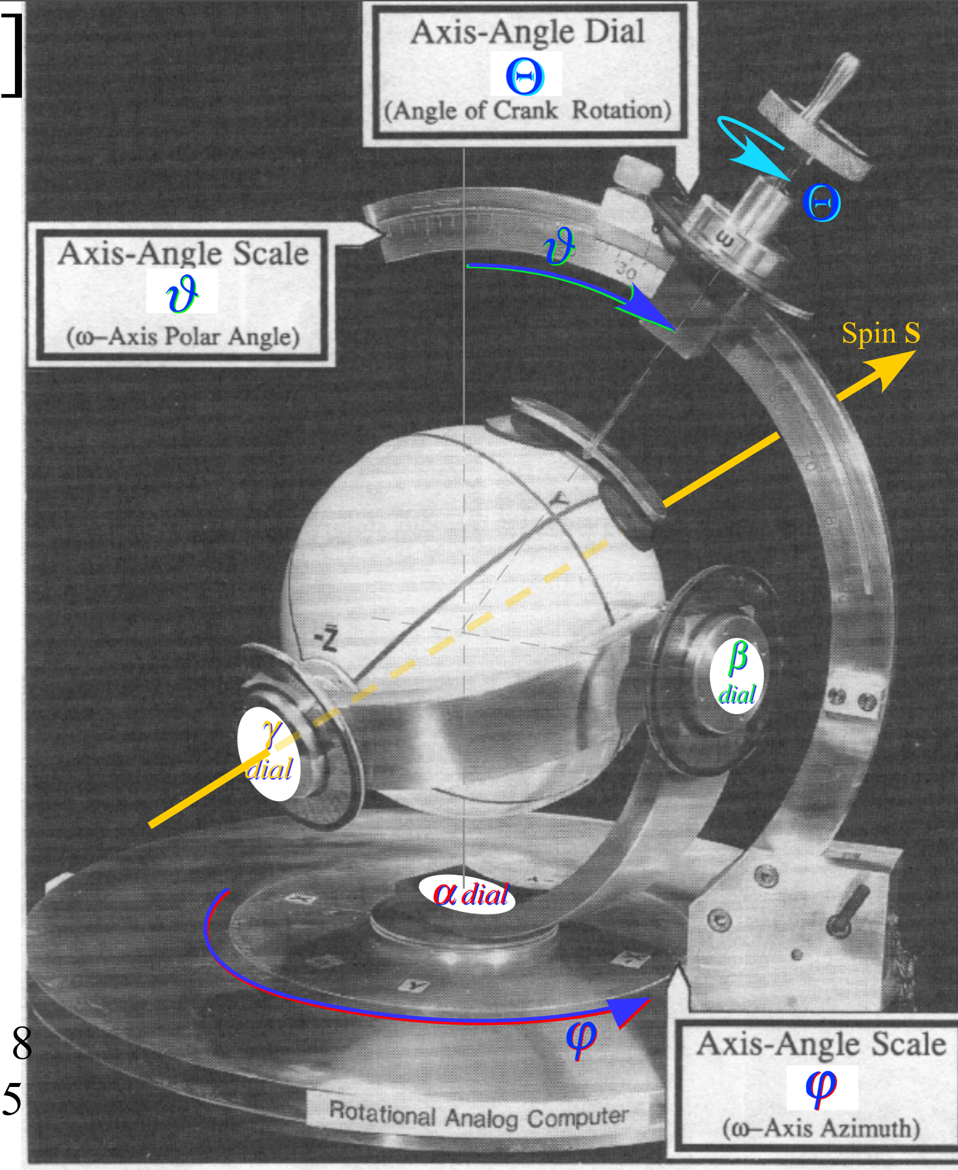
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







From Lecture 7
page 80 to 89



This Lecture 8
page 21 to 25

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$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

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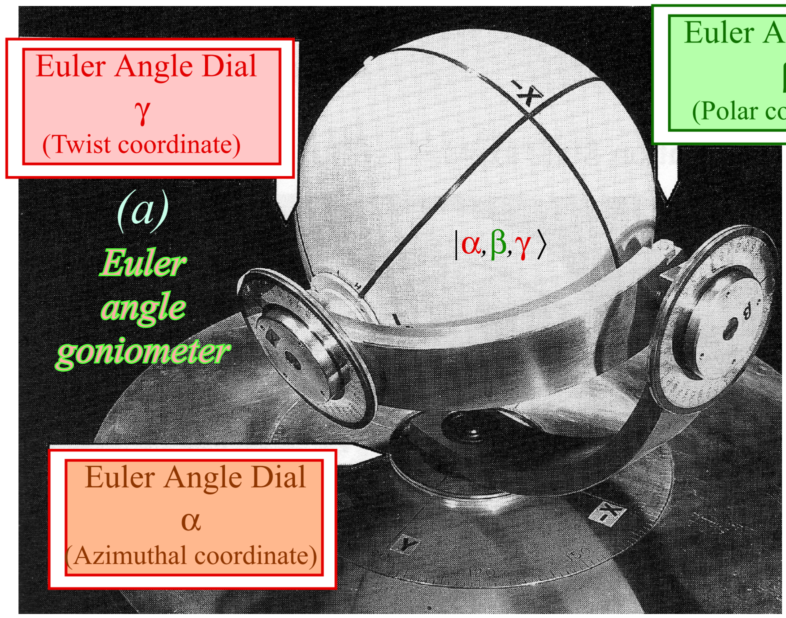
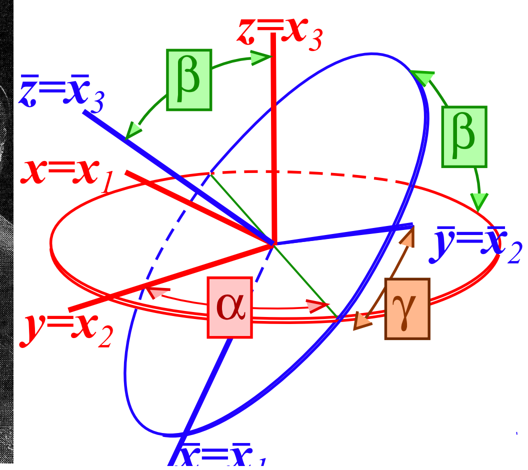
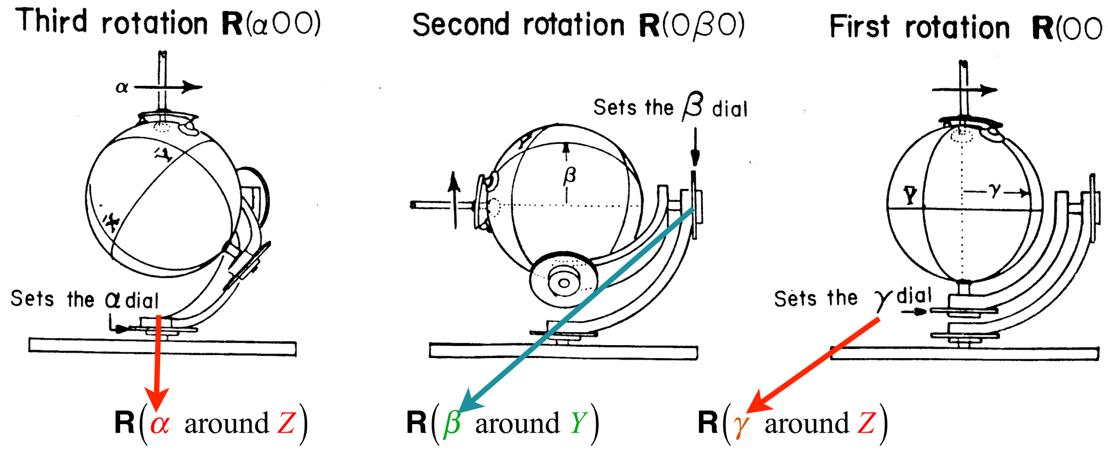
Euler R(αβγ) versus *Darboux* R[φϑΘ]



