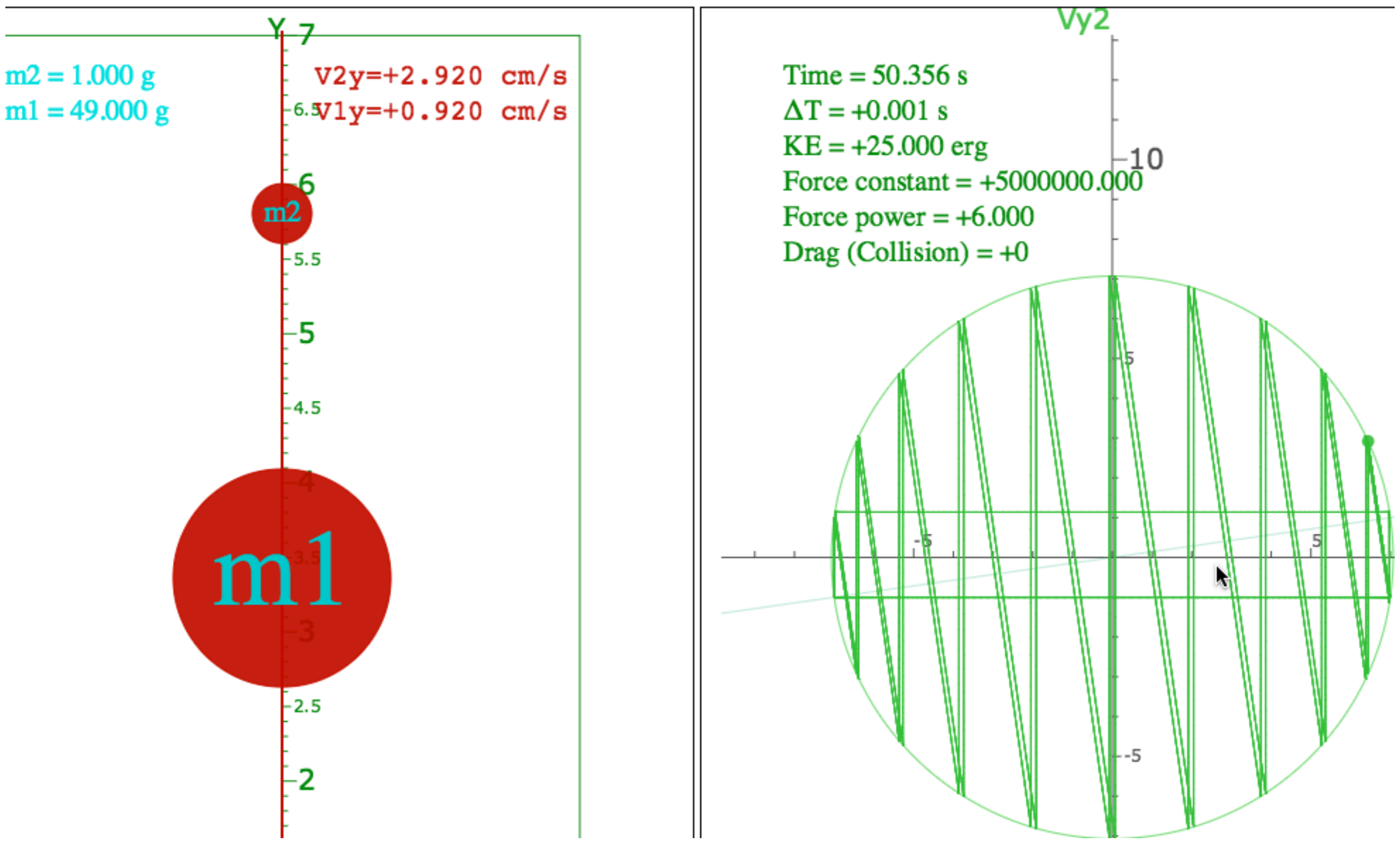
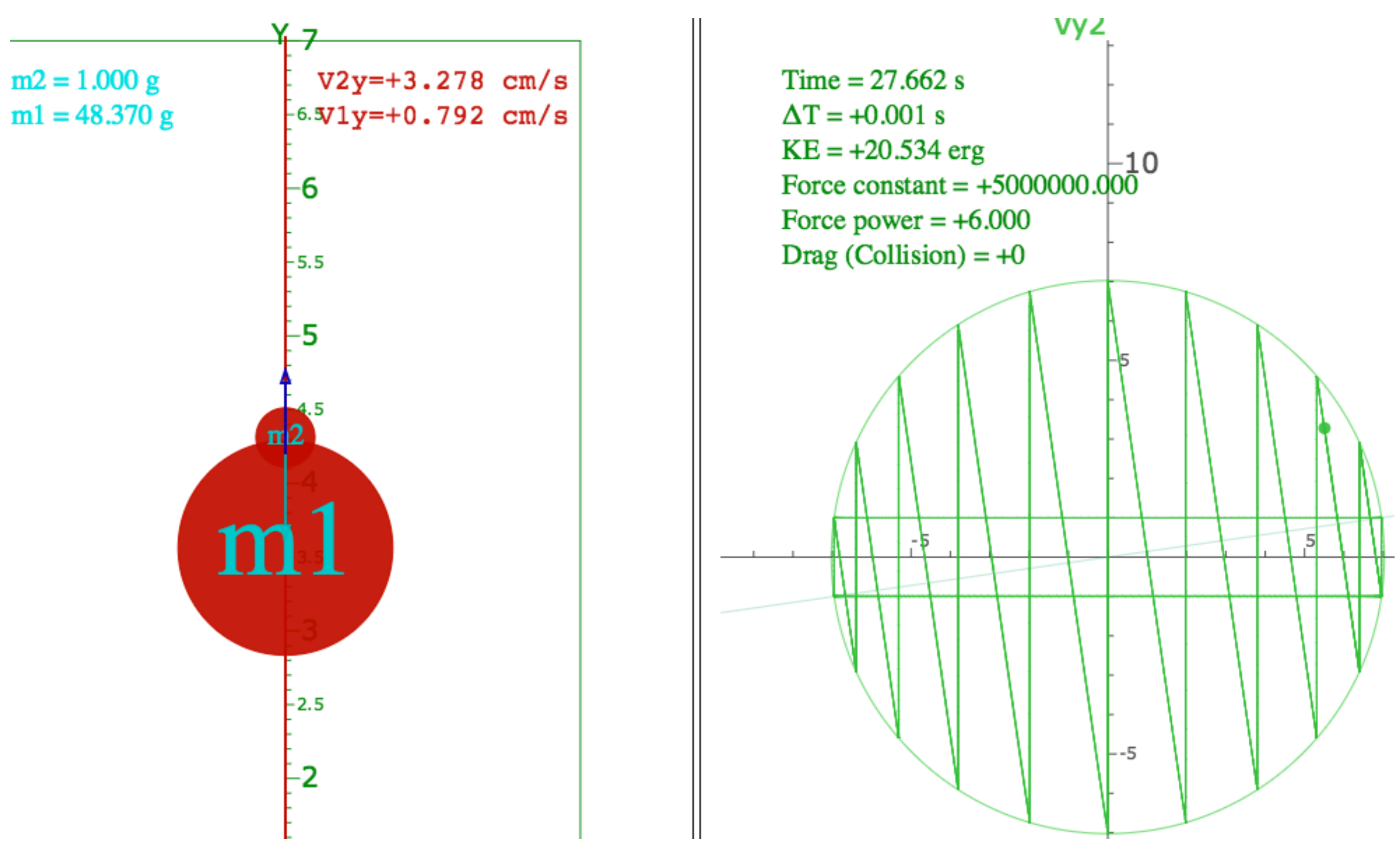
Exercise 4.1 Estrangian plot in text Fig. 4.2 (Details on p.64 of *Lect. 3to4x*. See p.131, too.) has mass ratio $M_1/m_2 = 49/1$ and has nearly (but not quite) periodic path plot. (Let the pen-mass be m_2=1 here.)



