Group Theory in Quantum Mechanics

Lecture 22 (4.18.13)

Octahedral $O_h \supset O \supset D_4 \supset C_4$ eiegensolution in coset spaces II

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)

(PSDS - Ch. 4)

Review Coset factored splitting of $O \supset D_4 \supset C_4$ projectors and levels

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ of $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ subgroup cosets:

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Fundamental $\mathbf{P}^{\mu}_{m,n}$ definitions:

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

Calculating and Factoring $\mathbf{P}^{T_1}_{1_4 0_4}$

Structure and applications of various subgroup chain ireps

$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ *subgroup cosets:*

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Fundamental $\mathbf{P}^{\mu}_{m,n}$ *definitions:*

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

Calculating and Factoring $\mathbf{P}^{T_1}_{1_4 0_4}$

Structure and applications of various subgroup chain ireps

$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Calculating $\mathbf{P}^{E}_{2_4 2_4}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	·	·	·
$A_2\downarrow C_4$	·	·	1	·
$E\downarrow C_4$	1	·	(1)	·
$T_1\downarrow C_4$	1	1	·	1
$T_2\downarrow C_4$	·	1	1	1

$O:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$$\mathbf{P}^{E}_{2_4 2_4}=\mathbf{p}_{2_4}\mathbf{P}^{E}=\mathbf{P}^{E}\mathbf{p}_{2_4}$$

$$=\sum_{g}^{\circ O}\frac{\ell^E}{\circ O}(\chi^E_g)\cdot\mathbf{g}\cdot(\mathbf{p}_{2_4})=\sum_{g}^{\circ O}\frac{2}{96}(\chi^E_g)\cdot\mathbf{g}\cdot(1\cdot\mathbf{1}+1\cdot\rho_z-1\cdot\mathbf{R}_z-1\cdot\tilde{\mathbf{R}}_z)$$

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\rho_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\}\quad \rho_x C_4=\{\rho_x,\rho_y,\mathbf{i}_4,\mathbf{i}_3\}\quad \mathbf{r}_1C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\}\quad \mathbf{r}_2C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\}\quad \tilde{\mathbf{r}}_1C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\}\quad \tilde{\mathbf{r}}_2C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

$$=\tfrac{1}{48}\chi^E_{\mathbf{1}}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{48}\chi^E_{\rho_x}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{48}\chi^E_{\mathbf{r}_1}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{48}\chi^E_{\mathbf{r}_2}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{r}}_1}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{r}}_2}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})$$

$$+\tfrac{1}{48}\chi^E_{\rho_z}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{48}\chi^E_{\rho_y}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{48}\chi^E_{\mathbf{r}_4}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{48}\chi^E_{\mathbf{r}_3}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{r}}_3}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{r}}_4}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})$$

$$+\tfrac{1}{48}\chi^E_{\mathbf{R}_z}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{48}\chi^E_{\mathbf{i}_4}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{48}\chi^E_{\mathbf{i}_1}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{48}\chi^E_{\mathbf{i}_2}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{R}}_x}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{48}\chi^E_{\mathbf{R}_x}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})$$

$$+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{R}}_z}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{48}\chi^E_{\mathbf{i}_3}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{48}\chi^E_{\mathbf{R}_y}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{48}\chi^E_{\tilde{\mathbf{R}}_y}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{48}\chi^E_{\mathbf{i}_6}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{48}\chi^E_{\mathbf{i}_5}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)$$

$$=\tfrac{1}{48}(+2)(1,+1,-1,-1)\ =\tfrac{1}{48}(+2)(1,+1,-1,-1)\ +\tfrac{1}{48}(-1)(1,+1,-1,-1)\ +\tfrac{1}{48}(-1)(1,+1,-1,-1)\ +\tfrac{1}{48}(-1)(1,+1,-1,-1)\ +\tfrac{1}{48}(-1)(1,+1,-1,-1)$$

$$+\tfrac{1}{48}(+2)(+1,1,-1,-1)\ +\tfrac{1}{48}(+2)(+1,1,-1,-1)\ +\tfrac{1}{48}(-1)(+1,1,-1,-1)\ +\tfrac{1}{48}(-1)(+1,1,-1,-1)\ +\tfrac{1}{48}(-1)(+1,1,-1,-1)\ +\tfrac{1}{48}(-1)(+1,1,-1,-1)$$

$$+\tfrac{1}{48}(0)(-1,-1,1,+1)\ +\tfrac{1}{48}(0)(-1,-1,1,+1)\ +\tfrac{1}{48}(0)(-1,-1,1,+1)\ +\tfrac{1}{48}(0)(-1,-1,1,+1)\ +\tfrac{1}{48}(0)(-1,-1,1,+1)\ +\tfrac{1}{48}(0)(-1,-1,1,+1)$$

$$+\tfrac{1}{48}(0)(-1,-1,+1,1)\ +\tfrac{1}{48}(0)(-1,-1,+1,1)\ +\tfrac{1}{48}(0)(-1,-1,+1,1)\ +\tfrac{1}{48}(0)(-1,-1,+1,1)\ +\tfrac{1}{48}(0)(-1,-1,+1,1)\ +\tfrac{1}{48}(0)(-1,-1,+1,1)$$

4, 4,-4,-4, *4, 4,-4,-4,* *-2,-2, 2, 2,* *-2,-2, 2, 2,* *-2,-2, 2, 2,* *-2,-2, 2, 2,*

$$\tfrac{1}{12}(1\mathbf{1}+1\rho_z-1\mathbf{R}_z-1\tilde{\mathbf{R}}_z+1\rho_x+1\rho_y-1\mathbf{i}_4-1\mathbf{i}_3\quad -\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_4+\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{R}_y\quad -\tfrac{1}{2}\mathbf{r}_2-\tfrac{1}{2}\mathbf{r}_3+\tfrac{1}{2}\mathbf{i}_2+\tfrac{1}{2}\tilde{\mathbf{R}}_y\quad -\tfrac{1}{2}\tilde{\mathbf{r}}_1-\tfrac{1}{2}\tilde{\mathbf{r}}_3+\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\mathbf{i}_6\quad -\tfrac{1}{2}\tilde{\mathbf{r}}_2-\tfrac{1}{2}\tilde{\mathbf{r}}_4+\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{i}_5)$$

$$\mathbf{P}^{E}_{2_4 2_4}=\tfrac{1}{12}(1\mathbf{1}-\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_2-\tfrac{1}{2}\mathbf{r}_3-\tfrac{1}{2}\mathbf{r}_4\quad -\tfrac{1}{2}\tilde{\mathbf{r}}_1-\tfrac{1}{2}\tilde{\mathbf{r}}_2-\tfrac{1}{2}\tilde{\mathbf{r}}_3-\tfrac{1}{2}\tilde{\mathbf{r}}_4\quad +1\rho_x+1\rho_y+1\rho_z\quad +\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{R}_y-1\mathbf{R}_z\quad +\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\tilde{\mathbf{R}}_y-1\tilde{\mathbf{R}}_z\quad +\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{i}_2-1\mathbf{i}_3-1\mathbf{i}_4+\tfrac{1}{2}\mathbf{i}_5+\tfrac{1}{2}\mathbf{i}_6)$$

Calculating $\mathbf{P}^{T_1}_{0_4 0_4}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	·	·	·
$A_2\downarrow C_4$	·	·	1	·
$E\downarrow C_4$	1	·	1	·
$T_1\downarrow C_4$	1	1	·	1
$T_2\downarrow C_4$	·	1	1	1

$O:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}$=1	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(1+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(1+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(1-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(1-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$$\mathbf{P}^{T_1}_{0_4 0_4}=\mathbf{p}_{0_4}\mathbf{P}^{T_1}=\mathbf{P}^{T_1}\mathbf{p}_{0_4}$$

$$=\sum_g^{\circ O}\frac{\ell^{T_1}}{\circ O}(\chi_g^{T_1})\cdot\mathbf{g}\cdot(\mathbf{p}_{0_4})=\sum_g^{\circ O}\frac{3}{96}(\chi_g^{T_1})\cdot\mathbf{g}\cdot(1\cdot\mathbf{1}+1\cdot\rho_z+1\cdot\mathbf{R}_z+1\cdot\tilde{\mathbf{R}}_z)$$

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\rho_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\}\quad \rho_x C_4=\{\rho_x,\rho_y,\mathbf{i}_4,\mathbf{i}_3\}\quad \mathbf{r}_1C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\}\quad \mathbf{r}_2C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\}\quad \tilde{\mathbf{r}}_1C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\}\quad \tilde{\mathbf{r}}_2C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

$$=\tfrac{1}{32}\chi_{\mathbf{1}}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})+\tfrac{1}{32}\chi_{\rho_x}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})+\tfrac{1}{32}\chi_{\mathbf{r}_1}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})+\tfrac{1}{32}\chi_{\mathbf{r}_2}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})+\tfrac{1}{32}\chi_{\tilde{\mathbf{r}}_1}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})+\tfrac{1}{32}\chi_{\tilde{\mathbf{r}}_2}^{T_1}(1,d^{04}_{\rho_z},d^{04}_{R_z},d^{04}_{\tilde{R}_z})$$

$$+\tfrac{1}{32}\chi_{\rho_z}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})+\tfrac{1}{32}\chi_{\rho_y}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})+\tfrac{1}{32}\chi_{\mathbf{r}_4}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})+\tfrac{1}{32}\chi_{\mathbf{r}_3}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})+\tfrac{1}{32}\chi_{\tilde{\mathbf{r}}_3}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})+\tfrac{1}{32}\chi_{\tilde{\mathbf{r}}_4}^{T_1}(d^{04}_{\rho_z},1,d^{04}_{\tilde{R}_z},d^{04}_{R_z})$$

$$+\tfrac{1}{32}\chi_{\mathbf{R}_z}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})+\tfrac{1}{32}\chi_{\mathbf{i}_4}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})+\tfrac{1}{32}\chi_{\mathbf{i}_1}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})+\tfrac{1}{32}\chi_{\mathbf{i}_2}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})+\tfrac{1}{32}\chi_{\tilde{\mathbf{R}}_x}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})+\tfrac{1}{32}\chi_{\mathbf{R}_x}^{T_1}(d^{04}_{\tilde{R}_z},d^{04}_{R_z},1,d^{04}_{\rho_z})$$

$$+\tfrac{1}{32}\chi_{\tilde{\mathbf{R}}_z}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)+\tfrac{1}{32}\chi_{\mathbf{i}_3}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)+\tfrac{1}{32}\chi_{\mathbf{R}_y}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)+\tfrac{1}{32}\chi_{\tilde{\mathbf{R}}_y}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)+\tfrac{1}{32}\chi_{\mathbf{i}_6}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)+\tfrac{1}{32}\chi_{\mathbf{i}_5}^{T_1}(d^{04}_{R_z},d^{04}_{\tilde{R}_z},d^{04}_{\rho_z},1)$$

$$=\tfrac{1}{32}(+3)(1,+1,+1,+1)+\tfrac{1}{32}(-1)(1,+1,+1,+1)+\tfrac{1}{32}(0)(1,+1,+1,+1)+\tfrac{1}{32}(0)(1,+1,+1,+1)+\tfrac{1}{32}(0)(1,+1,+1,+1)+\tfrac{1}{32}(0)(1,+1,+1,+1)$$

$$+\tfrac{1}{32}(-1)(+1,1,+1,+1)+\tfrac{1}{32}(-1)(+1,1,+1,+1)+\tfrac{1}{32}(0)(+1,1,+1,+1)+\tfrac{1}{32}(0)(+1,1,+1,+1)+\tfrac{1}{32}(0)(+1,1,+1,+1)+\tfrac{1}{32}(0)(+1,1,+1,+1)$$

$$+\tfrac{1}{32}(+1)(+1,+1,1,+1)+\tfrac{1}{32}(-1)(+1,+1,1,+1)+\tfrac{1}{32}(-1)(+1,+1,1,+1)+\tfrac{1}{32}(-1)(+1,+1,1,+1)+\tfrac{1}{32}(+1)(+1,+1,1,+1)+\tfrac{1}{32}(+1)(+1,+1,1,+1)$$

$$+\tfrac{1}{32}(+1)(+1,+1,+1,1)+\tfrac{1}{32}(-1)(+1,+1,+1,1)+\tfrac{1}{32}(+1)(+1,+1,+1,1)+\tfrac{1}{32}(+1)(+1,+1,+1,1)+\tfrac{1}{32}(-1)(+1,+1,+1,1)+\tfrac{1}{32}(-1)(+1,+1,+1,1)$$

4, 4, 0, 0, -4,-4,-4,-4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,

$$\tfrac{1}{8}(\underline{1}\mathbf{1}+\underline{1}\rho_z+\underline{1}\mathbf{R}_z+\underline{1}\tilde{\mathbf{R}}_z\ \underline{-1}\rho_x\ \underline{-1}\rho_y\ \underline{-1}\mathbf{i}_4\ \underline{-1}\mathbf{i}_3\ +\underline{0}\mathbf{r}_1+\underline{0}\mathbf{r}_4+\underline{0}\mathbf{i}_1+\underline{0}\mathbf{R}_y+\underline{0}\mathbf{r}_2+\underline{0}\mathbf{r}_3+\underline{0}\mathbf{i}_2+\underline{0}\tilde{\mathbf{R}}_y+\underline{0}\tilde{\mathbf{r}}_1+\underline{0}\tilde{\mathbf{r}}_3+\underline{0}\tilde{\mathbf{R}}_x+\underline{0}\mathbf{i}_6+\underline{0}\tilde{\mathbf{r}}_2+\underline{0}\tilde{\mathbf{r}}_4+\underline{0}\mathbf{R}_x+\underline{0}\mathbf{i}_5)$$

$$\mathbf{P}^{T_1}_{0_4 0_4}=\tfrac{1}{8}(\underline{1}\mathbf{1}\ +\underline{1}\rho_z\ \underline{-1}\rho_x\ \underline{-1}\rho_y\ +\underline{1}\mathbf{R}_z+\underline{1}\tilde{\mathbf{R}}_z\ \underline{-1}\mathbf{i}_4\ \underline{-1}\mathbf{i}_3)$$

Calculating $\mathbf{P}^{T_1}{}_{1_41_4}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	·	·	·
$A_2\downarrow C_4$	·	·	1	·
$E\downarrow C_4$	1	·	1	·
→ $T_1\downarrow C_4$	1	(1)	·	1
$T_2\downarrow C_4$	·	1	1	1

$O:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$$\mathbf{P}^{T_1}_{1_41_4}=\mathbf{p}_{1_4}\mathbf{P}^{T_1}=\mathbf{P}^{T_1}\mathbf{p}_{1_4}$$

$$=\sum_{g}^{\circ O}\frac{\ell^{T_1}}{\circ O}(\chi^{T_1}_g)\cdot\mathbf{g}\cdot(\mathbf{p}_{1_4})=\sum_{g}^{\circ O}\frac{3}{96}(\chi^{T_1}_g)\cdot\mathbf{g}\cdot(1\cdot\mathbf{1}-1\cdot\rho_z+i\cdot\mathbf{R}_z-i\cdot\tilde{\mathbf{R}}_z)$$

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\rho_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\}\quad \rho_xC_4=\{\rho_x,\rho_y,\mathbf{i}_4,\mathbf{i}_3\}\quad \mathbf{r}_1C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\}\quad \mathbf{r}_2C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\}\quad \tilde{\mathbf{r}}_1C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\}\quad \tilde{\mathbf{r}}_2C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

$$=\tfrac{1}{32}\chi^{T_1}_{\mathbf{1}}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_1}_{\rho_x}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{r}_1}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{r}_2}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{r}}_1}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{r}}_2}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})$$

$$+\tfrac{1}{32}\chi^{T_1}_{\rho_z}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_1}_{\rho_y}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{r}_4}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{r}_3}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{r}}_3}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{r}}_4}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})$$

$$+\tfrac{1}{32}\chi^{T_1}_{\mathbf{R}_z}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_4}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_1}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_2}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{R}}_x}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_1}_{\mathbf{R}_x}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})$$

$$+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{R}}_z}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_3}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_1}_{\mathbf{R}_y}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_1}_{\tilde{\mathbf{R}}_y}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_6}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_1}_{\mathbf{i}_5}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)$$

$$=\tfrac{1}{32}(+3)(1,-1,+i,-i)+\tfrac{1}{32}(-1)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)$$

$$+\tfrac{1}{32}(-1)(-1,1,-i,+i)+\tfrac{1}{32}(-1)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)$$

$$+\tfrac{1}{32}(+1)(-i,+i,1,-1)+\tfrac{1}{32}(-1)(-i,+i,1,-1)+\tfrac{1}{32}(-1)(-i,+i,1,-1)+\tfrac{1}{32}(-1)(-i,+i,1,-1)+\tfrac{1}{32}(+1)(-i,+i,1,-1)+\tfrac{1}{32}(+1)(-i,+i,1,-1)$$

$$+\tfrac{1}{32}(+1)(+i,-i,-1,1)+\tfrac{1}{32}(-1)(+i,-i,-1,1)+\tfrac{1}{32}(+1)(+i,-i,-1,1)+\tfrac{1}{32}(+1)(+i,-i,-1,1)+\tfrac{1}{32}(-1)(+i,-i,-1,1)+\tfrac{1}{32}(-1)(+i,-i,-1,1)$$

$+4,-4,\ 4i,-4i,\quad 0,\quad 0,\quad 0,\quad 0,\quad +2i,\ -2i,-2,\ +2,\quad +2i,\ -2i,-2,\ +2,\quad -2i,+2i,+2,\ -2\quad -2i,+2i,+2,\ -2.$

$$\tfrac{1}{8}(1\mathbf{1}-1\rho_z+i\mathbf{R}_z-i\tilde{\mathbf{R}}_z\quad+0\rho_x+0\rho_y+0\mathbf{i}_4+0\mathbf{i}_3\quad+\tfrac{i}{2}\mathbf{r}_1-\tfrac{i}{2}\mathbf{r}_4-\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{R}_y\quad+\tfrac{i}{2}\mathbf{r}_2-\tfrac{i}{2}\mathbf{r}_3-\tfrac{1}{2}\mathbf{i}_2+\tfrac{1}{2}\tilde{\mathbf{R}}_y\quad-\tfrac{i}{2}\tilde{\mathbf{r}}_1+\tfrac{i}{2}\tilde{\mathbf{r}}_3+\tfrac{1}{2}\tilde{\mathbf{R}}_x-\tfrac{1}{2}\mathbf{i}_6\quad-\tfrac{i}{2}\tilde{\mathbf{r}}_2+\tfrac{i}{2}\tilde{\mathbf{r}}_4+\tfrac{1}{2}\mathbf{R}_x-\tfrac{1}{2}\mathbf{i}_5)$$

$$\mathbf{P}^{T_1}_{1_41_4}=\tfrac{1}{8}(1\mathbf{1}+0\rho_x+0\rho_y-1\rho_z+\tfrac{i}{2}\mathbf{r}_1+\tfrac{i}{2}\mathbf{r}_2-\tfrac{i}{2}\mathbf{r}_3-\tfrac{i}{2}\mathbf{r}_4-\tfrac{i}{2}\tilde{\mathbf{r}}_1-\tfrac{i}{2}\tilde{\mathbf{r}}_2+\tfrac{i}{2}\tilde{\mathbf{r}}_3+\tfrac{i}{2}\tilde{\mathbf{r}}_4+\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{R}_y+i\mathbf{R}_z+\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\tilde{\mathbf{R}}_y-i\tilde{\mathbf{R}}_z-\tfrac{1}{2}\mathbf{i}_1-\tfrac{1}{2}\mathbf{i}_2+0\mathbf{i}_3+0\mathbf{i}_4-\tfrac{1}{2}\mathbf{i}_5-\tfrac{1}{2}\mathbf{i}_6)$$

Calculating $\mathbf{P}^{T_2}{}_{2_4 2_4}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	·	·	·
$A_2\downarrow C_4$	·	·	1	·
$E\downarrow C_4$	1	·	1	·
$T_1\downarrow C_4$	1	1	·	1
$T_2\downarrow C_4$	·	1	1	1

$O:\chi^{\mu}_{\mathbf{g}}$	g=1	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$$\mathbf{P}^{T_2}_{2_4 2_4}=\mathbf{p}_{2_4}\mathbf{P}^{T_2}=\mathbf{P}^{T_2}\mathbf{p}_{2_4}$$

$$=\sum_g^{\circ O}\frac{\ell^{T_2}}{\circ O}(\chi_g^{T_2})\cdot\mathbf{g}\cdot(\mathbf{p}_{2_4})=\sum_g^{\circ O}\frac{3}{96}(\chi_g^{T_2})\cdot\mathbf{g}\cdot(1\cdot\mathbf{1}+1\cdot\rho_z-1\cdot\mathbf{R}_z-1\cdot\tilde{\mathbf{R}}_z)$$

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\rho_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\}\quad \rho_x C_4=\{\rho_x,\rho_y,\mathbf{i}_4,\mathbf{i}_3\}\quad \mathbf{r}_1C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\}\quad \mathbf{r}_2C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\}\quad \tilde{\mathbf{r}}_1C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\}\quad \tilde{\mathbf{r}}_2C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

$$=\tfrac{1}{32}\chi^{T_2}_{\mathbf{1}}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\rho_x}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_1}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_2}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_1}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_2}(1,d^{24}_{\rho_z},d^{24}_{R_z},d^{24}_{\tilde{R}_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\rho_z}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\rho_y}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_4}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_3}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_3}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_4}(d^{24}_{\rho_z},1,d^{24}_{\tilde{R}_z},d^{24}_{R_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_z}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_4}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_1}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_2}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_x}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_x}(d^{24}_{\tilde{R}_z},d^{24}_{R_z},1,d^{24}_{\rho_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_z}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_3}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_y}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_y}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_6}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1))+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_5}(d^{24}_{R_z},d^{24}_{\tilde{R}_z},d^{24}_{\rho_z},1)$$

$$=\tfrac{1}{32}(+3)(1,+1,-1,-1)+\tfrac{1}{32}(-1)(1,+1,-1,-1)+\tfrac{1}{32}(0)(1,+1,-1,-1)+\tfrac{1}{32}(0)(1,+1,-1,-1)+\tfrac{1}{32}(0)(1,+1,-1,-1)+\tfrac{1}{32}(0)(1,+1,-1,-1)$$

$$+\tfrac{1}{32}(-1)(+1,1,-1,-1)+\tfrac{1}{32}(-1)(+1,1,-1,-1)+\tfrac{1}{32}(0)(+1,1,-1,-1)+\tfrac{1}{32}(0)(+1,1,-1,-1)+\tfrac{1}{32}(0)(+1,1,-1,-1)+\tfrac{1}{32}(0)(+1,1,-1,-1)$$

$$+\tfrac{1}{32}(-1)(-1,-1,1,+1)+\tfrac{1}{32}(+1)(-1,-1,1,+1)+\tfrac{1}{32}(+1)(-1,-1,1,+1)+\tfrac{1}{32}(+1)(-1,-1,1,+1)+\tfrac{1}{32}(-1)(-1,-1,1,+1)+\tfrac{1}{32}(-1)(-1,-1,1,+1)$$

$$+\tfrac{1}{32}(-1)(-1,-1,+1,1)+\tfrac{1}{32}(+1)(-1,-1,+1,1)+\tfrac{1}{32}(-1)(-1,-1,+1,1)+\tfrac{1}{32}(-1)(-1,-1,+1,1)+\tfrac{1}{32}(+1)(-1,-1,+1,1)+\tfrac{1}{32}(+1)(-1,-1,+1,1)$$

4, 4,-4,-4, *-4,-4, 4, 4,* *0, 0, 0, 0,* *0, 0, 0, 0,* *0, 0, 0, 0,* *0, 0, 0, 0,*

$$\tfrac{1}{8}(\underline{1}\mathbf{1}+\underline{1}\rho_z-\underline{1}\mathbf{R}_z-\underline{1}\tilde{\mathbf{R}}_z\ \underline{-1}\rho_x\underline{-1}\rho_y+\underline{1}\mathbf{i}_4+\underline{1}\mathbf{i}_3\ +\underline{0}\mathbf{r}_1+\underline{0}\mathbf{r}_4+\underline{0}\mathbf{i}_1+\underline{0}\mathbf{R}_y+\underline{0}\mathbf{r}_2+\underline{0}\mathbf{r}_3+\underline{0}\mathbf{i}_2+\underline{0}\tilde{\mathbf{R}}_y+\underline{0}\tilde{\mathbf{r}}_1+\underline{0}\tilde{\mathbf{r}}_3+\underline{0}\tilde{\mathbf{R}}_x+\underline{0}\mathbf{i}_6+\underline{0}\tilde{\mathbf{r}}_2+\underline{0}\tilde{\mathbf{r}}_4+\underline{0}\mathbf{R}_x+\underline{0}\mathbf{i}_5)$$

$$\mathbf{P}^{T_2}_{2_4 2_4}=\tfrac{1}{8}(\underline{1}\mathbf{1}\ +\underline{1}\rho_z\underline{-1}\rho_x\underline{-1}\rho_y\ -\underline{1}\mathbf{R}_z-\underline{1}\tilde{\mathbf{R}}_z\ +\underline{1}\mathbf{i}_4+\underline{1}\mathbf{i}_3\ )$$

Calculating $\mathbf{P}^{T_2}{}_{1_4 1_4}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	.	.	.
$A_2\downarrow C_4$	.	.	1	.
$E\downarrow C_4$	1	.	1	.
$T_1\downarrow C_4$	1	1	.	1
$T_2\downarrow C_4$	.	1	1	1

$O:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$$\mathbf{P}^{T_2}_{1_4 1_4}=\mathbf{p}_{1_4}\mathbf{P}^{T_2}=\mathbf{P}^{T_2}\mathbf{p}_{1_4}$$

$$=\sum_g^{\circ O}\frac{\ell^{T_2}}{\circ O}(\chi_g^{T_2})\cdot\mathbf{g}\cdot\left(\mathbf{p}_{1_4}\right)=\sum_g^{\circ O}\frac{3}{96}(\chi_g^{T_2})\cdot\mathbf{g}\cdot\left(1\cdot\mathbf{1}-1\cdot\rho_z+i\cdot\mathbf{R}_z-i\cdot\tilde{\mathbf{R}}_z\right)$$

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\rho_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\}\quad \rho_x C_4=\{\rho_x,\rho_y,\mathbf{i}_4,\mathbf{i}_3\}\quad \mathbf{r}_1C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\}\quad \mathbf{r}_2C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\}\quad \tilde{\mathbf{r}}_1C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\}\quad \tilde{\mathbf{r}}_2C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

$$=\tfrac{1}{32}\chi^{T_2}_{\mathbf{1}}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\rho_x}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_1}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_2}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_1}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_2}(1,d^{1_4}_{\rho_z},d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\rho_z}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\rho_y}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_4}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{r}_3}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_3}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{r}}_4}(d^{1_4}_{\rho_z},1,d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_z}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_4}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_1}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_2}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_x}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_x}(d^{1_4}_{\tilde{R}_z},d^{1_4}_{R_z},1,d^{1_4}_{\rho_z})$$

$$+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_z}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_3}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{R}_y}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\tilde{\mathbf{R}}_y}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_6}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)+\tfrac{1}{32}\chi^{T_2}_{\mathbf{i}_5}(d^{1_4}_{R_z},d^{1_4}_{\tilde{R}_z},d^{1_4}_{\rho_z},1)$$

$$=\tfrac{1}{32}(+3)(1,-1,+i,-i)+\tfrac{1}{32}(-1)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)+\tfrac{1}{32}(0)(1,-1,+i,-i)$$

$$+\tfrac{1}{32}(-1)(-1,1,-i,+i)+\tfrac{1}{32}(-1)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)+\tfrac{1}{32}(0)(-1,1,-i,+i)$$

$$+\tfrac{1}{32}(-1)(-i,+i,1,-1)+\tfrac{1}{32}(+1)(-i,+i,1,-1)+\tfrac{1}{32}(+1)(-i,+i,1,-1)+\tfrac{1}{32}(+1)(-i,+i,1,-1)+\tfrac{1}{32}(-1)(-i,+i,1,-1)+\tfrac{1}{32}(-1)(-i,+i,1,-1)$$

$$+\tfrac{1}{32}(-1)(+i,-i,-1,1)+\tfrac{1}{32}(+1)(+i,-i,-1,1)+\tfrac{1}{32}(-1)(+i,-i,-1,1)+\tfrac{1}{32}(-1)(+i,-i,-1,1)+\tfrac{1}{32}(+1)(+i,-i,-1,1)+\tfrac{1}{32}(+1)(+i,-i,-1,1)$$

+4,-4, 4i,-4i, *0,0,0,0,* *-2i,2i,2,-2,* *-2i,2i,2,-2,* *2i,-2i,-2,2,* *2i,-2i,-2,2.*

$$\tfrac{1}{8}(1\mathbf{1}-1\rho_z+i\mathbf{R}_z-i\tilde{\mathbf{R}}_z\quad+0\rho_x+0\rho_y+0\mathbf{i}_4+0\mathbf{i}_3\quad-\tfrac{i}{2}\mathbf{r}_1+\tfrac{i}{2}\mathbf{r}_4+\tfrac{1}{2}\mathbf{i}_1-\tfrac{1}{2}\mathbf{R}_y\quad-\tfrac{i}{2}\mathbf{r}_2+\tfrac{i}{2}\mathbf{r}_3+\tfrac{1}{2}\mathbf{i}_2-\tfrac{1}{2}\tilde{\mathbf{R}}_y\quad+\tfrac{i}{2}\tilde{\mathbf{r}}_1-\tfrac{i}{2}\tilde{\mathbf{r}}_3-\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\mathbf{i}_6\quad+\tfrac{i}{2}\tilde{\mathbf{r}}_2-\tfrac{i}{2}\tilde{\mathbf{r}}_4-\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{i}_5$$

$$\mathbf{P}^{T_2}_{1_4 1_4}=\tfrac{1}{8}(1\mathbf{1}\ +0\rho_x+0\rho_y-1\rho_z\ -\tfrac{i}{2}\mathbf{r}_1-\tfrac{i}{2}\mathbf{r}_2\ +\tfrac{i}{2}\mathbf{r}_3+\tfrac{i}{2}\mathbf{r}_4\ +\tfrac{i}{2}\tilde{\mathbf{r}}_1+\tfrac{i}{2}\tilde{\mathbf{r}}_2\ -\tfrac{i}{2}\tilde{\mathbf{r}}_3-\tfrac{i}{2}\tilde{\mathbf{r}}_4\ -\tfrac{1}{2}\mathbf{R}_x-\tfrac{1}{2}\mathbf{R}_y+i\mathbf{R}_z\ -\tfrac{1}{2}\tilde{\mathbf{R}}_x-\tfrac{1}{2}\tilde{\mathbf{R}}_y-i\tilde{\mathbf{R}}_z\ +\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{i}_2\ +0\mathbf{i}_3 0\mathbf{i}_4\ +\tfrac{1}{2}\mathbf{i}_5+\tfrac{1}{2}\mathbf{i}_6)$$

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ *subgroup cosets:*

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

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$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O\supset C_4 \sim T_d \supset C_{4i}$

Broken-class-ordered A_1*-sum:*

$$\mathbf{P}^{A_1}_{0_4 0_4} = \tfrac{1}{24}(1\cdot\mathbf{1} \; +1\mathbf{r}_1+1\mathbf{r}_2+1\mathbf{r}_3+1\mathbf{r}_4+1\tilde{\mathbf{r}}_1+1\tilde{\mathbf{r}}_2 \; +1\tilde{\mathbf{r}}_3+1\tilde{\mathbf{r}}_4 \; +1\rho_x+1\rho_y+1\rho_z \; +1\mathbf{R}_x+1\mathbf{R}_y \; +1\mathbf{R}_z \; +1\tilde{\mathbf{R}}_x+1\tilde{\mathbf{R}}_y \; +1\tilde{\mathbf{R}}_z \; +1\mathbf{i}_1+1\mathbf{i}_2 \; +1\mathbf{i}_3+1\mathbf{i}_4 \; +1\mathbf{i}_5+1\mathbf{i}_6)$$

Broken-class-ordered A_2*-sum:*

$$\mathbf{P}^{A_2}_{2_4 2_4} = \tfrac{1}{24}(1\cdot\mathbf{1} \; +1\mathbf{r}_1+1\mathbf{r}_2+1\mathbf{r}_3+1\mathbf{r}_4+1\tilde{\mathbf{r}}_1+1\tilde{\mathbf{r}}_2 \; +1\tilde{\mathbf{r}}_3+1\tilde{\mathbf{r}}_4 \; +1\rho_x+1\rho_y+1\rho_z \; -1\mathbf{R}_x-1\mathbf{R}_y \; -1\mathbf{R}_z \; -1\tilde{\mathbf{R}}_x-1\tilde{\mathbf{R}}_y \; -1\tilde{\mathbf{R}}_z \; -1\mathbf{i}_1-1\mathbf{i}_2 \; -1\mathbf{i}_3-1\mathbf{i}_4 \; -1\mathbf{i}_5-1\mathbf{i}_6)$$