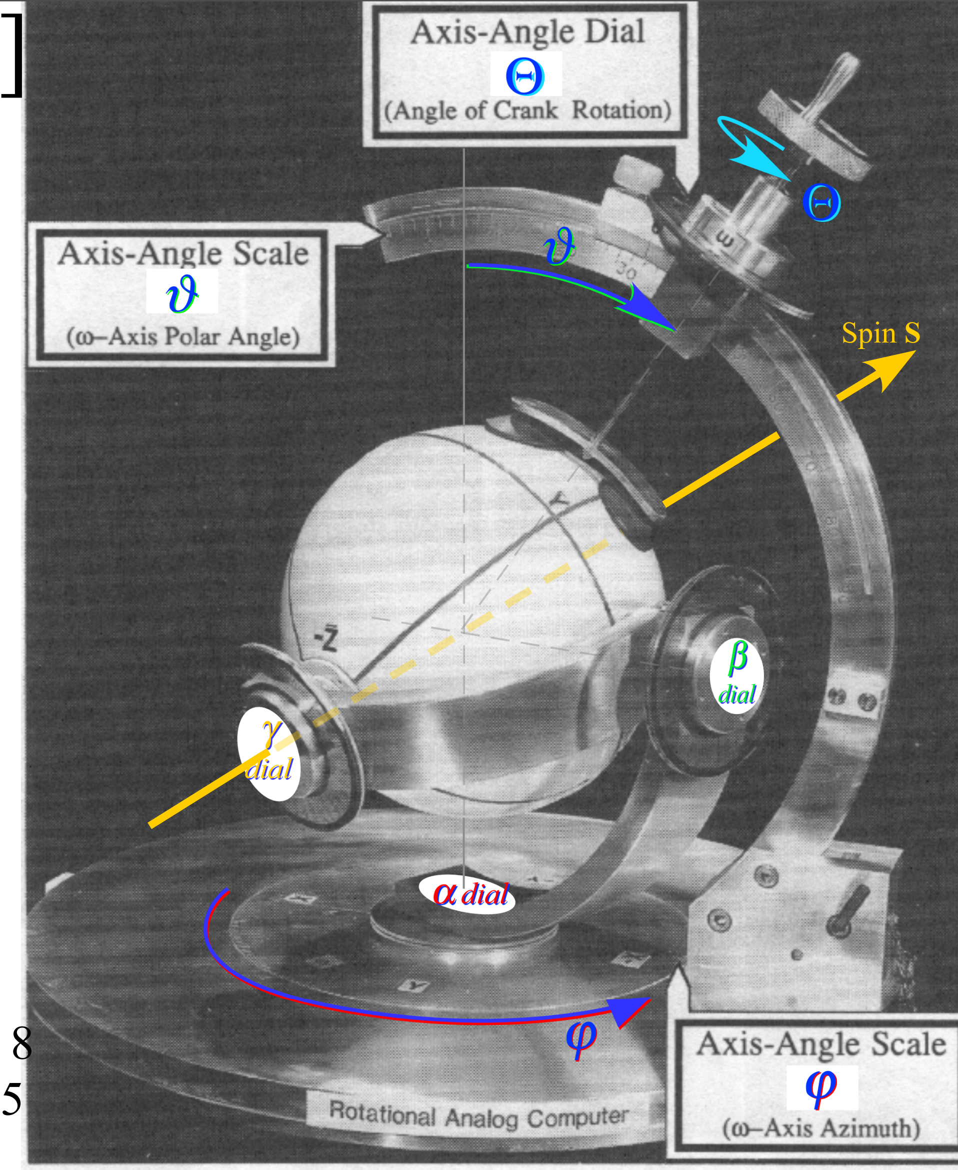
An astronomer's diagram





From Lecture 7 page 80 to 89

This Lecture 8 page 21 to 25



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y)\\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta)\\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

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$-p_1 = \sin[(\gamma+\alpha)/2]\ \cos\beta/2 = \hat{\Theta}_Z\ \sin\Theta/2 = \cos\vartheta\ \ \sin\Theta/2$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- formula*

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ *of spinor σ_μ-operators*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$*- geometry*

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *geometry*

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

→ *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

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Euler **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$

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This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

Euler R(αβγ) *related to Darboux* R[φϑΘ]

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This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

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$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

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$$\beta = 2\sin^{-1}(\sin\Theta/2\,\sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha - \gamma + \pi)/2$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

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Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\,\sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\,\sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\,\sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\,\sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\,\sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2 \qquad \cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha - \gamma + \pi)/2 \qquad \cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

Example: *Euler angles* $(\alpha=50°\ \beta=60°\ \gamma=70°)$

$$\varphi = (50° - 70° + 180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\ \sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\ \cos(50°+\gamma)/2] = 128.7°$$

Euler R(αβγ) *related to Darboux* R[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* (α=50° β=60° γ=70°)

$$\varphi = (50°-70°+180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: (αβγ) in terms of [φϑΘ]

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\ \sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2)) = 70.007°$$

