

Group Theory in Quantum Mechanics

Lecture 25 (4.30.13)

Rotational symmetry $U(2) \subset U(3)$ and $O(3)$

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)

(PSDS - Ch. 5, 7)

Review : 2-D \mathfrak{a} algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

Angular momentum raise-n-lower operators \mathbf{s}_+ and \mathbf{s}_-

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

Angular momentum commutation relations

Key Lie theorems

Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating $R(3)$ rotation and $U(2)$ representations

Applications of $R(3)$ rotation and $U(2)$ representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenlevels

Generalized Stern-Gerlach and transformation matrices

Angular momentum cones and high J properties

As of May 25, 2013

Links to the current Harter-Soft LearnIt web apps for Physics

(**Bold links** have default redirect pages. *Italics* are not yet meant for production. **Red** are in the final stages of testing.

Production Links - *For the students & general public*

BohrIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/BohrItWeb.html>"

BounceIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/BounceItWeb.html>"

BoxIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/BoxItWeb.html>"

CoulIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/CoulItWeb.html>"

Cycloidulum - Production; URL is "<http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html>"

LearnIt - Production; URL is "<http://www.uark.edu/ua/modphys>" or "<http://www.uark.edu/ua/modphys/markup/LearnItWeb.html>"

JerkIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/JerkItWeb.html>"

Pendulum - Production; URL is "<http://www.uark.edu/ua/modphys/markup/PendulumWeb.html>"

QuantIt - Production; URL is "<http://www.uark.edu/ua/modphys/markup/QuantItWeb.html>"

Relativity - Pirelli Entrant - Production; URL is "<http://www.uark.edu/ua/pirelli>" or "<http://www.uark.edu/ua/pirelli/html/default.html>"

Trebuchet Production; URL is "<http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html>"

Testing Links - *For internal use and testing by Harter & Heyoka*

BohrIt - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html>"

BounceIt - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html>"

BounceIt Title Page - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html>"

BoxIt - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html>"

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Cycloidulum - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html>"

Harter-Soft Web Apps - Quick Reference - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html>"

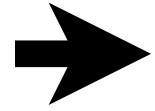
JerkIt - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html>"

ModernPhysics - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html>"

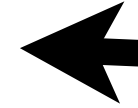
Pendulum - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html>"

QuantIt - Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/QuantItWeb.html>"

Trebuchet Testing; URL is "<http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html>"



Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations



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Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

	$ 00\rangle$	$ 01\rangle$	$ 10\rangle$	$ 02\rangle$	$ 11\rangle$	$ 20\rangle$	$ 03\rangle$	$ 12\rangle$	$ 21\rangle$	$ 30\rangle$...
$\langle 00 $	0										
$\langle 01 $		ω_-									
$\langle 10 $			ω_+								
$\langle 02 $				$2\omega_-$							
$\langle 11 $					$\omega_+ + \omega_-$						
$\langle 20 $						$2\omega_+$					
$\langle 03 $							$3\omega_-$				
$\langle 12 $								$\omega_+ + 2\omega_-$			
$\langle 21 $									$2\omega_+ + \omega_-$		
$\langle 30 $										$3\omega_+$	
\vdots											

$$\begin{aligned} \omega_+ - \omega_- &= \Omega \\ &= \sqrt{(2B)^2 + (2C)^2 + (A - D)^2} \\ &= A - D \end{aligned}$$

$$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

$$\mathbf{H}^A = A(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2) + D(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2)$$

$$\begin{aligned} \epsilon_{n_1 n_2}^A &= A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2) \\ &= \Omega_0(n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\nu + 1) + \Omega m \end{aligned}$$

Define *total quantum number* $\nu=2j$ and half-difference or *asymmetry quantum number* m

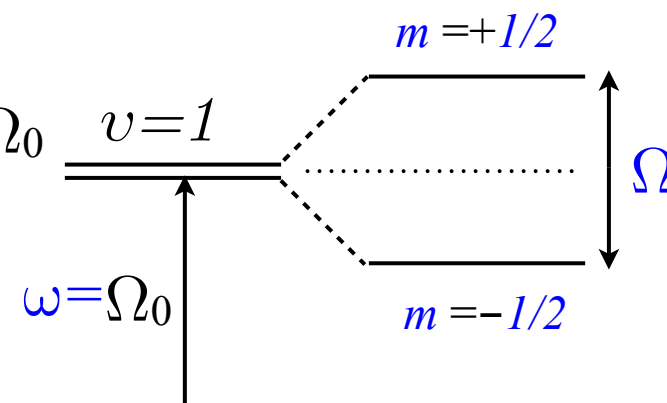
$$\nu = n_1 + n_2 = 2j$$

$$j = \frac{n_1 + n_2}{2} = \frac{\nu}{2}$$

$$m = \frac{n_1 - n_2}{2}$$

$\nu+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\frac{1}{2}\right)$$

$$\omega_- = \Omega_0 + \Omega\left(-\frac{1}{2}\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

	$ 00\rangle$	$ 01\rangle$	$ 10\rangle$	$ 02\rangle$	$ 11\rangle$	$ 20\rangle$	$ 03\rangle$	$ 12\rangle$	$ 21\rangle$	$ 30\rangle$...
$\langle 00 $	0										
$\langle 01 $		ω_-									
$\langle 10 $			ω_+								
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$\langle 11 $					$\omega_+ + \omega_-$						
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$\langle 30 $										$3\omega_+$	
\vdots											

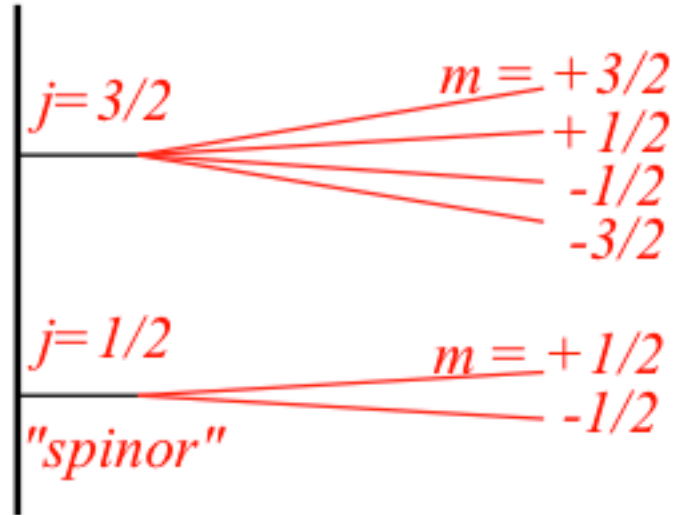
$$\omega_+ - \omega_- = \Omega$$

$$= \sqrt{(2B)^2 + (2C)^2 + (A - D)^2}$$

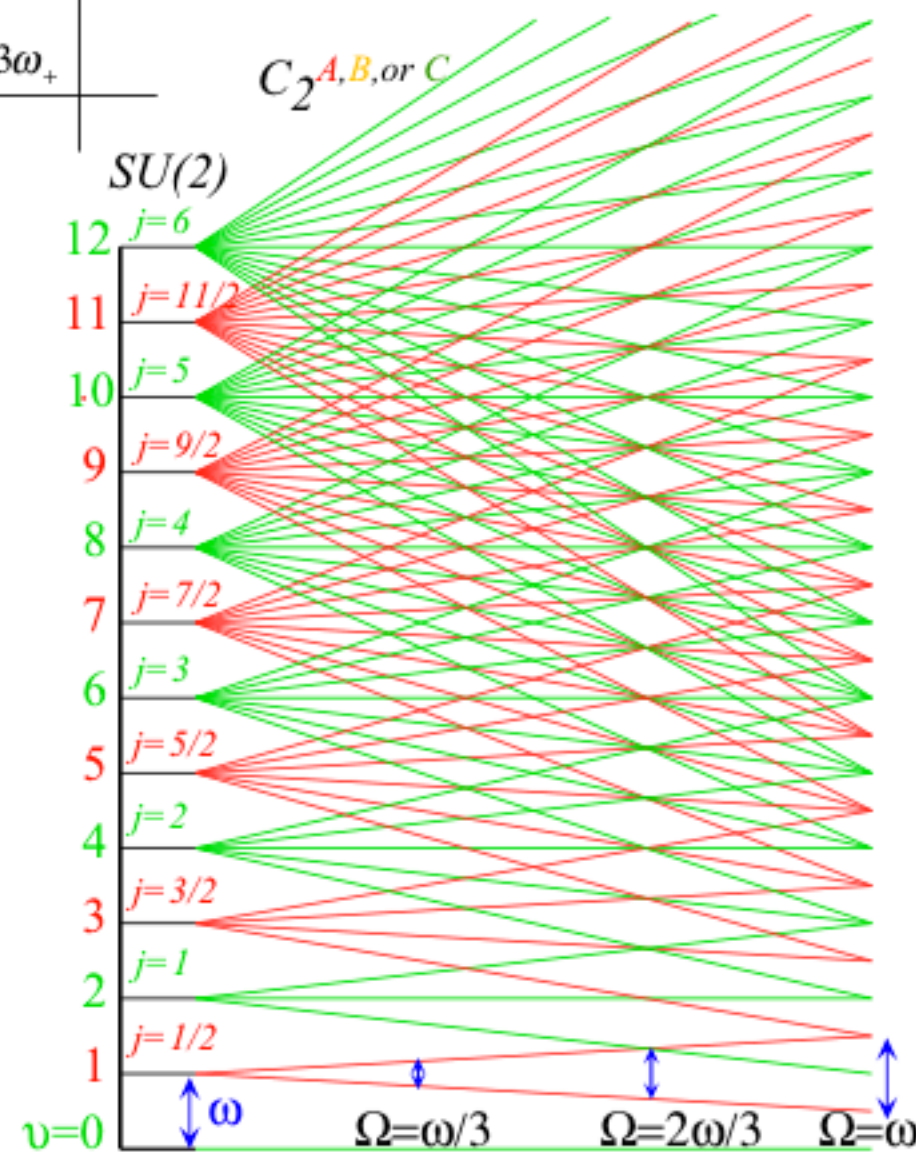
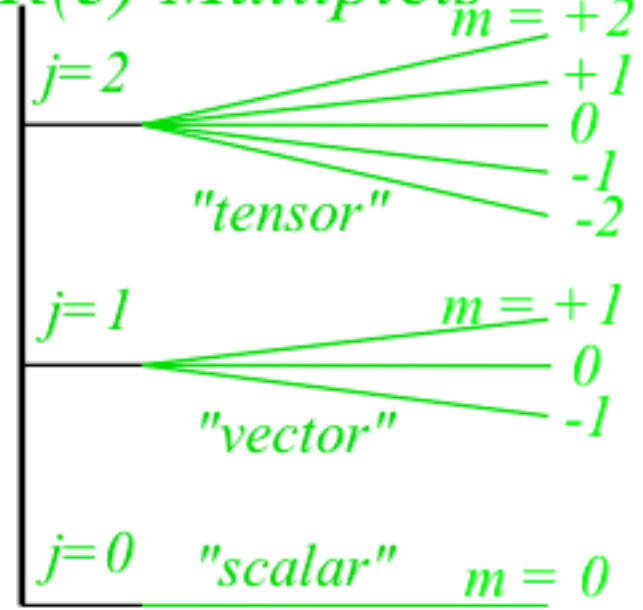
$$= A - D$$

$$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$$

SU(2) Multiplets



R(3) Multiplets

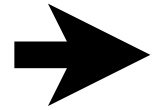


$C_2^{A,B, \text{ or } C}$

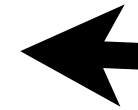
SU(2)

$v=0$

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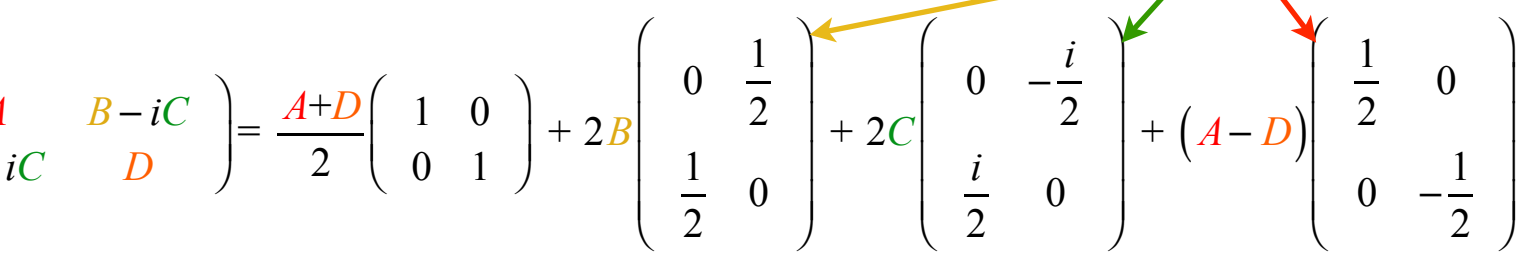
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Angular momentum generators by U(2) analysis

($\nu=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$


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($\nu=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D) \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B \begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C \begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D) \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

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($\nu=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B \begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C \begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D) \begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

Angular momentum generators by U(2) analysis

($\nu=1$) or ($j=1/2$) block \mathbf{H} matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$.

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($\nu=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

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($\nu=2j$) or ($2j+1$)-by- $(2j+1)$ block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j\text{-block}} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

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All j -block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal \mathbf{s}_Z

$$\langle \mathbf{H} \rangle^{j\text{-block}} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[(\Omega_X - i\Omega_Y) \langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y) \langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right] / 2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

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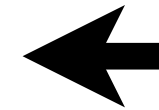
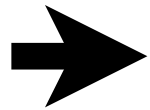
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$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with $j=1/2$ we see that \mathbf{S}_+ is an elementary projection operator $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to creation-destruction operator combinations $\mathbf{a}^\dagger \mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow, \quad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

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$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin \downarrow
creates up-spin \uparrow

to raise angular momentum by one \hbar unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

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$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin \uparrow
creates dn-spin \downarrow

to lower angular momentum by one \hbar unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

Angular momentum raise-n-lower operators \mathbf{s}_+ and \mathbf{s}_-

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

Angular momentum commutation relations

Key Lie theorems

Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating $R(3)$ rotation and $U(2)$ representations

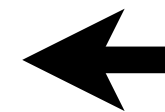
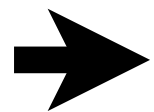
Applications of $R(3)$ rotation and $U(2)$ representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenstates

Generalized Stern-Gerlach and transformation matrices

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$U(2)$ boson oscillator states $|n_1, n_2\rangle$

Oscillator total quanta: $\nu = (n_1 + n_2)$

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Oscillator $\mathbf{a}^\dagger \mathbf{a} \dots$

$$\mathbf{a}_1^\dagger \mathbf{a}_2 |n_1 n_2\rangle = \sqrt{n_1 + 1} \sqrt{n_2} |n_1 + 1, n_2 - 1\rangle$$

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Oscillator $\mathbf{a}^\dagger \mathbf{a}$ give \mathbf{s}_+ matrices.

$$\mathbf{a}_1^\dagger \mathbf{a}_2 |n_1 n_2\rangle = \sqrt{n_1+1} \sqrt{n_2} |n_1+1 n_2-1\rangle \Rightarrow \mathbf{s}_+ \begin{pmatrix} j \\ m \end{pmatrix} = \sqrt{j+m+1} \sqrt{j-m} \begin{pmatrix} j \\ m+1 \end{pmatrix}$$

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$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

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Oscillator total quanta: $\nu = (n_1 + n_2)$ Rotor total momenta: $j = \nu/2$ and z-momenta: $m = (n_1 - n_2)/2$

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$j=2$ tensor \mathbf{s}_+ ...and \mathbf{s}_Z

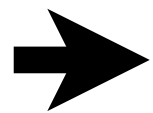
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Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

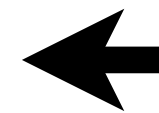
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Key Lie theorems



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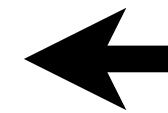
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Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk} \mathbf{e}_{mn} = \delta_{km} \mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

$$\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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U(2) Oscillator
eigensolutions:

$$\mathbf{H} |n_1 n_2\rangle = \sum_{n=1}^2 \omega_n \mathbf{a}_n^\dagger \mathbf{a}_n |n_1 n_2\rangle = (\omega_1 n_1 + \omega_2 n_2) |n_1 n_2\rangle = (\omega_1 (j+m) + \omega_2 (j-m)) \left| \begin{matrix} j \\ m \end{matrix} \right\rangle$$

Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

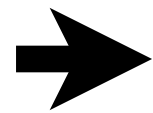
Angular momentum generators by $U(2)$ analysis

Angular momentum raise-n-lower operators \mathbf{s}_+ and \mathbf{s}_-

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

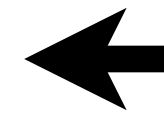
Angular momentum commutation relations

Key Lie theorems



Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle



Generating $R(3)$ rotation and $U(2)$ representations

Applications of $R(3)$ rotation and $U(2)$ representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenstates

Generalized Stern-Gerlach and transformation matrices

Angular momentum cones and high J properties

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Angular momentum squared $\mathbf{s}\cdot\mathbf{s}$ and *Z-component* \mathbf{s}_Z share eigenstates

$$\mathbf{s}\cdot\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

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$$D^{\frac{1}{2}}(\mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2) = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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(Normal order: *left ← creation, destruct → right.*)

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For Large j:

Magnitude of angular momentum $|\mathbf{s}|$ approaches $j + 1/2$: $|\mathbf{s}| \left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle = \sqrt{\mathbf{s} \cdot \mathbf{s}} \left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle = \sqrt{j(j + 1)} \left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle \cong \left(j + \frac{1}{2} \right) \left| \begin{smallmatrix} j \\ m \end{smallmatrix} \right\rangle$

Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

Angular momentum raise-n-lower operators \mathbf{s}_+ and \mathbf{s}_-

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

Angular momentum commutation relations

Key Lie theorems

Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating $R(3)$ rotation and $U(2)$ representations

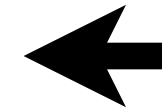
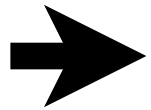
Applications of $R(3)$ rotation and $U(2)$ representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenstates

Generalized Stern-Gerlach and transformation matrices

Angular momentum cones and high J properties



Angular momentum uncertainty angle

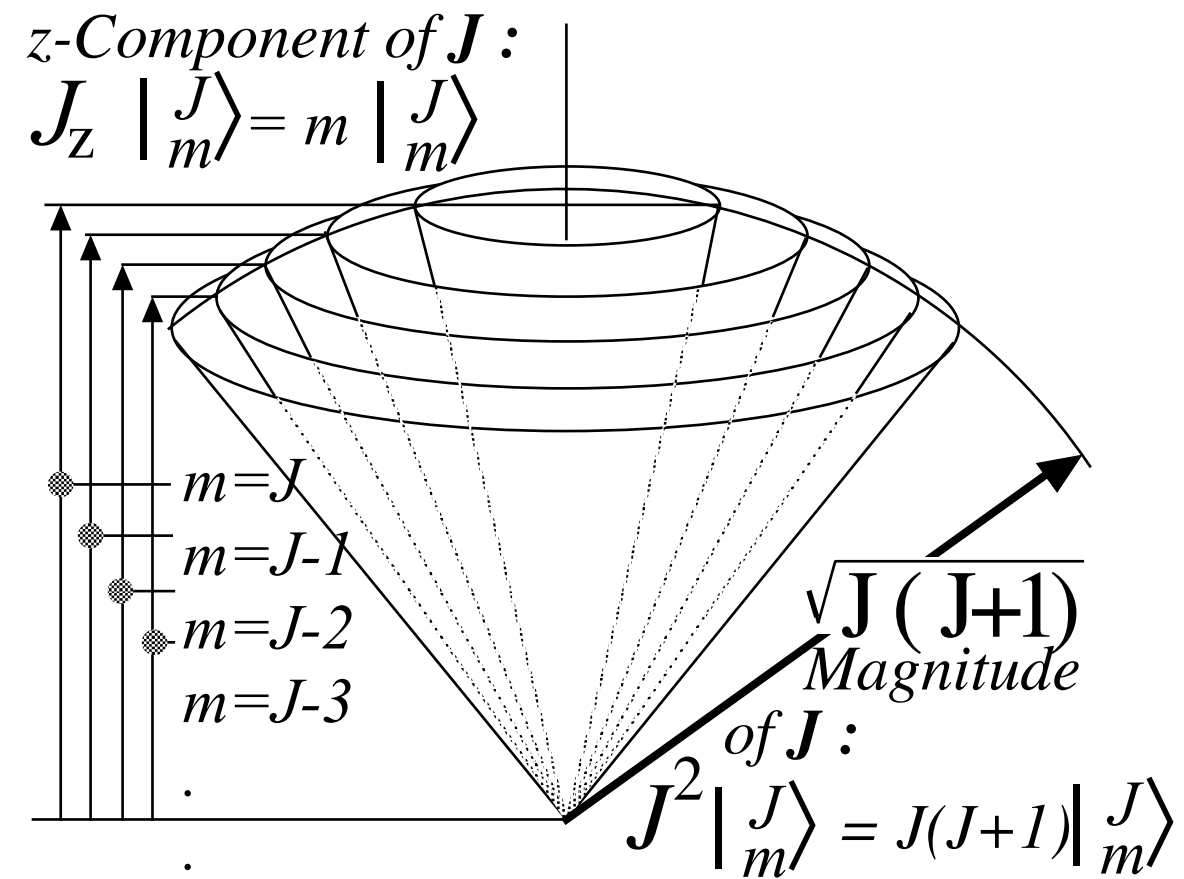
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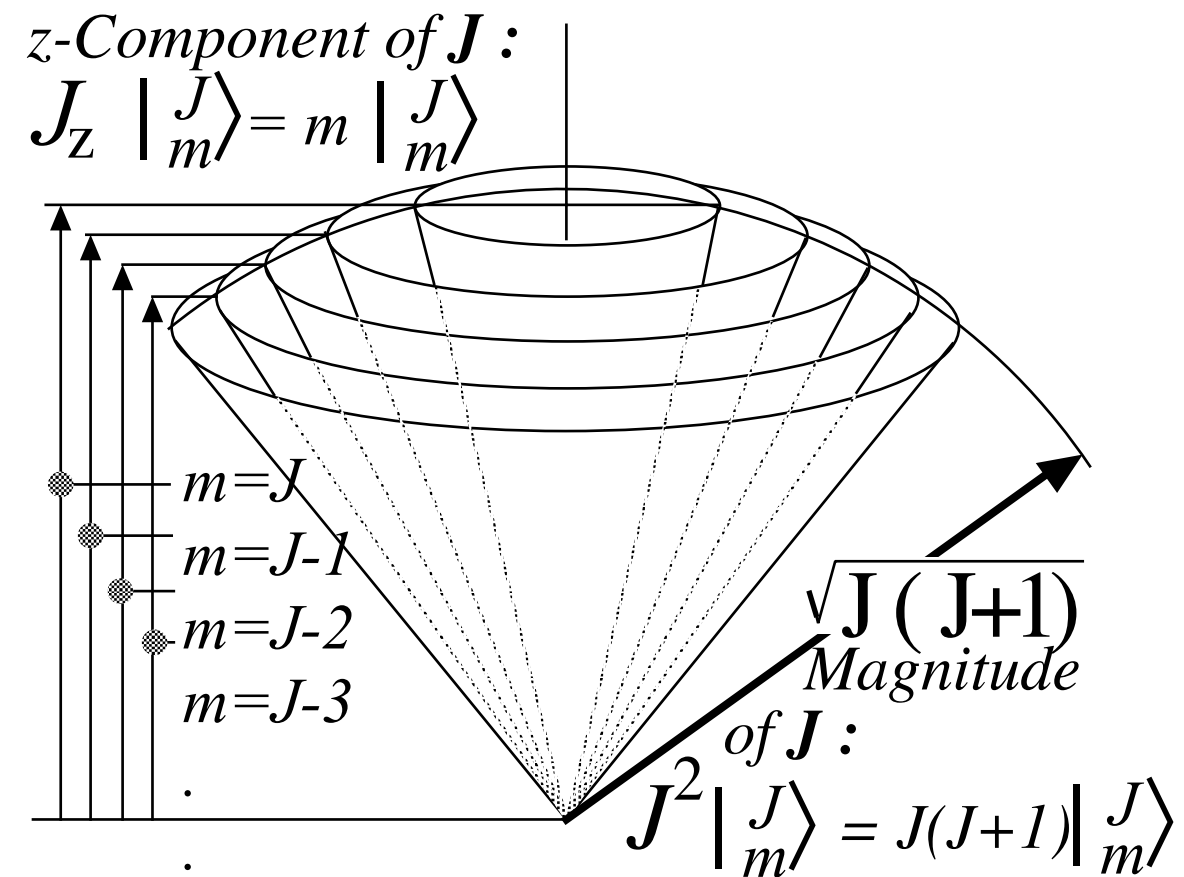
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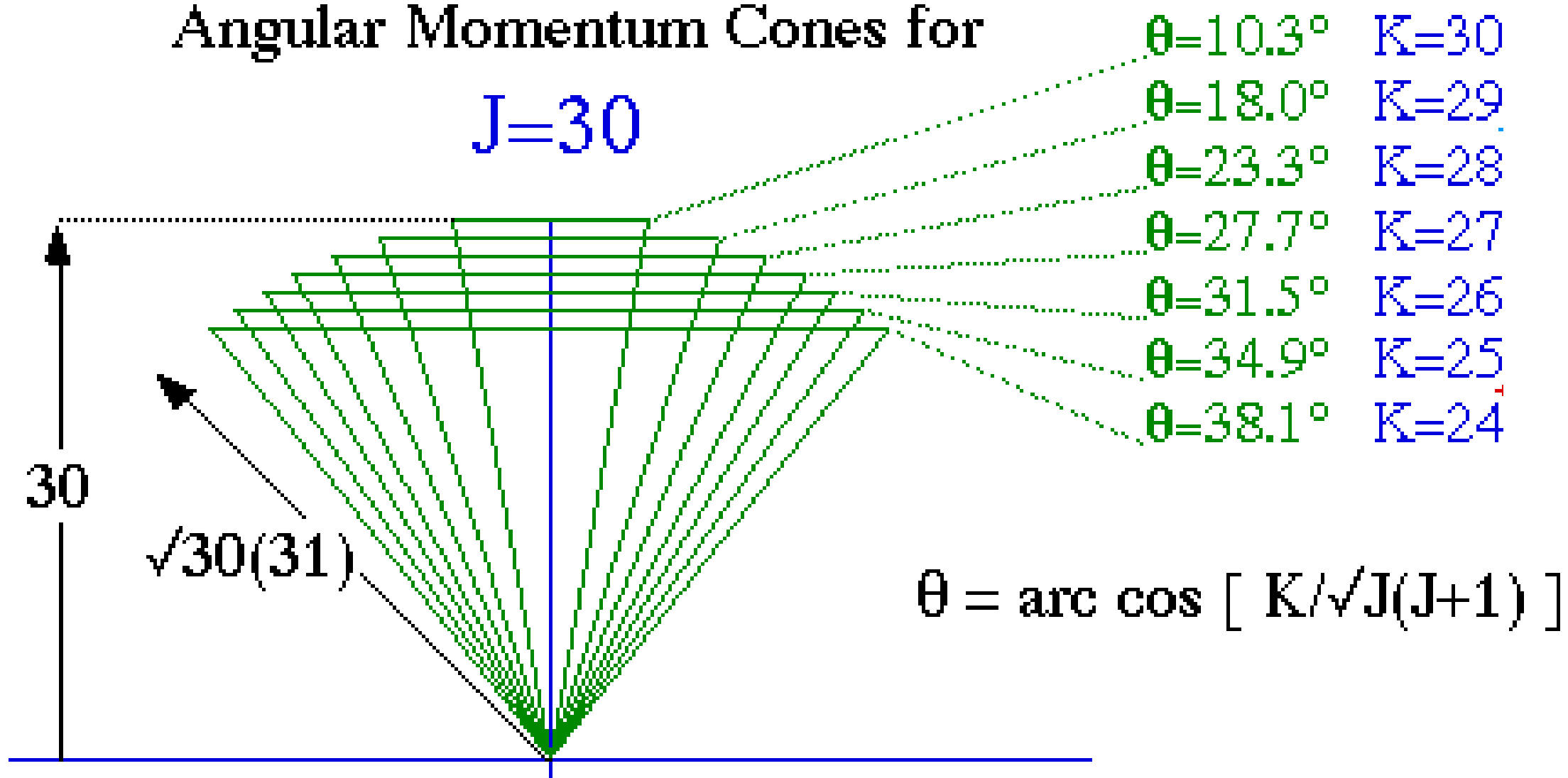
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Angular Momentum Cones for $J=30$



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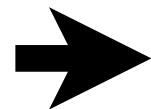
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Generating $R(3)$ rotation and $U(2)$ representations

A fundamental (spin-1/2) Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1' \rangle & \langle 1|2' \rangle \\ \langle 2|1' \rangle & \langle 2|2' \rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1 \rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2 \rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1 \rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2 \rangle \end{pmatrix} = \begin{pmatrix} D_{11}^{1/2}(\alpha\beta\gamma) & D_{12}^{1/2}(\alpha\beta\gamma) \\ D_{21}^{1/2}(\alpha\beta\gamma) & D_{22}^{1/2}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

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Corresponding transformation of the fundamental creation operators

$$\mathbf{a}_{1'}^\dagger = D_{11}^{1/2}(\alpha\beta\gamma)\mathbf{a}_1^\dagger + D_{21}^{1/2}(\alpha\beta\gamma)\mathbf{a}_2^\dagger = e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \mathbf{a}_1^\dagger + e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \mathbf{a}_2^\dagger$$

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Binomial expansion is a double sum over binomial coefficients: $\binom{n}{k} = n! / k!(n-k)!$

$$\mathbf{R}(\alpha\beta\gamma) \left| \begin{smallmatrix} j \\ n \end{smallmatrix} \right\rangle = \frac{\sum_{\ell} \sum_k \binom{j+n}{\ell} (D_{11}\mathbf{a}_1^\dagger)^\ell (D_{21}\mathbf{a}_2^\dagger)^{j+n-\ell} \binom{j-n}{k} (D_{12}\mathbf{a}_1^\dagger)^k (D_{22}\mathbf{a}_2^\dagger)^{j-n-k}}{\sqrt{(j+n)!(j-n)!}} |00\rangle = \frac{\sum_{\ell} \sum_k \binom{j+n}{\ell} \binom{j-n}{k} (D_{11}\mathbf{a}_1^\dagger)^\ell (D_{21}\mathbf{a}_2^\dagger)^{j+n-\ell} (D_{12}\mathbf{a}_1^\dagger)^k (D_{22}\mathbf{a}_2^\dagger)^{j-n-k}}{\sqrt{(j+n)!(j-n)!} \ell!(j+n-\ell)!k!(j-n-k)!} |00\rangle$$

Generating $R(3)$ rotation and $U(2)$ representations

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1' \rangle & \langle 1|2' \rangle \\ \langle 2|1' \rangle & \langle 2|2' \rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1 \rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2 \rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1 \rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2 \rangle \end{pmatrix} = \begin{pmatrix} D_{11}^{1/2}(\alpha\beta\gamma) & D_{12}^{1/2}(\alpha\beta\gamma) \\ D_{21}^{1/2}(\alpha\beta\gamma) & D_{22}^{1/2}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}_{1'}^\dagger = D_{11}^{1/2}(\alpha\beta\gamma) \mathbf{a}_1^\dagger + D_{21}^{1/2}(\alpha\beta\gamma) \mathbf{a}_2^\dagger = e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \mathbf{a}_1^\dagger + e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \mathbf{a}_2^\dagger$$

$$\mathbf{a}_{2'}^\dagger = D_{12}^{1/2}(\alpha\beta\gamma) \mathbf{a}_1^\dagger + D_{22}^{1/2}(\alpha\beta\gamma) \mathbf{a}_2^\dagger = -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \mathbf{a}_1^\dagger + e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \mathbf{a}_2^\dagger$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\nu=2j$)-oscillator state ($\nu=2j$)-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma) \left| \begin{matrix} j \\ n \end{matrix} \right\rangle = \frac{(\mathbf{a}_{1'}^\dagger)^{j+n} (\mathbf{a}_{2'}^\dagger)^{j-n}}{\sqrt{(j+n)!(j-n)!}} |00\rangle = \frac{(D_{11}\mathbf{a}_1^\dagger + D_{21}\mathbf{a}_2^\dagger)^{j+n} (D_{21}\mathbf{a}_1^\dagger + D_{22}\mathbf{a}_2^\dagger)^{j-n}}{\sqrt{(j+n)!(j-n)!}} |00\rangle$$

Binomial expansion is a double sum over binomial coefficients: $\binom{n}{k} = n! / k!(n-k)!$

$$\mathbf{R}(\alpha\beta\gamma) \left| \begin{matrix} j \\ n \end{matrix} \right\rangle = \frac{\sum_{\ell} \sum_k \binom{j+n}{\ell} (D_{11}\mathbf{a}_1^\dagger)^\ell (D_{21}\mathbf{a}_2^\dagger)^{j+n-\ell} \binom{j-n}{k} (D_{12}\mathbf{a}_1^\dagger)^k (D_{22}\mathbf{a}_2^\dagger)^{j-n-k}}{\sqrt{(j+n)!(j-n)!}} |00\rangle = \frac{\sum_{\ell} \sum_k \binom{j+n}{\ell} \binom{j-n}{k} (D_{11}\mathbf{a}_1^\dagger)^\ell (D_{21}\mathbf{a}_2^\dagger)^{j+n-\ell} (D_{12}\mathbf{a}_1^\dagger)^k (D_{22}\mathbf{a}_2^\dagger)^{j-n-k}}{\sqrt{(j+n)!(j-n)!} \ell!(j+n-\ell)!k!(j-n-k)!} |00\rangle$$

Let \mathbf{a}^\dagger -operator powers be $j \pm m$ forms : $j+m = \ell+k$, $j-m = 2j-\ell-k$ so $\ell \equiv j+m-k$ and $j+n-\ell \equiv n-m+k$

$$= \frac{\sum_{\ell} \sum_k \binom{j+n}{\ell} \binom{j-n}{k} (D_{11})^\ell (D_{21})^{j+n-\ell} (D_{12})^k (D_{22})^{j-n-k}}{\sqrt{(j+n)!(j-n)!} \ell!(j+n-\ell)!k!(j-n-k)!} (\mathbf{a}_1^\dagger)^{\ell+k} (\mathbf{a}_2^\dagger)^{2j-\ell-k} |00\rangle = \frac{\sum_m \sum_k \binom{j+n}{m} \binom{j-n}{k} (D_{11})^{j+m-k} (D_{21})^{n-m+k} (D_{12})^k (D_{22})^{j-n-k}}{\sqrt{(j+n)!(j-n)!} (j+m-k)!(n-m+k)!k!(j-n-k)!} (\mathbf{a}_1^\dagger)^{j+m} (\mathbf{a}_2^\dagger)^{j-m} |00\rangle$$

Generating $R(3)$ rotation and $U(2)$ representations

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

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Corresponding transformation of the fundamental creation operators

$$\mathbf{a}_{1'}^\dagger = D_{11}^{1/2}(\alpha\beta\gamma) \mathbf{a}_1^\dagger + D_{21}^{1/2}(\alpha\beta\gamma) \mathbf{a}_2^\dagger = e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \mathbf{a}_1^\dagger + e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \mathbf{a}_2^\dagger$$

$$\mathbf{a}_{2'}^\dagger = D_{12}^{1/2}(\alpha\beta\gamma) \mathbf{a}_1^\dagger + D_{22}^{1/2}(\alpha\beta\gamma) \mathbf{a}_2^\dagger = -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \mathbf{a}_1^\dagger + e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \mathbf{a}_2^\dagger$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\nu=2j$)-oscillator state

($\nu=2j$)-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma) \left| \begin{smallmatrix} j \\ n \end{smallmatrix} \right\rangle = \frac{(\mathbf{a}_{1'}^\dagger)^{j+n} (\mathbf{a}_{2'}^\dagger)^{j-n}}{\sqrt{(j+n)!(j-n)!}} |00\rangle = \frac{(D_{11}\mathbf{a}_1^\dagger + D_{21}\mathbf{a}_2^\dagger)^{j+n} (D_{21}\mathbf{a}_1^\dagger + D_{22}\mathbf{a}_2^\dagger)^{j-n}}{\sqrt{(j+n)!(j-n)!}} |00\rangle$$

This gives general *irreducible representation of $U(2)$* :

$$\left\langle \begin{smallmatrix} j \\ m \end{smallmatrix} \right| \mathbf{R}(\alpha\beta\gamma) \left| \begin{smallmatrix} j \\ n \end{smallmatrix} \right\rangle = D_{m,n}^j(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!} \sqrt{(j+m)!(j-m)!} \frac{\sum_k (D_{11})^{j+m-k} (D_{21})^{n-m+k} (D_{12})^k (D_{22})^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

And general *$SU(2)$ irreducible representation for Euler angles $(\alpha\beta\gamma)$* .

$$\left\langle \begin{smallmatrix} j \\ m \end{smallmatrix} \right| \mathbf{R}(\alpha\beta\gamma) \left| \begin{smallmatrix} j \\ n \end{smallmatrix} \right\rangle = D_{m,n}^j(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!} \sqrt{(j+m)!(j-m)!} \frac{\sum_k (-1)^k \left(\cos \frac{\beta}{2} \right)^{2j+m-n-2k} \left(\sin \frac{\beta}{2} \right)^{n-m+2k} e^{-i(m\alpha+n\gamma)}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

Angular momentum raise-n-lower operators \mathbf{s}_+ and \mathbf{s}_-

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

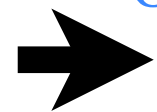
Angular momentum commutation relations

Key Lie theorems

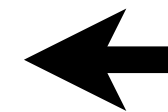
Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating $R(3)$ rotation and $U(2)$ representations



Applications of $R(3)$ rotation and $U(2)$ representations



Molecular and nuclear wavefunctions

Molecular and nuclear eigenlevels

Generalized Stern-Gerlach and transformation matrices

Angular momentum cones and high J properties

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

$$Y_m^\ell(\phi\theta) = D_{m,n=0}^{\ell*}(\phi\theta 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

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$$Y_m^\ell(\phi\theta) = D_{m,n=0}^{\ell*}(\phi\theta 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

$$Y_1^1(\phi,\theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi,\theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{-1}^1(\phi,\theta) = \sqrt{\frac{3}{4\pi}} e^{+i\phi} \frac{\sin\theta}{\sqrt{2}}$$

Applications of R(3) rotation and U(2) representations

Vector (j=l=1) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

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Center (n=0) column with the factor $\sqrt{\frac{2l+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^l .

$$Y_m^l(\phi, \theta) = D_{m, n=0}^{l*}(\phi, \theta, 0) \sqrt{\frac{2l+1}{4\pi}}$$

Dipole (j=l=1) wavefunctions

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

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Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

$$Y_m^\ell(\phi, \theta) = D_{m, n=0}^{\ell*}(\phi, \theta, 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix.

$$\begin{pmatrix} \langle 1 | 1 \rangle & \langle 1 | 1 \rangle & \langle -1 | 1 \rangle \\ \langle 0 | 1 \rangle & \langle 0 | 1 \rangle & \langle 0 | 1 \rangle \\ \langle -1 | 1 \rangle & \langle -1 | 1 \rangle & \langle -1 | 1 \rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{-1}^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{+i\phi} \frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

$$Y_m^\ell(\phi, \theta) = D_{m, n=0}^{\ell*}(\phi, \theta, 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \langle 1 | 1 \rangle & \langle 1 | 1 \rangle & \langle -1 | 1 \rangle \\ \langle 0 | 1 \rangle & \langle 0 | 1 \rangle & \langle 0 | 1 \rangle \\ \langle -1 | 1 \rangle & \langle -1 | 1 \rangle & \langle -1 | 1 \rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta \end{pmatrix}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{-1}^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{+i\phi} \frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

$$Y_m^\ell(\phi, \theta) = D_{m, n=0}^{\ell*}(\phi, \theta, 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \\ \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \\ \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{-1}^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{+i\phi} \frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

$$Y_m^\ell(\phi, \theta) = D_{m, n=0}^{\ell*}(\phi, \theta, 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \\ \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \\ \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ x \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ y \end{smallmatrix} \rangle & \langle \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} | \begin{smallmatrix} 1 \\ z \end{smallmatrix} \rangle \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} D_{x,x}^1(\alpha\beta\gamma) & D_{x,y}^1 & D_{x,z}^1 \\ D_{y,x}^1 & D_{y,y}^1 & D_{y,z}^1 \\ D_{z,x}^1 & D_{z,y}^1 & D_{z,z}^1 \end{pmatrix} = \begin{pmatrix} \cos\alpha \cos\beta \cos\gamma - \sin\alpha \sin\gamma & -\cos\alpha \cos\beta \sin\gamma - \sin\alpha \cos\gamma & \cos\alpha \sin\beta \\ \sin\alpha \cos\beta \cos\gamma + \cos\alpha \sin\gamma & -\sin\alpha \cos\beta \sin\gamma + \cos\alpha \cos\gamma & \sin\alpha \sin\beta \\ -\cos\gamma \sin\beta & \sin\gamma \sin\beta & \cos\beta \end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=l=1) representation

$$D^1(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} \frac{1+\cos\beta}{2} e^{-i\gamma} & e^{-i\alpha} \frac{-\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{-i\alpha} \frac{1-\cos\beta}{2} e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} e^{i\gamma} \\ e^{i\alpha} \frac{1-\cos\beta}{2} e^{-i\gamma} & e^{i\alpha} \frac{\sin\beta}{\sqrt{2}} e^{-i\gamma} & e^{i\alpha} \frac{1+\cos\beta}{2} e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2 \frac{\beta}{2} = \frac{1+\cos\beta}{2}$, $\sin^2 \frac{\beta}{2} = \frac{1-\cos\beta}{2}$, $\sin \frac{\beta}{2} \cos \frac{\beta}{2} = \frac{\sin\beta}{2}$,

$$Y_1^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{-i\phi} \frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{-1}^1(\phi, \theta) = \sqrt{\frac{3}{4\pi}} e^{+i\phi} \frac{\sin\theta}{\sqrt{2}}$$

Center (n=0) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y_m^ℓ .

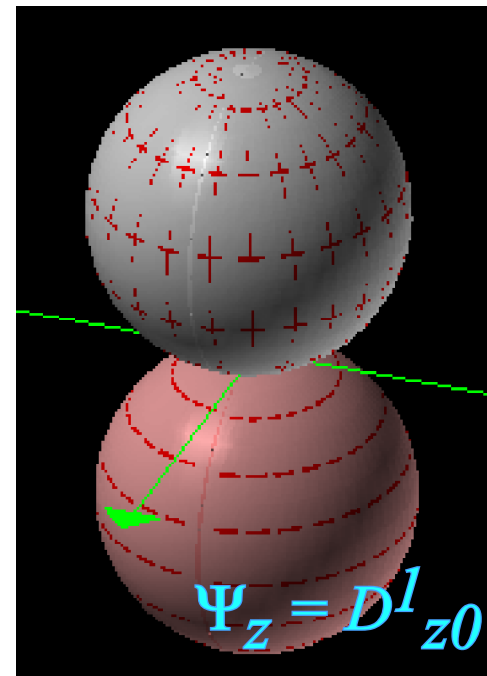
$$Y_m^\ell(\phi, \theta) = D_{m, n=0}^{\ell*}(\phi, \theta, 0) \sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=l=1) waves

$$D_{1,0}^{1*}(\phi, \theta) = -e^{i\phi} \frac{\sin\theta}{\sqrt{2}} = -\frac{\cos\phi \sin\theta + i \sin\phi \sin\theta}{\sqrt{2}} = -\frac{x+iy}{r\sqrt{2}}$$

$$D_{0,0}^{1*}(\phi, \theta) = \cos\theta = \frac{z}{r}$$

$$D_{-1,0}^{1*}(\phi, \theta) = e^{-i\phi} \frac{\sin\theta}{\sqrt{2}} = \frac{\cos\phi \sin\theta - i \sin\phi \sin\theta}{\sqrt{2}} = \frac{x-iy}{r\sqrt{2}}$$



j = 1
Standing
p-Waves

$$\Psi_x^1(\phi, \theta) = D_{x,z}^1(\phi, \theta, 0)$$

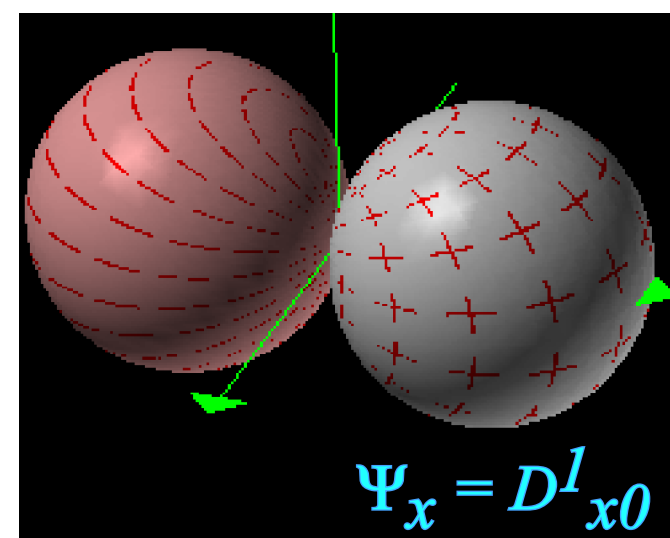
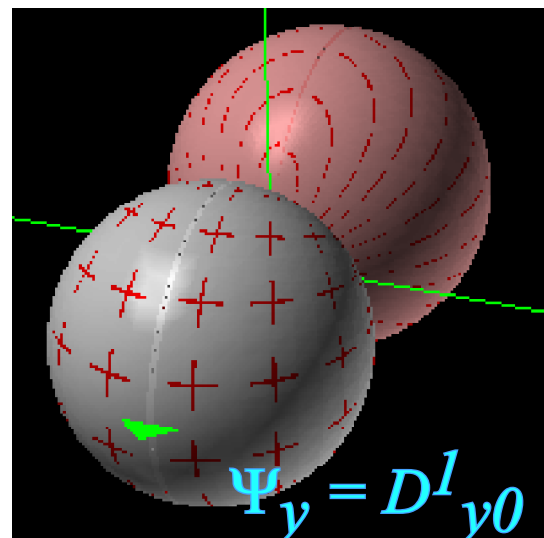
$$= \cos\phi \sin\theta$$

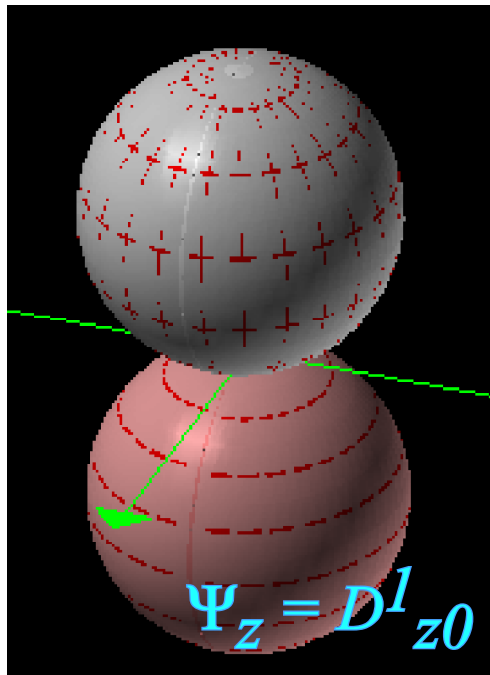
$$\Psi_y^1(\phi, \theta) = D_{y,z}^1(\phi, \theta, 0)$$

$$= \sin\phi \sin\theta$$

$$\Psi_z^1(\phi, \theta) = D_{z,z}^1(\phi, \theta, 0)$$

$$= \cos\theta$$

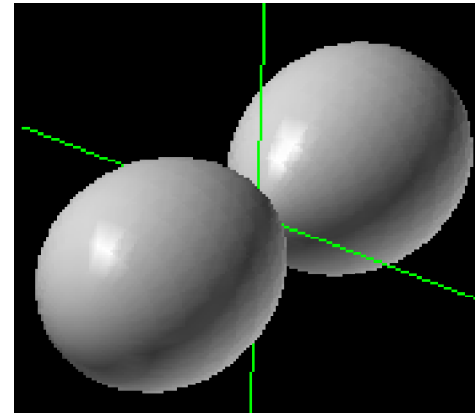




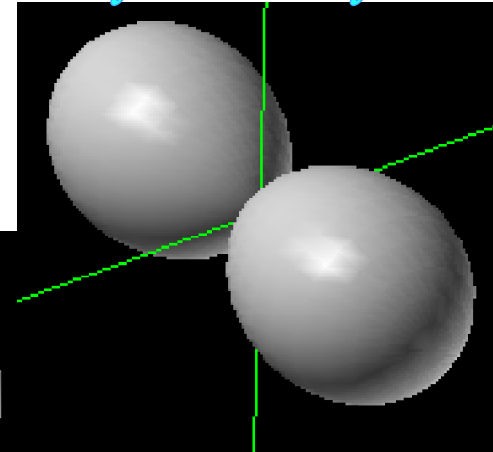
$j = 1$
Standing
 p -Waves

$$\Psi_z = D^1_{z0}$$

$$|\Psi_x|^2 = |D^1_{x0}|^2$$

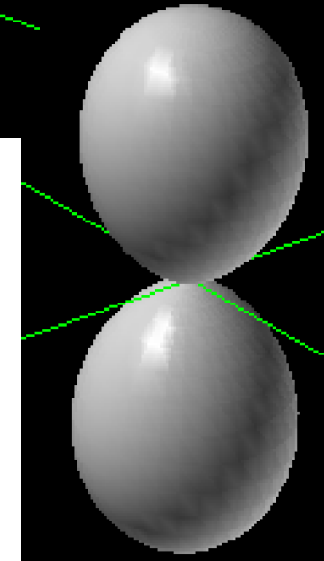


$$|\Psi_y|^2 = |D^1_{y0}|^2$$



Standing p -Wave
Distributions

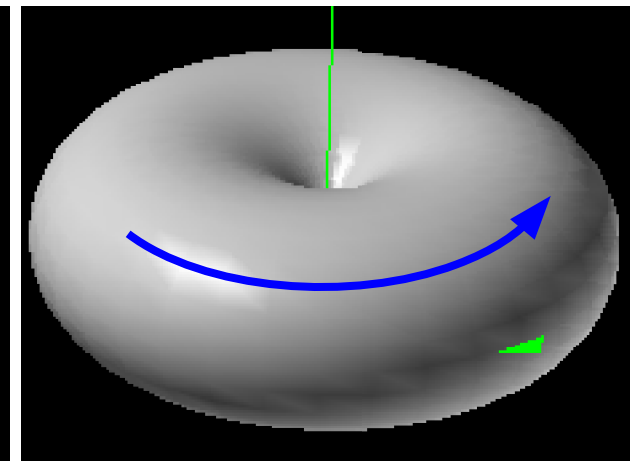
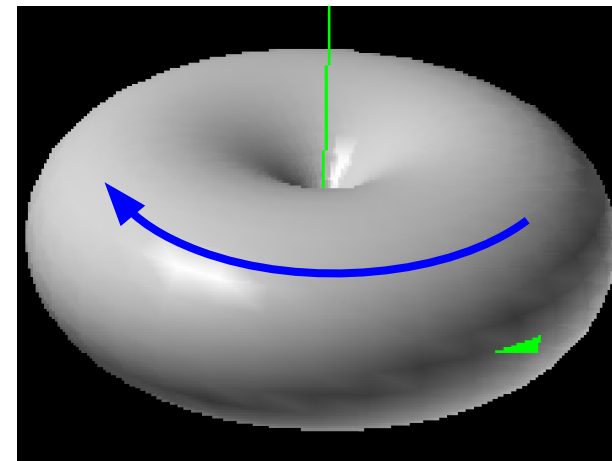
$$|\Psi_z|^2 = |D^1_{z0}|^2$$



Moving p -Wave
Distributions

$$|\Psi_{-1}|^2 = |D^1_{-10}|^2$$

$$|\Psi_1|^2 = |D^1_{10}|^2$$



$$\Psi_x^1(\phi, \theta) = D^1_{x,z}(\phi, \theta, 0) \\ = \cos \phi \sin \theta$$

$$\Psi_y^1(\phi, \theta) = D^1_{y,z}(\phi, \theta, 0) \\ = \sin \phi \sin \theta$$

$$\Psi_z^1(\phi, \theta) = D^1_{z,z}(\phi, \theta, 0) \\ = \cos \theta$$

Applications of $R(3)$ rotation and $U(2)$ representations

Tensor ($j=\ell=2$) representation

$$D^2(\alpha\beta 0) = \begin{pmatrix} e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{-i2\alpha} \sin^2\beta & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 \\ e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta - 1) & -\sqrt{\frac{3}{2}} e^{-i\alpha} \sin\beta \cos\beta & e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta + 1) & -e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta \\ \sqrt{\frac{3}{8}} \sin^2\beta & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \frac{3\cos^2\beta - 1}{2} & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \sqrt{\frac{3}{8}} \sin^2\beta \\ e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta + 1) & \sqrt{\frac{3}{2}} e^{i\alpha} \sin\beta \cos\beta & e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta - 1) & -e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta \\ e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{i2\alpha} \sin^2\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 \end{pmatrix}$$

Applications of $R(3)$ rotation and $U(2)$ representations

Tensor ($j=\ell=2$) representation

$$D^2(\alpha\beta 0) = \begin{pmatrix} e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{-i2\alpha} \sin^2\beta & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 \\ e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta-1) & -\sqrt{\frac{3}{2}} e^{-i\alpha} \sin\beta \cos\beta & e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta+1) & -e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta \\ \sqrt{\frac{3}{8}} \sin^2\beta & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \sqrt{\frac{3}{8}} \sin^2\beta \\ e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta+1) & \sqrt{\frac{3}{2}} e^{i\alpha} \sin\beta \cos\beta & e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta-1) & -e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta \\ e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{i2\alpha} \sin^2\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 \end{pmatrix}$$

Spherical 2^k -multipole functions X_q^k or X -functions are D^* -functions times the k^{th} power of radius (r^k).

$$\sqrt{4\pi/5} Y_{m=2}^{\ell=2}(\phi\theta) = D_{2,0}^{2*}(\phi\theta 0) = \sqrt{\frac{3}{8}} e^{i2\phi} \sin^2\theta = \sqrt{\frac{3}{8}} \frac{(x+iy)^2}{r^2}$$

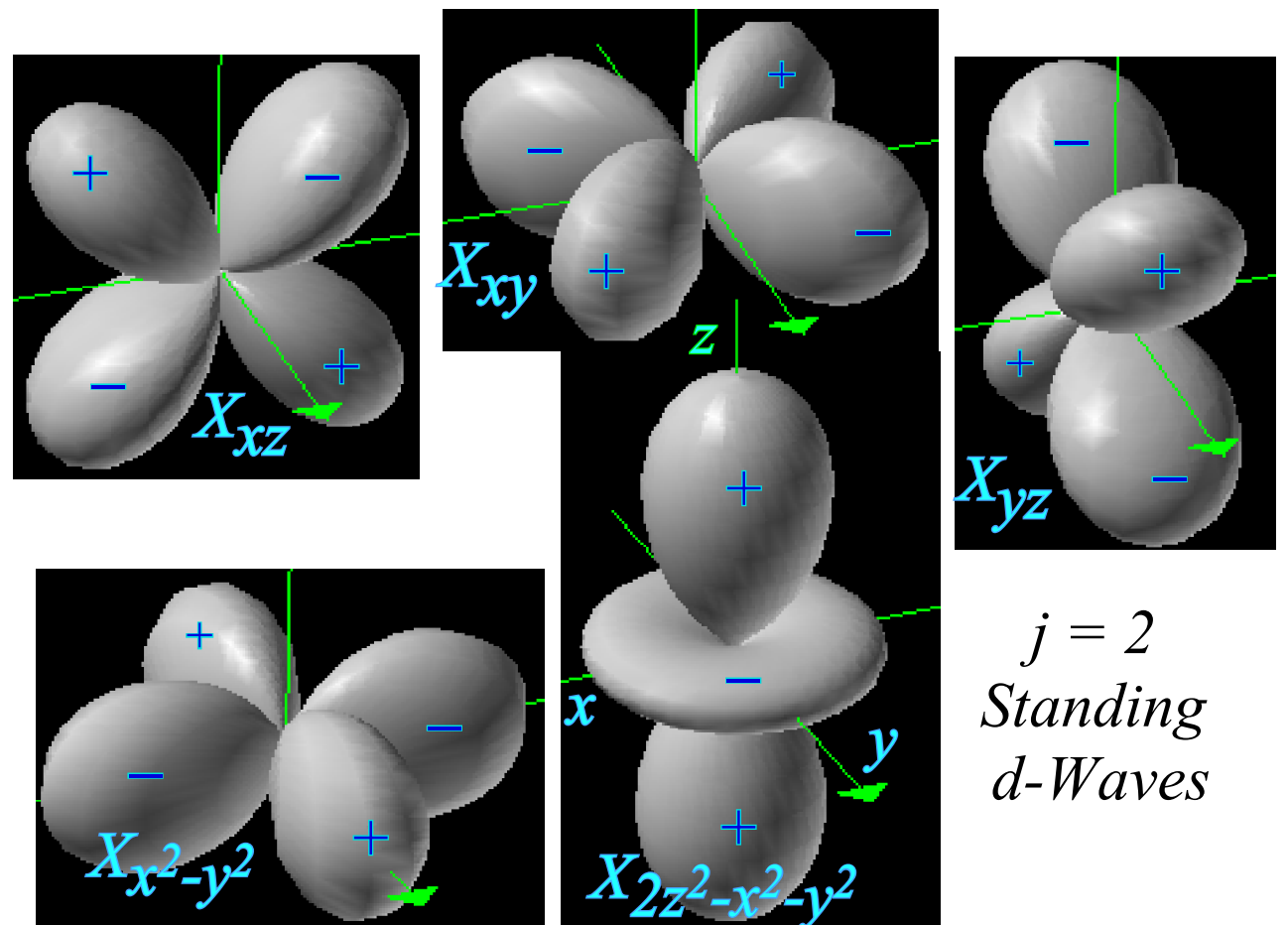
$$X_q^k = r^k D_{q,0}^{k*} = \sqrt{\frac{4\pi}{2k+1}} r^k Y_q^k$$

$$\sqrt{4\pi/5} Y_{m=1}^{\ell=2}(\phi\theta) = D_{1,0}^{2*}(\phi\theta 0) = -\sqrt{\frac{3}{2}} e^{i\phi} \sin\theta \cos\theta = -\sqrt{\frac{3}{2}} \frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5} Y_{m=0}^{\ell=2}(\phi\theta) = D_{0,0}^{2*}(\phi\theta 0) = \frac{3\cos^2\theta-1}{2} = \frac{3z^2-r^2}{2r^2}$$

$$\sqrt{4\pi/5} Y_{m=-1}^{\ell=2}(\phi\theta) = D_{-1,0}^{2*}(\phi\theta 0) = \sqrt{\frac{3}{2}} e^{-i\phi} \sin\theta \cos\theta = \sqrt{\frac{3}{2}} \frac{(x-iy)z}{r^2}$$

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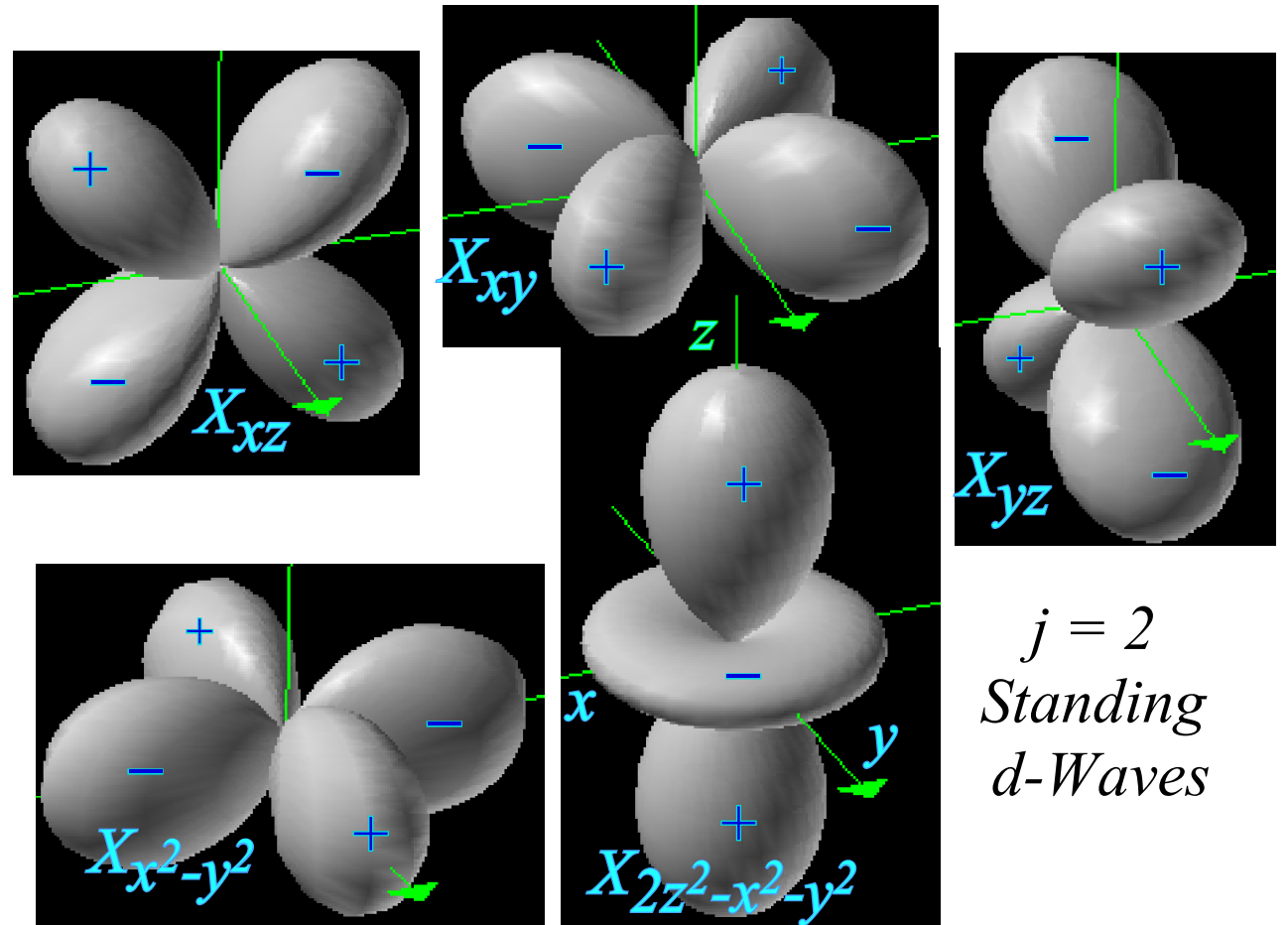
Applications of $R(3)$ rotation and $U(2)$ representations

Tensor ($j=\ell=2$) representation

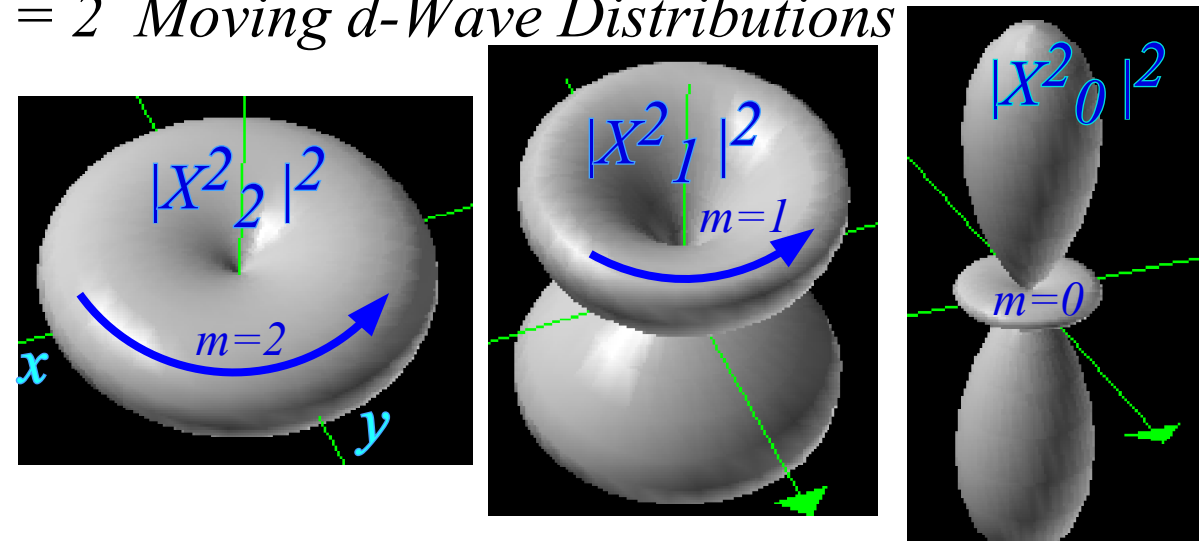
Spherical 2^k -multipole functions X_q^k or X -functions are D^* -functions times the k^{th} power of radius (r^k).

$$\begin{aligned} \sqrt{4\pi/5} Y_{m=2}^{\ell=2}(\phi\theta) &= D_{2,0}^{2*}(\phi\theta) = \sqrt{\frac{3}{8}} e^{i2\phi} \sin^2 \theta = \sqrt{\frac{3}{8}} \frac{(x+iy)^2}{r^2} \\ \sqrt{4\pi/5} Y_{m=1}^{\ell=2}(\phi\theta) &= D_{1,0}^{2*}(\phi\theta) = -\sqrt{\frac{3}{2}} e^{i\phi} \sin \theta \cos \theta = -\sqrt{\frac{3}{2}} \frac{(x+iy)z}{r^2} \\ \sqrt{4\pi/5} Y_{m=0}^{\ell=2}(\phi\theta) &= D_{0,0}^{2*}(\phi\theta) = \frac{3\cos^2 \theta - 1}{2} = \frac{3z^2 - r^2}{2r^2} \\ \sqrt{4\pi/5} Y_{m=-1}^{\ell=2}(\phi\theta) &= D_{-1,0}^{2*}(\phi\theta) = \sqrt{\frac{3}{2}} e^{-i\phi} \sin \theta \cos \theta = \sqrt{\frac{3}{2}} \frac{(x-iy)z}{r^2} \\ \sqrt{4\pi/5} Y_{m=-2}^{\ell=2}(\phi\theta) &= D_{-2,0}^{2*}(\phi\theta) = \sqrt{\frac{3}{8}} e^{-i2\phi} \sin^2 \theta = \sqrt{\frac{3}{8}} \frac{(x-iy)^2}{r^2} \end{aligned}$$

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$j = 2$ Moving d -Wave Distributions



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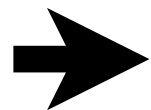
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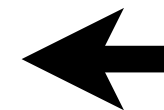


Molecular and nuclear wavefunctions

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For $SU(2)$ and $R(3)$, sum over rotations is an integral over Euler angles $(\alpha\beta\gamma)$.

For integral- $j=0, 1, 2, \dots$ the $R(3)$ integral over polar angle β ranges from 0 to π .

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Same applies to the generators \mathbf{s}_Z or \mathbf{J}_Z of $SU(2)$ or $R(3)$.

$$\mathbf{s}_Z |j_{m,n}\rangle = m |j_{m,n}\rangle \quad \bar{\mathbf{s}}_Z |j_{m,n}\rangle = -n |j_{m,n}\rangle$$

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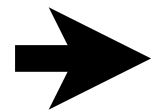
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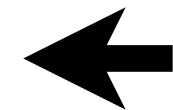
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$$\mathbf{H}_{\text{symmetric top}} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + A\mathbf{J}_{\bar{Z}}^2 \quad (\text{Molecular spin is labeled } \mathbf{J} \text{ instead of } \mathbf{s})$$

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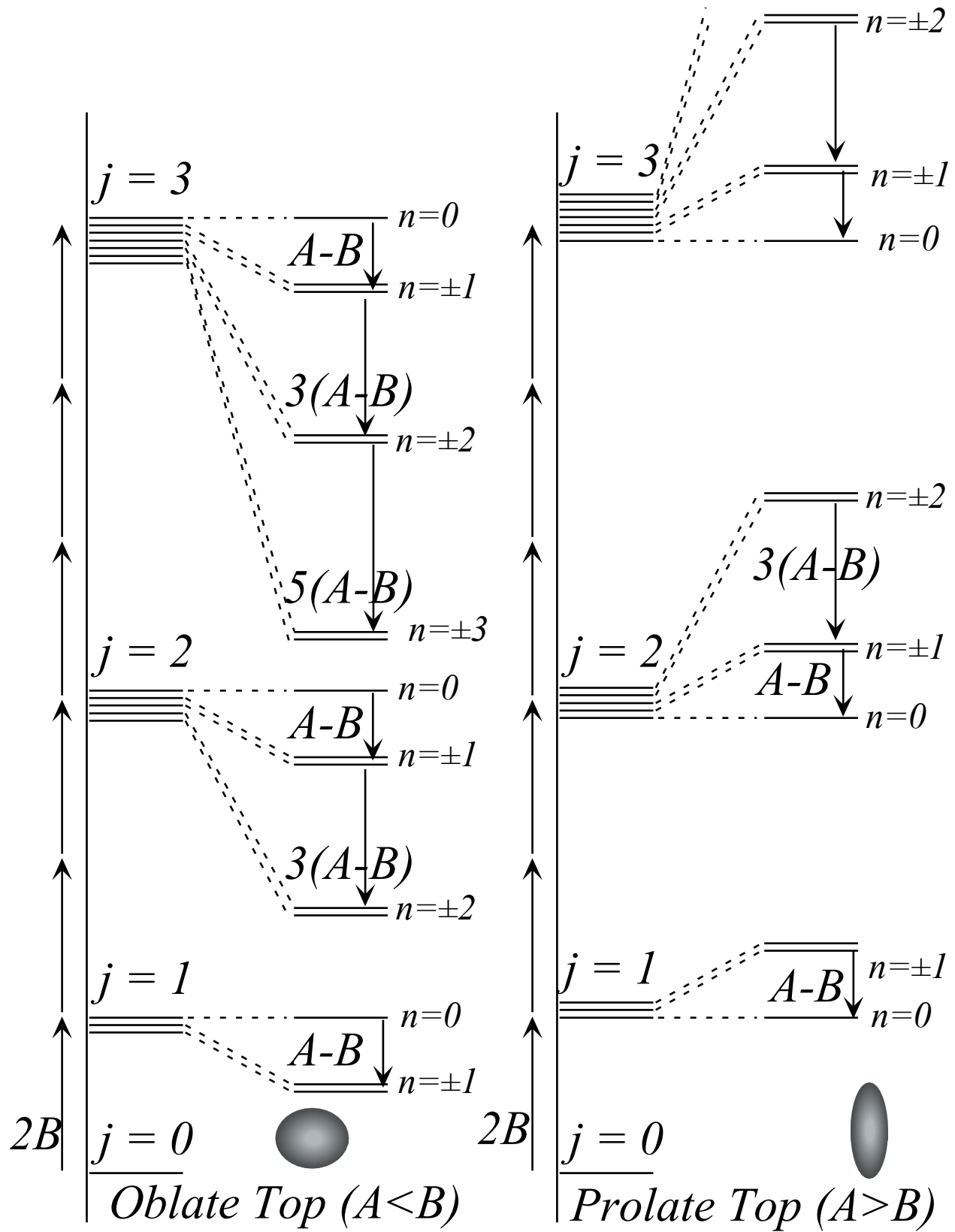
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Eigenvalue and energy level spectrum is shown next.

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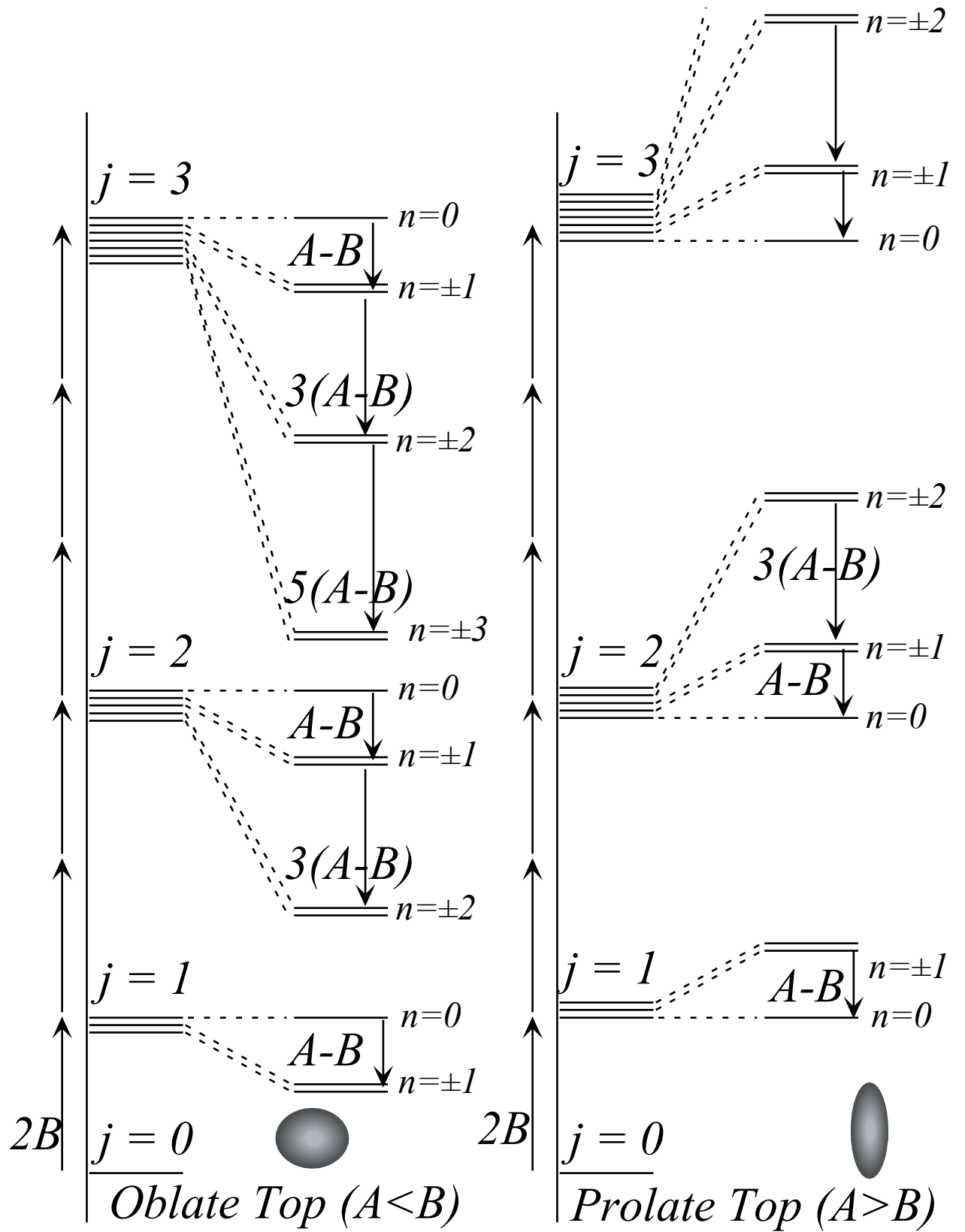


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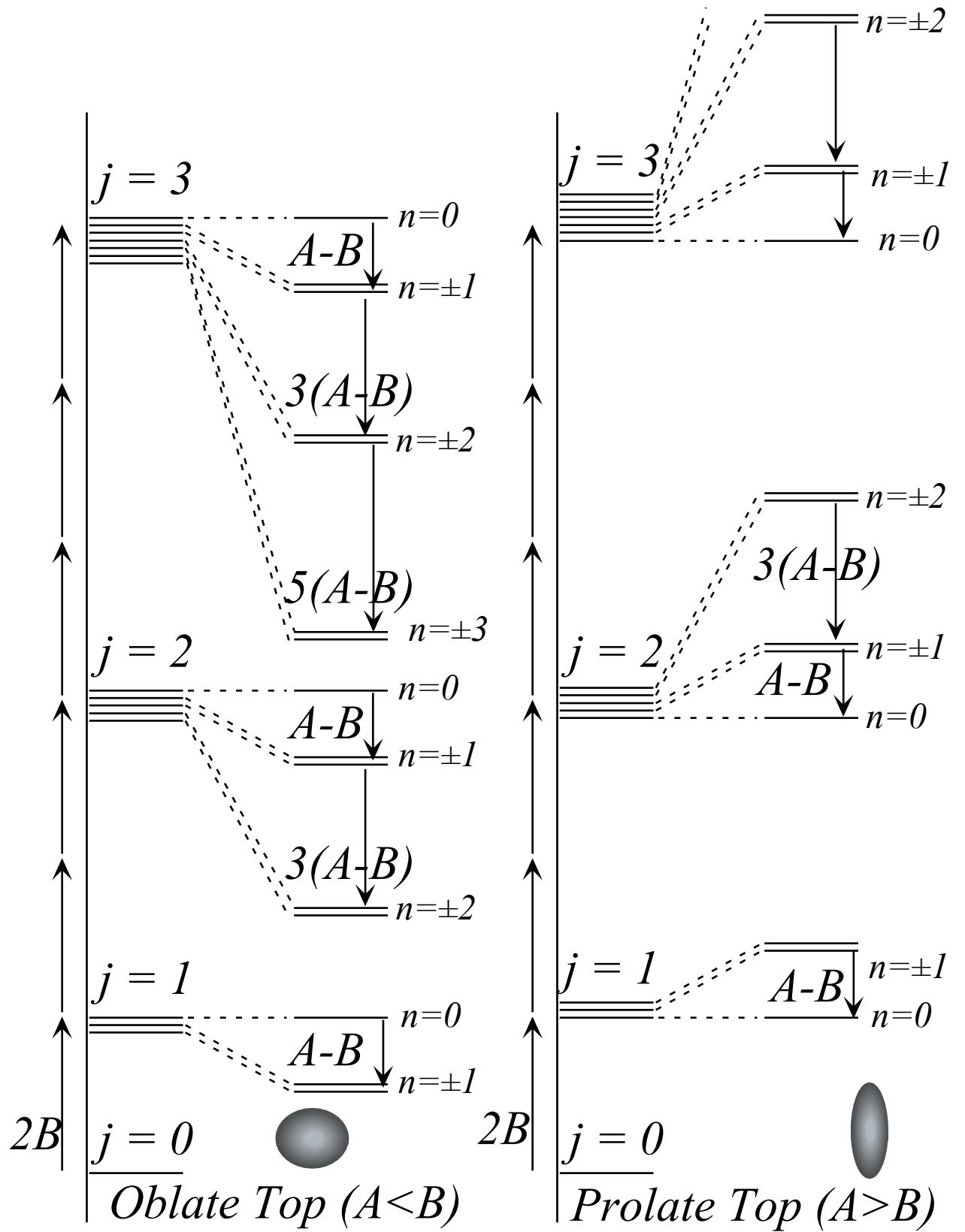
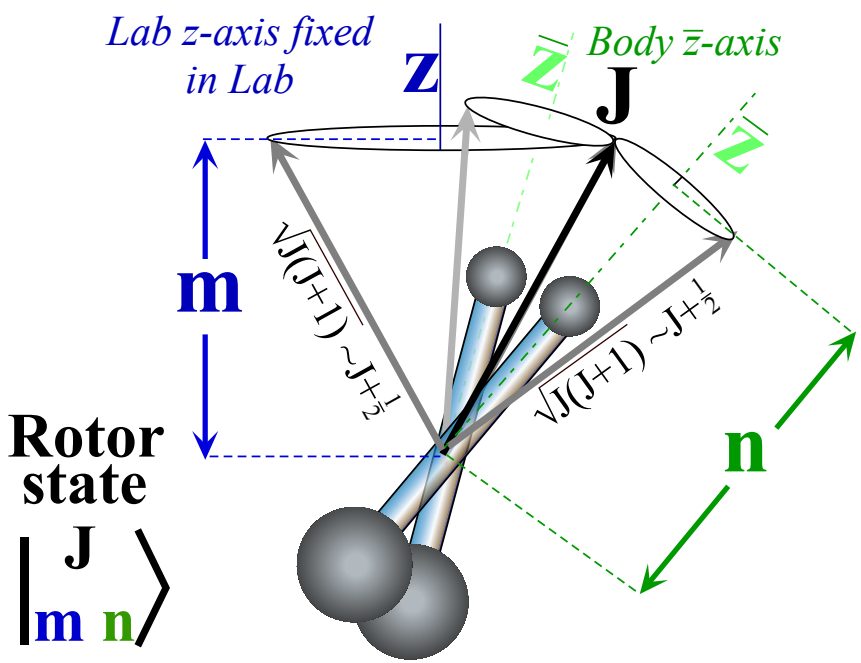


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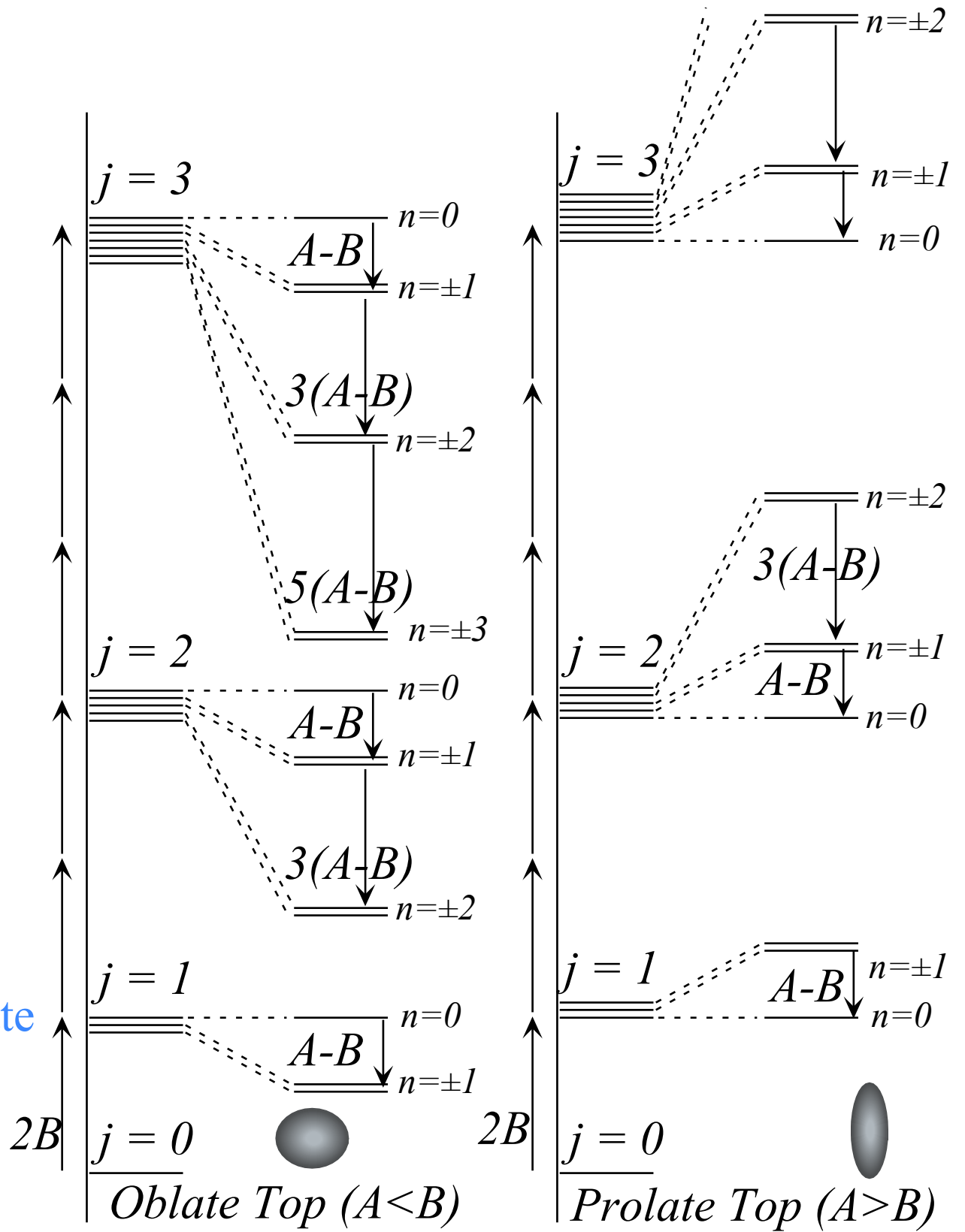
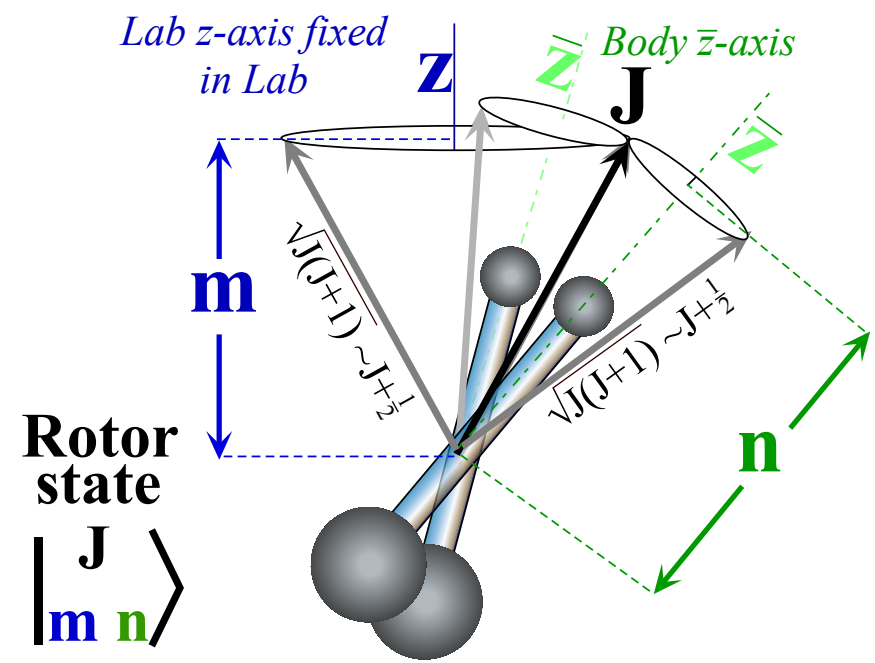


$$\mathbf{H}_{\text{symmetric top}} = B\mathbf{J}_X^2 + B\mathbf{J}_Y^2 + B\mathbf{J}_Z^2 + (A - B)\mathbf{J}_Z^2 = B\mathbf{J} \cdot \mathbf{J} + (A - B)\mathbf{J}_Z^2$$

Eigensolution equations:

$$\begin{aligned} \mathbf{H}_{\text{symmetric top}} |j, m, n\rangle & \\ &= B\mathbf{J} \cdot \mathbf{J} + (A - B)\mathbf{J}_Z^2 |j, m, n\rangle \\ &= [BJ(J + 1) + (A - B)n^2] |j, m, n\rangle \end{aligned}$$

Mock-Mach-Multiplicity is $(2j+1)^2$ for each j



Even $n=0$ levels are $2j+1$ -fold degenerate
If n is non-zero the degeneracy is $4j+2$.

Applications of R(3) rotation and U(2) representations

Molecular and nuclear eigenlevels

Introducing Racah tensor notation

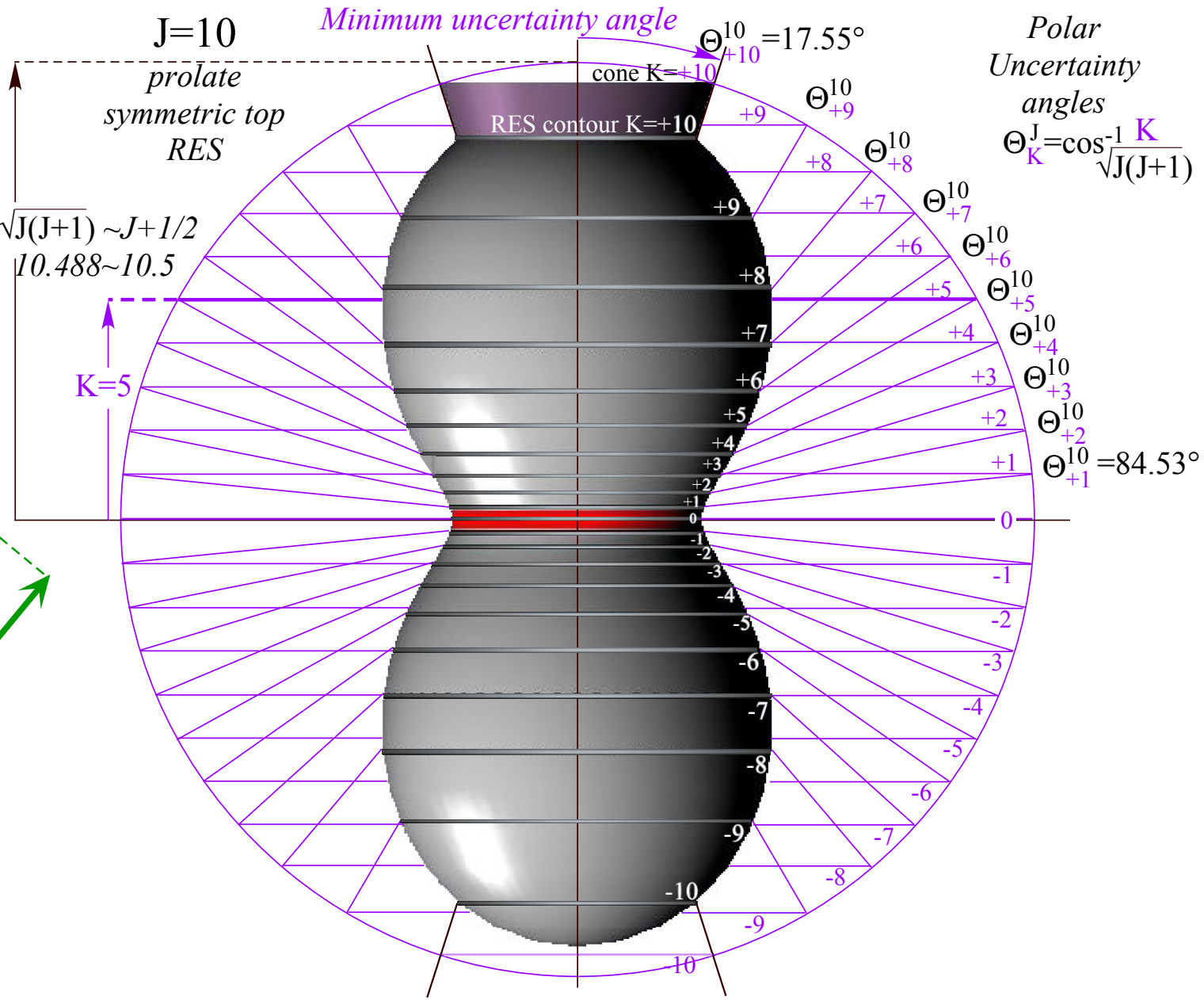
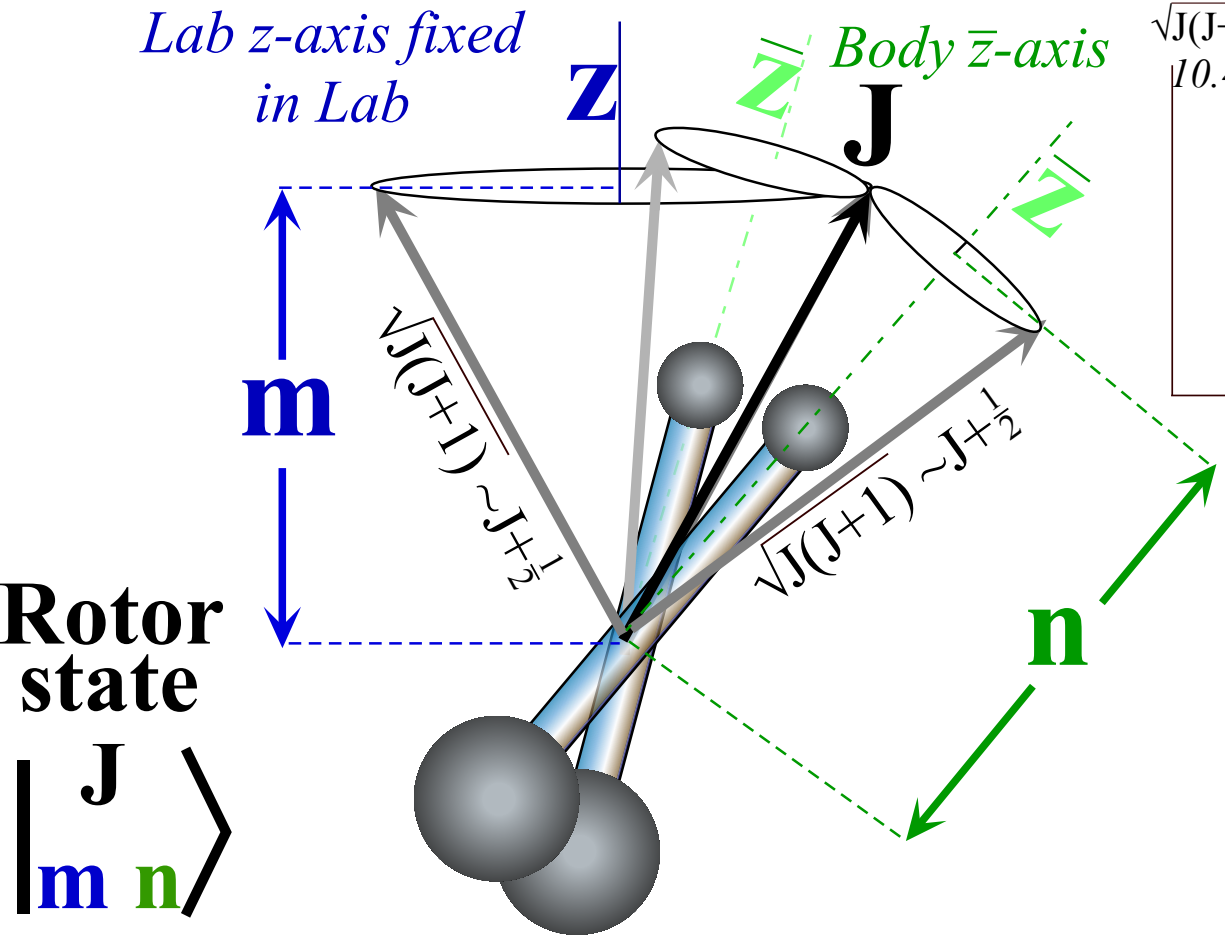
Eigensolution equations:

$$\begin{aligned}
 & \mathbf{H}_{\text{symmetric top}} \left| \begin{matrix} j \\ m, n \end{matrix} \right\rangle \\
 &= B \mathbf{J} \cdot \mathbf{J} + (A - B) \mathbf{J}_{\bar{z}}^2 \left| \begin{matrix} j \\ m, n \end{matrix} \right\rangle \\
 &= \left[BJ(J + 1) + (A - B)n^2 \right] \left| \begin{matrix} j \\ m, n \end{matrix} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 T_0^0 &= \mathbf{J} \cdot \mathbf{J} = \langle J \rangle^2 = (J_x^2 + J_y^2 + J_z^2), \\
 T_0^2 &= \frac{1}{2} \langle J \rangle^2 (3 \cos^2 \beta - 1) = \frac{1}{2} (2J_z^2 - J_x^2 - J_y^2), \\
 H &= B T_0^0 + \frac{2}{3} (A - B) T_0^2
 \end{aligned}$$

Mock-Mach-Multiplicity is $(2j+1)^2$ for each j

Lab z-axis fixed in Lab **Z** Body \bar{z} -axis **J**



Review : 2-D $\mathfrak{su}(2)$ algebra of $U(2)$ representations

Angular momentum generators by $U(2)$ analysis

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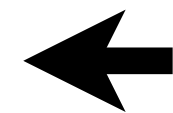
Applications of $R(3)$ rotation and $U(2)$ representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenlevels

➔ Generalized Stern-Gerlach and transformation matrices

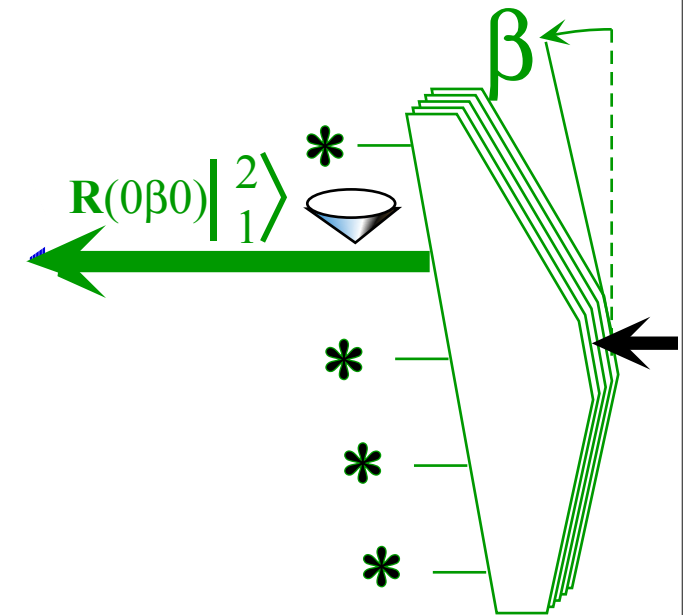
Angular momentum cones and high J properties



Applications of $R(3)$ rotation and $U(2)$ representations

Generalized Stern-Gerlach and transformation matrices

Polarization analysis Suppose a spin- j state $\mathbf{R}(0\beta 0) |^{j=2}_{m=1}\rangle$ exits an analyzer rotated by β



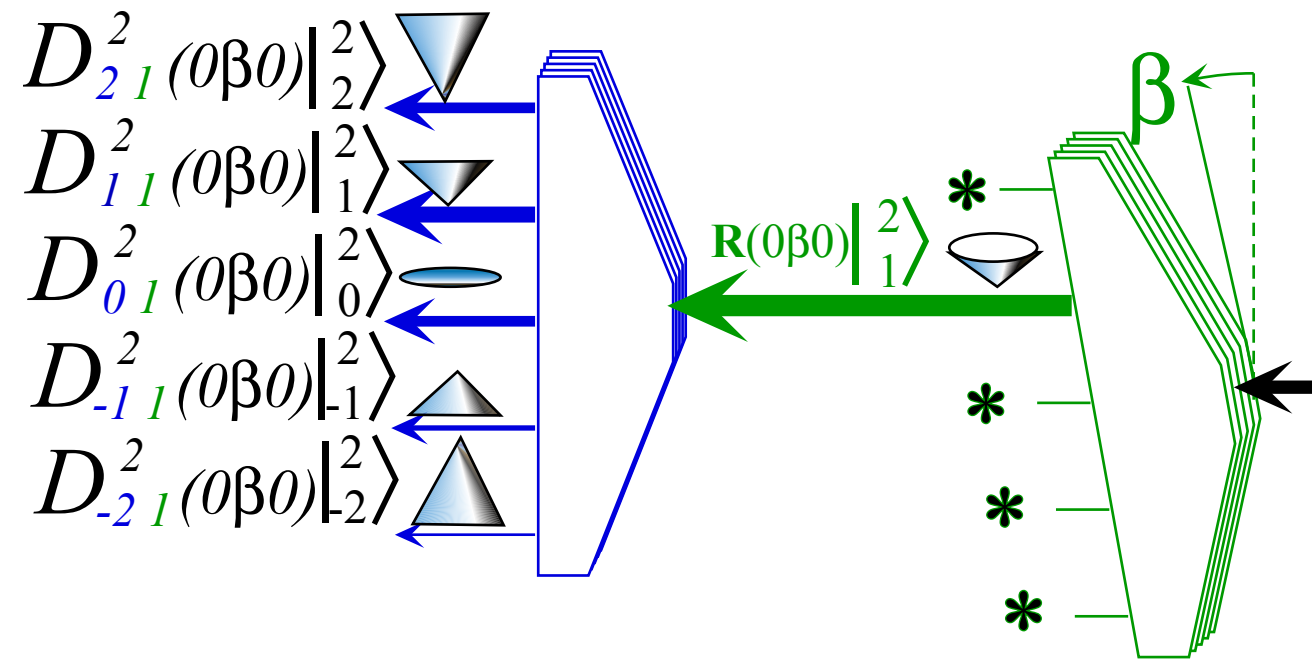
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Polarization analysis Suppose a spin- j state $\mathbf{R}(0\beta 0) |^{j=2}_{m=1}\rangle$ exits an analyzer rotated by β

and then enters a vertical ($\beta=0$) analyzer and forced to choose from unrotated states $|^{j=2}_{m'}\rangle$

$$\begin{aligned} \mathbf{R}(0\beta 0) |^j_m\rangle &= \sum_{m'=-j}^j |^j_{m'}\rangle \langle^j_{m'} | \mathbf{R}(0\beta 0) |^j_m\rangle \\ &= \sum_{m'=-j}^j |^j_{m'}\rangle D^j_{m'm}(0\beta 0) \end{aligned}$$

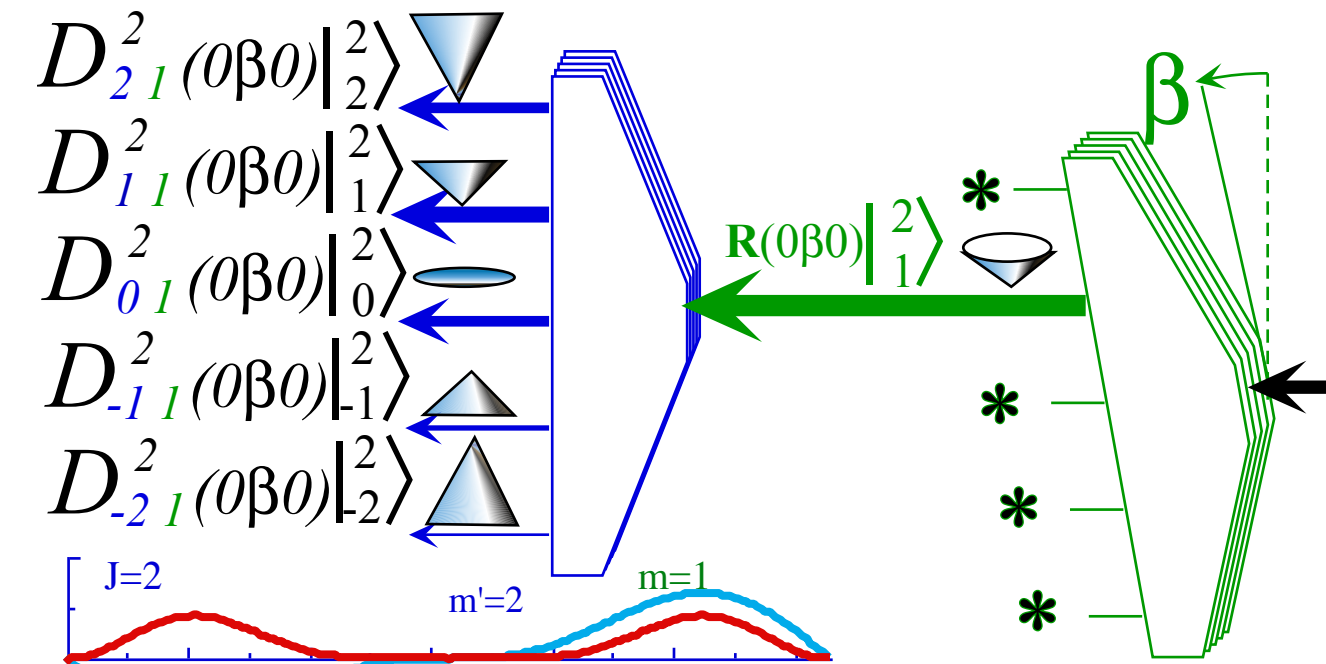


Applications of $R(3)$ rotation and $U(2)$ representations

Generalized Stern-Gerlach and transformation matrices

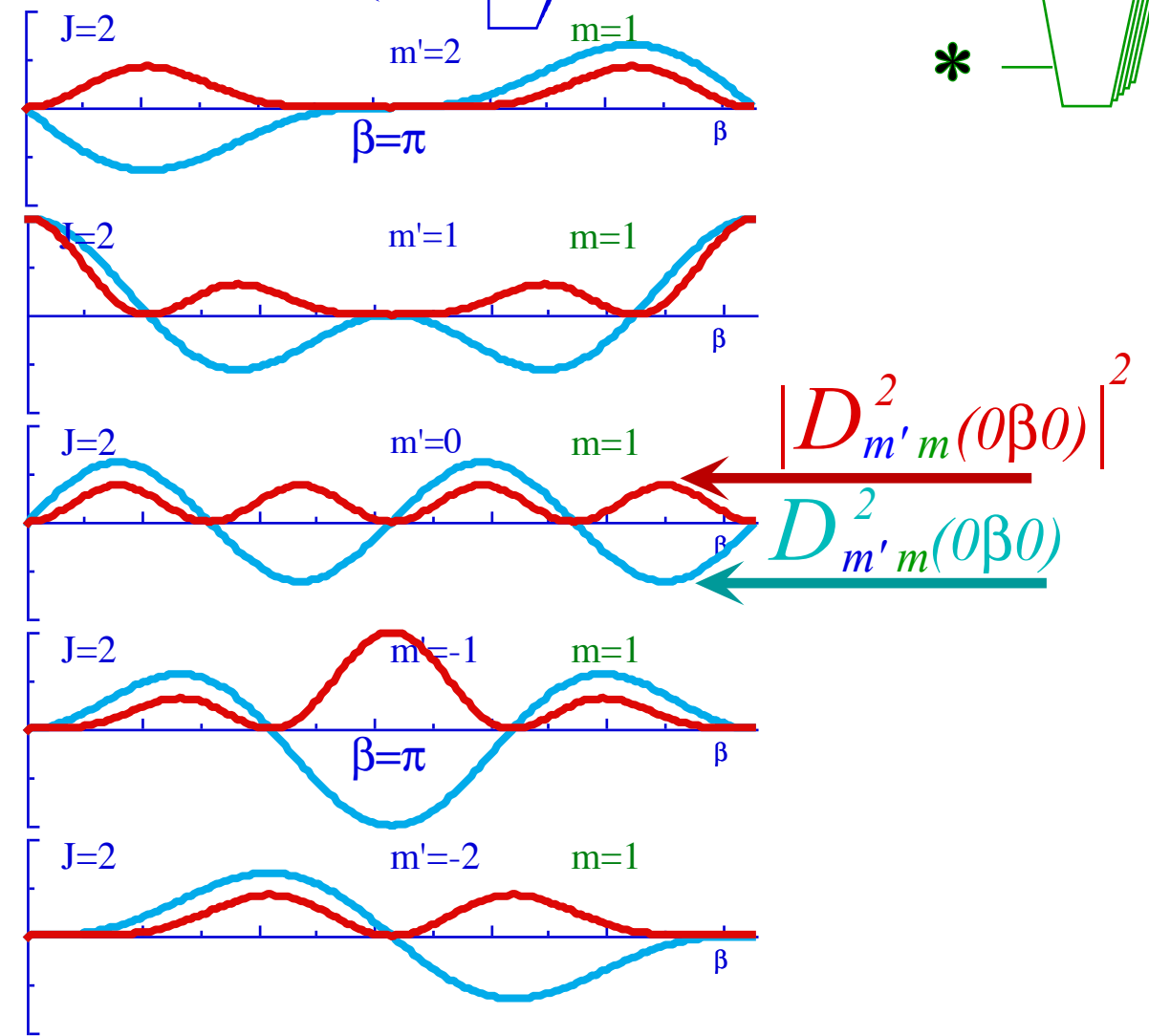
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Overlap of state $\mathbf{R}(\alpha\beta\gamma) |^2_1\rangle$ with unrotated $|^{j=2}_{m'}\rangle$ is the corresponding D-matrix element.

$$\langle^{j'}_{m'} | \mathbf{R}(\alpha\beta\gamma) |^2_1\rangle = \delta^{j'2} D^2_{m'1}(\alpha\beta\gamma) = \langle^{j'}_{m'} |^2_1\rangle_R$$

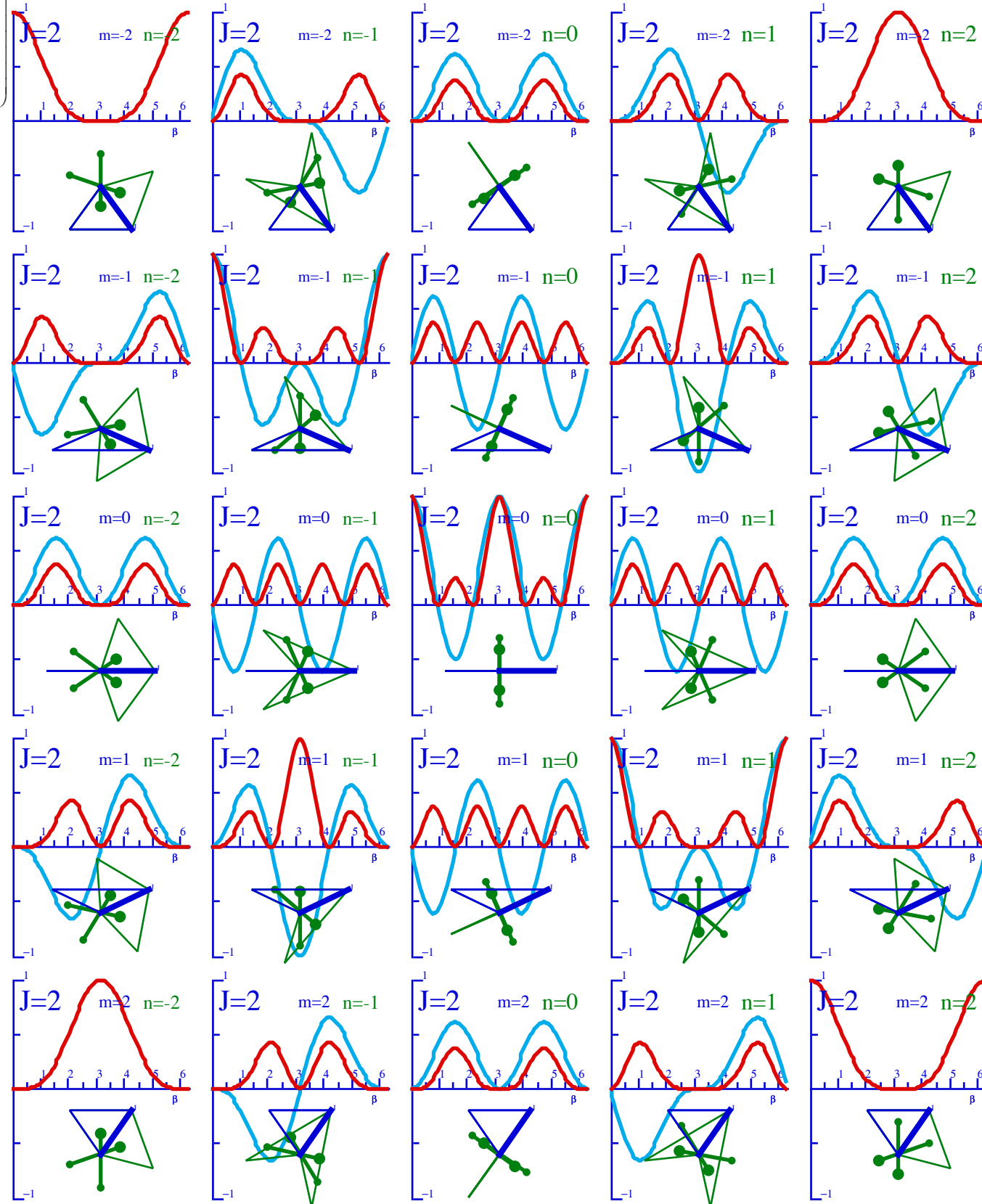


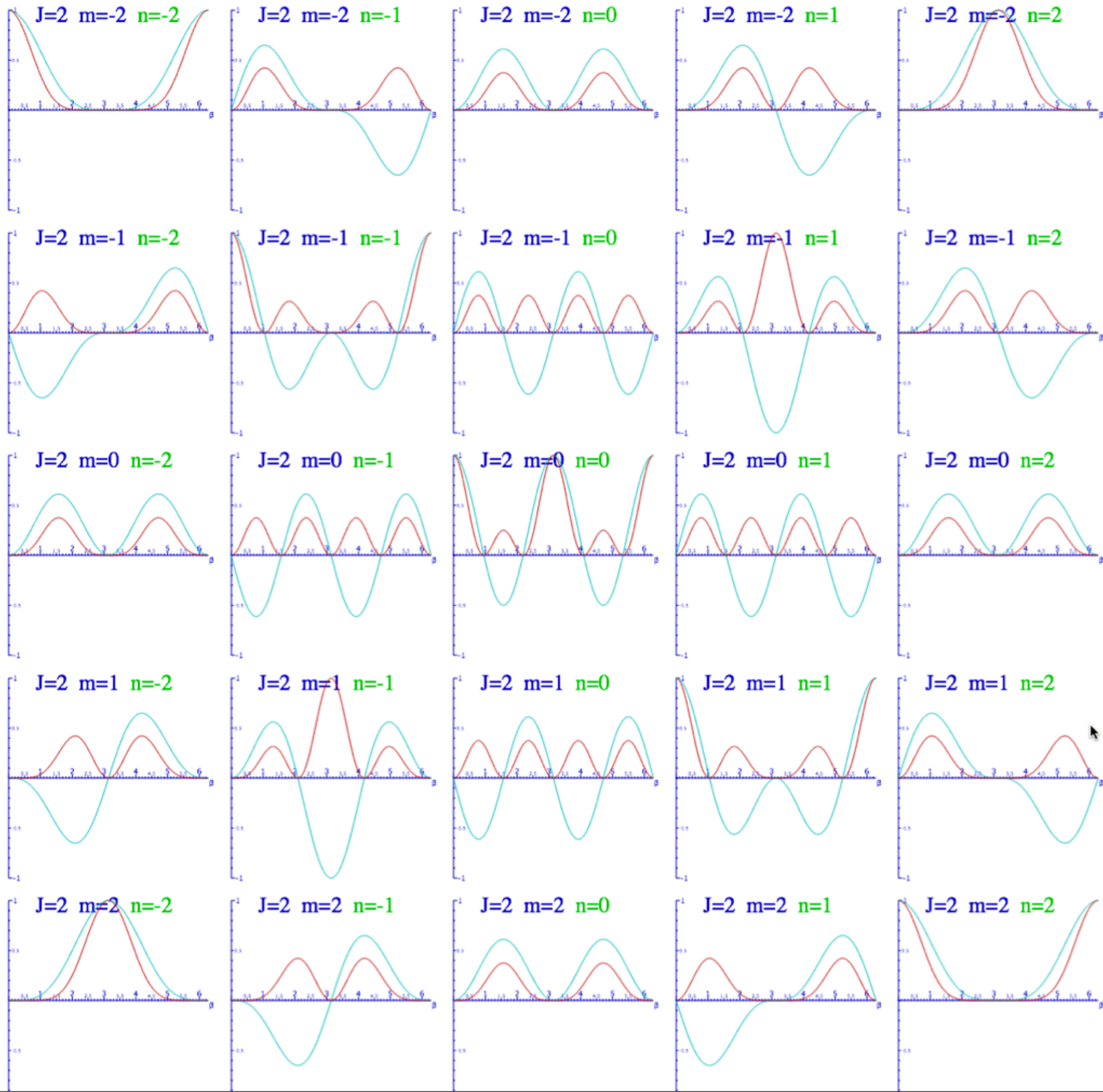
$$D^2(\alpha\beta 0) = \begin{pmatrix} e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{-i2\alpha} \sin^2\beta & e^{-i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 \\ e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{-i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta-1) & -\sqrt{\frac{3}{2}} e^{-i\alpha} \sin\beta \cos\beta & e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta+1) & -e^{-i\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta \\ \sqrt{\frac{3}{8}} \sin^2\beta & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}} \sin\beta \cos\beta & \sqrt{\frac{3}{8}} \sin^2\beta \\ e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i\alpha} \left(\frac{1-\cos\beta}{2}\right) (2\cos\beta+1) & \sqrt{\frac{3}{2}} e^{i\alpha} \sin\beta \cos\beta & e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) (2\cos\beta-1) & -e^{i\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta \\ e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha} \left(\frac{1-\cos\beta}{2}\right) \sin\beta & \sqrt{\frac{3}{8}} e^{i2\alpha} \sin^2\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right) \sin\beta & e^{i2\alpha} \left(\frac{1+\cos\beta}{2}\right)^2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{R}(0\beta 0) \left| \begin{matrix} j \\ m \end{matrix} \right\rangle &= \sum_{m'=-j}^j \left| \begin{matrix} j \\ m' \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m' \end{matrix} \right| \mathbf{R}(0\beta 0) \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \\ &= \sum_{m'=-j}^j \left| \begin{matrix} j \\ m' \end{matrix} \right\rangle D_{m'm}^j(0\beta 0) \end{aligned}$$

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$$\left\langle \begin{matrix} j' \\ m' \end{matrix} \right| \mathbf{R}(\alpha\beta\gamma) \left| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle = \delta^{j'2} D_{m'1}^2(\alpha\beta\gamma) = \left\langle \begin{matrix} j' \\ m' \end{matrix} \right| \left| \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle_R$$





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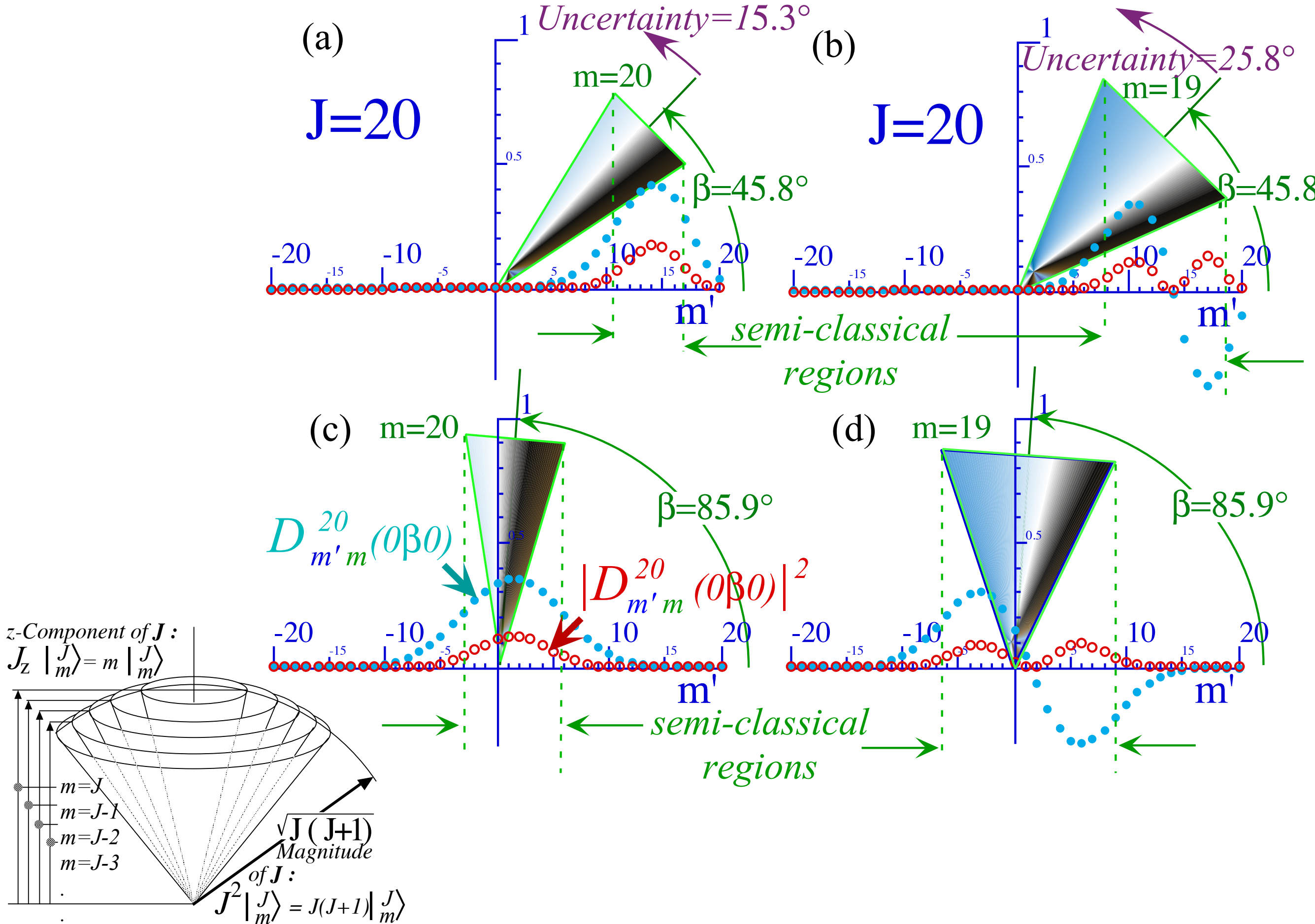
Molecular and nuclear wavefunctions

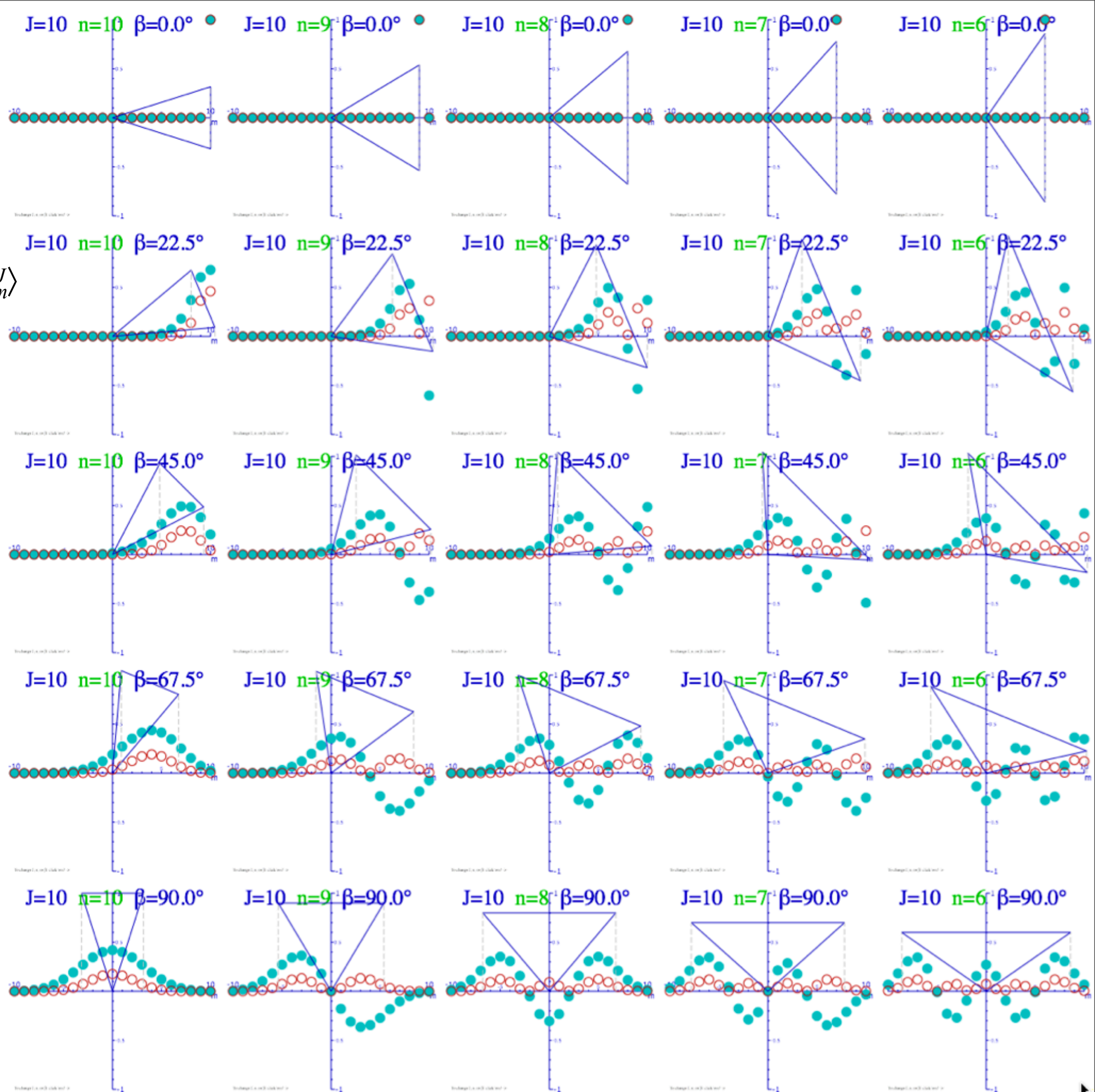
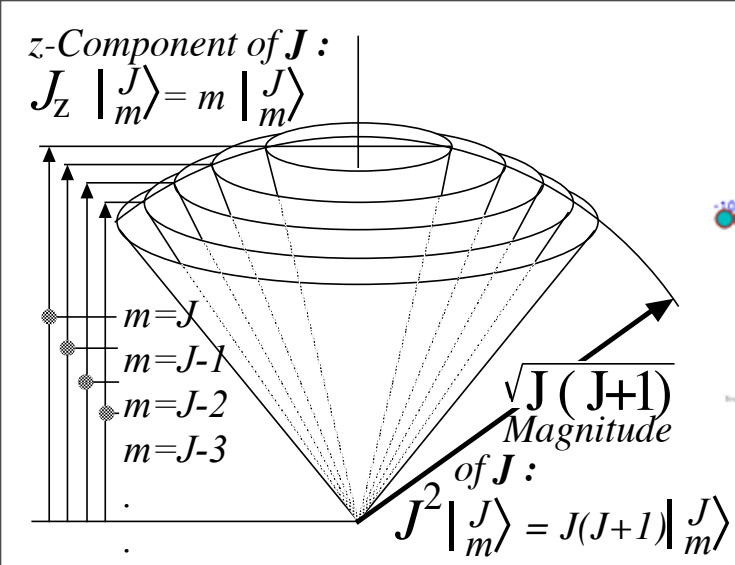
Molecular and nuclear eigenlevels

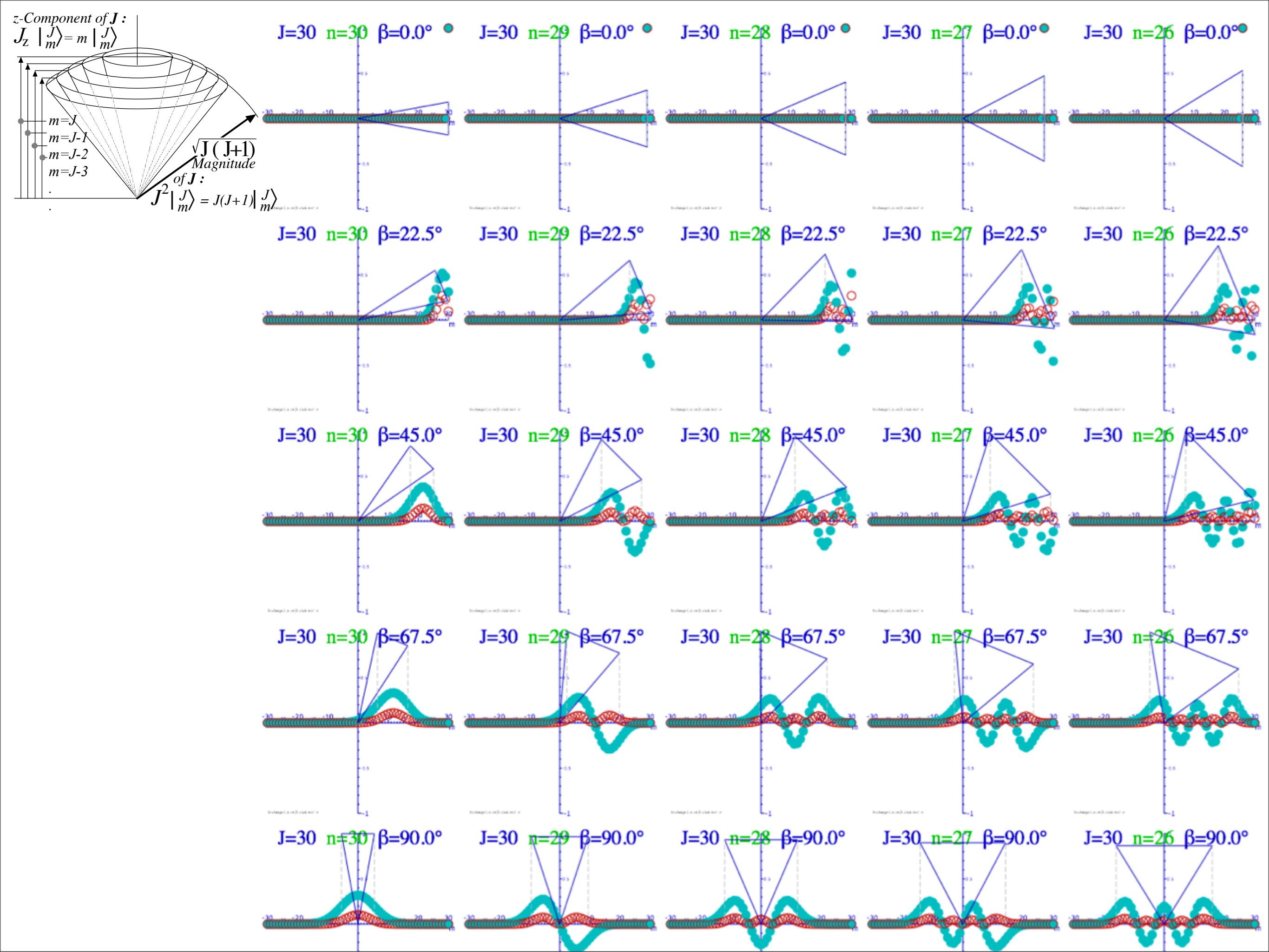
Generalized Stern-Gerlach and transformation matrices

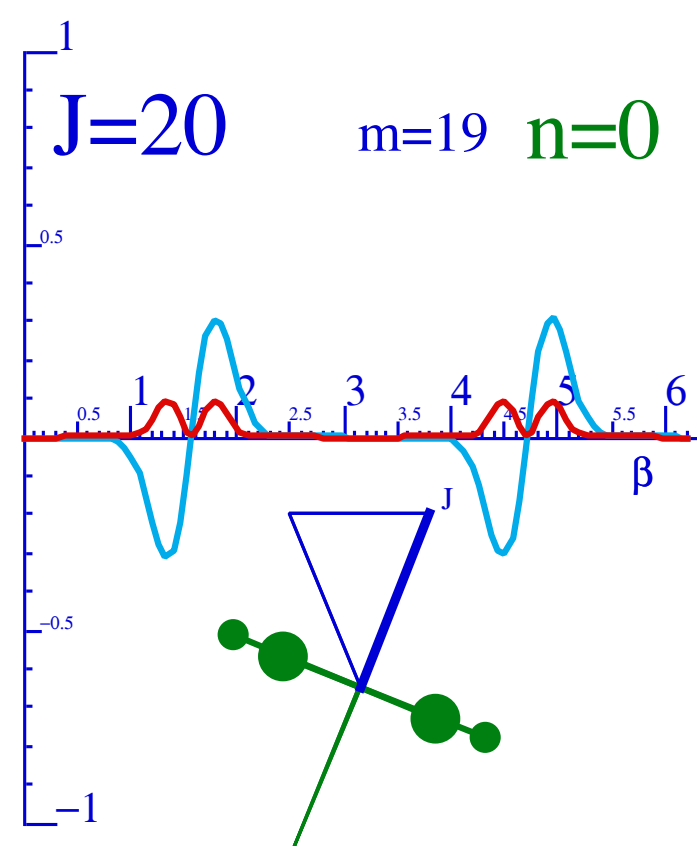
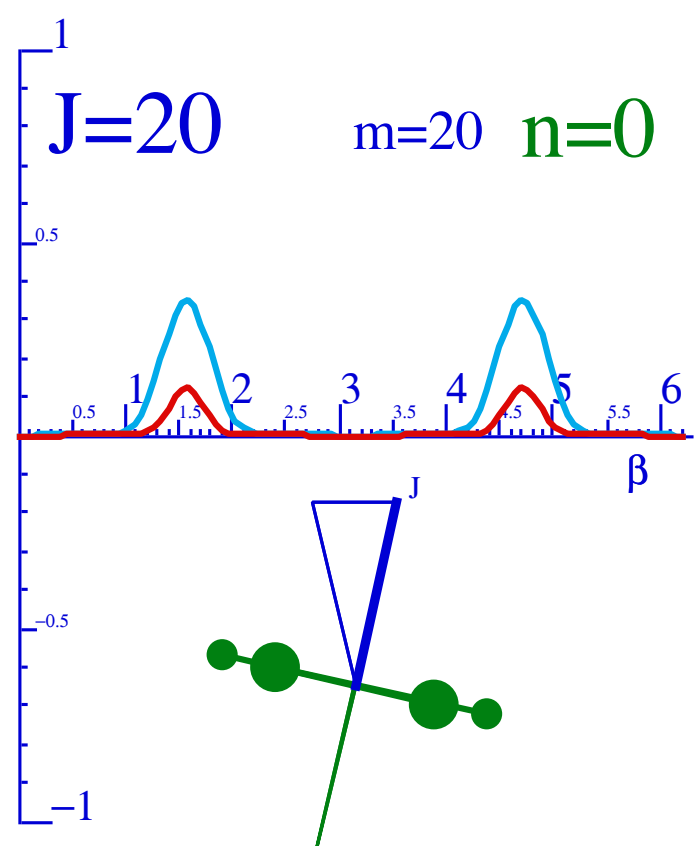
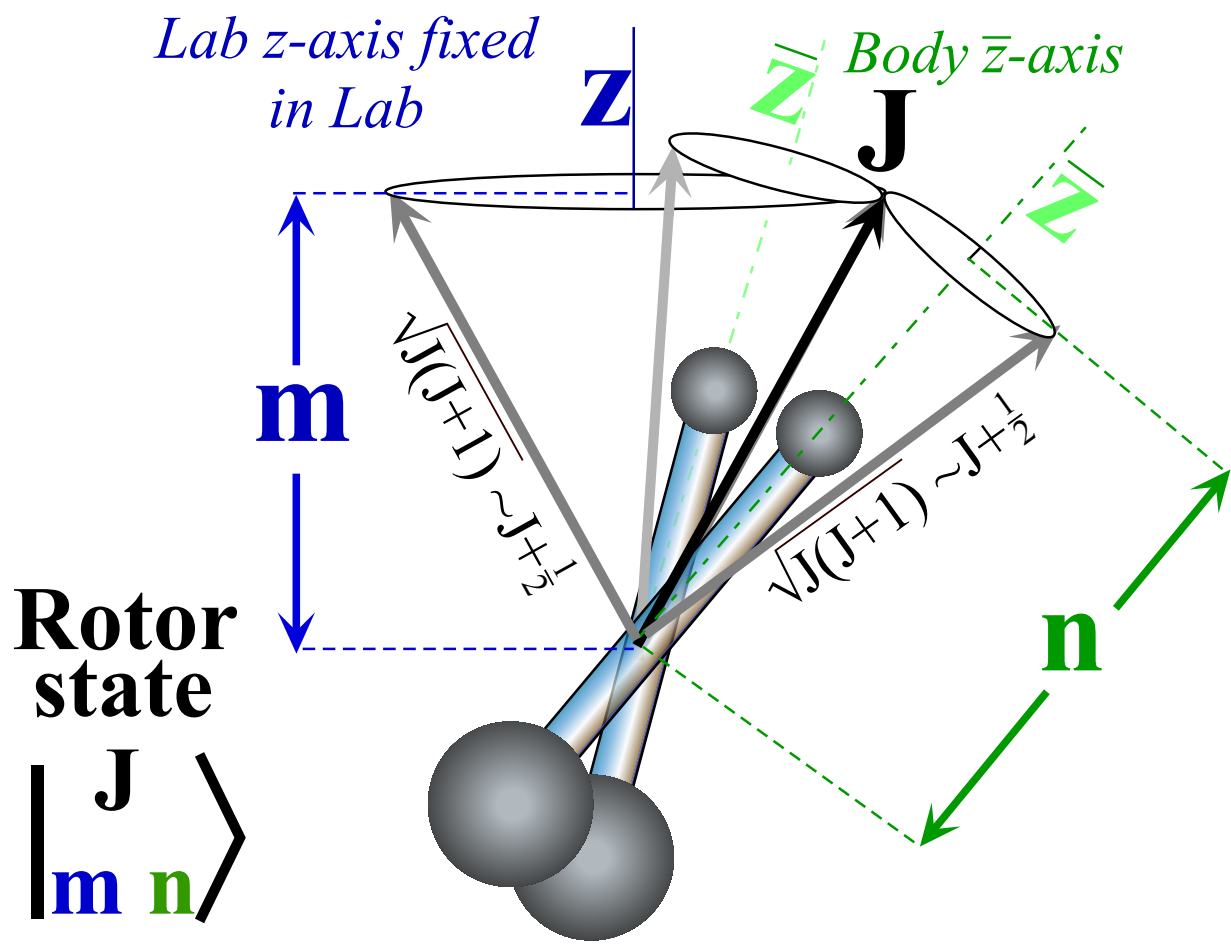
 *Angular momentum cones and high J properties* 

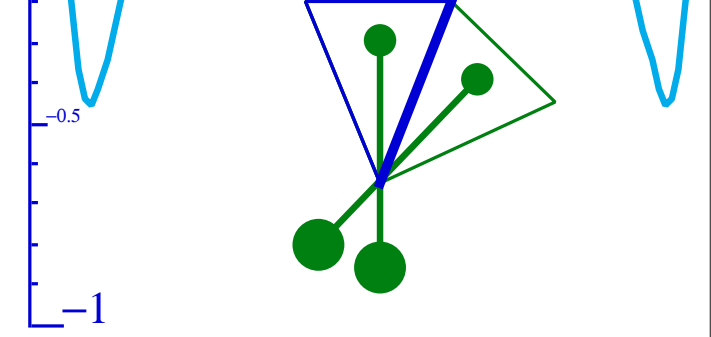
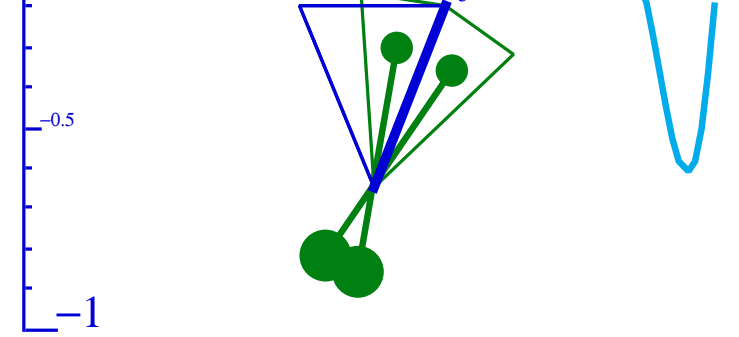
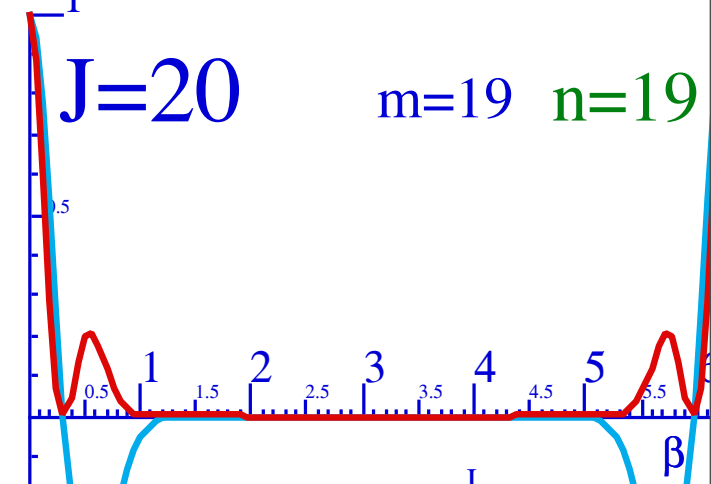
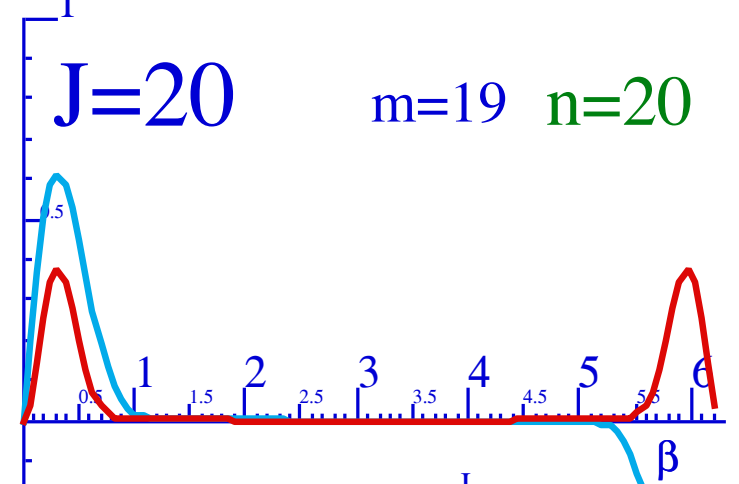
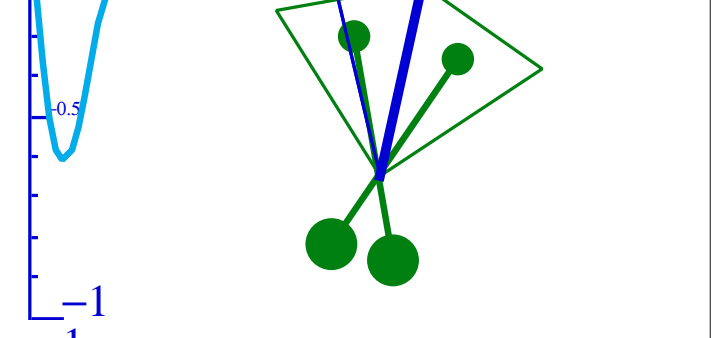
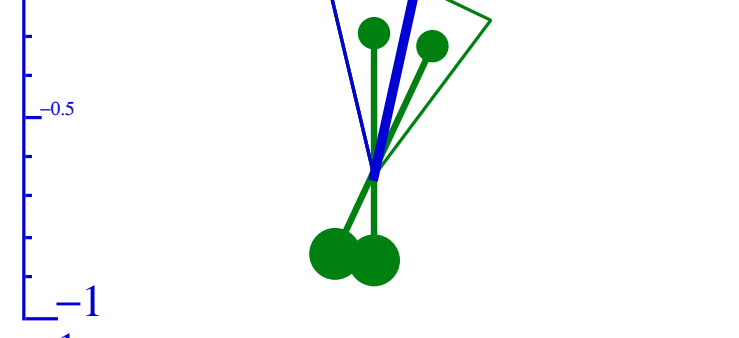
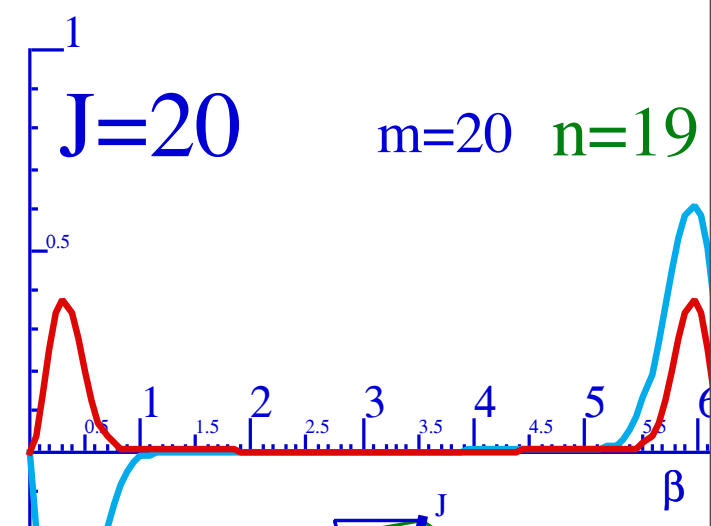
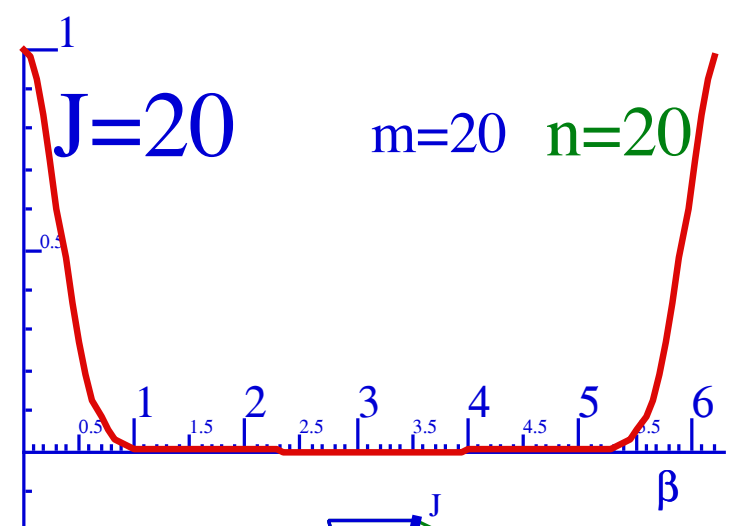
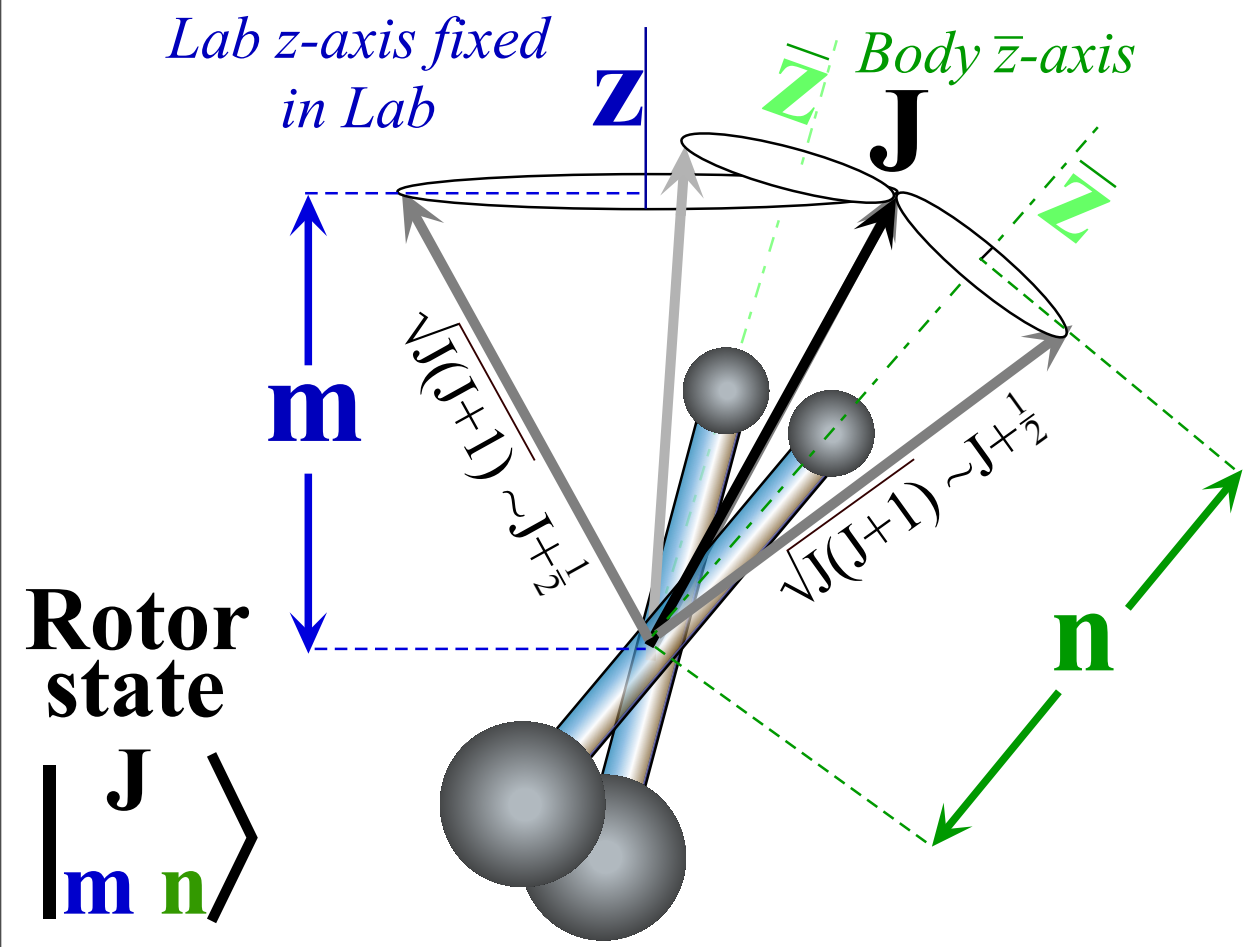
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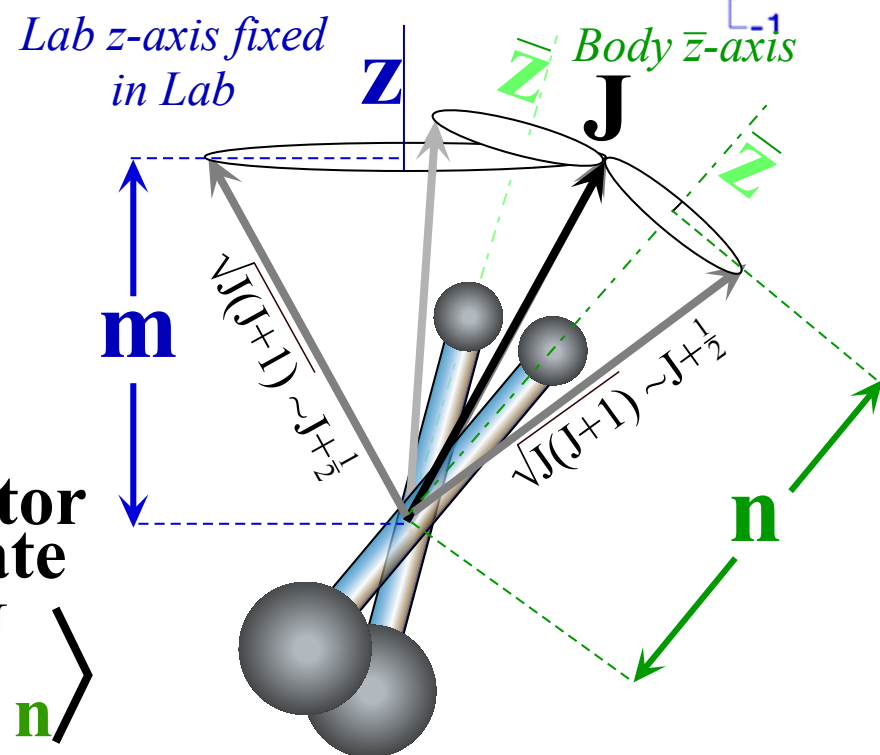
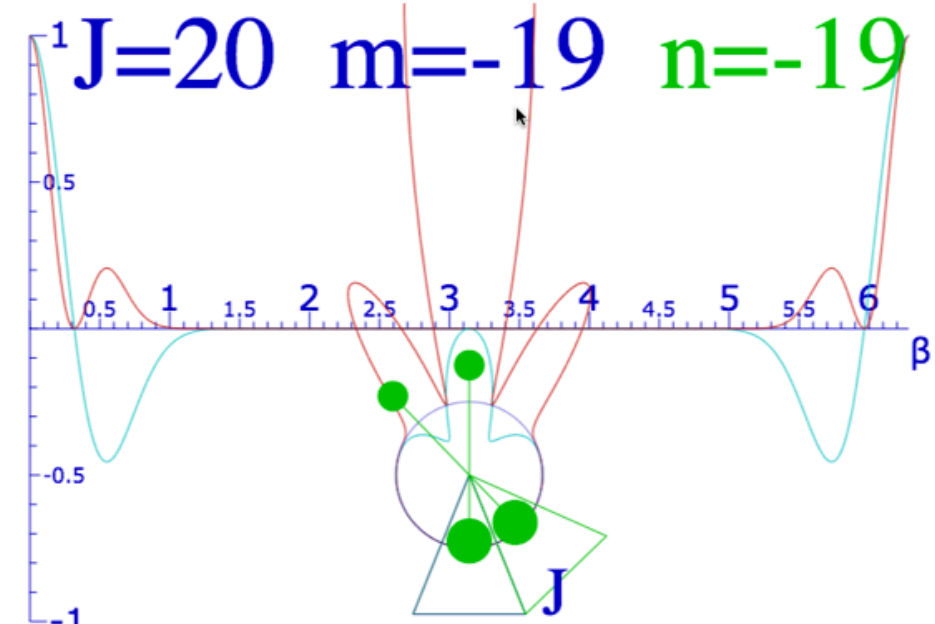
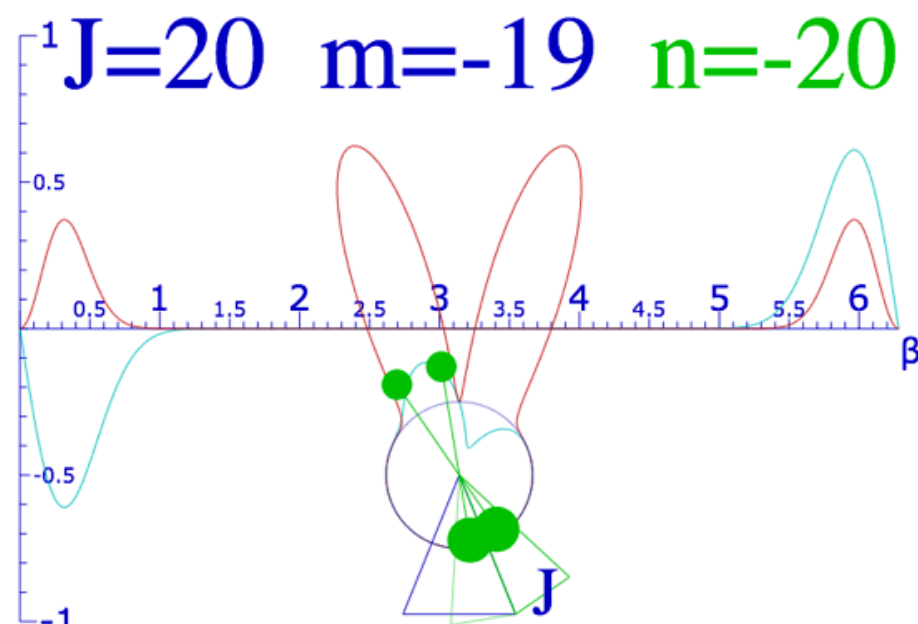
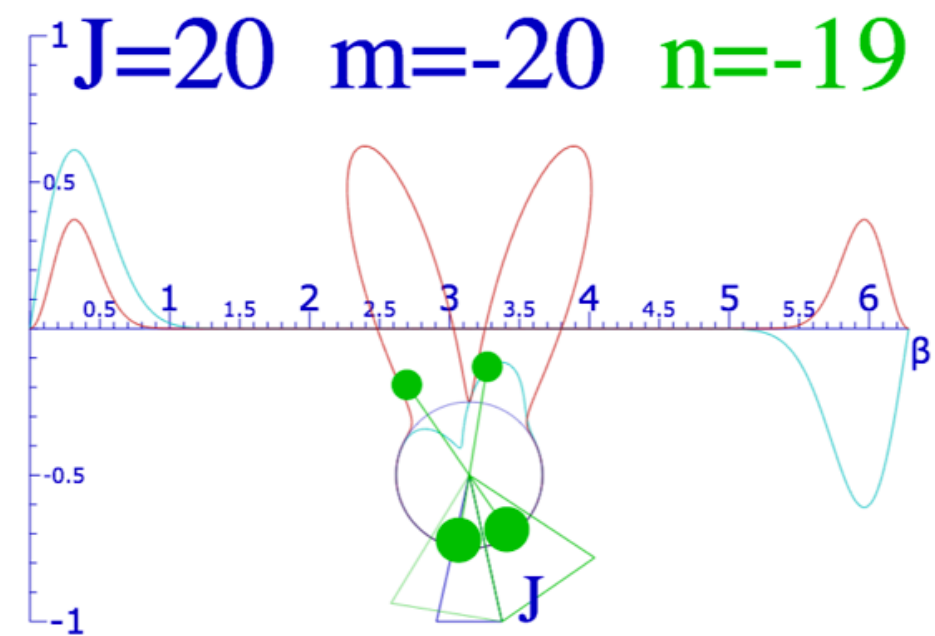
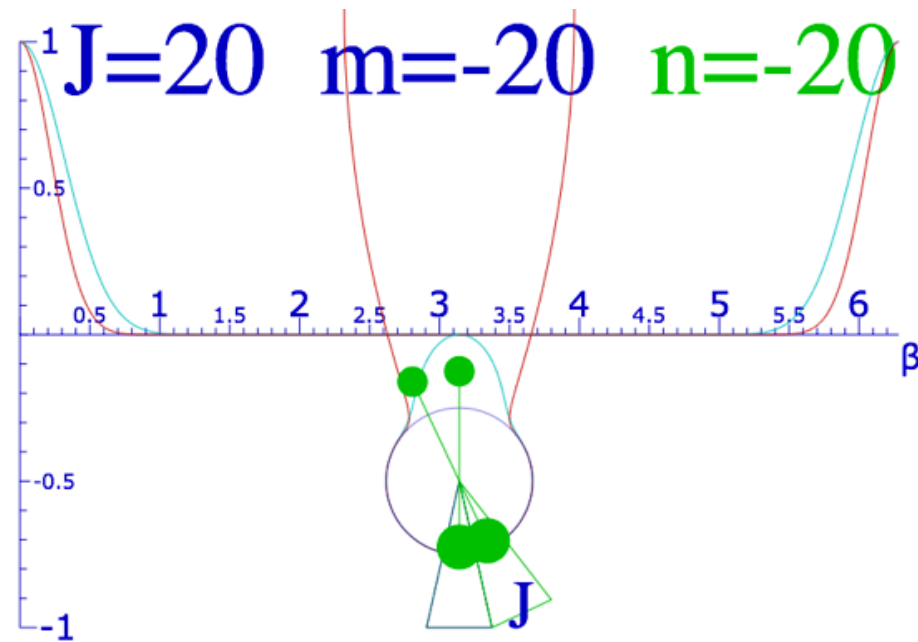












As of May 25, 2013

Links to the current Harter-Soft LearnIt web apps for Physics

(**Bold links** have default redirect pages. *Italics* are not yet meant for production. **Red** are in the final stages of testing.)

Production Links - *For the students & general public*

[BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"](http://www.uark.edu/ua/modphys/markup/BohrItWeb.html)

[BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"](http://www.uark.edu/ua/modphys/markup/BounceItWeb.html)

[BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"](http://www.uark.edu/ua/modphys/markup/BoxItWeb.html)

[CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"](http://www.uark.edu/ua/modphys/markup/CoulItWeb.html)

[Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"](http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html)

[LearnIt - Production; URL is "**http://www.uark.edu/ua/modphys**" or "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"](http://www.uark.edu/ua/modphys/markup/LearnItWeb.html)

[JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"](http://www.uark.edu/ua/modphys/markup/JerkItWeb.html)

[Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"](http://www.uark.edu/ua/modphys/markup/PendulumWeb.html)

[QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"](http://www.uark.edu/ua/modphys/markup/QuantItWeb.html)

[Relativity - *Pirelli Entrant* - Production; URL is "**http://www.uark.edu/ua/pirelli**" or "http://www.uark.edu/ua/pirelli/html/default.html"](http://www.uark.edu/ua/modphys/markup/RelativityWeb.html)

[Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"](http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html)

Testing Links - *For internal use and testing by Harter & Heyoka*

[BohrIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html)

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[BounceIt Title Page - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html"](http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html)

[BoxIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html)

[CoulIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html)

[Cycloidulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html)

[Harter-Soft Web Apps - Quick Reference - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html)

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[ModernPhysics - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html"](http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html)

[Pendulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html"](http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html)

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