Broken-class-ordered E*-sum:*

$$\mathbf{P}^{E}_{0_4 0_4} = \tfrac{1}{12}(1\cdot\mathbf{1} \; -\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_2 \; -\tfrac{1}{2}\mathbf{r}_3-\tfrac{1}{2}\mathbf{r}_4 \; -\tfrac{1}{2}\tilde{\mathbf{r}}_1-\tfrac{1}{2}\tilde{\mathbf{r}}_2 \; -\tfrac{1}{2}\tilde{\mathbf{r}}_3-\tfrac{1}{2}\tilde{\mathbf{r}}_4 \; +1\rho_x+1\rho_y+1\rho_z \; -\tfrac{1}{2}\mathbf{R}_x-\tfrac{1}{2}\mathbf{R}_y \; +1\mathbf{R}_z \; -\tfrac{1}{2}\tilde{\mathbf{R}}_x-\tfrac{1}{2}\tilde{\mathbf{R}}_y \; +1\tilde{\mathbf{R}}_z \; -\tfrac{1}{2}\mathbf{i}_1-\tfrac{1}{2}\mathbf{i}_2 \; +1\mathbf{i}_3+1\mathbf{i}_4 \; -\tfrac{1}{2}\mathbf{i}_5-\tfrac{1}{2}\mathbf{i}_6)$$

$$\mathbf{P}^{E}_{2_4 2_4} = \tfrac{1}{12}(1\cdot\mathbf{1} \; -\tfrac{1}{2}\mathbf{r}_1-\tfrac{1}{2}\mathbf{r}_2 \; -\tfrac{1}{2}\mathbf{r}_3-\tfrac{1}{2}\mathbf{r}_4 \; -\tfrac{1}{2}\tilde{\mathbf{r}}_1-\tfrac{1}{2}\tilde{\mathbf{r}}_2 \; -\tfrac{1}{2}\tilde{\mathbf{r}}_3-\tfrac{1}{2}\tilde{\mathbf{r}}_4 \; +1\rho_x+1\rho_y+1\rho_z \; +\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{R}_y \; -1\mathbf{R}_z \; +\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\tilde{\mathbf{R}}_y \; -1\tilde{\mathbf{R}}_z \; +\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{i}_2 \; -1\mathbf{i}_3-1\mathbf{i}_4 \; +\tfrac{1}{2}\mathbf{i}_5+\tfrac{1}{2}\mathbf{i}_6)$$

Broken-class-ordered T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; +\tfrac{i}{2}\mathbf{r}_1+\tfrac{i}{2}\mathbf{r}_2 \; -\tfrac{i}{2}\mathbf{r}_3-\tfrac{i}{2}\mathbf{r}_4 \; -\tfrac{i}{2}\tilde{\mathbf{r}}_1-\tfrac{i}{2}\tilde{\mathbf{r}}_2 \; +\tfrac{i}{2}\tilde{\mathbf{r}}_3+\tfrac{i}{2}\tilde{\mathbf{r}}_4 \; +0\rho_x+0\rho_y \; -1\rho_z \; +\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{R}_y \; +i\mathbf{R}_z \; +\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\tilde{\mathbf{R}}_y \; -i\tilde{\mathbf{R}}_z \; -\tfrac{1}{2}\mathbf{i}_1-\tfrac{1}{2}\mathbf{i}_2 \; +0\mathbf{i}_3+0\mathbf{i}_4 \; -\tfrac{1}{2}\mathbf{i}_5-\tfrac{1}{2}\mathbf{i}_6)$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; -\tfrac{i}{2}\mathbf{r}_1-\tfrac{i}{2}\mathbf{r}_2 \; +\tfrac{i}{2}\mathbf{r}_3+\tfrac{i}{2}\mathbf{r}_4 \; +\tfrac{i}{2}\tilde{\mathbf{r}}_1+\tfrac{i}{2}\tilde{\mathbf{r}}_2 \; -\tfrac{i}{2}\tilde{\mathbf{r}}_3-\tfrac{i}{2}\tilde{\mathbf{r}}_4 \; +0\rho_x+0\rho_y \; -1\rho_z \; +\tfrac{1}{2}\mathbf{R}_x+\tfrac{1}{2}\mathbf{R}_y \; -i\mathbf{R}_z \; +\tfrac{1}{2}\tilde{\mathbf{R}}_x+\tfrac{1}{2}\tilde{\mathbf{R}}_y \; +i\tilde{\mathbf{R}}_z \; -\tfrac{1}{2}\mathbf{i}_1-\tfrac{1}{2}\mathbf{i}_2 \; +0\mathbf{i}_3+0\mathbf{i}_4 \; -\tfrac{1}{2}\mathbf{i}_5-\tfrac{1}{2}\mathbf{i}_6)$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; +0+0 \; +0+0 \; +0+0 \; +0+0 \; -1\rho_x-1\rho_y \; +1\rho_z \; +0 \; +0 \; +1\mathbf{R}_z \; +0 \; +0 \; +1\tilde{\mathbf{R}}_z \; +0+0 \; -1\mathbf{i}_3 \; -1\mathbf{i}_4 \; +0+0)$$

Broken-class-ordered T_2*-sum:*

$$\mathbf{P}^{T_2}_{1_4 1_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; -\tfrac{i}{2}\mathbf{r}_1-\tfrac{i}{2}\mathbf{r}_2 \; +\tfrac{i}{2}\mathbf{r}_3+\tfrac{i}{2}\mathbf{r}_4 \; +\tfrac{i}{2}\tilde{\mathbf{r}}_1+\tfrac{i}{2}\tilde{\mathbf{r}}_2 \; -\tfrac{i}{2}\tilde{\mathbf{r}}_3-\tfrac{i}{2}\tilde{\mathbf{r}}_4 \; +0\rho_x+0\rho_y-1\rho_z \; -\tfrac{1}{2}\mathbf{R}_x-\tfrac{1}{2}\mathbf{R}_y \; +i\mathbf{R}_z \; -\tfrac{1}{2}\tilde{\mathbf{R}}_x-\tfrac{1}{2}\tilde{\mathbf{R}}_y \; -i\tilde{\mathbf{R}}_z \; +\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{i}_2 \; +0\mathbf{i}_3+0\mathbf{i}_4+\tfrac{1}{2}\mathbf{i}_5+\tfrac{1}{2}\mathbf{i}_6)$$

$$\mathbf{P}^{T_2}_{3_4 3_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; +\tfrac{i}{2}\mathbf{r}_1+\tfrac{i}{2}\mathbf{r}_2 \; -\tfrac{i}{2}\mathbf{r}_3-\tfrac{i}{2}\mathbf{r}_4 \; -\tfrac{i}{2}\tilde{\mathbf{r}}_1-\tfrac{i}{2}\tilde{\mathbf{r}}_2 \; +\tfrac{i}{2}\tilde{\mathbf{r}}_3+\tfrac{i}{2}\tilde{\mathbf{r}}_4 \; +0\rho_x+0\rho_y-1\rho_z \; -\tfrac{1}{2}\mathbf{R}_x-\tfrac{1}{2}\mathbf{R}_y \; -i\mathbf{R}_z \; -\tfrac{1}{2}\tilde{\mathbf{R}}_x-\tfrac{1}{2}\tilde{\mathbf{R}}_y \; +i\tilde{\mathbf{R}}_z \; +\tfrac{1}{2}\mathbf{i}_1+\tfrac{1}{2}\mathbf{i}_2 \; +0\mathbf{i}_3+0\mathbf{i}_4+\tfrac{1}{2}\mathbf{i}_5+\tfrac{1}{2}\mathbf{i}_6)$$

$$\mathbf{P}^{T_2}_{2_4 2_4} = \tfrac{1}{8}(1\cdot\mathbf{1} \; +0+0 \; +0+0 \; +0+0 \; +0+0 \; -1\rho_x-1\rho_y+1\rho_z \; +0+0 \; -1\mathbf{R}_z \; +0+0 \; -1\tilde{\mathbf{R}}_z \; +0+0 \; +1\mathbf{i}_4+1\mathbf{i}_3 \; +0+0)$$

$O:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

➡ *Factoring out* $O \supset C_4$ *subgroup cosets:* ⬅

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

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$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ of $O\supset C_4 \sim T_d \supset C_{4i}$

Factoring out $O\supset C_4$ subgroup cosets:

$$\mathbf{1}C_4=\mathbf{1}\{\mathbf{1},\boldsymbol{\rho}_z,\mathbf{R}_z,\tilde{\mathbf{R}}_z\} \quad \boldsymbol{\rho}_x C_4=\{\boldsymbol{\rho}_x,\boldsymbol{\rho}_y,\mathbf{i}_4,\mathbf{i}_3\} \quad \mathbf{r}_1 C_4=\{\mathbf{r}_1,\mathbf{r}_4,\mathbf{i}_1,\mathbf{R}_y\} \quad \mathbf{r}_2 C_4=\{\mathbf{r}_2,\mathbf{r}_3,\mathbf{i}_2,\tilde{\mathbf{R}}_y\} \quad \tilde{\mathbf{r}}_1 C_4=\{\tilde{\mathbf{r}}_1,\tilde{\mathbf{r}}_3,\tilde{\mathbf{R}}_x,\mathbf{i}_6\} \quad \tilde{\mathbf{r}}_2 C_4=\{\tilde{\mathbf{r}}_2,\tilde{\mathbf{r}}_4,\mathbf{R}_x,\mathbf{i}_5\}$$

Coset-factored A_1-sum:

$$\mathbf{P}^{A_1}_{0_4 0_4}=\tfrac{1}{12}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4}+(1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{0_4}+(1)\cdot\mathbf{r}_1\mathbf{p}_{0_4}+(1)\cdot\mathbf{r}_2\mathbf{p}_{0_4}+(1)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4}+(1)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Coset-factored A_2-sum:

$$\mathbf{P}^{A_2}_{2_4 2_4}=\tfrac{1}{12}[(1)\cdot\mathbf{1}\mathbf{p}_{2_4}+(1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{2_4}+(1)\cdot\mathbf{r}_1\mathbf{p}_{2_4}+(1)\cdot\mathbf{r}_2\mathbf{p}_{2_4}+(1)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{2_4}+(1)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{2_4}]$$

Coset-factored E-sum:

$$\mathbf{P}^{E}_{0_4 0_4}=\tfrac{1}{12}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4}+(1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{0_4}+(-\tfrac{1}{2})\cdot\mathbf{r}_1\mathbf{p}_{0_4}+(-\tfrac{1}{2})\cdot\mathbf{r}_2\mathbf{p}_{0_4}+(-\tfrac{1}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4}+(-\tfrac{1}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

$$\mathbf{P}^{E}_{2_4 2_4}=\tfrac{1}{12}[(1)\cdot\mathbf{1}\mathbf{p}_{2_4}+(1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{2_4}+(-\tfrac{1}{2})\cdot\mathbf{r}_1\mathbf{p}_{2_4}+(-\tfrac{1}{2})\cdot\mathbf{r}_2\mathbf{p}_{2_4}+(-\tfrac{1}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{2_4}+(-\tfrac{1}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{2_4}]$$

Coset-factored T_1-sum:

$$\mathbf{P}^{T_1}_{1_4 1_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4}+(0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{1_4}+(+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4}+(+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4}+(0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{3_4}+(-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4}+(-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4}+(-1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{0_4}+(0)\cdot\mathbf{r}_1\mathbf{p}_{0_4}+(0)\cdot\mathbf{r}_2\mathbf{p}_{0_4}+(0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4}+(0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Coset-factored T_2-sum:

$$\mathbf{P}^{T_2}_{1_4 1_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4}+(0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4}+(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_2}_{3_4 3_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4}+(0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4}+(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_2}_{2_4 2_4}=\tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{2_4}+(1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{2_4}+(0)\cdot\mathbf{r}_1\mathbf{p}_{2_4}+(0)\cdot\mathbf{r}_2\mathbf{p}_{2_4}+(0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{2_4}+(0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{2_4}]$$

$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=$$

$$\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$C_4:\chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_z$	ρ_z	$\tilde{\mathbf{R}}_z$
$\mu=0_4$	1	1	1	1
1_4	1	$-i$	-1	i
2_4	1	-1	1	-1
3_4	1	$-i$	-1	$-i$

C_4 subgroup correlation to O

$O\supset C_4$	$(0)_4$	$(1)_4$	$(2)_4$	$(3)_4=(-1)_4$
A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

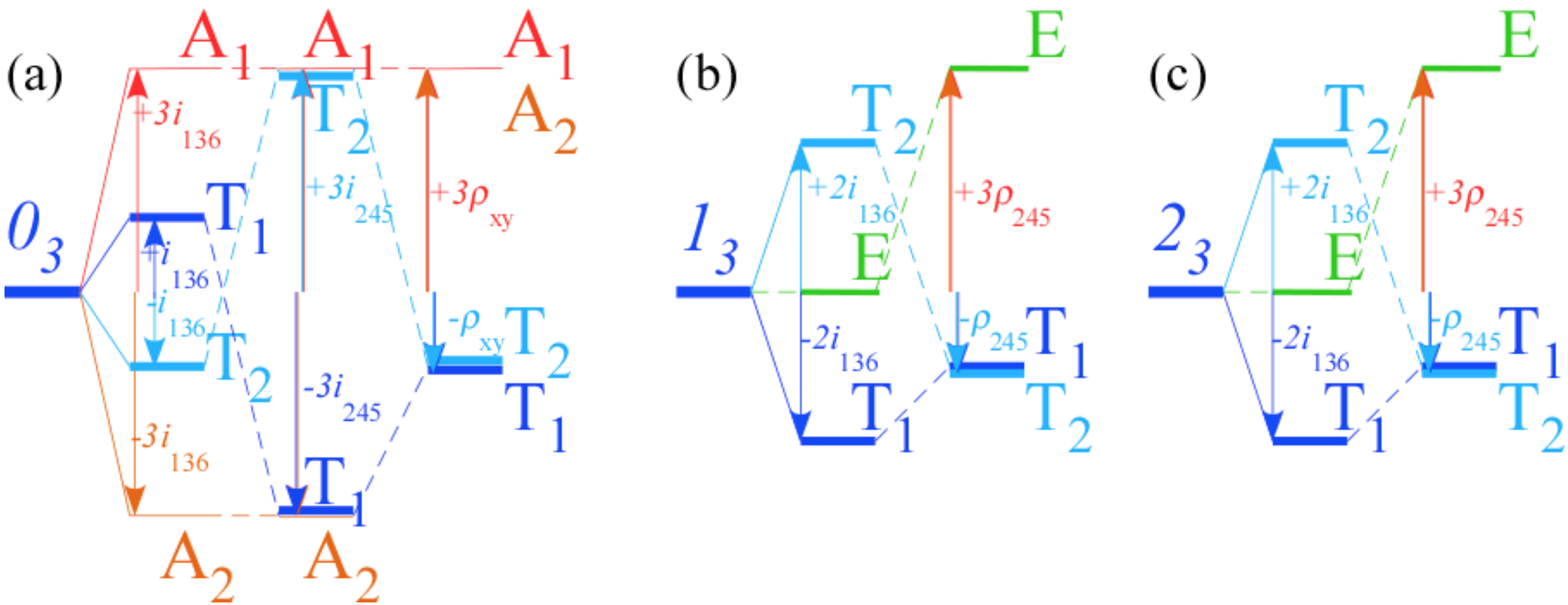
C_4 Projectors to split octahedral P^α

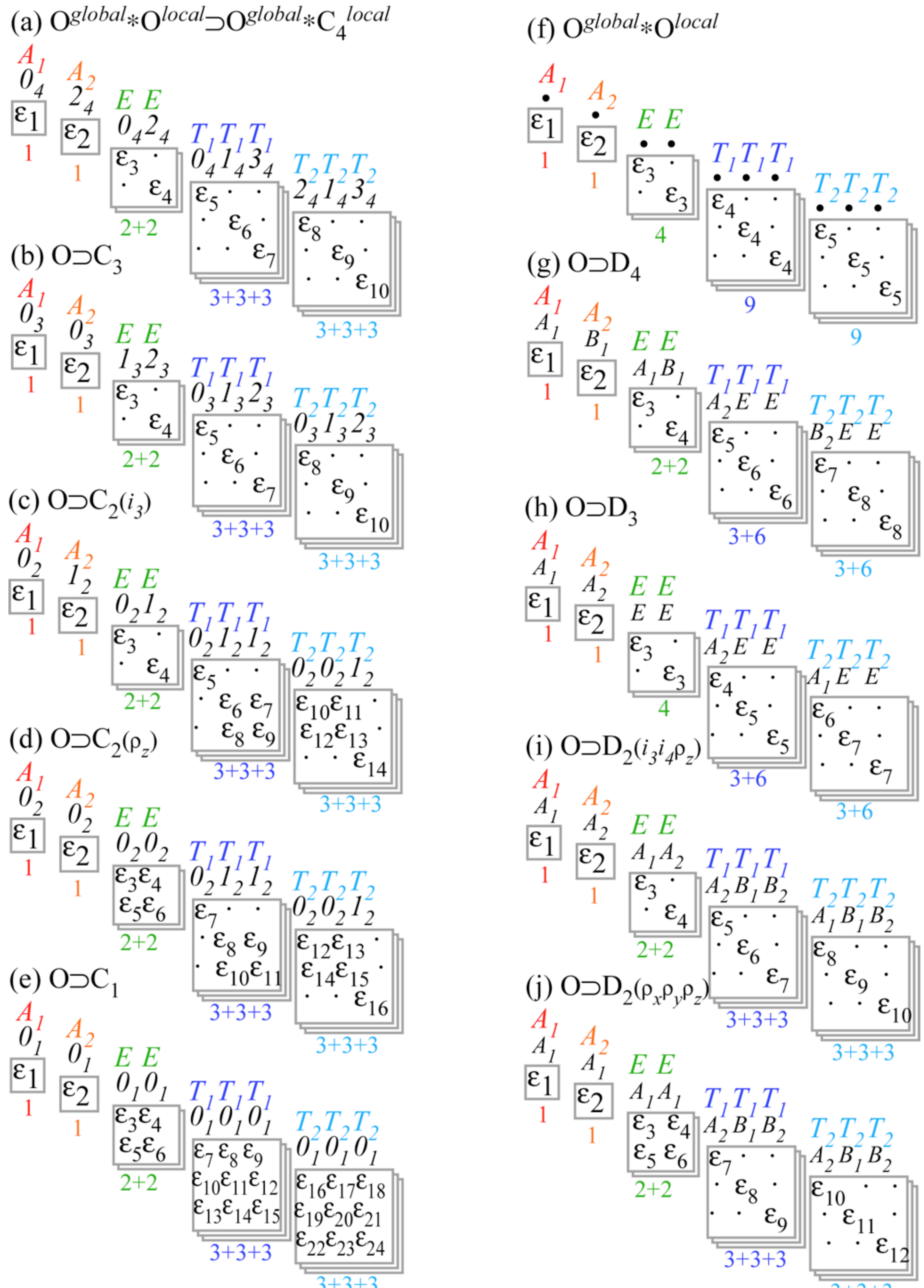
$$\mathbf{p}_{m_4}=\sum_{p=0}^{3}\frac{e^{2\pi i m\cdot p/4}}{4}\mathbf{R}_z^p=\begin{cases}\mathbf{p}_{0_4}=(\mathbf{1}+\mathbf{R}_z+\rho_z+\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{1_4}=(\mathbf{1}+i\mathbf{R}_z-\rho_z-i\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{2_4}=(\mathbf{1}-\mathbf{R}_z+\rho_z-\tilde{\mathbf{R}}_z)/4\\ \mathbf{p}_{3_4}=(\mathbf{1}-i\mathbf{R}_z-\rho_z+i\tilde{\mathbf{R}}_z)/4\end{cases}$$

$\mathbf{1}\cdot\mathbf{P}^\alpha=$	$(\mathbf{p}_{0_4}$	$+\mathbf{p}_{1_4}$	$+\mathbf{p}_{2_4}$	$+\mathbf{p}_{3_4})\cdot\mathbf{P}^\alpha$
$\mathbf{1}\cdot\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_40_4}$	$+0$	$+0$	$+0$
$\mathbf{1}\cdot\mathbf{P}^{A_2}=$	0	$+0$	$+\mathbf{P}^{A_2}_{2_42_4}$	$+0$
$\mathbf{1}\cdot\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_40_4}$	$+0$	$+\mathbf{P}^{E}_{2_42_4}$	$+0$
$\mathbf{1}\cdot\mathbf{P}^{T_1}=$	$\mathbf{P}^{T_1}_{0_40_4}$	$+\mathbf{P}^{T_1}_{1_41_4}$	$+0$	$+\mathbf{P}^{T_1}_{3_43_4}$
$\mathbf{1}\cdot\mathbf{P}^{T_2}=$	0	$+\mathbf{P}^{T_2}_{1_41_4}$	$+\mathbf{P}^{T_2}_{2_42_4}$	$+\mathbf{P}^{T_2}_{3_43_4}$

10 split $O\supset C_4$ octahedral P^α related to 10 split sub-classes

$\mathbf{P}^{(\alpha)}_{n_4n_4}(O\supset C_4)$	$\mathbf{1}$	$r_1r_2\tilde{r}_3\tilde{r}_4$	$\tilde{r}_1\tilde{r}_2r_3r_4$	$\rho_x\rho_y$	ρ_z	$R_x\tilde{R}_xR_y\tilde{R}_y$	R_z	$\tilde{R}_z$	$i_1i_2i_5i_6$	i_3i_4
$24\cdot\mathbf{P}^{A_1}_{0_40_4}$	1	1	1	1	1	1	1	1	1	1
$24\cdot\mathbf{P}^{A_2}_{2_42_4}$	1	1	1	1	1	-1	-1	-1	-1	-1
$12\cdot\mathbf{P}^{E}_{0_40_4}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	1
$12\cdot\mathbf{P}^{E}_{2_42_4}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$+\frac{1}{2}$	-1	-1	$+\frac{1}{2}$	-1
$8\cdot\mathbf{P}^{T_1}_{1_41_4}$	1	$-\frac{i}{2}$	$+\frac{i}{2}$	0	-1	$+\frac{1}{2}$	$-i$	$+i$	$-\frac{1}{2}$	0
$8\cdot\mathbf{P}^{T_1}_{3_43_4}$	1	$+\frac{i}{2}$	$-\frac{i}{2}$	0	-1	$+\frac{1}{2}$	$+i$	$-i$	$-\frac{1}{2}$	0
$8\cdot\mathbf{P}^{T_1}_{0_40_4}$	1	0	0	-1	1	0	1	1	0	-1
$8\cdot\mathbf{P}^{T_2}_{1_41_4}$	1	$+\frac{i}{2}$	$-\frac{i}{2}$	0	-1	$-\frac{1}{2}$	$-i$	$+i$	$+\frac{1}{2}$	0
$8\cdot\mathbf{P}^{T_2}_{3_43_4}$	1	$-\frac{i}{2}$	$+\frac{i}{2}$	0	-1	$-\frac{1}{2}$	$+i$	$-i$	$+\frac{1}{2}$	0
$8\cdot\mathbf{P}^{T_2}_{2_42_4}$	1	0	0	-1	1	0	-1	-1	0	1


(a)
A1
A1
A1
A2
+3i 136
0 3
T1
+i 136
-i 136
T2
T2
+3i 245
+3ρ xy
-ρ xy
T2
T1
-3i 245
-3i 136
T1
A2
A2
(b)
E
T2
1 3
+2i 136
E
+3ρ 245
-2i 136
-ρ 245
T1
T1
T2
(c)
E
T2
2 3
+2i 136
E
+3ρ 245
-2i 136
-ρ 245
T1
T1
T2


(a) O^global*O^local⊃O^global*C_4^local
(b) O⊃C_3
(c) O⊃C_2(i_3)
(d) O⊃C_2(ρ_z)
(e) O⊃C_1
(f) O^global*O^local
(g) O⊃D_4
(h) O⊃D_3
(i) O⊃D_2(i_3i_4ρ_z)
(j) O⊃D_2(ρ_xρ_yρ_z)
2+2
3+3+3
3+6
4
9

Effects of broken or transition local symmetry for i-class

$$D^{A_1}_{0_4 0_4}(i_k \mathbf{i}_k) = i_1 + i_2 + i_3 + i_4 + i_5 + i_6$$
$$D^{A_2}_{2_4 2_4}(i_k \mathbf{i}_k) = -(i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$$

$D^E(i_k \mathbf{i}_k) =$

	0_4	2_4
0_4	$-\frac{1}{2}(i_1 + i_2 + i_5 + i_6) + i_3 + i_4$	$\frac{\sqrt{3}}{2}(i_1 + i_2 - i_5 - i_6)$
2_4	$h.c.$	$\frac{1}{2}(i_1 + i_2 + i_5 + i_6) - i_3 - i_4$

$D^{T_1^*}(i_k \mathbf{i}_k)$	1_4	3_4	0_4
1_4	$-\frac{1}{2}(i_1 + i_2 + i_5 + i_6)$	$-\frac{1}{2}(i_1 + i_2 - i_5 - i_6) - i(i_3 - i_4)$	$-\frac{1}{\sqrt{2}}(i_1 - i_2) + \frac{i}{\sqrt{2}}(i_5 - i_6)$
3_4	$h.c.$	$-\frac{1}{2}(i_1 + i_2 + i_5 + i_6)$	$+\frac{1}{\sqrt{2}}(i_1 - i_2) + \frac{i}{\sqrt{2}}(i_5 - i_6)$
0_4	$h.c.$	$h.c.$	$-(i_3 + i_4)$

$D^{T_2^*}(i_k \mathbf{i}_k)$	1_4	3_4	2_4
1_4	$+\frac{1}{2}(i_1 + i_2 + i_5 + i_6)$	$+\frac{1}{2}(i_1 + i_2 - i_5 - i_6) - i(i_3 - i_4)$	$+\frac{1}{\sqrt{2}}(i_1 - i_2) + \frac{i}{\sqrt{2}}(i_5 - i_6)$
3_4	$h.c.$	$+\frac{1}{2}(i_1 + i_2 + i_5 + i_6)$	$-\frac{1}{\sqrt{2}}(i_1 - i_2) + \frac{i}{\sqrt{2}}(i_5 - i_6)$
0_4	$h.c.$	$h.c.$	$+(i_3 + i_4)$

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ *subgroup cosets:*

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Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Fundamental $\mathbf{P}^{\mu}_{m,n}$ *definitions:*

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

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$O_h \supset D_{4h} \supset C_{4v}$

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Hint: Sub-group chain factoring helps. Since $\mathbf{P}^{\mu}$ is *all-commuting*: $\mathbf{p}_{m_4}\mathbf{P}^{\mu} = \mathbf{P}^{\mu}_{m_4 m_4} = \mathbf{P}^{\mu}\mathbf{p}_{m_4}$

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This reduces to a smaller object $\mathbf{p}_{m_4}\mathbf{g}\mathbf{p}_{n_4}$ to calculate: $\mathbf{P}^{\mu}_{m_4 m_4}\mathbf{g}\mathbf{P}^{\mu}_{n_4 n_4} = \mathbf{P}^{\mu}\mathbf{p}_{m_4}\mathbf{g}\mathbf{p}_{n_4}$

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

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Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

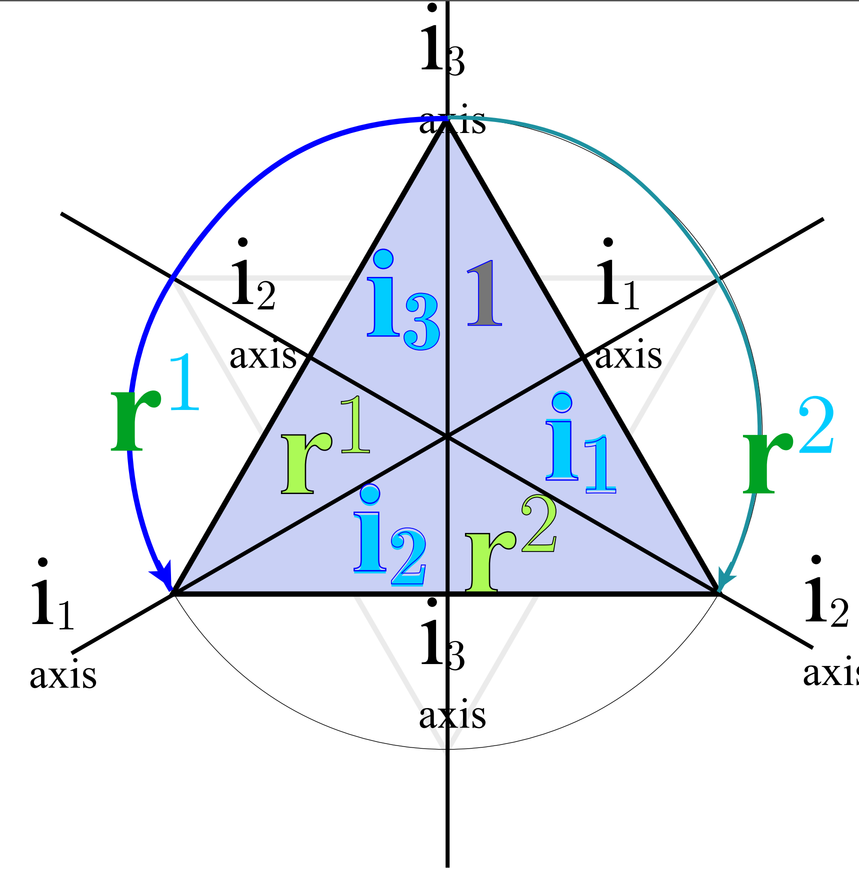
Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu *}_{mn}(\mathbf{g})\mathbf{g}$$

$D_3 : \chi^{\alpha}_{k}$	χ^{α}_{1}	χ^{α}_{r}	χ^{α}_{i}
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	-1
$\alpha = E$	2	-1	0

Given: $\mathbf{P}^{E} = \frac{1}{3}(2\mathbf{c}_1 - \mathbf{c}_r + 0)$

$$= \frac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)$$



Do $C_2 = \{\mathbf{1}, \mathbf{i}_3\}$ splitting:

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E}\mathbf{p}_{0_2} = \frac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\frac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$$

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Then find nilpotent proportional to: $\mathbf{P}^{E}_{1_2 0_2} = \mathbf{P}^{E}\mathbf{p}_{1_2}\mathbf{r}\mathbf{p}_{0_2}$

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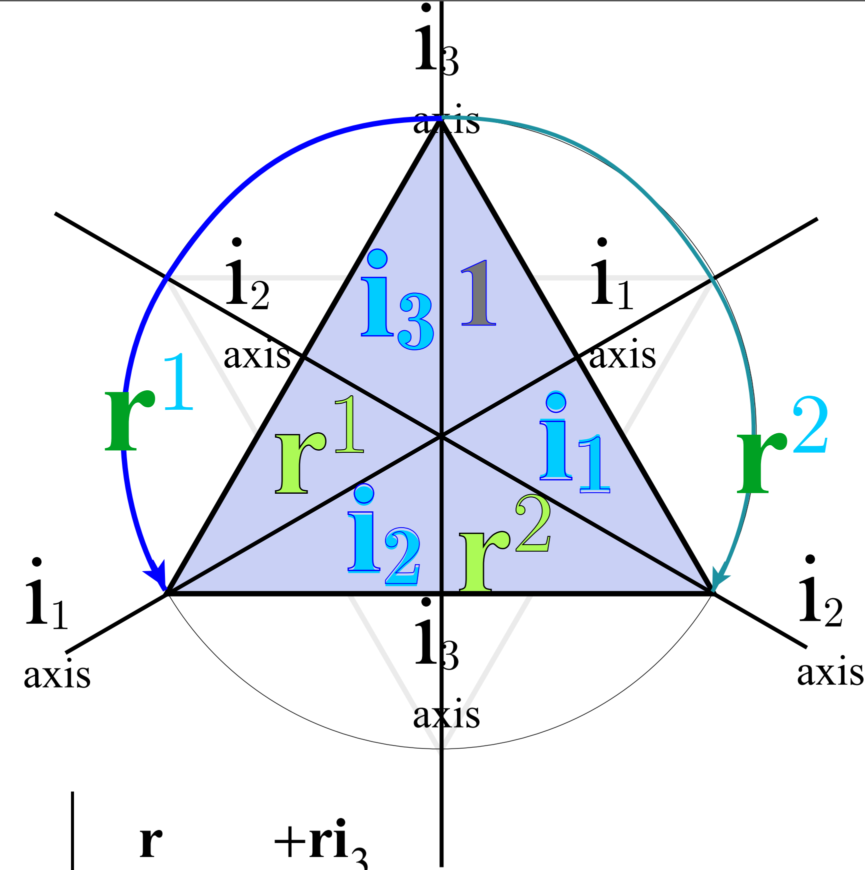

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	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

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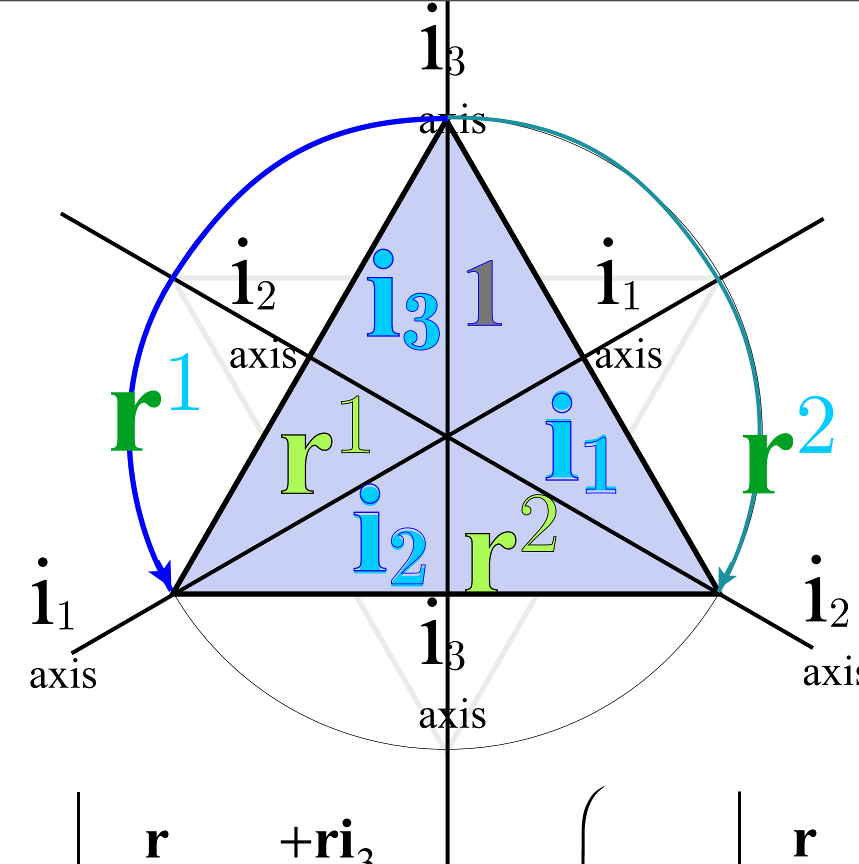

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Then find nilpotent proportional to: $\mathbf{P}^E_{1_2 0_2} = \mathbf{P}^E \mathbf{p}_{1_2} \mathbf{r} \mathbf{p}_{0_2} = \mathbf{P}^E \, \frac{1}{2}\cdot\frac{1}{2}$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