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ *of group-operators*

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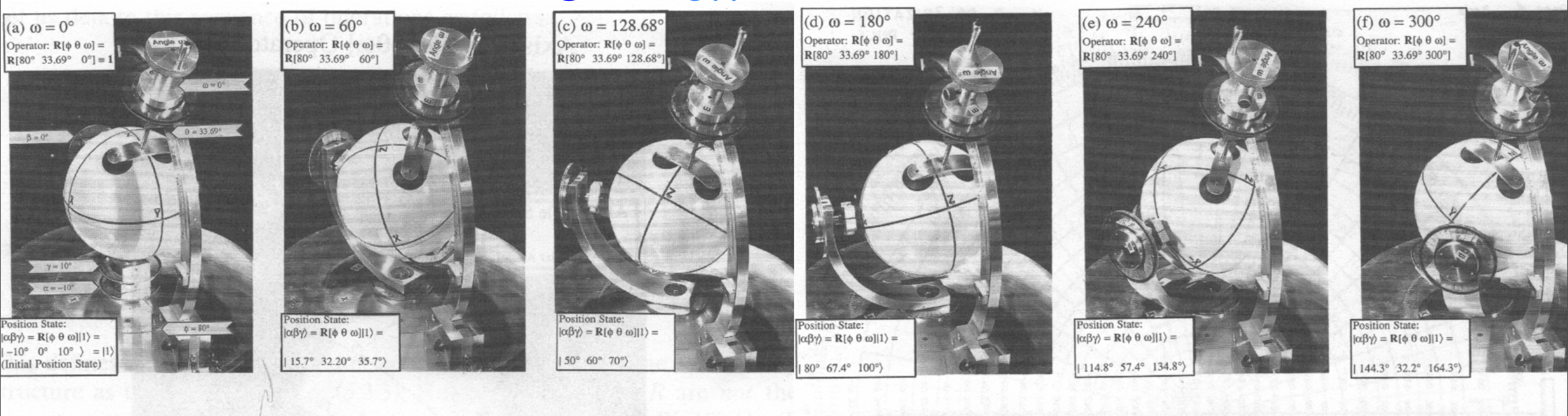
➡ *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

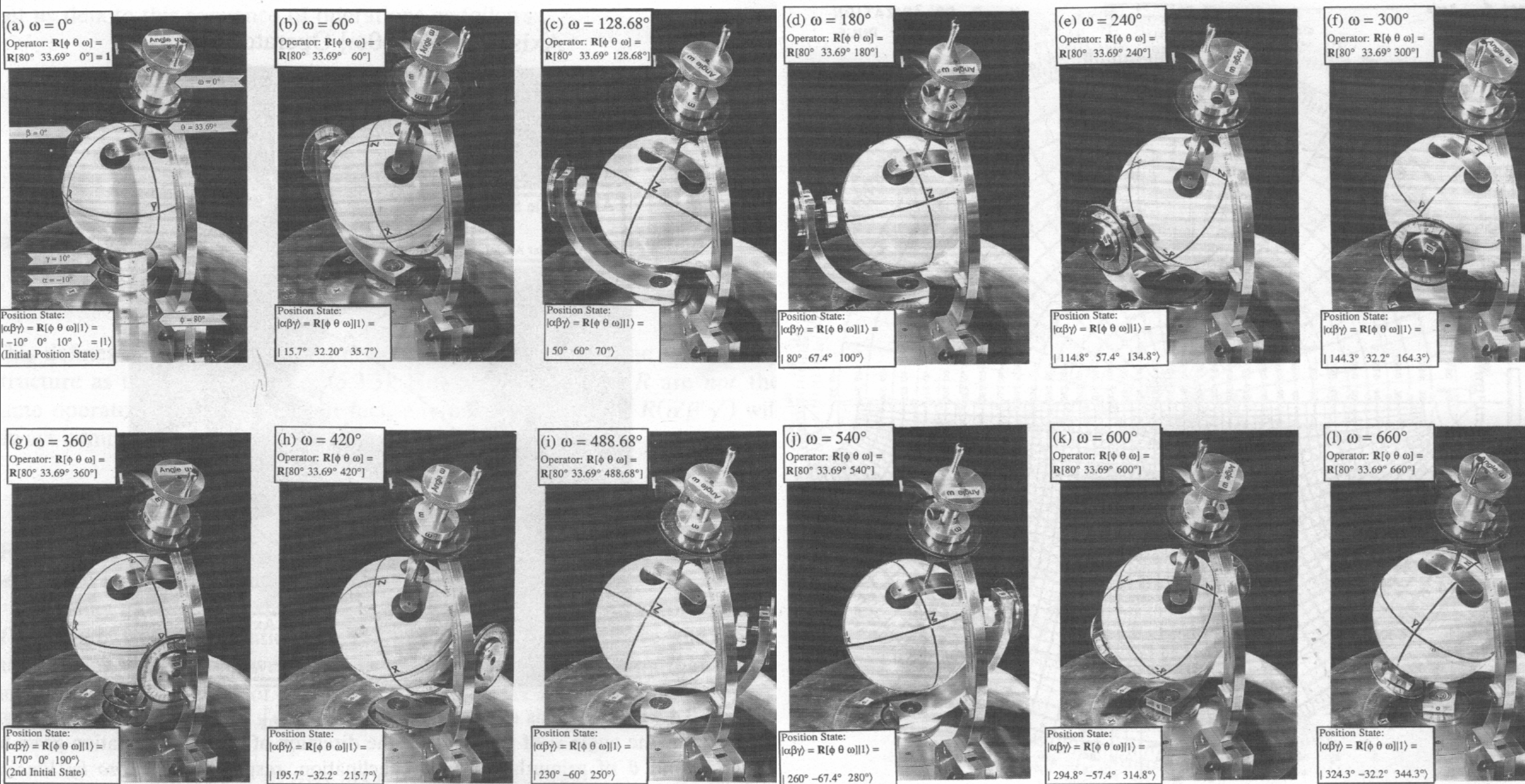
Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

$\Theta=0°$ $\Theta=60°$ $\Theta=128.7°$ $\Theta=180°$ $\Theta=240°$ $\Theta=300°$



