

Lecture 17 Extra Topics

Thur. 3.xx.2016

Matrix algebra of quantum 2-state eigenmodes and dynamics

(Ch. 3-4 of Unit 2)

More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect. 16 to 17 p. 56-85)

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

A proof of completeness using Lagrange interpolation formula (Truer-than-True)

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (ABD-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase



ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial^2_t\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

2D-HO Hamilton equation approach to ABCD Hamilton-Pauli spinor theory (Lect. 16 p.27-33)

More algebraic theory of matrix diagonalization



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Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

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2D-HO Hamilton equations

Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

With example matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1})$$

$$\mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

$$\mathbf{p}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_1 \mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M} - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Multiplication properties of \mathbf{p}_j :

$$\mathbf{p}_j \mathbf{p}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{p}_j - \varepsilon_m \mathbf{p}_j) = \mathbf{p}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{p}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases}$$

Last step:

make *Idempotent Projectors*: $\mathbf{P}_k = \frac{\mathbf{p}_k}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} = \frac{\prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1})}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)}$
(Idempotent means: $\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$)

$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases}$$

implies: $\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M}$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = |\varepsilon_1\rangle \langle \varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} = |\varepsilon_2\rangle \langle \varepsilon_2|$$

The \mathbf{P}_j are *Mutually Ortho-Normal* as are bra-ket $\langle \varepsilon_j|$ and $|\varepsilon_j\rangle$ inside \mathbf{P}_j 's

Eigen-bra-ket projectors of matrix:

$$\begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

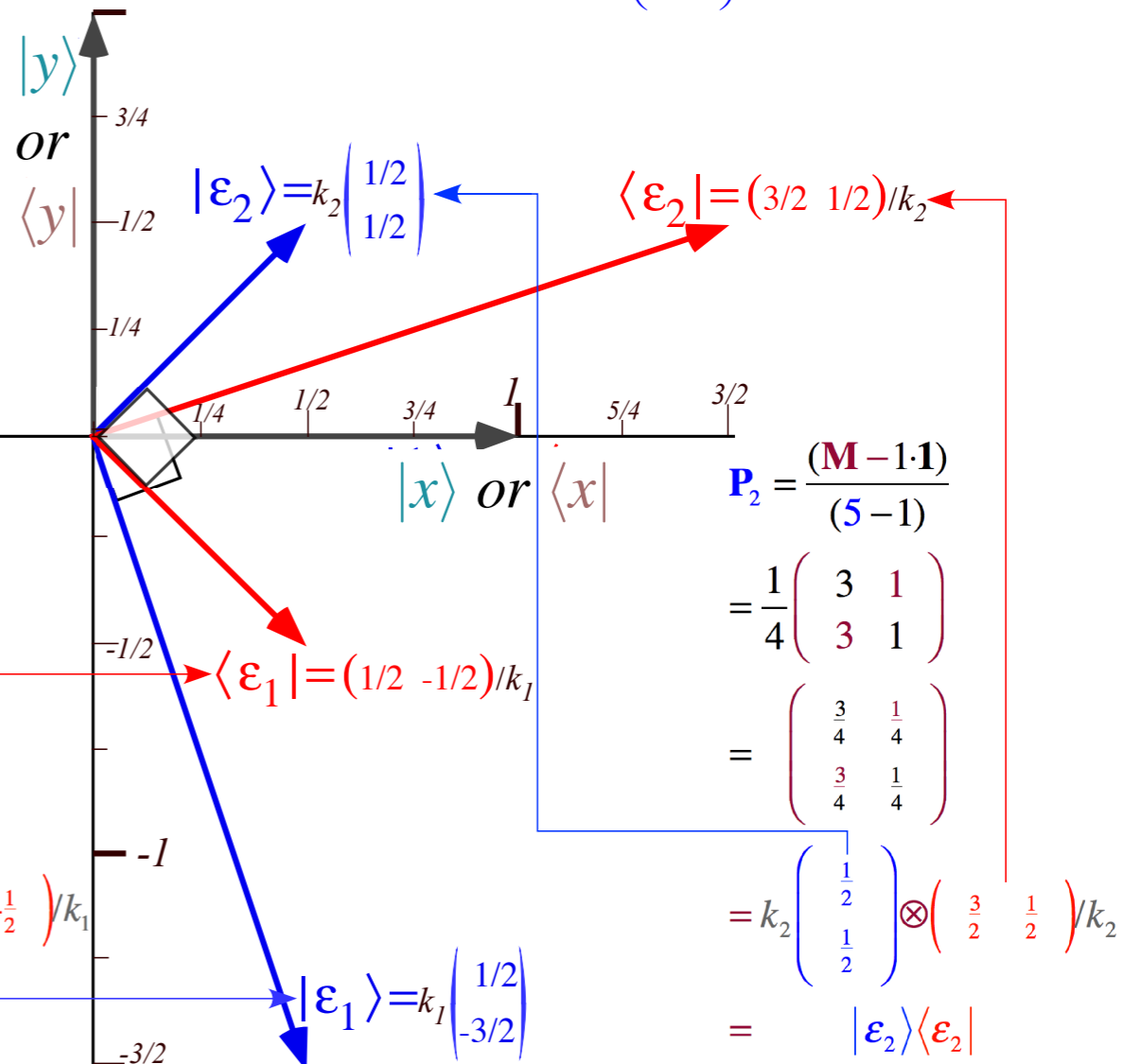
...and the \mathbf{P}_j satisfy a *Completeness Relation*:

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2| + \dots + |\varepsilon_n\rangle \langle \varepsilon_n|$$

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$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} / k_2 = |\varepsilon_2\rangle \langle \varepsilon_2|$$



Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

Multiplication properties of \mathbf{p}_j :

$$\mathbf{p}_j \mathbf{p}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{p}_j - \varepsilon_m \mathbf{p}_j) = \mathbf{p}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{p}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases}$$

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$$\mathbf{P}_j \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases} \quad \begin{aligned} \mathbf{M} \mathbf{P}_k &= \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M} \\ \text{implies:} \\ \mathbf{M} \mathbf{P}_k &= \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M} \end{aligned}$$

The \mathbf{P}_j are *Mutually Ortho-Normal* as are bra-ket $\langle \varepsilon_j |$ and $|\varepsilon_j \rangle$ inside \mathbf{P}_j 's

$$\begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = |\varepsilon_1 \rangle \langle \varepsilon_1 | + |\varepsilon_2 \rangle \langle \varepsilon_2 | + \dots + |\varepsilon_n \rangle \langle \varepsilon_n |$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1 \rangle \langle \varepsilon_1 | + |\varepsilon_2 \rangle \langle \varepsilon_2 |$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1 \mathbf{P}_1 + 5 \mathbf{P}_2 = 1 |\varepsilon_1 \rangle \langle \varepsilon_1 | + 5 |\varepsilon_2 \rangle \langle \varepsilon_2 | = 1 \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5 \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

With example matrix

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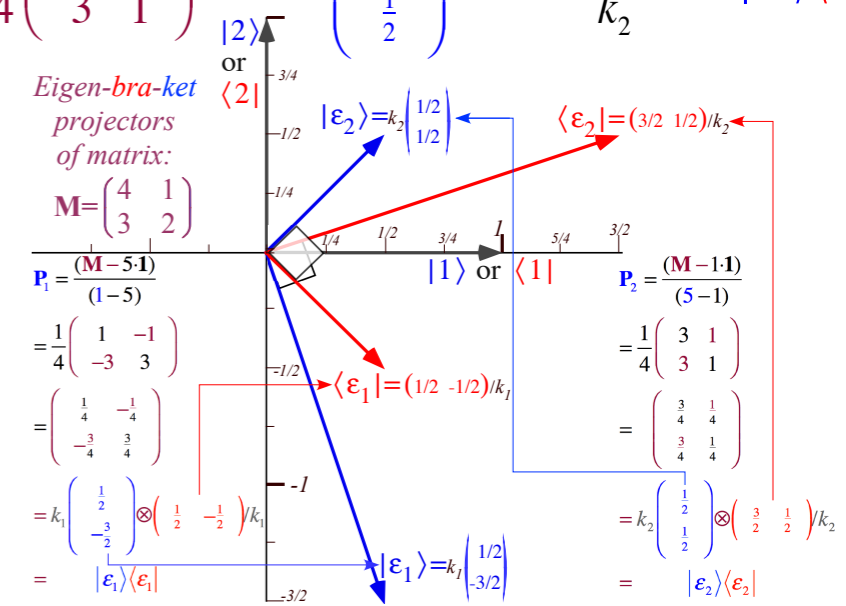
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Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1 \rangle \langle \varepsilon_1 |$$

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Eigen-operators $\mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{P}_k$ then give *Spectral Decomposition* of operator \mathbf{M}

$$\mathbf{M} = \mathbf{M} \mathbf{P}_1 + \mathbf{M} \mathbf{P}_2 + \dots + \mathbf{M} \mathbf{P}_n = \varepsilon_1 \mathbf{P}_1 + \varepsilon_2 \mathbf{P}_2 + \dots + \varepsilon_n \mathbf{P}_n$$

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$$\mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \mathbf{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M}$$

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...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of \mathbf{M}

$$f(\mathbf{M}) = f(\varepsilon_1) \mathbf{P}_1 + f(\varepsilon_2) \mathbf{P}_2 + \dots + f(\varepsilon_n) \mathbf{P}_n$$

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Examples:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 + 3 \cdot 5^{50} & 5^{50} - 1 \\ 3 \cdot 5^{50} - 3 & 5^{50} + 3 \end{pmatrix}$$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm \sqrt{1} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm \sqrt{5} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{3\sqrt{5}}{4} & -\frac{1}{4} + \frac{\sqrt{5}}{4} \\ -\frac{3}{4} + \frac{3\sqrt{5}}{4} & \frac{3}{4} + \frac{\sqrt{5}}{4} \end{pmatrix}$$

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$\{|x\rangle, |y\rangle\}$ -orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$ -completeness