$= \frac{1}{4}\,\mathbf{P}^E$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{i}_2$
$-\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$

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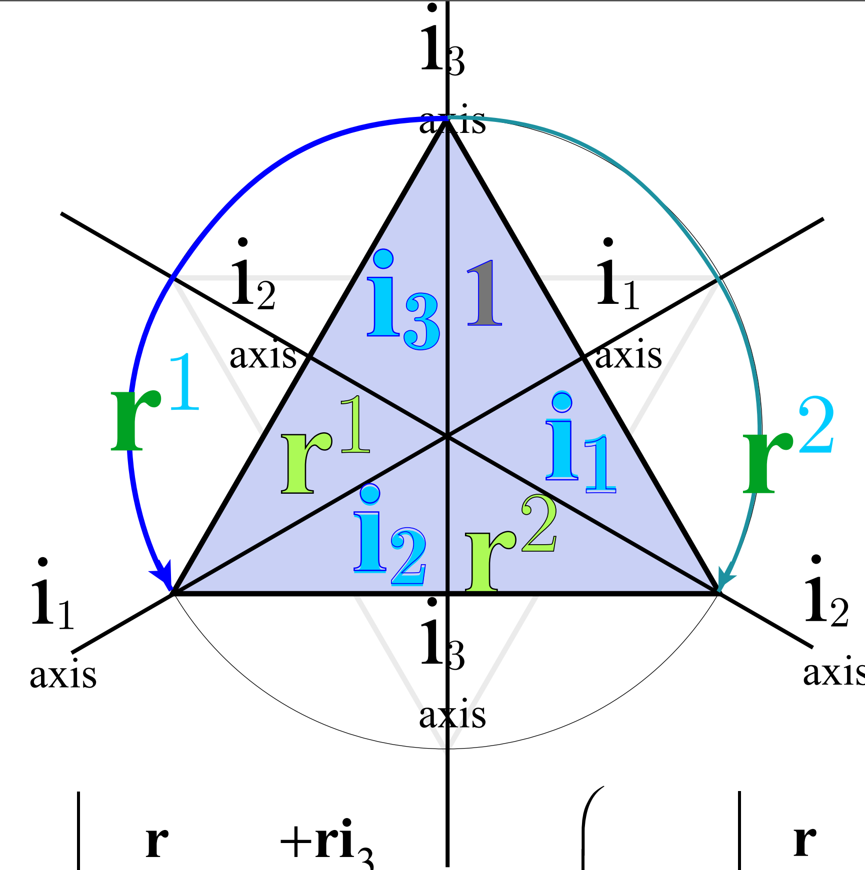

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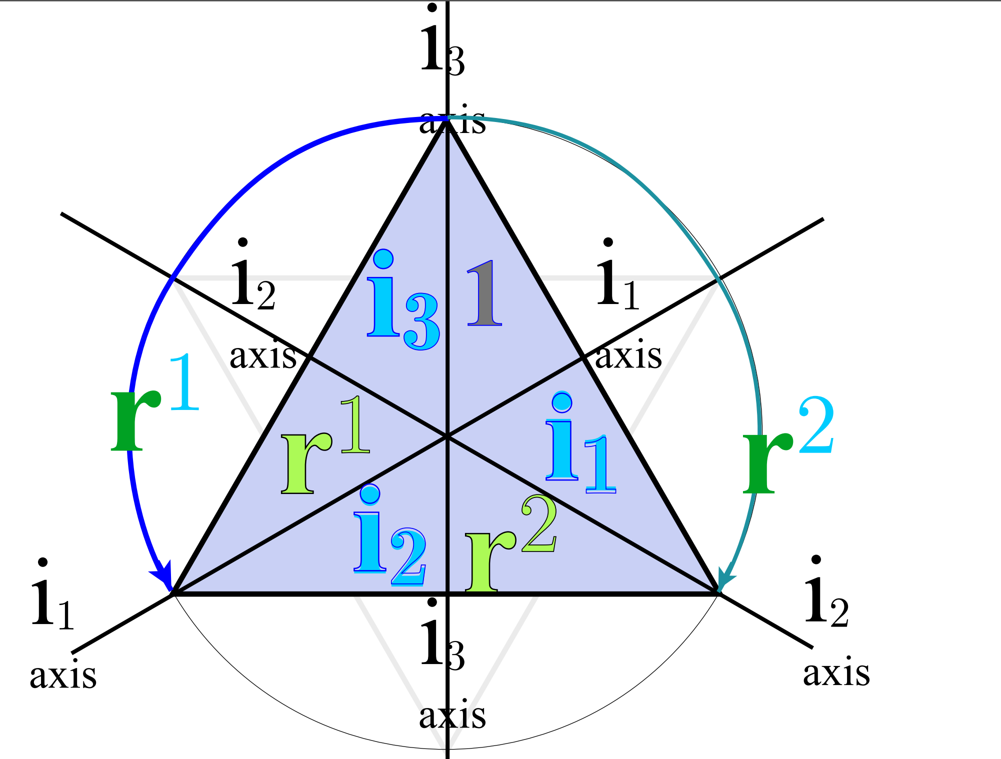

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$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

$= \frac{1}{4}\mathbf{P}^E$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{i}_2$
$-\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$

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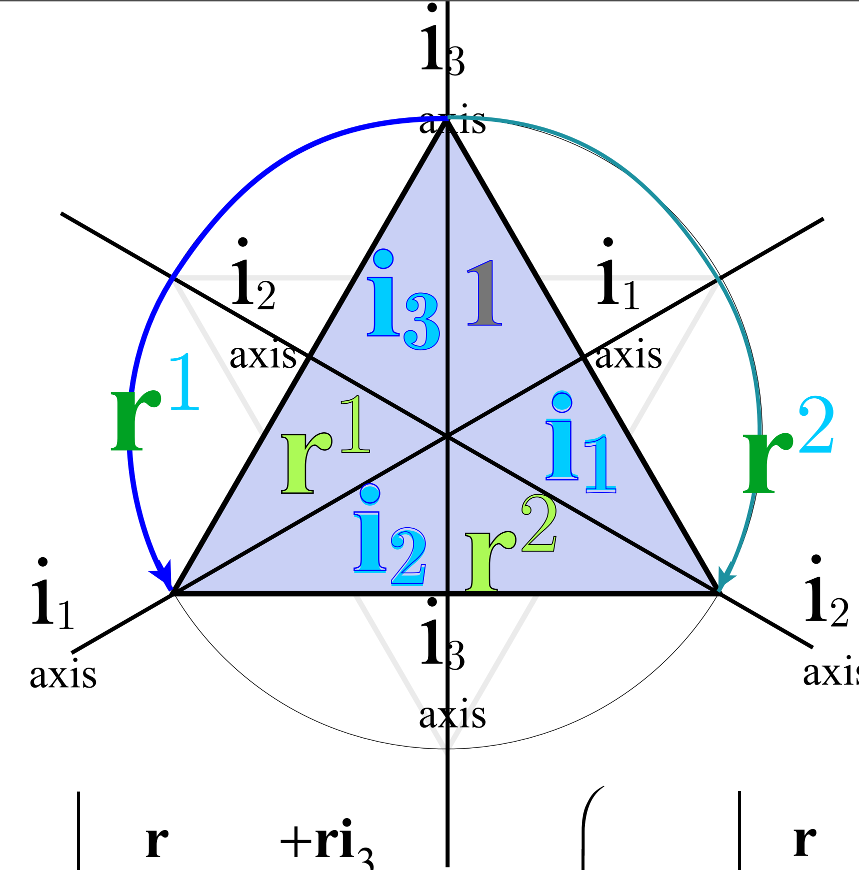

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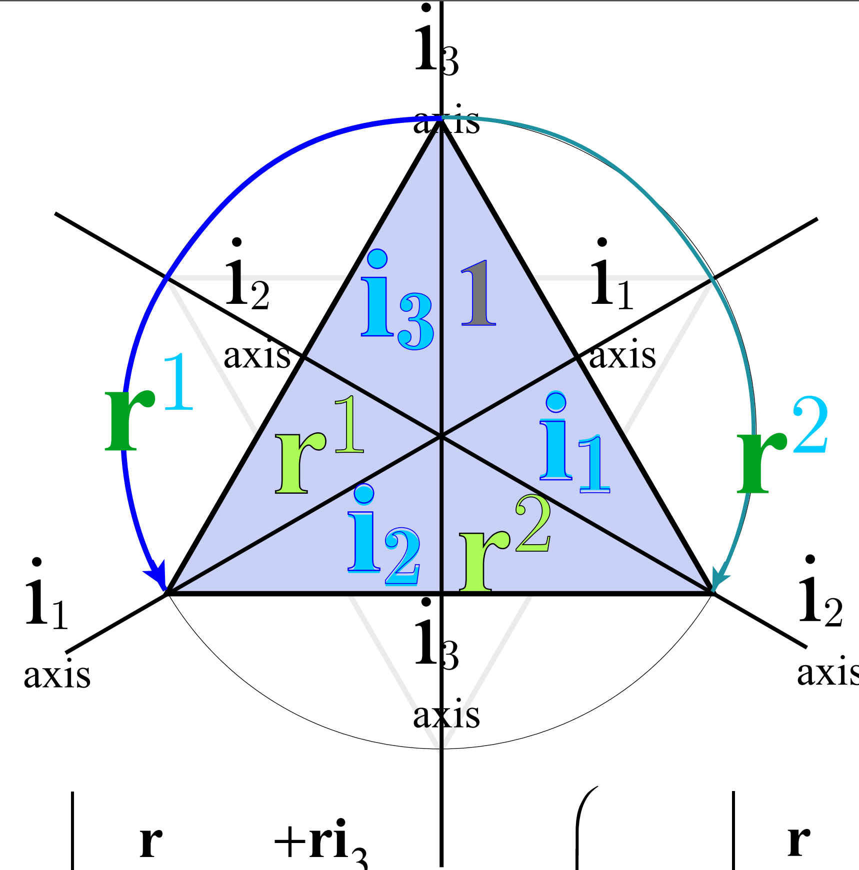

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$= \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$

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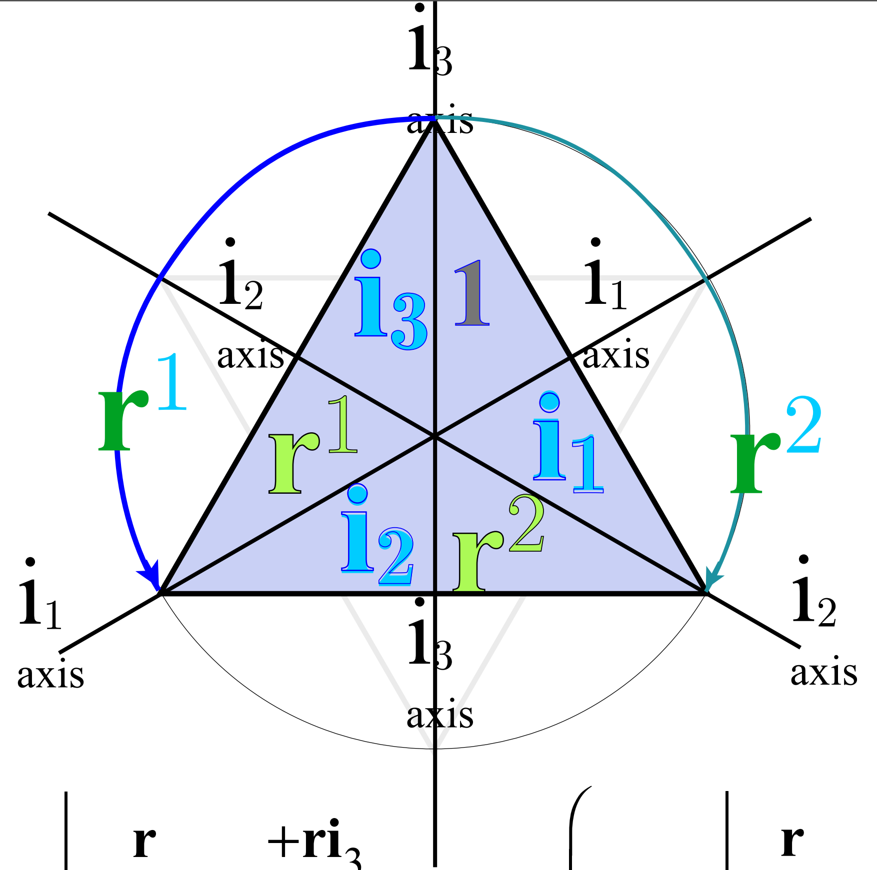

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$= \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$

$$\mathbf{P}^E_{0_2 0_2} = (?)^2 \cdot \left(\begin{array}{c|cccc} & +\mathbf{r} & -\mathbf{r}^2 & -\mathbf{i}_1 & +\mathbf{i}_2 \\ \hline +\mathbf{r}^2 & +\mathbf{1} & -\mathbf{r} & -\mathbf{i}_2 & +\mathbf{i}_3 \\ -\mathbf{r} & -\mathbf{r}^2 & +\mathbf{1} & +\mathbf{i}_3 & -\mathbf{i}_1 \\ -\mathbf{i}_1 & -\mathbf{i}_2 & +\mathbf{i}_3 & +\mathbf{1} & -\mathbf{r} \\ +\mathbf{i}_2 & +\mathbf{i}_3 & -\mathbf{i}_1 & -\mathbf{r}^2 & +\mathbf{1} \end{array}\right)$$

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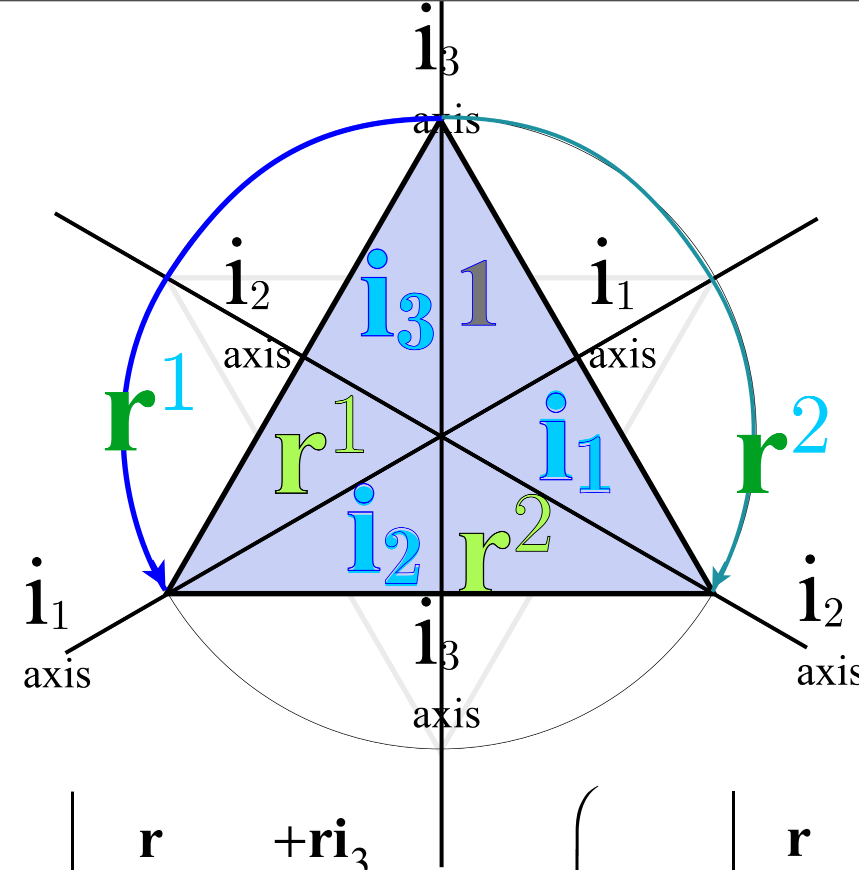

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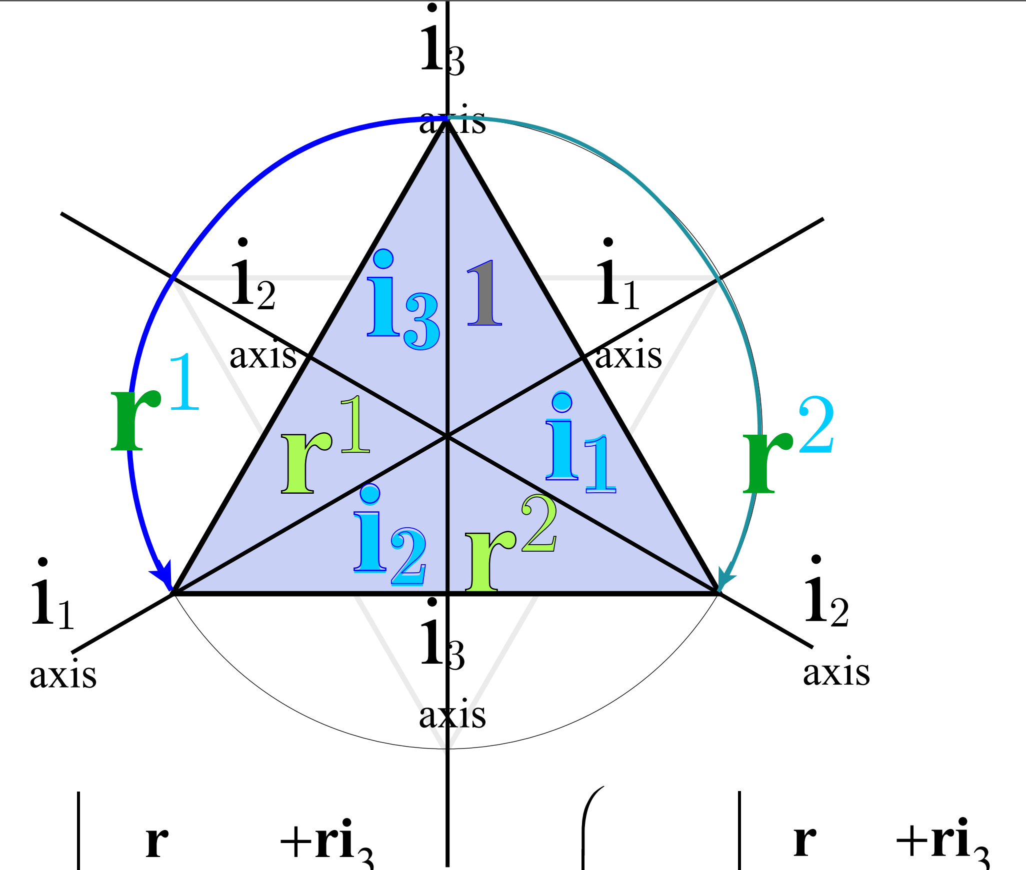

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$= \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$

$(?)^2\cdot 4 = \frac{1}{3}$

$\mathbf{P}^{E}_{0_2 0_2} = (?)^2\cdot$

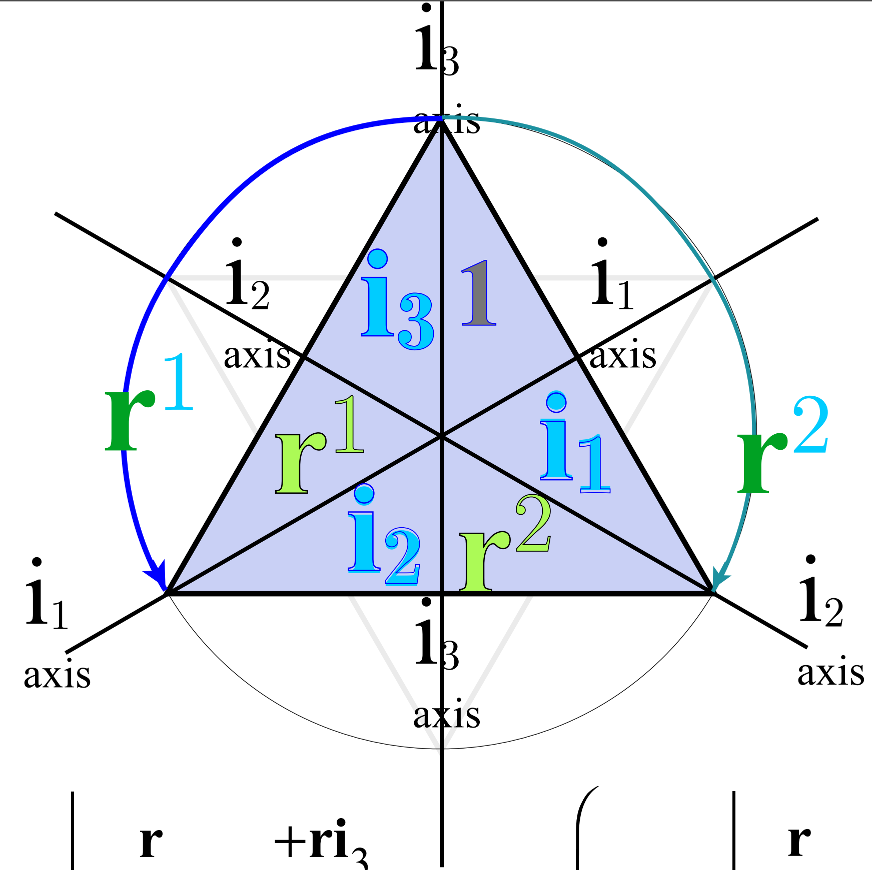
	$+\mathbf{r}$	$-\mathbf{r}^2$	$-\mathbf{i}_1$	$+\mathbf{i}_2$
$+\mathbf{r}^2$	$+\mathbf{1}$	$-\mathbf{r}$	$-\mathbf{i}_2$	$+\mathbf{i}_3$
$-\mathbf{r}$	$-\mathbf{r}^2$	$+\mathbf{1}$	$+\mathbf{i}_3$	$-\mathbf{i}_1$
$-\mathbf{i}_1$	$-\mathbf{i}_2$	$+\mathbf{i}_3$	$+\mathbf{1}$	$-\mathbf{r}$
$+\mathbf{i}_2$	$+\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$	$+\mathbf{1}$

$= (?)^2\cdot(+4\mathbf{1} - 2\mathbf{r} - 2\mathbf{r}^2 - 2\mathbf{i}_1 - 2\mathbf{i}_2 + 4\mathbf{i}_3)$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{\mu}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu *}_{mn}(\mathbf{g})\mathbf{g}$$



$D_3 : \chi^{\alpha}_{k}$	χ^{α}_{1}	χ^{α}_{r}	χ^{α}_{i}
$\alpha = A_1$	1	1	1
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Given: $\mathbf{P}^{E} = \frac{1}{3}(2\mathbf{c}_1 - \mathbf{c}_r + 0)$

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Do $C_2 = \{\mathbf{1}, \mathbf{i}_3\}$ splitting:

$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E}\mathbf{p}_{0_2} = \frac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\frac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$

$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E}\mathbf{p}_{1_2} = \frac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\frac{1}{2}(\mathbf{1} - \mathbf{i}_3) = \frac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)$

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	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

$= \frac{1}{4}\mathbf{P}^{E}$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{i}_2$
$-\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$

or: $\mathbf{P}^{E}_{0_2 1_2} = (?)\cdot(\mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 + \mathbf{i}_2)$

so: $\mathbf{P}^{E\dagger}_{0_2 1_2} = \mathbf{P}^{E}_{1_2 0_2} = (?)^{*}(\mathbf{r}^2 - \mathbf{r} - \mathbf{i}_1 + \mathbf{i}_2)$

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$(?)^2\cdot 4 = \frac{1}{3}$

$\mathbf{P}^{E}_{0_2 0_2} = (?)^2\cdot$

	$+\mathbf{r}$	$-\mathbf{r}^2$	$-\mathbf{i}_1$	$+\mathbf{i}_2$
$+\mathbf{r}^2$	$+\mathbf{1}$	$-\mathbf{r}$	$-\mathbf{i}_2$	$+\mathbf{i}_3$
$-\mathbf{r}$	$-\mathbf{r}^2$	$+\mathbf{1}$	$+\mathbf{i}_3$	$-\mathbf{i}_1$
$-\mathbf{i}_1$	$-\mathbf{i}_2$	$+\mathbf{i}_3$	$+\mathbf{1}$	$-\mathbf{r}$
$+\mathbf{i}_2$	$+\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$	$+\mathbf{1}$

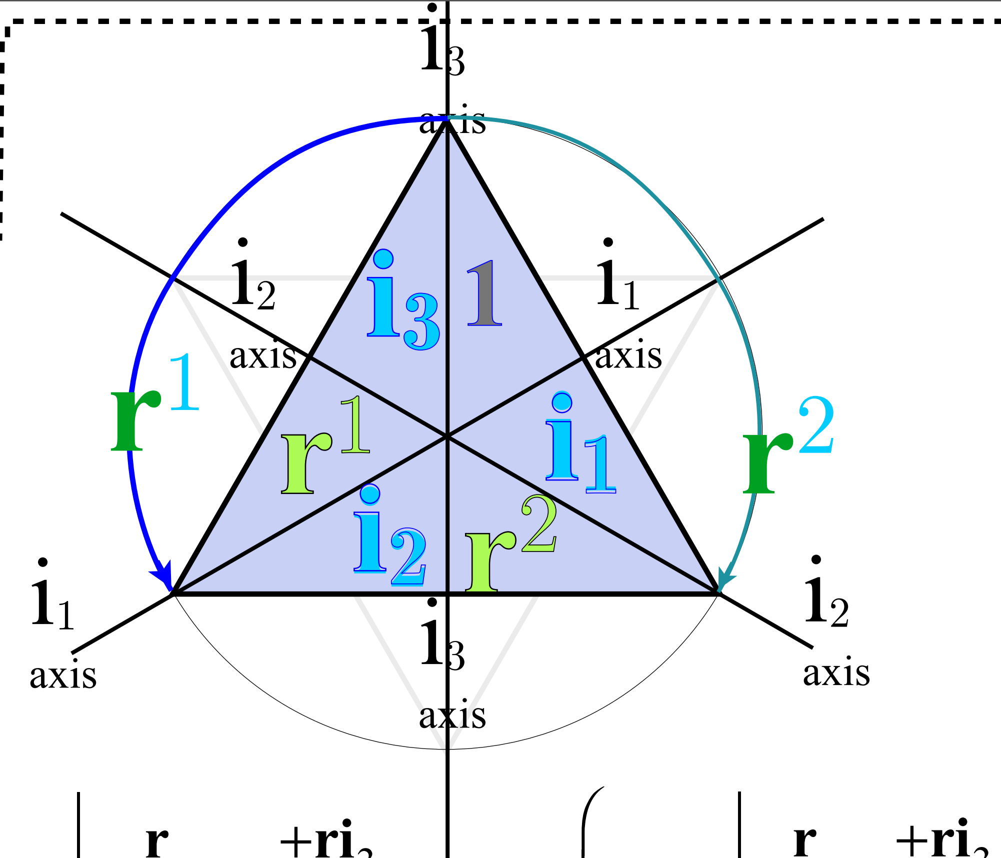
$= (?)^2\cdot(+4\mathbf{1} - 2\mathbf{r} - 2\mathbf{r}^2 - 2\mathbf{i}_1 - 2\mathbf{i}_2 + 4\mathbf{i}_3)$

Solving for (?)-factor: $(?) = \pm\sqrt{3}/6$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu *}_{mn}(\mathbf{g})\mathbf{g}$$



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$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E}\mathbf{p}_{0_2} = \tfrac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\tfrac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E}\mathbf{p}_{1_2} = \tfrac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\tfrac{1}{2}(\mathbf{1} - \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)$$

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	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

$= \frac{1}{4}\,\mathbf{P}^{E}$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{i}_2$
$-\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$

or: $\mathbf{P}^{E}_{0_2 1_2} = (?)\cdot(\mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 + \mathbf{i}_2)$

so: $\mathbf{P}^{E\dagger}_{0_2 1_2} = \mathbf{P}^{E}_{1_2 0_2} = (?)^*(\mathbf{r}^2 - \mathbf{r} - \mathbf{i}_1 + \mathbf{i}_2)$

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$\mathbf{P}^{E}_{0_2 0_2} = (?)^2\cdot$

	$+\mathbf{r}$	$-\mathbf{r}^2$	$-\mathbf{i}_1$	$+\mathbf{i}_2$
$+\mathbf{r}^2$	$+\mathbf{1}$	$-\mathbf{r}$	$-\mathbf{i}_2$	$+\mathbf{i}_3$
$-\mathbf{r}$	$-\mathbf{r}^2$	$+\mathbf{1}$	$+\mathbf{i}_3$	$-\mathbf{i}_1$
$-\mathbf{i}_1$	$-\mathbf{i}_2$	$+\mathbf{i}_3$	$+\mathbf{1}$	$-\mathbf{r}$
$+\mathbf{i}_2$	$+\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$	$+\mathbf{1}$

$(?)^2\cdot 4 = \frac{1}{3}$

$= (?)^2\cdot(+4\mathbf{1} - 2\mathbf{r} - 2\mathbf{r}^2 - 2\mathbf{i}_1 - 2\mathbf{i}_2 + 4\mathbf{i}_3)$

Solving for (?)-factor: $(?) = \pm\sqrt{3}/6$

$(?)^2\cdot 4 = \frac{1}{3} = \frac{1}{3}D^{E^*}_{0_2 0_2}(1)$

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Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

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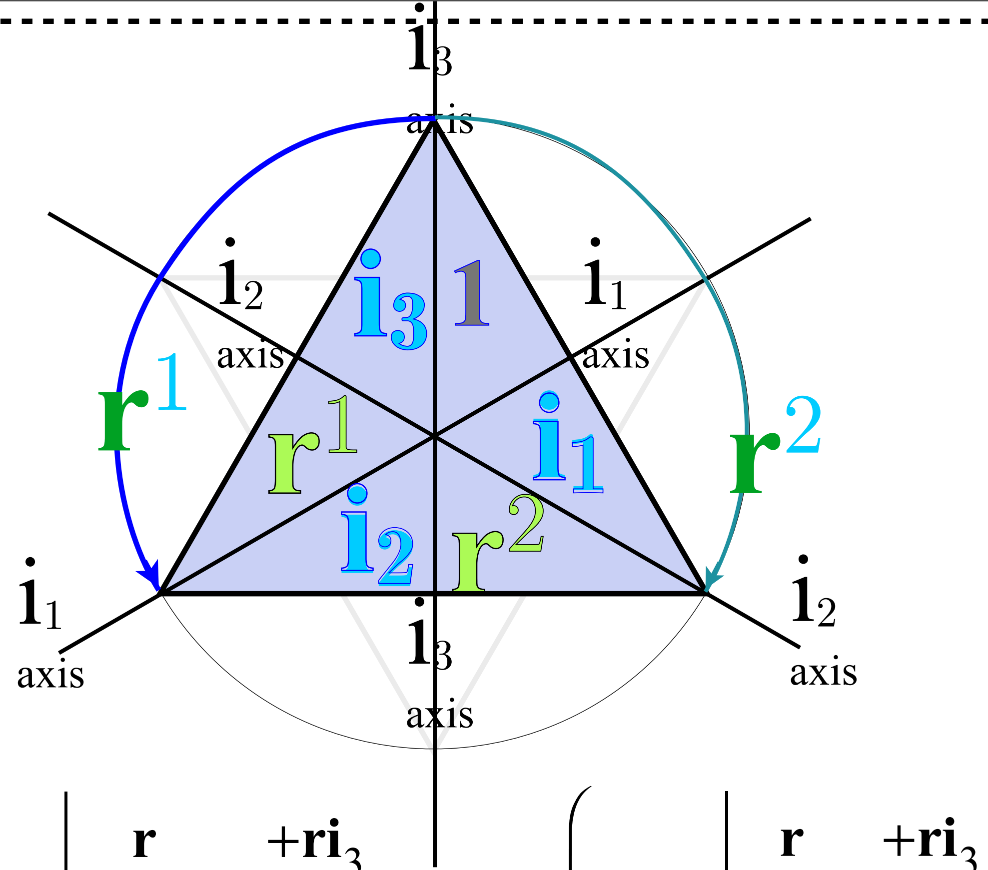

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or: $\mathbf{P}^{E}_{0_2 1_2} = (?)\cdot(\mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 + \mathbf{i}_2)$

so: $\mathbf{P}^{E\dagger}_{0_2 1_2} = \mathbf{P}^{E}_{1_2 0_2} = (?)^{*}(\mathbf{r}^2 - \mathbf{r} - \mathbf{i}_1 + \mathbf{i}_2)$

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$$\mathbf{P}^{E}_{0_2 0_2} = (?)^2\cdot\begin{pmatrix} & +\mathbf{r} & -\mathbf{r}^2 & -\mathbf{i}_1 & +\mathbf{i}_2 \\ +\mathbf{r}^2 & +\mathbf{1} & -\mathbf{r} & -\mathbf{i}_2 & +\mathbf{i}_3 \\ -\mathbf{r} & -\mathbf{r}^2 & +\mathbf{1} & +\mathbf{i}_3 & -\mathbf{i}_1 \\ -\mathbf{i}_1 & -\mathbf{i}_2 & +\mathbf{i}_3 & +\mathbf{1} & -\mathbf{r} \\ +\mathbf{i}_2 & +\mathbf{i}_3 & -\mathbf{i}_1 & -\mathbf{r}^2 & +\mathbf{1} \end{pmatrix} = (?)^2\cdot(+4\mathbf{1} - 2\mathbf{r} - 2\mathbf{r}^2 - 2\mathbf{i}_1 - 2\mathbf{i}_2 + 4\mathbf{i}_3)$$

$(?)^2\cdot 4 = \frac{1}{3}$

Solving for (?)-factor: $(?) = \pm\sqrt{3}/6$ Gives all $\mathbf{P}^{E}$

$(?)^2\cdot 4 = \frac{1}{3} = \frac{1}{3}D^{E*}_{0_2 0_2}(1)$

$$\mathbf{P}^{E}_{0_2 1_2} = \tfrac{1}{3}\left(\tfrac{\sqrt{3}}{2}\mathbf{r} - \tfrac{\sqrt{3}}{2}\mathbf{r}^2 - \tfrac{\sqrt{3}}{2}\mathbf{i}_1 + \tfrac{\sqrt{3}}{2}\mathbf{i}_2\right)$$

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Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

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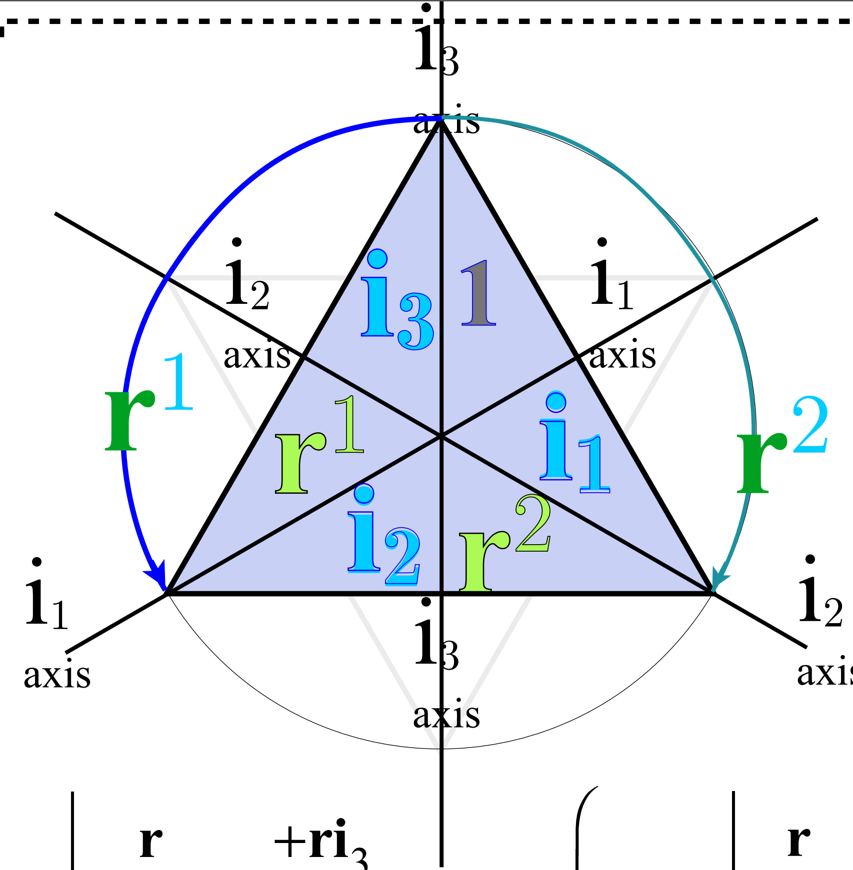

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	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$-\mathbf{i}_3$	$-\mathbf{i}_3\mathbf{r}$	$-\mathbf{i}_3\mathbf{r}\mathbf{i}_3$

$= \frac{1}{4}\mathbf{P}^{E}$

	$\mathbf{r}$	$+\mathbf{r}\mathbf{i}_3$
$\mathbf{1}$	$\mathbf{r}$	$+\mathbf{i}_2$
$-\mathbf{i}_3$	$-\mathbf{i}_1$	$-\mathbf{r}^2$

or: $\mathbf{P}^{E}_{0_2 1_2} = (?)\cdot(\mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 + \mathbf{i}_2)$

so: $\mathbf{P}^{E\dagger}_{0_2 1_2} = \mathbf{P}^{E}_{1_2 0_2} = (?)^{*}(\mathbf{r}^2 - \mathbf{r} - \mathbf{i}_1 + \mathbf{i}_2)$

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$\mathbf{P}^{E}_{0_2 0_2} = (?)^2 \cdot$

	$+\mathbf{r}$	$-\mathbf{r}^2$	$-\mathbf{i}_1$	$+\mathbf{i}_2$
$+\mathbf{r}^2$	$+\mathbf{1}$	$-\mathbf{r}$	$-\mathbf{i}_2$	$+\mathbf{i}_3$
$-\mathbf{r}$	$-\mathbf{r}^2$	$+\mathbf{1}$	$+\mathbf{i}_3$	$-\mathbf{i}_1$
$-\mathbf{i}_1$	$-\mathbf{i}_2$	$+\mathbf{i}_3$	$+\mathbf{1}$	$-\mathbf{r}$
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Solving for (?)-factor: $(?) = \pm\sqrt{3}/6$ Gives all $\mathbf{P}^E$ and D^E to $\pm$

$(?)^2\cdot 4 = \frac{1}{3} = \frac{1}{3} D^{E*}_{0_2 0_2}(1)$

$\pm\,\mathbf{P}^{E}_{0_2 1_2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\mathbf{r} - \frac{\sqrt{3}}{2}\mathbf{r}^2 - \frac{\sqrt{3}}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_2\right)$

$\pm\, D^{E*}_{0_2 1_2}(r) = \frac{\sqrt{3}}{2}$, *etc.*

$\underline{\pm}\mathbf{P}^{E}_{0_2 1_2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\mathbf{r} - \frac{\sqrt{3}}{2}\mathbf{r}^2 - \frac{\sqrt{3}}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_2\right)$ Now, to set ± signs of off-diagonal components...