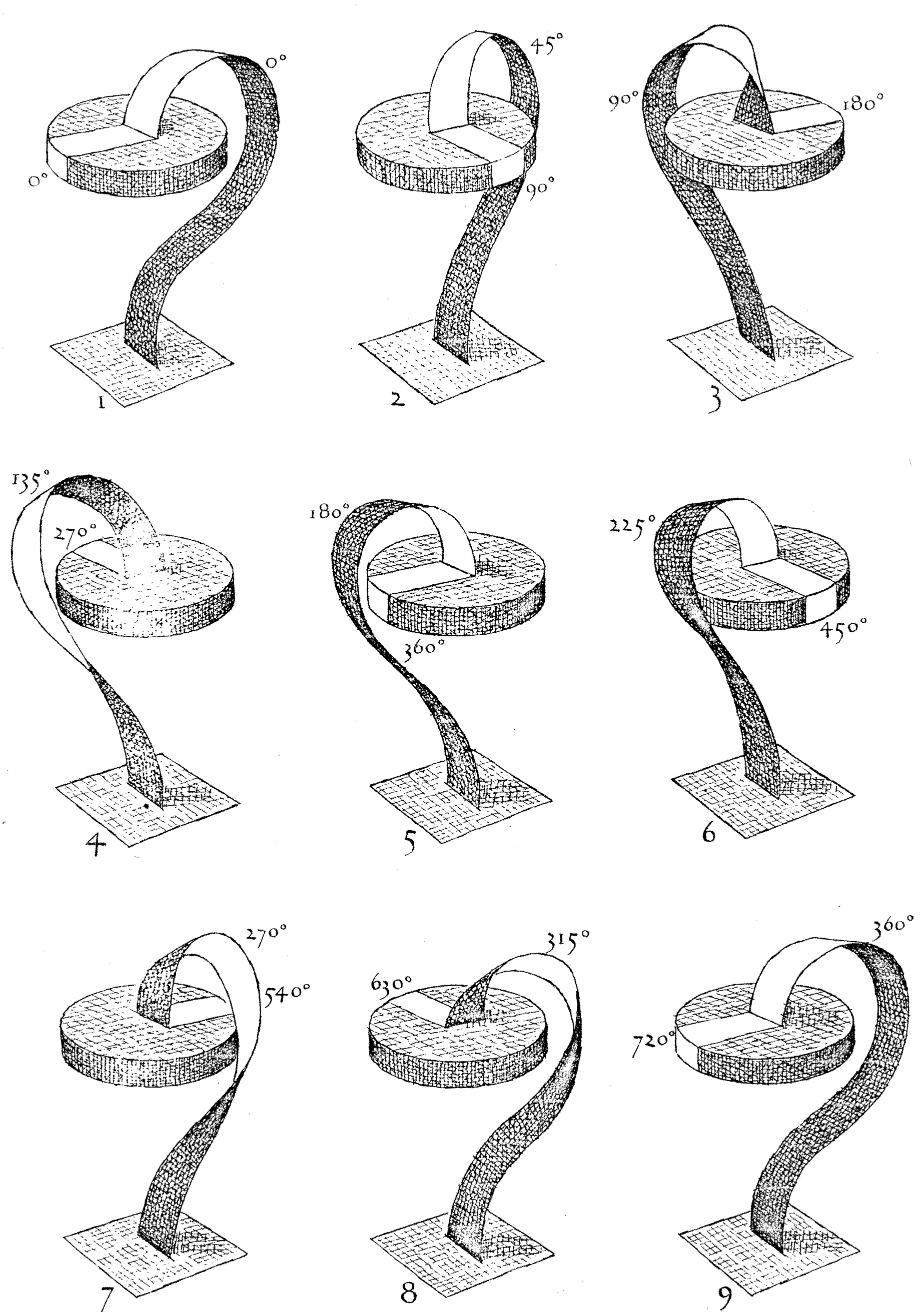
Euler **R**(αβγ) *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

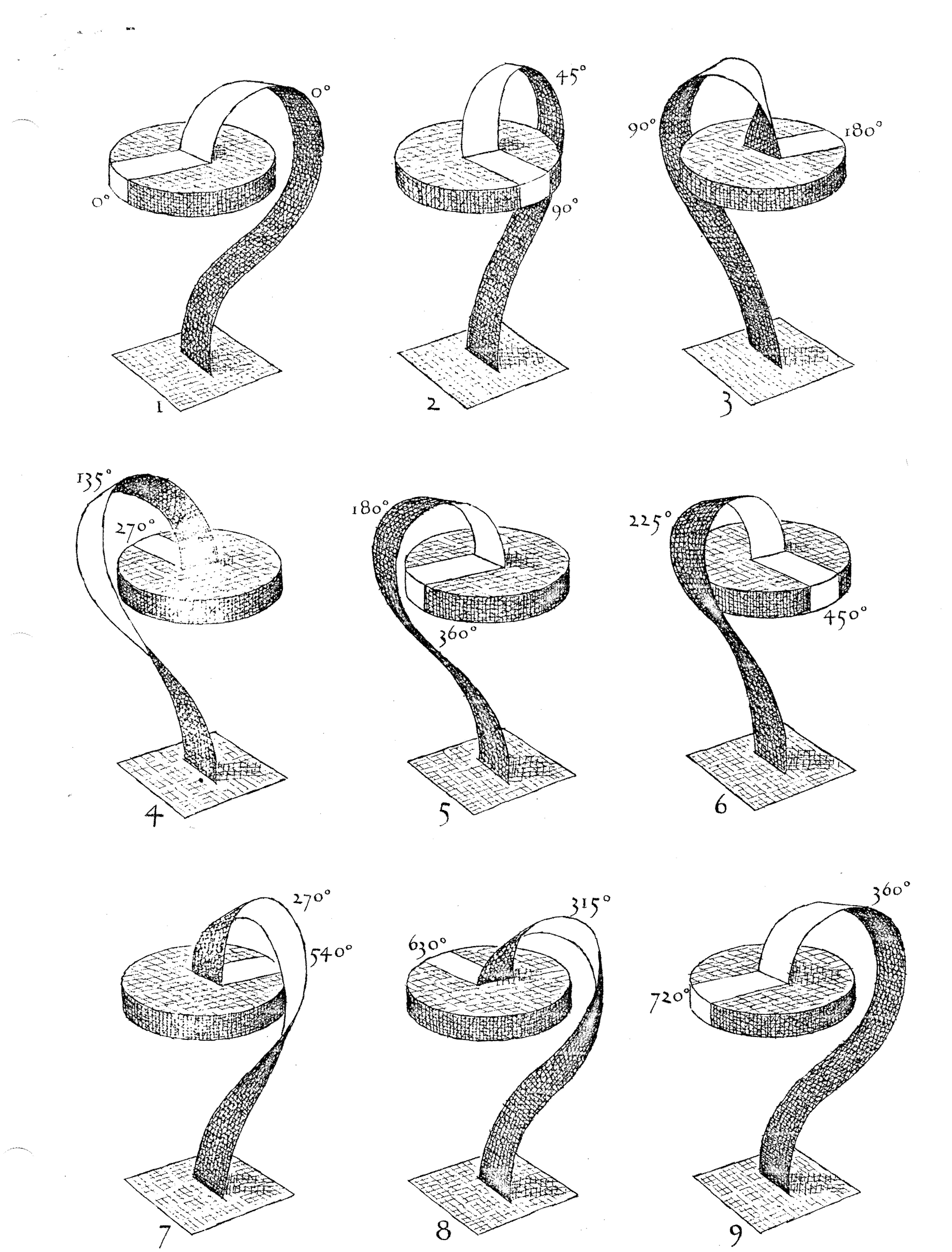
Some "real-world" applications of
the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