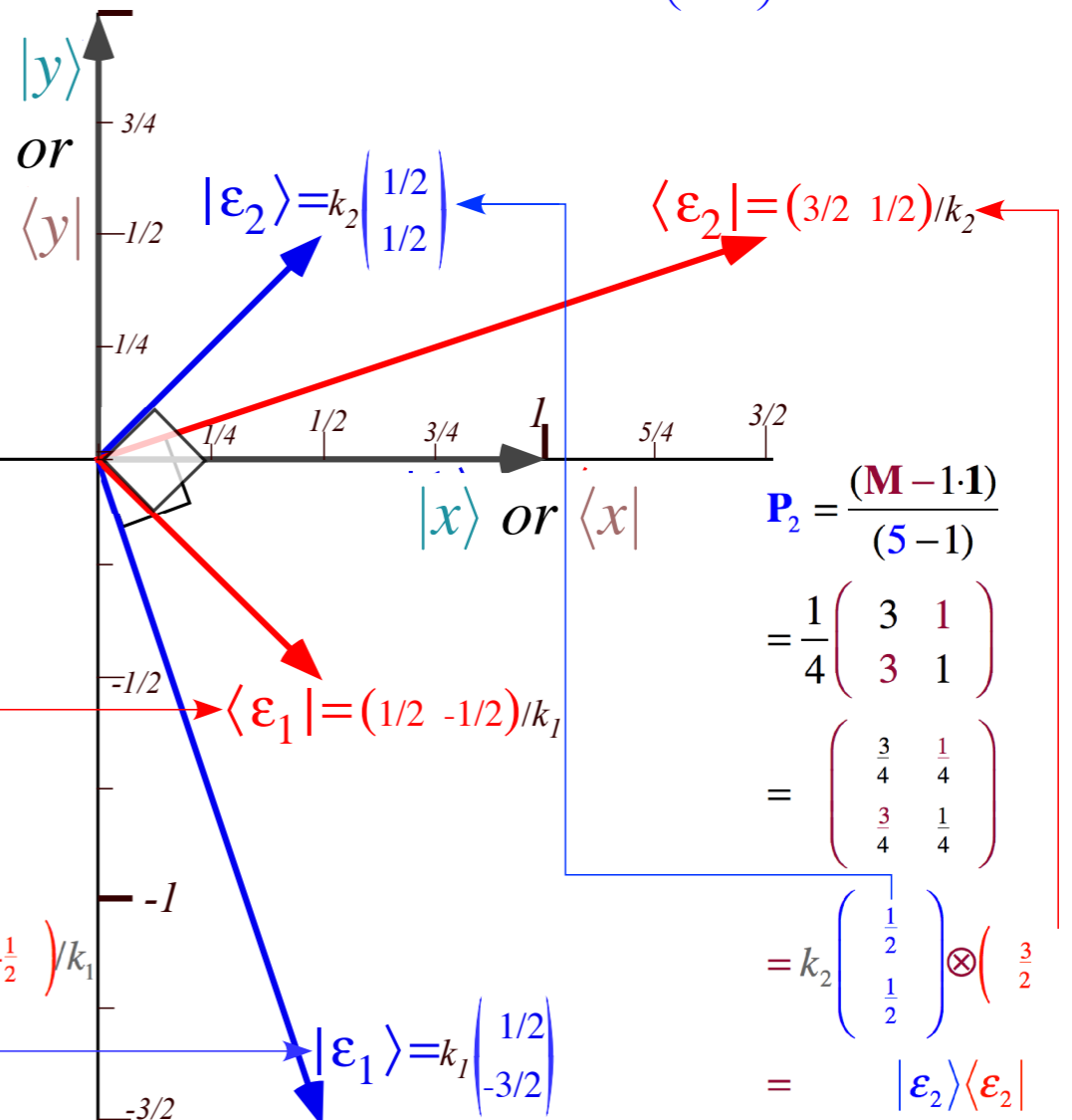
$$\langle x | y \rangle = \delta_{x,y} = \langle x | \mathbf{1} | y \rangle = \langle x | \varepsilon_1 \rangle \langle \varepsilon_1 | y \rangle + \langle x | \varepsilon_2 \rangle \langle \varepsilon_2 | y \rangle.$$

$\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$ -orthonormality with $\{|x\rangle, |y\rangle\}$ -completeness

$$\langle \varepsilon_i | \varepsilon_j \rangle = \delta_{i,j} = \langle \varepsilon_i | \mathbf{1} | \varepsilon_j \rangle = \langle \varepsilon_i | x \rangle \langle x | \varepsilon_j \rangle + \langle \varepsilon_i | y \rangle \langle y | \varepsilon_j \rangle$$

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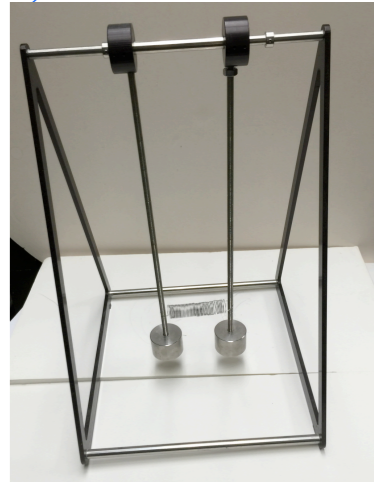
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Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

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Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k = \begin{cases} \mathbf{0} & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases}$$

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State vector representations of orthonormality are quite **similar** to representations of completeness.

Like 2-sides of the same coin.

$$\{|x\rangle, |y\rangle\}\text{-orthonormality with } \{|\epsilon_1\rangle, |\epsilon_2\rangle\}\text{-completeness}$$
$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\epsilon_1\rangle\langle\epsilon_1|y\rangle + \langle x|\epsilon_2\rangle\langle\epsilon_2|y\rangle.$$

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Dirac δ -function

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However Schrodinger wavefunction notation $\psi(x) = \langle x | \psi \rangle$ shows quite a difference...

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$$\langle \epsilon_i | \epsilon_j \rangle = \delta_{i,j} = \dots + \psi_i^*(x) \psi_j(x) + \psi_2(y) \psi_2^*(y) + \dots \rightarrow \int dx \psi_i^*(x) \psi_j(x)$$

However Schrodinger wavefunction notation $\psi(x) = \langle x | \psi \rangle$ shows quite a difference...

...particularly in the orthonormality integral.

More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

➔ *A proof of completeness using Lagrange interpolation formula (Truer-than-True)*

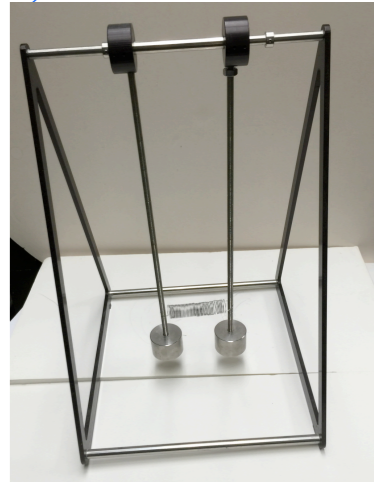
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (ABD-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase



ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

2D-HO Hamilton equations

Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

A Proof of Projector Completeness (Truer-than-true by Lagrange interpolation)

Compare matrix completeness relation and functional spectral decompositions

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = \sum_{\epsilon_k} \mathbf{P}_k = \sum_{\epsilon_k} \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)} \quad f(\mathbf{M}) = f(\epsilon_1)\mathbf{P}_1 + f(\epsilon_2)\mathbf{P}_2 + \dots + f(\epsilon_n)\mathbf{P}_n = \sum_{\epsilon_k} f(\epsilon_k)\mathbf{P}_k = \sum_{\epsilon_k} f(\epsilon_k) \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)}$$

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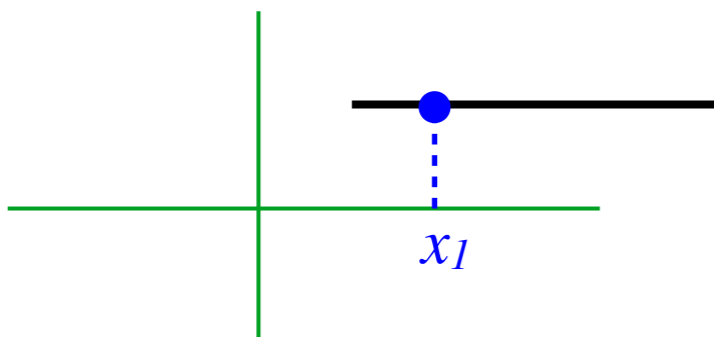
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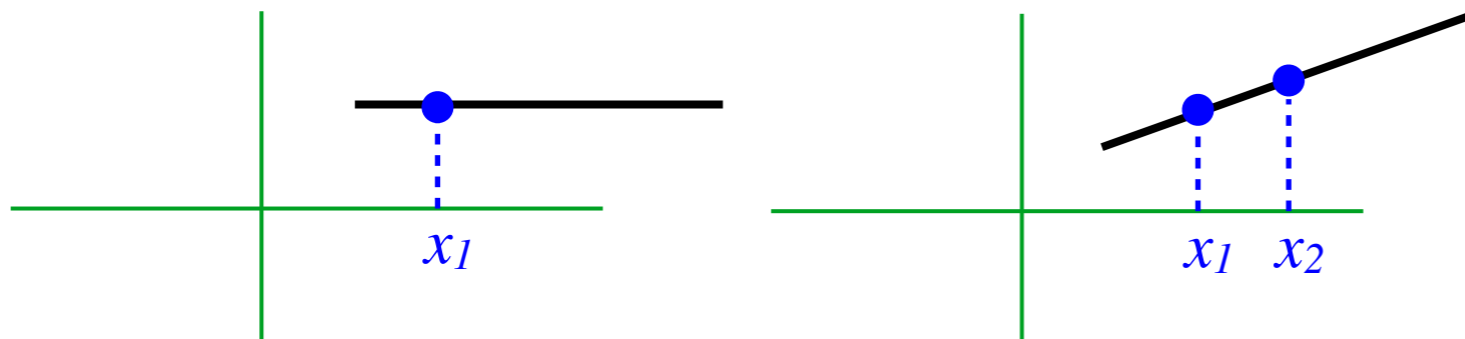
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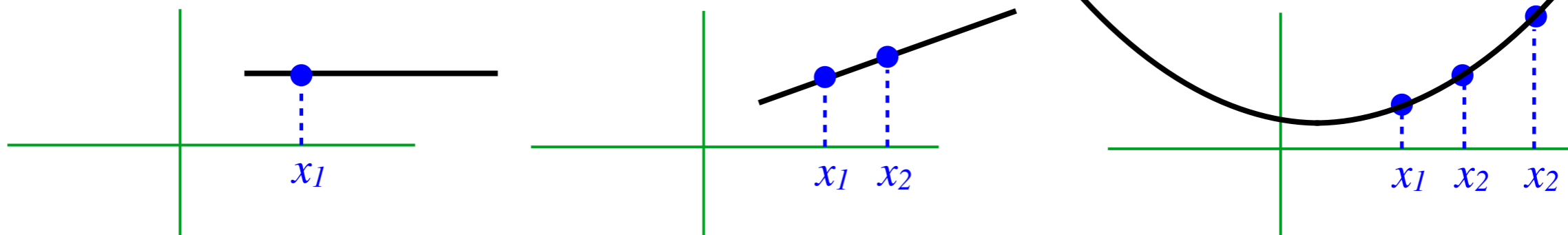
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All distinct values $\epsilon_1 \neq \epsilon_2 \neq \dots \neq \epsilon_N$ satisfy $\sum \mathbf{P}_k = \mathbf{1}$.

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Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$. So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1 = \sum_{m=1}^N P_m(x) \quad x = \sum_{m=1}^N x_m P_m(x) \quad x^2 = \sum_{m=1}^N x_m^2 P_m(x)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula \rightarrow *Completeness formula* as $x \rightarrow \mathbf{M}$ and as $x_k \rightarrow \epsilon_k$ and as $P_k(x_k) \rightarrow \mathbf{P}_k$

All distinct values $\epsilon_1 \neq \epsilon_2 \neq \dots \neq \epsilon_N$ satisfy $\sum \mathbf{P}_k = \mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1 + \mathbf{P}_2 = \frac{\prod_{j \neq 1} (\mathbf{M} - \epsilon_j \mathbf{1})}{\prod_{j \neq 1} (\epsilon_1 - \epsilon_j)} + \frac{\prod_{j \neq 2} (\mathbf{M} - \epsilon_j \mathbf{1})}{\prod_{j \neq 2} (\epsilon_2 - \epsilon_j)} = \frac{(\mathbf{M} - \epsilon_2 \mathbf{1})}{(\epsilon_1 - \epsilon_2)} + \frac{(\mathbf{M} - \epsilon_1 \mathbf{1})}{(\epsilon_2 - \epsilon_1)} = \frac{(\mathbf{M} - \epsilon_2 \mathbf{1}) - (\mathbf{M} - \epsilon_1 \mathbf{1})}{(\epsilon_1 - \epsilon_2)} = \frac{-\epsilon_2 \mathbf{1} + \epsilon_1 \mathbf{1}}{(\epsilon_1 - \epsilon_2)} = \mathbf{1} \text{ (for all } \epsilon_j \text{)}$$

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \dots + \mathbf{P}_n = \sum_{\epsilon_k} \mathbf{P}_k = \sum_{\epsilon_k} \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)} \quad f(\mathbf{M}) = f(\epsilon_1)\mathbf{P}_1 + f(\epsilon_2)\mathbf{P}_2 + \dots + f(\epsilon_n)\mathbf{P}_n = \sum_{\epsilon_k} f(\epsilon_k)\mathbf{P}_k = \sum_{\epsilon_k} f(\epsilon_k) \frac{\prod_{m \neq k} (\mathbf{M} - \epsilon_m \mathbf{1})}{\prod_{m \neq k} (\epsilon_k - \epsilon_m)}$$

with *Lagrange interpolation formula* of function $f(x)$ approximated by its value at N points x_1, x_2, \dots, x_N .

$$L(f(x)) = \sum_{k=1}^N f(x_k) \cdot P_k(x) \quad \text{where: } P_k(x) = \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

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However, only *select* values ϵ_k work for eigen-forms $\mathbf{M}\mathbf{P}_k = \epsilon_k \mathbf{P}_k$ or orthonormality $\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k$.

More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

A proof of completeness using Lagrange interpolation formula (Truer-than-True)

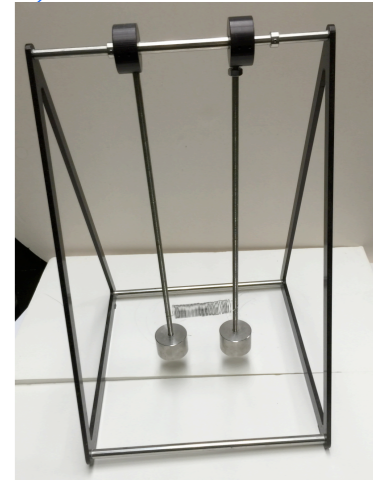
➔ *Diagonalizing Transformations (D-Ttran) from projectors*

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (ABD-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase



ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

2D-HO Hamilton equations

Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

(REVIEW) 2D classical HO compared to $U(2)$ quantum 2-state system

Introducing $ABCD$ Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A, \sigma_B, \sigma_C\} = \{\sigma_Z, \sigma_X, \sigma_Y\}$

σ_A -products 3D vector analysis and “Crazy-Thing-Theorem”

Eigensolutions by general matrix-algebra with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent ($\mathbf{P} \cdot \mathbf{P} = \mathbf{P}$) projectors (how eigenvalues \Rightarrow eigenvectors)

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

$U(2) \supset C_2$ $ABCD$ group theory method to find 2D-HO eigenmodes and eigenvalues

Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

Circular-chiral-cycloton (C-Type) symmetry

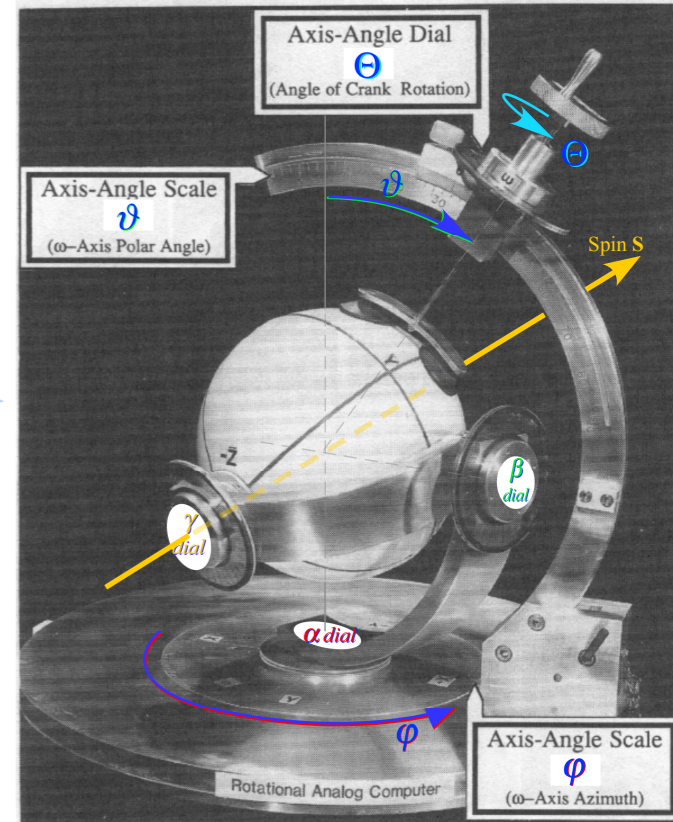
Mixed $ABCD$ symmetry examples

More theory of matrix diagonalization

Discussion of orthogonality vs. completeness vis-a-vis Operator vs. State

Lagrange functional interpolation formula

➔ Diagonalizing Transformations (D-Tran) from projectors



Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2\rangle\langle\epsilon_2|$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$
$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran **rows**, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into inverse d-tran **columns**.

Diagonalizing Transformations (D-Tran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2\rangle\langle\epsilon_2|$$

Load distinct bras $\langle\epsilon_1|$ and $\langle\epsilon_2|$ into d-tran **rows**, kets $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ into inverse d-tran **columns**.

$$\left\{ \langle\epsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\epsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}, \quad \left\{ |\epsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\epsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$$

$(\epsilon_1, \epsilon_2) \leftarrow (1, 2)$ *d-Tran matrix*

$(1, 2) \leftarrow (\epsilon_1, \epsilon_2)$ *INVERSE d-Tran matrix*

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Diagonalizing Transformations (D-Tran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2\rangle\langle\epsilon_2|$$

Load distinct bras $\langle\epsilon_1|$ and $\langle\epsilon_2|$ into d-tran **rows**, kets $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ into inverse d-tran **columns**.

$$\left\{ \langle\epsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\epsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}, \quad \left\{ |\epsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\epsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$$

$(\epsilon_1, \epsilon_2) \leftarrow (1, 2)$ *d-Tran matrix*

$(1, 2) \leftarrow (\epsilon_1, \epsilon_2)$ *INVERSE d-Tran matrix*

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\epsilon_1|\mathbf{K}|\epsilon_1\rangle & \langle\epsilon_1|\mathbf{K}|\epsilon_2\rangle \\ \langle\epsilon_2|\mathbf{K}|\epsilon_1\rangle & \langle\epsilon_2|\mathbf{K}|\epsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Diagonalizing Transformations (D-Tran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\mathbf{1})}{(1-5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\mathbf{1})}{(5-1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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$(\varepsilon_1, \varepsilon_2) \leftarrow (1, 2)$ *d-Tran matrix*

$(1, 2) \leftarrow (\varepsilon_1, \varepsilon_2)$ *INVERSE d-Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

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Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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Diagonalizing Transformations (D-Tran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\mathbf{1})}{(1-5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\mathbf{1})}{(5-1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\epsilon_2\rangle\langle\epsilon_2|$$

Load distinct bras $\langle\epsilon_1|$ and $\langle\epsilon_2|$ into d-tran **rows**, kets $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$ into inverse d-tran **columns**.

$$\left\{ \langle\epsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\epsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}, \quad \left\{ |\epsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\epsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}$$

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$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

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$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are “easy”

$$\begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\epsilon_1|\mathbf{1}|\epsilon_1\rangle & \langle\epsilon_1|\mathbf{1}|\epsilon_2\rangle \\ \langle\epsilon_2|\mathbf{1}|\epsilon_1\rangle & \langle\epsilon_2|\mathbf{1}|\epsilon_2\rangle \end{pmatrix} \begin{pmatrix} \langle\epsilon_1|x\rangle & \langle\epsilon_1|y\rangle \\ \langle\epsilon_2|x\rangle & \langle\epsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix}^\dagger = \begin{pmatrix} \langle x|\epsilon_1\rangle^* & \langle y|\epsilon_1\rangle^* \\ \langle x|\epsilon_2\rangle^* & \langle y|\epsilon_2\rangle^* \end{pmatrix} = \begin{pmatrix} \langle x|\epsilon_1\rangle & \langle x|\epsilon_2\rangle \\ \langle y|\epsilon_1\rangle & \langle y|\epsilon_2\rangle \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

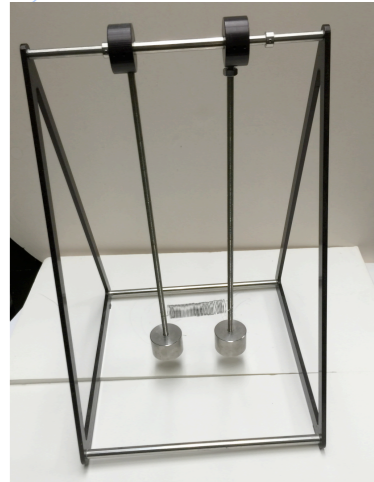
Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

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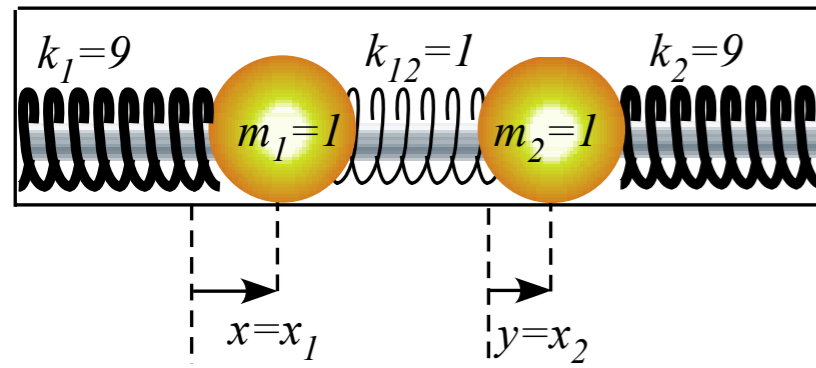
2D-HO eigensolution example with asymmetric (ABD-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase



ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$
2D-HO Hamilton equations

Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

Analyzing 2D-HO beats and mixed mode eigen-solutions



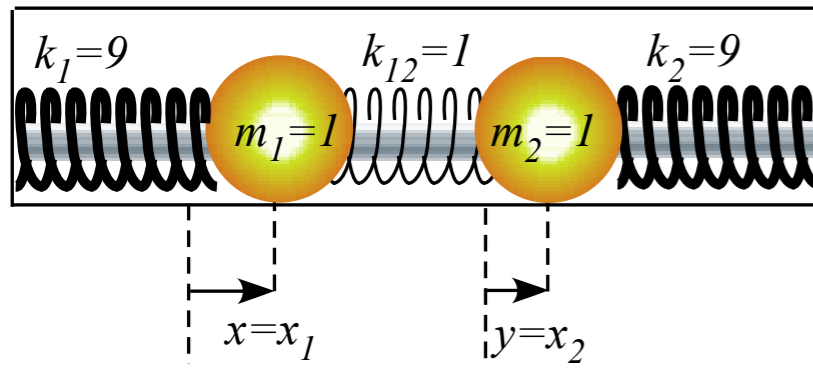
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$
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The \mathbf{K} secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K - 9)(K - 11) = (K - K_1)(K - K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 9, \quad K_2 = \omega_0^2(\epsilon_2) = 11,$

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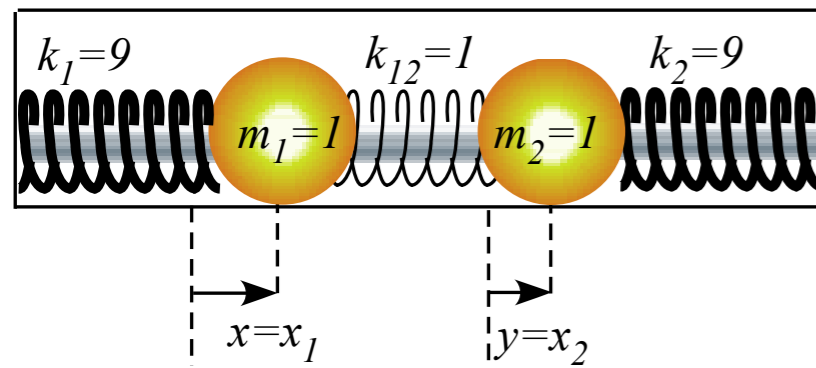
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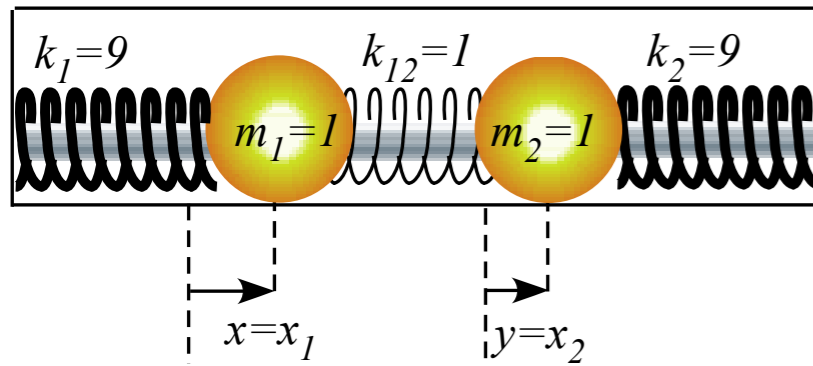
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Eigenbra vectors: $\langle\epsilon_1| = \left(1/\sqrt{2} \quad +1/\sqrt{2} \right)$, $\langle\epsilon_2| = \left(1/\sqrt{2} \quad -1/\sqrt{2} \right)$

Analyzing 2D-HO beats and mixed mode eigen-solutions



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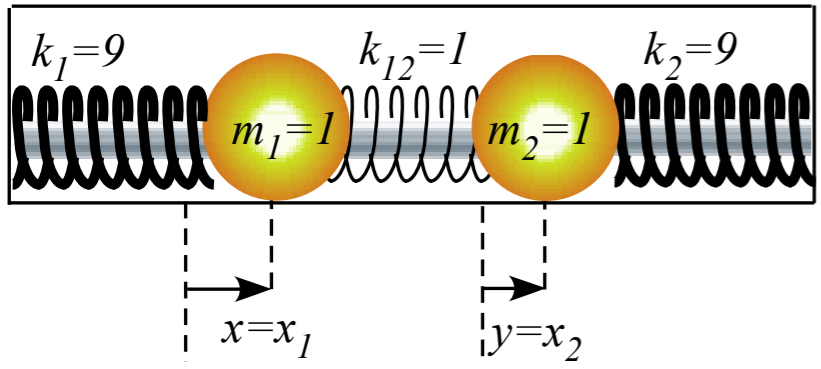
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Mixed mode dynamics

$$|x(t)\rangle = |\epsilon_1\rangle \langle\epsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\epsilon_2\rangle \langle\epsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

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Analyzing 2D-HO beats and mixed mode eigen-solutions



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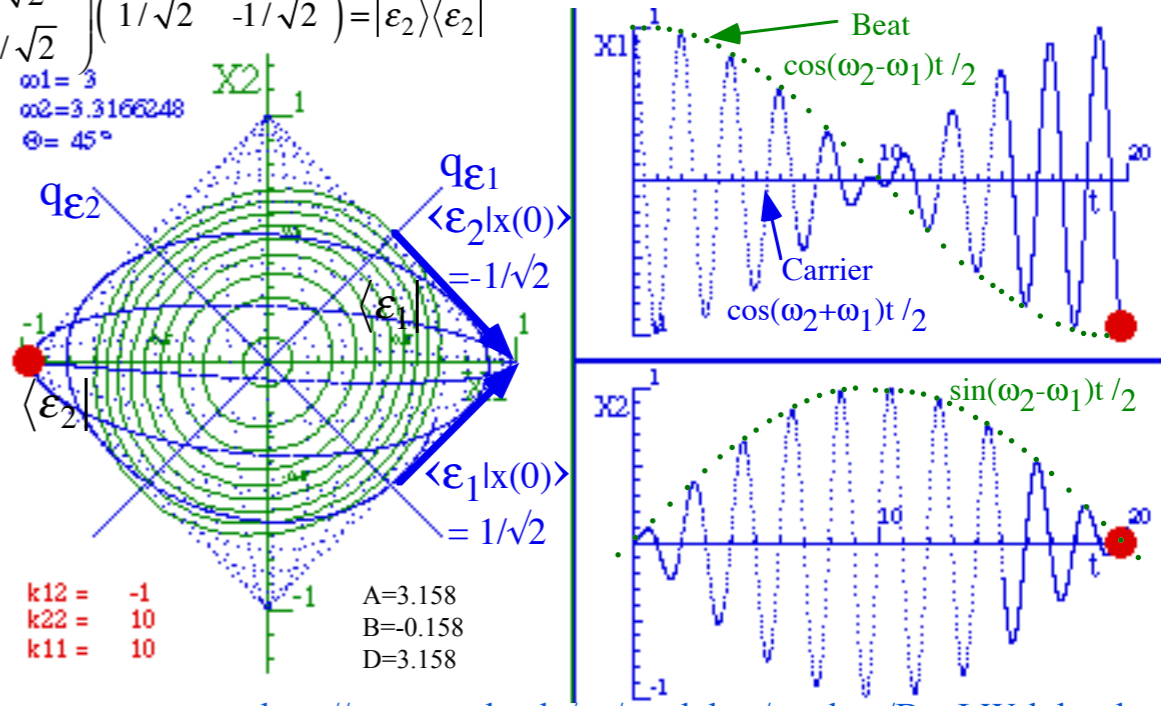
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100% modulation (SWR=0) $\frac{e^{ia} + e^{ib}}{2} = e^{\frac{i(a+b)}{2}} \frac{e^{\frac{i(a-b)}{2}} + e^{-\frac{i(a-b)}{2}}}{2}$

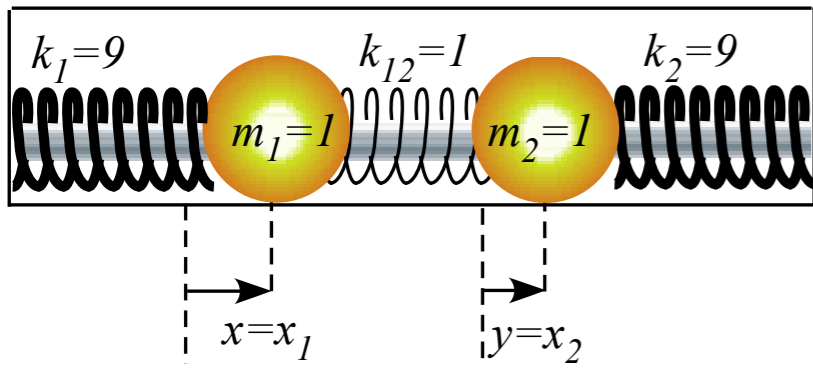
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\omega_1 t} + e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t} - e^{-i\omega_2 t}}{2} \end{pmatrix} = \frac{e^{-\frac{i(\omega_1 + \omega_2)t}}{2}} \begin{pmatrix} e^{-\frac{i(\omega_1 - \omega_2)t}{2}} + e^{\frac{i(\omega_1 - \omega_2)t}{2}} \\ e^{-\frac{i(\omega_1 - \omega_2)t}{2}} - e^{\frac{i(\omega_1 - \omega_2)t}{2}} \end{pmatrix}$$



<http://www.uark.edu/ua/modphys/markup/BoxItWeb.html>

BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



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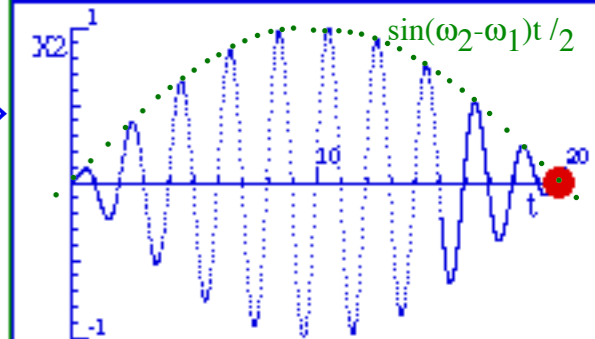
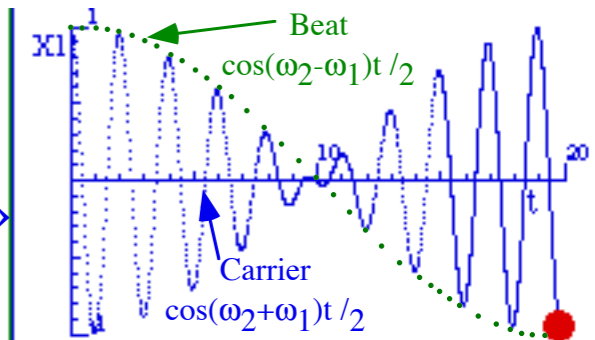
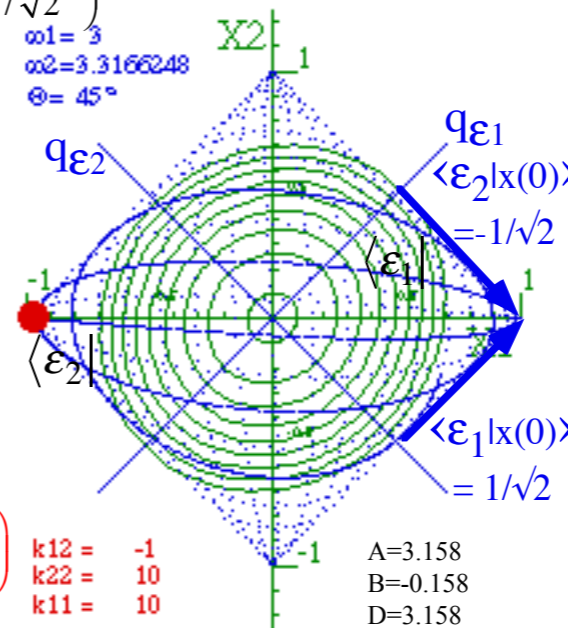
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Note the i phase

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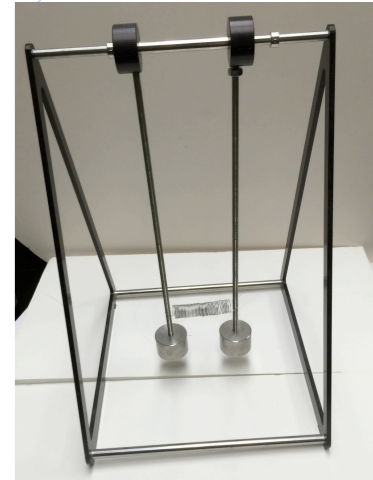
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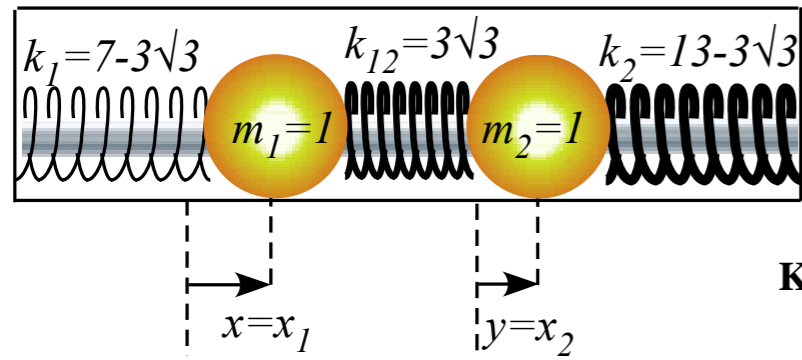


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Spectral decomposition of 2D-HO mode dynamics for lower symmetry

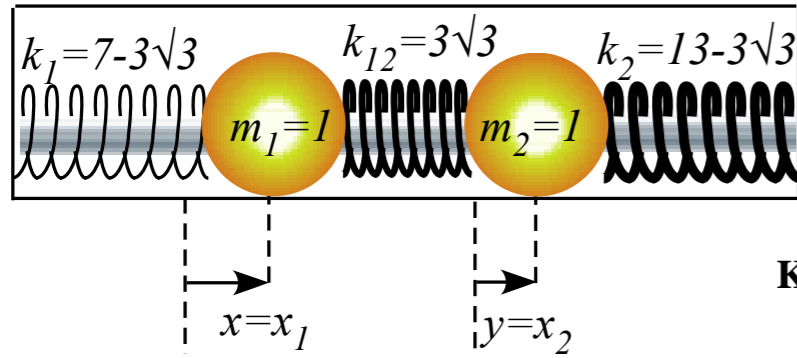


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$$\text{Det}(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$$

$$\text{Trace}(\mathbf{K}) = 7 + 13 = 20$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



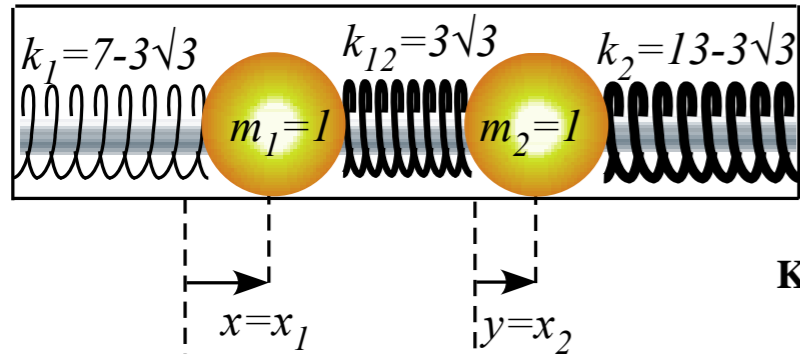
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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0$

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



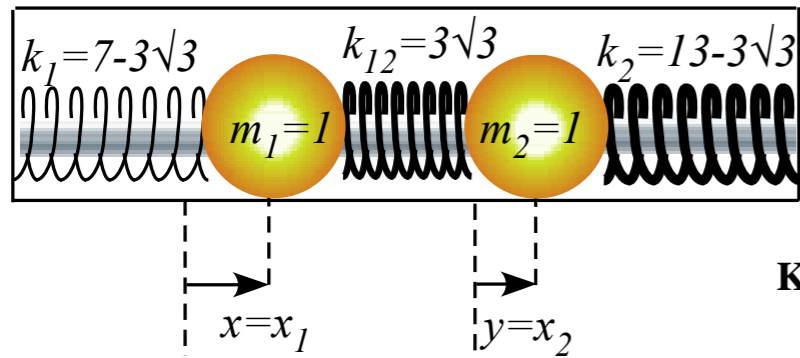
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The \mathbf{K} secular equation $K^2 - \text{Trace}(\mathbf{K})K + \text{Det}(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K - 4)(K - 16)$

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Spectral decomposition of 2D-HO mode dynamics for lower symmetry

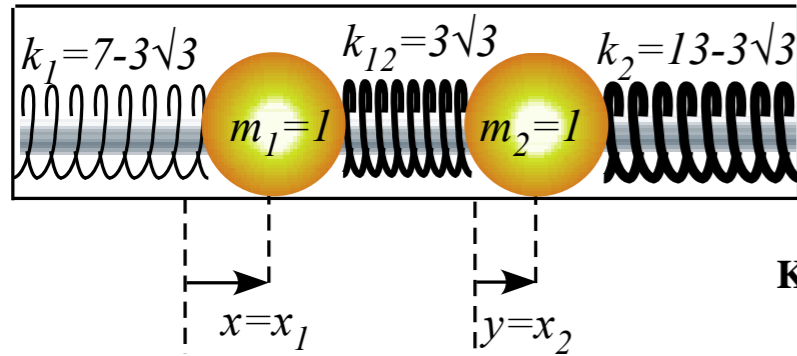


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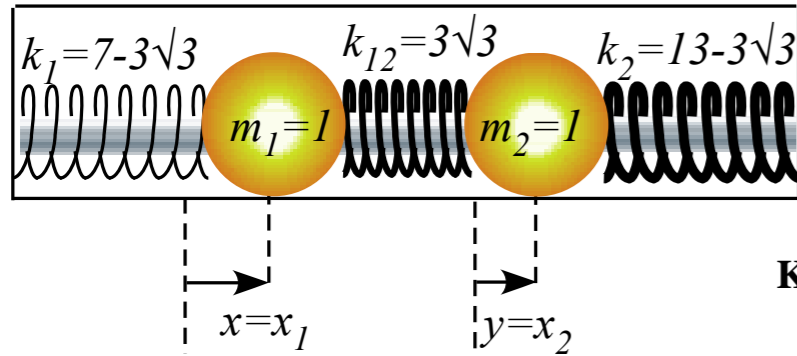
Eigenvalues K_k and squared eigenfrequencies $\omega_0(\epsilon_k)^2$ $K_1 = \omega_0^2(\epsilon_1) = 4,$ $K_2 = \omega_0^2(\epsilon_2) = 16,$

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$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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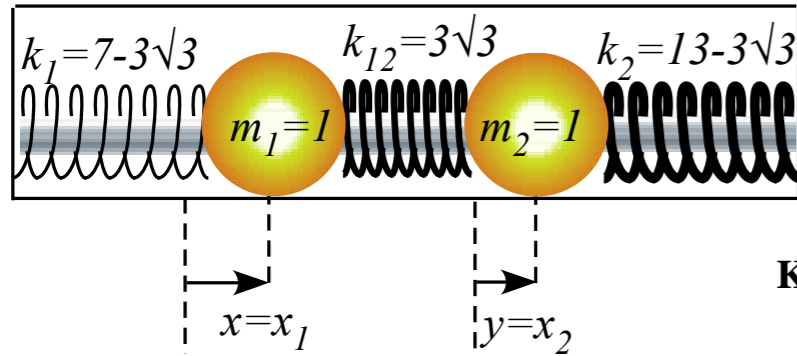
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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

A proof of completeness using Lagrange interpolation formula (Truer-than-True)

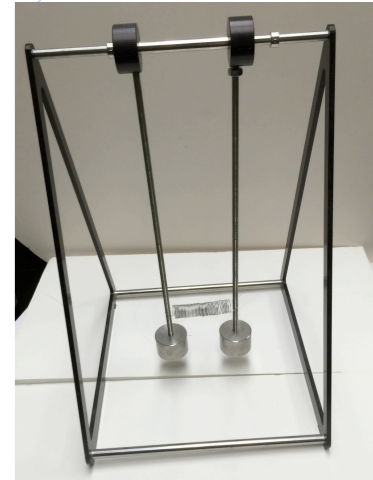
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (ABD-Type) symmetry

➔ *Initial state projection, mixed mode beat dynamics with variable phase*

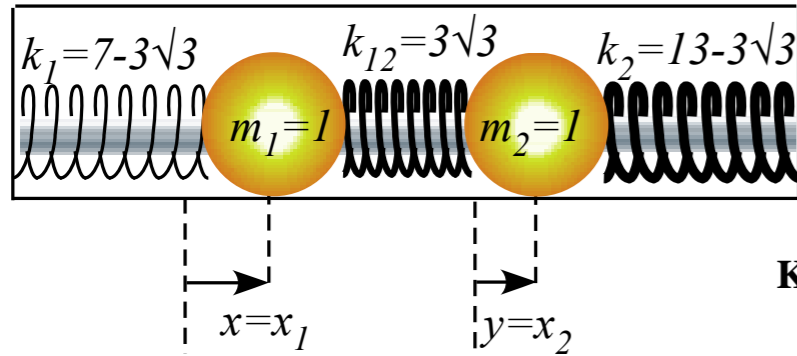


ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

2D-HO Hamilton equations

Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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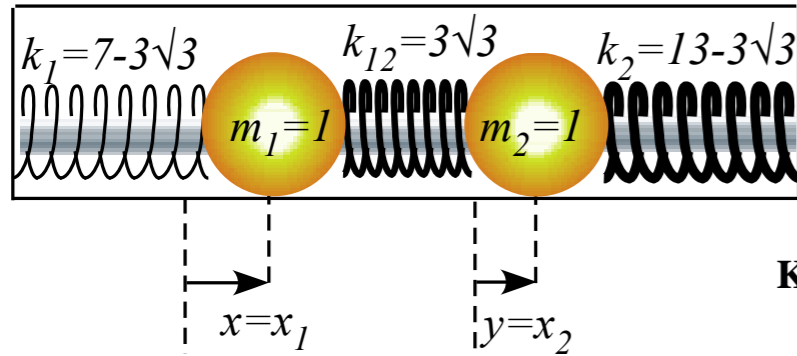
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Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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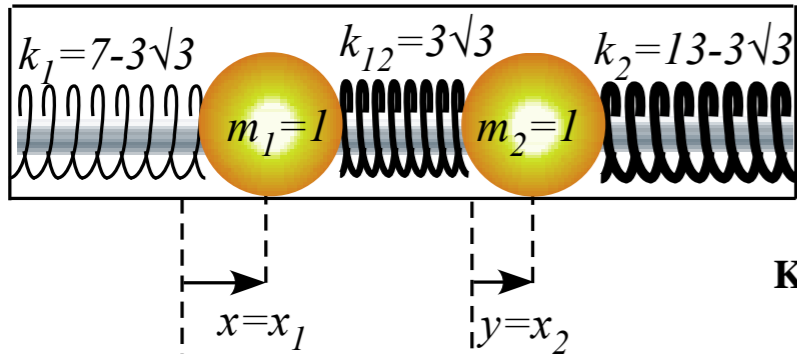
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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \left(\frac{\sqrt{3}}{2}\right) + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \left(-\frac{1}{2}\right)$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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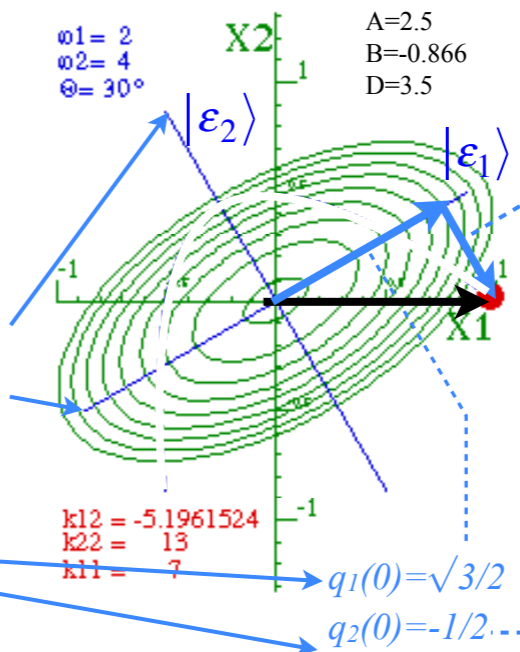
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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

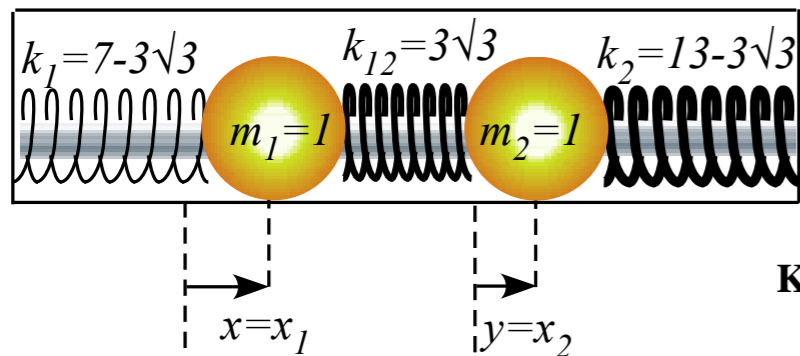


(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

$k_{12} = -5.1961524$
 $k_{22} = 13$
 $k_{11} = 7$

$q_1(0) = \sqrt{3}/2$
 $q_2(0) = -1/2$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

Using $\cos 4t = 2 \cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2} (2 \cos^2 2t - 1) = -\frac{4}{3} [q_1(t)]^2 + \frac{1}{2}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

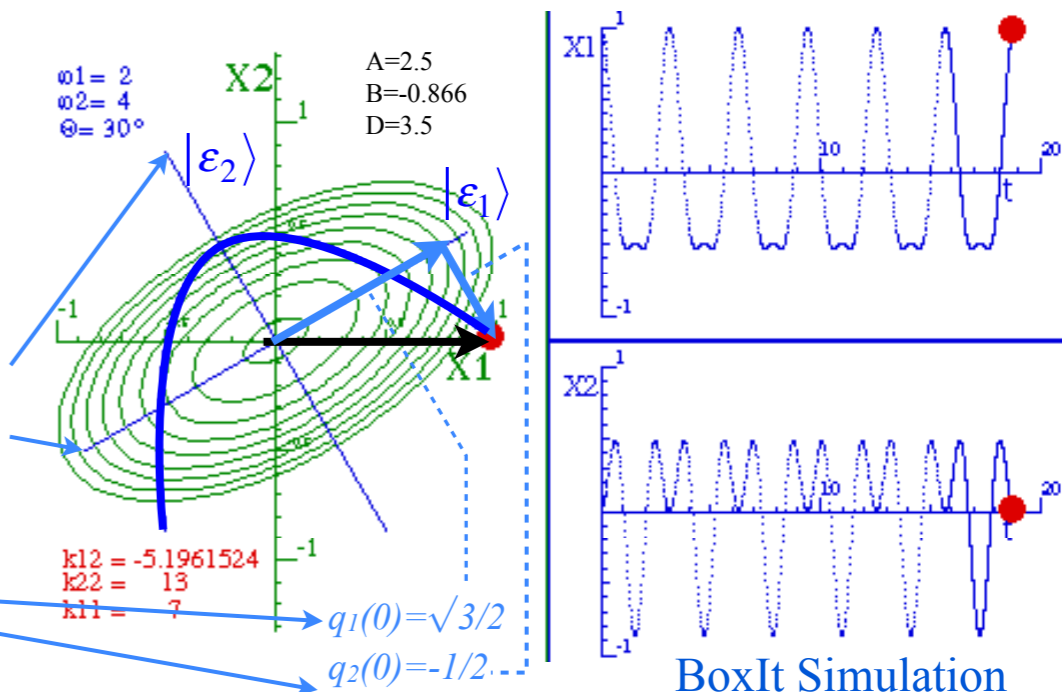
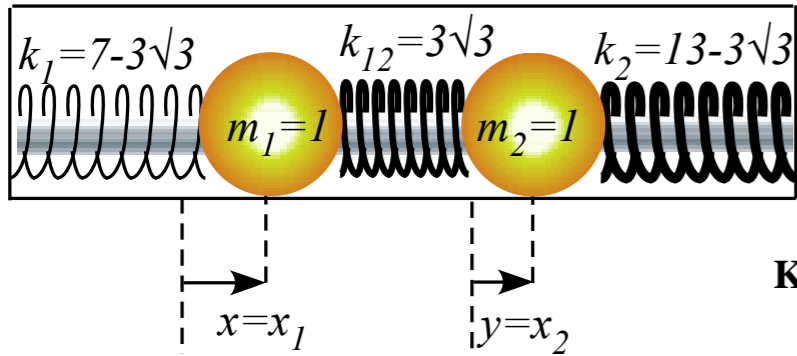


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential ($V=\text{const.}$) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\epsilon_1)=2.0$, $\omega_0(\epsilon_2)=4.0$) and zero initial velocity.
<http://www.uark.edu/ua/modphys/markup/BoxItWeb.html>

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$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7 - 16 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 16 \end{pmatrix}}{4 - 16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 & \\ & 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\epsilon_1\rangle\langle\epsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 7 - 4 & -3\sqrt{3} \\ -3\sqrt{3} & 13 - 4 \end{pmatrix}}{16 - 4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 & \\ & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = |\epsilon_2\rangle\langle\epsilon_2|$$

Eigenbra vectors: $\langle\epsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}$, $\langle\epsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

Spectral decomposition of initial state $\mathbf{x}(0) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_1 + \mathbf{P}_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -1/2 \end{pmatrix}$$

(Note projection of $\mathbf{x}(0)$ onto eigen-axes)

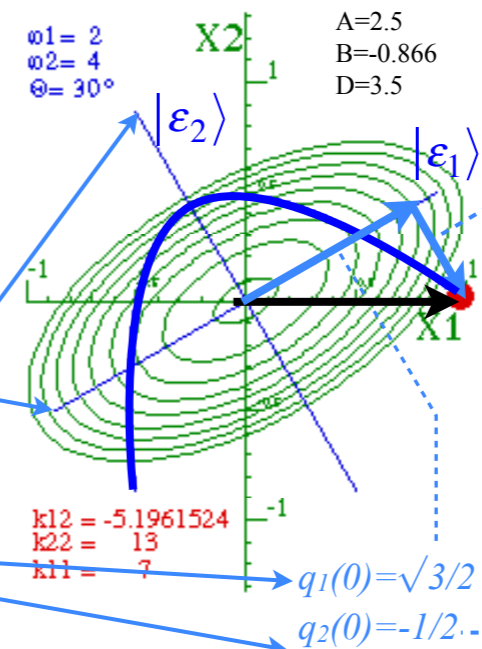
$$\begin{pmatrix} q_1(t) = \frac{\sqrt{3}}{2} \cos 2t, & q_2(t) = -\frac{1}{2} \cos 4t \end{pmatrix}$$

Using $\cos 4t = 2 \cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2} (2 \cos^2 2t - 1) = -\frac{4}{3} [q_1(t)]^2 + \frac{1}{2}$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example of a Tschebycheff Polynomial order 2



BoxIt Simulation

Pafnuty Chebyshev



Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshev, Tchebychev or Tchebycheff, or Tschebyshev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

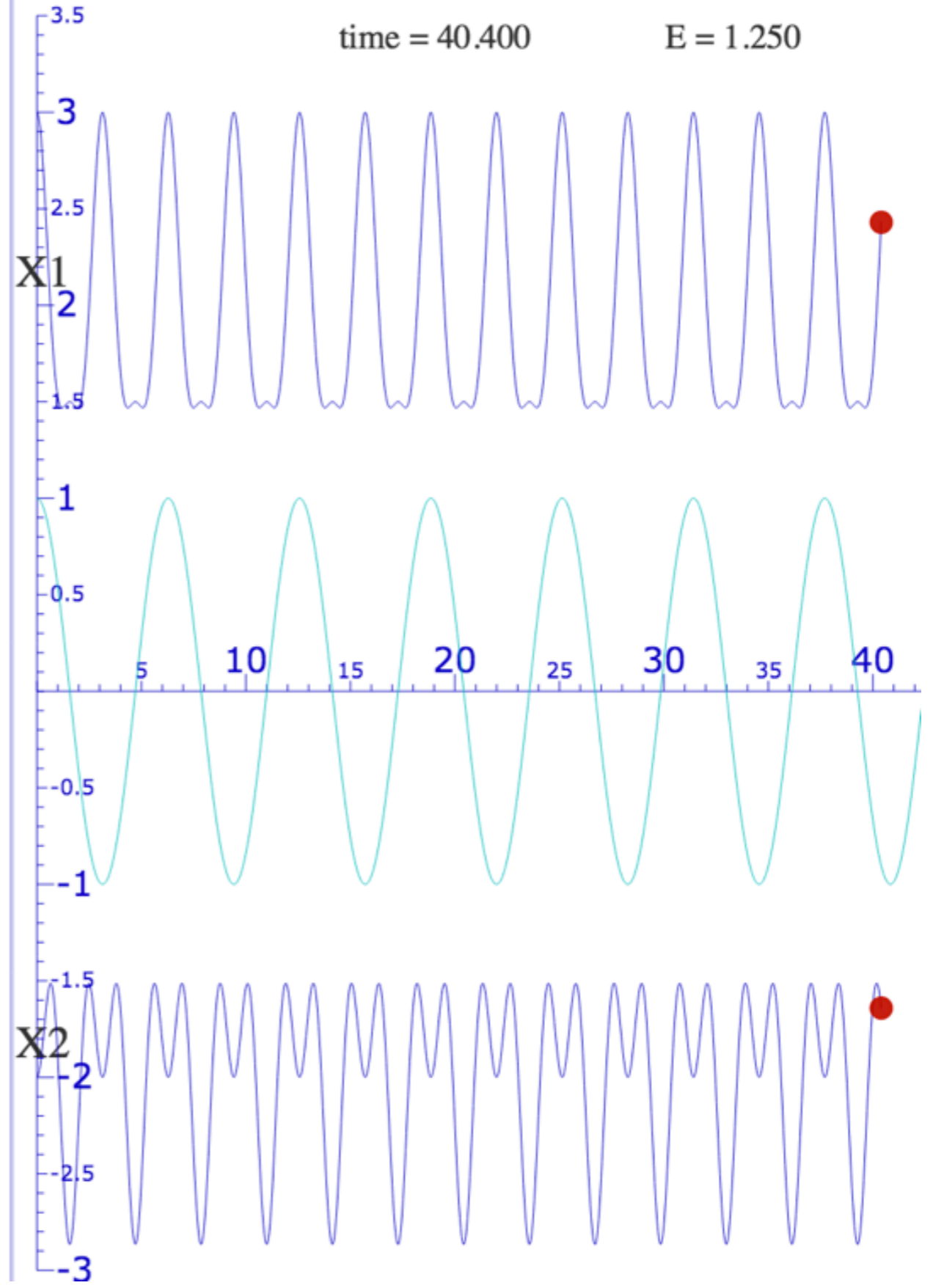
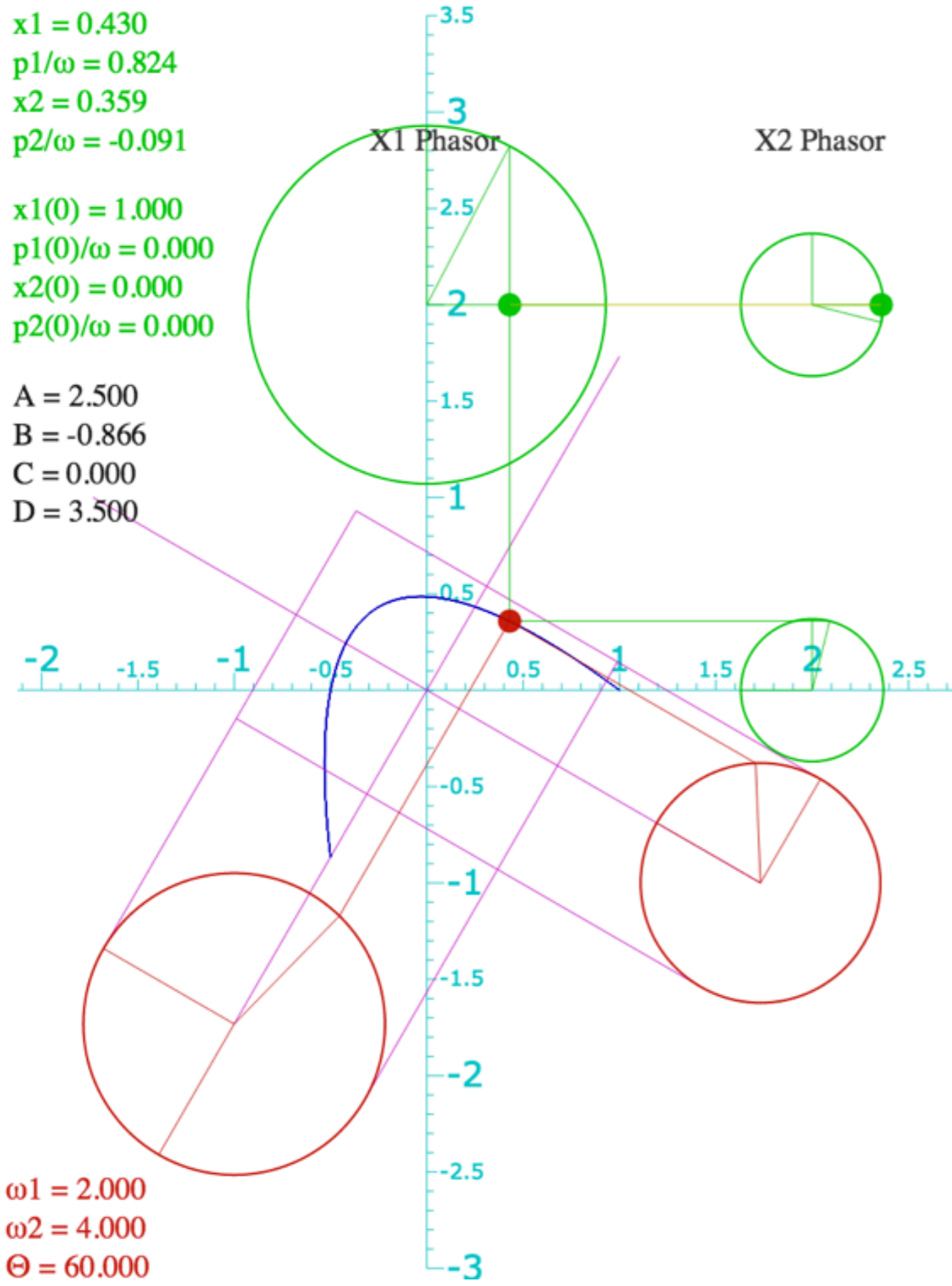
Died: December 8, 1894, Saint Petersburg

$x1 = 0.430$
 $p1/\omega = 0.824$
 $x2 = 0.359$
 $p2/\omega = -0.091$

$x1(0) = 1.000$
 $p1(0)/\omega = 0.000$
 $x2(0) = 0.000$
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$A = 2.500$
 $B = -0.866$
 $C = 0.000$
 $D = 3.500$

$\omega1 = 2.000$
 $\omega2 = 4.000$
 $\Theta = 60.000$



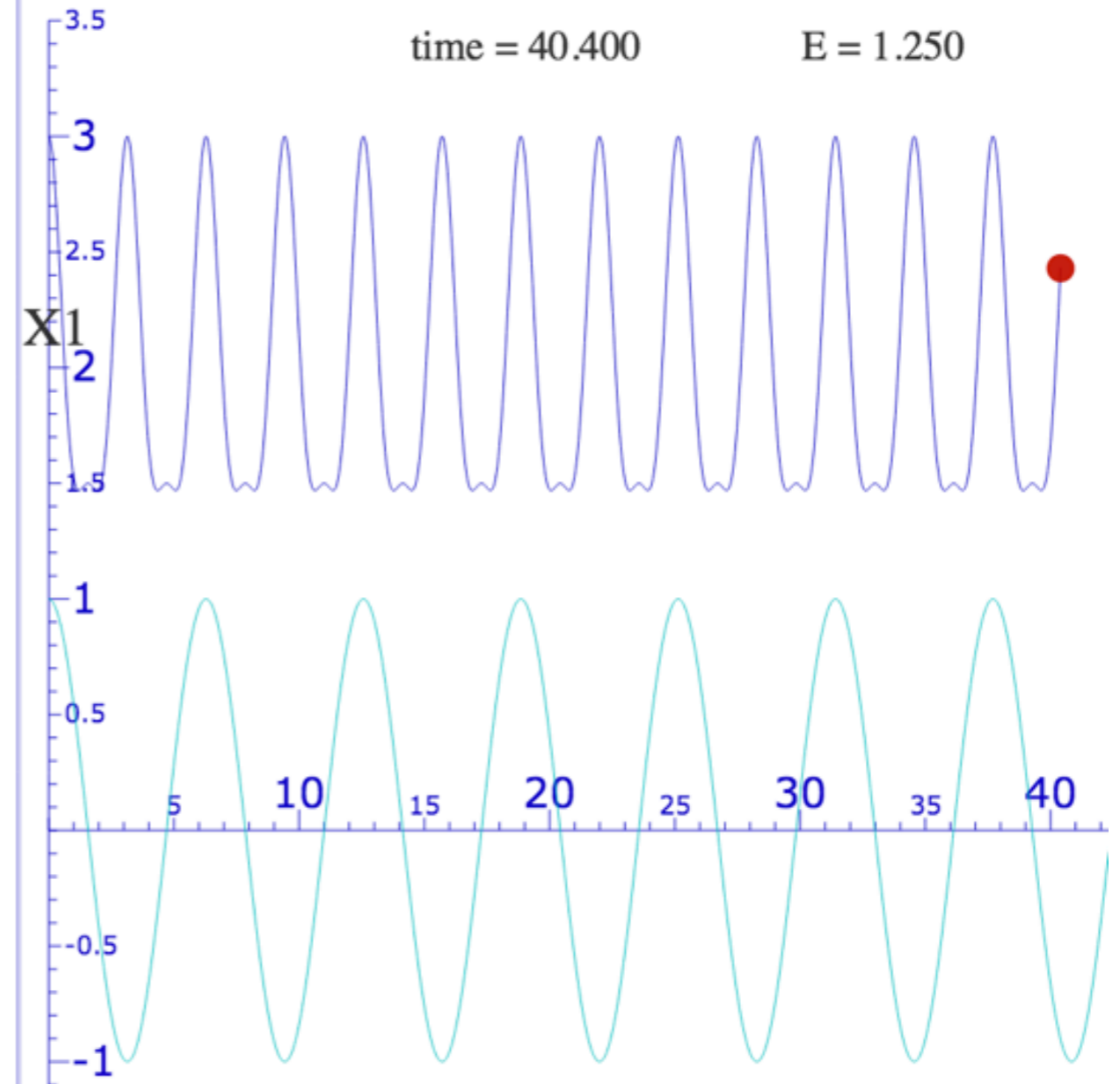
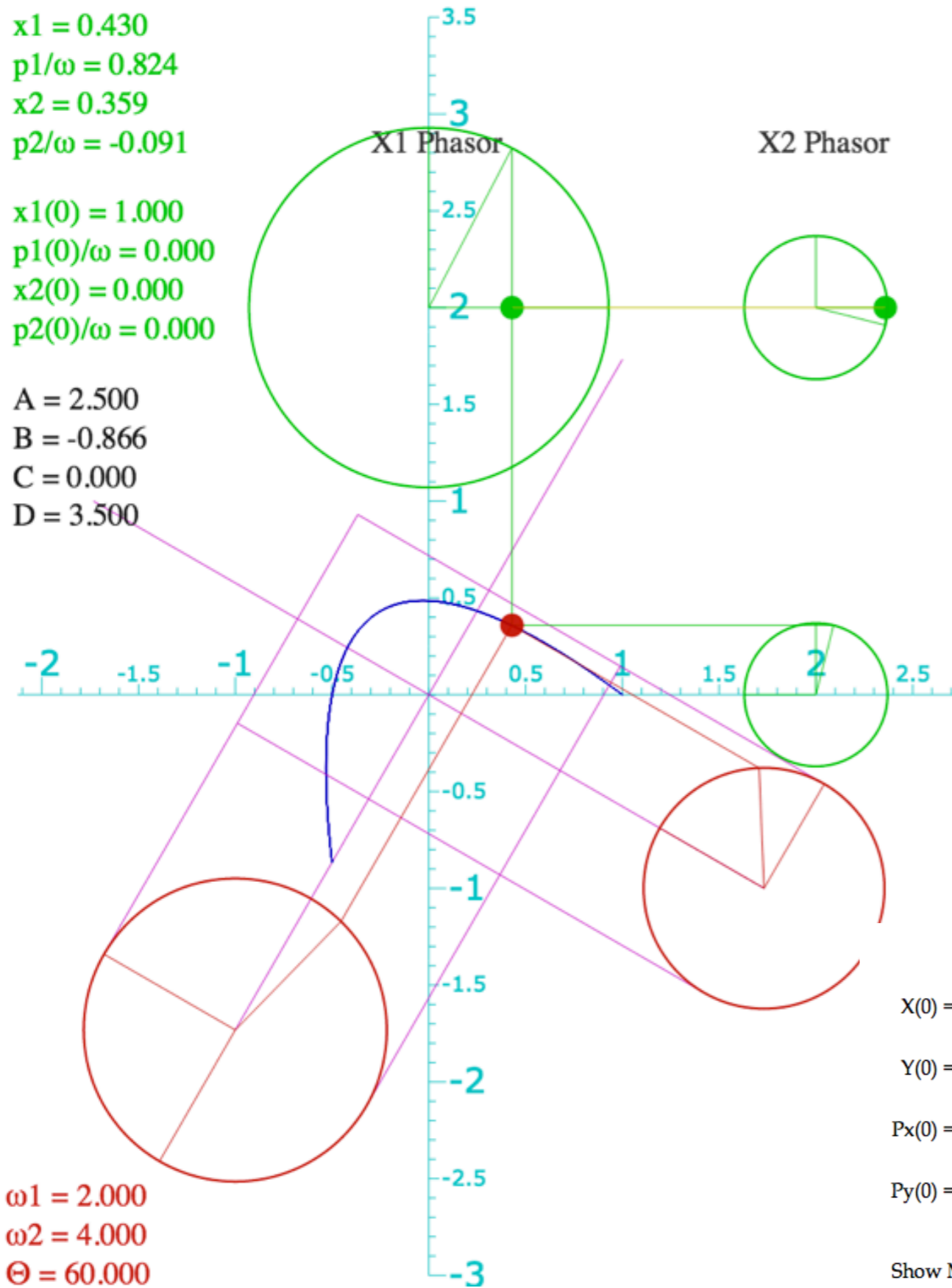
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Start Resume Reset T=0 Erase Paths Speed = x10[^]

$X(0) =$ $A =$ Number of Derivatives =

$Y(0) =$ $B =$

$Px(0) =$ $C =$

$Py(0) =$ $D =$

wantVectorHeads, wantTimeRateTangents

Show Multi-Phasor View Draw PE Levels Left Phasor Rides on Right Phasor

Show the YXT Phasor View Draw Box Lines Left Phasor Rides on Right Phasor

Draw Main Phasors Draw Modal Phasors Normalize Phasors Print $\omega1:\omega2$ fractions

Draw Vector Heads Draw Time Rate Tangents

More algebraic theory of matrix diagonalization

Review of matrix ortho-completeness (Lect.16 to 17 p. 56-85)

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

A proof of completeness using Lagrange interpolation formula (Truer-than-True)

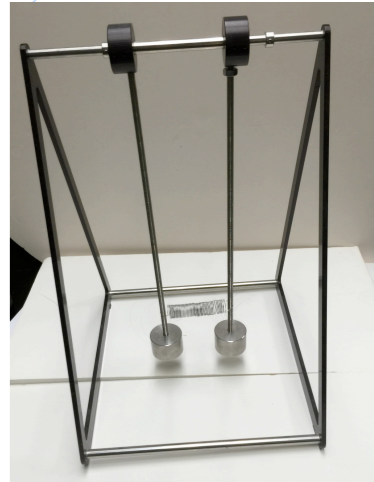
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed $\pi/2$ phase

2D-HO eigensolution example with asymmetric (ABD-Type) symmetry

Initial state projection, mixed mode beat dynamics with variable phase



➔ *ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$*
2D-HO Hamilton equations
Another approach to ABCD Hamilton-Pauli spinor equations (Lect. 16 p.27-33)

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*Both have 4 parameters
($2^2 = 2+2$)*

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*For $C=0$
Is form of 2D Hooke
harmonic oscillator*

$$\frac{\partial^2}{\partial t^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$\begin{aligned} \dot{x}_1 &= \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 & \dot{p}_1 &= -\frac{\partial H_c}{\partial x_1} = -(Ax_1 + Bx_2 + Cp_2) \\ \dot{x}_2 &= \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 & \dot{p}_2 &= -\frac{\partial H_c}{\partial x_2} = -(Bx_1 + Dx_2 - Cp_1) \end{aligned}$$

Finally a 2nd time derivative (Assume constant A, B, D , and *let $C=0$*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$

$$\begin{aligned} \ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A(Ax_1 + Bx_2 + Cp_2) - B(Bx_1 + Dx_2 - Cp_1) - C(Bp_1 + Dp_2 + Cx_1) \\ &= -(A^2 + B^2 + C^2)x_1 - (AB + BD)x_2 - C(A + D)p_2 \end{aligned}$$

$$\begin{aligned} \ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B(Ax_1 + Bx_2 + Cp_2) - D(Bx_1 + Dx_2 - Cp_1) + C(Ap_1 + Bp_2 - Cx_2) \\ &= -(AB + BD)x_1 - (B^2 + D^2 + C^2)x_2 + C(A + D)p_1 \end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For $C=0$ Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus **Classical 2DHO:** $\partial^2_t \mathbf{x} = -\mathbf{K} \cdot \mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

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First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

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Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like “square-root” of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle}$

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Recall simple derivation in Lect. 16 p.32-33)