$\underline{\pm}D^{E^*}_{0_2 1_2}(r) = \frac{\sqrt{3}}{2}, etc.$

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Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

$D_3 : \chi_k^{\alpha}$	χ_1^{α}	χ_r^{α}	χ_i^{α}
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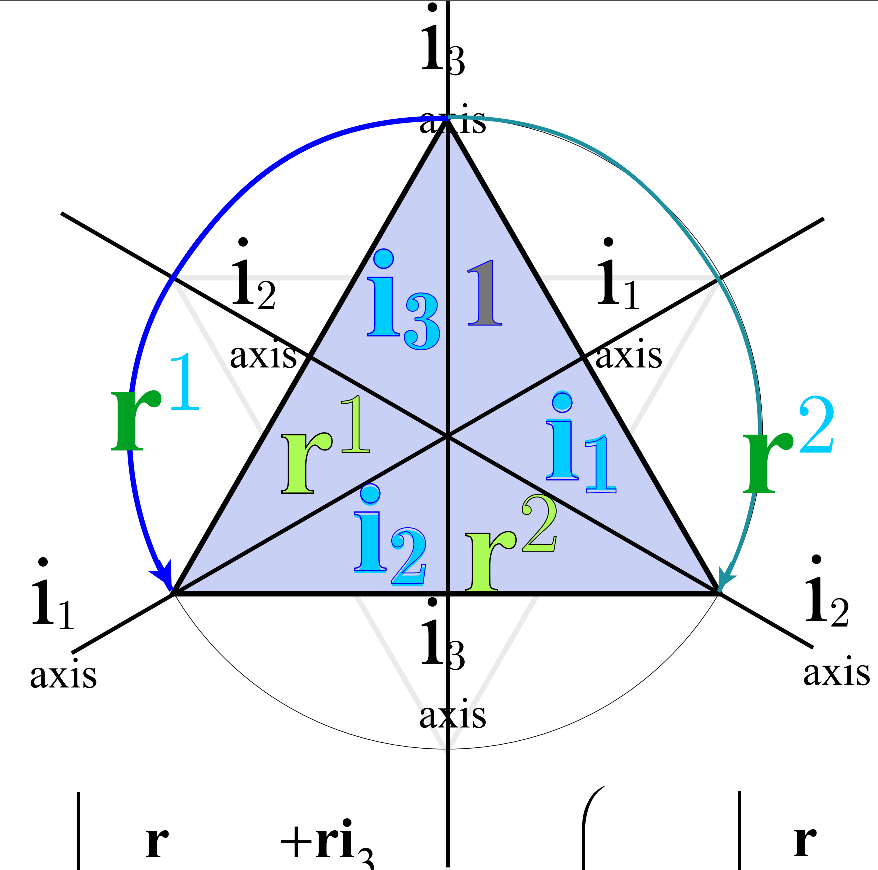

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$$\mathbf{P}^E_{0_2 0_2} = \mathbf{P}^E \mathbf{p}_{0_2} = \tfrac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\tfrac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$$

$$\mathbf{P}^E_{1_2 1_2} = \mathbf{P}^E \mathbf{p}_{1_2} = \tfrac{1}{3}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)\tfrac{1}{2}(\mathbf{1} - \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r} - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)$$

Then find nilpotent proportional to: $\mathbf{P}^E_{1_2 0_2} = \mathbf{P}^E \mathbf{p}_{1_2} \mathbf{r} \mathbf{p}_{0_2} = \mathbf{P}^E \frac{1}{2}\cdot\frac{1}{2} \left(\begin{array}{c|cc} & \mathbf{r} & +\mathbf{r}\mathbf{i}_3 \\ \hline \mathbf{1} & \mathbf{r} & +\mathbf{r}\mathbf{i}_3 \\ -\mathbf{i}_3 & -\mathbf{i}_3\mathbf{r} & -\mathbf{i}_3\mathbf{r}\mathbf{i}_3 \end{array} \right) = \frac{1}{4}\mathbf{P}^E \left(\begin{array}{c|cc} & \mathbf{r} & +\mathbf{r}\mathbf{i}_3 \\ \hline \mathbf{1} & \mathbf{r} & +\mathbf{i}_2 \\ -\mathbf{i}_3 & -\mathbf{i}_1 & -\mathbf{r}^2 \end{array} \right)$

$\pm\mathbf{P}^E_{0_2 1_2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\mathbf{r} - \frac{\sqrt{3}}{2}\mathbf{r}^2 - \frac{\sqrt{3}}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_2\right)$ Now, to set $\pm$ signs...

Make group space vectors:

$$\left|\mathbf{P}^E_{0_2 0_2}\right\rangle = \tfrac{1}{2\sqrt{3}}\left(2\left|\mathbf{1}\right\rangle - \left|\mathbf{r}\right\rangle - \left|\mathbf{r}^2\right\rangle - \left|\mathbf{i}_1\right\rangle - \left|\mathbf{i}_2\right\rangle + 2\left|\mathbf{i}_3\right\rangle\right)$$

$$\left|\mathbf{P}^E_{1_2 0_2}\right\rangle = \tfrac{1}{2}\left(0\left|\mathbf{1}\right\rangle + \left|\mathbf{r}\right\rangle - \left|\mathbf{r}^2\right\rangle - \left|\mathbf{i}_1\right\rangle + \left|\mathbf{i}_2\right\rangle + 0\left|\mathbf{i}_3\right\rangle\right)$$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}{}_{m,n}$

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

$D_3 : \chi_k^{\alpha}$	χ_1^{α}	χ_r^{α}	χ_i^{α}
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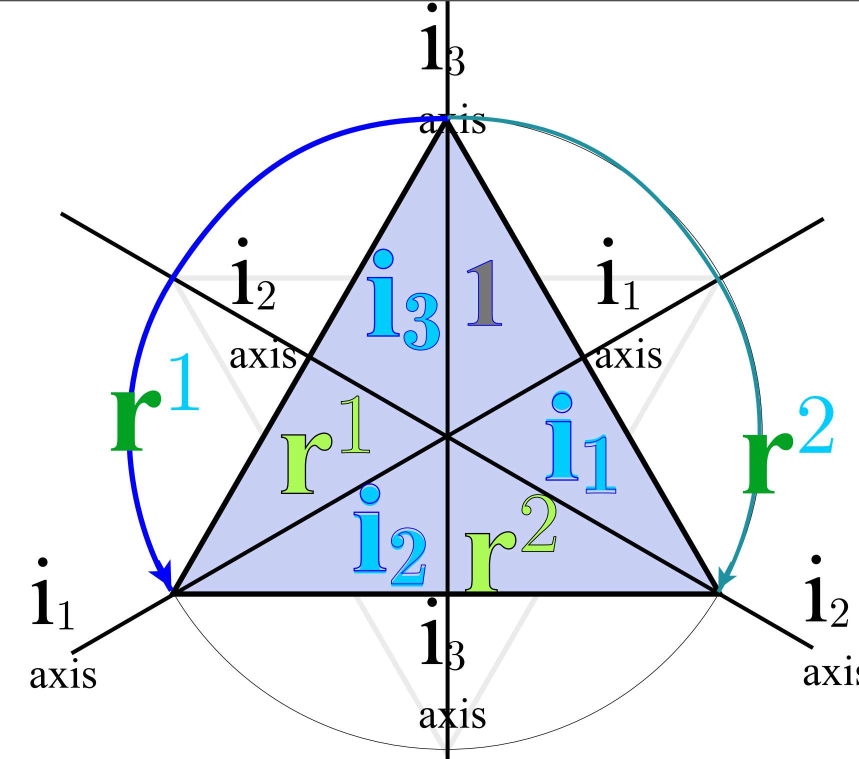

Do $C_2 = \{\mathbf{1}, \mathbf{i}_3\}$ splitting:

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Do desired $\mathbf{g}=\mathbf{r}$ transformation:

$$\mathbf{r}\left|\mathbf{P}^E_{0_2 0_2}\right\rangle = \tfrac{1}{2\sqrt{3}}(2|\mathbf{r}\rangle - |\mathbf{r}^2\rangle - |\mathbf{1}\rangle - |\mathbf{i}_3\rangle - |\mathbf{i}_1\rangle + 2|\mathbf{i}_2\rangle)$$

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Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

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$D_3 : \chi^\alpha_k$	χ^α_1	χ^α_r	χ^α_i
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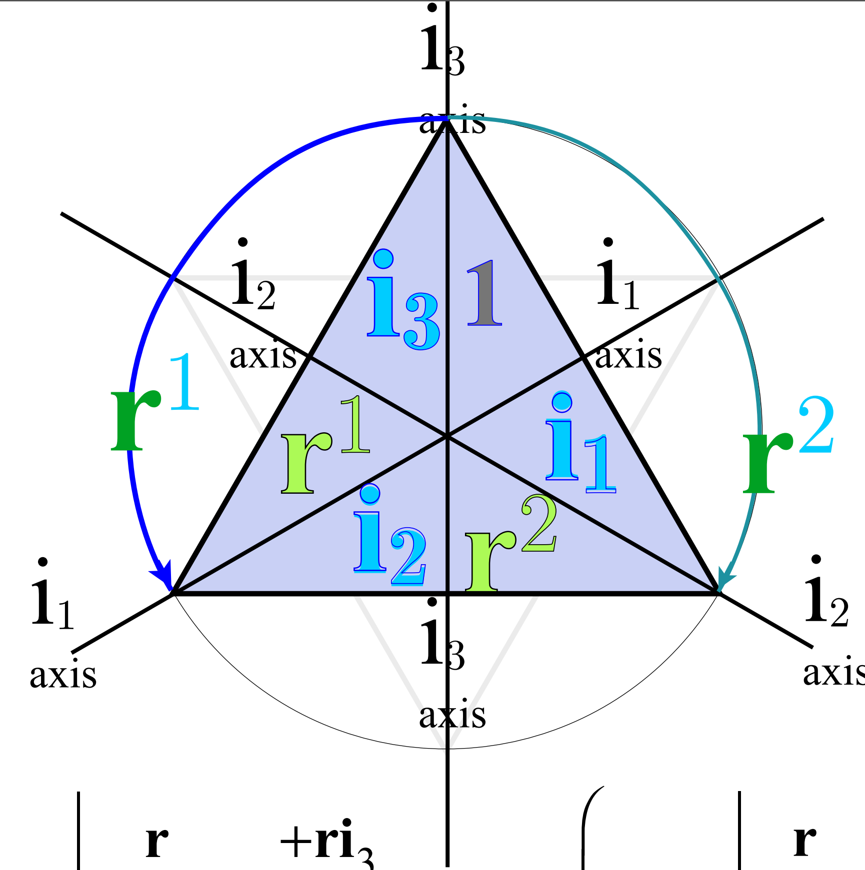

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Set up to find matrix of $\mathbf{g}=\mathbf{r}$ transformation:

$$\mathbf{r}\left|\mathbf{P}^E_{0_2 0_2}\right\rangle = \tfrac{1}{2\sqrt{3}}\left(-\left|\mathbf{1}\right\rangle + 2\left|\mathbf{r}\right\rangle - \left|\mathbf{r}^2\right\rangle - \left|\mathbf{i}_1\right\rangle + 2\left|\mathbf{i}_2\right\rangle - \left|\mathbf{i}_3\right\rangle\right)$$

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Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

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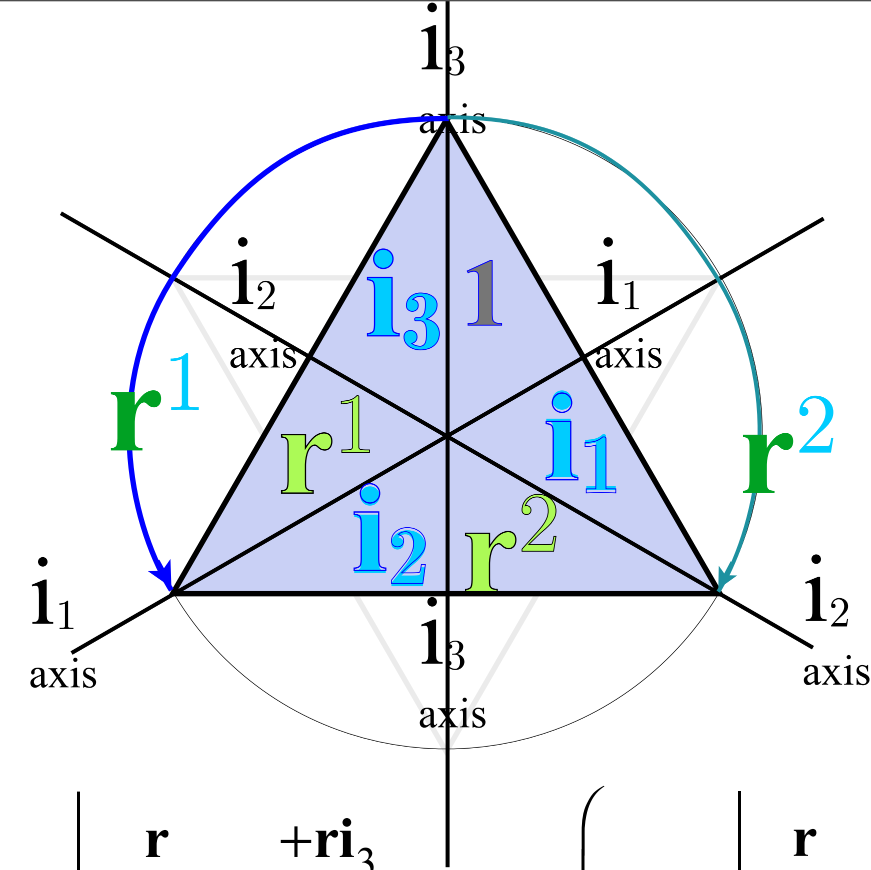

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$$\left\langle\mathbf{P}^E_{1_2 0_2}\right|\mathbf{r}\left|\mathbf{P}^E_{0_2 0_2}\right\rangle = \frac{1}{2}(0+1-1-1+1+0)\cdot\frac{1}{2\sqrt{3}}(-1+2-1-1+2-1) = \sqrt{3}/2$$

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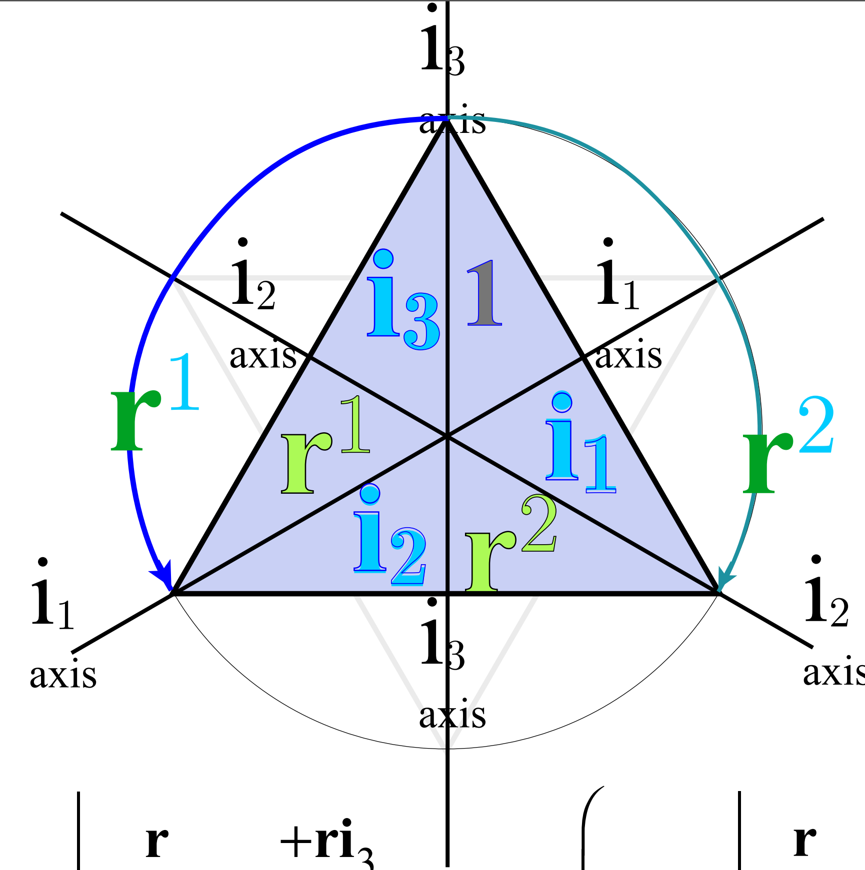
Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$



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The D_{01} ± sign was (-).

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ *subgroup cosets:*

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Fundamental $\mathbf{P}^{\mu}_{m,n}$ *definitions:*

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

➡ *Calculating and Factoring* $\mathbf{P}^{T_1}_{1_4 0_4}$ ⬅

Structure and applications of various subgroup chain ireps

$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Coset-factored T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4} + (0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4} + (0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4} + (-1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Coset-factored T₁*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4} + (0)\cdot\rho_x\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4} + (0)\cdot\rho_x\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4} + (-1)\cdot\rho_x\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Coset-factored T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4} + (0)\cdot\rho_x\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4} + (0)\cdot\rho_x\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4} + (-1)\cdot\rho_x\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4} =$

	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$-\rho_z$	$-\mathbf{r}_3$	$-\mathbf{r}_2$	$-\tilde{\mathbf{R}}_y$	$-\mathbf{i}_2$
$+i\mathbf{R}_z$	$+i\mathbf{i}_6$	$+i\tilde{\mathbf{R}}_x$	$+i\tilde{\mathbf{r}}_1$	$+i\tilde{\mathbf{r}}_3$
$-i\tilde{\mathbf{R}}_z$	$-i\mathbf{R}_x$	$-i\mathbf{i}_5$	$-i\tilde{\mathbf{r}}_4$	$-i\tilde{\mathbf{r}}_2$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}{}_{m,n}$

Coset-factored T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{1_4} \quad +(0)\cdot\rho_x\mathbf{p}_{1_4} \quad +(+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} \quad +(+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} \quad +(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} \quad +(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{3_4} \quad +(0)\cdot\rho_x\mathbf{p}_{3_4} \quad +(-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} \quad +(-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} \quad +(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} \quad +(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{0_4} \quad +(-1)\cdot\rho_x\mathbf{p}_{0_4} \quad +(0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} \quad +(0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} \quad +(0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} \quad +(0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4} =$

	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$-\rho_z$	$-\mathbf{r}_3$	$-\mathbf{r}_2$	$-\tilde{\mathbf{R}}_y$	$-\mathbf{i}_2$
$+i\mathbf{R}_z$	$+i\mathbf{i}_6$	$+i\tilde{\mathbf{R}}_x$	$+i\tilde{\mathbf{r}}_1$	$+i\tilde{\mathbf{r}}_3$
$-i\tilde{\mathbf{R}}_z$	$-i\mathbf{R}_x$	$-i\mathbf{i}_5$	$-i\tilde{\mathbf{r}}_4$	$-i\tilde{\mathbf{r}}_2$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}{}_{m,n}$

Coset-factored T1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{1_4} \quad + (0)\cdot\rho_x\mathbf{p}_{1_4} \quad + (+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} \quad + (+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} \quad + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} \quad + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{3_4} \quad + (0)\cdot\rho_x\mathbf{p}_{3_4} \quad + (-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} \quad + (-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} \quad + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} \quad + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{0_4} \quad + (-1)\cdot\rho_x\mathbf{p}_{0_4} \quad + (0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} \quad + (0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} \quad + (0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} \quad + (0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4} =$

	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$-\rho_z$	$-\mathbf{r}_3$	$-\mathbf{r}_2$	$-\tilde{\mathbf{R}}_y$	$-\mathbf{i}_2$
$+i\mathbf{R}_z$	$+i\mathbf{i}_6$	$+i\tilde{\mathbf{R}}_x$	$+i\tilde{\mathbf{r}}_1$	$+i\tilde{\mathbf{r}}_3$
$-i\tilde{\mathbf{R}}_z$	$-i\mathbf{R}_x$	$-i\mathbf{i}_5$	$-i\tilde{\mathbf{r}}_4$	$-i\tilde{\mathbf{r}}_2$

$$= (\mathbf{r}_1 + \mathbf{r}_4 + \mathbf{i}_1 + \mathbf{R}_y) - (\mathbf{r}_2 + \mathbf{r}_3 + \mathbf{i}_2 + \tilde{\mathbf{R}}_y) + i(\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_3 + \tilde{\mathbf{R}}_x + \mathbf{i}_6) - i(\tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_4 + \mathbf{R}_x + \mathbf{i}_5)$$

$$= \mathbf{r}_1\mathbf{p}_{0_4} - \mathbf{r}_2\mathbf{p}_{0_4} + i\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} - i\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}$$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Coset-factored T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{1_4} \quad +(0)\cdot\rho_x\mathbf{p}_{1_4} \quad +(+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} \quad +(+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} \quad +(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} \quad +(-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{3_4} \quad +(0)\cdot\rho_x\mathbf{p}_{3_4} \quad +(-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} \quad +(-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} \quad +(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} \quad +(+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1}\mathbf{p}_{0_4} \quad +(-1)\cdot\rho_x\mathbf{p}_{0_4} \quad +(0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} \quad +(0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} \quad +(0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} \quad +(0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{0_4 0_4} = D^{T_1}_{1_4 0_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4} =$

	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_4$	$\mathbf{i}_1$	$\mathbf{R}_y$
$-\rho_z$	$-\mathbf{r}_3$	$-\mathbf{r}_2$	$-\tilde{\mathbf{R}}_y$	$-\mathbf{i}_2$
$+i\mathbf{R}_z$	$+i\mathbf{i}_6$	$+i\tilde{\mathbf{R}}_x$	$+i\tilde{\mathbf{r}}_1$	$+i\tilde{\mathbf{r}}_3$
$-i\tilde{\mathbf{R}}_z$	$-i\mathbf{R}_x$	$-i\mathbf{i}_5$	$-i\tilde{\mathbf{r}}_4$	$-i\tilde{\mathbf{r}}_2$

$$= (\mathbf{r}_1+\mathbf{r}_4+\mathbf{i}_1+\mathbf{R}_y) - (\mathbf{r}_2+\mathbf{r}_3+\mathbf{i}_2+\tilde{\mathbf{R}}_y) + i(\tilde{\mathbf{r}}_1+\tilde{\mathbf{r}}_3+\tilde{\mathbf{R}}_x+\mathbf{i}_6) - i(\tilde{\mathbf{r}}_2+\tilde{\mathbf{r}}_4+\mathbf{R}_x+\mathbf{i}_5)$$

$$= \mathbf{r}_1\mathbf{p}_{0_4} - \mathbf{r}_2\mathbf{p}_{0_4} + i\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} - i\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}$$

Result is nicely factored:

$$\mathbf{P}^{T_1}_{1_4 0_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{0_4} \sim (?)\cdot(\mathbf{r}_1\mathbf{p}_{0_4} - \mathbf{r}_2\mathbf{p}_{0_4} + i\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} - i\tilde{\mathbf{r}}_2\mathbf{p}_{0_4})$$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Coset-factored T_1*-sum:*

$$\mathbf{P}^{T_1}_{1_4 1_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{1_4} + (0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{1_4} + (+\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{1_4} + (-\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{1_4}]$$

$$\mathbf{P}^{T_1}_{3_4 3_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{3_4} + (0)\cdot\boldsymbol{\rho}_x\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_1\mathbf{p}_{3_4} + (-\tfrac{i}{2})\cdot\mathbf{r}_2\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + (+\tfrac{i}{2})\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{3_4}]$$

$$\mathbf{P}^{T_1}_{0_4 0_4} = \tfrac{1}{8}[(1)\cdot\mathbf{1p}_{0_4} + (-1)\cdot\boldsymbol{\rho}_x\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_1\mathbf{p}_{0_4} + (0)\cdot\mathbf{r}_2\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_1\mathbf{p}_{0_4} + (0)\cdot\tilde{\mathbf{r}}_2\mathbf{p}_{0_4}]$$

Calculating: $\mathbf{P}^{T_1}_{1_4 1_4}\mathbf{r}_1\mathbf{P}^{T_1}_{3_4 3_4} = D^{T_1}_{1_4 3_4}(\mathbf{r}_1)\mathbf{P}^{T_1}_{1_4 3_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{3_4}$

Then find nilpotent proportional to: $\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{3_4} =$

	$\mathbf{r}_1$	$-\mathbf{r}_4$	$-i\mathbf{i}_1$	$+i\mathbf{R}_y$
$\mathbf{1}$	$\mathbf{r}_1$	$-\mathbf{r}_4$	$-i\mathbf{i}_1$	$+i\mathbf{R}_y$
$-\boldsymbol{\rho}_z$	$-\mathbf{r}_3$	$+\mathbf{r}_2$	$+i\tilde{\mathbf{R}}_y$	$-i\mathbf{i}_2$
$+i\mathbf{R}_z$	$+i\mathbf{i}_6$	$-i\tilde{\mathbf{R}}_x$	$+\tilde{\mathbf{r}}_1$	$-\tilde{\mathbf{r}}_3$
$-i\tilde{\mathbf{R}}_z$	$-i\mathbf{R}_x$	$+i\mathbf{i}_5$	$-\tilde{\mathbf{r}}_4$	$+\tilde{\mathbf{r}}_2$

$$= (\mathbf{r}_1 - \mathbf{r}_4 - i\mathbf{i}_1 + i\mathbf{R}_y) + (\mathbf{r}_2 - \mathbf{r}_3 - i\mathbf{i}_2 + i\tilde{\mathbf{R}}_y) + (\tilde{\mathbf{r}}_1 - \tilde{\mathbf{r}}_3 - i\tilde{\mathbf{R}}_x + i\mathbf{i}_6) + (\tilde{\mathbf{r}}_2 - \tilde{\mathbf{r}}_4 - i\mathbf{R}_x + i\mathbf{i}_5)$$

$$= \mathbf{r}_1\mathbf{p}_{3_4} + \mathbf{r}_2\mathbf{p}_{3_4} + \tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + \tilde{\mathbf{r}}_2\mathbf{p}_{3_4}$$

Result is nicely factored:

$$\mathbf{P}^{T_1}_{1_4 3_4} = \mathbf{P}^{T_1}\mathbf{p}_{1_4}\mathbf{r}_1\mathbf{p}_{3_4} \sim (?)\cdot(\mathbf{r}_1\mathbf{p}_{3_4} + \mathbf{r}_2\mathbf{p}_{3_4} + \tilde{\mathbf{r}}_1\mathbf{p}_{3_4} + \tilde{\mathbf{r}}_2\mathbf{p}_{3_4})$$

Review Coset factored splitting of $O \supset D_4 \supset C_4$ *projectors and levels*

Irreducible idempotent projectors $\mathbf{P}^{\mu}_{m,m}$ *of* $O \supset C_4 \sim T_d \supset C_{4i}$

Calculating $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Factoring out $O \supset C_4$ *subgroup cosets:*

Factoring $\mathbf{P}^{E}_{0_4 0_4}$ $\mathbf{P}^{E}_{2_4 2_4}$ $\mathbf{P}^{T_1}_{0_4 0_4}$ $\mathbf{P}^{T_1}_{1_4 1_4}$ $\mathbf{P}^{T_2}_{2_4 2_4}$