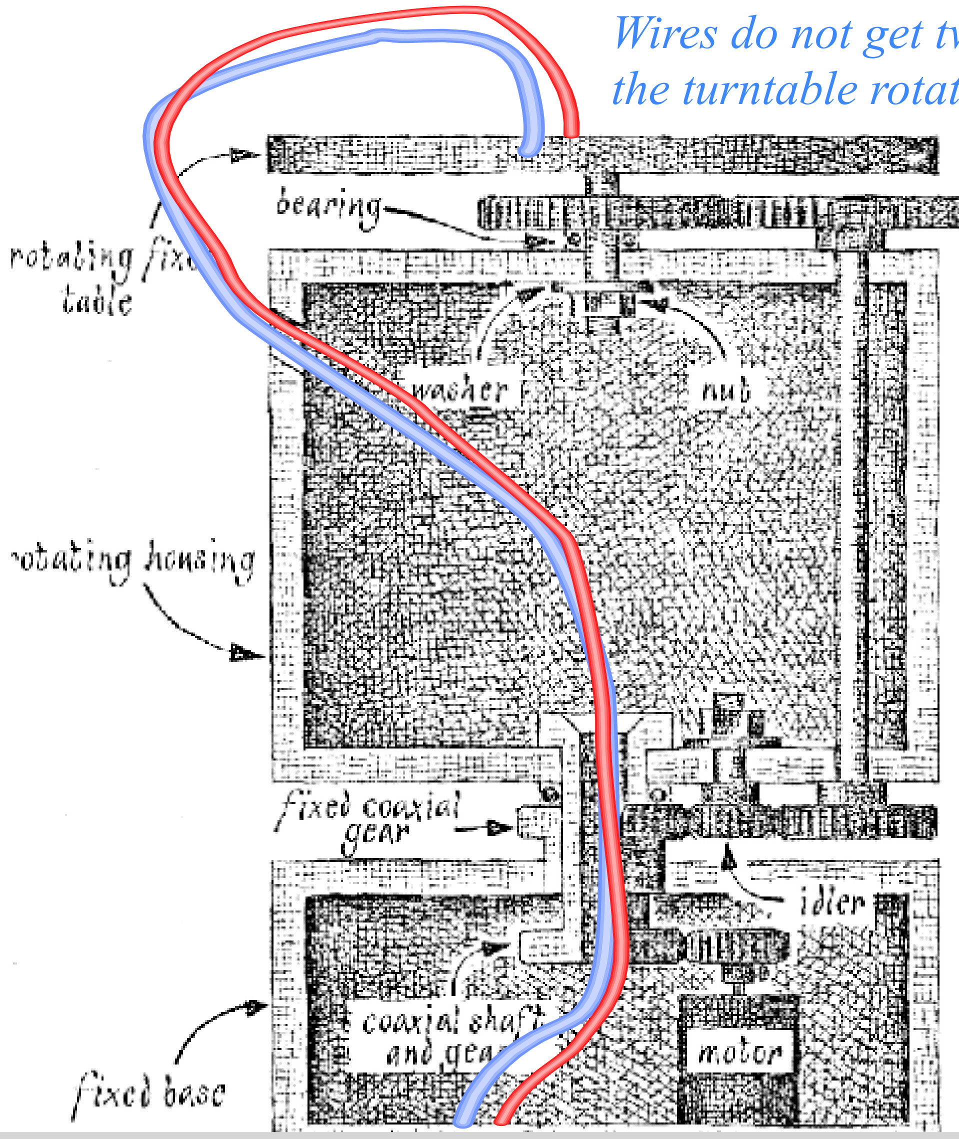
From Scientific American
December 1975-p.120-125

Some "real-world" applications of the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

Wires do not get twisted up as the turntable rotates



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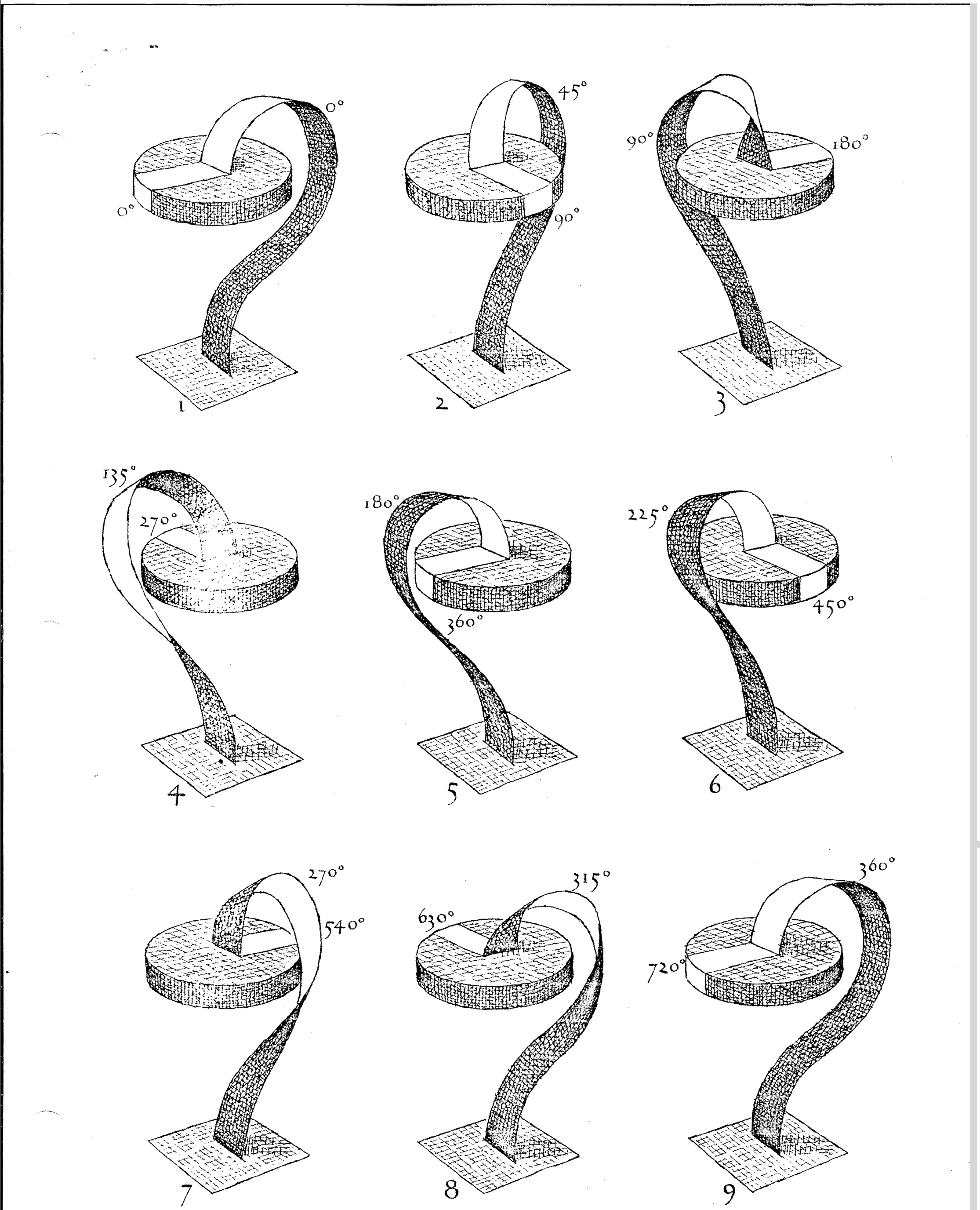