Derive a closed formula for value of M_1= 48.37...(to at least 7 figures) having *exactly* periodic behavior.

Simplest formula should relate to tangent of a desired Estrangian rotation half-angle $\theta/2$ for mass M_1 .

Changing to M_1= 48.37 gives more nearly periodic paths shown below. (Seems perfect but it's not.)
(Experiment using BounceIt on web. [lhttp://www.uark.edu/ua/modphys/markup/BounceItWeb.php](http://www.uark.edu/ua/modphys/markup/BounceItWeb.php))



Solution to *Pseudo-Rotations*

Solving the mass ratio equation (5.10b) for m_1/m_2 in terms of angle θ:

$$\cos\theta \equiv \left(\frac{m_1 - m_2}{M}\right) \quad and: \ \sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{M}\right)$$

This will simplify to:

$$\cos\theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} \quad and: \ \frac{m_1}{m_2} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2} \quad or: \ \cot\frac{\theta}{2} = \sqrt{\frac{m_1}{m_2}}$$

Given angle $\theta = \pi/11$ or $\theta/2 = \pi/22$ and m_2=1 predicts

$$m_1 = \tan^2\frac{\pi}{22} = 48.3741500787$$

Solution to *Pseudo-Rotations*

Pseudo-Vibrations

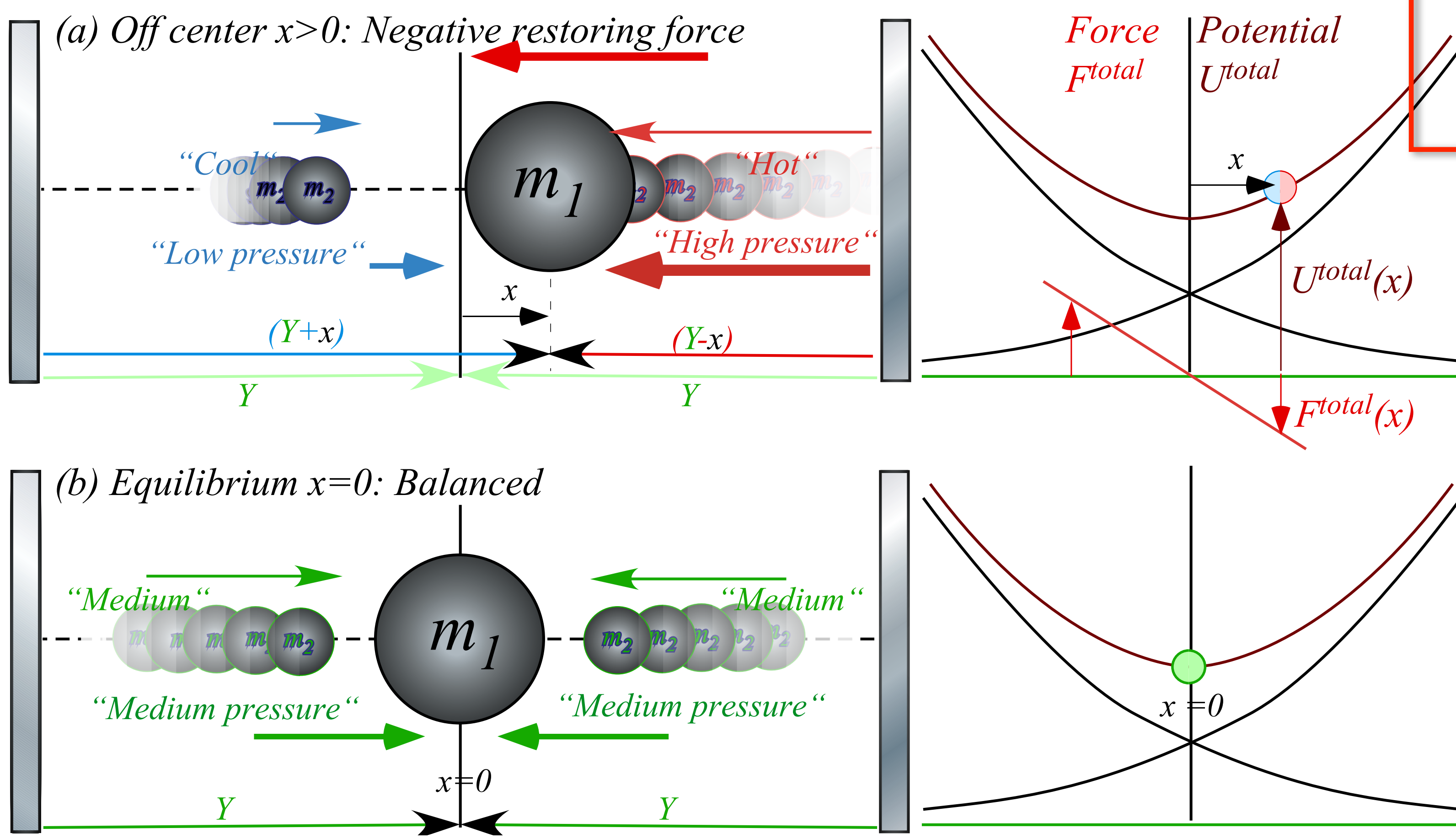
Exercise 4.2 On p.50-55 of *Lect. 5to6*, is shown pseudo-harmonic motion of the large mass $M=50kg$ attacked on either side by a pair of tiny masses $m=0.1kg$ each traveling back and forth in a range of $Y_0=3.5m$ at an average speed of $20m/sec$. The calculation seems to come up a period about $\sqrt{3}$ times too big for mass M. Explain what was overlooked and derive an improved formula for the period.

Example of oscillator with opposing Isothermal potentials

1D-Isothermal Force Law (assume v_2 is constant for all Y):

$$F = \frac{m_2 v_2^2}{Y} = \frac{const.}{Y}$$

$$F^{phys} = \frac{m_2 v_2^2}{Y} = -\frac{\Delta U}{\Delta Y} \quad \textit{implies :} \quad U = -m_2 v_2^2 \ln(Y)$$



$$F = \frac{m_2 \left(\bar{v}_2^{IN} Y_0\right)^2}{Y^3} = \frac{const.}{Y^3}$$

1D-Adiabatic Force Law

Fig. 5.3
Unit 1

Two opposing 1D-Adiabatic Force Law fields

$$F^{total} = \frac{f}{(Y_0+x)^3} - \frac{f}{(Y_0-x)^3} = f\left[\frac{1}{Y_0^3} - \frac{3x}{Y_0^4} + \frac{6x^2}{Y_0^5} - \frac{10x^3}{Y_0^6} + \cdots\right] - f\left[\frac{1}{Y_0^3} + \frac{3x}{Y_0^4} + \frac{6x^2}{Y_0^5} + \frac{10x^3}{Y_0^6} + \cdots\right]$$

$$(Y_0+x)^{-3} = Y_0^{-3} - 3xY_0^{-4} + \frac{3\cdot 4}{2}x^2 Y_0^{-5} - \frac{3\cdot 4\cdot 5}{2\cdot 3}x^3 Y_0^{-6}\ldots \qquad (Y_0-x)^{-3} = Y_0^{-3} + 3xY_0^{-4} + \frac{3\cdot 4}{2}x^2 Y_0^{-5} + \frac{3\cdot 4\cdot 5}{2\cdot 3}x^3 Y_0^{-6}\ldots$$