Irreducible nilpotent projectors $\mathbf{P}^{\mu}_{m,n}$

Fundamental $\mathbf{P}^{\mu}_{m,n}$ *definitions:*

Review of $D_3 \supset C_2 \sim C_{3v} \supset C_v$

Calculating and Factoring $\mathbf{P}^{T_1}_{1_4 0_4}$

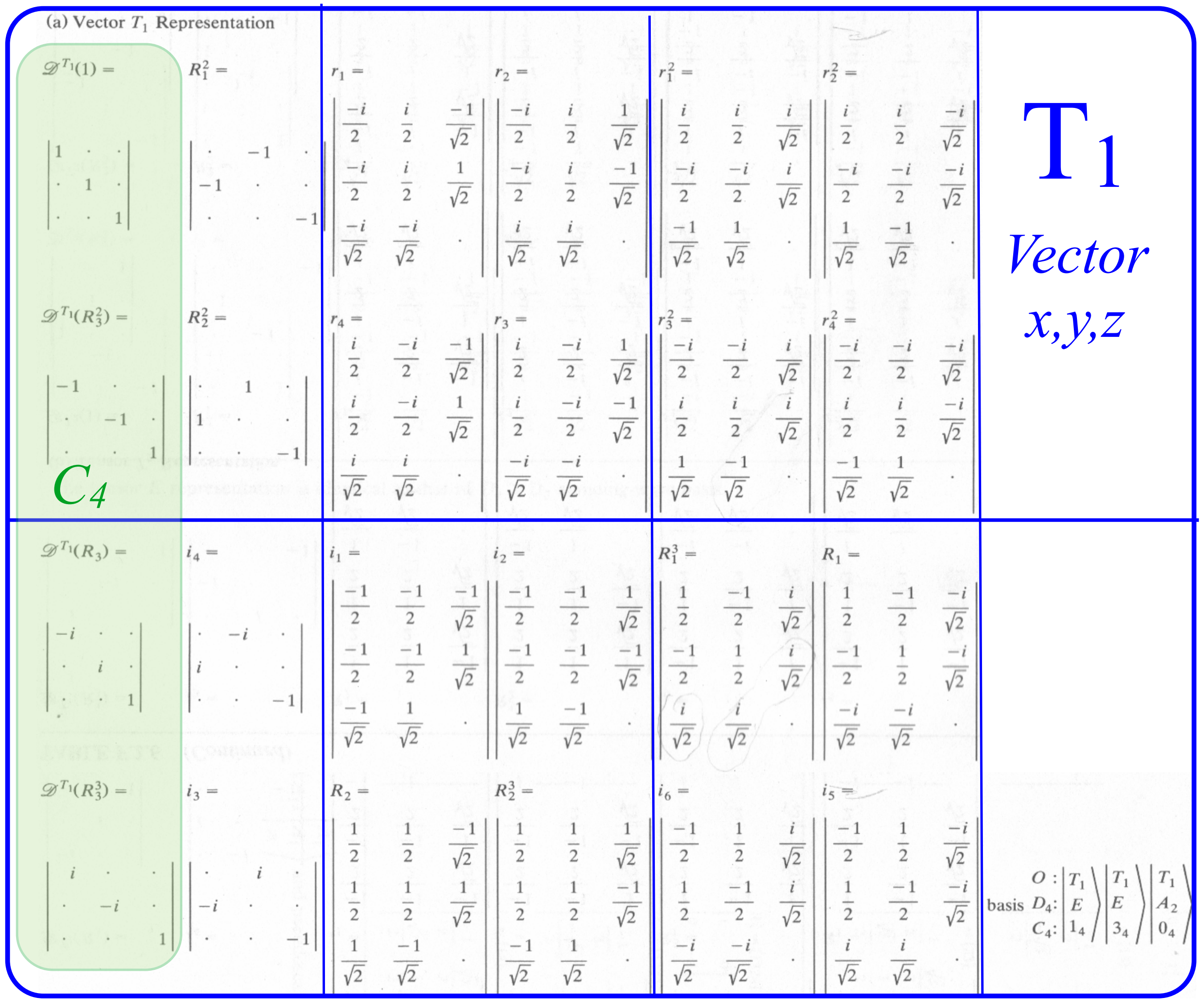
➔ *Structure and applications of various subgroup chain ireps* ⟵

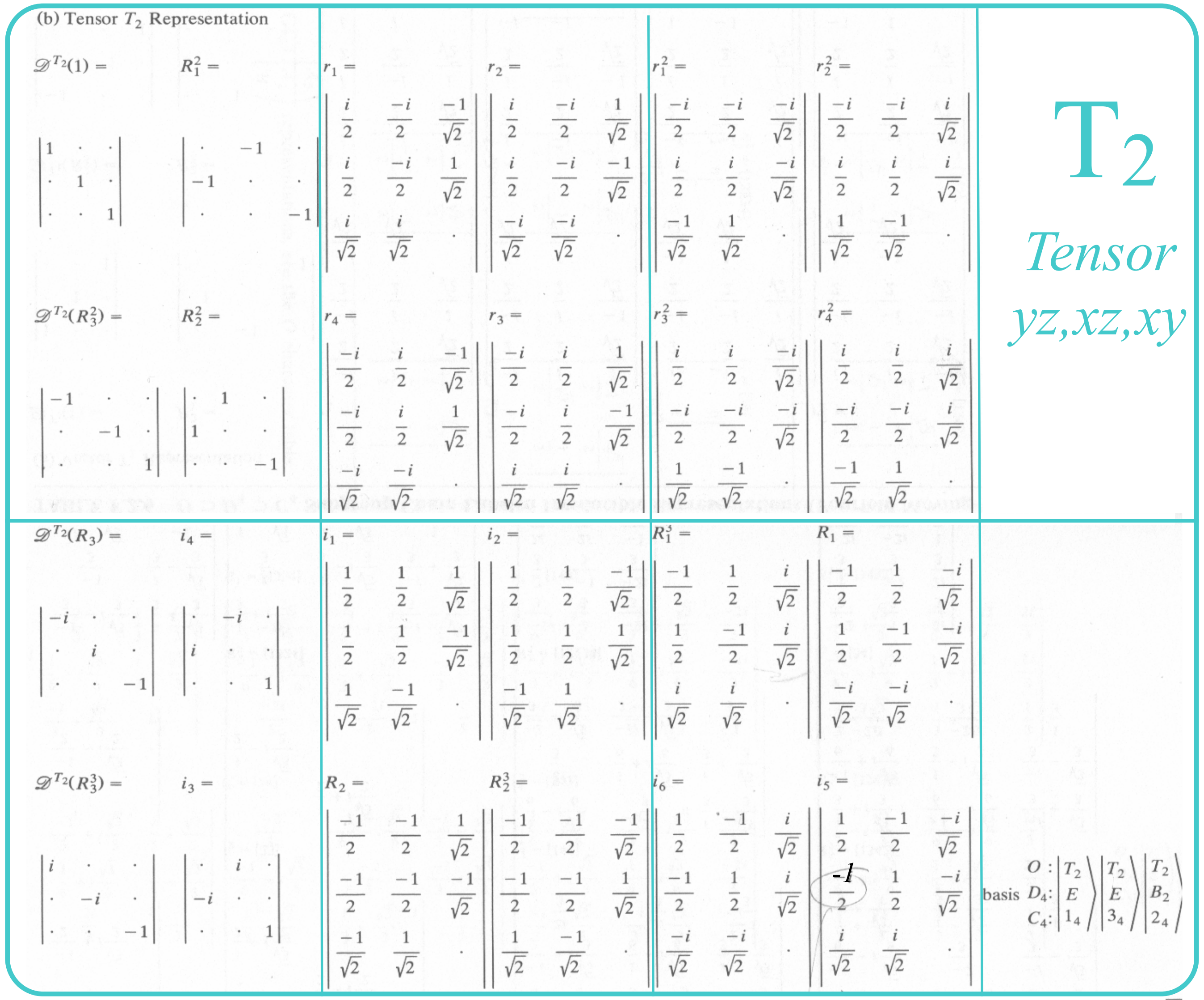
$O_h \supset D_{4h} \supset C_{4v}$

$O_h \supset D_{3h} \supset C_{3v}$

$O_h \supset C_{2v}$

Ireps for O⊃D₄⊃C₄ subgroup chain





E — Tensor $x^2+y^2-2z^2$, $(x^2-y^2)\sqrt{3}$

$1=[1][2][3][4]$: $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $R_1^2=[13][24]$: $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $r_1=[132]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_2=[124]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_1^2=[123]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_2^2=[142]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$

$R_3^2=[12][34]$: $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $R_2^2=[14][23]$: $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $r_4=[234]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_3=[124]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_3^2=[134]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$ $r_4^2=[243]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&-\frac12\end{pmatrix}$

$R_3=[1423]$: $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $i_4=[12]$: $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $i_1=[14]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $i_2=[23]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $R_1^3=[1432]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $R_1=[1234]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$

$R_3^3=[1324]$: $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $i_3=[34]$: $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $R_2=[1243]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $R_2^3=[1342]$: $\begin{pmatrix}-\frac12&+\frac{\sqrt3}{2}\\+\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $i_6=[24]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$ $i_5=[13]$: $\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&+\frac12\end{pmatrix}$

basis: O: E, E; D_4: A_1, B_1; C_4: 0_4, 2_4

$O: \chi^{\mu}_{\mathbf{g}}$	g=1	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

Ireps for O⊃D_4⊃D_2 subgroup chain

T_1 — *Vector* x,y,z

D_2:

$\mathscr{D}^{T_2}(1)=\begin{vmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_1^2=\begin{vmatrix}1&\cdot&\cdot\\\cdot&-1&\cdot\\\cdot&\cdot&-1\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3^2)=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&-1&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_2^2=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&-1\end{vmatrix}$

$r_1=\begin{vmatrix}\cdot&\cdot&1\\-1&\cdot&\cdot\\\cdot&-1&\cdot\end{vmatrix}$ $r_2=\begin{vmatrix}\cdot&\cdot&-1\\-1&\cdot&\cdot\\\cdot&1&\cdot\end{vmatrix}$ $r_1^2=\begin{vmatrix}\cdot&-1&\cdot\\\cdot&\cdot&-1\\1&\cdot&\cdot\end{vmatrix}$ $r_2^2=\begin{vmatrix}\cdot&-1&\cdot\\\cdot&\cdot&1\\-1&\cdot&\cdot\end{vmatrix}$

$r_4=\begin{vmatrix}\cdot&\cdot&1\\1&\cdot&\cdot\\\cdot&1&\cdot\end{vmatrix}$ $r_3=\begin{vmatrix}\cdot&\cdot&-1\\1&\cdot&\cdot\\\cdot&-1&\cdot\end{vmatrix}$ $r_3^2=\begin{vmatrix}\cdot&1&\cdot\\\cdot&\cdot&-1\\-1&\cdot&\cdot\end{vmatrix}$ $r_4^2=\begin{vmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\1&\cdot&\cdot\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3)=\begin{vmatrix}\cdot&-1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&-1\end{vmatrix}$ $i_4=\begin{vmatrix}\cdot&-1&\cdot\\-1&\cdot&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $i_1=\begin{vmatrix}\cdot&\cdot&-1\\\cdot&1&\cdot\\-1&\cdot&\cdot\end{vmatrix}$ $i_2=\begin{vmatrix}\cdot&\cdot&1\\\cdot&1&\cdot\\1&\cdot&\cdot\end{vmatrix}$ $R_1^3=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&-1&\cdot\end{vmatrix}$ $R_1=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&\cdot&-1\\\cdot&1&\cdot\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3^3)=\begin{vmatrix}\cdot&1&\cdot\\-1&\cdot&\cdot\\\cdot&\cdot&-1\end{vmatrix}$ $i_3=\begin{vmatrix}\cdot&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_2=\begin{vmatrix}\cdot&\cdot&-1\\\cdot&-1&\cdot\\1&\cdot&\cdot\end{vmatrix}$ $R_2^3=\begin{vmatrix}\cdot&\cdot&1\\\cdot&-1&\cdot\\-1&\cdot&\cdot\end{vmatrix}$ $i_6=\begin{vmatrix}1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{vmatrix}$ $i_5=\begin{vmatrix}1&\cdot&\cdot\\\cdot&\cdot&-1\\\cdot&-1&\cdot\end{vmatrix}$

basis: $\left|\begin{matrix}O\\D_4\\D_2\end{matrix}\right.$ $\begin{vmatrix}T_1\\E\\B_1\end{vmatrix}\rangle$ $\begin{vmatrix}T_1\\E\\B_2\end{vmatrix}\rangle$ $\begin{vmatrix}T_1\\A_2\\A_2\end{vmatrix}\rangle$

T_2 — *Tensor* yz,xz,xy

$\mathscr{D}^{T_2}(1)=\begin{vmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_1^2=\begin{vmatrix}1&\cdot&\cdot\\\cdot&-1&\cdot\\\cdot&\cdot&-1\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3^2)=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&-1&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_2^2=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&-1\end{vmatrix}$

$r_1=\begin{vmatrix}\cdot&\cdot&1\\-1&\cdot&\cdot\\\cdot&-1&\cdot\end{vmatrix}$ $r_2=\begin{vmatrix}\cdot&\cdot&-1\\-1&\cdot&\cdot\\\cdot&1&\cdot\end{vmatrix}$ $r_1^2=\begin{vmatrix}\cdot&-1&\cdot\\\cdot&\cdot&-1\\1&\cdot&\cdot\end{vmatrix}$ $r_2^2=\begin{vmatrix}\cdot&-1&\cdot\\\cdot&\cdot&1\\-1&\cdot&\cdot\end{vmatrix}$

$r_4=\begin{vmatrix}\cdot&\cdot&1\\1&\cdot&\cdot\\\cdot&1&\cdot\end{vmatrix}$ $r_3=\begin{vmatrix}\cdot&\cdot&-1\\1&\cdot&\cdot\\\cdot&-1&\cdot\end{vmatrix}$ $r_3^2=\begin{vmatrix}\cdot&1&\cdot\\\cdot&\cdot&-1\\-1&\cdot&\cdot\end{vmatrix}$ $r_4^2=\begin{vmatrix}\cdot&1&\cdot\\\cdot&\cdot&1\\1&\cdot&\cdot\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3)=\begin{vmatrix}\cdot&-1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&-1\end{vmatrix}$ $i_4=\begin{vmatrix}\cdot&-1&\cdot\\-1&\cdot&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $i_1=\begin{vmatrix}\cdot&\cdot&-1\\\cdot&1&\cdot\\-1&\cdot&\cdot\end{vmatrix}$ $i_2=\begin{vmatrix}\cdot&\cdot&1\\\cdot&1&\cdot\\1&\cdot&\cdot\end{vmatrix}$ $R_1^3=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&-1&\cdot\end{vmatrix}$ $R_1=\begin{vmatrix}-1&\cdot&\cdot\\\cdot&\cdot&-1\\\cdot&1&\cdot\end{vmatrix}$

$\mathscr{D}^{T_2}(R_3^3)=\begin{vmatrix}\cdot&1&\cdot\\-1&\cdot&\cdot\\\cdot&\cdot&-1\end{vmatrix}$ $i_3=\begin{vmatrix}\cdot&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\end{vmatrix}$ $R_2=\begin{vmatrix}\cdot&\cdot&-1\\\cdot&-1&\cdot\\1&\cdot&\cdot\end{vmatrix}$ $R_2^3=\begin{vmatrix}\cdot&\cdot&1\\\cdot&-1&\cdot\\-1&\cdot&\cdot\end{vmatrix}$ $i_6=\begin{vmatrix}1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{vmatrix}$ $i_5=\begin{vmatrix}1&\cdot&\cdot\\\cdot&\cdot&-1\\\cdot&-1&\cdot\end{vmatrix}$

basis: $\left|\begin{matrix}O\\D_4\\D_2\end{matrix}\right.$ $\begin{vmatrix}T_2\\E\\B_1\end{vmatrix}\rangle$ $\begin{vmatrix}T_2\\E\\B_2\end{vmatrix}\rangle$ $\begin{vmatrix}T_2\\B_2\\A_2\end{vmatrix}\rangle$

E — *Tensor* $x^2+y^2-2z^2$, $(x^2-y^2)\sqrt{3}$

$\mathscr{D}^E(1)=\begin{vmatrix}1&0\\0&1\end{vmatrix}$ $R_1^2=\begin{vmatrix}1&0\\0&1\end{vmatrix}$ $\mathscr{D}^E(R_3^2)=\begin{vmatrix}1&0\\0&1\end{vmatrix}$ $R_2^2=\begin{vmatrix}1&0\\0&1\end{vmatrix}$

$r_1=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_2=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_1^2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_2^2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$

$r_4=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_3=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_3^2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$ $r_4^2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{-1}{2}\end{vmatrix}$

$\mathscr{D}^E(R_3)=\begin{vmatrix}1&0\\0&-1\end{vmatrix}$ $i_4=\begin{vmatrix}1&0\\0&-1\end{vmatrix}$ $i_1=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $i_2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $R_1^3=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $R_1=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$

$\mathscr{D}^E(R_3^3)=\begin{vmatrix}1&0\\0&-1\end{vmatrix}$ $i_3=\begin{vmatrix}1&0\\0&-1\end{vmatrix}$ $R_2=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $R_2^3=\begin{vmatrix}\frac{-1}{2}&\frac{\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $i_6=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$ $i_5=\begin{vmatrix}\frac{-1}{2}&\frac{-\sqrt3}{2}\\\frac{-\sqrt3}{2}&\frac{1}{2}\end{vmatrix}$

basis: $\left|\begin{matrix}O\\D_4\\D_2\end{matrix}\right.$ $\begin{vmatrix}E\\A_1\\A_1\end{vmatrix}\rangle$ $\begin{vmatrix}E\\B_1\\A_1\end{vmatrix}\rangle$

O: $\chi^\mu_{\mathbf{g}}$	**g**=1	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

Ireps for O⊃D_3⊃C_2 subgroup chain

$\mathscr{D}^{T_1}(1) = \begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$

C_2

$i_4 = [12]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$

$R_1^2 = [13][24]$ $\begin{vmatrix} \cdot & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{-2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_3 = [1423]$ $\begin{vmatrix} \cdot & \frac{-\sqrt{3}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{2}{3} & \frac{-\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

$r_1 = [132]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$

$i_5 = [13]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & \cdot \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$

$r_4 = [234]$ $\begin{vmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{6} & \frac{\sqrt{2}}{3} \\ \cdot & \frac{-\sqrt{8}}{3} & \frac{-1}{3} \end{vmatrix}$

$i_6 = [24]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{6} & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{3}}{6} & \frac{-5}{6} & \frac{-\sqrt{2}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

$r_1^2 = [123]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$

$i_2 = [23]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$

$r_2^2 = [142]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{6} & \frac{5}{6} & \frac{\sqrt{2}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_2^3 = [1342]$ $\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{2} & \frac{1}{6} & \frac{-\sqrt{2}}{3} \\ \cdot & \frac{\sqrt{8}}{3} & \frac{1}{3} \end{vmatrix}$

$R_2^2 = [14][23]$ $\begin{vmatrix} \cdot & \frac{-\sqrt{3}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{3} & \frac{-2}{3} & \frac{\sqrt{2}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_3^3 = [1324]$ $\begin{vmatrix} \cdot & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{3} & \frac{2}{3} & \frac{-\sqrt{2}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

$R_3^2 = [12][34]$ $\begin{vmatrix} -1 & \cdot & \cdot \\ \cdot & \frac{1}{3} & \frac{-\sqrt{8}}{3} \\ \cdot & \frac{-\sqrt{8}}{3} & \frac{-1}{3} \end{vmatrix}$

$i_3 = [34]$ $\begin{vmatrix} -1 & \cdot & \cdot \\ \cdot & \frac{-1}{3} & \frac{\sqrt{8}}{3} \\ \cdot & \frac{\sqrt{8}}{3} & \frac{1}{3} \end{vmatrix}$

$r_2 = [124]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{6} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{6} & \frac{5}{6} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_1 = [1234]$ $\begin{vmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{6} & \frac{-\sqrt{2}}{3} \\ \cdot & \frac{\sqrt{8}}{3} & \frac{1}{3} \end{vmatrix}$

$r_3 = [143]$ $\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{6} & \frac{-1}{6} & \frac{-\sqrt{8}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_1^3 = [1432]$ $\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdot \\ \frac{-\sqrt{3}}{6} & \frac{1}{6} & \frac{\sqrt{8}}{3} \\ \frac{\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

$r_3^2 = [134]$ $\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{3}}{2} & \frac{-1}{6} & \frac{\sqrt{2}}{3} \\ \cdot & \frac{-\sqrt{8}}{3} & \frac{-1}{3} \end{vmatrix}$

$i_1 = [14]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{3}}{6} & \frac{-5}{6} & \frac{-\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

$r_4^2 = [243]$ $\begin{vmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & \cdot \\ \frac{-\sqrt{3}}{6} & \frac{-1}{6} & \frac{-\sqrt{8}}{3} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{3} & \frac{-1}{3} \end{vmatrix}$

$R_2 = [1243]$ $\begin{vmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & \cdot \\ \frac{\sqrt{3}}{6} & \frac{1}{6} & \frac{\sqrt{8}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \end{vmatrix}$

T_1 *Vector* *u,v,w*

$$\text{basis:}\ \begin{matrix} O \\ D_3 \\ C_2 \end{matrix} \left| \begin{matrix} T_1 \\ E \\ 0_2 \end{matrix} \right\rangle \left| \begin{matrix} T_1 \\ E \\ 1_2 \end{matrix} \right\rangle \left| \begin{matrix} T_1 \\ A_2 \\ 1_2 \end{matrix} \right\rangle$$

$\mathscr{D}^{T_2}(1) = \begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$

$i_4 = [12]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$

$R_1^2 = [13][24]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{-2}{3} & \frac{-\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} & \frac{-\sqrt{3}}{3} & \cdot \end{vmatrix}$

$R_3 = [1423]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{-2}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} & \frac{-\sqrt{3}}{3} & \cdot \end{vmatrix}$

$r_1 = [132]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \cdot & \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$

$i_5 = [13]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \cdot & \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_4 = [234]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{-\sqrt{2}}{3} & \frac{-1}{6} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{vmatrix}$

$i_6 = [24]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{5}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{vmatrix}$

$r_1^2 = [123]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \cdot & \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$

$i_2 = [23]$ $\begin{vmatrix} 1 & \cdot & \cdot \\ \cdot & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \cdot & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_2^2 = [142]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{5}{6} & \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{-1}{2} \end{vmatrix}$

$R_2^3 = [1342]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{-\sqrt{2}}{3} & \frac{-1}{6} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{-1}{2} \end{vmatrix}$

$R_2^2 = [14][23]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{-2}{3} & \frac{\sqrt{3}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & \cdot \end{vmatrix}$

$R_3^3 = [1324]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{-2}{3} & \frac{\sqrt{3}}{3} \\ \frac{-\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & \cdot \end{vmatrix}$

$R_3^2 = [12][34]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & \cdot \\ \cdot & \cdot & -1 \end{vmatrix}$

$i_3 = [34]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & \cdot \\ \cdot & \cdot & 1 \end{vmatrix}$

$r_2 = [124]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{5}{6} & \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{3}}{6} & \frac{-1}{2} \end{vmatrix}$

$R_1 = [1234]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{-\sqrt{2}}{3} & \frac{-1}{6} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{3}}{6} & \frac{-1}{2} \end{vmatrix}$

$r_3 = [143]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{8}}{3} & \frac{-1}{6} & \frac{-\sqrt{3}}{6} \\ \cdot & \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$R_1^3 = [1432]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{\sqrt{8}}{3} & \frac{-1}{6} & \frac{-\sqrt{3}}{6} \\ \cdot & \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$

$r_3^2 = [134]$ $\begin{vmatrix} \frac{-1}{3} & \frac{\sqrt{8}}{3} & \cdot \\ \frac{-\sqrt{2}}{3} & \frac{-1}{6} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{3}}{6} & \frac{1}{2} \end{vmatrix}$

$i_1 = [14]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{3} & \frac{5}{6} & \frac{-\sqrt{3}}{6} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{3}}{6} & \frac{1}{2} \end{vmatrix}$

$r_4^2 = [243]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{8}}{3} & \frac{-1}{6} & \frac{\sqrt{3}}{6} \\ \cdot & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$R_2 = [1243]$ $\begin{vmatrix} \frac{-1}{3} & \frac{-\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{8}}{3} & \frac{-1}{6} & \frac{\sqrt{3}}{6} \\ \cdot & \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$

T_2 *Tensor* *vw,uw,uv*

$$\text{basis:}\ \begin{matrix} O \\ D_3 \\ C_2 \end{matrix} \left| \begin{matrix} T_2 \\ B_2 \\ 0_2 \end{matrix} \right\rangle \left| \begin{matrix} T_2 \\ E \\ 0_2 \end{matrix} \right\rangle \left| \begin{matrix} T_2 \\ E \\ 1_2 \end{matrix} \right\rangle$$

$\mathscr{D}^E(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, C_2 $i_4 = [12]$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$r_1 = [132]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$ $i_5 = [13]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_1^2 = [123]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $i_2 = [23]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$R_1^2 = [13][24]$ $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ $R_3 = [1423]$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$r_4 = [234]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix}$ $i_6 = [24]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_2^2 = [142]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $R_2^3 = [1342]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$R_2^2 = [14][23]$ $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ $R_3^3 = [1324]$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$r_2 = [124]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $R_1 = [1234]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_3^2 = [134]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $i_1 = [14]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$R_3^2 = [12][34]$ $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ $i_3 = [34]$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

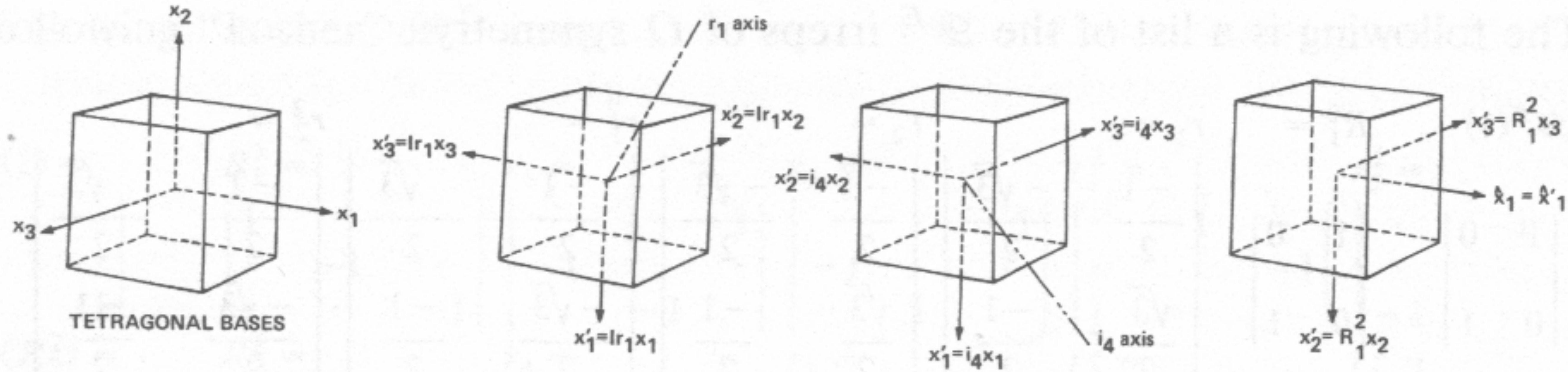
$r_3 = [143]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $R_1^3 = [1432]$ $\begin{vmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

$r_4^2 = [243]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{vmatrix}$ $R_2 = [1243]$ $\begin{vmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix}$

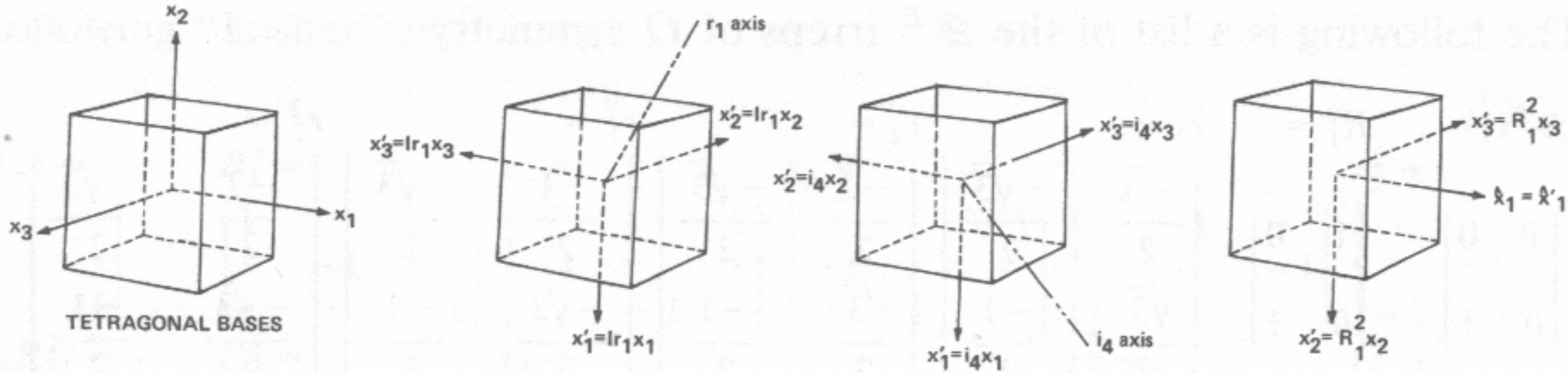
E *Tensor*
$u^2+v^2-2w^2$
$(u^2-v^2)\sqrt{3}$

basis: $\left| \begin{matrix} O \\ D_3 \\ C_2 \end{matrix} \;\begin{matrix} E \\ E \\ 0_2 \end{matrix} \right\rangle$, $\left| \begin{matrix} E \\ E \\ 1_2 \end{matrix} \right\rangle$

$O: \chi^{\mu}_{\mathbf{g}}$	$\mathbf{g}$=1	$\mathbf{r}_{1-4}$ $\tilde{\mathbf{r}}_{1-4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$ $\tilde{\mathbf{R}}_{xyz}$	$\mathbf{i}_{1-6}$
$\mu=A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1



$$D^{T_{1u}}(Ir_1) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad D^{T_{1u}}(i_4) = \begin{pmatrix} \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{pmatrix} \qquad D^{T_{1u}}(R_1^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

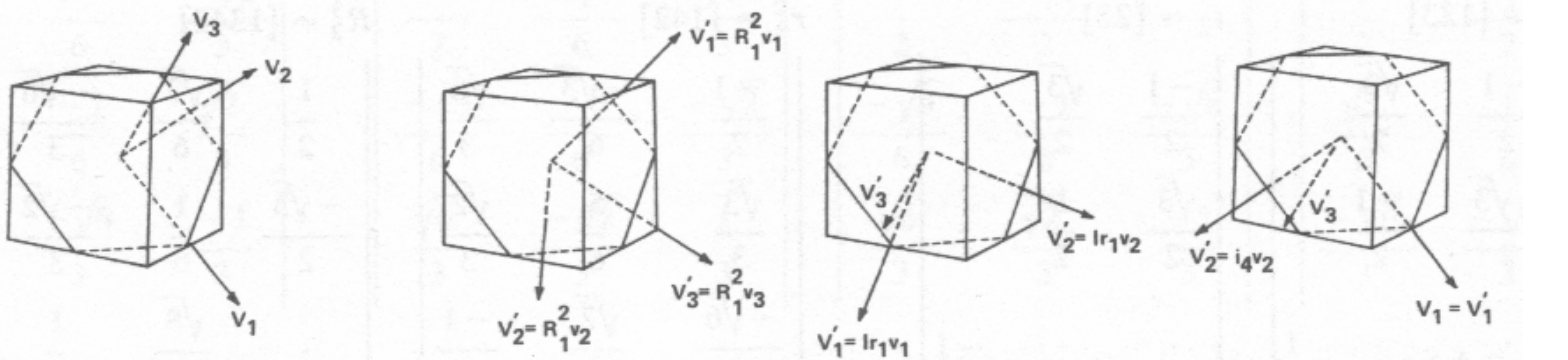


TETRAGONAL BASES

$$D^{T1u}(Ir_1) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad D^{T1u}(i_4) = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{pmatrix} \qquad D^{T1u}(R_1^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

TRIGONAL BASES

$$D^{T1u}(R_1^2) = \begin{pmatrix} 0 & \sqrt{3}/3 & \sqrt{6}/3 \\ \sqrt{3}/3 & -2/3 & \sqrt{2}/3 \\ \sqrt{6}/3 & \sqrt{2}/3 & -1/3 \end{pmatrix} \qquad D^{T1u}(Ir_1) = \begin{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{pmatrix} \qquad D^{T1u}(i_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

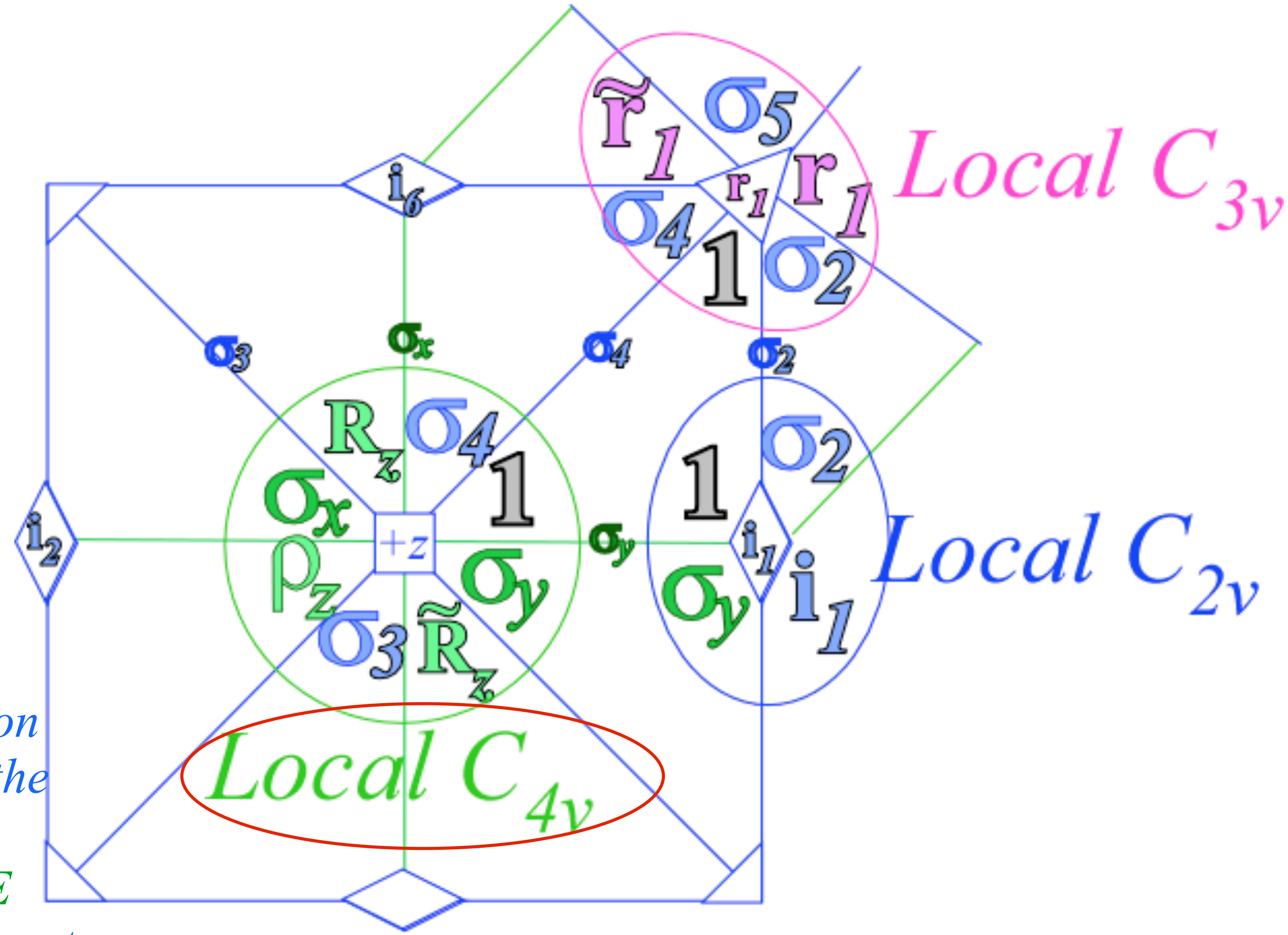


$O \supset C_4$	0_4	1_4	2_4	3_4
$A_1 \downarrow C_4$	1	·	·	·
$A_2 \downarrow C_4$	·	·	1	·
$E \downarrow C_4$	1	·	1	·
$T_1 \downarrow C_4$	1	1	·	1
$T_2 \downarrow C_4$	·	1	1	1