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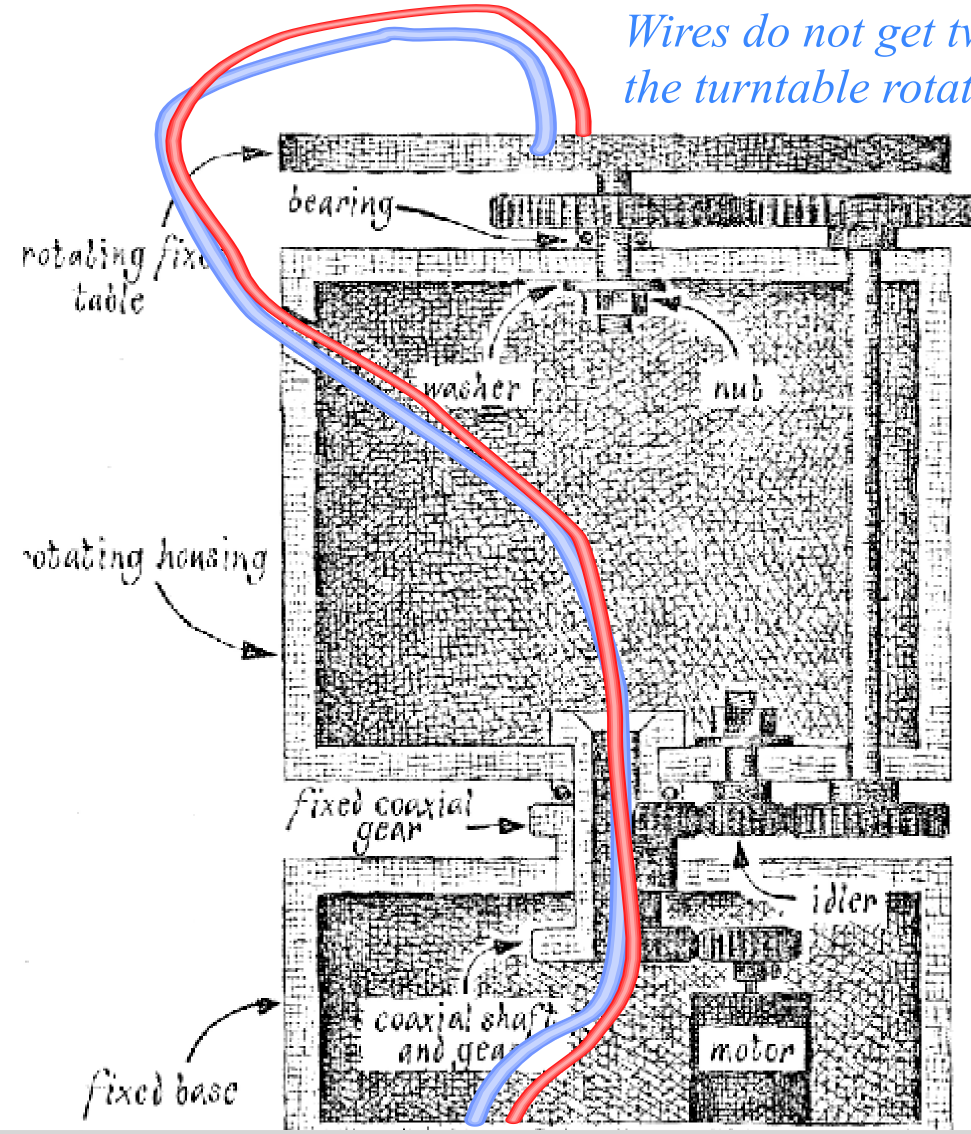

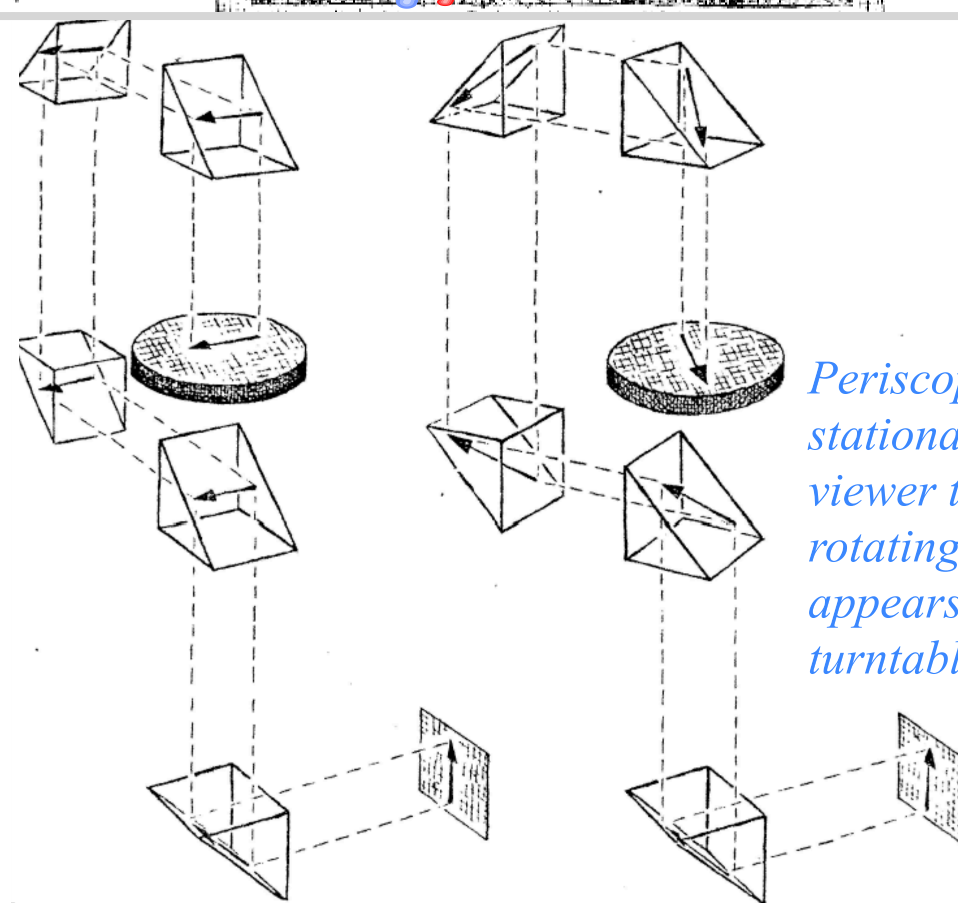

Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Review:How "Crazy-Thing"-Theorem makes spinor and vector representation matrices

Half-angle $\Theta/2=\varphi$ replacement and Darboux crank axis operators

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Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

➡ *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma) \leftrightarrow$ **R**$[\varphi\vartheta\Theta]$

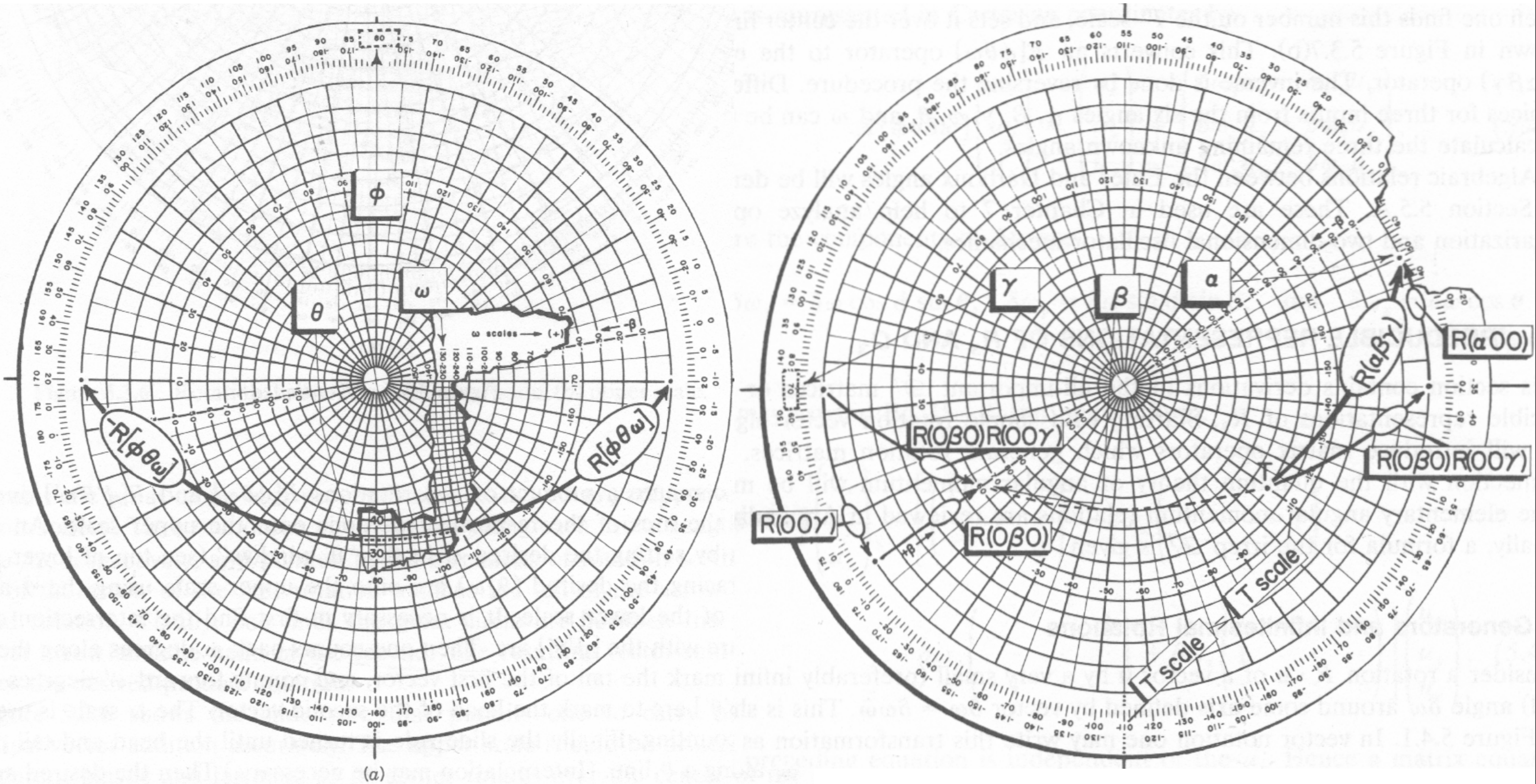

Figure 5.3.7 Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.