Binomial Theorem

$$(Y_0+x)^n = Y_0^n + nY_0^{n-1}x + \frac{n(n-1)}{2}Y_0^{n-2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}Y_0^{n-3}x^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}Y_0^{n-4}x^4\ldots$$

Example of oscillator with opposing Isothermal potentials

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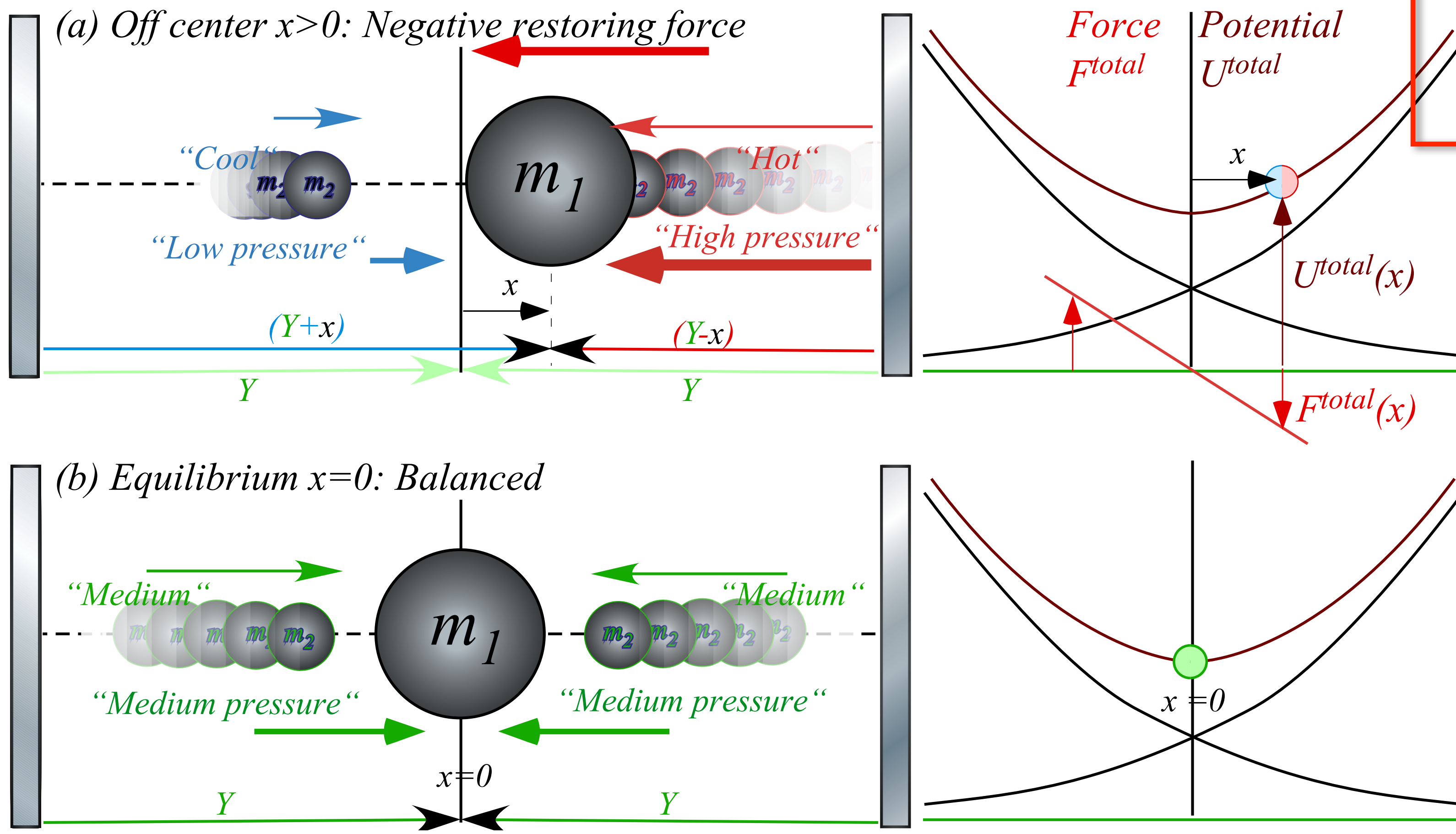

$$F = \frac{m_2 \left(\bar{v}_2^{IN} Y_0\right)^2}{Y^3} = \frac{const.}{Y^3}$$

1D-Adiabatic Force Law

Fig. 5.3
Unit 1

Two opposing 1D-Adiabatic Force Law fields

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$$(Y_0+x)^{-3} = Y_0^{-3} - 3xY_0^{-4} + \frac{3\cdot 4}{2}x^2 Y_0^{-5} - \frac{3\cdot 4\cdot 5}{2\cdot 3}x^3 Y_0^{-6} \ldots \qquad (Y_0-x)^{-3} = Y_0^{-3} + 3xY_0^{-4} + \frac{3\cdot 4}{2}x^2 Y_0^{-5} + \frac{3\cdot 4\cdot 5}{2\cdot 3}x^3 Y_0^{-6} \ldots$$

Binomial Theorem

$$(Y_0+x)^n = Y_0^n + nY_0^{n-1}x + \frac{n(n-1)}{2}Y_0^{n-2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}Y_0^{n-3}x^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}Y_0^{n-4}x^4 \ldots$$

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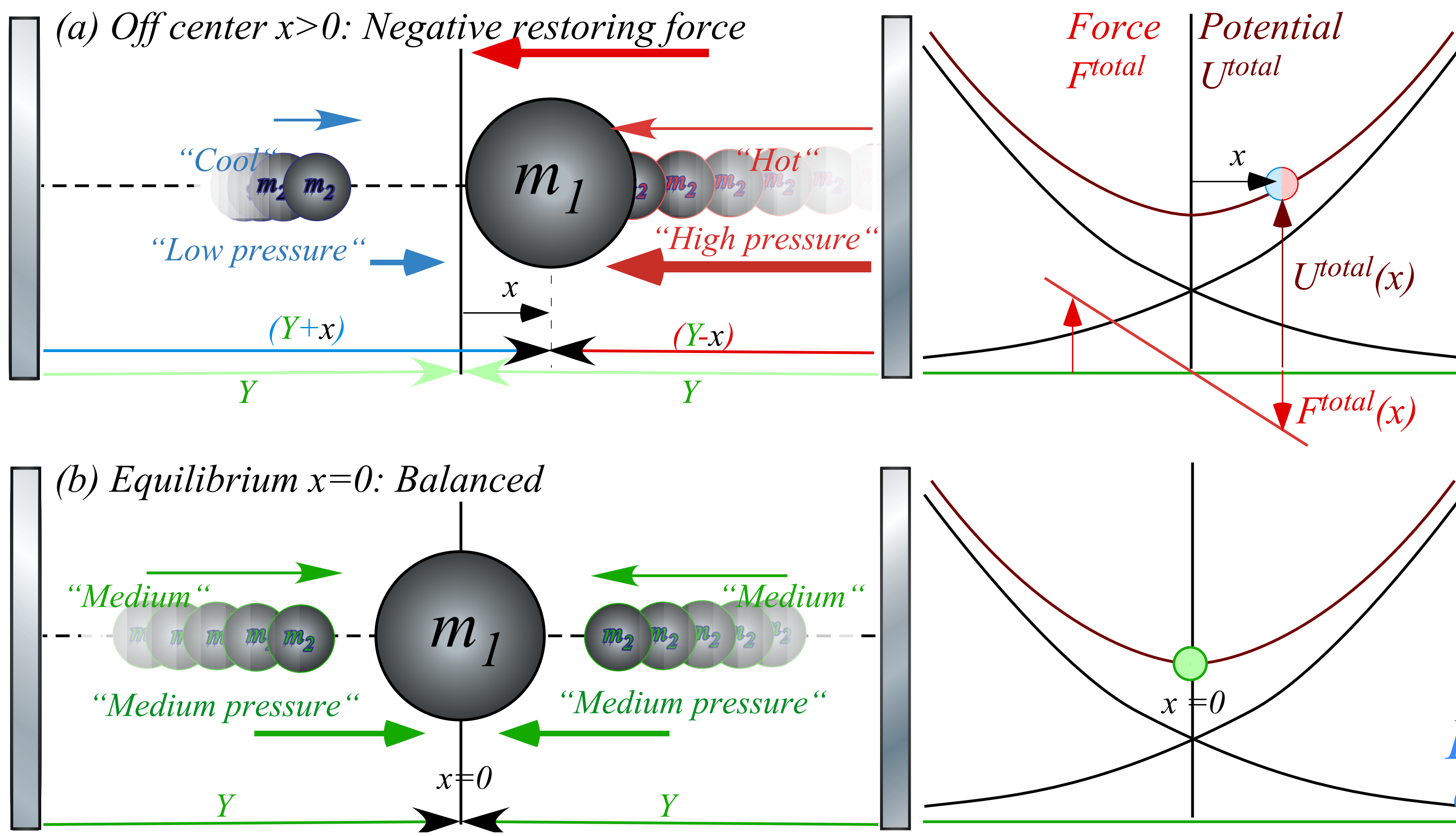

Fig. 5.3
Unit 1

Harmonic oscillator term

Anharmonic oscillator terms...

Two opposing 1D-Adiabatic Force Law fields

$$F^{total} = \frac{f}{(Y_0+x)^3} - \frac{f}{(Y_0-x)^3} = f\left[-\frac{3x}{Y_0^4} - \frac{10x^3}{Y_0^6}\cdots\right] + f\left[-\frac{3x}{Y_0^4} - \frac{10x^3}{Y_0^6} + \cdots\right] = -2f\frac{3x}{Y_0^4} - 2f\frac{10x^3}{Y_0^6} - \ldots$$

where:

Adiabatic harmonic force constant: $k = 2 \cdot 3f / Y_0^4 = 2 \cdot 3m_2 \left(\bar{v}_2^{IN}\right)^2 / Y_0^2$.

$$f = m_2 \left(\bar{v}_2^{IN} Y_0\right)^2 = const.$$

$$F = \frac{m_2 \left(\bar{v}_2^{IN} Y_0\right)^2}{Y^3} = \frac{f}{Y^3}$$

1D-Adiabatic Force Law

Example of oscillator with opposing Isothermal potentials

1D-Isothermal Force Law (assume v_2 is constant for all Y):

$$F = \frac{m_2 v_2^2}{Y} = \frac{const.}{Y}$$

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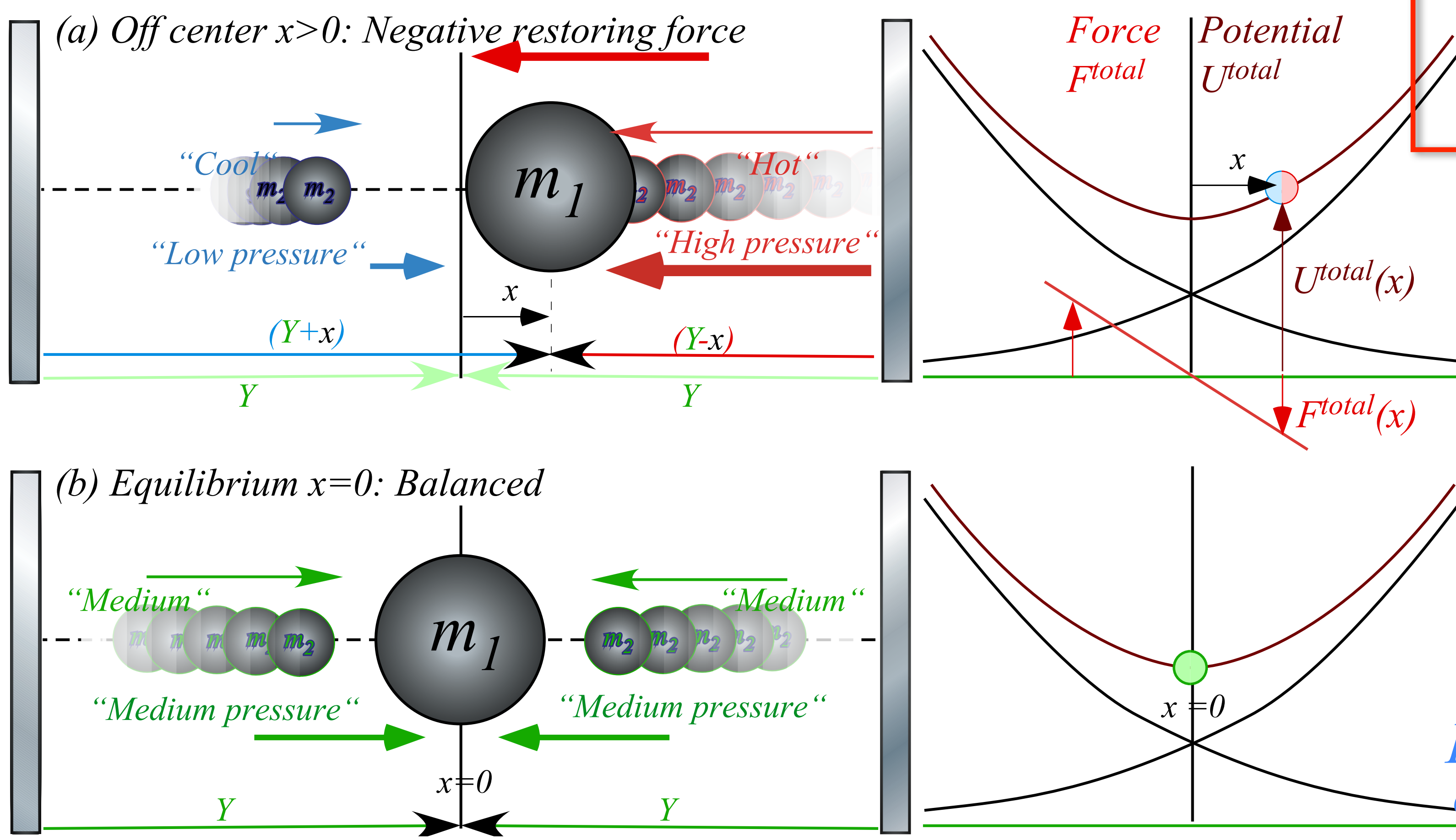

$$F = \frac{m_2 \left(\bar{v}_2^{IN} Y_0\right)^2}{Y^3} = \frac{const.}{Y^3}$$

1D-Adiabatic Force Law

Fig. 5.3 Unit 1

Anharmonic oscillator terms...

Harmonic oscillator term

Two opposing 1D-Adiabatic Force Law fields

$$F^{total} = \frac{f}{(Y_0+x)^3} - \frac{f}{(Y_0-x)^3} = f\left[-\frac{3x}{Y_0^4} - \frac{10x^3}{Y_0^6}\cdots\right] + f\left[-\frac{3x}{Y_0^4} - \frac{10x^3}{Y_0^6} + \cdots\right] = -2f\frac{3x}{Y_0^4} - 2f\frac{10x^3}{Y_0^6} - \ldots$$

where:

Adiabatic harmonic force constant : $k = 2\cdot 3f/Y_0^4 = 2\cdot 3m_2\left(\bar{v}_2^{IN}\right)^2/Y_0^2.$

$$f = m_2\left(\bar{v}_2^{IN} Y_0\right)^2 = const.$$

HO ∡frequency: $\omega = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{2\cdot 3m_2}{m_1}}\frac{\bar{v}_2^{IN}}{Y_0} = 2\pi\upsilon$

Adiabatic frequency is √3 times faster than isothermal frequency

Switch
$m_1 = m_3$
with
m_2
to match formula

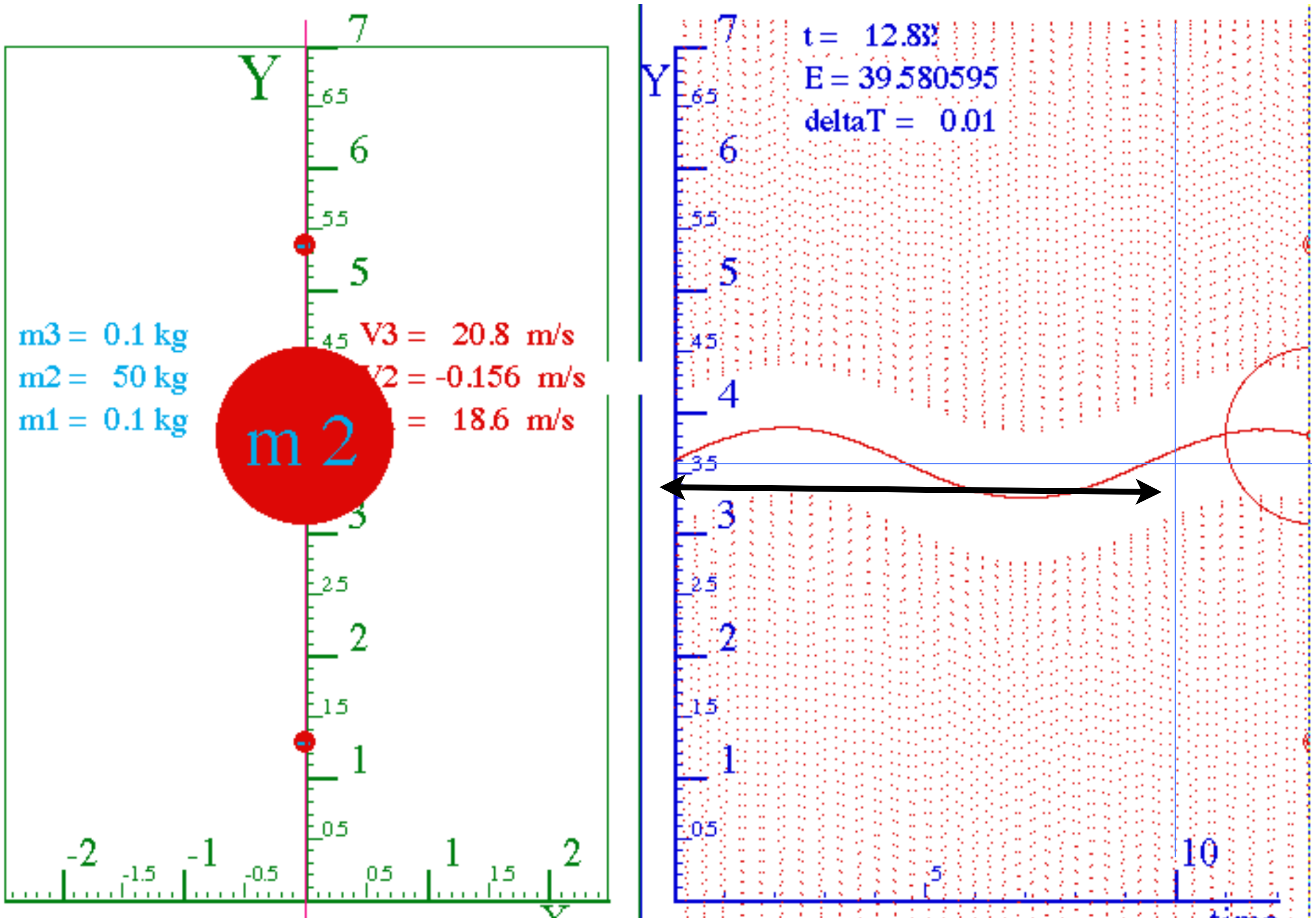

Fig. 5.3 Unit 1

Simulation of an adiabatic case

[BounceIt Superball Collision Web Simulator: 1:500:1 mass ratios (Small Amplitude)]

Sample problem: *Compute period given* $m_1=50$, $m_2=0.1=m_3$, $v_2=20$, $Y_0=3.5$

Adiabatic Period : $\tau = \frac{1}{\upsilon} = 2\pi\sqrt{\frac{m_1}{k}} = 2\pi\sqrt{\frac{m_1}{2\cdot 3m_2}\frac{Y_0}{v_2}}$

Adiabatic Period is $\sqrt{3}$ *times shorter :*

$$\tau = 2\pi\sqrt{\frac{m_1}{2\cdot 3m_2}\frac{Y_0}{v_2}} = 6.28\sqrt{\frac{50}{2\cdot 3\cdot(0.1)}\frac{3.5}{20}}$$

$=10.03$ *That's only a little too big!*

Adiabatic Frequency

HO ∡frequency: $\omega = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{2\cdot 3m_2}{m_1}\frac{v_2}{Y_0}} = 2\pi\upsilon$

Switch $m_1=m_3$ with m_2 to match formula

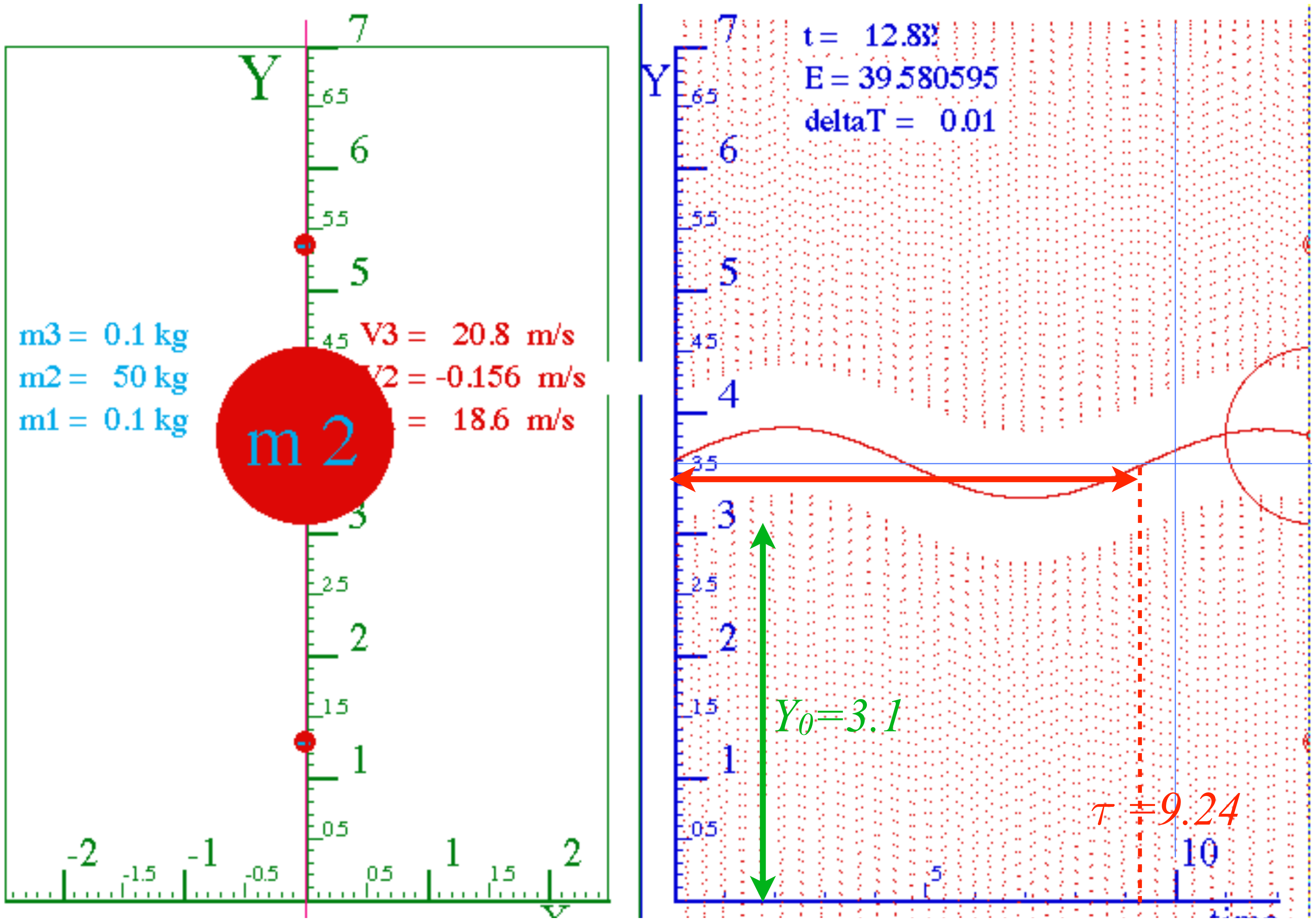

Fig. 5.3 Unit 1

Simulation of an adiabatic case

[BounceIt Superball Collision Web Simulator: 1:500:1 mass ratios (Small Amplitude)]

Sample problem: *Compute period given* $m_1=50$, $m_2=0.1=m_3$, $v_2=20$, $Y_0=3.5$

Perhaps we overestimated Y_0 range…. Let's try $Y_0=3.1$

Adiabatic Period is $\sqrt{3}$ *times shorter :*

$$\tau = 2\pi\sqrt{\frac{m_1}{2\cdot 3m_2}\frac{Y_0}{v_2}} = 6.28\sqrt{\frac{50}{2\cdot 3\cdot(0.1)}\frac{3.1}{20}} = 9.24$$

That's better!

Adiabatic Period : $\tau = \frac{1}{\upsilon} = 2\pi\sqrt{\frac{m_1}{k}} = 2\pi\sqrt{\frac{m_1}{2\cdot 3m_2}\frac{Y_0}{v_2}}$

Adiabatic Frequency

HO ∡frequency: $\omega = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{2\cdot 3m_2}{m_1}\frac{v_2}{Y_0}} = 2\pi\upsilon$