$O_h \supset C_{4v}$	A'	B'	A''	B''	E
$A_{1g} \downarrow C_{4v}$	1	·	·	·	·
$A_{2g} \downarrow C_{4v}$	·	1	·	·	·
$E_g \downarrow C_{4v}$	1	1	·	·	·
$T_{1g} \downarrow C_{4v}$	·	·	1	·	1
$T_{2g} \downarrow C_{4v}$	·	·	·	1	1
$A_{1g} \downarrow C_{4v}$	·	·	1	·	·
$A_{2u} \downarrow C_{4v}$	·	·	·	1	·
$E_u \downarrow C_{4v}$	·	·	1	1	·
$T_{1u} \downarrow C_{4v}$	1	·	·	·	1
$T_{2u} \downarrow C_{4v}$	·	1	·	·	1

,

$O_h \supset C_{4v}$ correlation predicts the parity of the $A_1 T_1 E$ cluster is not uniformly even (g) or odd (u): $A_{1g} T_{1u} E_g$



$0_4\uparrow O$ cluster
Symmetry$_{parity}$
$A_{1g}T_{1u}E_g$

$$
\begin{pmatrix}
\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle & \cdots & \langle 1|\mathbf{H}|6\rangle \\
\langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle & \cdots & \langle 2|\mathbf{H}|6\rangle \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\langle 6|\mathbf{H}|1\rangle & \langle 6|\mathbf{H}|2\rangle & \cdots & \langle 6|\mathbf{H}|6\rangle
\end{pmatrix}
=
\begin{pmatrix}
H & T & S & S & S & S \\
T & H & S & S & S & S \\
S & S & H & T & S & S \\
S & S & T & H & S & S \\
S & S & S & S & H & T \\
S & S & S & S & T & H
\end{pmatrix}
$$

$$E^{A_1} = H + T + 4S$$
$$E^{T_1} = H - T$$
$$E^{E} = H + T - 2S$$

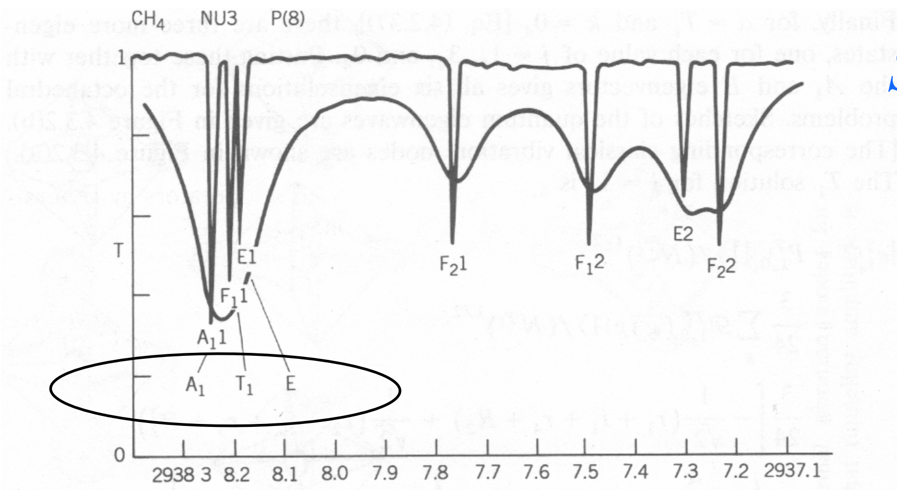
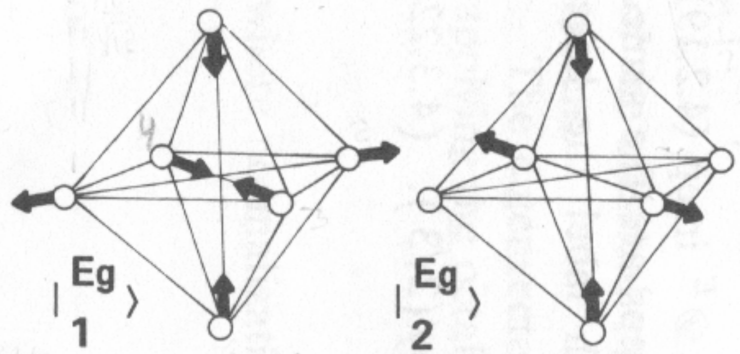
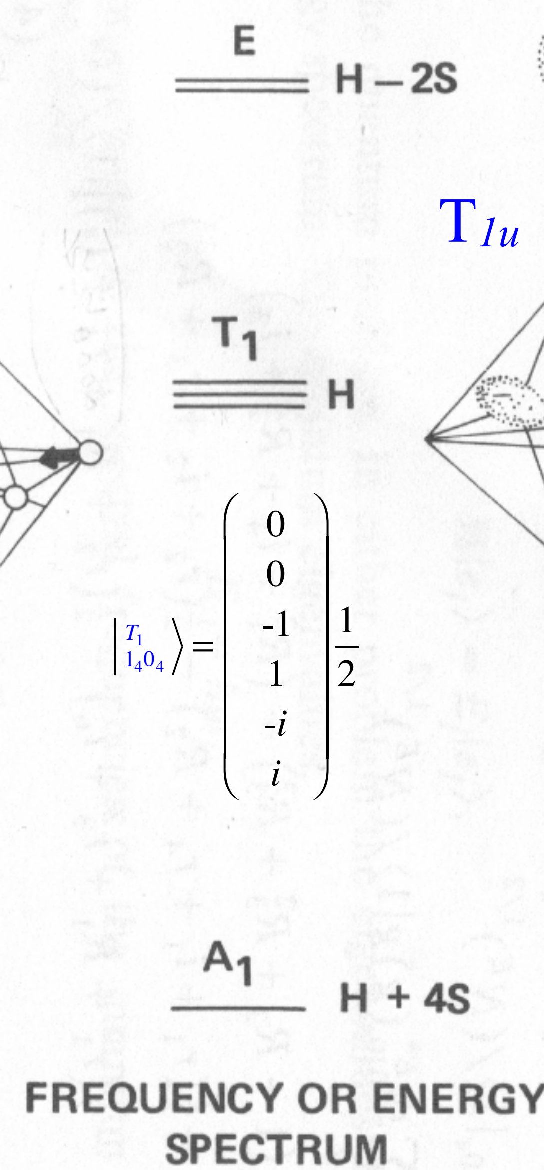


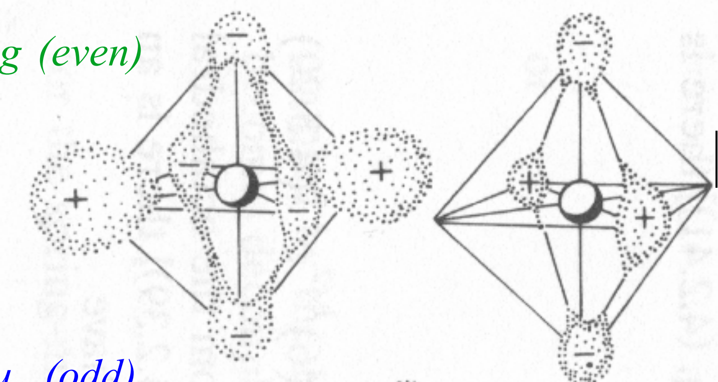
Figure 4.3.3 Evidence of an (A_1T_1E) spectral cluster in methane laser spectra. (Courtesy of Dr. Allan Pine, MIT Lincoln Laboratories, from *Journal of Optical Society of America* **66**, 97 (1976)). The ordering and approximate spacing of the A_1T_1 and E lines is consistent with that of Figure 4.3.2.

$$\left|{E \atop 0_4 0_4}\right\rangle = \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{2\sqrt{3}}$$



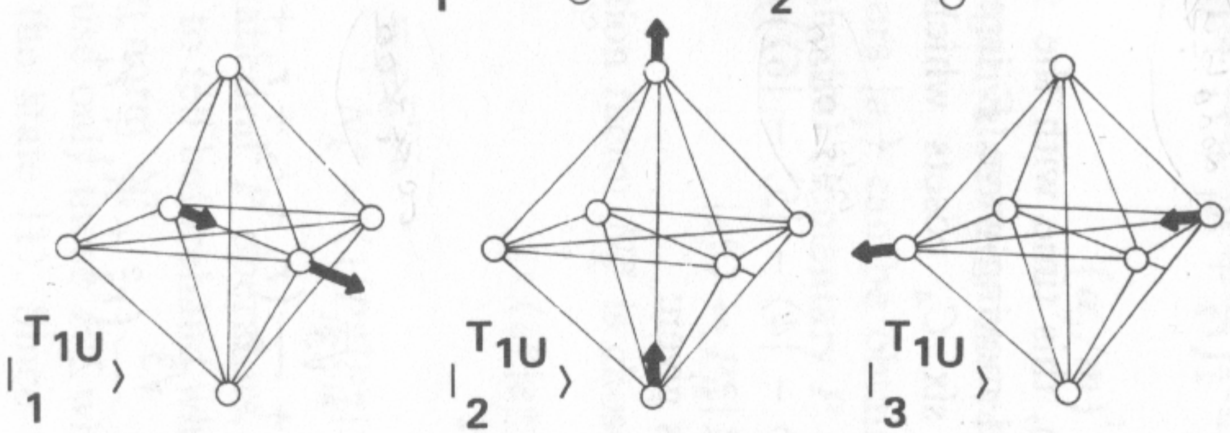


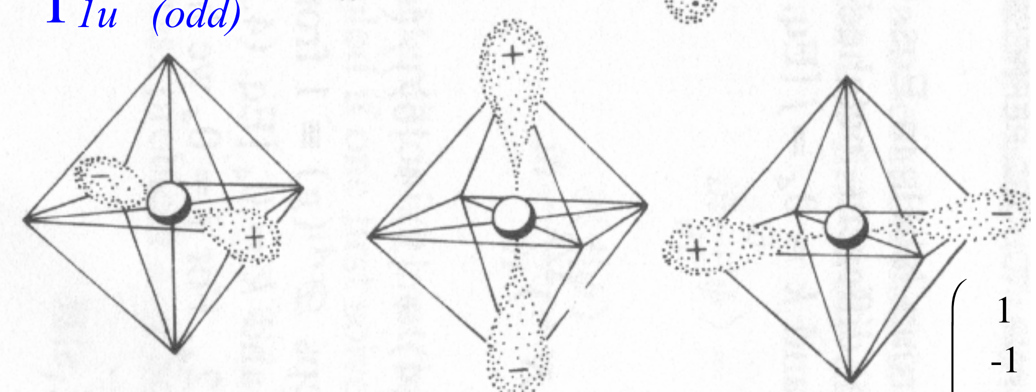
E_g *(even)*



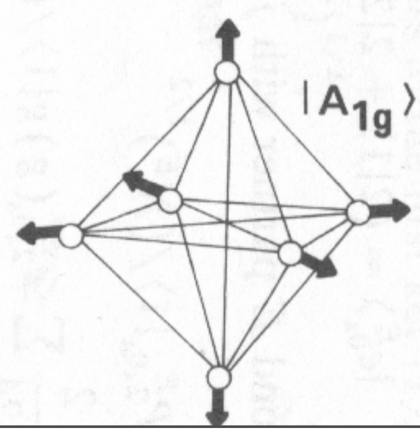
$$\left|{E \atop 2_4 0_4}\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{2}$$

T_{1u} *(odd)*

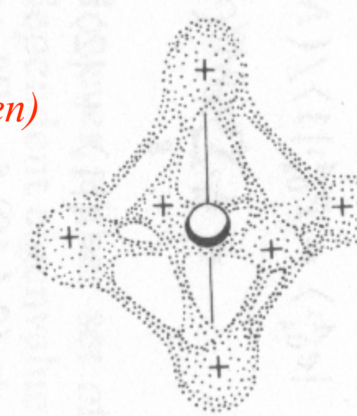




$$\left|{T_1 \atop 1_4 0_4}\right\rangle = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -i \\ i \end{pmatrix} \frac{1}{2} \qquad \left|{T_1 \atop 0_4 0_4}\right\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$



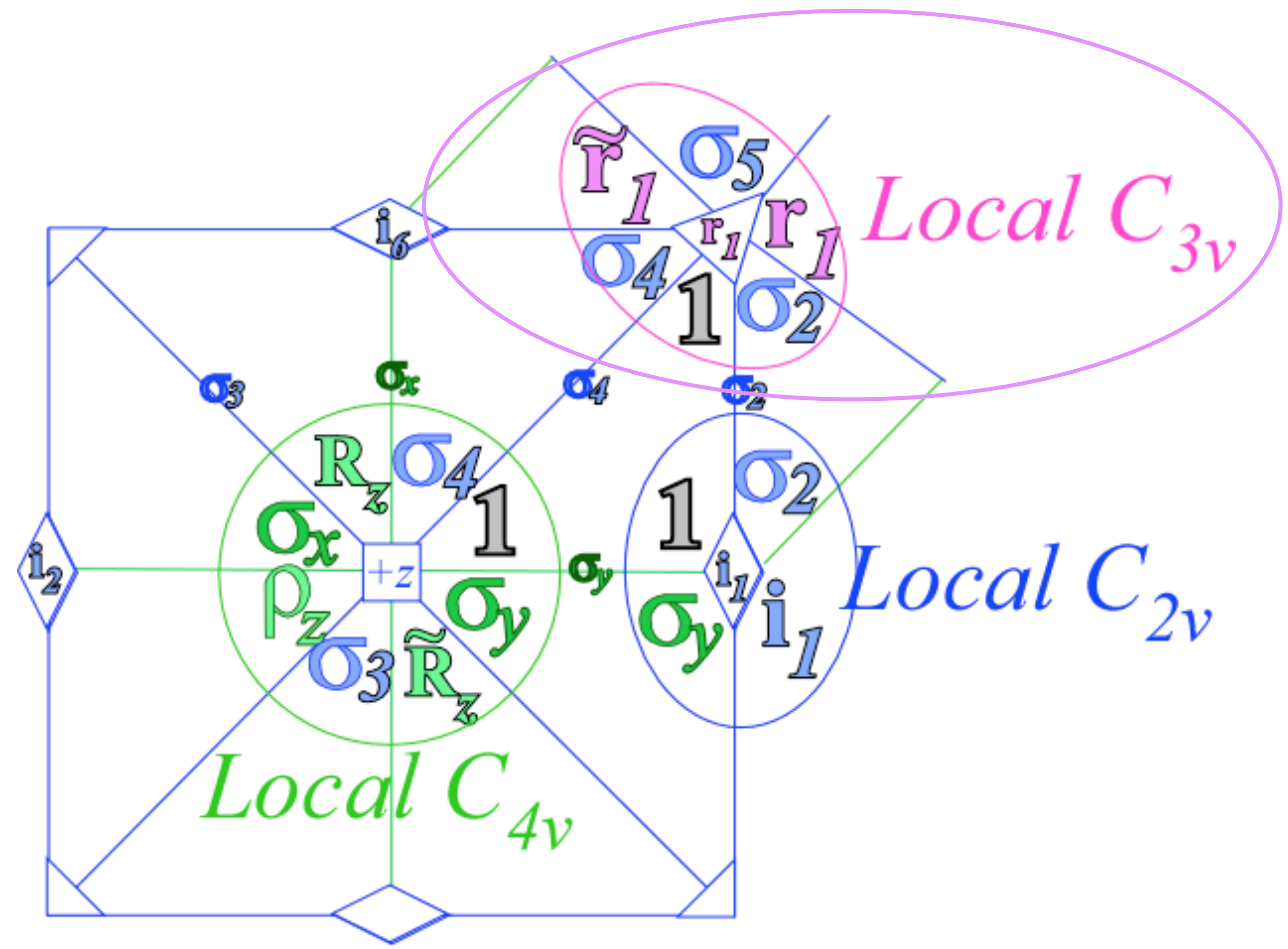
A_{1g} *(even)*

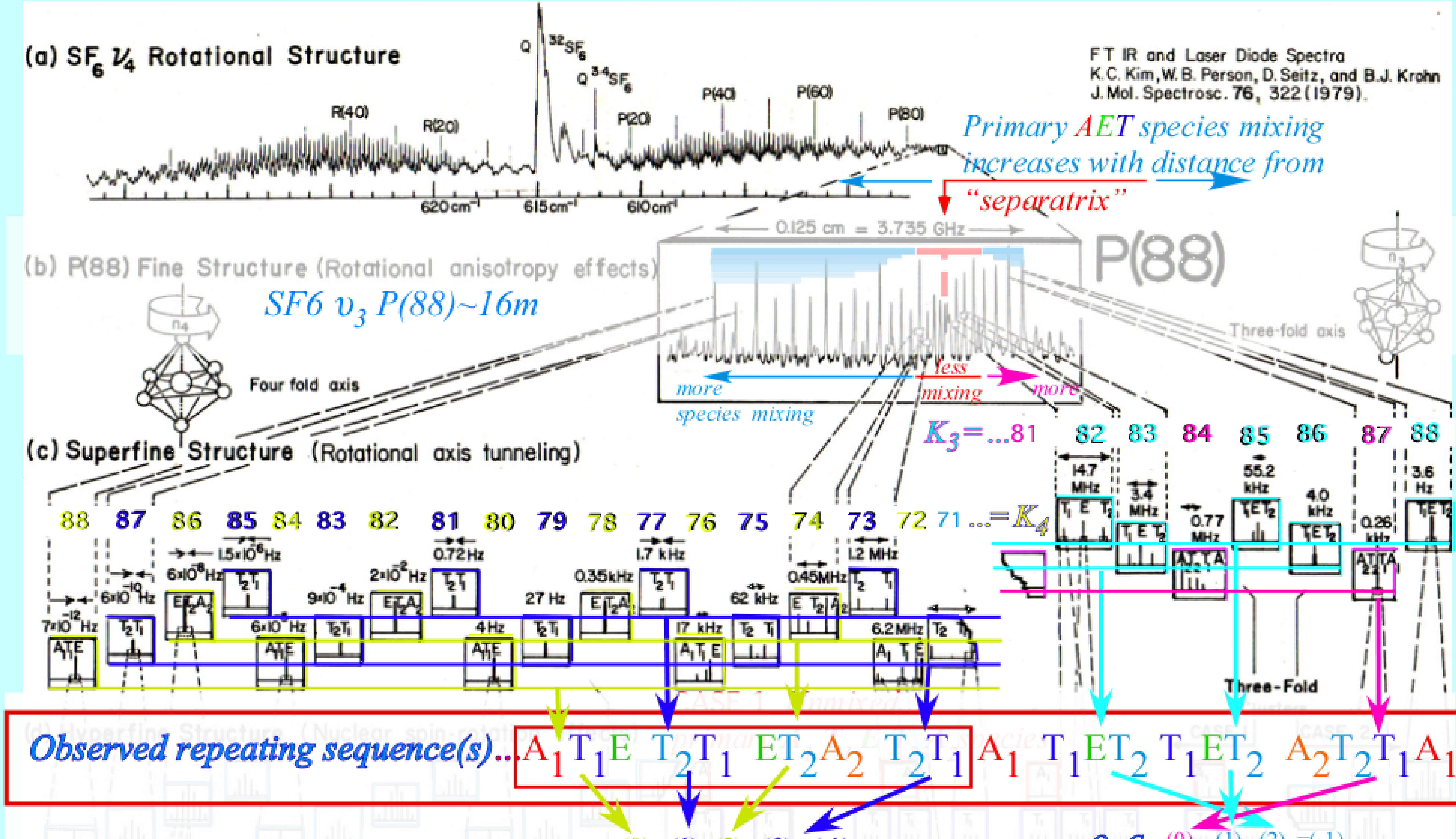


$$\left|{A_1 \atop 0_4 0_4}\right\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}}$$

$O\supset C_3$	0_3	1_3	2_3
$A_1{\downarrow}C_3$	1	·	·
$A_2{\downarrow}C_3$	1	·	·
$E{\downarrow}C_3$	·	1	1
$T_1{\downarrow}C_3$	1	1	1
$T_2{\downarrow}C_3$	1	1	1

$O_h\supset C_{3v}$	A'	A''	E
$A_{1g}{\downarrow}C_{3v}$	1	·	·
$A_{2g}{\downarrow}C_{3v}$	·	1	·
$E_g{\downarrow}C_{3v}$	·	·	1
$T_{1g}{\downarrow}C_{3v}$	·	1	1
$T_{2g}{\downarrow}C_{3v}$	1	·	1
$A_{1g}{\downarrow}C_{3v}$	·	1	·
$A_{2u}{\downarrow}C_{3v}$	1	·	·
$E_u{\downarrow}C_{3v}$	·	·	1
$T_{1u}{\downarrow}C_{3v}$	1	·	1
$T_{2u}{\downarrow}C_{3v}$	·	1	1



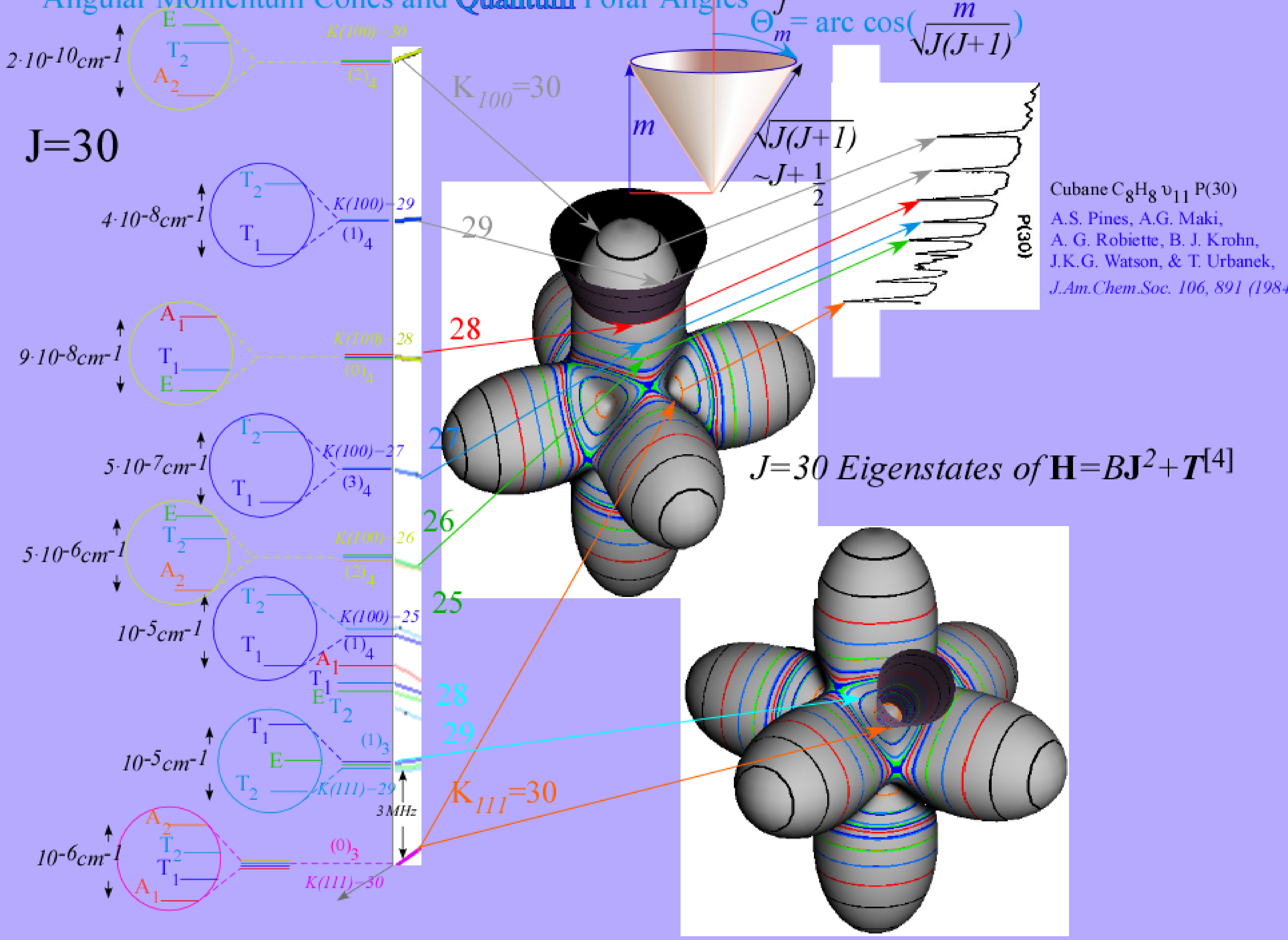


	(0)4	(1)4	(2)4	(3)4
A1	1	•	•	•
A2	•	•	1	•
E	1	•	1	•
T1	1	1	•	1
T2	•	1	1	1

	(0)3	(1)3	(2)3
A1	1	•	•
A2	1	•	•
E	•	1	1
T1	1	1	1
T2	1	1	1

Angular Momentum Cones and Quantum Polar Angles

$$\Theta_m^J = \arccos\left(\frac{m}{\sqrt{J(J+1)}}\right)$$



Cubane C_8H_8 υ_{11} P(30)
A.S. Pines, A.G. Maki, A. G. Robiette, B. J. Krohn, J.K.G. Watson, & T. Urbanek, *J.Am.Chem.Soc. 106, 891 (1984)*

J=30 Eigenstates of $\mathbf{H}=B\mathbf{J}^2+T^{[4]}$

$O\supset C_4$	0_4	1_4	2_4	3_4
$A_1\downarrow C_4$	1	·	·	·
$A_2\downarrow C_4$	·	·	1	·
$E\downarrow C_4$	1	·	1	·
$T_1\downarrow C_4$	1	1	·	1
$T_2\downarrow C_4$	·	1	1	1

$O\supset C_3$	0_3	1_3	2_3
$A_1\downarrow C_3$	1	·	·
$A_2\downarrow C_3$	1	·	·
$E\downarrow C_3$	·	1	1
$T_1\downarrow C_3$	1	1	1
$T_2\downarrow C_3$	1	1	1

$O\supset C_2(\mathbf{i}_1)$	0_2	1_2
$A_1\downarrow C_2$	1	·
$A_2\downarrow C_2$	·	1
$E\downarrow C_2$	1	1
$T_1\downarrow C_2$	1	2
$T_2\downarrow C_2$	2	1

$O\supset C_2(\rho_z)$	0_2	1_2
$A_1\downarrow C_2$	1	·
$A_2\downarrow C_2$	1	·
$E\downarrow C_2$	2	·
$T_1\downarrow C_2$	1	2
$T_2\downarrow C_2$	1	2

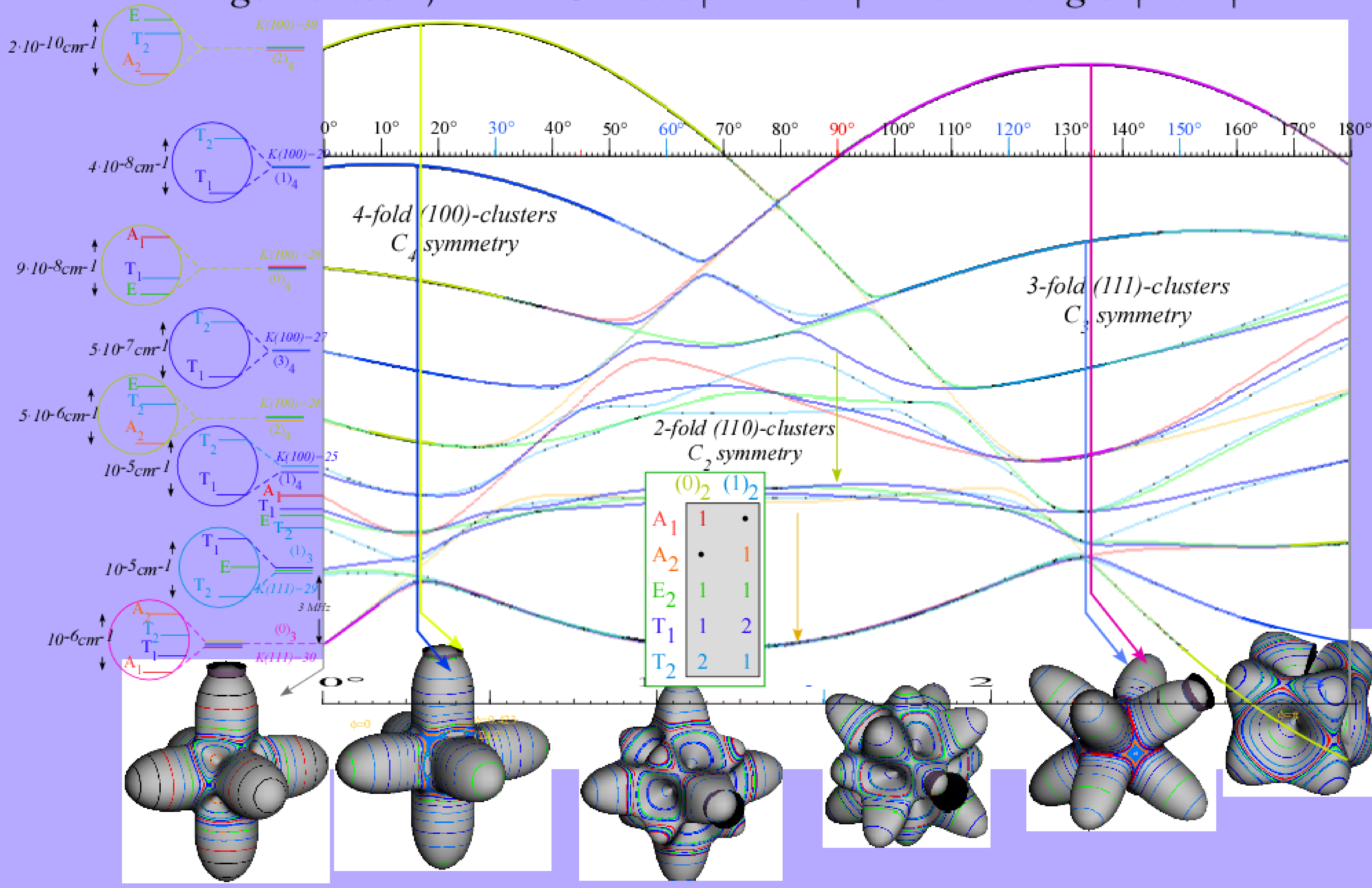
$O_h\supset C_{4v}$	A'	B'	A''	B''	E
$A_{1g}\downarrow C_{4v}$	1	·	·	·	·
$A_{2g}\downarrow C_{4v}$	·	1	·	·	·
$E_g\downarrow C_{4v}$	1	1	·	·	·
$T_{1g}\downarrow C_{4v}$	·	·	1	·	1
$T_{2g}\downarrow C_{4v}$	·	·	·	1	1
$A_{1g}\downarrow C_{4v}$	·	·	1	·	·
$A_{2u}\downarrow C_{4v}$	·	·	·	1	·
$E_u\downarrow C_{4v}$	·	·	1	1	·
$T_{1u}\downarrow C_{4v}$	1	·	·	·	1
$T_{2u}\downarrow C_{4v}$	·	1	·	·	1

$O_h\supset C_{3v}$	A'	A''	E
$A_{1g}\downarrow C_{3v}$	1	·	·
$A_{2g}\downarrow C_{3v}$	·	1	·
$E_g\downarrow C_{3v}$	·	·	1
$T_{1g}\downarrow C_{3v}$	·	1	1
$T_{2g}\downarrow C_{3v}$	1	·	1
$A_{1g}\downarrow C_{3v}$	·	1	·
$A_{2u}\downarrow C_{3v}$	1	·	·
$E_u\downarrow C_{3v}$	·	·	1
$T_{1u}\downarrow C_{3v}$	1	·	1
$T_{2u}\downarrow C_{3v}$	·	1	1

$O_h\supset C^i_{2v}$	A'	B'	A''	B''
$A_{1g}\downarrow C^i_{2v}$	1	·	·	·
$A_{2g}\downarrow C^i_{2v}$	·	1	·	·
$E_g\downarrow C^i_{2v}$	1	1	·	·
$T_{1g}\downarrow C^i_{2v}$	·	1	1	1
$T_{2g}\downarrow C^i_{2v}$	1	·	1	1
$A_{1g}\downarrow C^i_{2v}$	·	·	1	·
$A_{2u}\downarrow C^i_{2v}$	·	·	·	1
$E_u\downarrow C^i_{2v}$	·	·	1	1
$T_{1u}\downarrow C^i_{2v}$	1	1	·	1
$T_{2u}\downarrow C^i_{2v}$	1	1	1	·

$O_h\supset C^z_{2v}$	A'	B'	A''	B''
$A_{1g}\downarrow C^z_{2v}$	1	·	·	·
$A_{2g}\downarrow C^z_{2v}$	1	·	·	·
$E_g\downarrow C^z_{2v}$	2	·	·	·
$T_{1g}\downarrow C^z_{2v}$	·	1	1	1
$T_{2g}\downarrow C^z_{2v}$	·	1	1	1
$A_{1g}\downarrow C^z_{2v}$	·	·	1	·
$A_{2u}\downarrow C^z_{2v}$	·	·	1	·
$E_u\downarrow C^z_{2v}$	·	·	2	·
$T_{1u}\downarrow C^z_{2v}$	1	1	·	1
$T_{2u}\downarrow C^z_{2v}$	1	1	·	1