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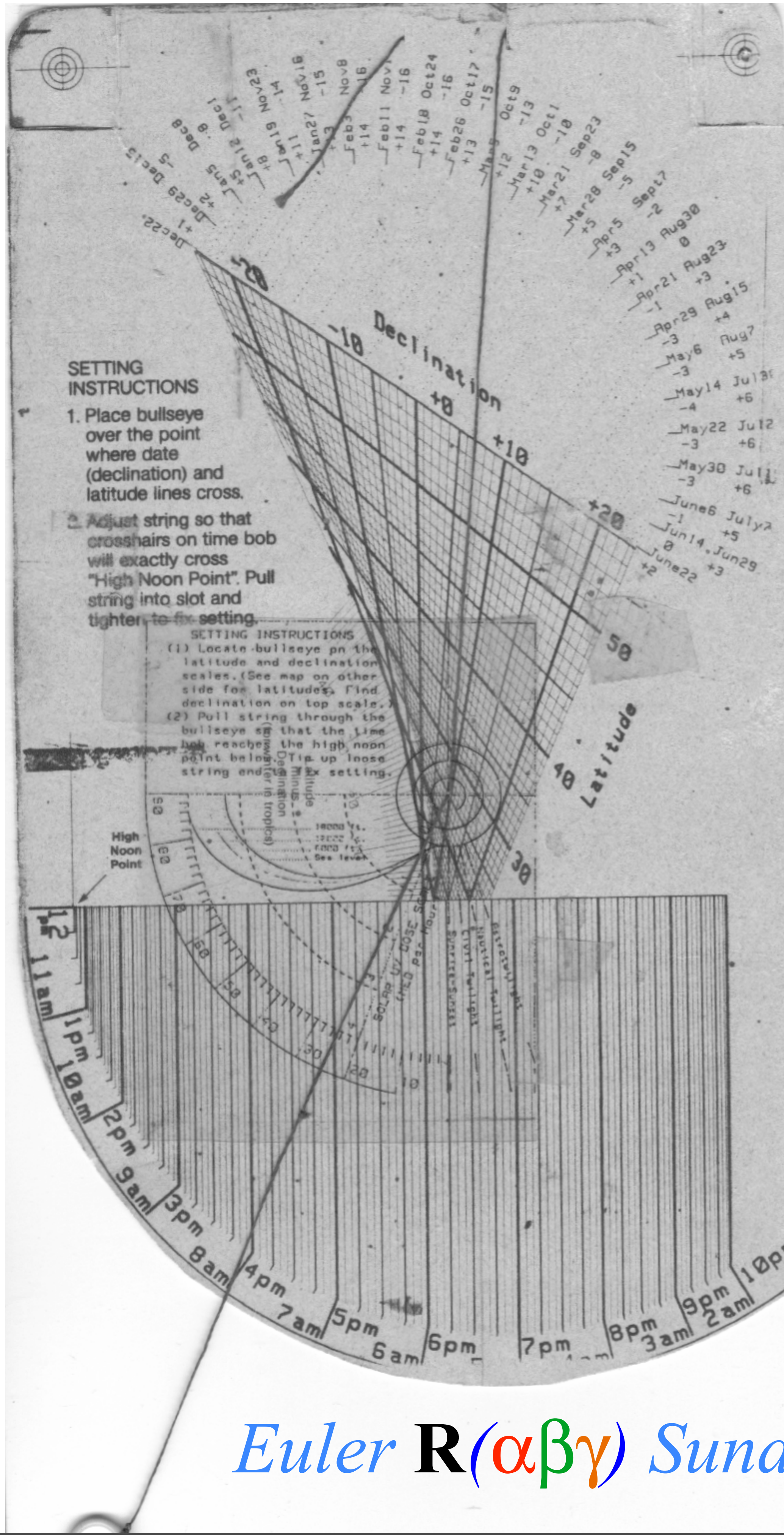
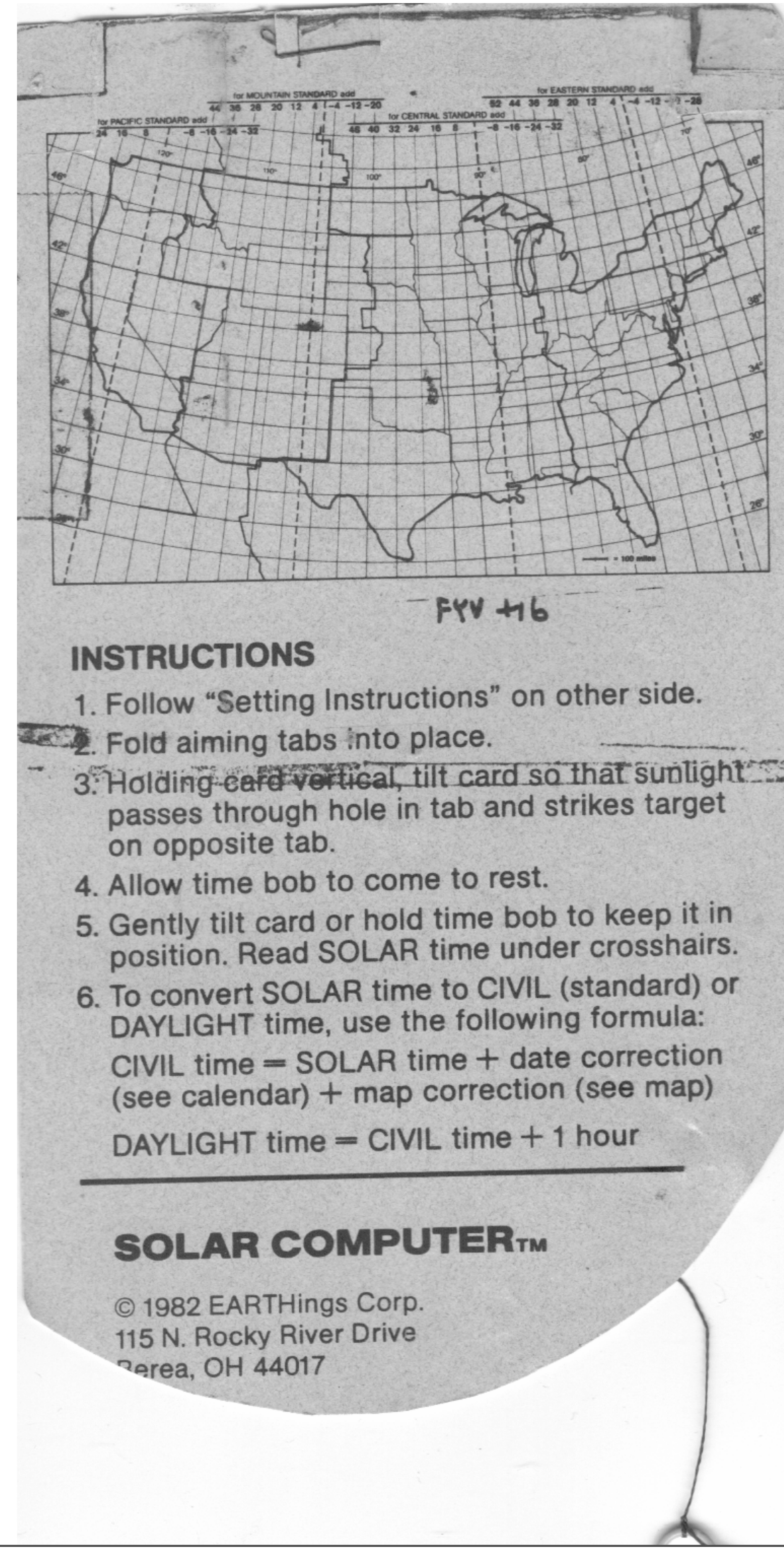
R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

→ *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *Sundial*





Euler **R**(αβγ) *Sundial*