Group Theory in Quantum Mechanics

Lecture 7 (2.5.13)

Spectral Analysis of U(2) Operators

(Quantum Theory for Computer Age - Ch. 10 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5)

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver & three famous 2-state systems

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

- Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$
- Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - Spinor arithmetic like complex arithmetic
  - Spinor vector algebra like complex vector algebra
  - Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)
- Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
    - 2D Spinor vs 3D vector rotation
    - NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ in $\mathbf{B}$ field
- Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$,and $\mathbf{R}(0,0,\gamma)$
  - Spin-1 (3D-real vector) case
  - Spin-1/2 (2D-complex spinor) case
- 3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states
  - Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$
  - Polarization ellipse and spinor state dynamics

➡ *Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver*

Review: How symmetry groups become eigen-solvers

Suppose you need to diagonalize a complicated operator $\mathbf{K}$ and knew that $\mathbf{K}$ commutes with some other operators $\mathbf{G}$ and $\mathbf{H}$ for which irreducible projectors are more easily found.

$$\mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger\mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K}$$

$$\mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger\mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K}$$

(Here assuming *unitary* $\mathbf{G}^\dagger=\mathbf{G}^{-1}$ and $\mathbf{H}^\dagger=\mathbf{H}^{-1}$.)

This means $\mathbf{K}$ is *invariant* to the transformation by $\mathbf{G}$ and $\mathbf{H}$
and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

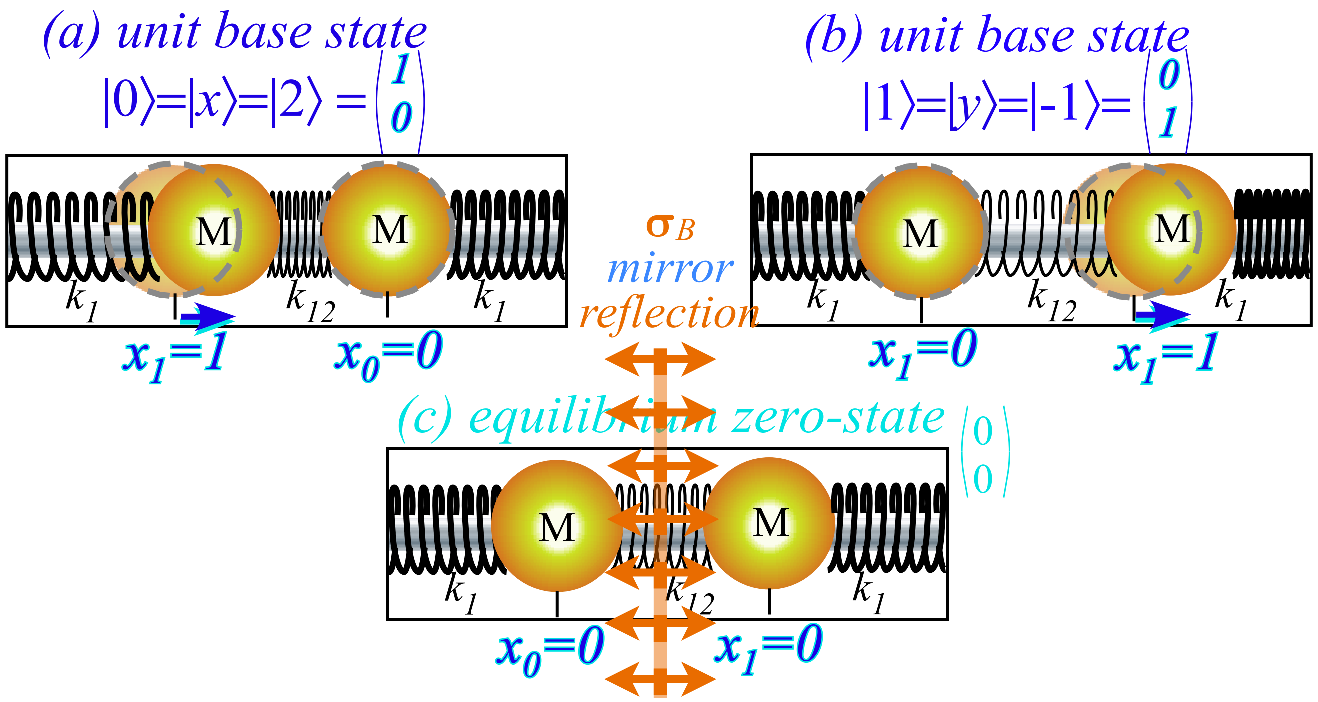
The group $\mathscr{G}_\mathbf{K} = \{\mathbf{1}, \mathbf{G}, \mathbf{H},..\}$ so formed by such operators is called a *symmetry group* for $\mathbf{K}$.

In certain ideal cases a $\mathbf{K}$-matrix $\langle\mathbf{K}\rangle$ is a linear combination of matrices $\langle\mathbf{1}\rangle$,$\langle\mathbf{G}\rangle$,$\langle\mathbf{H}\rangle$,... from $\mathscr{G}_\mathbf{K}$.
Then spectral resolution of $\{\langle\mathbf{1}\rangle,\langle\mathbf{G}\rangle,\langle\mathbf{H}\rangle,..\}$ also resolves $\langle\mathbf{K}\rangle$.

We will study ideal cases first. More general cases are built from this idea.

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0,x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M} \\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0,x_1\} \longrightarrow \{x_1,x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

$K_{11} \equiv K \equiv K_{22}$ and: $K_{12} \equiv k \equiv K_{12} = -k_{12}$ (Let: $M=1$)

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} K & k \\ k & K \end{pmatrix} = K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{K} = K\cdot\mathbf{1} + k\cdot\sigma_B$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table:*

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle=\mathbf{1}|0\rangle\equiv|\mathbf{1}\rangle\ ,\ |1\rangle=\sigma_B|0\rangle\equiv|\sigma_B\rangle\}$

$$\begin{pmatrix} \langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2=1$

or: $\sigma_B{}^2-1=0=(\sigma_B-1)(\sigma_B+1)$

with eigenvalues:
$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+,\mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^\pm$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^+ = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \;=\; K\cdot\mathbf{1} \;-\; k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - k_{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

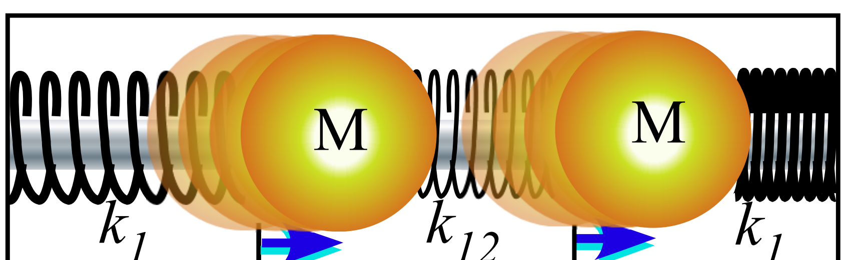
Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

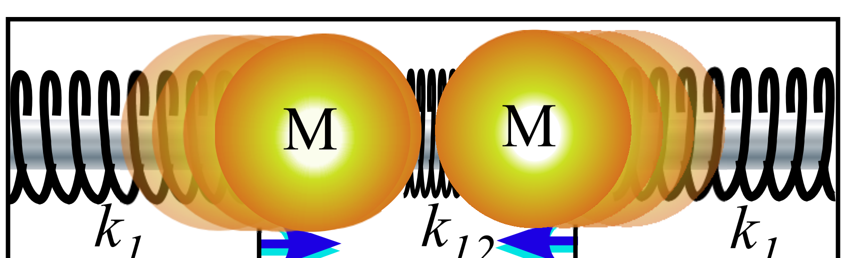
$\varepsilon^+(\mathbf{K}) = K - k_{12} = k_1$, $\quad \varepsilon^-(\mathbf{K}) = K + k_{12} = k_1+2k_{12}$

Even mode $|+\rangle=|0_2\rangle = \begin{pmatrix}1\\ 1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

Odd mode $|-\rangle=|1_2\rangle = \begin{pmatrix}1\\ -1\end{pmatrix}/\sqrt{2}$

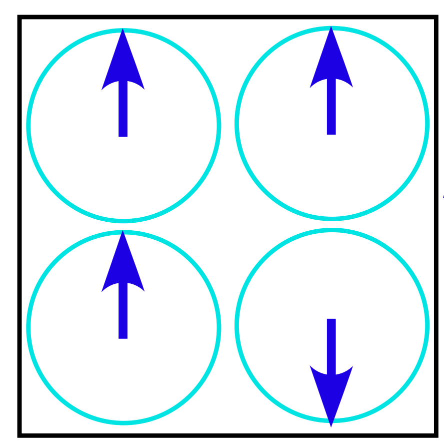


$x_0=1/\sqrt{2}$ $\quad x_1=-1/\sqrt{2}$

C_2 *mode phase character tables*

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = -\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K}\left|\mathbf{x}\right\rangle$$

$$\begin{pmatrix}\langle x_1|\ddot{\mathbf{x}}\rangle\\ \langle x_2|\ddot{\mathbf{x}}\rangle\end{pmatrix} = -\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle\\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{x}\rangle\\ \langle x_2|\mathbf{x}\rangle\end{pmatrix}$$

$$\begin{pmatrix}\ddot{x}_+\\ \ddot{x}_-\end{pmatrix} = -\begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}\begin{pmatrix}x_+\\ x_-\end{pmatrix}$$

$$\left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K}\left|\mathbf{x}\right\rangle$$

$$\begin{pmatrix}\langle +|\ddot{\mathbf{x}}\rangle\\ \langle -|\ddot{\mathbf{x}}\rangle\end{pmatrix} = -\begin{pmatrix}\langle +|\mathbf{K}|+\rangle & \langle +|\mathbf{K}|-\rangle\\ \langle -|\mathbf{K}|+\rangle & \langle -|\mathbf{K}|-\rangle\end{pmatrix}\begin{pmatrix}\langle +|\mathbf{x}\rangle\\ \langle -|\mathbf{x}\rangle\end{pmatrix}$$

C_2 Symmetric 2DHO ***un****coupled dynamics*

Eigenbra vectors: $\langle +| = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$, $\langle -| = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$

Eigenket vectors: $|+\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$, $|-\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}$

$$\begin{pmatrix}M\ddot{x}_+ + (k_1)x_+\\ M\ddot{x}_- + (k_1+2k_{12})x_-\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Each mode runs independently

(+)*-mode at frequency* $\omega_+ = \sqrt{(k_1/M)}$

(−)*-mode at frequency* $\omega_- = \sqrt{(k_1+2k_{12})/M}$

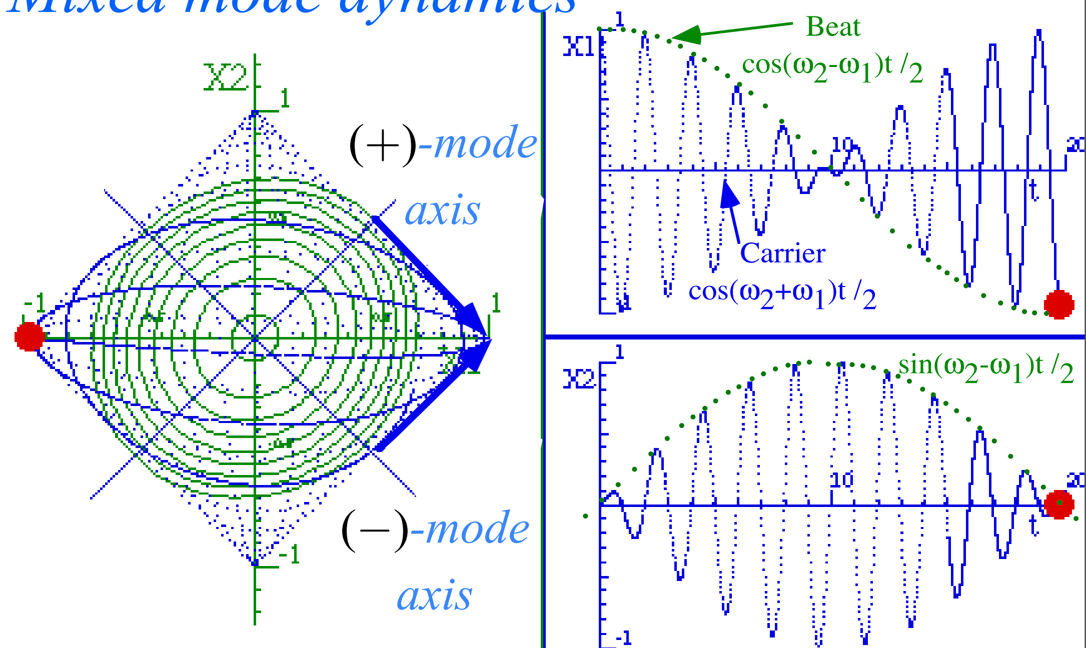
Spectral decomposition of initial state $\mathbf{x}(0)=(x_1\ \ x_2)=(1,0)$:

$$\mathbf{1}\cdot\mathbf{x}(0) = (\mathbf{P}_+ + \mathbf{P}_-)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathbf{x}(0) = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\left(\frac{1}{\sqrt{2}}\right) + \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\ \frac{-1}{2}\end{pmatrix}$$

$$\text{so: } \mathbf{x}(t) = e^{-i\omega_+ t}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix} + e^{-i\omega_- t}\begin{pmatrix}\frac{1}{2}\\ \frac{-1}{2}\end{pmatrix}$$

Mixed mode dynamics



100% AM modulation results

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

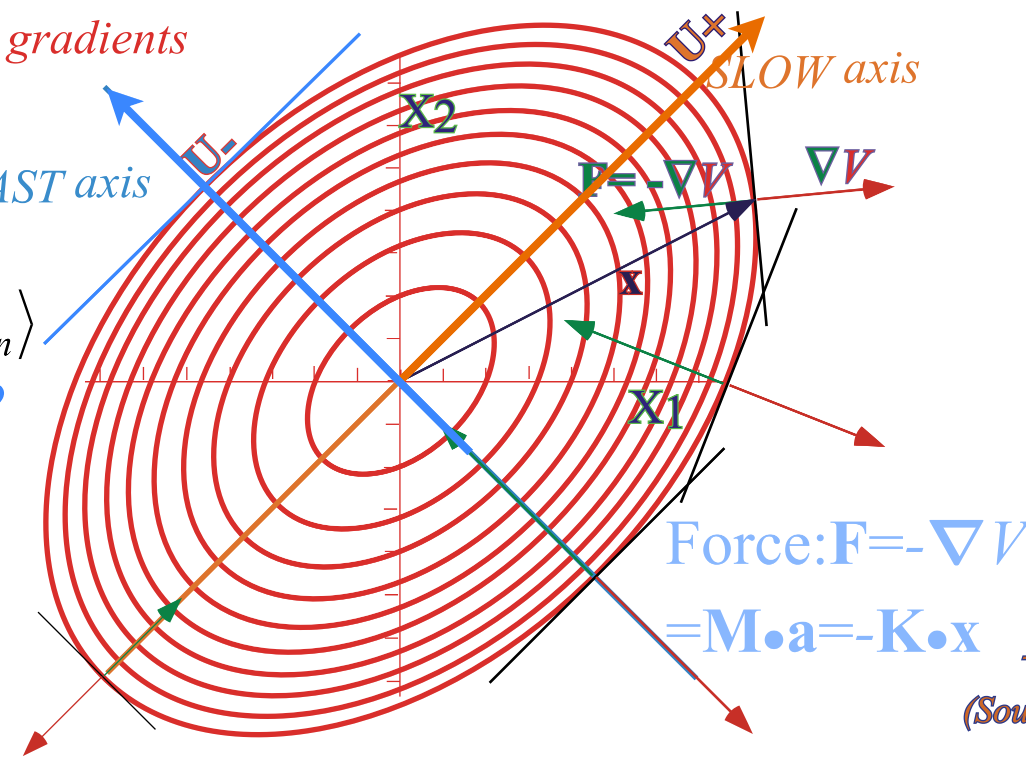
$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{pmatrix}\dfrac{e^{-i\omega_+ t}+e^{-i\omega_- t}}{2}\\ \dfrac{e^{-i\omega_+ t}-e^{-i\omega_- t}}{2}\end{pmatrix} = \frac{e^{-i\frac{(\omega_+ + \omega_-)}{2}t}}{2}\begin{pmatrix}e^{-i\frac{(\omega_+ - \omega_-)}{2}t} + e^{i\frac{(\omega_+ - \omega_-)}{2}t}\\ e^{-i\frac{(\omega_+ - \omega_-)}{2}t} - e^{i\frac{(\omega_+ - \omega_-)}{2}t}\end{pmatrix} = e^{-i\frac{(\omega_+ + \omega_-)}{2}t}\begin{pmatrix}\cos\dfrac{(\omega_- - \omega_+)t}{2}\\ i\sin\dfrac{(\omega_- - \omega_+)t}{2}\end{pmatrix}$$

Note the ***i*** *phase*

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!
Only extremal axes work. (major or minor axes)

Force: $\mathbf{F} = -\nabla V$
$= \mathbf{M} \bullet \mathbf{a} = -\mathbf{K} \bullet \mathbf{x}$

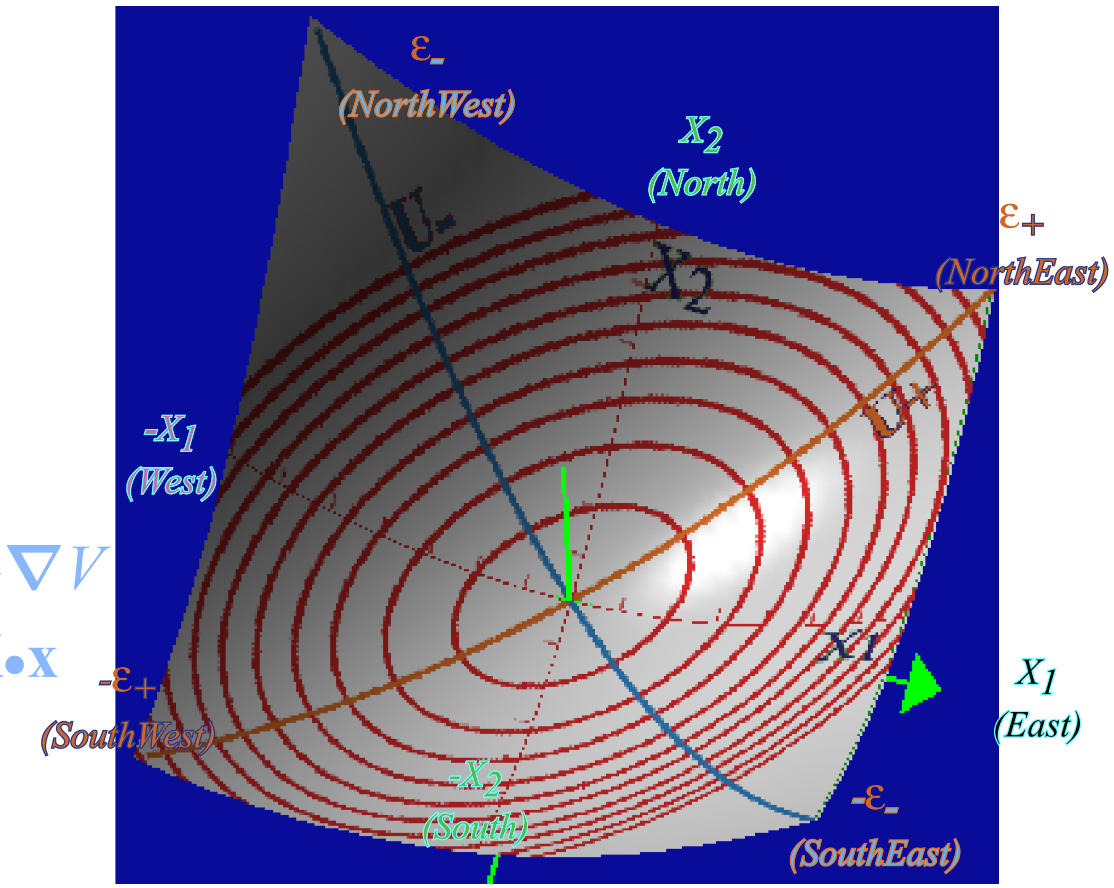
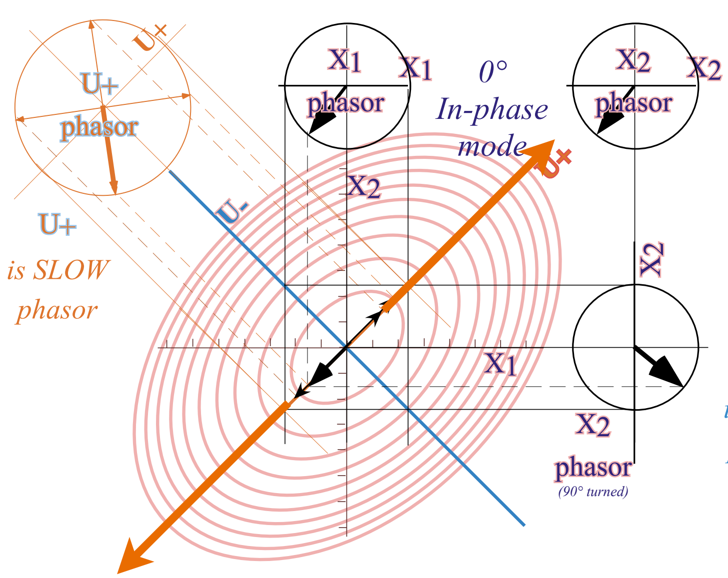
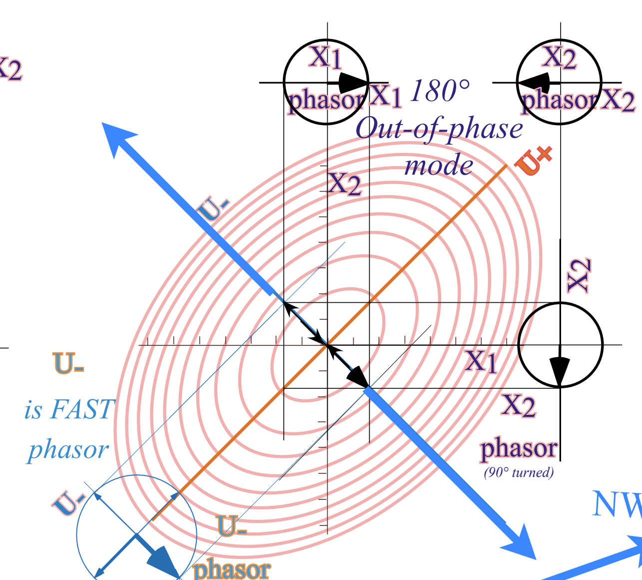