Eigenvalues of $\mathbf{H}=B\mathbf{J}^2+\cos\phi\,\mathbf{T}^{[4]}+\sin\phi\,\mathbf{T}^{[6]}$ *vs. mix angle* ϕ*:* $0<\phi<\pi$



Ireps for O⊃D_4⊃C_4 *subgroup chain*

T_1

Vector

x,y,z

T_2

Tensor

yz,xz,xy

E

Tensor

$x^2+y^2-2z^2$

$(x^2-y^2)\sqrt{3}$

$\ell^{A_1}= 1$

$\ell^{A_2}= 1$

$\ell^{E} = 2$

$\ell^{T_1}= 3$

$\ell^{T_2}= 3$

Example: $G=O$

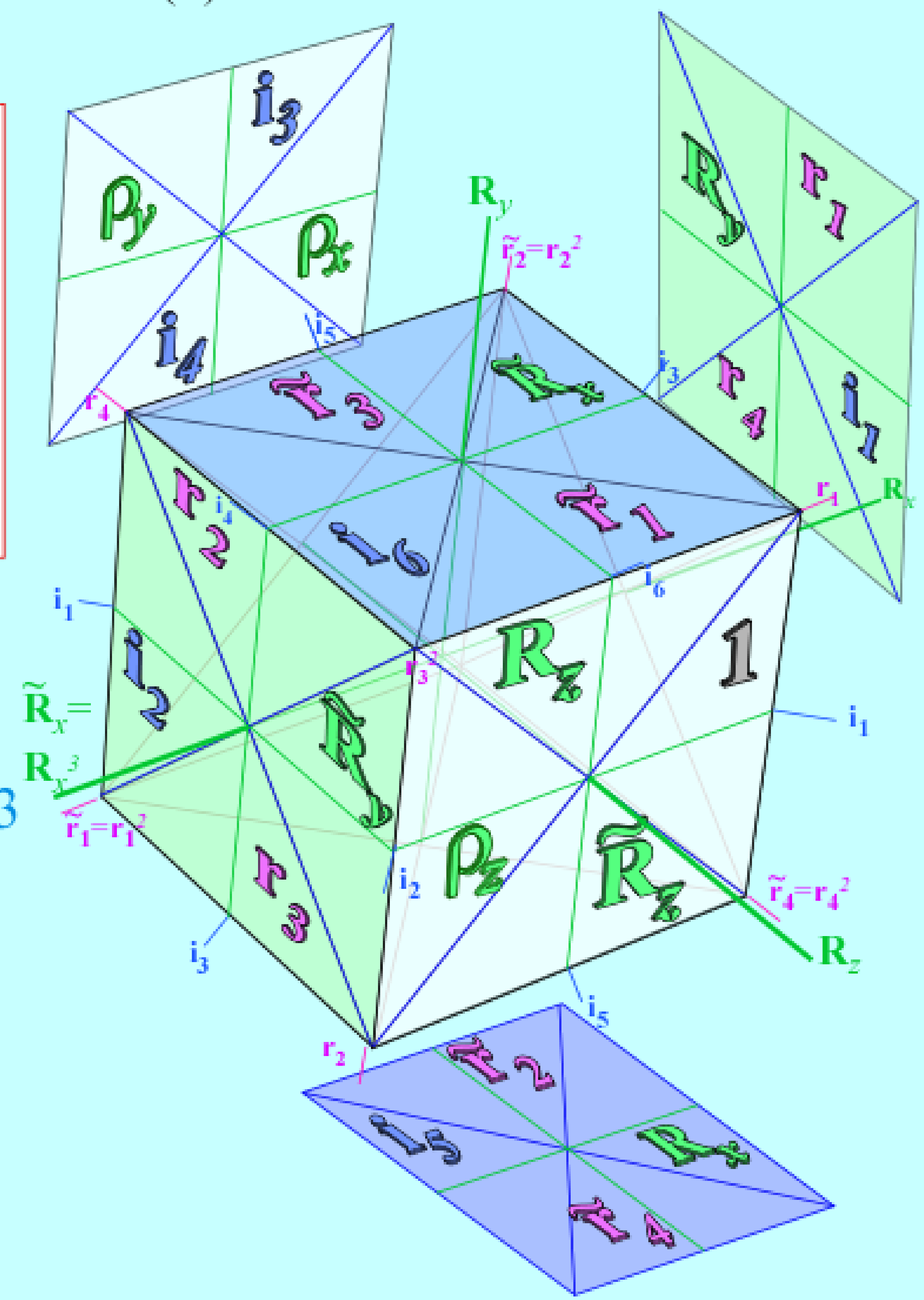
Cubic-Octahedral Group O

Centrum: $\kappa(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^0 =1^0+1^0+2^0+3^0+3^0=5$

Rank: $\rho(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^1 =1^1+1^1+2^1+3^1+3^1=10$

Order: $^o(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^0 =1^2+1^2+2^2+3^2+3^2=24$

	O group $\chi^{\alpha}_{\kappa_g}$	$g=1$	r_{1-4} $\tilde{r}_{1-4}$	ρ_{xyz}	R_{xyz} $\tilde{R}_{xyz}$	i_{1-6}
s-orbital r^2	$\alpha = A_1$	1	1	1	1	1
	A_2	1	1	1	−1	−1
d-orbitals $\{x^2+y^2-2z^2, x^2-y^2\}$	E	2	−1	2	0	0
p-orbitals $\{x, y, z\}$	T_1	3	0	−1	1	−1
$\{xz,yz,xy\}$ d-orbitals	T_2	3	0	−1	−1	1



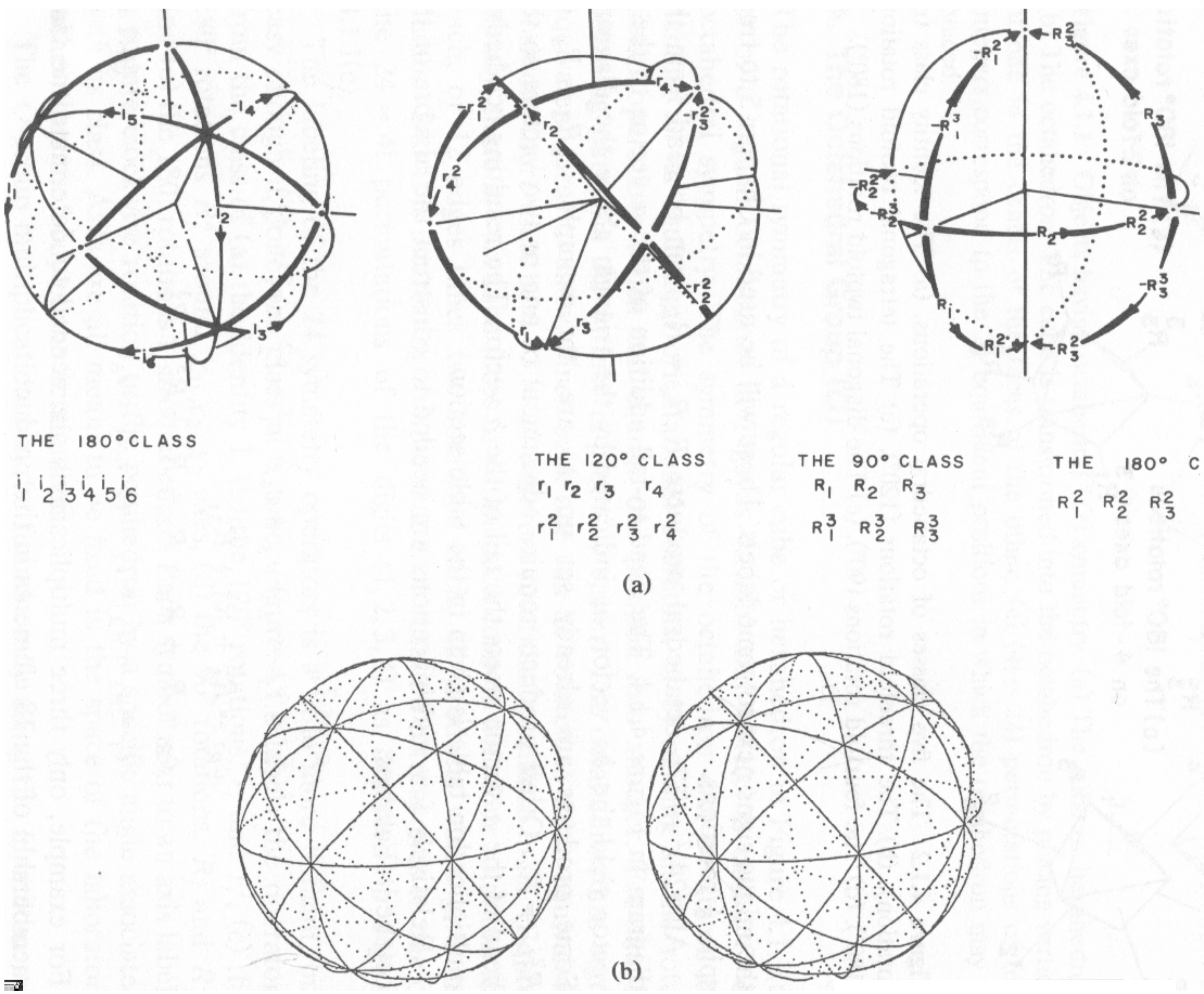
$O\supset C_4$

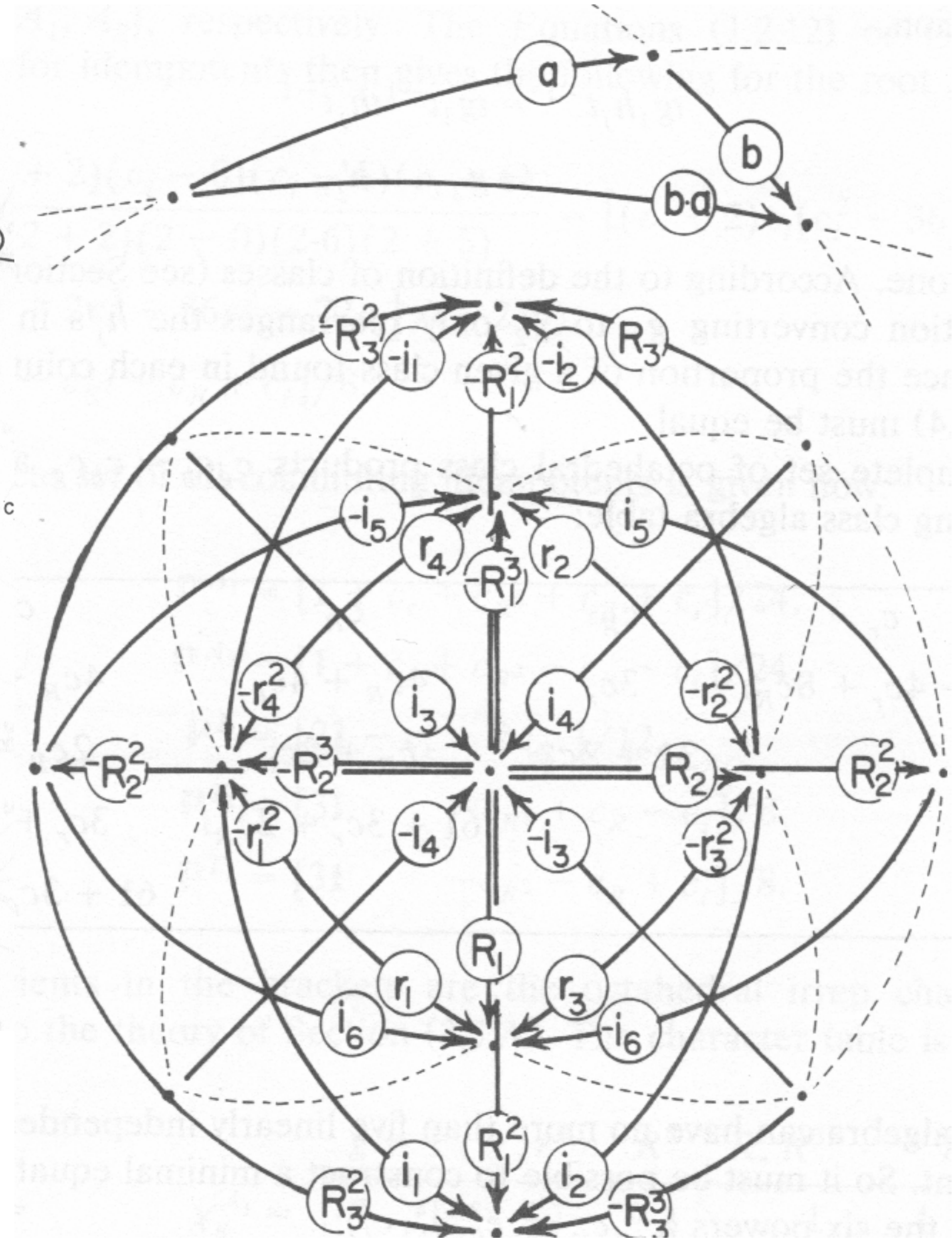
	$(0)_4$	$(1)_4$	$(2)_4$	$(3)_4=(-1)_4$
A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

$O\supset C_3$

	$(0)_3$	$(1)_3$	$(2)_3=(-1)_3$
A_1	1	•	•
A_2	1	•	•
E	•	1	1
T_1	1	1	1
T_2	1	1	1

Octahedral O and spin-O⊂U(2) rotation nomogram from Fig. 4.1.3-4 *Principles of Symmetry, Dynamics and Spectroscopy*





Octahedral O⊃D₄⊃C₄ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{z180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, 1, 1, 1, 1.$

$A_2(O)\downarrow D_4 = 1, 1, -1, 1, -1.$

$E(O)\downarrow D_4 = 2, 2, 0, 2, 0.$

$T_2(O)\downarrow D_4 = 3, -1, 1, -1, -1.$

$T_2(O)\downarrow D_4 = 3, -1, -1, -1, 1.$

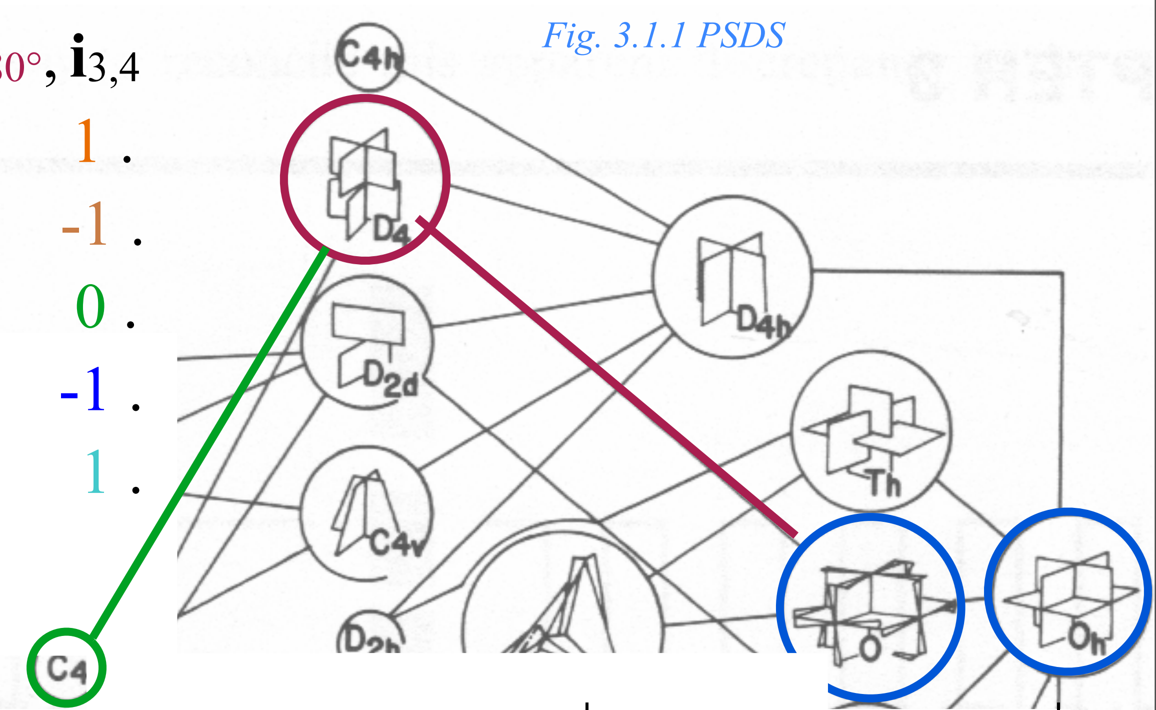


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

D₄↓C₄ subduction

$\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1, 1, 1, 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1, -1, 1, -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1, 1, 1, 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1, -1, 1, -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2, 0, -2, 0. \quad =(1)_4\oplus(3)_4$

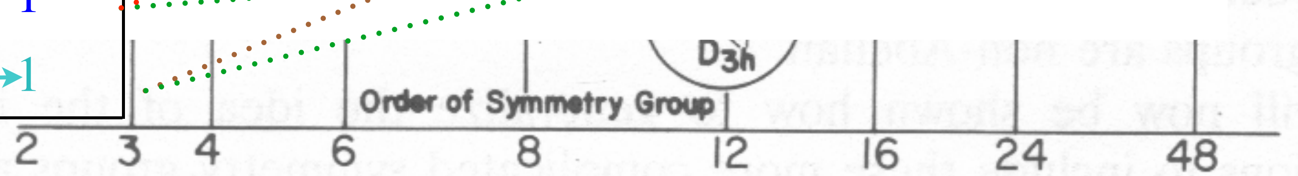
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

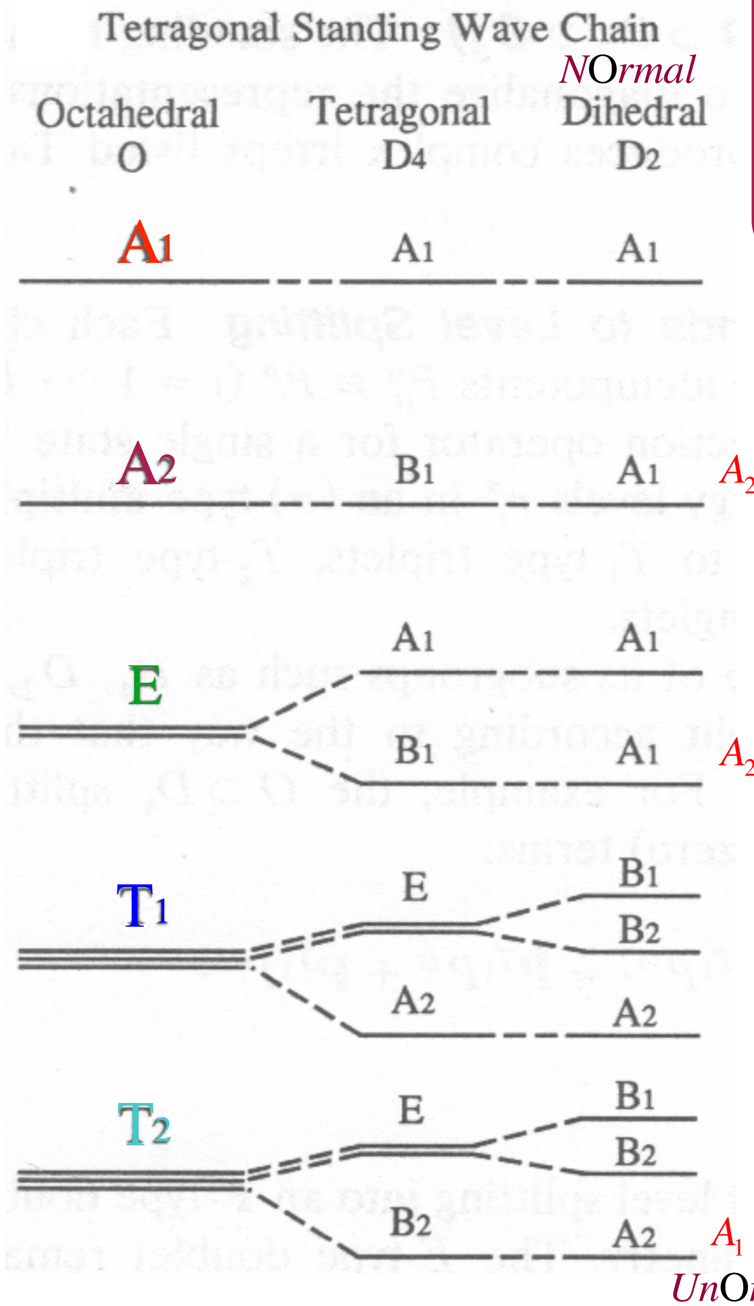
O↓C₄ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1



$O_h \supset O \supset D_4 \supset C_4$ subgroup splitting



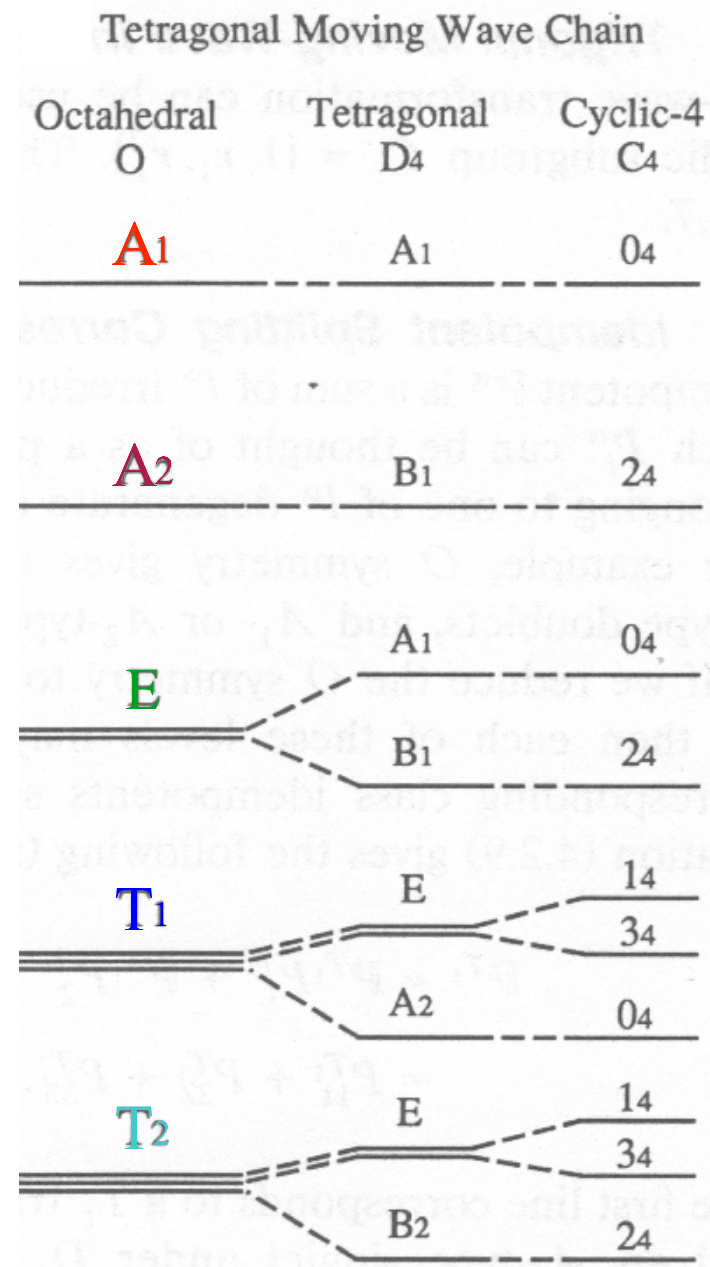
D_4	$\mathbf{1}$	ρ_z	$\mathbf{R}_z$	$\rho_{x,y}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

*NO*rmal $D_2 = \{1, \mathbf{R}_3^2, \mathbf{R}_1^2, \mathbf{R}_2^2\}$

$D_4 \downarrow D_2$	A_1	B_1	A_2	B_2
A_1	1	·	·	·
B_1	1	·	·	·
A_2	·	·	1	·
B_2	·	·	1	·
E	·	1	·	1

*Un*Ormal $D_2 = \{1, \mathbf{R}_3^2, \mathbf{i}_3, \mathbf{i}_4\}$

$D_4 \downarrow D_2$	A_1	B_1	A_2	B_2
A_1	1	·	·	·
B_1	·	·	1	·
A_2	·	·	1	·
B_2	1	·	·	·
E	·	1	·	1



$D_2^{Nm}\{\ \mathbf{1},\ \mathbf{R}_z^2,\ \mathbf{R}_x^2,\ \mathbf{R}_y^2\}$

$D_2^{Un}\{\ \mathbf{1},\ \mathbf{R}_z^2,\ \mathbf{i}_3,\ \mathbf{i}_4\}$

A_1	1	1	1	1
B_1	1	-1	1	-1
A_2	1	1	-1	-1
B_2	1	-1	-1	1

$-1_4 =$

$D_4 \downarrow C_4$	0_4	1_4	2_4	3_4
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1

		$\mathbf{r}, \tilde{\mathbf{r}}_i$	ρ_{xyz}	$\mathbf{R}, \tilde{\mathbf{R}}_{xyz}$	
O	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{R}^2$	$\mathbf{R}^3$	$\mathbf{i}_k$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

*NO*rmal $D_2 = \{1, \mathbf{R}_3^2, \mathbf{R}_1^2, \mathbf{R}_2^2\}$ *Un*Ormal $D_2 = \{1, \mathbf{R}_3^2, \mathbf{i}_3, \mathbf{i}_4\}$

$O \downarrow D_2$	A_1	B_1	A_2	B_2
A_1	1	·	·	·
A_2	1	·	·	·
E	2	·	·	·
T_1	·	1	1	1
T_2	·	1	1	1

$O \downarrow D_2$	A_1	B_1	A_2	B_2
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	·	1	1	1
T_2	1	1	·	1

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$-1_4 =$

$O \downarrow C_4$	0_4	1_4	2_4	3_4
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	1	1	1

$O_h \supset O \supset D_4 \supset C_{4v} \supset C_{2v}$ *subgroup splitting*

$\downarrow C_{4v}$	A'	B'	A''	B''	E
$\mathscr{D}^{A_{1g}}$	1	·	·	·	·
$\mathscr{D}^{A_{2g}}$	·	1	·	·	·
$\mathscr{D}^{E_g}$	1	1	·	·	·
$\mathscr{D}^{T_{1g}}$	·	·	1	·	1
$\mathscr{D}^{T_{2g}}$	·	·	·	1	1
$\mathscr{D}^{A_{1u}}$	·	·	1	·	·
$\mathscr{D}^{A_{2u}}$	·	·	·	1	·
$\mathscr{D}^{E_u}$	·	·	1	1	·
$\mathscr{D}^{T_{1u}}$	1	·	·	·	1
$\mathscr{D}^{T_{2u}}$	·	1	·	·	1

$\downarrow C_{2v}$	A'	B'	A''	B''
$\mathscr{D}^{A_{1g}}$	1	·	·	·
$\mathscr{D}^{A_{2g}}$	·	1	·	·
$\mathscr{D}^{E_g}$	1	1	·	·
$\mathscr{D}^{T_{1g}}$	·	1	1	1
$\mathscr{D}^{T_{2g}}$	1	·	1	1
$\mathscr{D}^{A_{1u}}$	·	·	1	·
$\mathscr{D}^{A_{2u}}$	·	·	·	1
$\mathscr{D}^{E_u}$	·	·	1	1
$\mathscr{D}^{T_{1u}}$	1	1	·	1
$\mathscr{D}^{T_{2u}}$	1	1	1	·

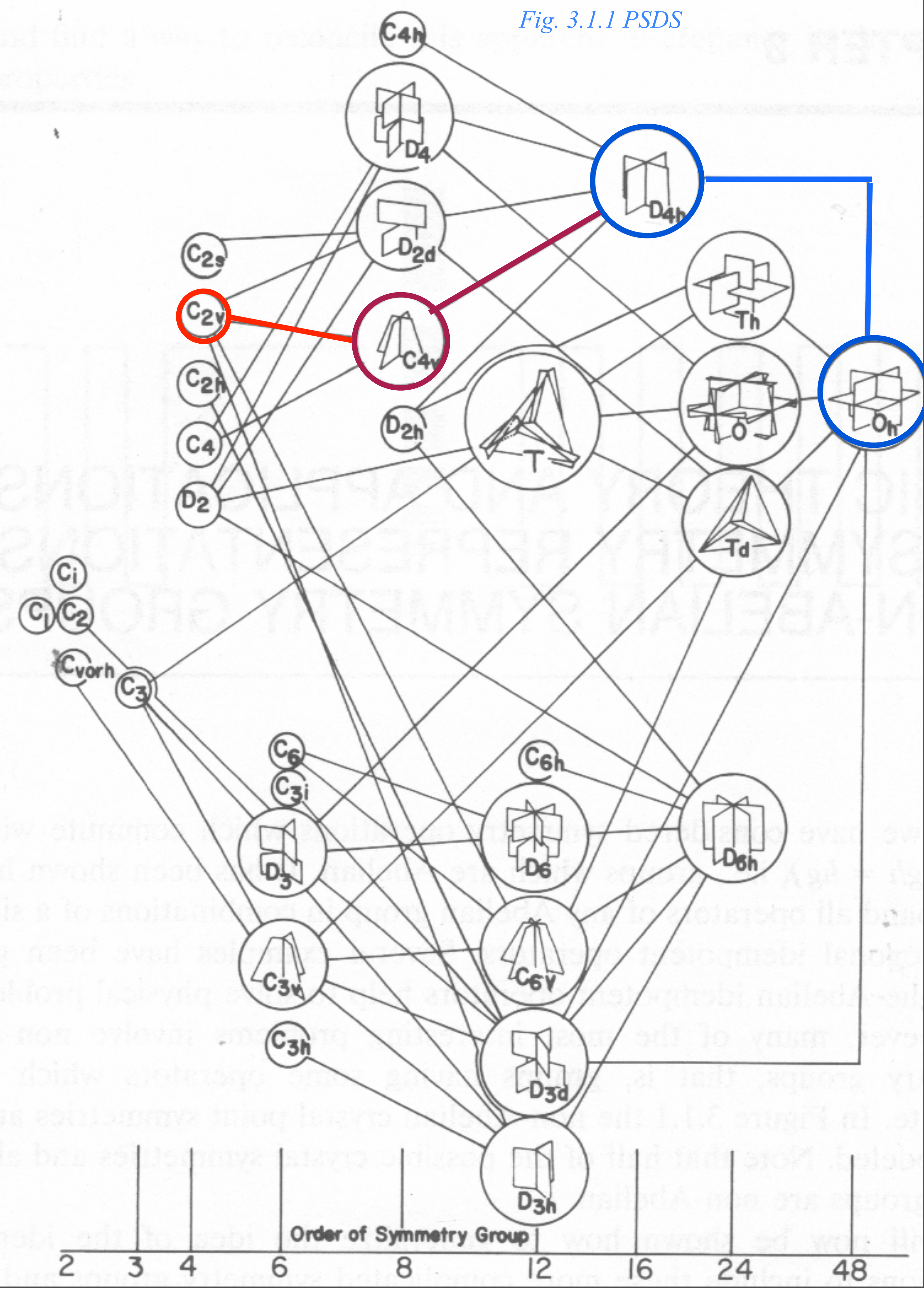


Fig. 3.1.1 PSDS

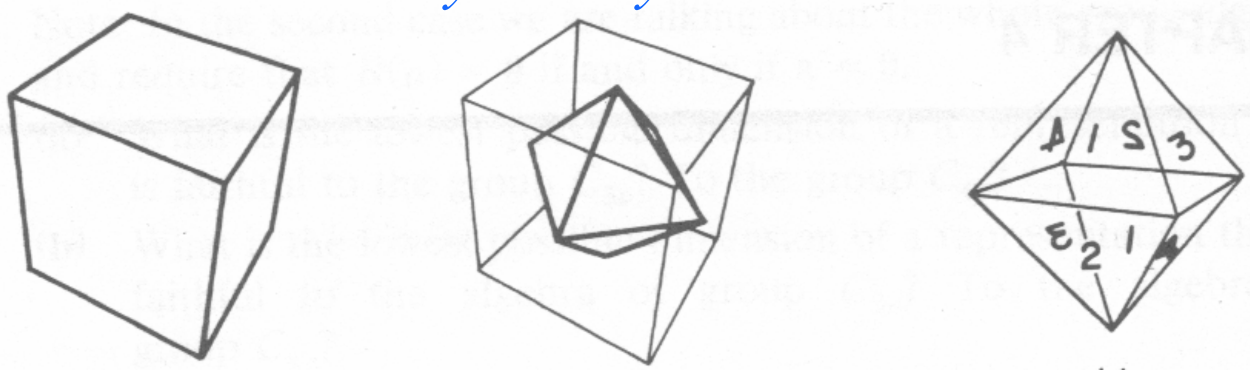
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_4$	$\tilde{\mathbf{r}}_1$	$\tilde{\mathbf{r}}_2$	$\tilde{\mathbf{r}}_3$	$\tilde{\mathbf{r}}_4$	ρ_x	ρ_y	ρ_z	$\mathbf{R}_x$	$\mathbf{R}_y$	$\mathbf{R}_z$	$\tilde{\mathbf{R}}_x$	$\tilde{\mathbf{R}}_y$	$\tilde{\mathbf{R}}_z$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$
ρ_z	$\mathbf{r}_3$	$\mathbf{r}_4$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\tilde{\mathbf{r}}_4$	$\tilde{\mathbf{r}}_3$	$\tilde{\mathbf{r}}_2$	$\tilde{\mathbf{r}}_1$	ρ_y	ρ_x	$\mathbf{1}$	$\mathbf{i}_6$	$\mathbf{i}_2$	$\tilde{\mathbf{R}}_z$	$\mathbf{i}_5$	$\mathbf{i}_1$	$\mathbf{R}_z$	$\tilde{\mathbf{R}}_y$	$\mathbf{R}_y$	$\mathbf{i}_4$	$\mathbf{i}_3$	$\tilde{\mathbf{R}}_x$	$\mathbf{R}_x$
$\mathbf{R}_z$	$\mathbf{i}_6$	$\mathbf{i}_5$	$\mathbf{R}_x$	$\tilde{\mathbf{R}}_x$	$\tilde{\mathbf{R}}_y$	$\mathbf{R}_y$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\tilde{\mathbf{R}}_z$	$\mathbf{r}_1$	$\tilde{\mathbf{r}}_3$	ρ_z	$\mathbf{r}_2$	$\tilde{\mathbf{r}}_4$	$\mathbf{1}$	$\tilde{\mathbf{r}}_1$	$\tilde{\mathbf{r}}_2$	ρ_y	ρ_x	$\mathbf{r}_4$	$\mathbf{r}_3$
$\tilde{\mathbf{R}}_z$	$\mathbf{R}_x$	$\tilde{\mathbf{R}}_x$	$\mathbf{i}_6$	$\mathbf{i}_5$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{R}_y$	$\tilde{\mathbf{R}}_y$	$\mathbf{i}_4$	$\mathbf{i}_3$	$\mathbf{R}_z$	$\mathbf{r}_3$	$\tilde{\mathbf{r}}_2$	$\mathbf{1}$	$\mathbf{r}_4$	$\tilde{\mathbf{r}}_1$	ρ_z	$\tilde{\mathbf{r}}_4$	$\tilde{\mathbf{r}}_3$	ρ_x	ρ_y	$\mathbf{r}_2$	$\mathbf{r}_1$