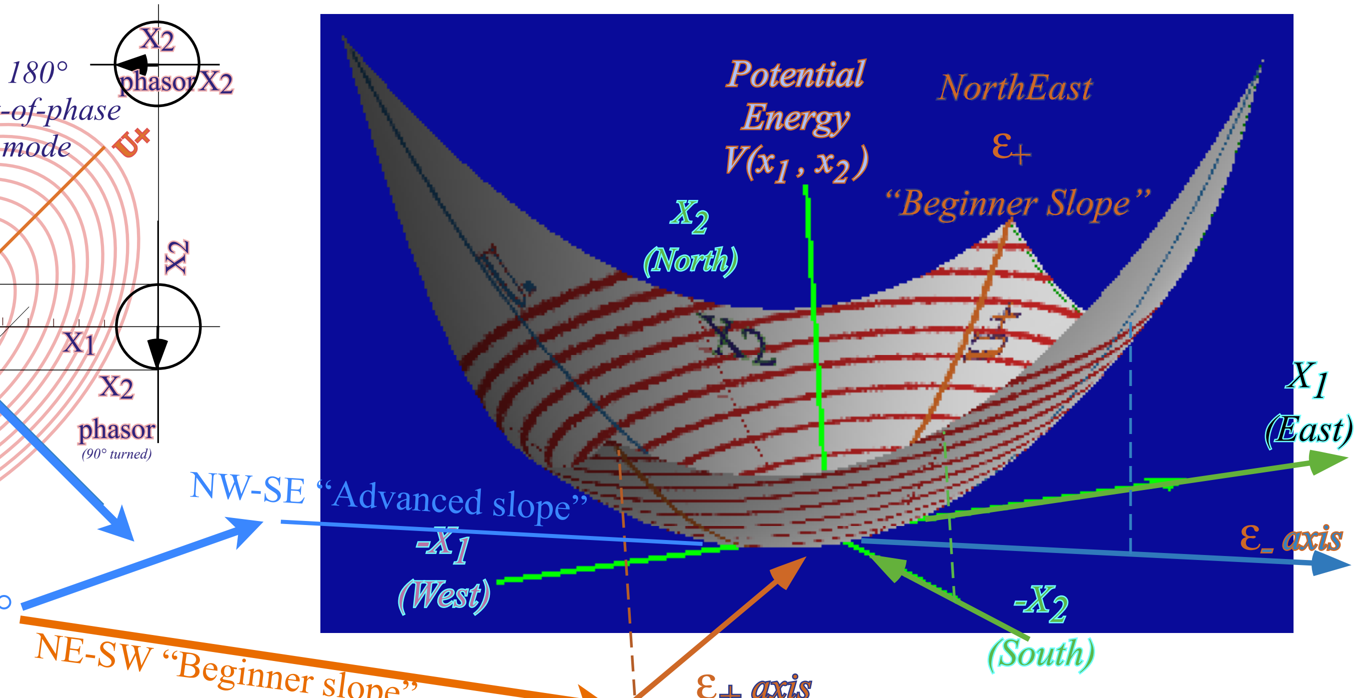
Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode

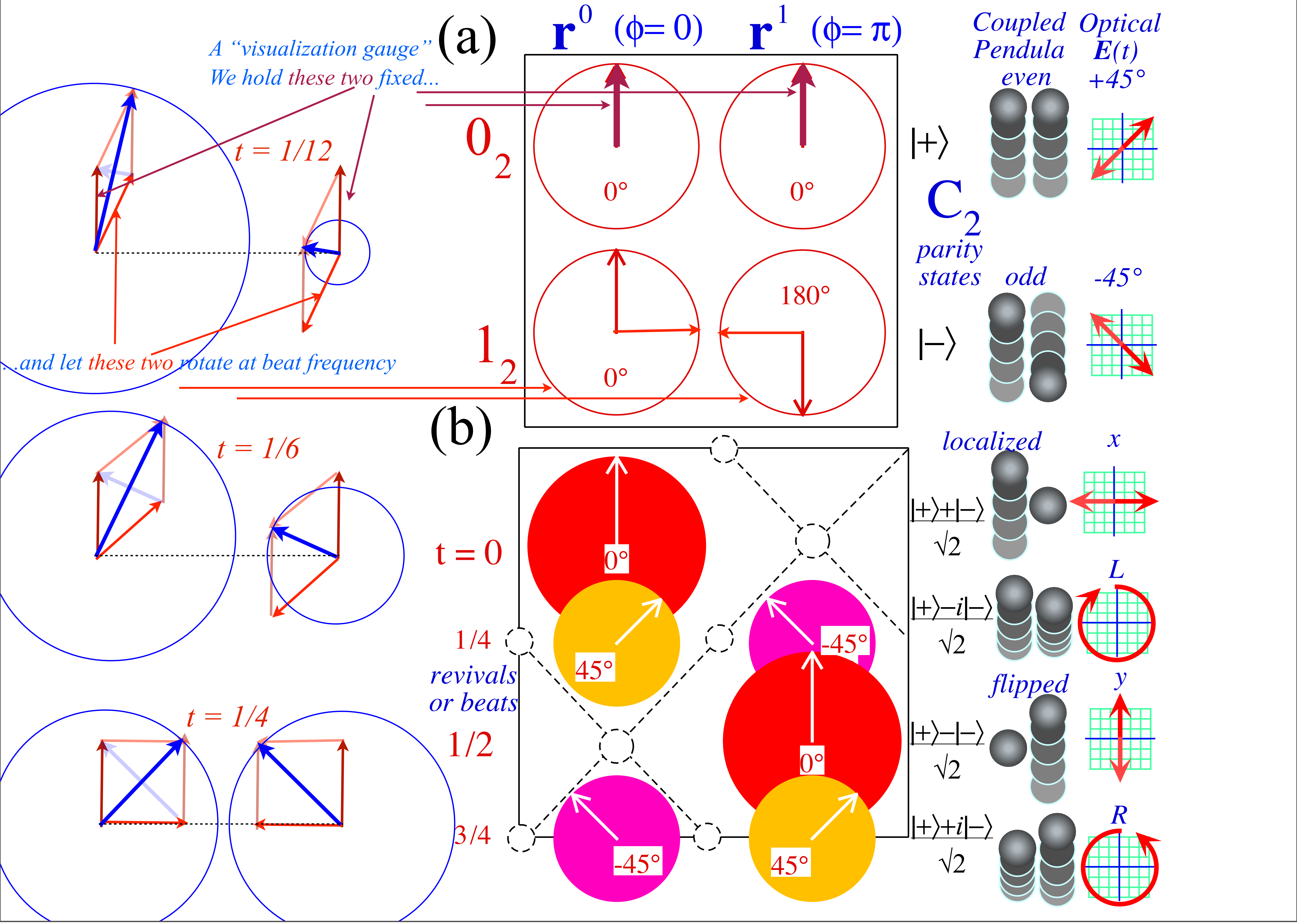




With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

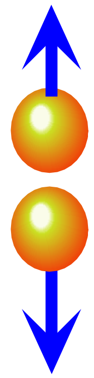
2D-HO beats and mixed mode geometry



Some of the most famous 2-state systems and their two-complex-component coordinates.

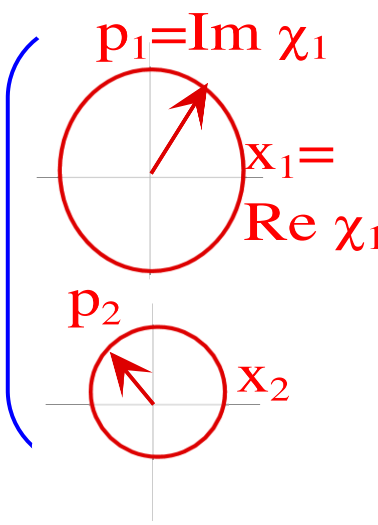
(a) Electron Spin-1/2-Polarization

Spin-up
$|1\rangle = |\uparrow\rangle$



Spin-dn
$|2\rangle = |\downarrow\rangle$

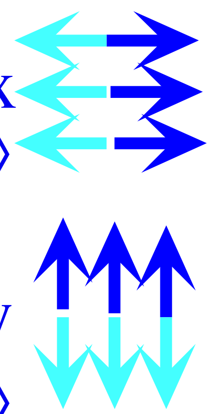
$$|\chi\rangle = \begin{pmatrix} \chi\uparrow \\ \chi\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

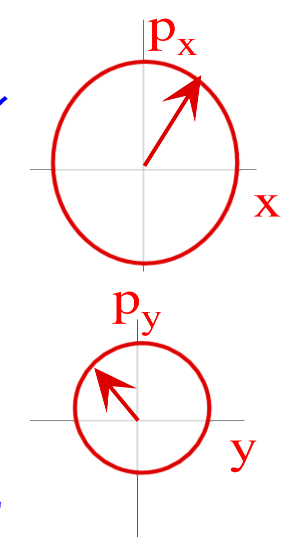
(b) Photon Spin-1-Polarization

Plane-x
$|1\rangle = |x\rangle$



Plane-y
$|2\rangle = |y\rangle$

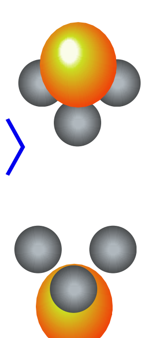
$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

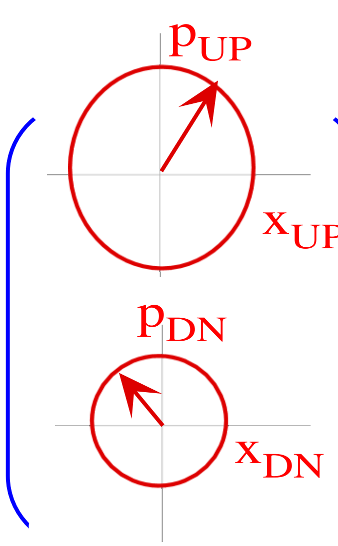
(c) Ammonia (NH_3) Inversion States

N-UP
$|1\rangle = |UP\rangle$



N-DN
$|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, *Balance* $S_B = S_X$, *and Chirality* $S_C = S_Y$

Polarization ellipse and spinor state dynamics

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

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Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t}=\begin{pmatrix} A & B \\ B & D \end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix} A & B \\ B & D \end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}$$

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Finally a 2[nd] time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2[nd]-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2\\ &=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)\\ &=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1\\ &=-B\left(Ax_1+Bx_2+Cp_2\right)-D\left(Bx_1+Dx_2-Cp_1\right)+C\left(Ap_1+Bp_2-Cx_2\right)\\ &=-\left(AB+BD\right)x_1-\left(B^2+D^2+C^2\right)x_2+C\left(A+D\right)p_1\end{aligned}$$

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\equiv\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t}=\begin{pmatrix}A & B\\ B & D\end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix}A & B\\ B & D\end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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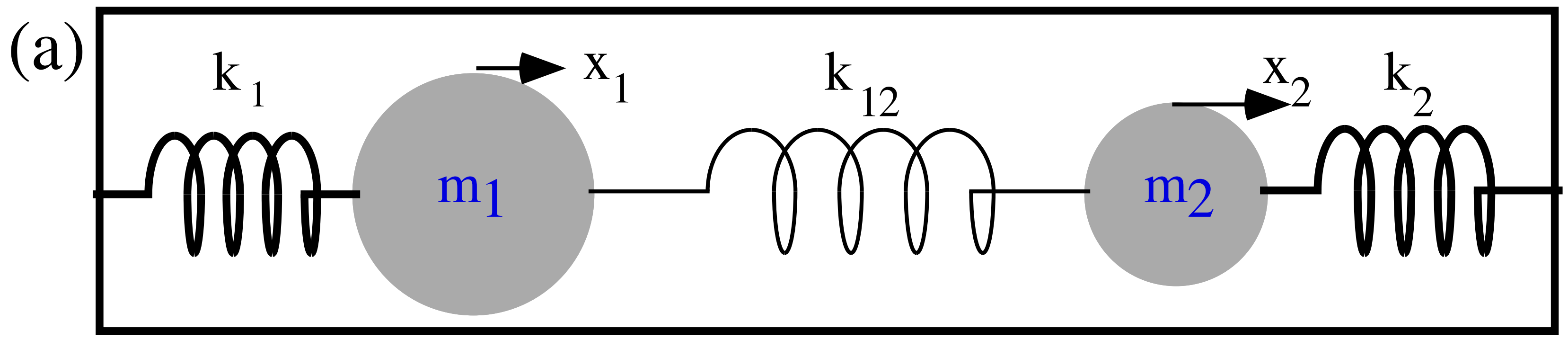
$$-\ddot{x}_1 = K_{11}x_1 + K_{12}x_2$$
$$-\ddot{x}_2 = K_{21}x_1 + K_{22}x_2$$



$$m_1K_{11} = A^2 + B^2 = k_1 + k_{12}, \quad m_1K_{12} = AB + BD = -k_{12},$$
$$m_2K_{21} = AB + BD = -k_{12}, \quad m_2K_{22} = B^2 + D^2 = k_2 + k_{12}.$$

Review: How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

- Spinor arithmetic like complex arithmetic
- Spinor vector algebra like complex vector algebra
- Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

- The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
  - 2D Spinor vs 3D vector rotation
  - NMR Hamiltonian: 3D Spin Moment **m** in **B** field

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

- Spin-1 (3D-real vector) case
- Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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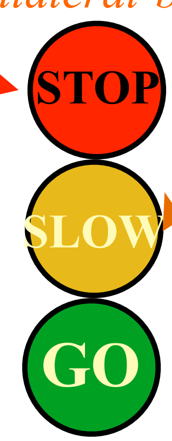

Motivation for coloring scheme:
The Traffic Signal

Standing waves

Moving waves

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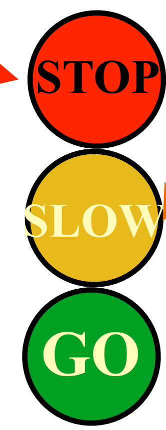

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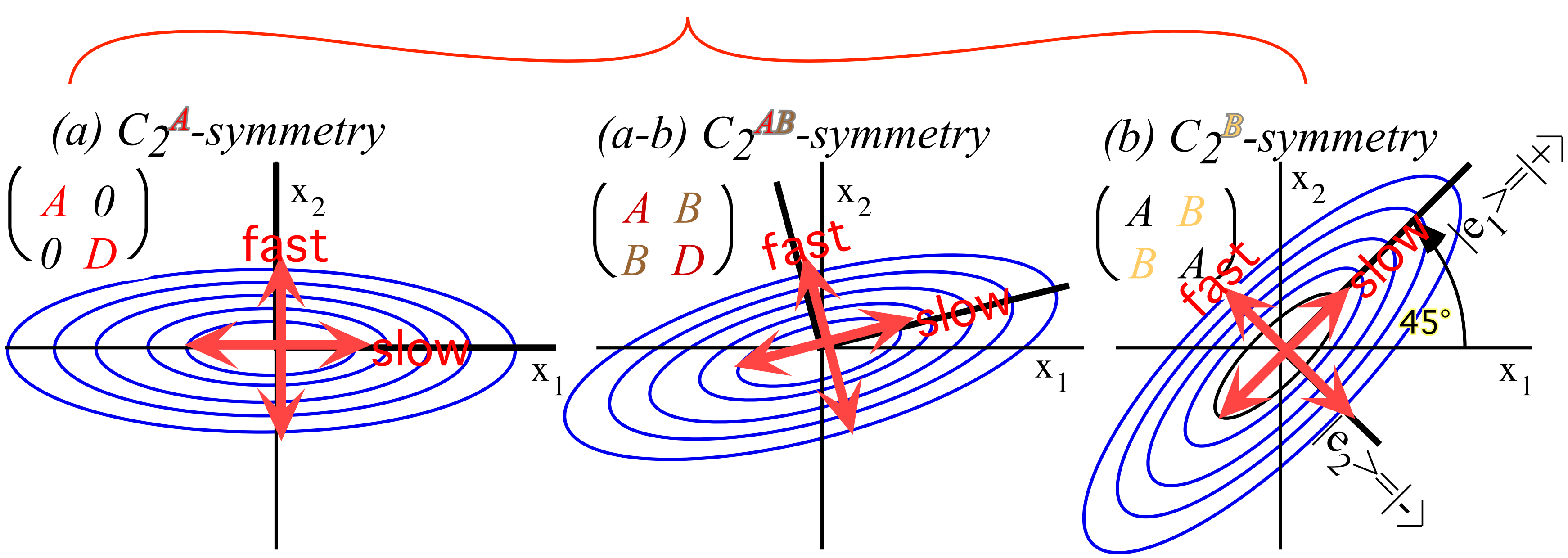

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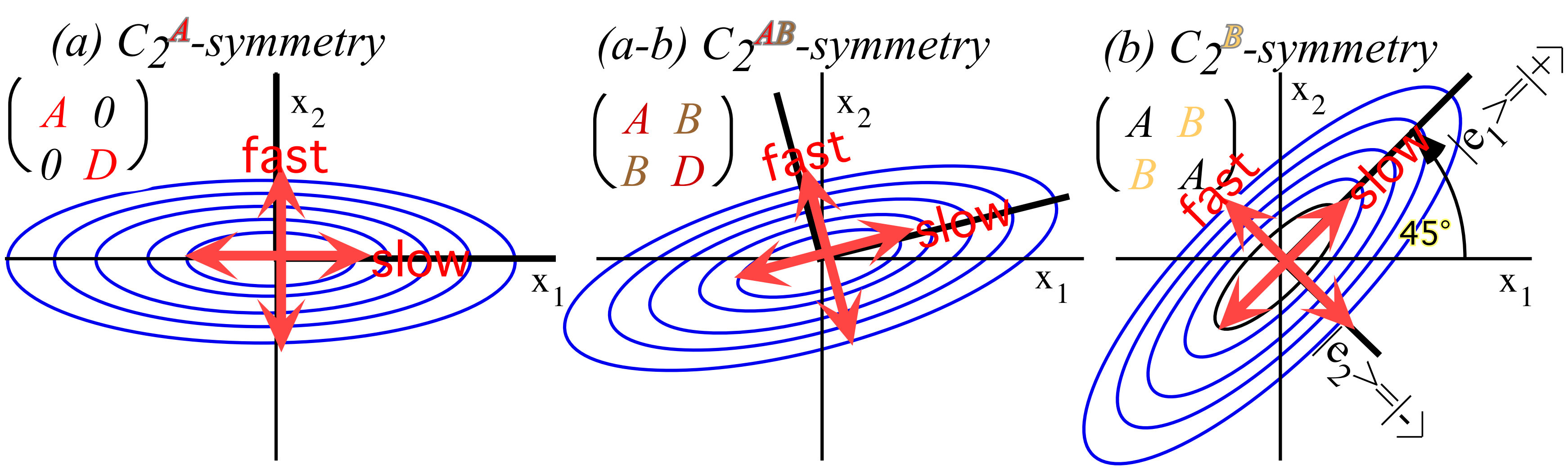
Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623



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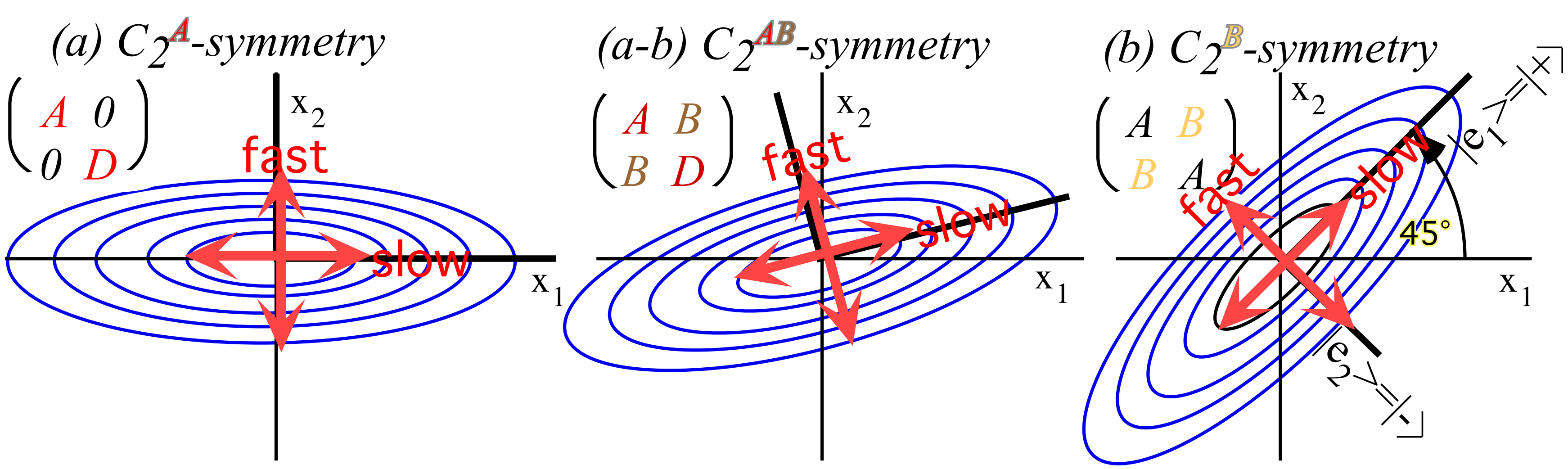

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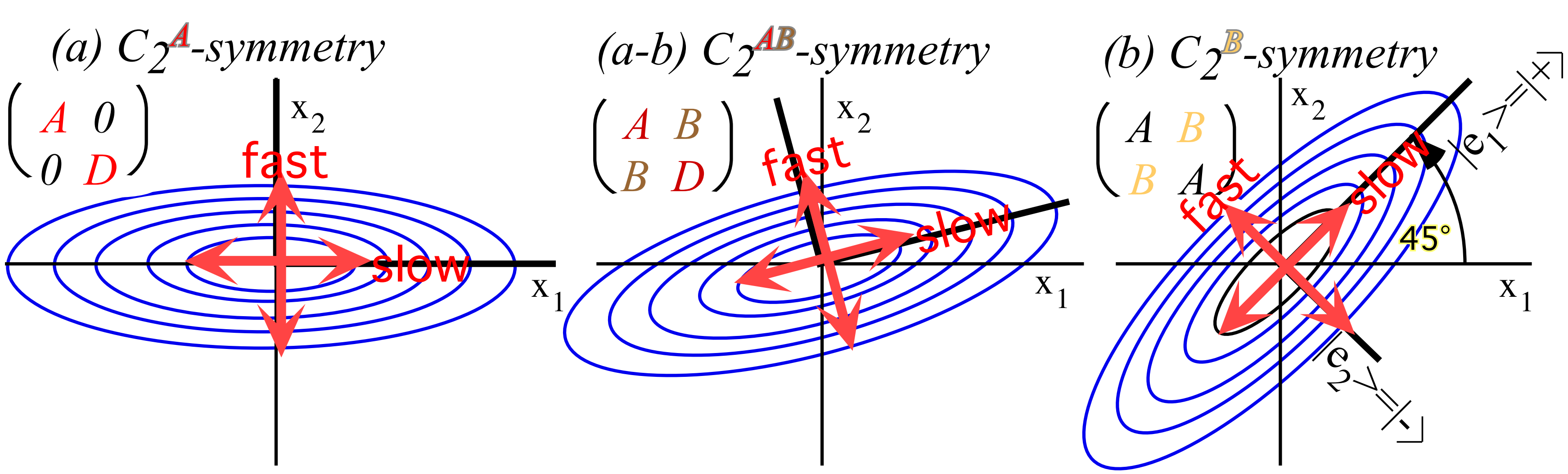

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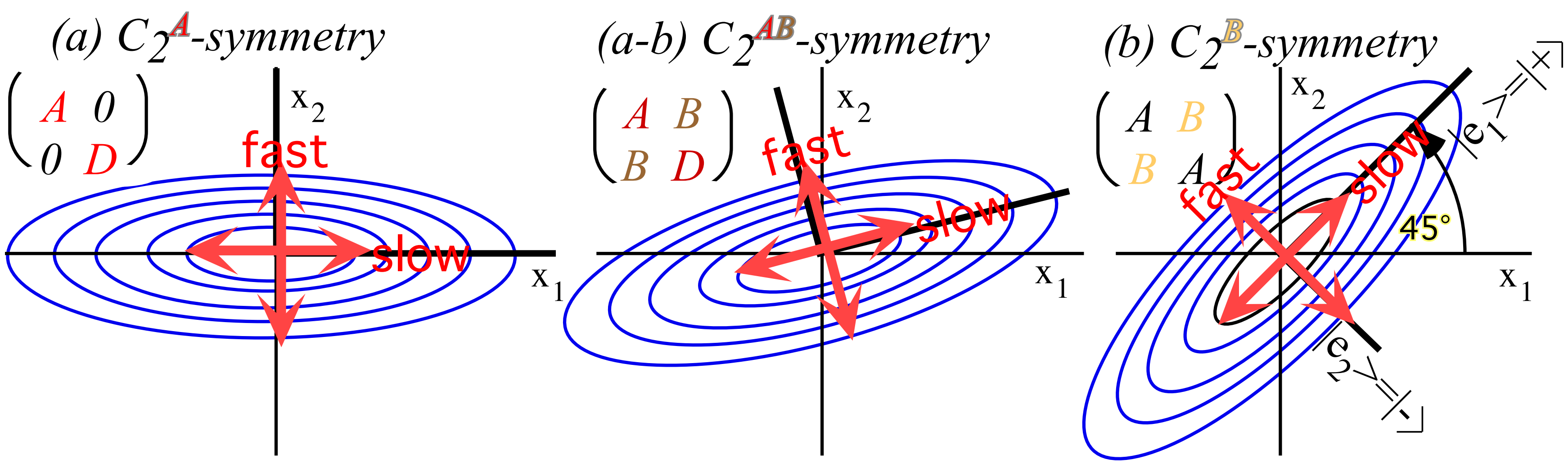

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OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

ABCD Time evolution operator

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$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

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$$\begin{matrix} & a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z & & a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ = & +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z & = & +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ & +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z & & +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

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$$\sigma_Z \cdot \sigma_X: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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To finish we need another symmetry property called *anti-commutation*: $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ *,etc.*

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So: $\sigma_a^2 = \mathbf{1}$

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

→ *Deriving* $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
→ *Spinor vector algebra*	*like*	*complex vector algebra*
Spinor exponentials	*like*	*complex exponentials* *("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

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$$\underset{\sigma_Z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\cdot\underset{\sigma_X}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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Write the product in Gibbs notation. (This is where Gibbs *got* his {**i,j,k**} notation!)

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(Recall complex variable result.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
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Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment* $\mathbf{m}$ *in* $\mathbf{B}$ *field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

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This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\boldsymbol{\sigma}_A + \varphi_B\boldsymbol{\sigma}_B + \varphi_Z\boldsymbol{\sigma}_Z \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\ \sigma_\varphi\sin\varphi$

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$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0\cdot t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

$$=(\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi)e^{-i\omega_0\cdot t}$$

ABCD Time evolution operator

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ powerful.

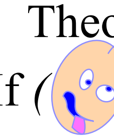
Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall Euler - DeMoivre Theorem.)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$

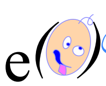
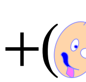
$$-i(\varphi + \frac{1}{3!}\varphi^3 \cdots) \quad -i(\sin\varphi)$$

Note even powers of *(-i)* are ±*1* and odd powers of *(-i)* are ±*i*.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

The Crazy Thing Theorem:

If ()^2= **-1**

Then:

e()$^\varphi$ =**1**cosφ +()sinφ

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$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteqdot (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi\ -\ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi\ -\ i\ \sigma_\varphi \sin\varphi$

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$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

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$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

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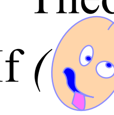
If ()^2= -**1**

Then:

$e^{(\cdot)\varphi} = \mathbf{1}\cos\varphi + (\cdot)\sin\varphi$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteqdot (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$$

generalizes to:

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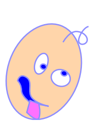
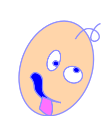
Here:  = *-i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$*
- *Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$*
  - *Spinor arithmetic like complex arithmetic*
  - *Spinor vector algebra like complex vector algebra*
  - *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*
- *Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$*
  - *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
    - *2D Spinor vs 3D vector rotation*
    - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*
- *Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$*
  - *Spin-1 (3D-real vector) case*
  - *Spin-1/2 (2D-complex spinor) case*
- *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*
  - *Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$*
  - *Polarization ellipse and spinor state dynamics*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

ABCD Time evolution operator

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

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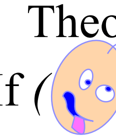
$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

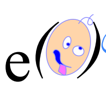
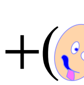
$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 1:
A or Z rotation

The
Crazy Thing
Theorem:

If ()^2 = -**1**

Then:

e$^{(}$  $^{)\varphi}$ = **1** cos φ + () sin φ

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

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$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

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$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

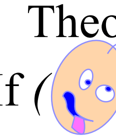
Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

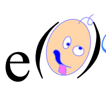
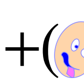
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e()φ = **1**cosφ + ()sinφ

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$\sigma_A = \sigma_Z$ $\quad$ $\sigma_B = \sigma_X$ $\quad$ $\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$$

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$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1:
A or Z
rotation

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2:
C or Y
rotation

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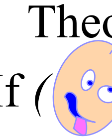
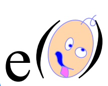
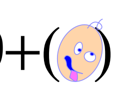
If ()^2 = -**1**
Then:
e()θ = **1**cosθ+()sinθ

Example 3:
Any $\varphi = \omega t$-axial
rotation

Let: $\vec{\varphi} = \vec{\omega}\cdot t$

$$e^{-i(\sigma\bullet\vec{\varphi})t} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi = \mathbf{1}\cos\varphi - i\ (\sigma\bullet\hat{\varphi})\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\ (\sigma_A\hat{\varphi}_A)\sin\varphi \quad - i\ (\sigma_B\hat{\varphi}_B)\sin\varphi \quad - i\ (\sigma_C\hat{\varphi}_C)\sin\varphi$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix}0&1\\1&0\end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix}\cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi\\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi\end{pmatrix}$$

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

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$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

We test these operators by making them rotate each other....

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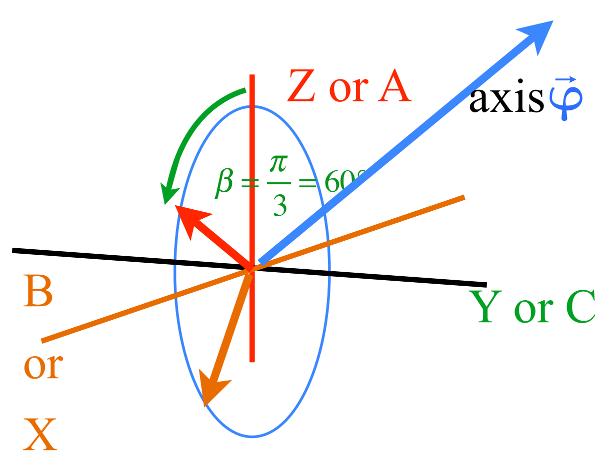

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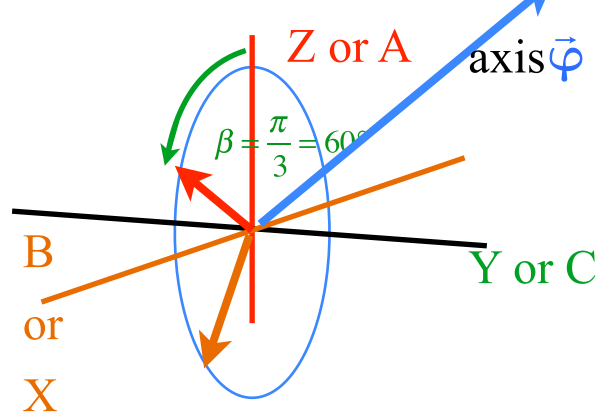

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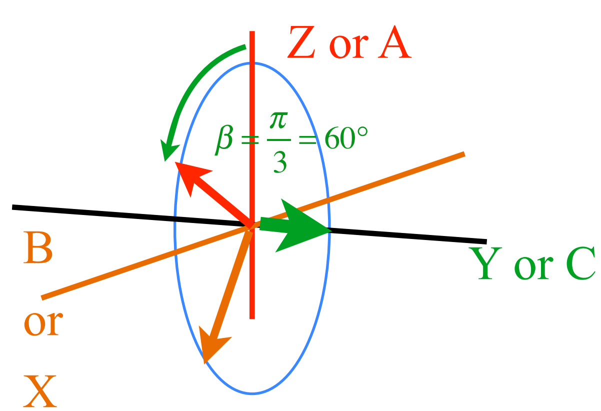

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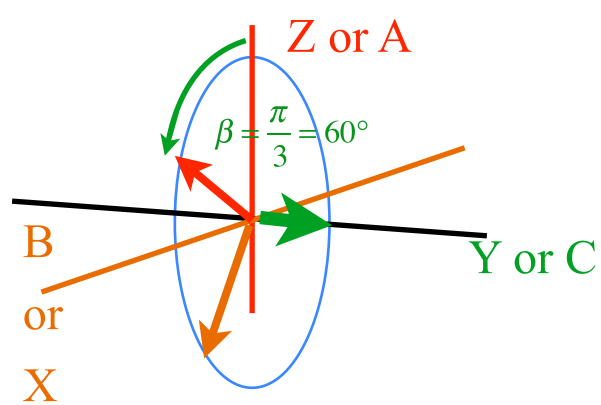

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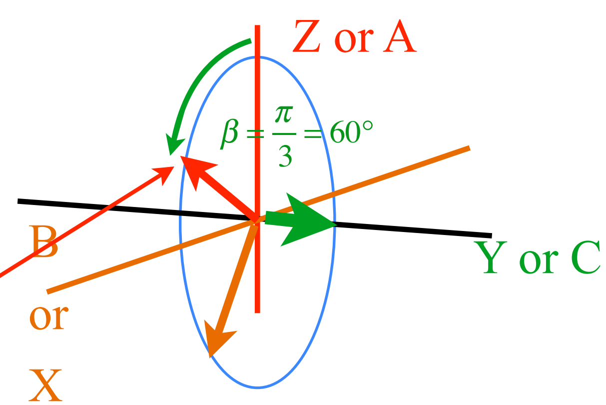

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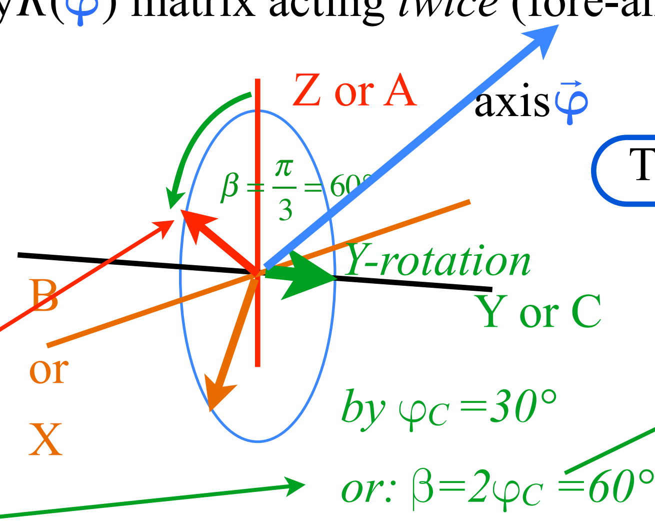
$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\sin 2\varphi_C$$
$$= \quad \boldsymbol{\sigma}_A\ \cos 2\varphi_C + \ \boldsymbol{\sigma}_B\ \sin 2\varphi_C$$



The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

ABCD Time evolution operator

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$|\Psi(t)\rangle = e^{-i\mathbf{H}\cdot t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot t - iB\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot t - iC\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$

generalizes to:

$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi \ - \ i\ \sigma_\varphi \sin\varphi$

where: $\vec{\varphi} = \begin{pmatrix}\varphi_A\\ \varphi_B\\ \varphi_C\end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2}\\ B\\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A = \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C = \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2: C or Y rotation

3D axis vector $\vec{\varphi}=\vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi \varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_μ-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft-1) to give: $\sigma_\mu^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_\mu R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_\mu R^\dagger(\vec{\varphi})$

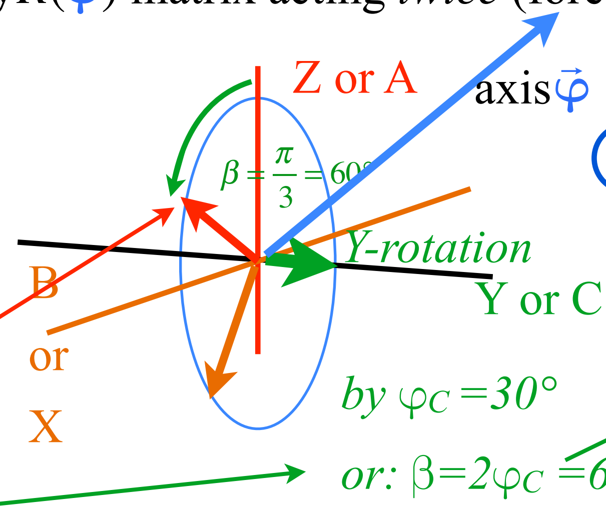
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A \cos 2\varphi_C + \boldsymbol{\sigma}_B \sin 2\varphi_C$$



The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

$\vec{\varphi}=\vec{\omega}\cdot t$ equal to $\vec{\omega}$ only at $t=1$ but $\hat{\varphi}=\hat{\omega}$ always.

$$\hat{\varphi} = \begin{pmatrix}\hat{\varphi}_A\\ \hat{\varphi}_B\\ \hat{\varphi}_C\end{pmatrix} = \begin{pmatrix}\varphi_A\\ \varphi_B\\ \varphi_C\end{pmatrix}\frac{1}{\sqrt{\varphi_A^2+\varphi_B^2+\varphi_C^2}} = \begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\frac{1}{\sqrt{\omega_A^2+\omega_B^2+\omega_C^2}}$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

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generalizes to:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\,\sigma_\varphi \sin\varphi$$

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$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A = \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

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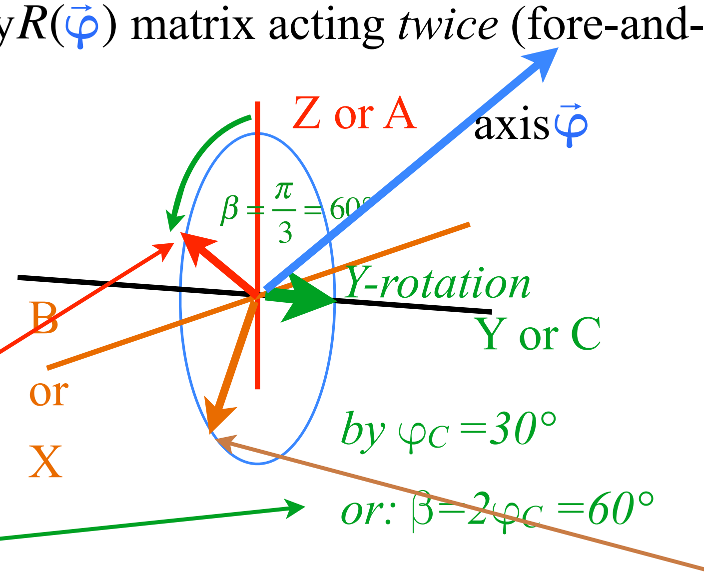
Example 2: C or Y rotation

3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_μ-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft^{-1}) to give: $\sigma_\mu^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_\mu R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_\mu R^\dagger(\vec{\varphi})$

$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\sin 2\varphi_C$$
$$= \boldsymbol{\sigma}_A\cos 2\varphi_C + \boldsymbol{\sigma}_B\sin 2\varphi_C$$



$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C\\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}-1&0\\0&1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\cos 2\varphi_C$$
$$= -\boldsymbol{\sigma}_A\sin 2\varphi_C + \boldsymbol{\sigma}_B\cos 2\varphi_C$$

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

➡ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

➡ *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1 = \sigma_0, \sigma_B = \sigma_X, \sigma_C = \sigma_Y, \sigma_A = \sigma_Z\}$

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$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\,\mathbf{1}+\Omega_A\,\mathbf{S}_A+\Omega_B\,\mathbf{S}_B+\Omega_C\,\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

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The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1} = \sigma_0, \mathsf{S}_B = \mathsf{S}_X, \mathsf{S}_C = \mathsf{S}_Y, \mathsf{S}_A = \mathsf{S}_Z\}$

(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

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Notation for 2D Spinor space

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \sigma_\omega$$

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t} = e^{-i\omega_0\cdot t}e^{-i\sigma_\omega \omega\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\sigma_\omega \sin\omega\cdot t\right)$$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\ \sigma_0 + \omega_A\ \sigma_A + \omega_B\ \sigma_B + \omega_C\ \sigma_C = \omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\ \mathbf{1} + \Omega_A\ \mathbf{S}_A + \Omega_B\ \mathbf{S}_B + \Omega_C\ \mathbf{S}_C = \Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\sigma_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\sigma_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

Notation for 2D Spinor space

$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\,\mathbf{1}+\Omega_A\,\mathbf{S}_A+\Omega_B\,\mathbf{S}_B+\Omega_C\,\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+(A-D)\begin{pmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{pmatrix}+2B\begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix}+2C\begin{pmatrix}0&-\frac{i}{2}\\\frac{i}{2}&0\end{pmatrix}$$

Notation for 3D Vector space

0^{th} *component unchanged*

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

"Crank" vector

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1}=\sigma_0, \mathsf{S}_B=\mathsf{S}_X, \mathsf{S}_C=\mathsf{S}_Y, \mathsf{S}_A=\mathsf{S}_Z\}$

(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

$$\vec{\varphi}=\begin{pmatrix}\varphi_A\\\varphi_B\\\varphi_C\end{pmatrix}=\vec{\omega}\cdot t=\begin{pmatrix}\omega_A\\\omega_B\\\omega_C\end{pmatrix}\cdot t=\begin{pmatrix}\frac{A-D}{2}\\B\\C\end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix}\omega_A\\\omega_B\\\omega_C\end{pmatrix}\cdot t=\begin{pmatrix}\frac{A-D}{2}\\B\\C\end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix}A & B-iC\\B+iC & D\end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

"Crank" vector

$$\vec{\Theta}=\begin{pmatrix}\Theta_A\\\Theta_B\\\Theta_C\end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix}\Omega_A\\\Omega_B\\\Omega_C\end{pmatrix}\cdot t=\begin{pmatrix}A-D\\2B\\2C\end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix}\Omega_A\\\Omega_B\\\Omega_C\end{pmatrix}\cdot t=\begin{pmatrix}A-D\\2B\\2C\end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

→ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

→ *2D Spinor vs 3D vector rotation*

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, *Balance* $S_B = S_X$, *and Chirality* $S_C = S_Y$

Polarization ellipse and spinor state dynamics

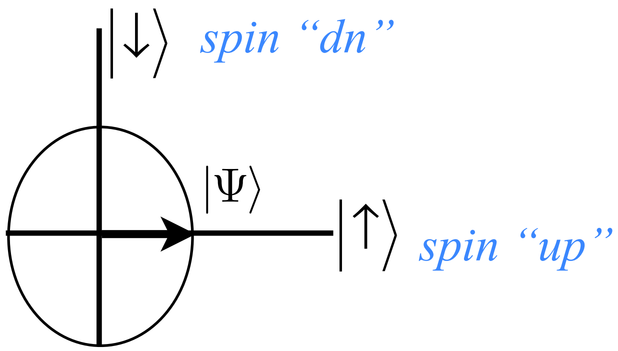
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor $\{|\uparrow\rangle,|\downarrow\rangle\}$*-space (complex)*

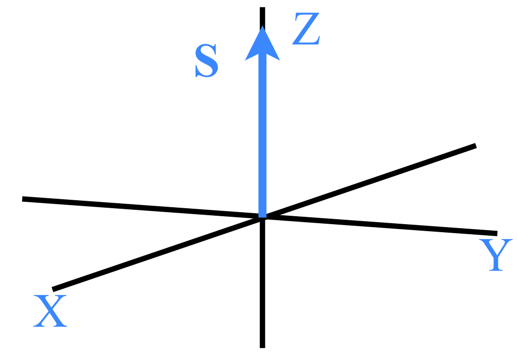
R(3):3D Spin Vector $\{S_X, S_Y, S_Z\}$*-space (real)*

State vector $|\Psi\rangle=|\uparrow\rangle\langle\uparrow|\Psi\rangle+|\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S}=|X\rangle\langle X|\mathbf{S}\rangle+|Y\rangle\langle Y|\mathbf{S}\rangle+|Z\rangle\langle Z|\mathbf{S}\rangle$



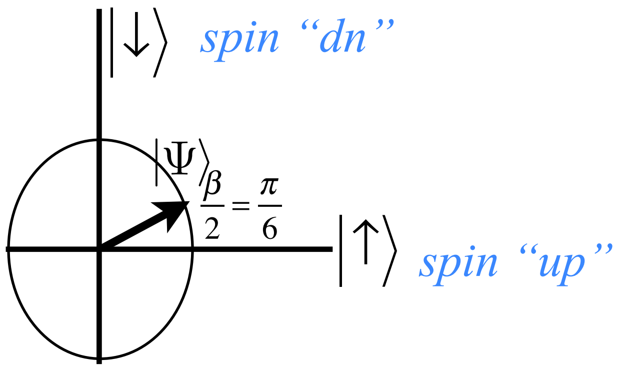
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$$



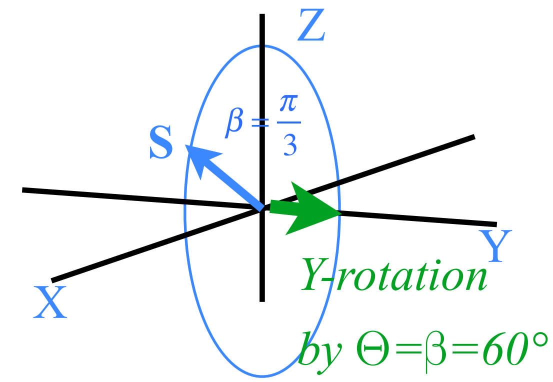
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

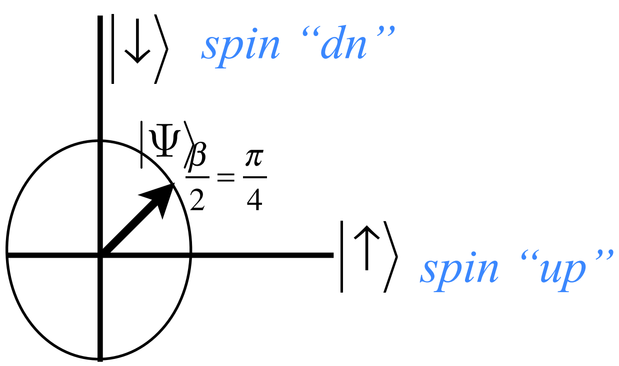
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$



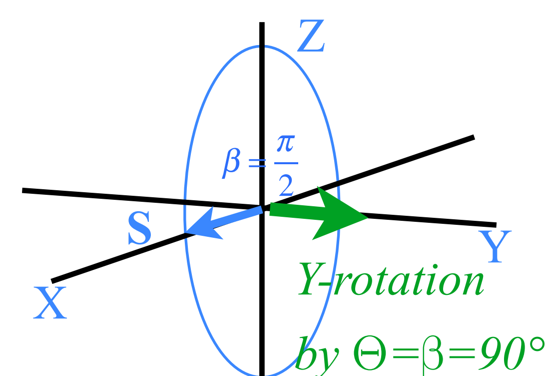
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



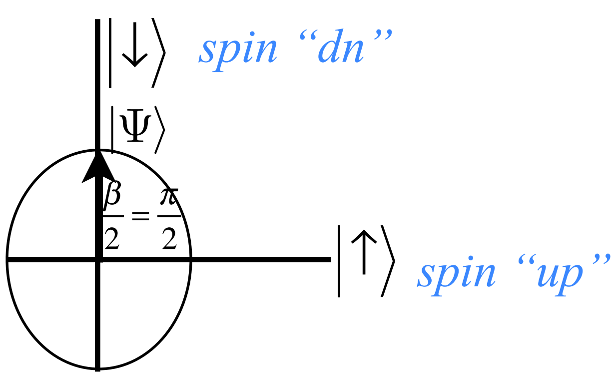
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



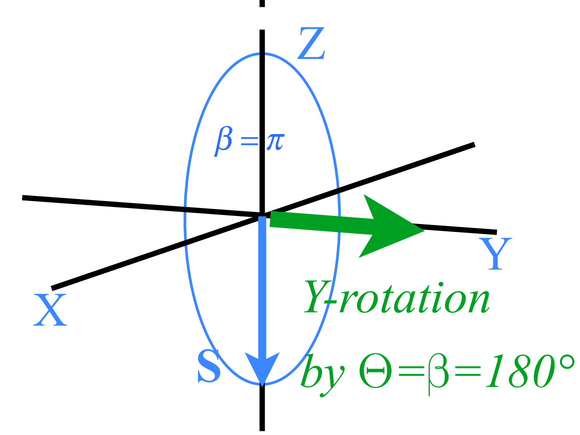
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast"

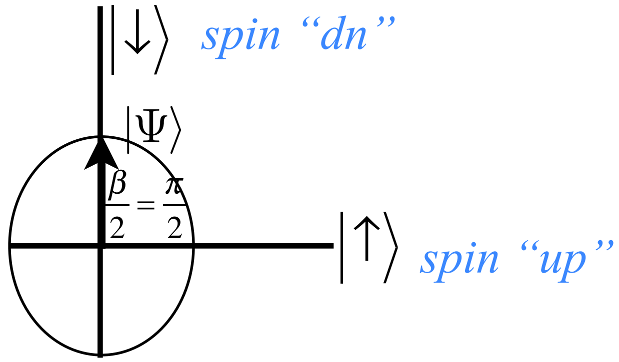
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

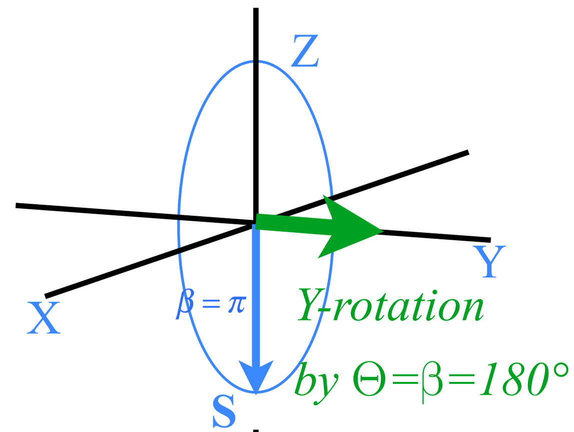
R(3):3D Spin Vector {Sx, Sy, Sz}-space (real)

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

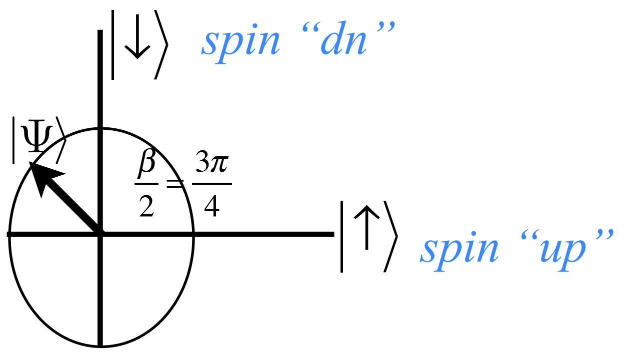
Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



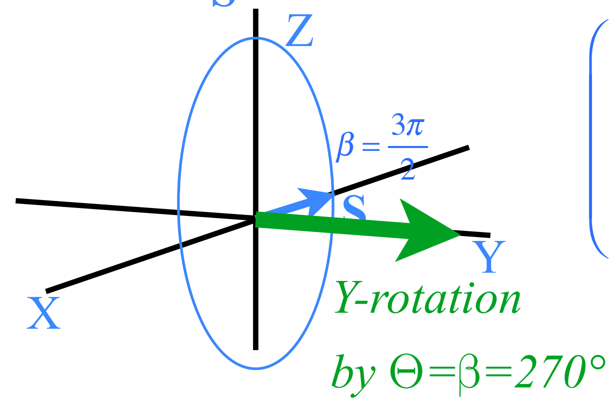
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$



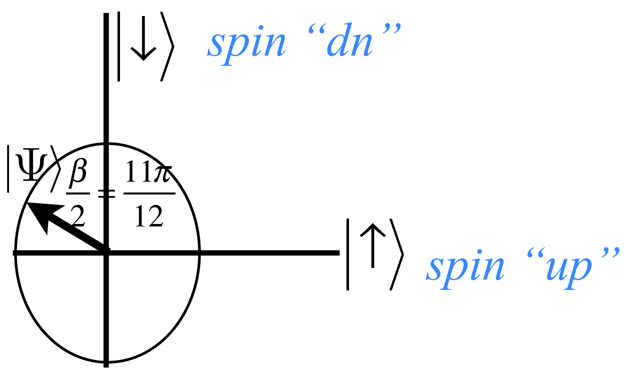
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$



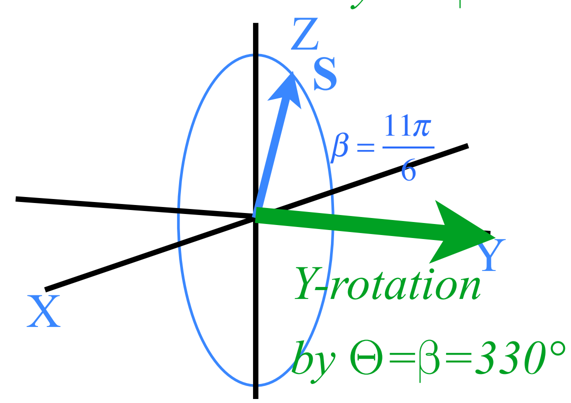
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



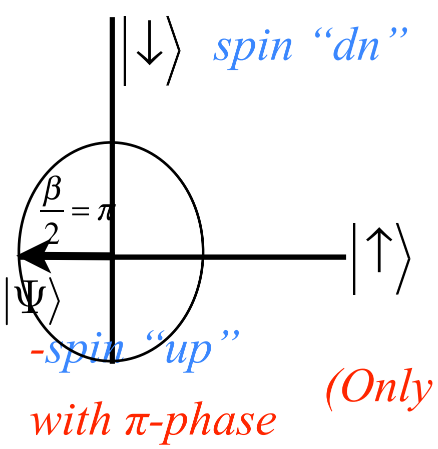
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$

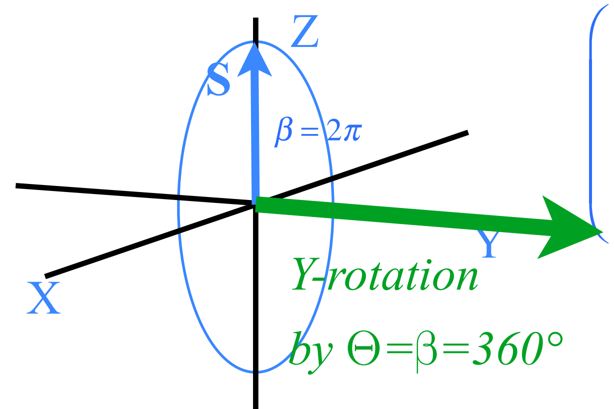


$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} =$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}-1\\ 0\end{pmatrix}$$

(Only "half-way" home after 2π =360° rotation)



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

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Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

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Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

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Polarization ellipse and spinor state dynamics

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma}=\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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$$= gB_Z\ \sigma_A \quad + \quad gB_X\ \sigma_X \quad + gB_Y\ \sigma_Y \quad = \vec{\omega}\bullet\vec{\sigma} = \omega\sigma_\omega$$

Notation for 2D Spinor space

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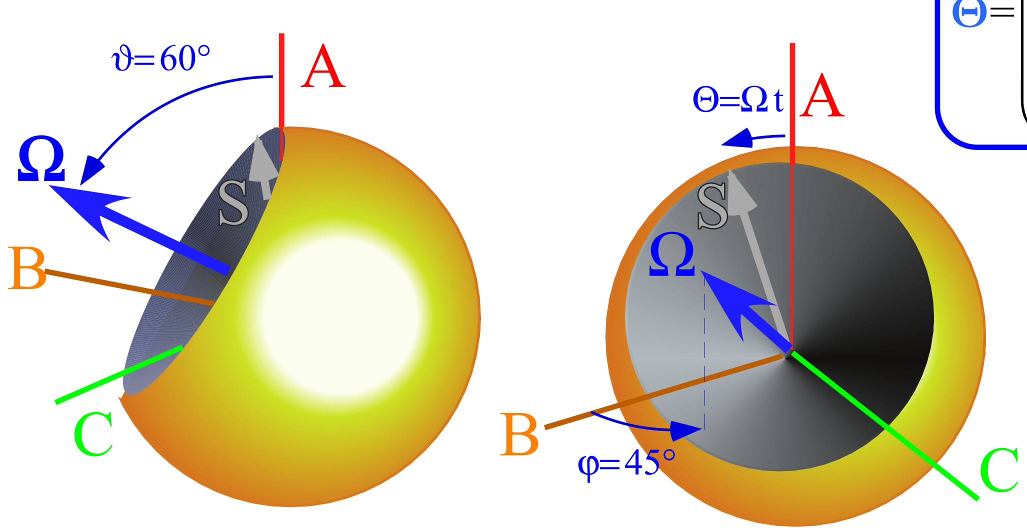
The driving $\Theta=\Omega t$ crank vector defined by *ABCD* of Hamiltonian H .

Notation for 3D Vector space

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$



Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ whirling Stokes state vector S in ABC-space.

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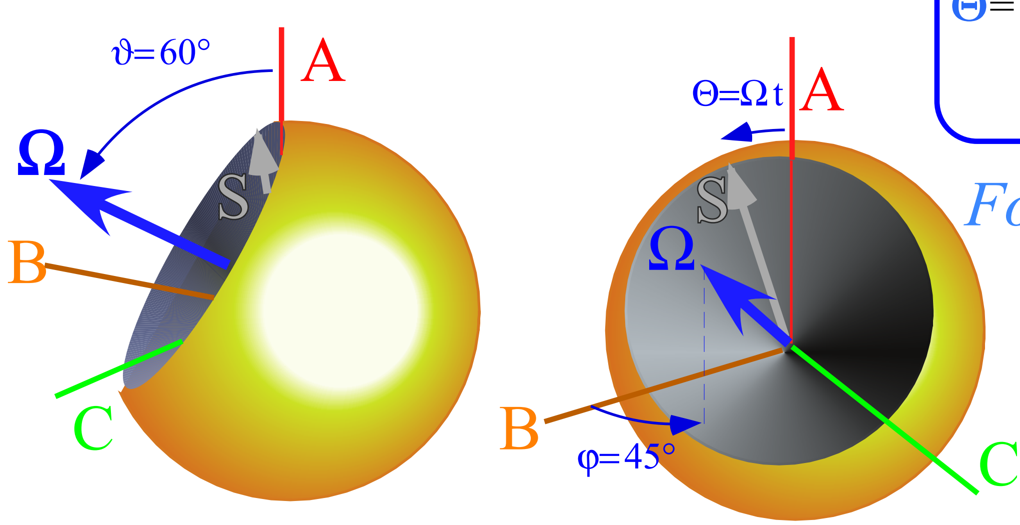
For fermion spin that $\mathbf{\Omega}$ *is the* g**B**-*field!*



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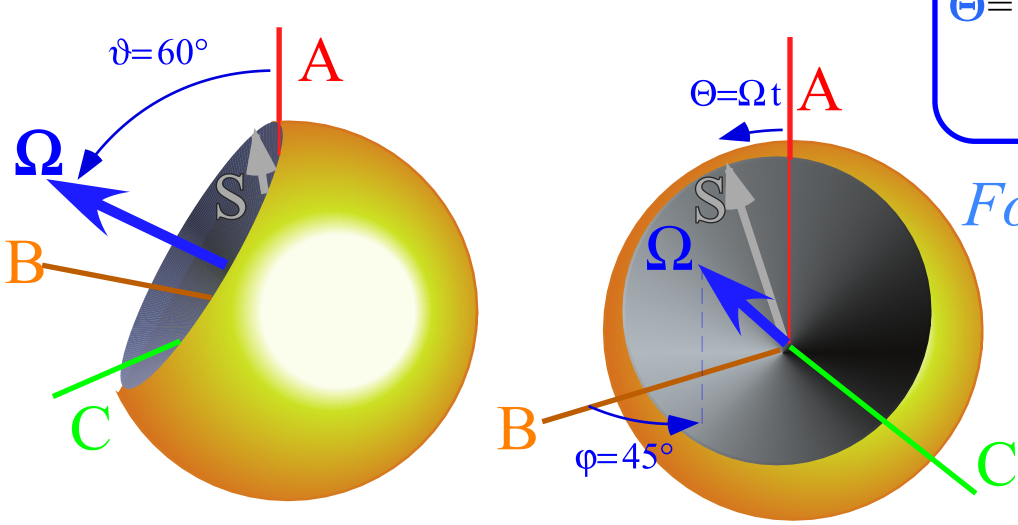
For fermion spin that $\mathbf{\Omega}$ *is the g***B***-field!*

*Q: But, how is a spin state-$|\psi\rangle$ or spin vector-***S** *defined?*



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$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X - igB_Y \\ gB_X + igB_Y & -gB_Z \end{pmatrix} = gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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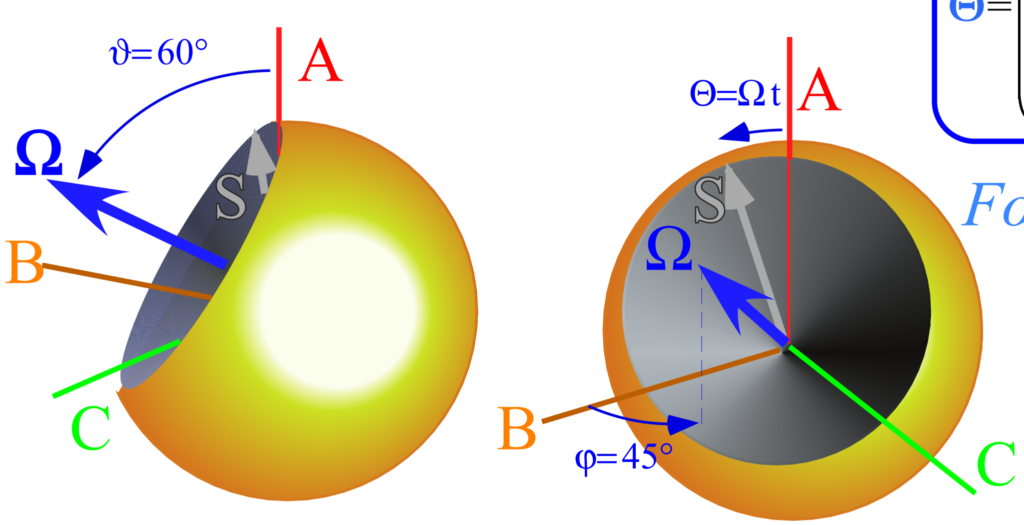

For fermion spin that $\mathbf{\Omega}$ *is the g***B***-field!*

Q: But, how is a spin state-$|\psi\rangle$ *or spin vector-***S** *defined?*

A: By U(2) group operator $|\psi(t)\rangle = \mathbf{R}[\boldsymbol{\Theta}]|\psi(0)\rangle$.

Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* ***S*** *in ABC-space.*

Review: How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$*
- *Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$*
  - *Spinor arithmetic like complex arithmetic*
  - *Spinor vector algebra like complex vector algebra*
  - *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*
- *Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$*
  - *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
    - *2D Spinor vs 3D vector rotation*
    - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*
- *Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$*
  - *Spin-1 (3D-real vector) case*
  - *Spin-1/2 (2D-complex spinor) case*
- *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*
  - *Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$*
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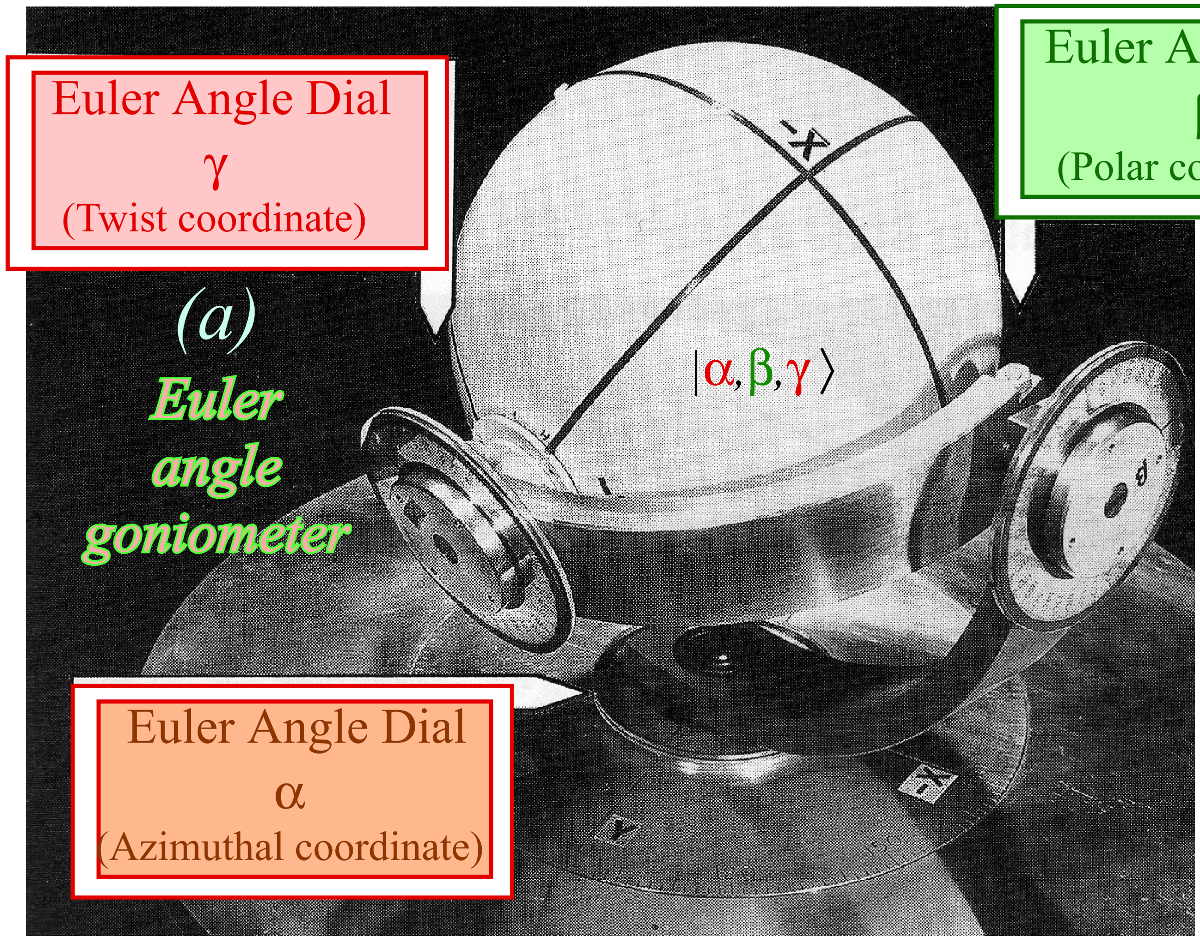
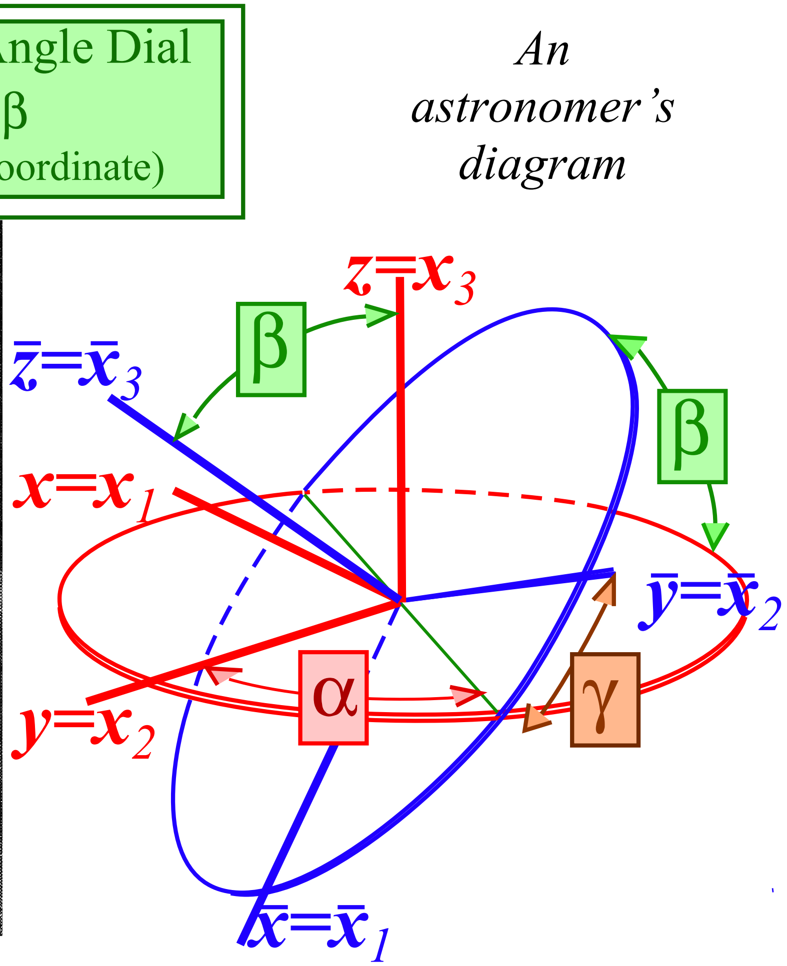
Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

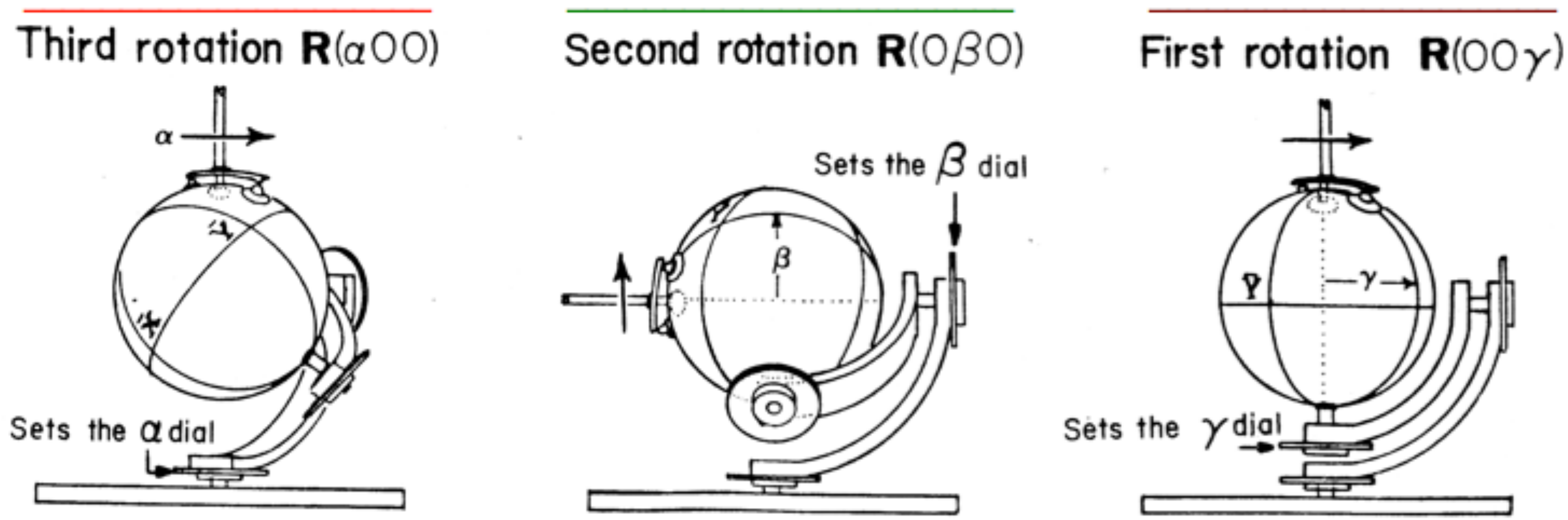
Spin-1 (3D-real vector) case





Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

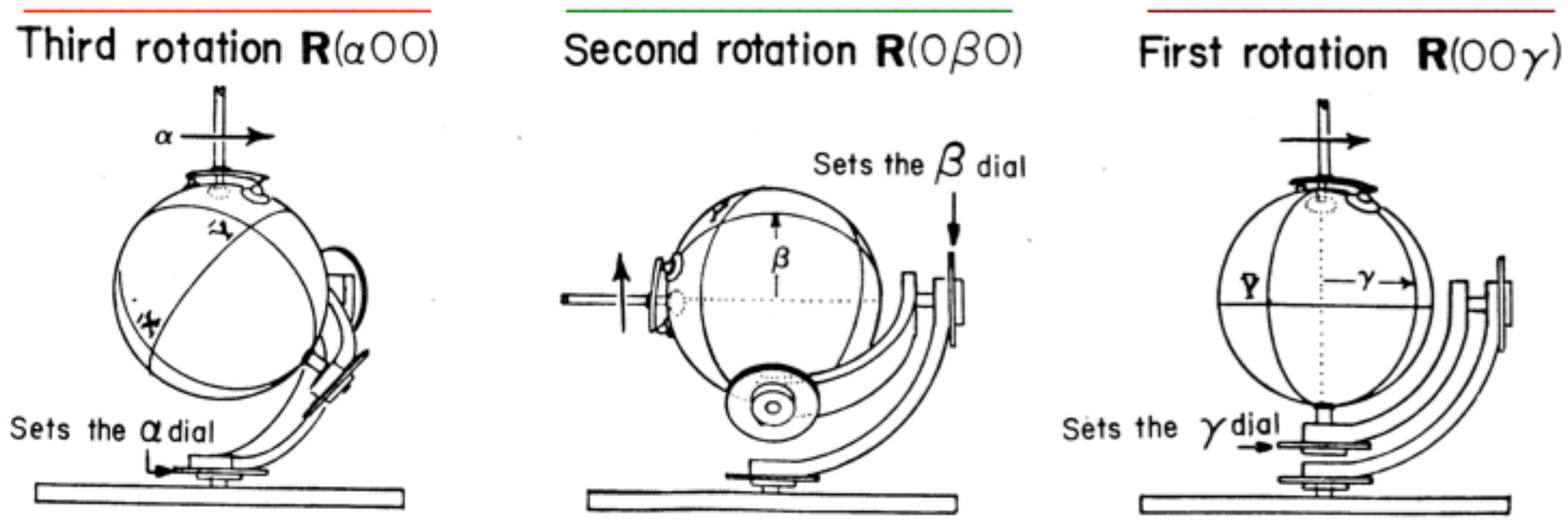
Spin-1 (3D-real vector) case



$$\left\langle R(\alpha\beta\gamma)\right\rangle = \left\langle R(\alpha 0 0)\right\rangle \qquad \left\langle R(0\beta 0)\right\rangle \qquad \left\langle R(0 0\gamma)\right\rangle$$

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case



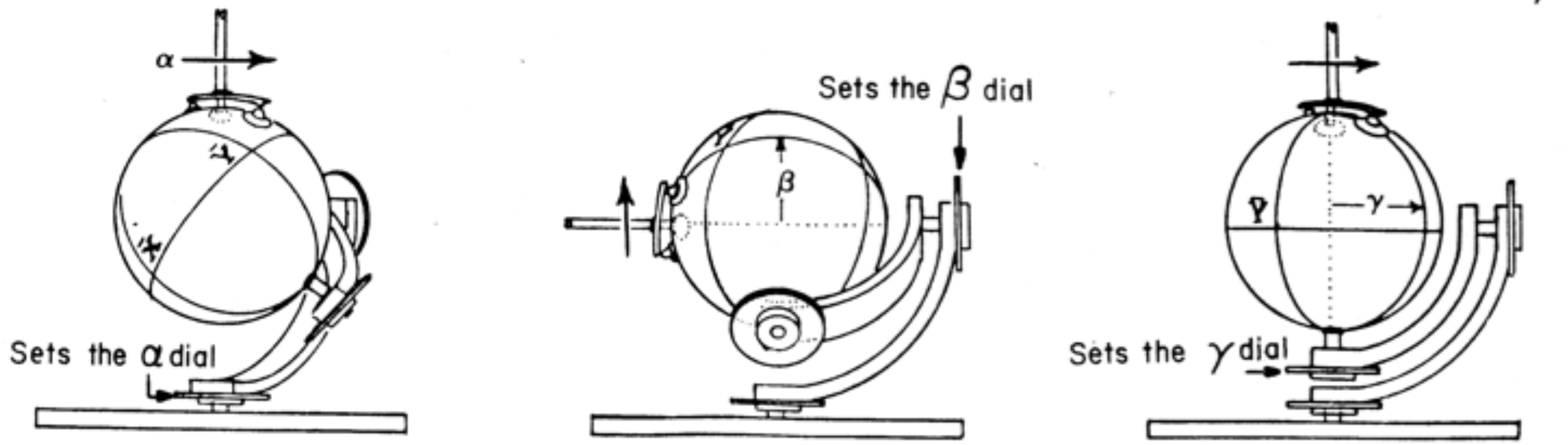
$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \qquad \langle R(0\beta 0)\rangle \qquad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

Third rotation **R**$(\alpha 00)$ | Second rotation **R**$(0\beta 0)$ | First rotation **R**(00γ)



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \langle R(0\beta 0)\rangle \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

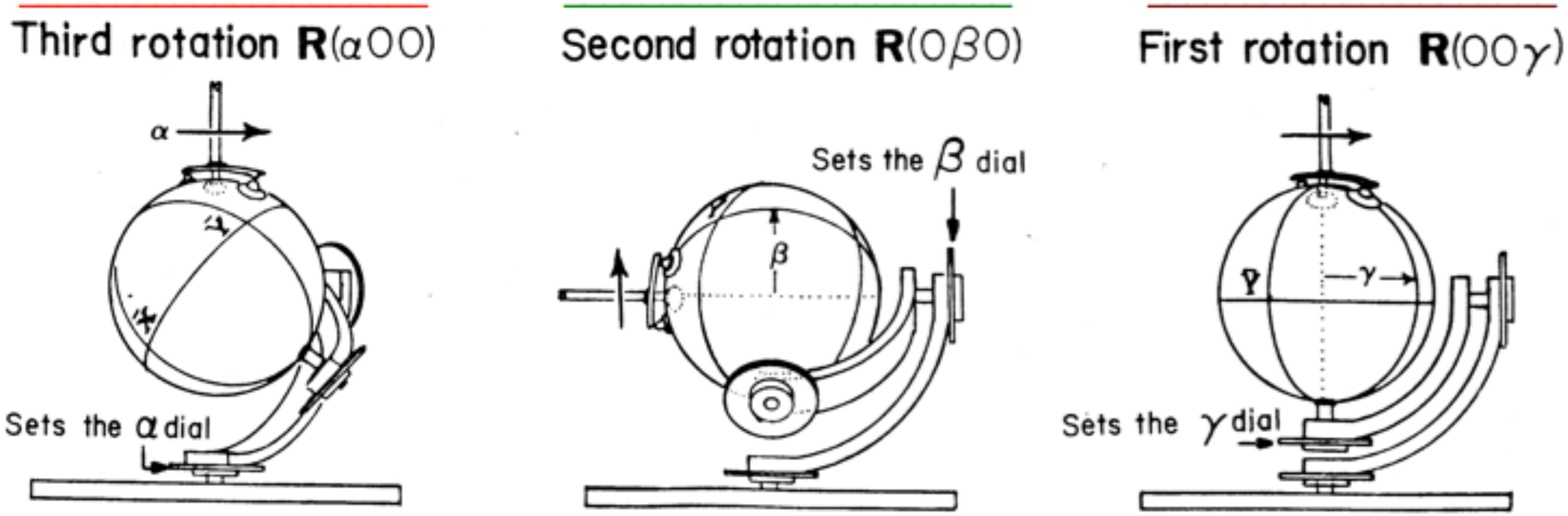
$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X| \\ \langle \mathbf{e}_Y| \\ \langle \mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \boxed{\cos\alpha\sin\beta} \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \boxed{\sin\alpha\sin\beta} \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \boxed{\cos\beta} \end{pmatrix}$$

Note lab frame polar coordinates

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 0 0)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

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Note lab-frame polar coordinates of Z(body)

...and body-frame polar coordinates

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

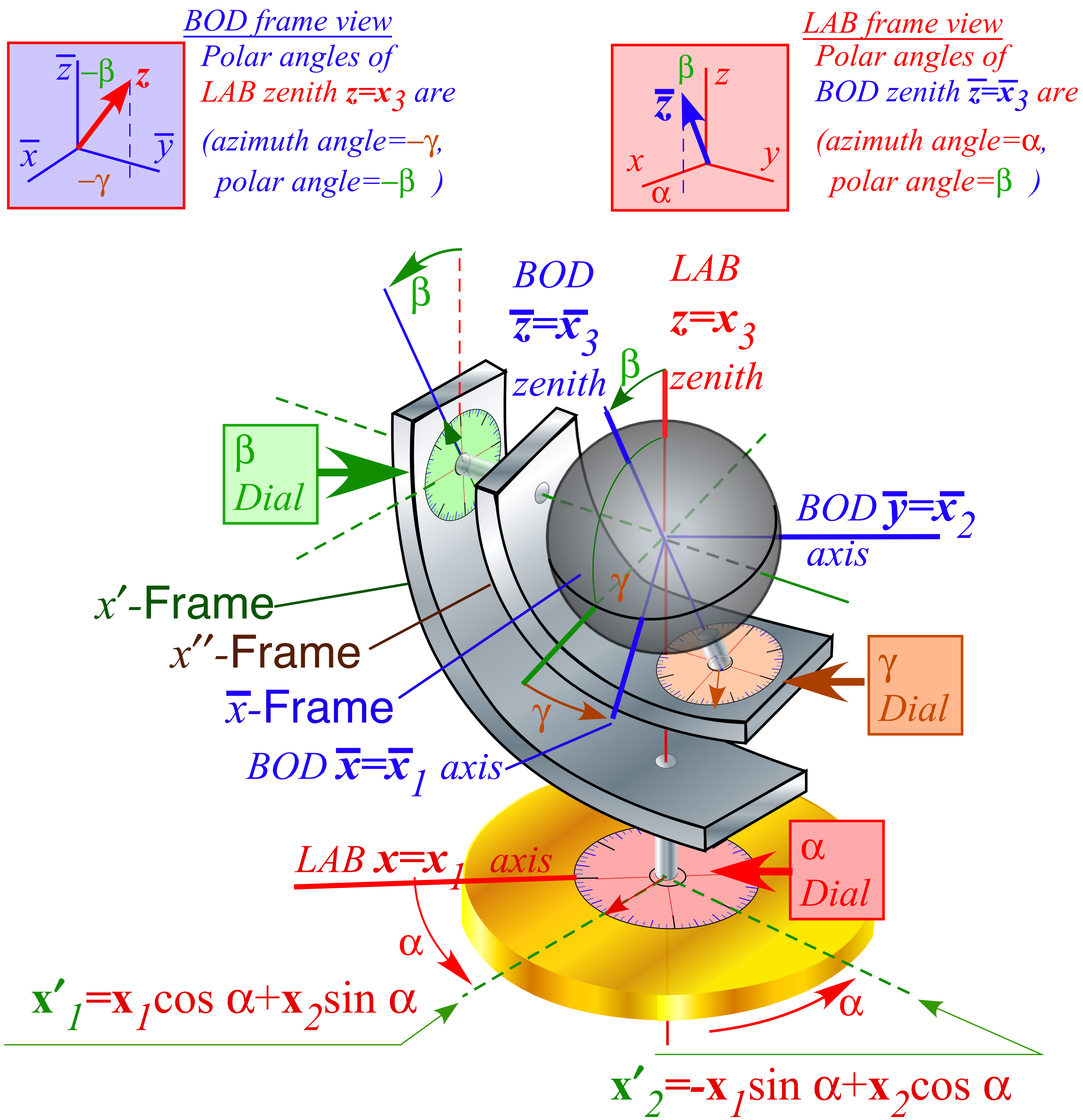

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

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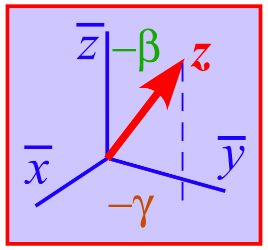
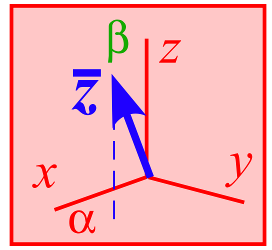
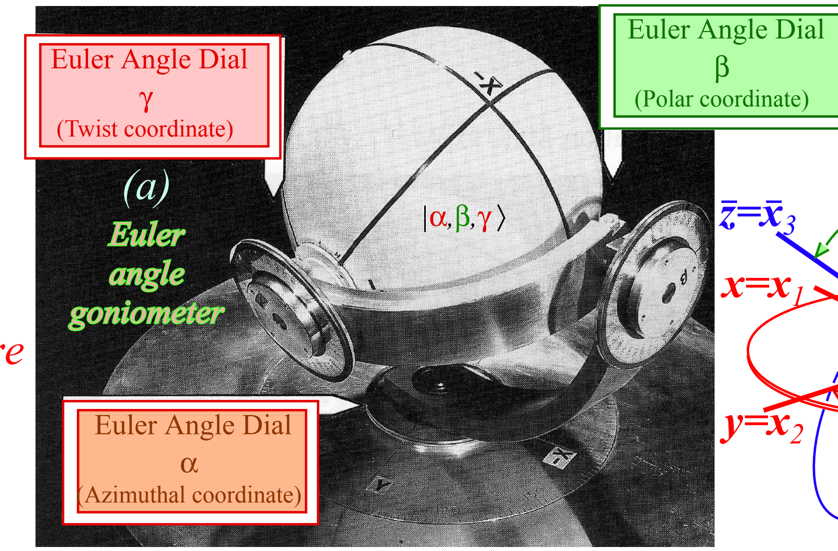


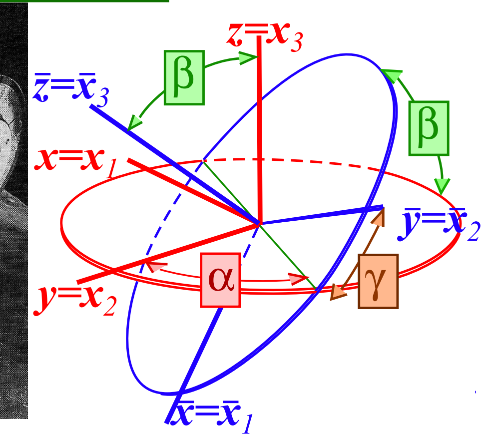
BOD frame view
Polar angles of LAB zenith $z=x_3$ *are*
(azimuth angle=−γ, polar angle=−β)

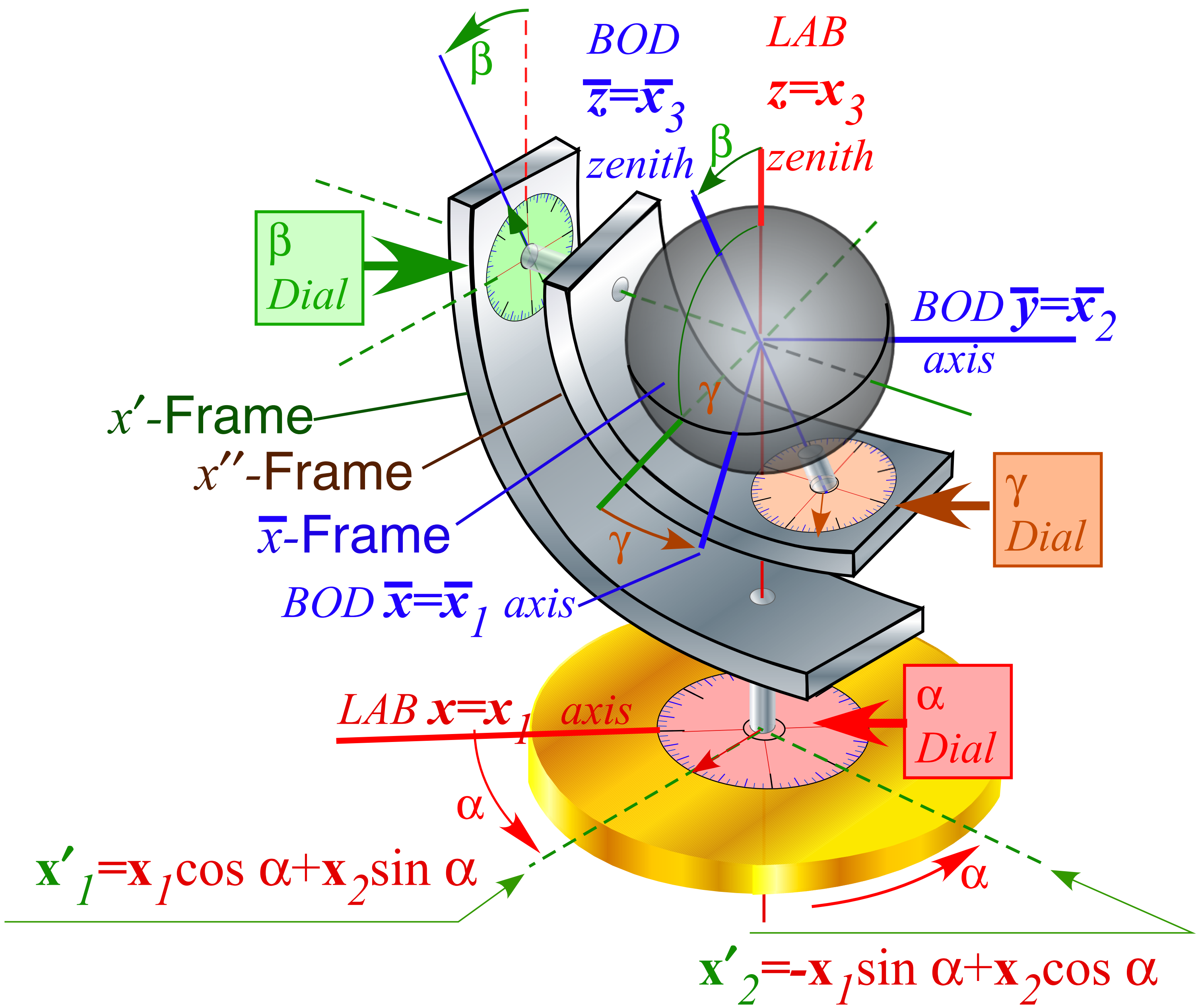


LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are*
(azimuth angle=α, polar angle=β)



An astronomer's diagram





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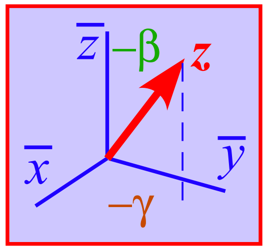
Spin-1 (3D-real vector) case



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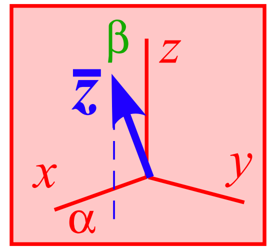
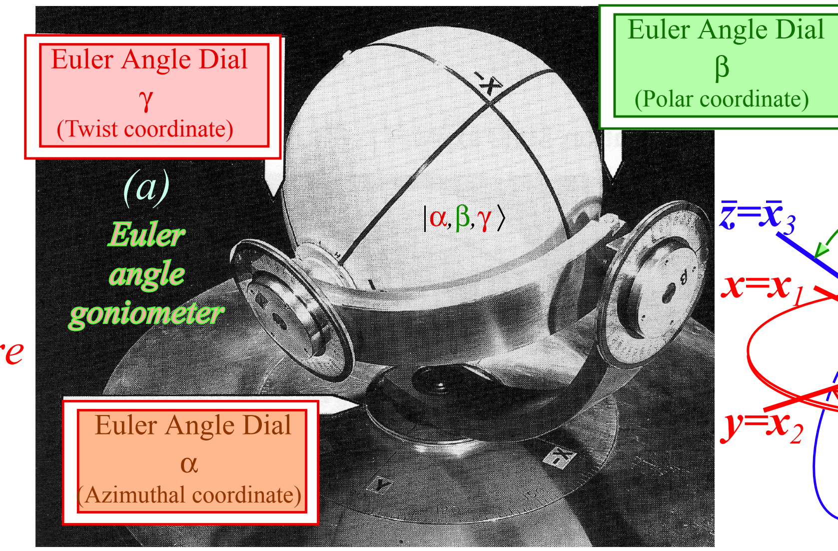
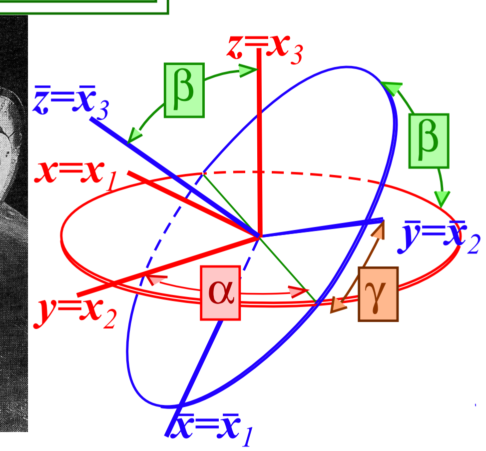


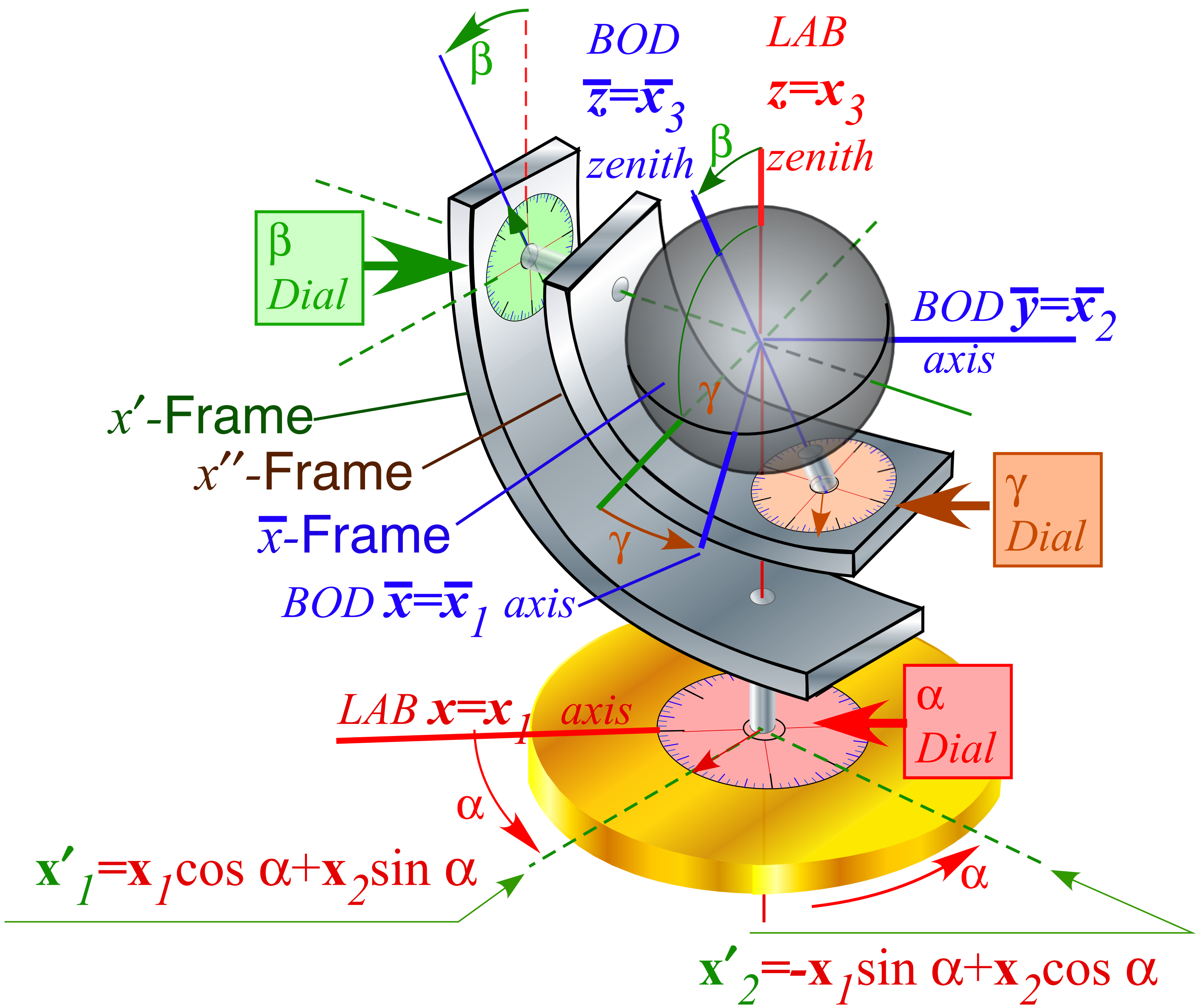
LAB frame view

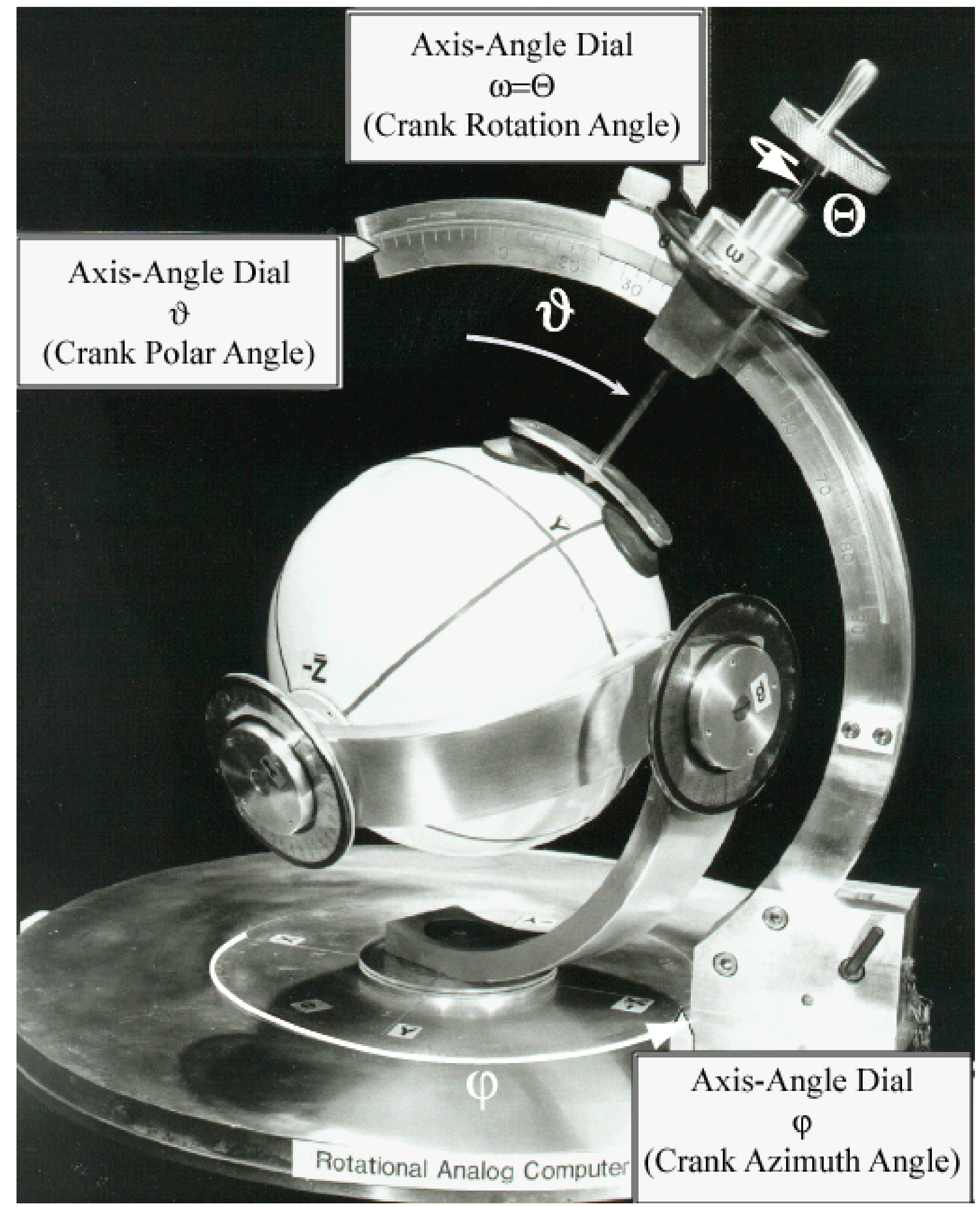
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are*

(azimuth angle=α, polar angle=β)









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Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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- *Spinor vector algebra like complex vector algebra*
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The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment $\mathbf{m}$ *in* $\mathbf{B}$ *field*

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Spin-1 (3D-real vector) case

➡ *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

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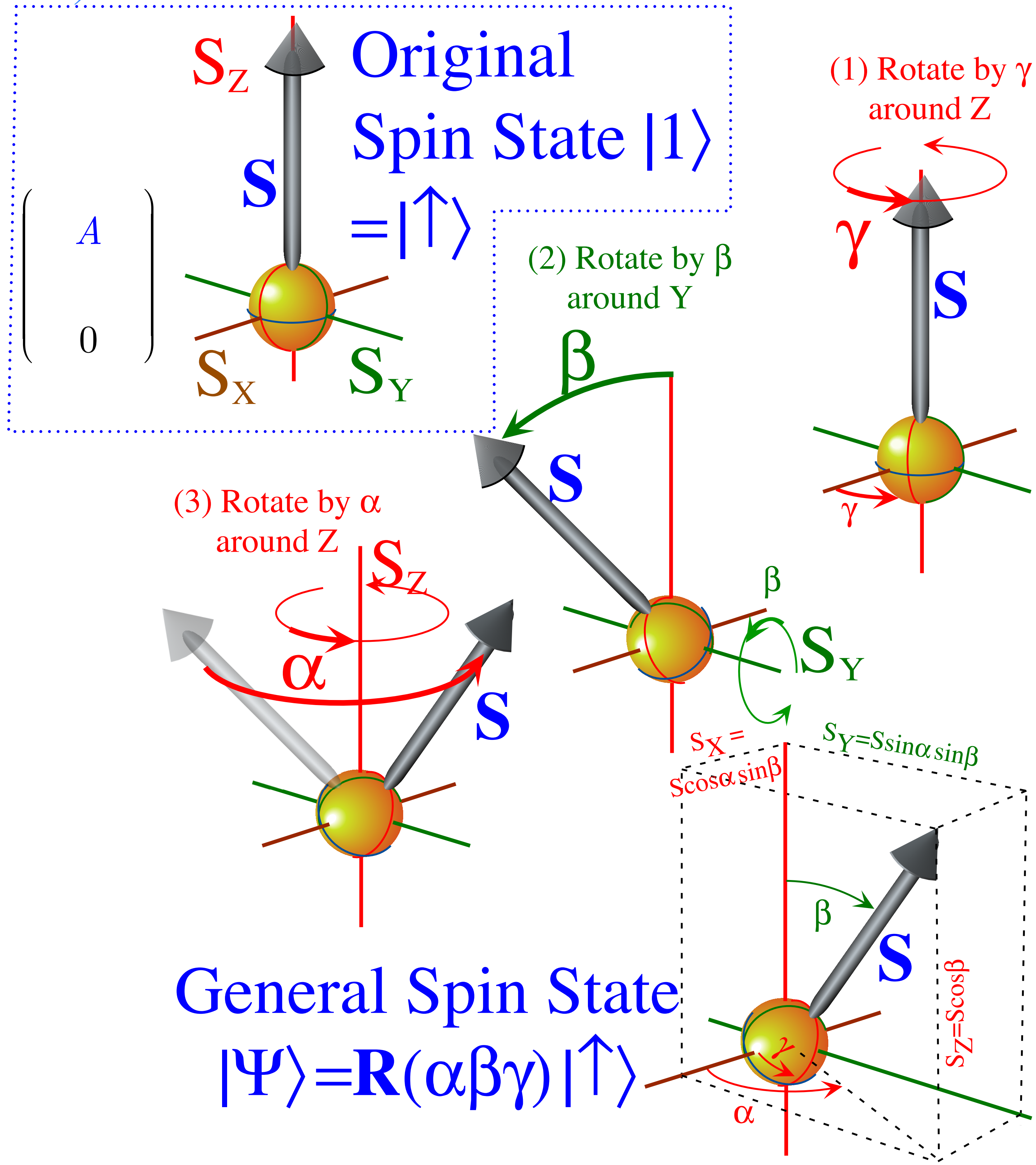
Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$



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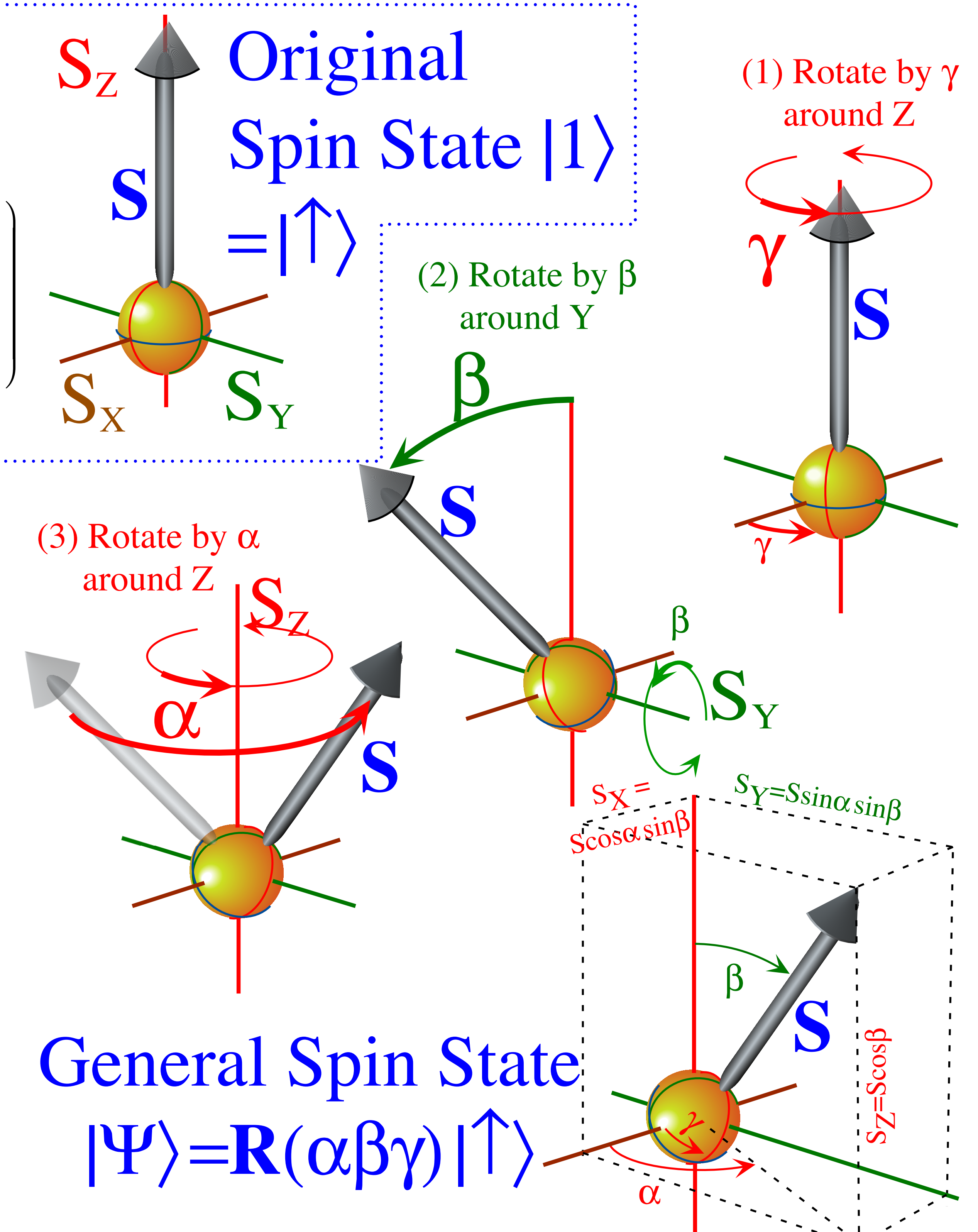
$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



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Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
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➡ *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*

➡ *Asymmetry* $S_A = S_Z$, *Balance* $S_B = S_X$, *and Chirality* $S_C = S_Y$

Polarization ellipse and spinor state dynamics

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\textit{Asymmetry}\ S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$$

$$\textit{Balance}\ S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$$

$$\textit{Chirality}\ S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

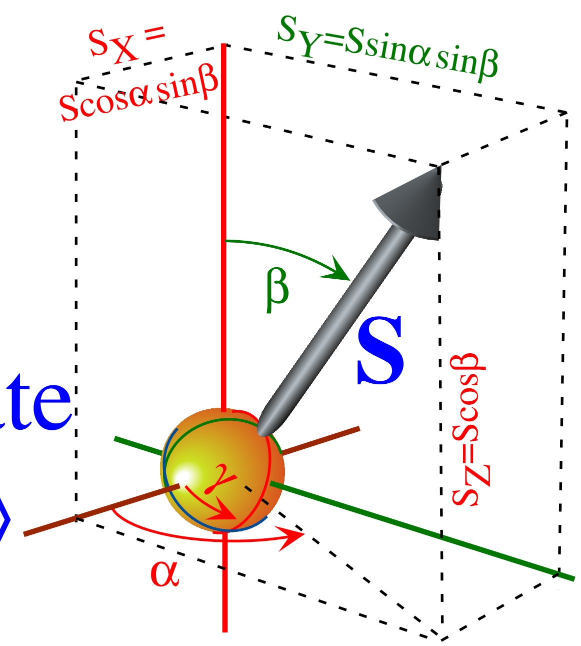
Asymmetry $S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\uparrow\rangle$



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Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
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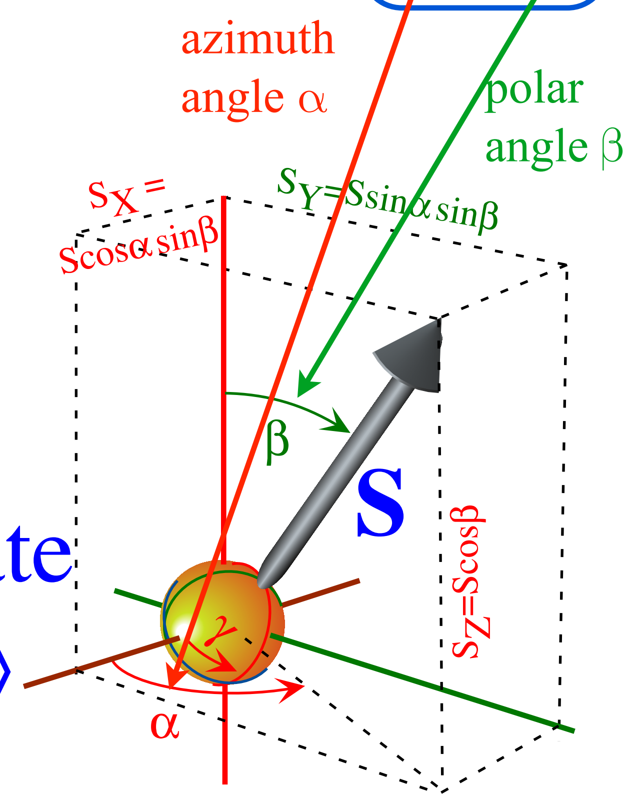
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

Asymmetry
$$S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$$

Balance
$$S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

Chirality
$$S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot - \sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

General Spin State $|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

Asymmetry $$S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$$

Balance $$S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

Chirality $$S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2} \cdot - \sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

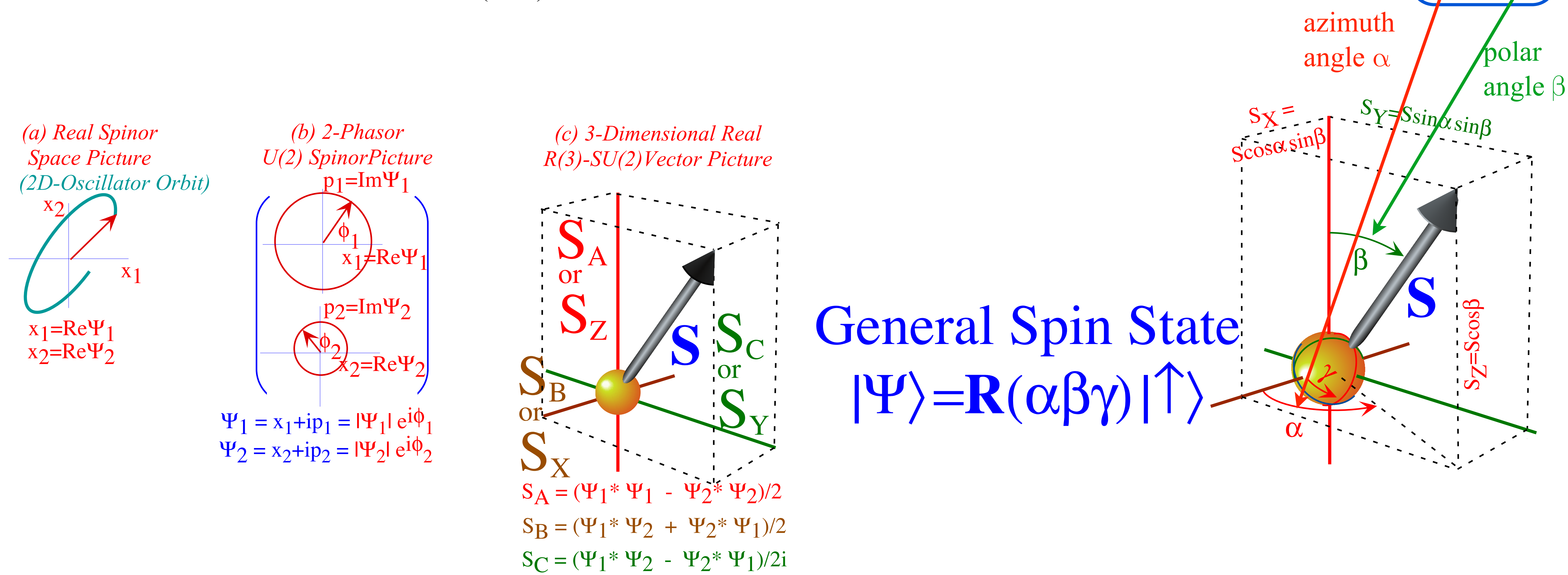


Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

Review:How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

- *Hamilton-Pauli spinor symmetry (* $\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$
- *Deriving* $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *Spinor arithmetic like complex arithmetic*
  - *Spinor vector algebra like complex vector algebra*
  - *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*
- *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
  - *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
    - *2D Spinor vs 3D vector rotation*
    - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*
- *Euler's state definition using rotations* $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$
  - *Spin-1 (3D-real vector) case*
  - *Spin-1/2 (2D-complex spinor) case*
- → *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*
  - *Asymmetry* $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$
  - → *Polarization ellipse and spinor state dynamics*

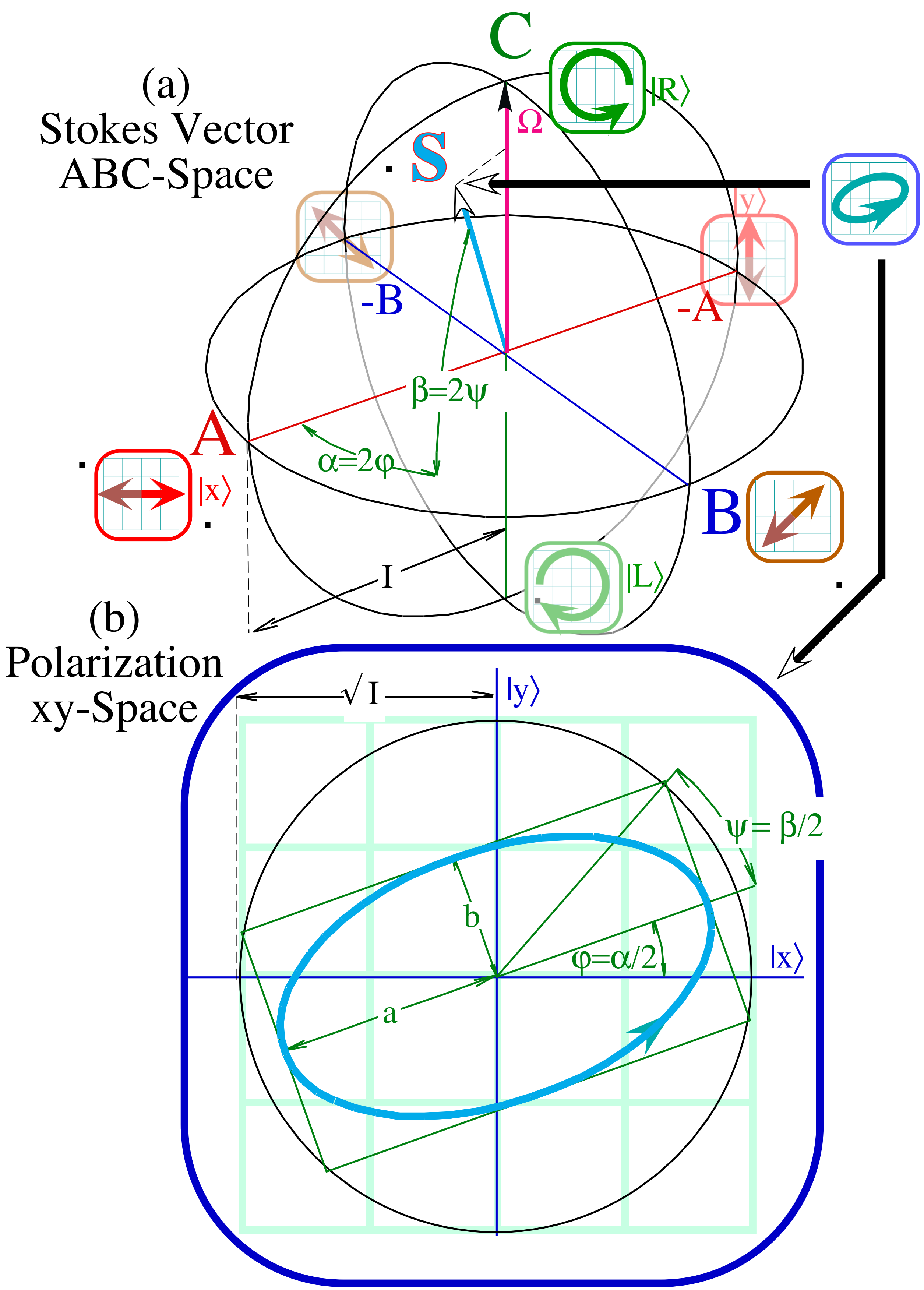


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics

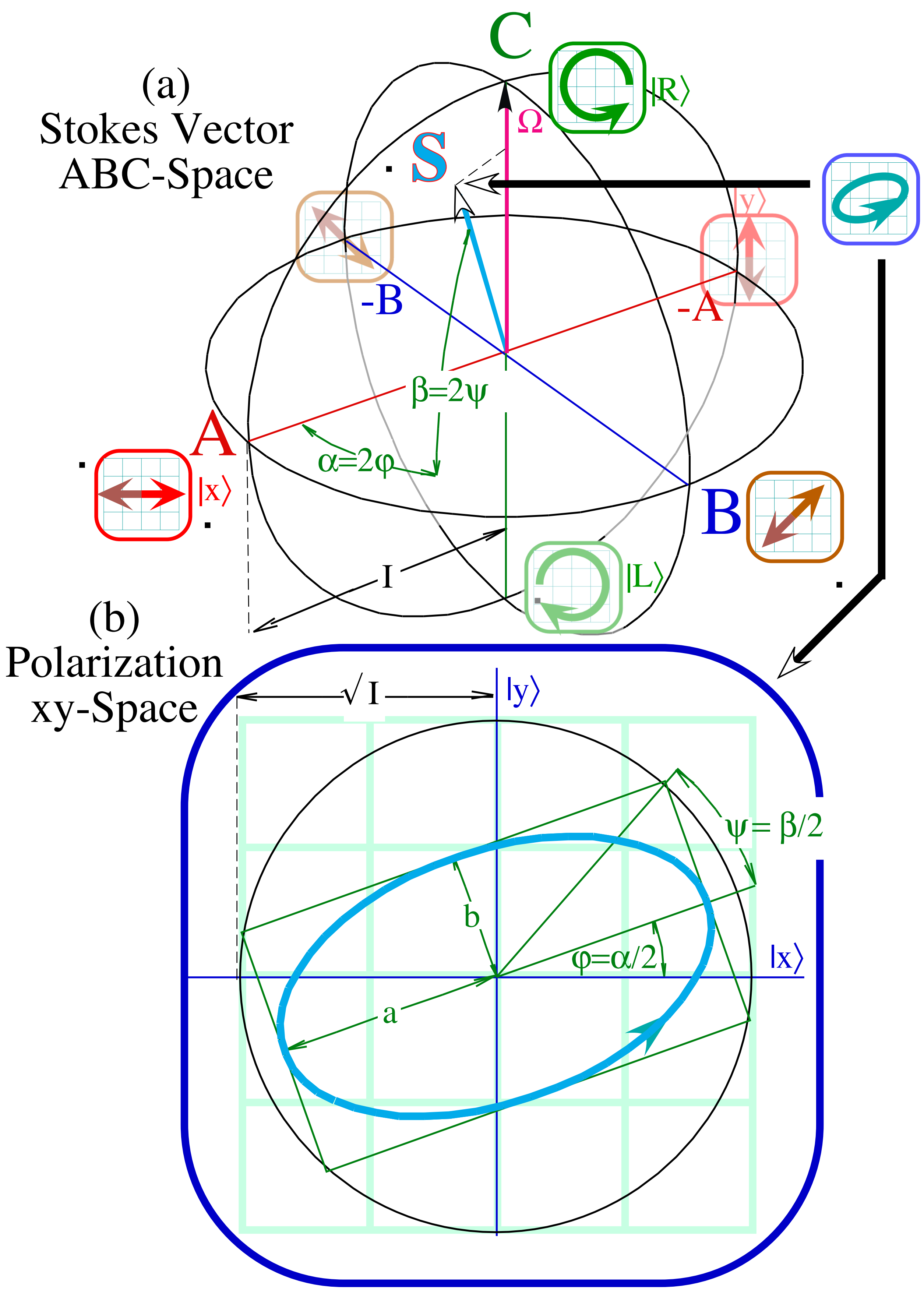


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

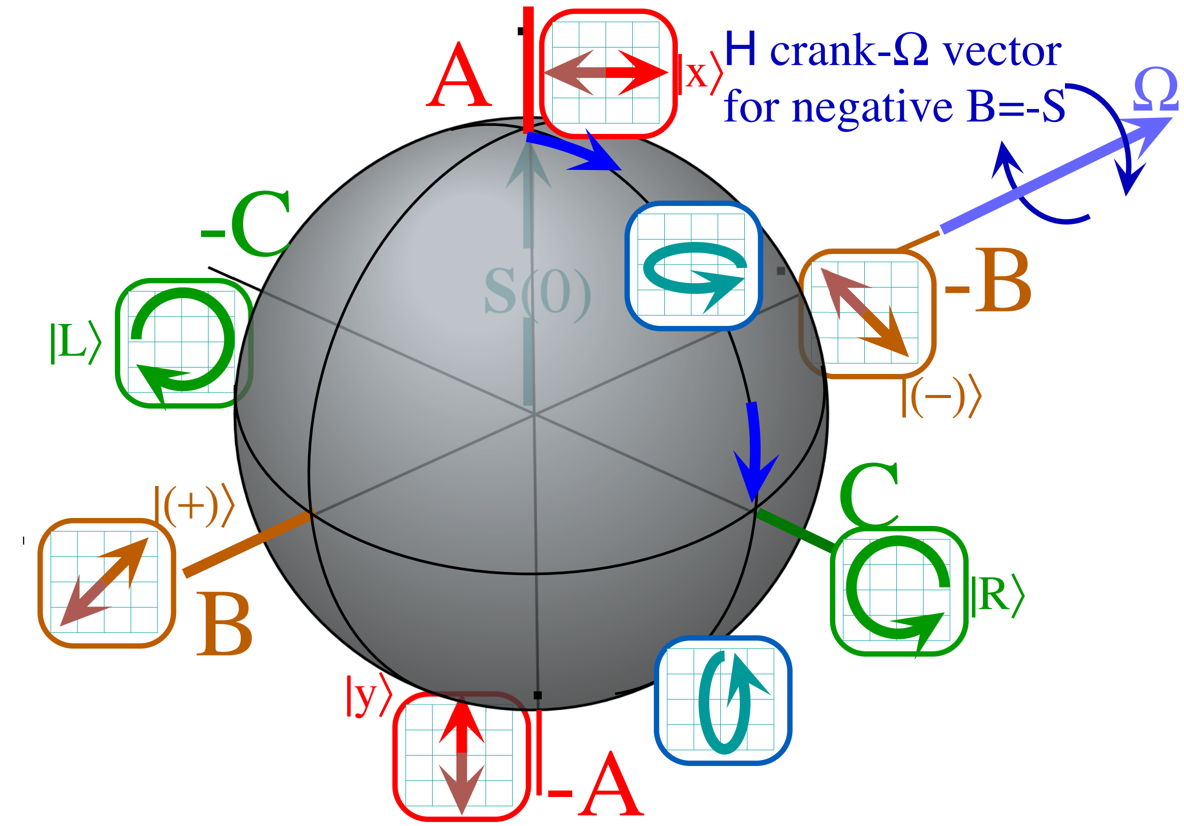


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Polarization ellipse and spinor state dynamics

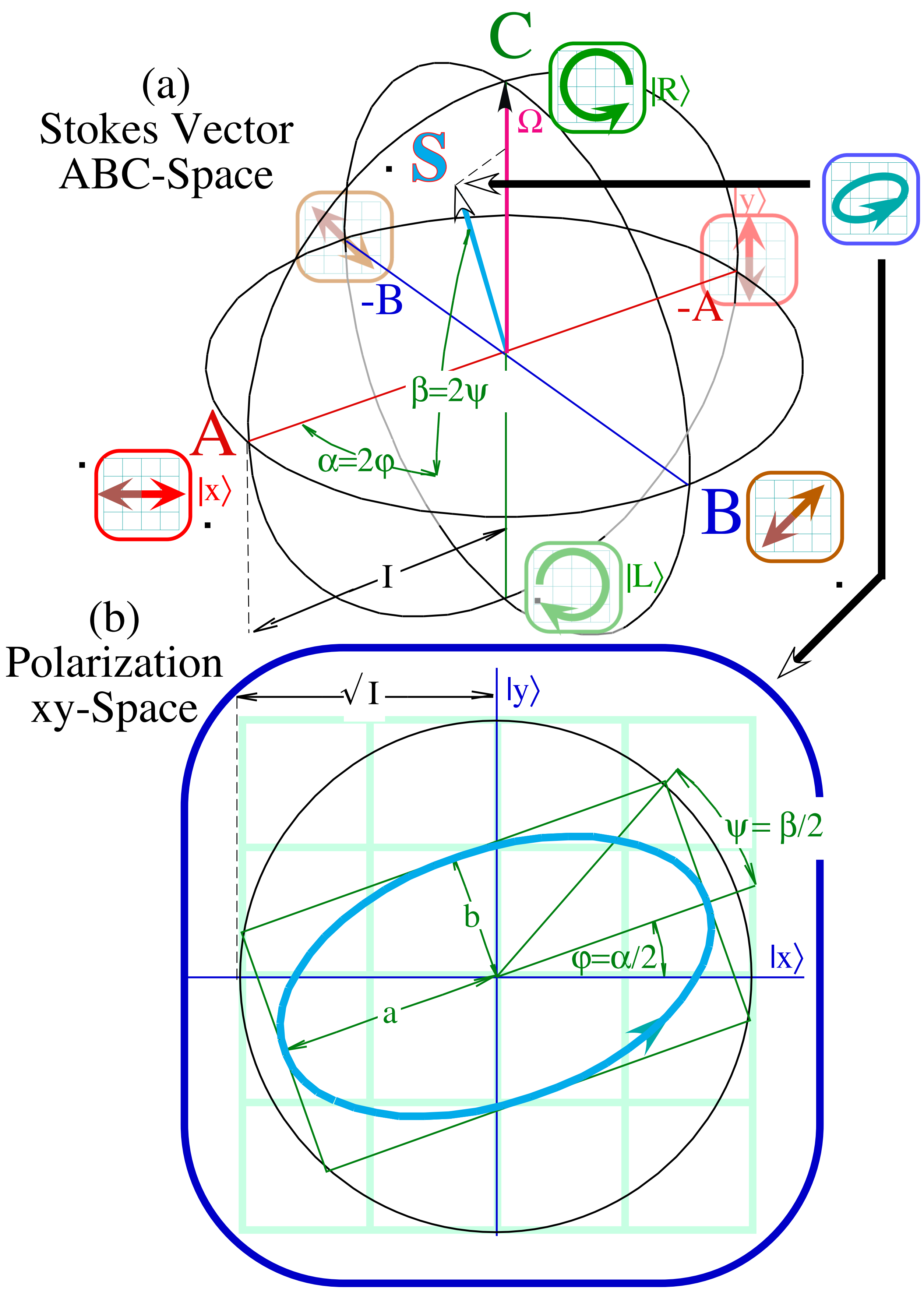


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

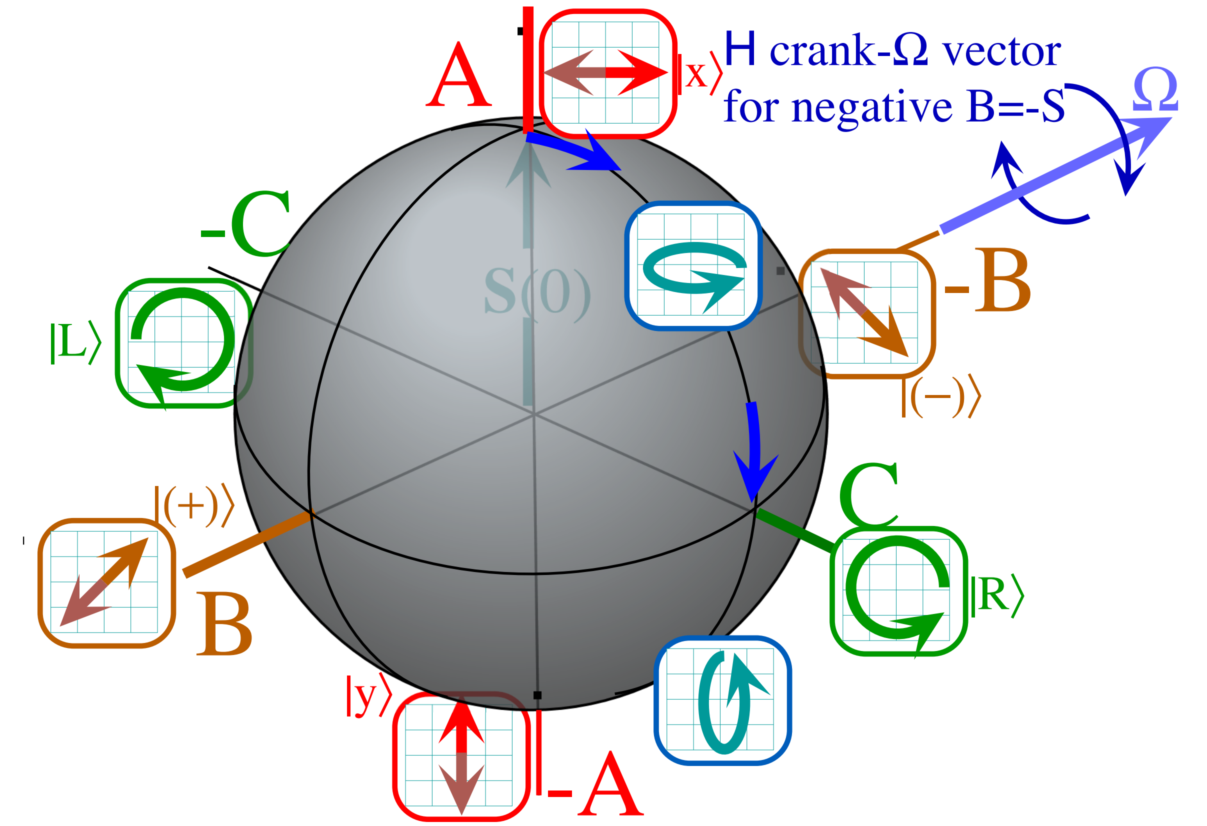
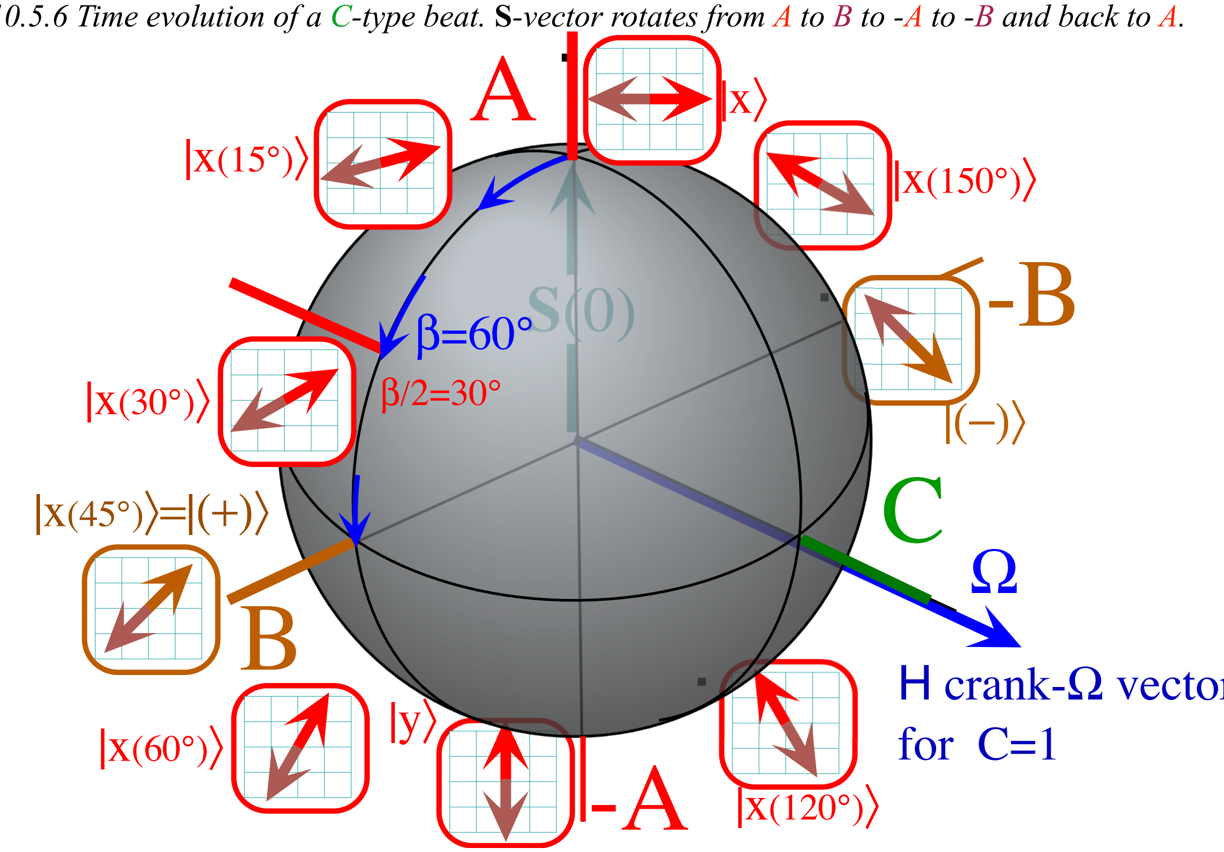


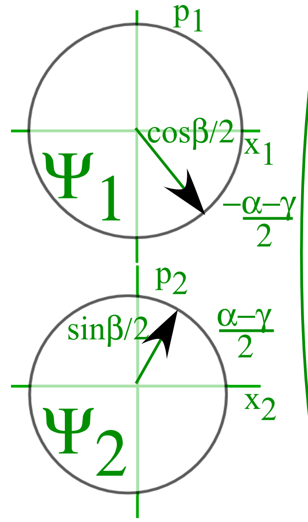
Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

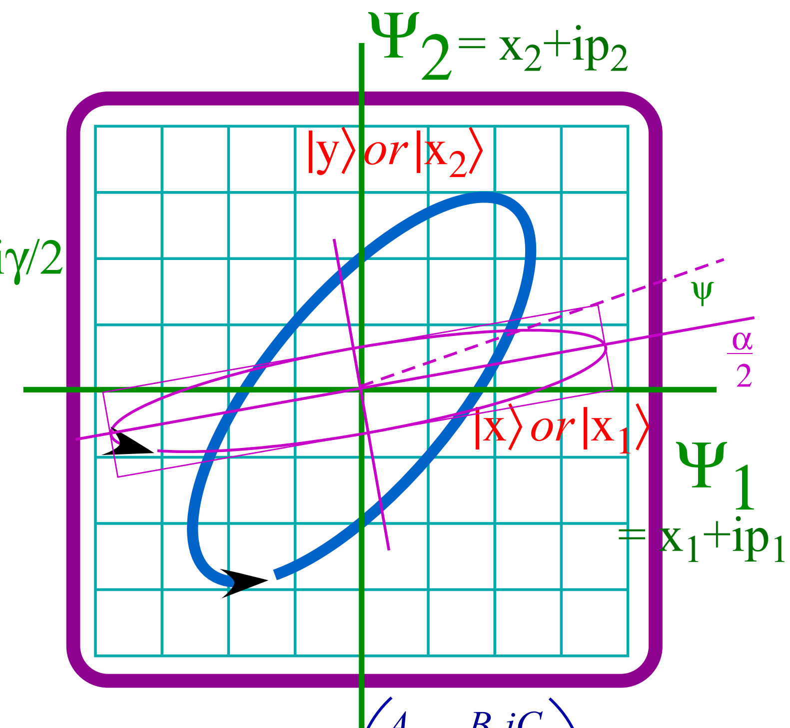


U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle$=



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2 \\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector
S

$$\begin{pmatrix}s_B\\ s_C\\ s_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\mathbf{\Omega} = \begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$