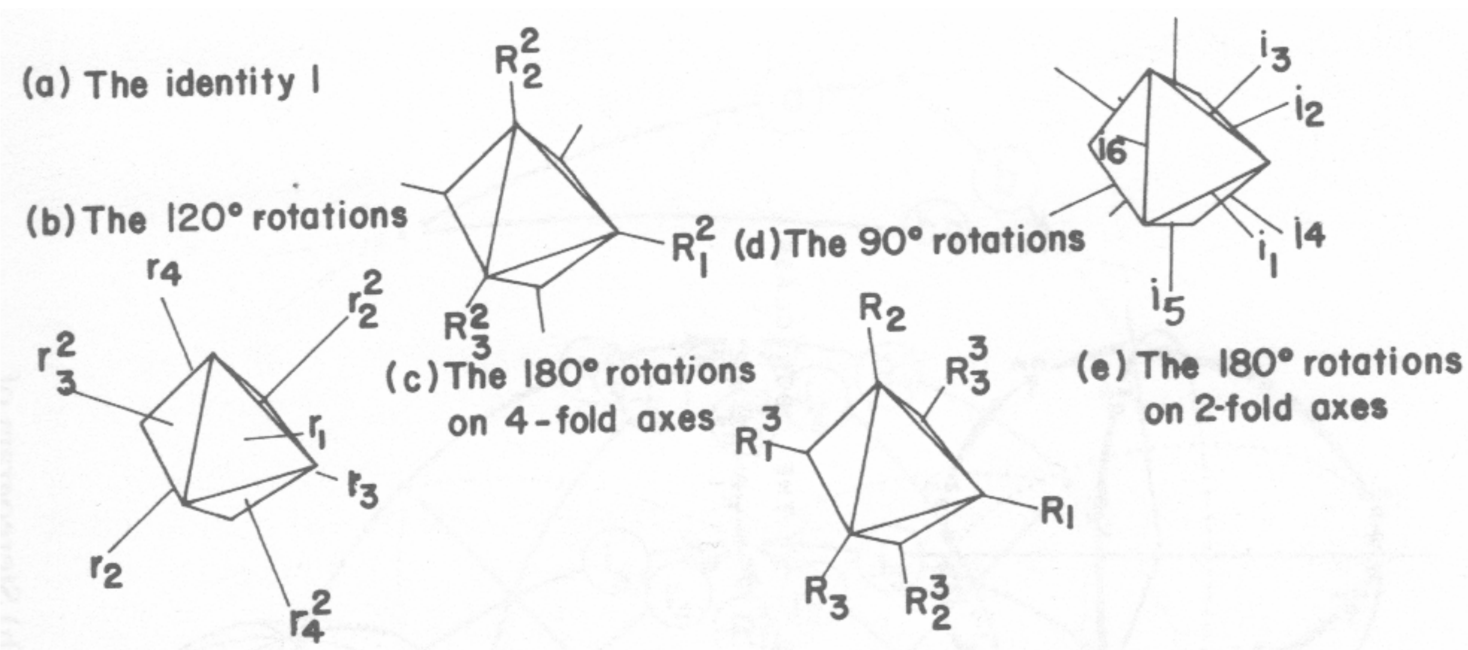
$\mathbf{1}$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_4$	$\tilde{\mathbf{r}}_1$	$\tilde{\mathbf{r}}_2$	$\tilde{\mathbf{r}}_3$	$\tilde{\mathbf{r}}_4$	ρ_x	ρ_y	ρ_z	$\mathbf{R}_x$	$\mathbf{R}_y$	$\mathbf{R}_z$	$\tilde{\mathbf{R}}_x$	$\tilde{\mathbf{R}}_y$	$\tilde{\mathbf{R}}_z$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\mathbf{i}_5$	$\mathbf{i}_6$
ρ_z	$\mathbf{r}_3$	$\mathbf{r}_4$	$\mathbf{r}_1$	$\mathbf{r}_2$	$\tilde{\mathbf{r}}_4$	$\tilde{\mathbf{r}}_3$	$\tilde{\mathbf{r}}_2$	$\tilde{\mathbf{r}}_1$	ρ_y	ρ_x	$\mathbf{1}$	$\mathbf{i}_6$	$\mathbf{i}_2$	$\tilde{\mathbf{R}}_z$	$\mathbf{i}_5$	$\mathbf{i}_1$	$\mathbf{R}_z$	$\tilde{\mathbf{R}}_y$	$\mathbf{R}_y$	$\mathbf{i}_4$	$\mathbf{i}_3$	$\tilde{\mathbf{R}}_x$	$\mathbf{R}_x$
$\mathbf{R}_z$	$\mathbf{i}_6$	$\mathbf{i}_5$	$\mathbf{R}_x$	$\tilde{\mathbf{R}}_x$	$\tilde{\mathbf{R}}_y$	$\mathbf{R}_y$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_4$	$\tilde{\mathbf{R}}_z$	$\mathbf{r}_1$	$\tilde{\mathbf{r}}_3$	ρ_z	$\mathbf{r}_2$	$\tilde{\mathbf{r}}_4$	$\mathbf{1}$	$\tilde{\mathbf{r}}_1$	$\tilde{\mathbf{r}}_2$	ρ_y	ρ_x	$\mathbf{r}_4$	$\mathbf{r}_3$
$\tilde{\mathbf{R}}_z$	$\mathbf{R}_x$	$\tilde{\mathbf{R}}_x$	$\mathbf{i}_6$	$\mathbf{i}_5$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{R}_y$	$\tilde{\mathbf{R}}_y$	$\mathbf{i}_4$	$\mathbf{i}_3$	$\mathbf{R}_z$	$\mathbf{r}_3$	$\tilde{\mathbf{r}}_2$	$\mathbf{1}$	$\mathbf{r}_4$	$\tilde{\mathbf{r}}_1$	ρ_z	$\tilde{\mathbf{r}}_4$	$\tilde{\mathbf{r}}_3$	ρ_x	ρ_y	$\mathbf{r}_2$	$\mathbf{r}_1$

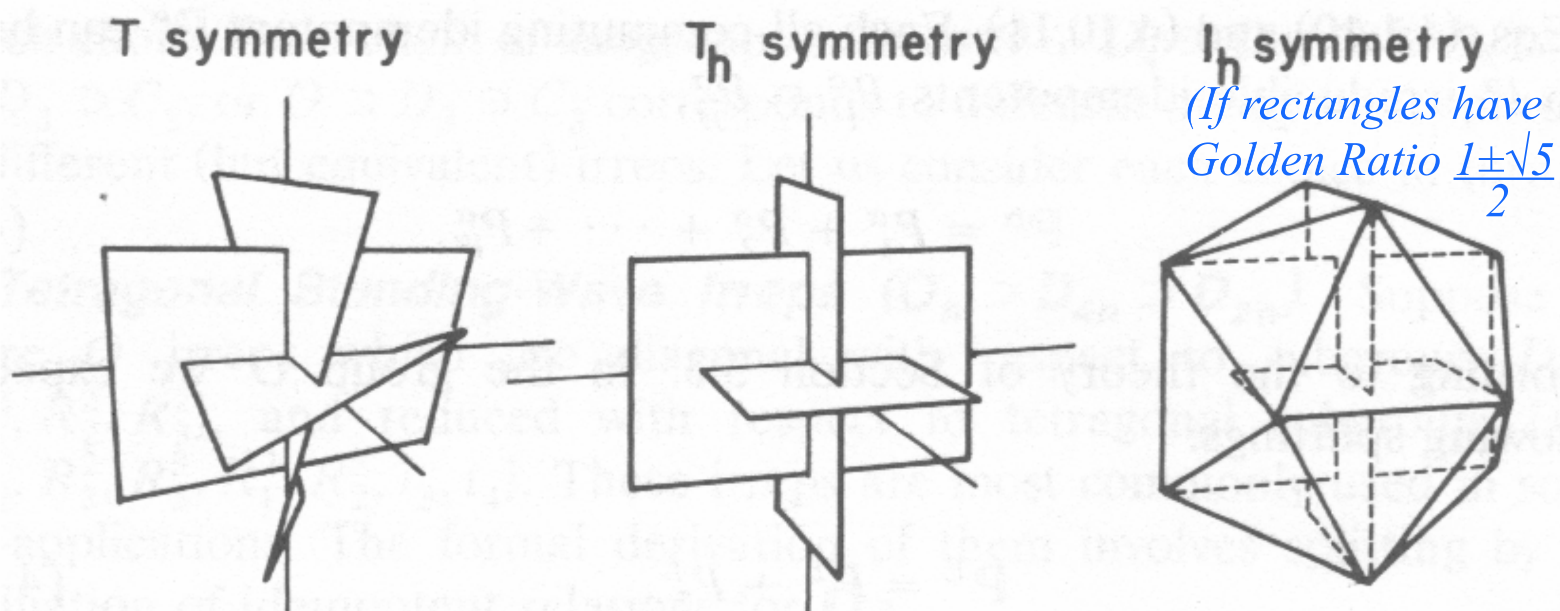
Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry

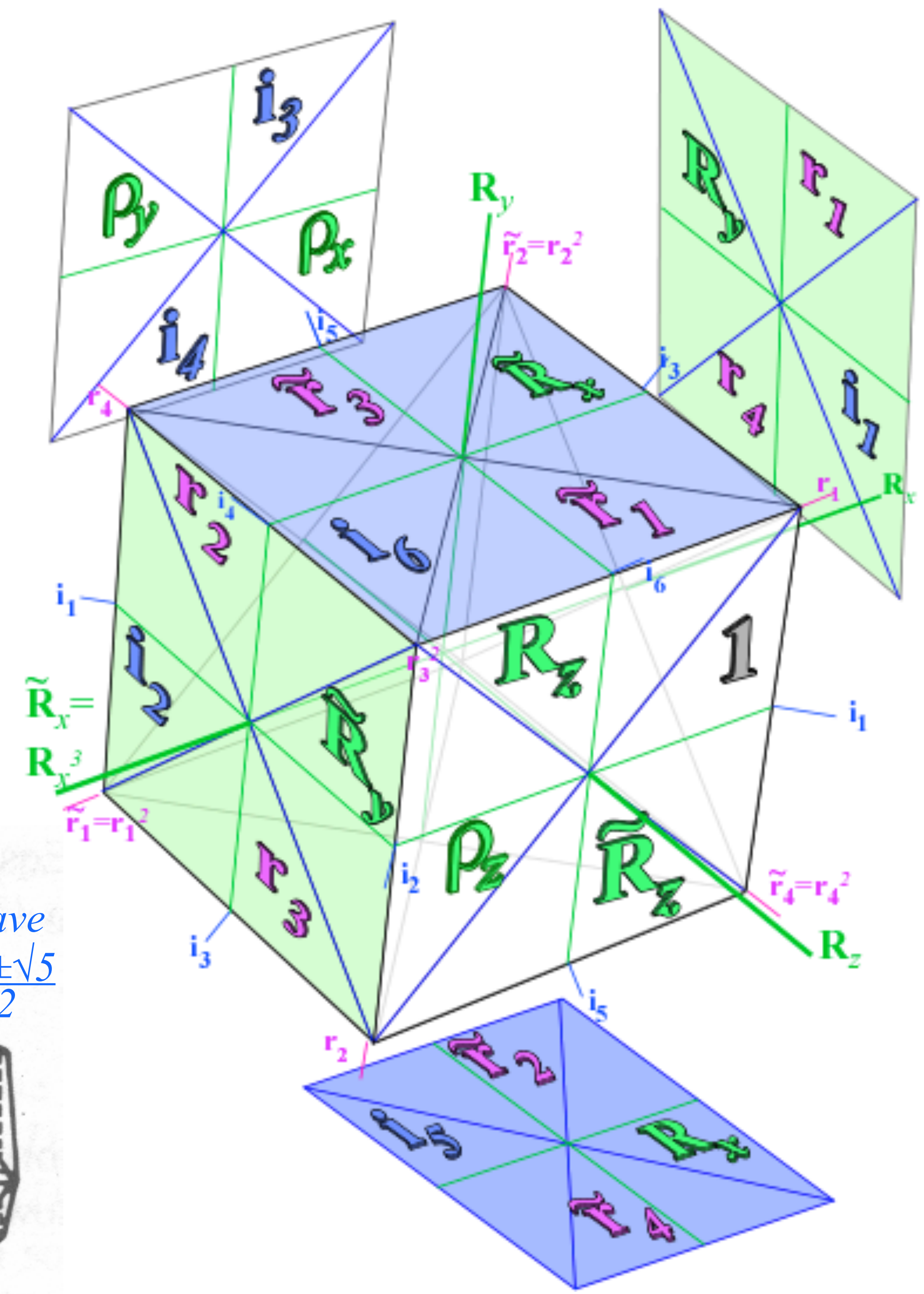
Order °O=6 hexahedron squares · 4 pts =24
=8 octahedron triangles · 3 pts =24
=12 lines · 2 pts =24 positions





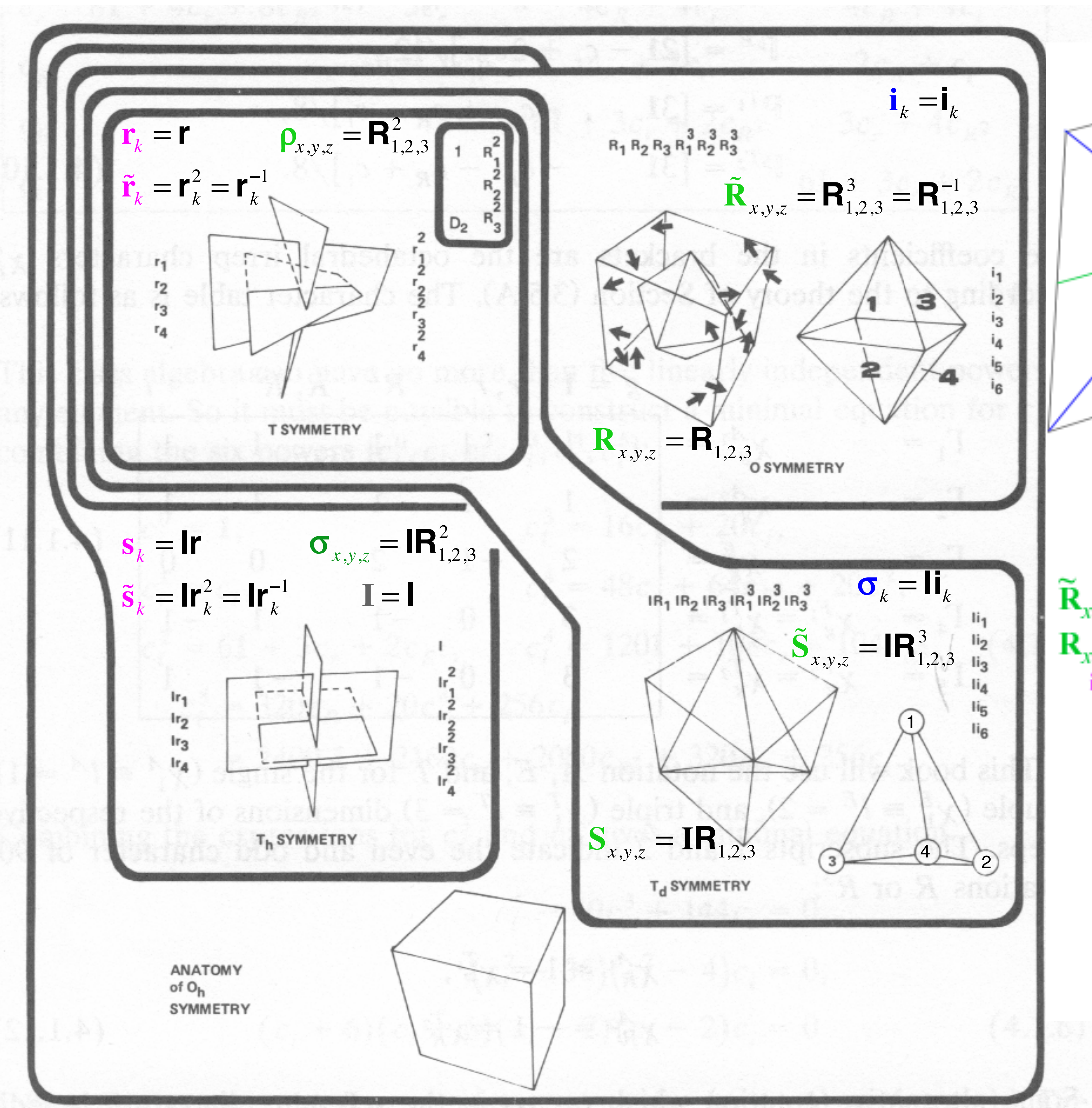


(If rectangles have Golden Ratio $\frac{1\pm\sqrt{5}}{2}$



Introduction to octahedral tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral groups $O_h \supset O \sim T_d$ and $O_h \supset T_h \supset T$



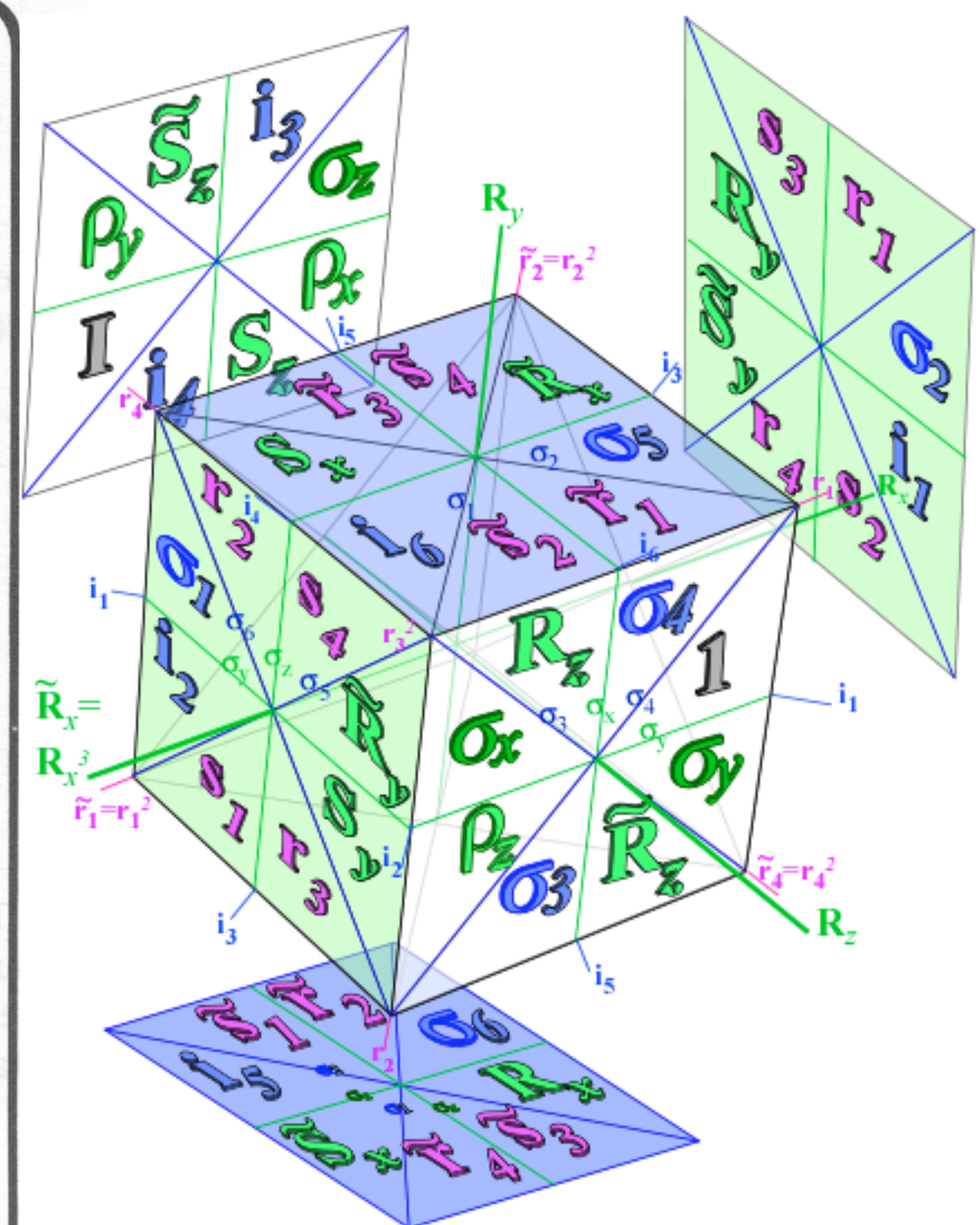
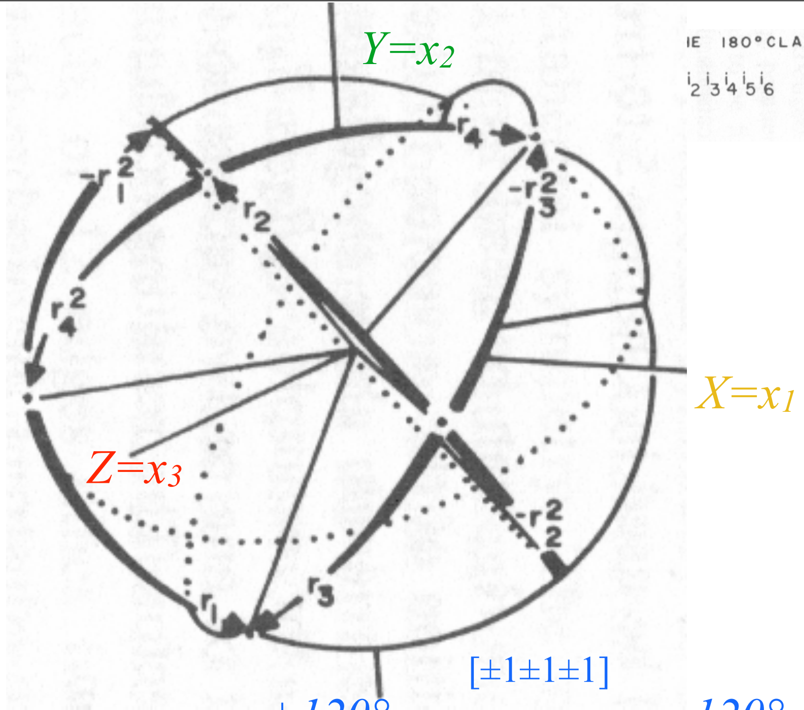
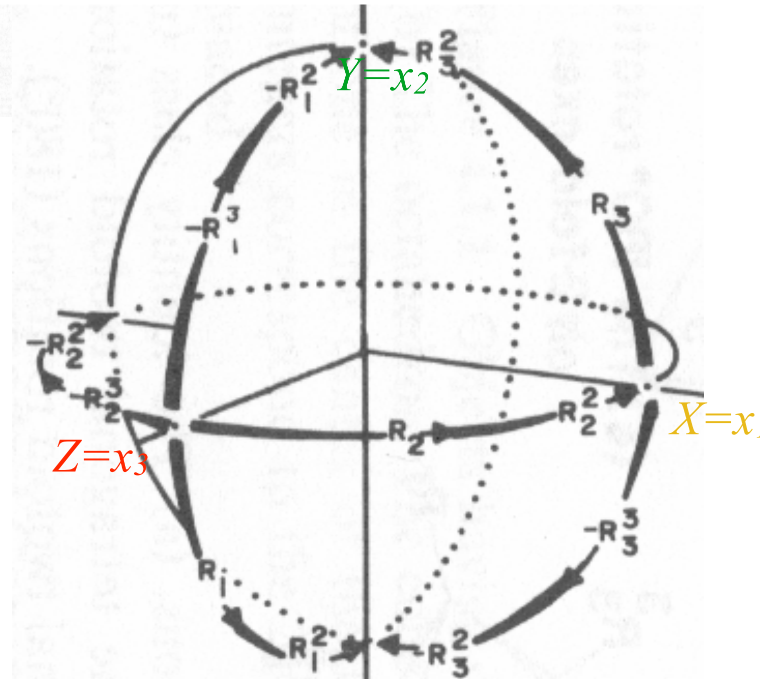
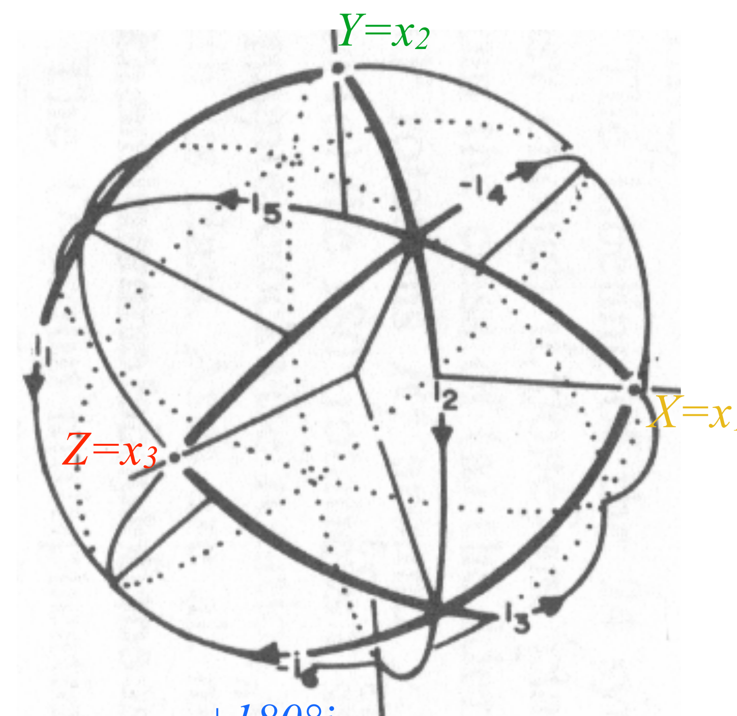


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T, T_h, T_d, and O. The Abelian D_2 subgroup of T is indicated also.

Fig. 4.1.5 from Principles of Symmetry, Dynamics and Spectroscopy







+120°: [1 1 1] [$\bar{1}$ $\bar{1}$ 1] [1 $\bar{1}$ $\bar{1}$] [$\bar{1}$ 1 $\bar{1}$] -120°: [$\bar{1}$ $\bar{1}$ $\bar{1}$] [1 1 $\bar{1}$] [$\bar{1}$ 1 1] [1 $\bar{1}$ 1] ±180° XYZ: [1 0 0] [0 1 0] [0 0 1] +90° XYZ: [1 0 0] [0 1 0] [0 0 1] -90° XYZ: [$\bar{1}$ 0 0] [0 $\bar{1}$ 0] [0 0 $\bar{1}$] ±180° i_k: [1 0 1] [1 0 $\bar{1}$] [1 1 0] [1 $\bar{1}$ 0] [0 1 $\bar{1}$] [0 1 1]

1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	i_3	i_6	i_1	$-R_3$	$-R_1$	$-R_2$	R_1^3	i_5	R_2^3	i_2	$-i_4$	R_3^3
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	R_3	$-R_1^3$	i_2	i_3	$-i_5$	R_2^3	i_6	$-R_1$	R_2	$-i_1$	R_3^3	i_4
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	$-i_4$	R_1	$-R_2^3$	R_3^3	i_6	i_2	i_5	$-R_1^3$	i_1	R_2	$-i_3$	R_3
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	$-R_3^3$	$-i_5$	R_2	$-i_4$	R_1^3	i_1	R_1	i_6	$-i_2$	R_2^3	R_3	i_3
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	$-i_1$	$-i_3$	$-i_6$	$-R_3$	$-i_4$	$-R_1$	i_5	$-i_2$	$-R_2$
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	i_2	$-i_3$	$-R_1$	R_2	$-R_3^3$	$-i_5$	i_4	$-R_3$	$-R_1^3$	$-i_6$	R_2^3	$-i_1$
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	$-R_2$	$-i_4$	$-i_6$	i_2	R_3	$-R_1^3$	$-i_3$	$-R_3^3$	i_5	R_1	$-i_1$	$-R_2^3$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	$-i_1$	$-R_3$	$-i_5$	$-R_2^3$	$-i_4$	R_1	$-R_3^3$	i_3	$-i_6$	R_1^3	R_2	$-i_2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	R_1^3	i_1	$-i_4$	$-R_1$	i_2	$-i_3$	$-R_2$	$-R_2^3$	R_3^3	R_3	$-i_6$	i_5
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	$-i_5$	R_2^3	i_3	$-i_6$	$-R_2$	$-i_4$	$-i_2$	i_1	$-R_3$	R_3^3	R_1	R_1^3
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	i_6	i_2	R_3^3	$-i_5$	$-i_1$	$-R_3$	R_2^3	$-R_2$	i_4	$-i_3$	R_1^3	$-R_1$
R_1	i_1	$-R_2^3$	$-i_2$	R_2	R_3^3	$-i_3$	$-R_3$	i_4	R_1^3	i_6	i_5	R_1^2	r_1	$-r_4^2$	-1	$-r_3$	r_2^2	$-r_4$	r_2	r_1^2	$-r_3^2$	$-R_2^2$	R_3^2
R_2	i_3	R_3	$-R_3^3$	i_4	R_1^3	i_5	$-i_6$	$-R_1$	$-i_2$	R_2^3	i_1	$-r_2^2$	R_2^2	r_1	r_3^2	-1	$-r_4$	R_1^2	R_3^2	$-r_2$	$-r_3$	$-r_4^2$	r_1^2
R_3	i_6	i_5	R_1	$-R_1^3$	R_2^3	$-R_2$	$-i_2$	$-i_1$	i_3	i_4	R_3^3	r_1	$-r_3^2$	R_3^2	$-r_2$	r_4^2	-1	r_1^2	r_2^2	R_2^2	$-R_1^2$	$-r_4$	$-r_3$
R_1^3	$-R_2$	$-i_2$	R_2^3	i_1	$-i_3$	$-R_3^3$	i_4	R_3	$-R_1$	i_5	$-i_6$	-1	$-r_4$	r_3^2	$-R_1^2$	r_2	$-r_1^2$	$-r_1$	r_3	r_2^2	$-r_4^2$	$-R_3^2$	$-R_2^2$
R_2^3	$-R_3$	i_3	i_4	R_3^3	$-i_6$	R_1	$-R_1^3$	i_5	$-i_1$	$-R_2$	$-i_2$	r_4^2	-1	$-r_2$	$-r_1^2$	$-R_2^2$	r_3	$-R_3^2$	R_1^2	$-r_1$	$-r_4$	$-r_2^2$	r_3^2
R_3^3	$-R_1$	R_1^3	i_6	i_5	$-i_1$	$-i_2$	R_2	$-R_2^3$	i_4	$-i_3$	$-R_3$	$-r_3$	r_2^2	-1	r_4	$-r_1^2$	$-R_3^2$	r_4^2	r_3^2	$-R_1^2$	$-R_2^2$	$-r_2$	$-r_1$
i_1	R_3^3	$-i_4$	i_3	R_3	$-R_1$	$-i_6$	$-i_5$	$-R_3^3$	R_3^3	i_2	$-R_2$	r_1^2	R_3^2	$-r_4$	r_4^2	$-R_1^2$	$-r_1$	-1	$-R_2^2$	$-r_3$	r_2	r_3^2	r_2^2
i_2	i_4	R_3^3	R_3	$-i_3$	$-i_5$	R_1^3	R_1	$-i_6$	R_2	$-i_1$	R_2^3	$-r_3^2$	$-R_1^2$	$-r_3$	$-r_2^2$	$-R_3^2$	$-r_2$	R_2^2	-1	r_4	$-r_1$	r_1^2	r_4^2
i_3	R_1^3	R_1	$-i_5$	i_6	$-R_2$	$-R_2^3$	$-i_1$	i_2	$-R_3$	R_3^3	$-i_4$	$-r_2$	r_1^2	R_1^2	$-r_1$	r_2^2	$-R_2^2$	r_3^2	$-r_4^2$	-1	R_3^2	r_3	$-r_4$
i_4	$-i_5$	i_6	$-R_1^3$	$-R_1$	$-i_2$	i_1	$-R_2^3$	$-R_2$	$-R_3^3$	$-R_3$	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	$-r_2^2$	r_1^2	$-R_3^3$	-1	r_1	$-r_2$
i_5	i_2	$-R_2$	i_1	$-R_2^3$	i_4	$-R_3$	i_3	$-R_3^3$	i_6	$-R_1^3$	$-R_1$	R_3^2	r_2	r_2^2	R_2^2	r_4	r_4^2	$-r_3$	$-r_1$	$-r_3^2$	$-r_1^2$	-1	$-R_1^2$
i_6	R_2^3	i_1	R_2	i_2	$-R_3$	$-i_4$	$-R_3^3$	$-i_3$	$-i_5$	$-R_1$	R_1^3	R_2^2	$-r_3$	r_1^2	$-R_3^2$	$-r_1$	r_3^2	$-r_2$	$-r_4$	r_4^2	r_2^2	R_1^2	-1

Octahedral O and spin-O⊂U(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy