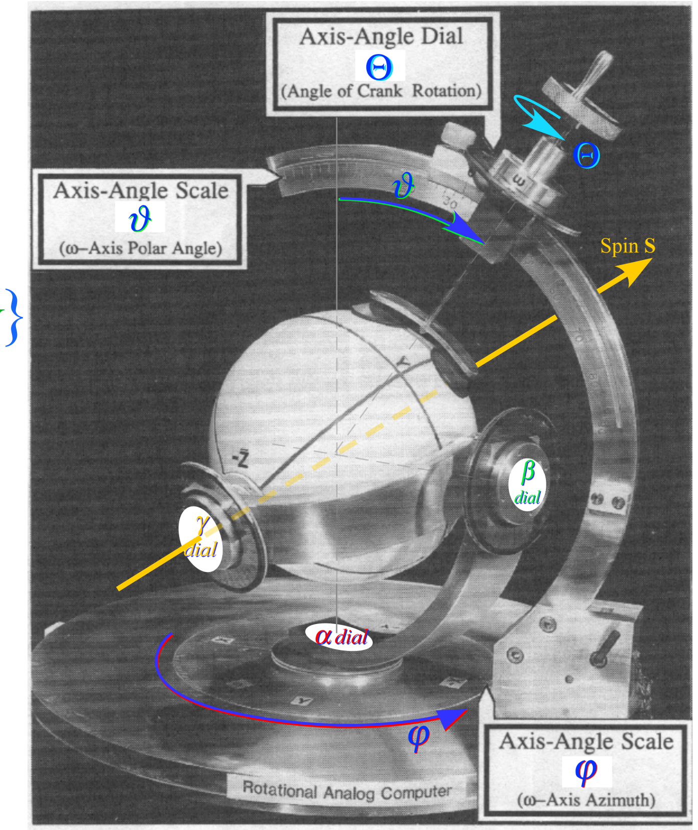
Lecture 18
Thur. 3.17.2016

Mechanical analogs of quantum 2-state eigenmodes and dynamics

(Ch. 3-4 of Unit 2)



(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

σ_A*-products 3D vector analysis and "Crazy-Thing-Theorem"*

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent ($\mathbf{P \cdot P = P}$ *) projectors (how eigenvalues⇒eigenvectors)*

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

U(2)⊃C₂ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

Circular-chiral-cyclotron (C-Type) symmetry

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

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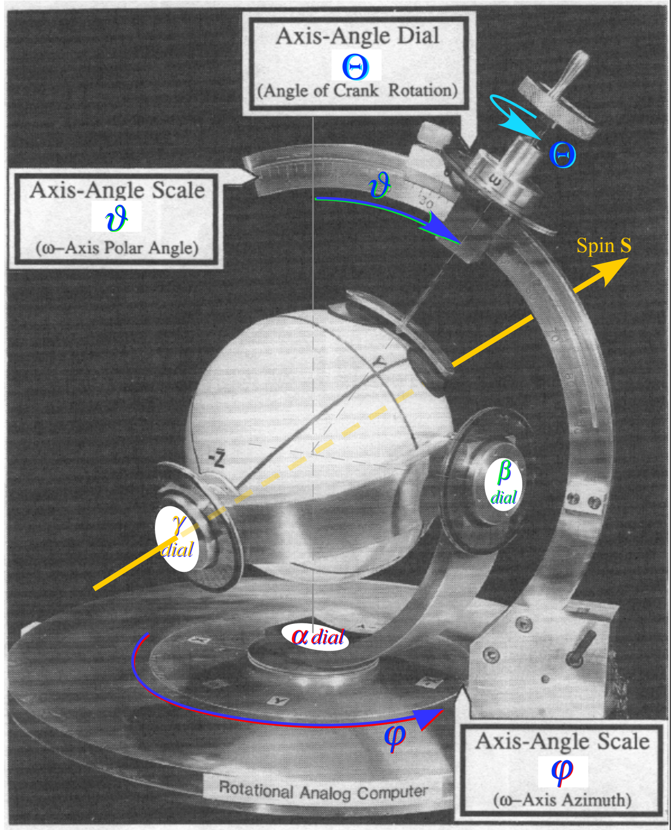

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Classical Newton-Hooke-Stokes equation $|\ddot{\mathbf{z}}\rangle = -\mathbf{K}\cdot|\mathbf{z}\rangle$ *Quantum Schrodinger-Pauli equation* $i\hbar|\dot{\Psi}\rangle = \mathbf{H}\cdot|\Psi\rangle$

$$\frac{d^2}{dt^2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{versus} \quad \hbar i \frac{d}{dt}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

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2D classical HO same as the U(2) quantum 2-state system

...if we set **K**-spring matrix to squared quantum operator $\mathbf{H}^2$

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...if we set square root $\sqrt{\mathbf{K}}$*-spring matrix to quantum operator* $\mathbf{H}$

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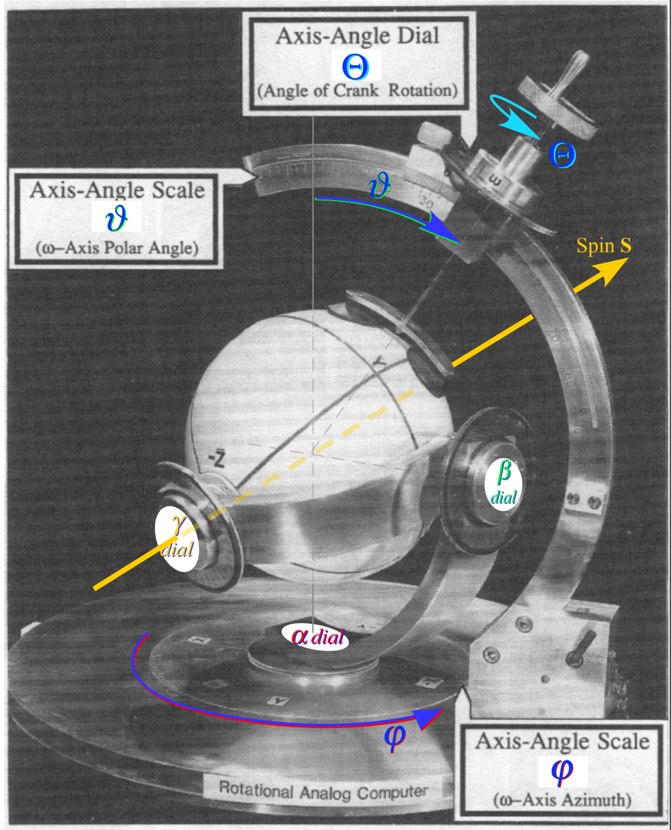
How the heck do you take a square root of a MATRIX?? (stay tuned!)



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Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

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$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

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CAUTION (stretched waves)

GO (moving waves)

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Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

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In 1843 Hamilton invents *quaternions* {**1, i, j, k**}. σ_μ related by *i*-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

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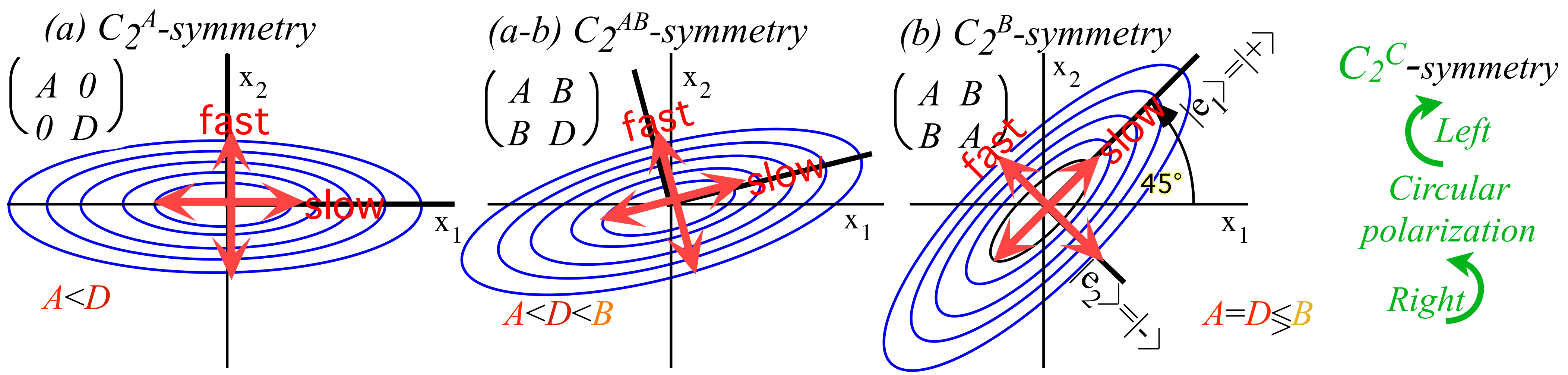

Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

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Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

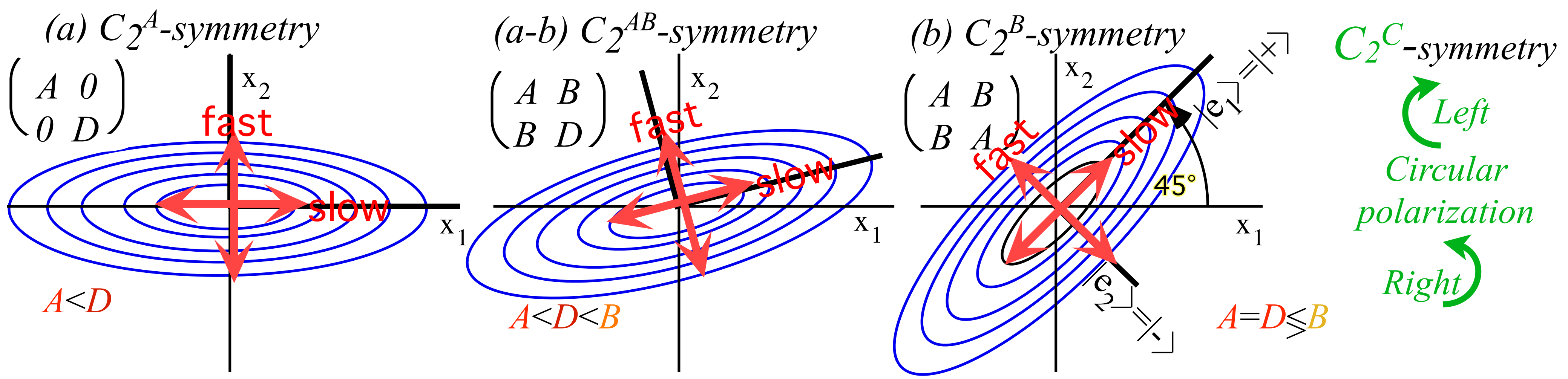


Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

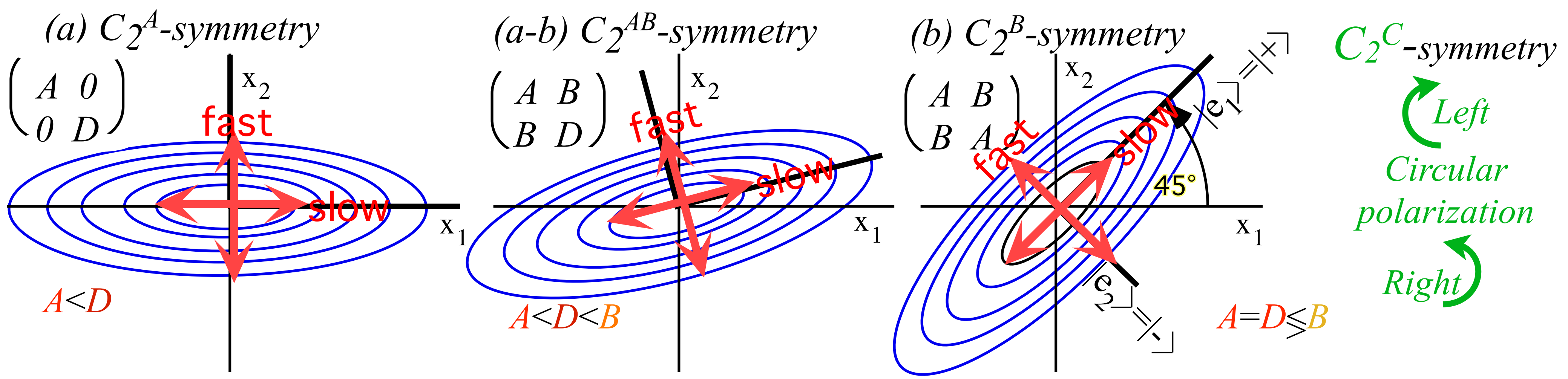
The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

In 1843 Hamilton invents *quaternions* {**1, i, j, k**}. σ_μ related by *i*-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X^2=\sigma_Y^2=\sigma_Z^2=+1$) (Each makes a cyclic C_2 group $C_2^A=\{1,\sigma_A\}$, $C_2^B=\{1,\sigma_B\}$, or $C_2^C=\{1,\sigma_C\}$.)



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(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

➔ *Algebra of Hamilton/Pauli hypercomplex operators* $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

σ_A-products 3D vector analysis and "Crazy-Thing-Theorem"

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Idempotent ($\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$) projectors (how eigenvalues⇒eigenvectors)

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Asymmetric-diagonal (AD-Type) symmetry

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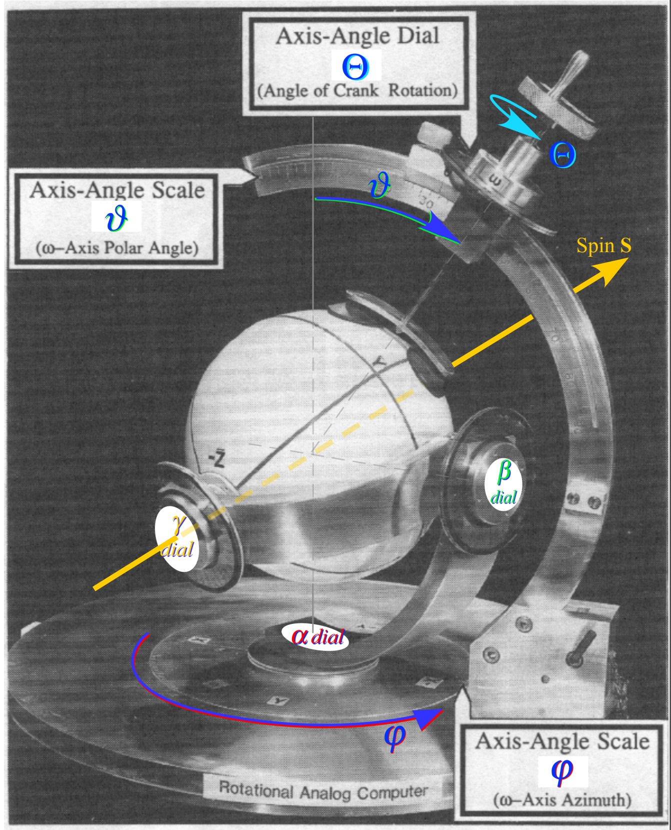

σ_N-products and 3D vector analysis

Spinor operators $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ have powerful symmetry relations.

Each σ_N squares to **one** (unit matrix **1**$= (\sigma_N)^2$). Quaternions square to *minus*-**one** (–**1=i·i** *etc.*) like $i=\sqrt{-1}$.

$\sigma_Z \square \sigma_X$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$\varepsilon-tensor\ form$

$$\sigma_K \sigma_L = i\varepsilon_{KLM}\sigma_M + \delta_{KL}\mathbf{1}$$

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	σ_X	σ_Y	σ_Z
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$$= \begin{matrix} a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z \\ +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z \\ +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z \end{matrix} = \begin{matrix} a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

Due to symmetry property called *anti-commutation*: $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ *,etc.*

$$\sigma_a{}^2 = \begin{matrix} a_X{}^2\mathbf{1} & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ -a_X a_Y\sigma_X\sigma_Y & +a_Y{}^2\mathbf{1} & +a_Y a_Z\sigma_Y\sigma_Z \\ -a_X a_Z\sigma_X\sigma_Z & -a_Y a_Z\sigma_Y\sigma_Z & +a_Z{}^2\mathbf{1} \end{matrix}$$

σ_N-products and 3D vector analysis

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(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing *ABCD* Hamilton Pauli spinor symmetry expansion

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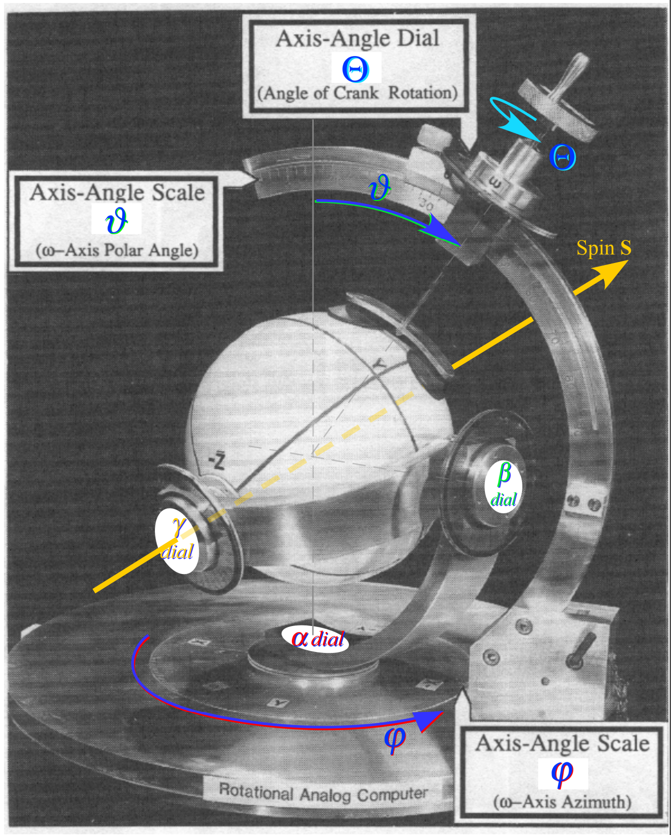

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$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
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σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$\varepsilon - tensor\ form$

$$\sigma_K \sigma_L = i\varepsilon_{KLM}\sigma_M + \delta_{KL}\mathbf{1}$$

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Same for spinor components based on *any* unit vector $\hat{\mathbf{a}} = (a_X, a_Y, a_Z)$ for which $\hat{\mathbf{a}} \bullet \hat{\mathbf{a}} = 1 = a_X^2 + a_Y^2 + a_Z^2$.

To see this just try it out on any $\hat{\mathbf{a}}$-component: $\sigma_a = \boldsymbol{\sigma} \bullet \hat{\mathbf{a}} = a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z$.

$$\sigma_a{}^2 = (\boldsymbol{\sigma} \bullet \hat{\mathbf{a}})(\boldsymbol{\sigma} \bullet \hat{\mathbf{a}}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)$$

$$= \begin{matrix} a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z \\ +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z \\ +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z \end{matrix} = \begin{matrix} a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

Due to symmetry property called *anti-commutation*: $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ *,etc.*

$$\sigma_a{}^2 = \begin{matrix} a_X{}^2\mathbf{1} & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ -a_X a_Y\sigma_X\sigma_Y & +a_Y{}^2\mathbf{1} & +a_Y a_Z\sigma_Y\sigma_Z \\ -a_X a_Z\sigma_X\sigma_Z & -a_Y a_Z\sigma_Y\sigma_Z & +a_Z{}^2\mathbf{1} \end{matrix} = (a_X{}^2 + a_Y{}^2 + a_Z{}^2)\mathbf{1} = \mathbf{1}$$

$$Result: \quad \sigma_a{}^2 = \mathbf{1}$$

σ_N-products and 3D vector analysis

Spinor operators $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ have powerful symmetry relations.

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Spinors do vector analysis for *any* 3D-vectors $\mathbf{a} = (a_X, a_Y, a_Z)$ and $\mathbf{b} = (b_X, b_Y, b_Z)$

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$$\begin{array}{llll} & a_X b_X \mathbf{1} & +a_X b_Y \sigma_X \sigma_Y & -a_X b_Z \sigma_Z \sigma_X & & +i(a_Y b_Z - a_Z b_Y)\sigma_X \\ = & -a_Y b_X \sigma_X \sigma_Y & +a_Y b_Y \mathbf{1} & +a_Y b_Z \sigma_Y \sigma_Z & = (a_X b_X + a_Y b_Y + a_Z b_Z)\mathbf{1} & +i(a_Z b_X - a_X b_Z)\sigma_Y \\ & +a_Z b_X \sigma_Z \sigma_X & -a_Z b_X \sigma_Y \sigma_Z & +a_Z b_Z \mathbf{1} & & +i(a_X b_Y - a_Y b_X)\sigma_Z \end{array}$$

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$$\begin{matrix} & a_X b_X \mathbf{1} & +a_X b_Y \sigma_X \sigma_Y & -a_X b_Z \sigma_Z \sigma_X & & +i(a_Y b_Z - a_Z b_Y)\sigma_X \\ = & -a_Y b_X \sigma_X \sigma_Y & +a_Y b_Y \mathbf{1} & +a_Y b_Z \sigma_Y \sigma_Z & = (a_X b_X + a_Y b_Y + a_Z b_Z)\mathbf{1} & +i(a_Z b_X - a_X b_Z)\sigma_Y \\ & +a_Z b_X \sigma_Z \sigma_X & -a_Z b_X \sigma_Y \sigma_Z & +a_Z b_Z \mathbf{1} & & +i(a_X b_Y - a_Y b_X)\sigma_Z \end{matrix}$$

Write the product in Gibbs notation. (This is where Gibbs *got* his {**i,j,k**} notation!)

$$\sigma_a \sigma_b = (\sigma \bullet \mathbf{a})(\sigma \bullet \mathbf{b}) = \quad (\mathbf{a} \bullet \mathbf{b})\mathbf{1} \quad + \quad i(\mathbf{a} \times \mathbf{b}) \bullet \sigma$$

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$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=i\sigma_Y$$

$\sigma_X \cdot \sigma_Z$
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=-i\sigma_Y$$

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$\sigma_X \cdot \sigma_Y$
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}i&0\\0&-i\end{pmatrix}=i\sigma_Z$$

$\sigma_Y \cdot \sigma_X$
$$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}-i&0\\0&i\end{pmatrix}=-i\sigma_Z$$

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$$= \begin{matrix} a_Xb_X\mathbf{1} & +a_Xb_Y\sigma_X\sigma_Y & -a_Xb_Z\sigma_Z\sigma_X \\ -a_Yb_X\sigma_X\sigma_Y & +a_Yb_Y\mathbf{1} & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & -a_Zb_X\sigma_Y\sigma_Z & +a_Zb_Z\mathbf{1}\end{matrix} = (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} \begin{matrix} +i(a_Yb_Z - a_Zb_Y)\sigma_X \\ +i(a_Zb_X - a_Xb_Z)\sigma_Y \\ +i(a_Xb_Y - a_Yb_X)\sigma_Z\end{matrix}$$

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$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = \quad (\mathbf{a}\bullet\mathbf{b})\mathbf{1} \quad + \quad i(\mathbf{a}\times\mathbf{b})\bullet\sigma$$

(Recall (1.10.29). in complex variable Ch. 10.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
$$= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) = \ (\mathbf{A}\bullet\mathbf{B}) \ + \ i(\mathbf{A}\times\mathbf{B})_Z$$

(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

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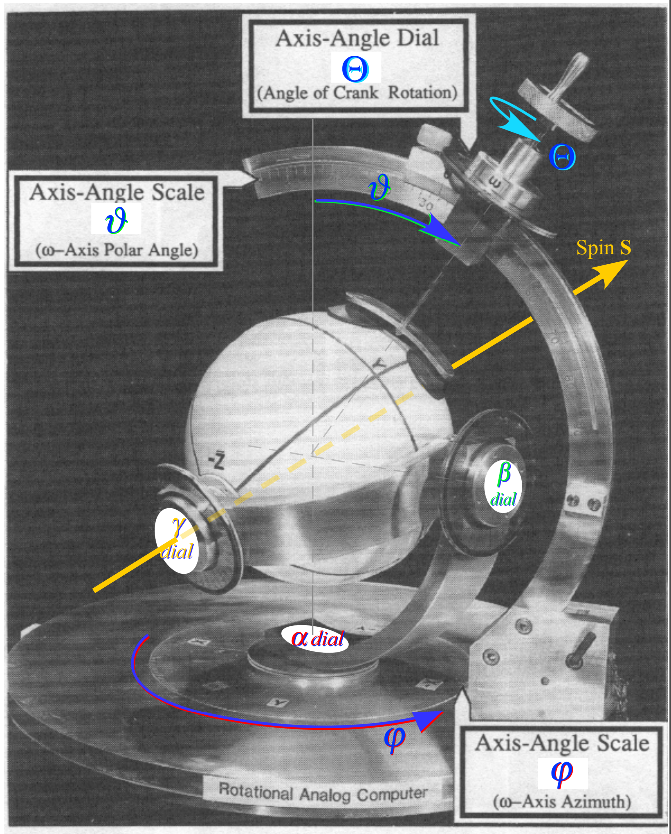

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent ($\mathbf{P \cdot P = P}$) projectors (how eigenvalues⇒eigenvectors)

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

$U(2) \supset C_2$ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

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Mixed ABCD symmetry examples

More theory of matrix diagonalization

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

The "Crazy-Thing-Theorem"

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall (1.10.17).)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$
$$-i(\varphi + \frac{1}{3!}\varphi^3 \cdots) \qquad -i(\sin\varphi)$$

Note even powers of *(-i)* are ±*1* and odd powers of *(-i)* are ±*i*.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

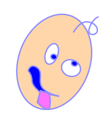
$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteqdot (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi\ -\ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi\ -\ i\ \sigma_\varphi \sin\varphi$

Here:  = -*i*

Crazy thing is just -√-1

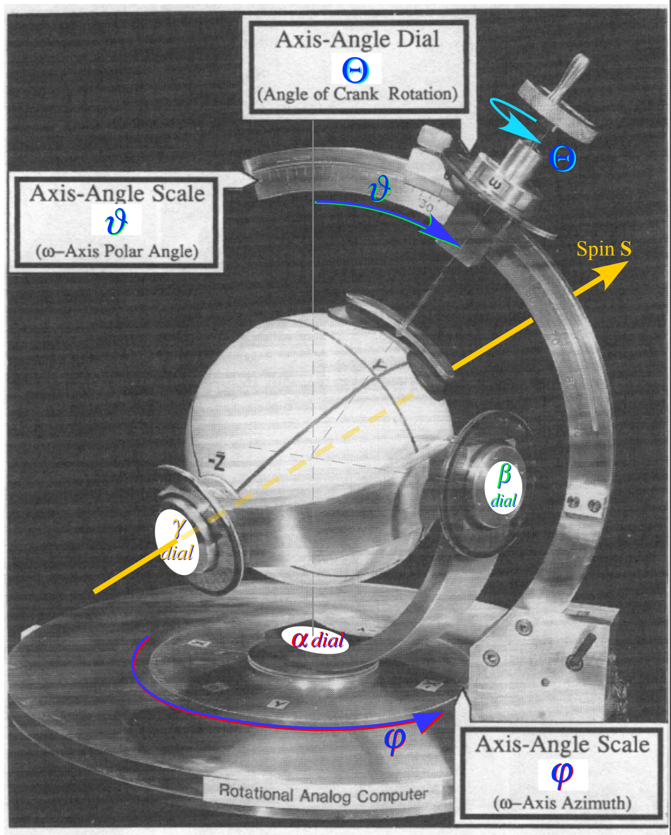
Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\text{crazy thing})\varphi}$ = **1**cosφ + (crazy thing)sinφ



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Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\varepsilon\begin{pmatrix} x \\ y \end{pmatrix}\quad \text{or:}\quad \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

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Trying to solve by Kramer's inversion:

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$$0 = \det|\mathbf{M} - \varepsilon \cdot \mathbf{1}| = \det\left|\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \varepsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|$$

$$0 = (4-\varepsilon)(2-\varepsilon) - 1\cdot 3 = 8 - 6\varepsilon + \varepsilon^2 - 1\cdot 3 = \varepsilon^2 - 6\varepsilon + 5$$

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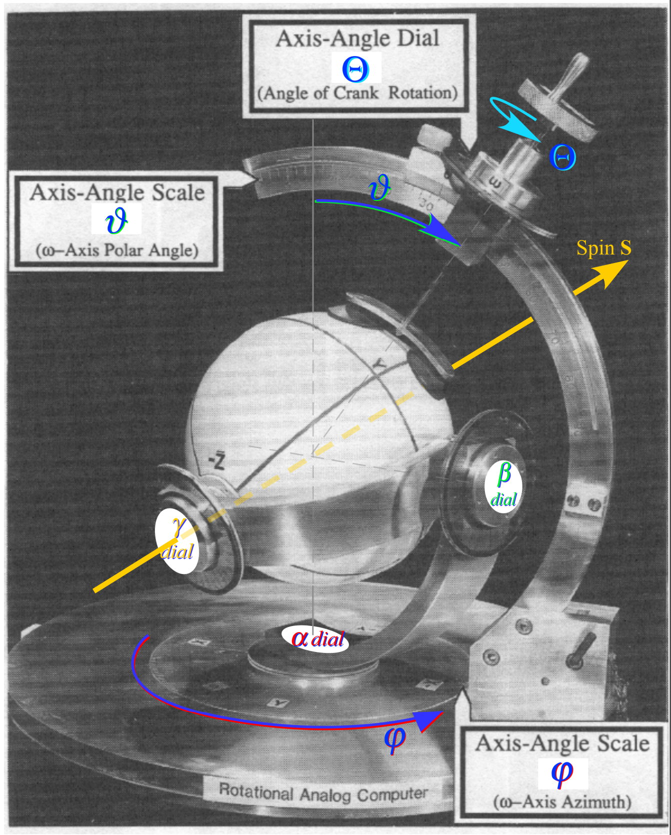

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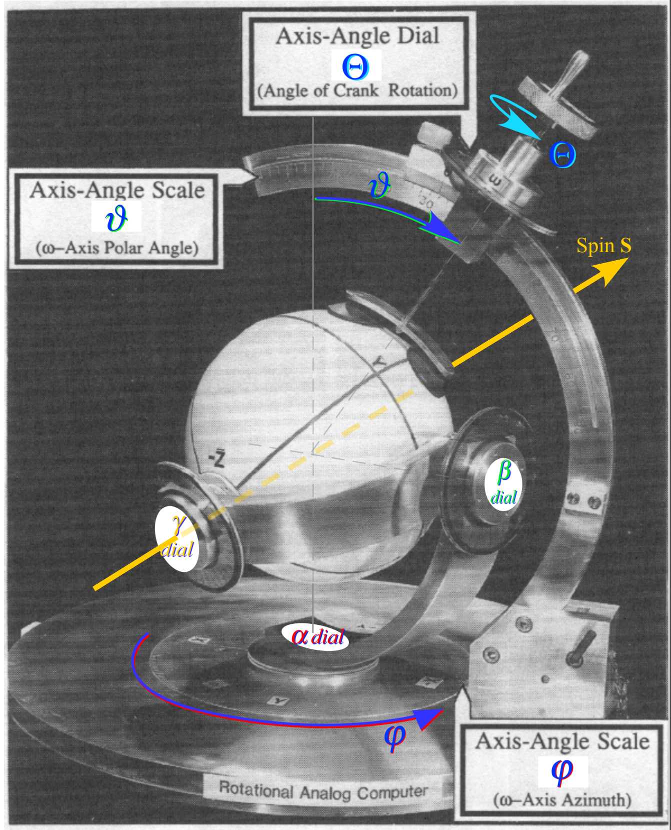

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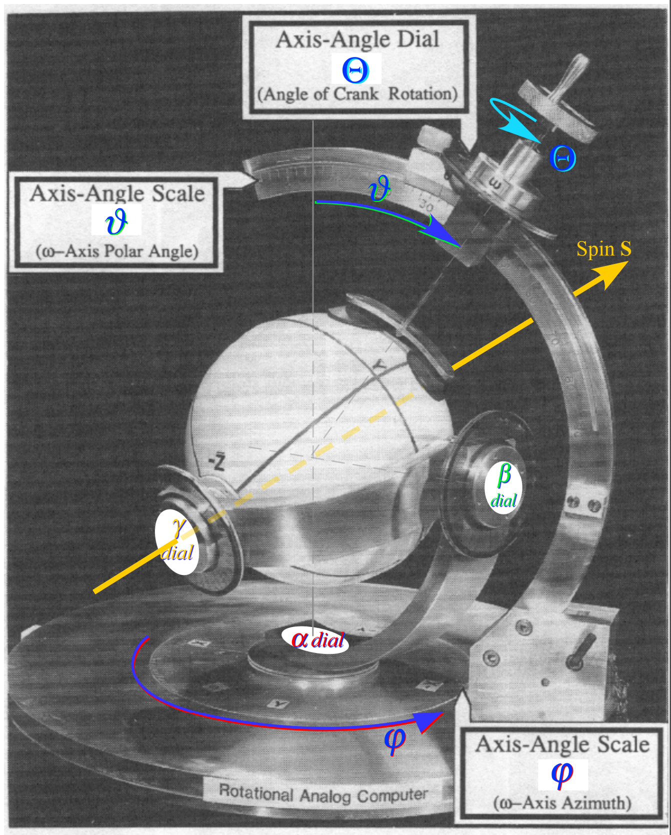

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Replace j^{th} HC-factor by (**1**) to make *projection operators* .

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
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Notice $\mathbf{p}_k$ *commutes with* **M**,..
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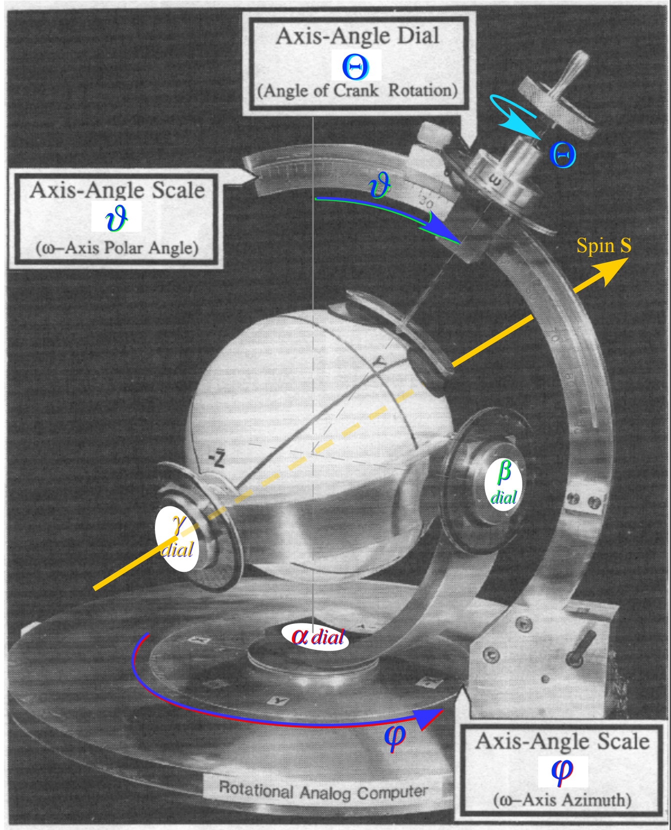

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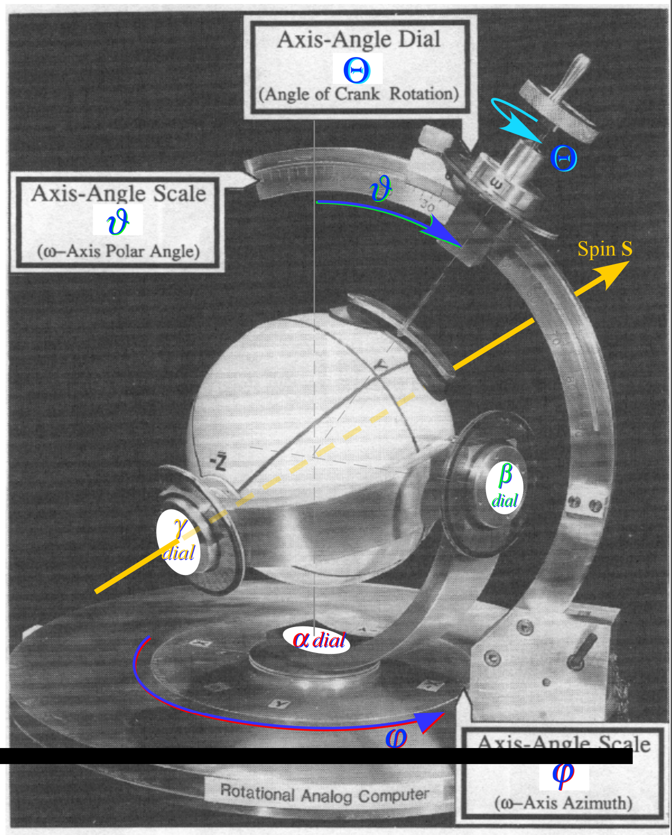

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"Gauge" scale factors that only affect plots

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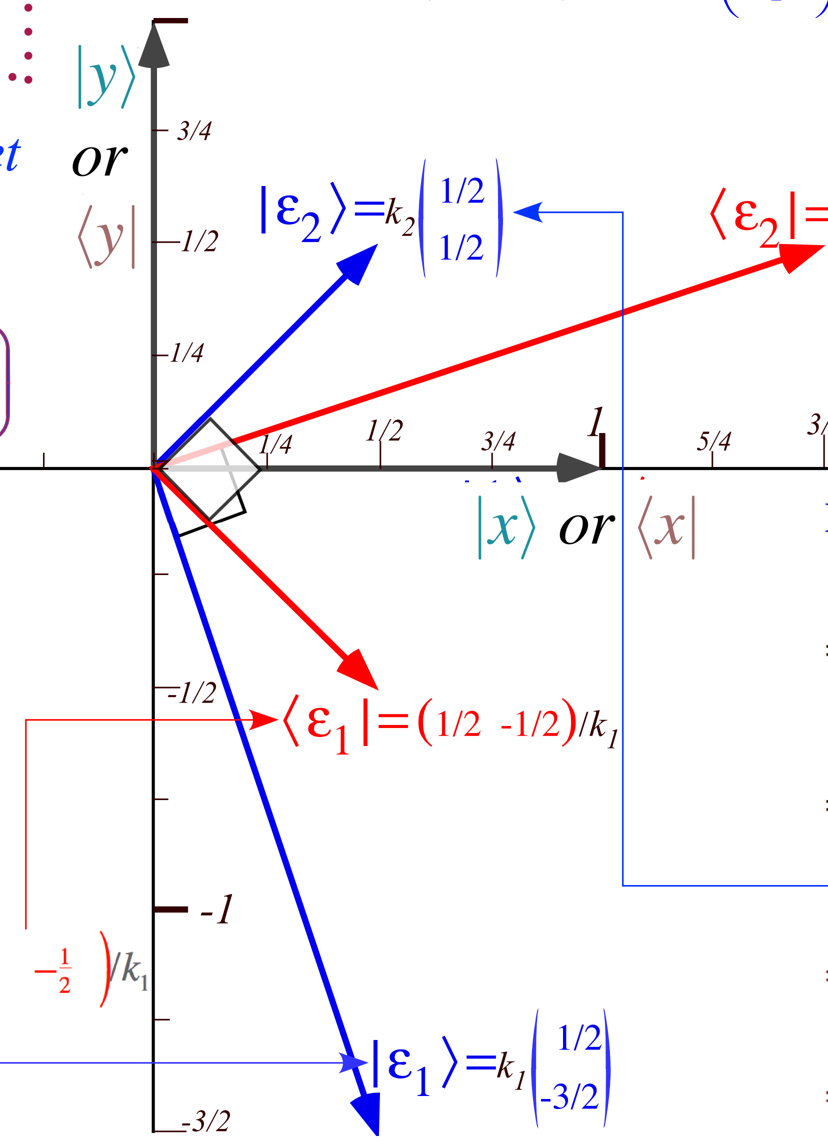

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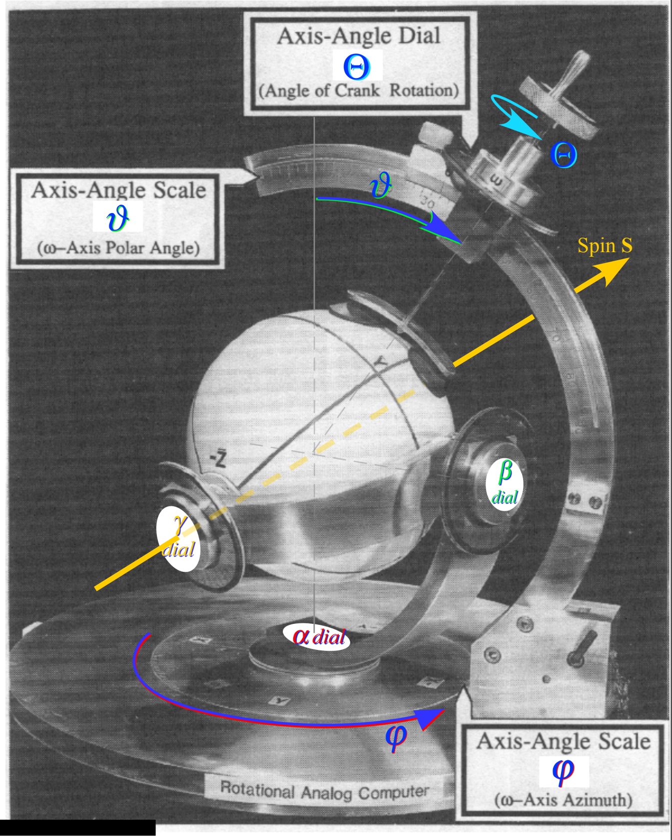

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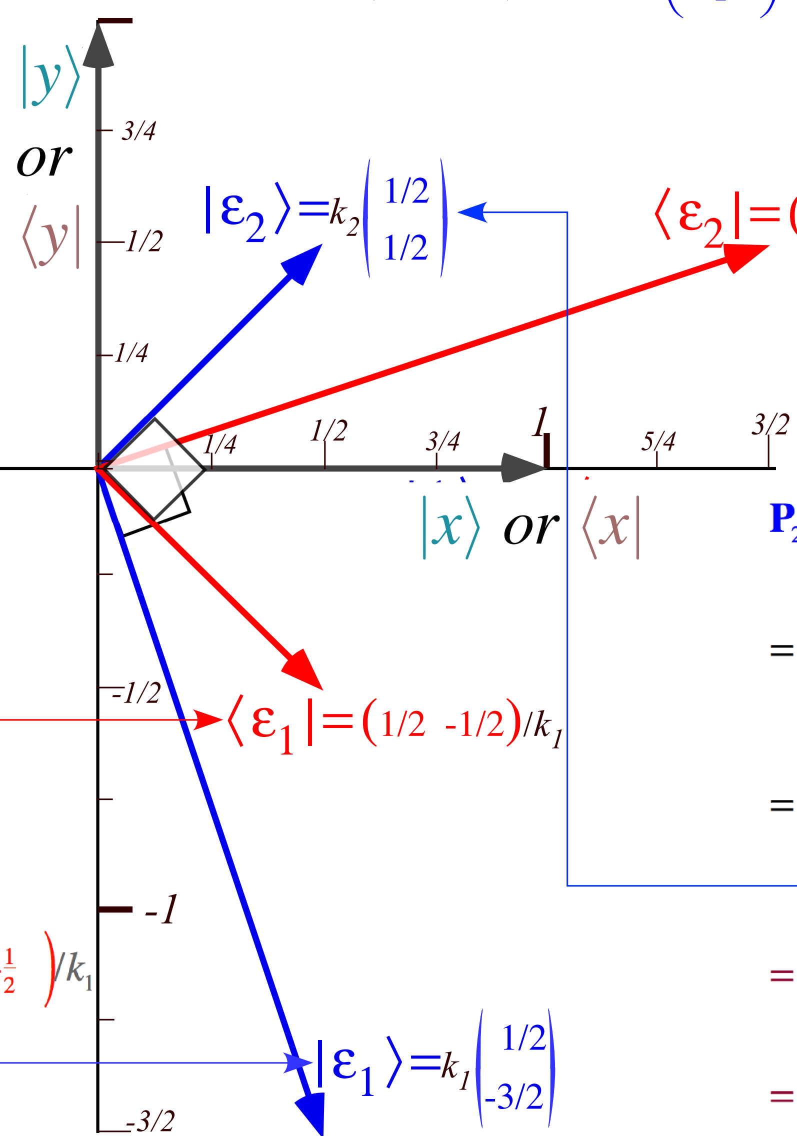

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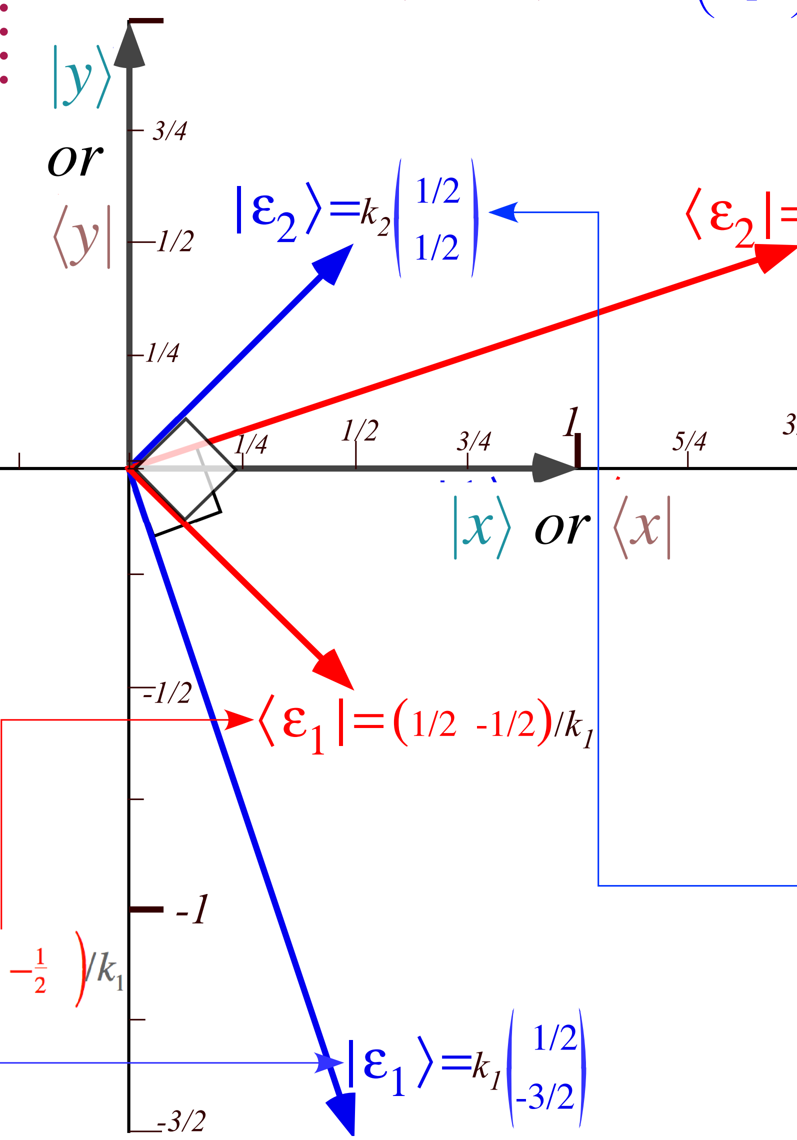

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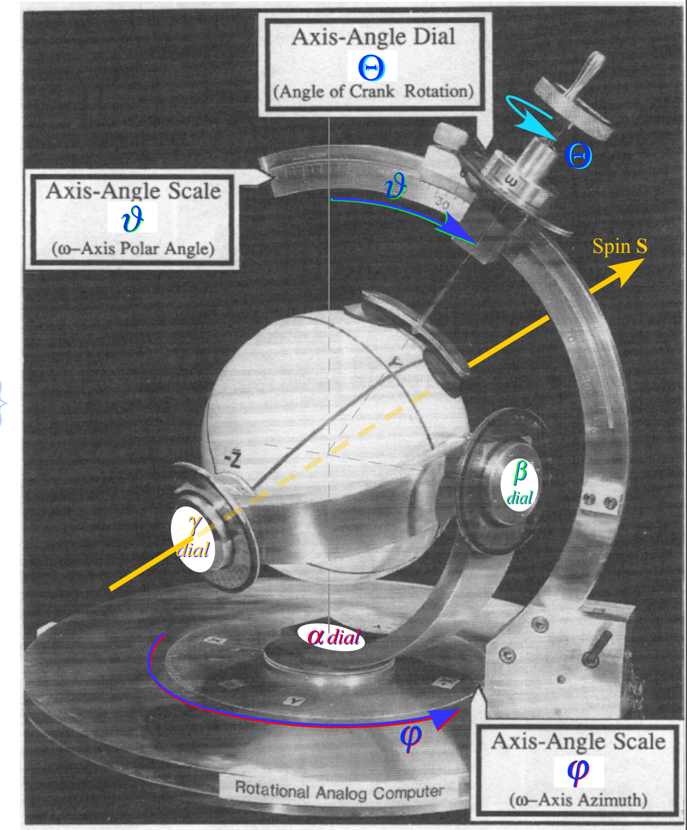

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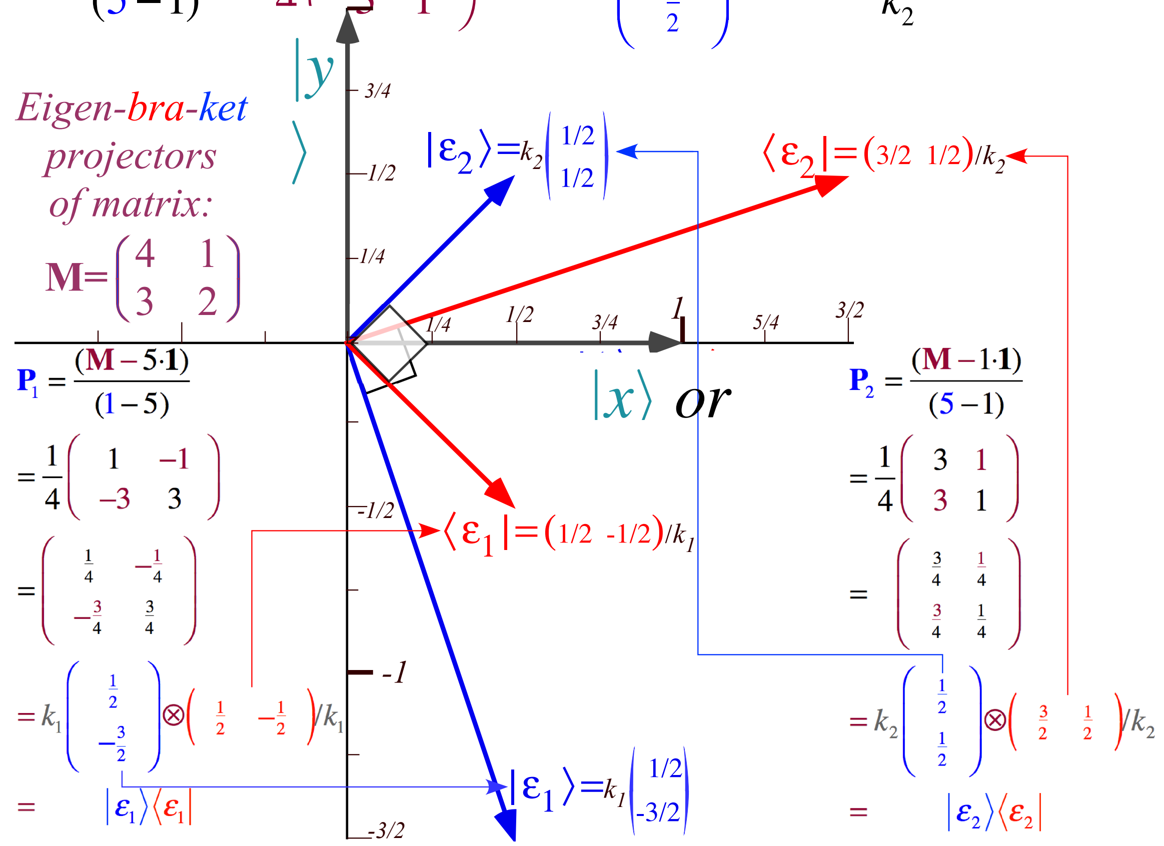

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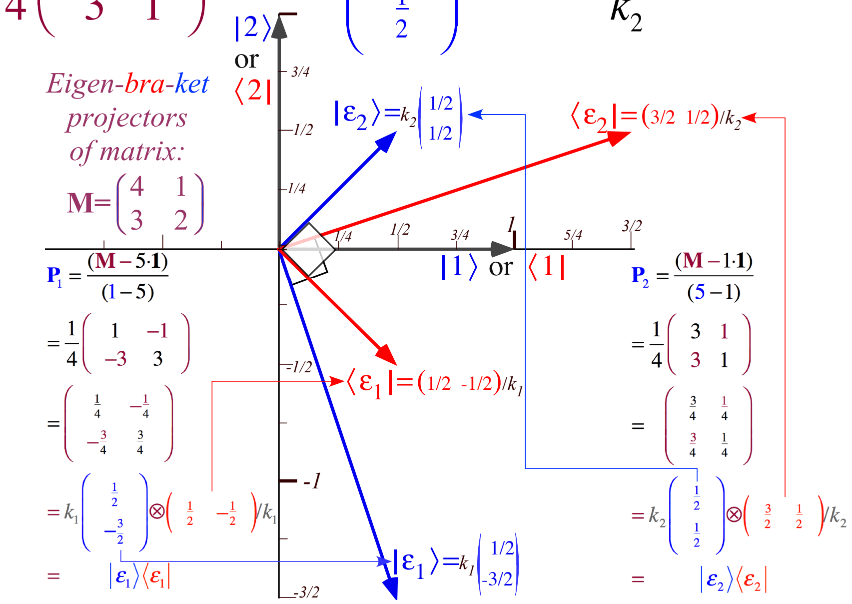

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Eigenvector orthonormality and completeness

Spectral Decompositions

➤ Functional spectral decomposition ←

$U(2)\supset C_2$ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

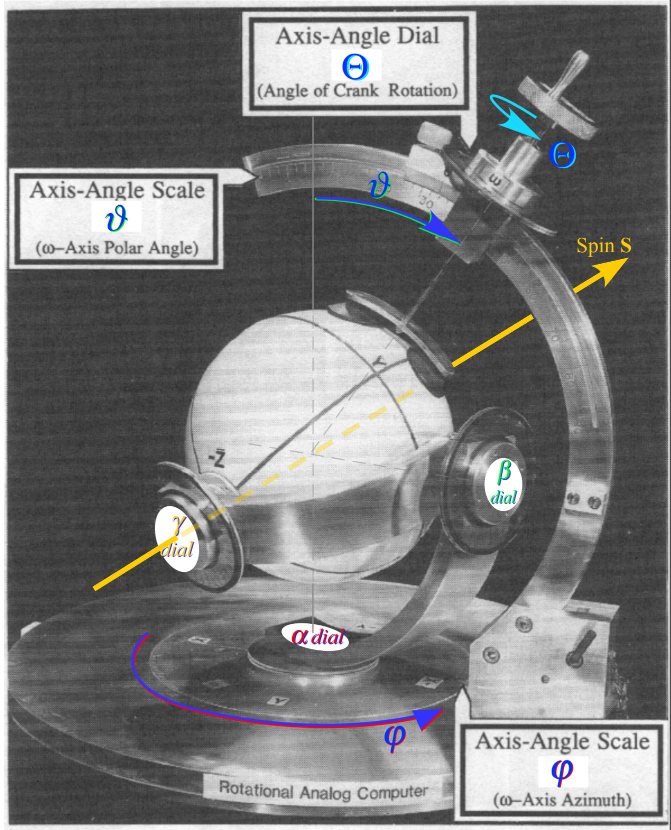
Circular-chiral-cycloton (C-Type) symmetry

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors



Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

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Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}\otimes\frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
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$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + ... + f(\varepsilon_n)\mathbf{P}_n$$

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Example:

$$\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1+3\cdot5^{50}&5^{50}-1\\3\cdot5^{50}-3&5^{50}+3\end{pmatrix}$$

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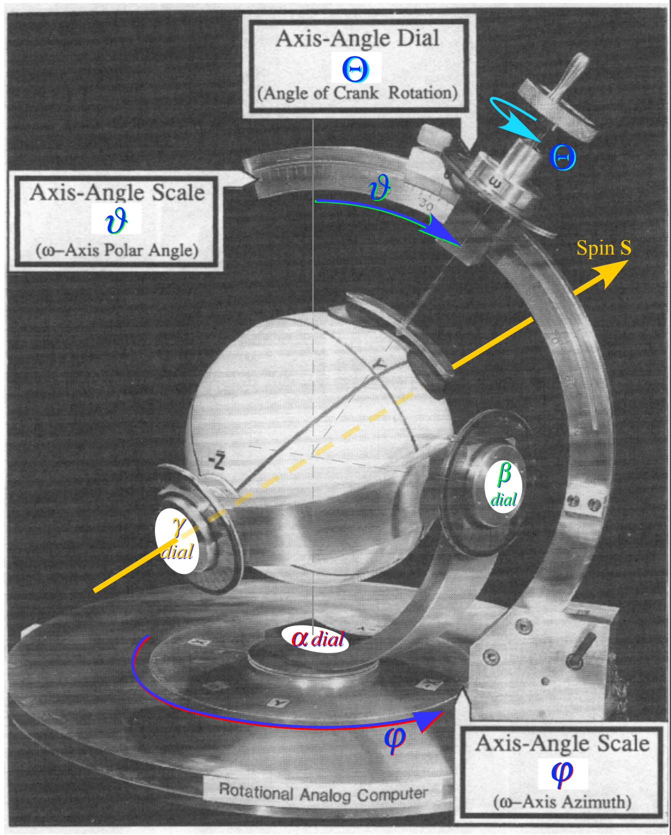
$$\sqrt{\mathbf{M}}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=\pm\sqrt{1}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}\pm\sqrt{5}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$



(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

σ_A-products 3D vector analysis and "Crazy-Thing-Theorem"

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

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Each H-matrix $\mathbf{H}=\boldsymbol{\sigma}_a+\omega_0\mathbf{1}$ *has a 3D-term* $\boldsymbol{\sigma}_a=\sigma\bullet\mathbf{a}$

$$\mathbf{H}=\frac{A-D}{2}\,\boldsymbol{\sigma}_A+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\,\boldsymbol{\sigma}_0$$

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Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A{}^2+\omega_B{}^2+\omega_C{}^2}=\sqrt{\frac{(A-D)^2}{4}+B^2+C^2}$

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$$\mathbf{H}=\frac{A-D}{2}\,\boldsymbol{\sigma}_A+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\,\boldsymbol{\sigma}_0$$

$$=\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\omega_A\,\boldsymbol{\sigma}_A+\omega_B\,\boldsymbol{\sigma}_B+\omega_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\,\mathbf{1}$$

Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A{}^2+\omega_B{}^2+\omega_C{}^2}=\sqrt{\frac{(A-D)^2}{4}+B^2+C^2}$

$$\frac{\mathbf{H}}{\omega_{ABCD}}=\frac{A-D}{2\omega_{ABCD}}\,\boldsymbol{\sigma}_A+\frac{B}{\omega_{ABCD}}\,\boldsymbol{\sigma}_B+\frac{C}{\omega_{ABCD}}\,\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\,\boldsymbol{\sigma}_0$$

$$=\hat{\omega}_A\,\boldsymbol{\sigma}_A+\hat{\omega}_B\,\boldsymbol{\sigma}_B+\hat{\omega}_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\,\mathbf{1}$$

$$=\hat{\omega}_A\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\hat{\omega}_B\begin{pmatrix}0&1\\1&0\end{pmatrix}+\hat{\omega}_C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2\omega_{ABCD}}\,\mathbf{1}$$

$$=\begin{pmatrix}\hat{\omega}_A&\hat{\omega}_B-i\hat{\omega}_C\\\hat{\omega}_B+i\hat{\omega}_C&-\hat{\omega}_A\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\sigma_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\hat{\omega}_0\mathbf{1}$$

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$$=\begin{pmatrix}\hat{\omega}_A&\hat{\omega}_B-i\hat{\omega}_C\\\hat{\omega}_B+i\hat{\omega}_C&-\hat{\omega}_A\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\sigma_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\hat{\omega}_0\mathbf{1}$$

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$$=\hat{\omega}_A\,\boldsymbol{\sigma}_A+\hat{\omega}_B\,\boldsymbol{\sigma}_B+\hat{\omega}_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

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$$\mathbf{h}(\mathbf{h}-\mathbf{1})=(-1)(\mathbf{h}-\mathbf{1})$$

$$\mathbf{h}(\mathbf{h}+\mathbf{1})=(+1)(\mathbf{h}+\mathbf{1})$$

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high eigenfrequency $\hat{\omega}_0+\omega_{ABCD}$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD-}=\tfrac{1}{2}\begin{pmatrix}\hat{\omega}_A-1\\\hat{\omega}_B+i\hat{\omega}_C\end{pmatrix}$$

low eigenfrequency $\hat{\omega}_0-\omega_{ABCD}$

Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A^2+\omega_B^2+\omega_C^2}=\sqrt{\frac{(A-D)^2}{4}+B^2+C^2}$

$$\frac{\mathbf{H}}{\omega_{ABCD}}=\frac{A-D}{2\omega_{ABCD}}\boldsymbol{\sigma}_A+\frac{B}{\omega_{ABCD}}\boldsymbol{\sigma}_B+\frac{C}{\omega_{ABCD}}\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\boldsymbol{\sigma}_0$$

$$=\hat{\omega}_A\boldsymbol{\sigma}_A+\hat{\omega}_B\boldsymbol{\sigma}_B+\hat{\omega}_C\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

$$=\hat{\omega}_A\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\hat{\omega}_B\begin{pmatrix}0&1\\1&0\end{pmatrix}+\hat{\omega}_C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

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high eigenfrequency $\hat{\omega}_0+\omega_{ABCD}$

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Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A{}^2+\omega_B{}^2+\omega_C{}^2}=\sqrt{\frac{(A-D)^2}{4}+B^2+C^2}$

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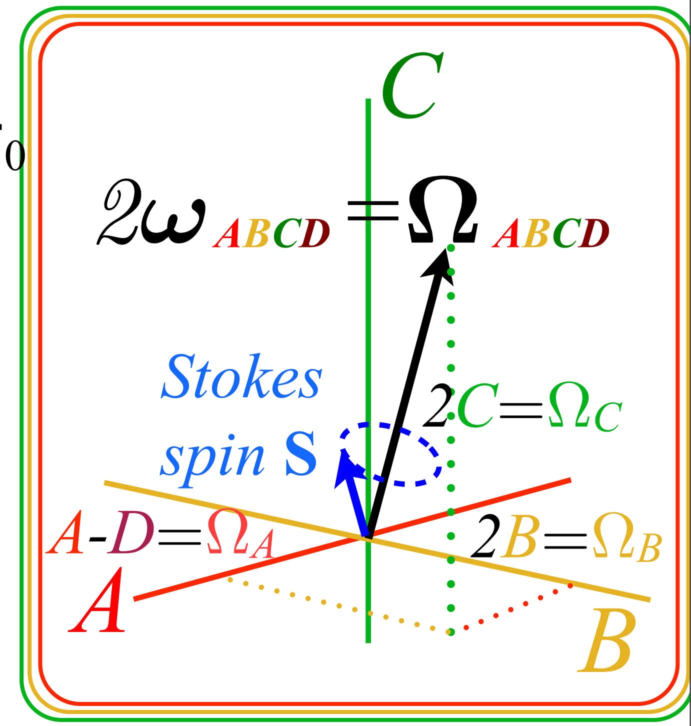
$$=\begin{pmatrix}\hat{\omega}_A&\hat{\omega}_B-i\hat{\omega}_C\\\hat{\omega}_B+i\hat{\omega}_C&-\hat{\omega}_A\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\hat{\omega}_0\mathbf{1}$$



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(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

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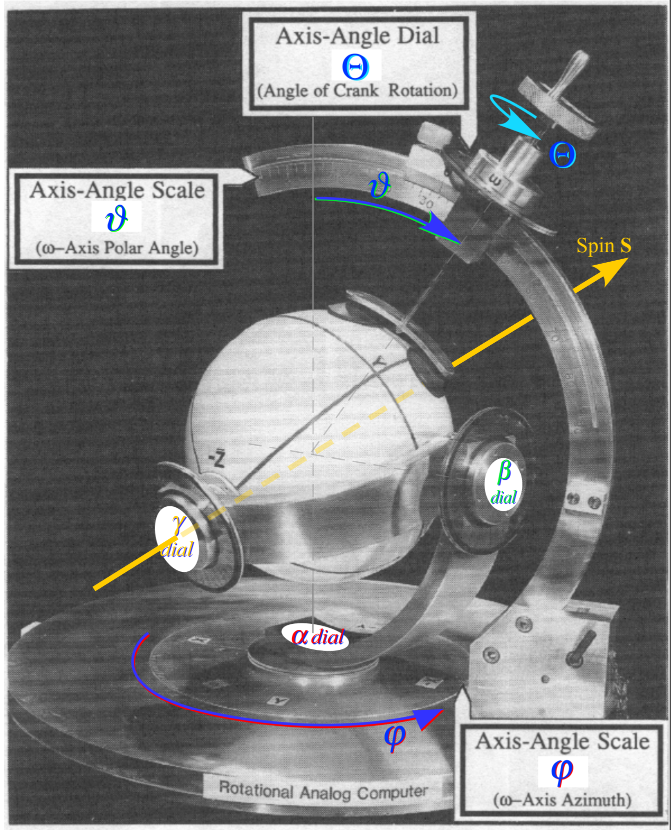

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

*Idempotent (**P·P=P**) projectors (how eigenvalues⇒eigenvectors)*

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

U(2)⊃C₂ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

➤ *Asymmetric-diagonal (AD-Type) symmetry* ⟵

Bilateral-balanced (B-Type) symmetry

Circular-chiral-cyclotron (C-Type) symmetry

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

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A-Type H-matrix $\mathbf{H}=\boldsymbol{\sigma}_A+\omega_0\mathbf{1}$

$$\mathbf{H}=\frac{A-D}{2}\boldsymbol{\sigma}_A+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\boldsymbol{\sigma}_0$$

$$=\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\omega_A\,\boldsymbol{\sigma}_A+\omega_B\,\boldsymbol{\sigma}_B+\omega_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\mathbf{1}$$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD+}=\tfrac{1}{2}\begin{pmatrix}\hat{\omega}_A+1\\0\end{pmatrix}$$

high eigenfrequency $\hat{\omega}_0+\omega_A$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD-}=\tfrac{1}{2}\begin{pmatrix}0\\-\hat{\omega}_A-1\end{pmatrix}$$

low eigenfrequency $\hat{\omega}_0-\omega_A$

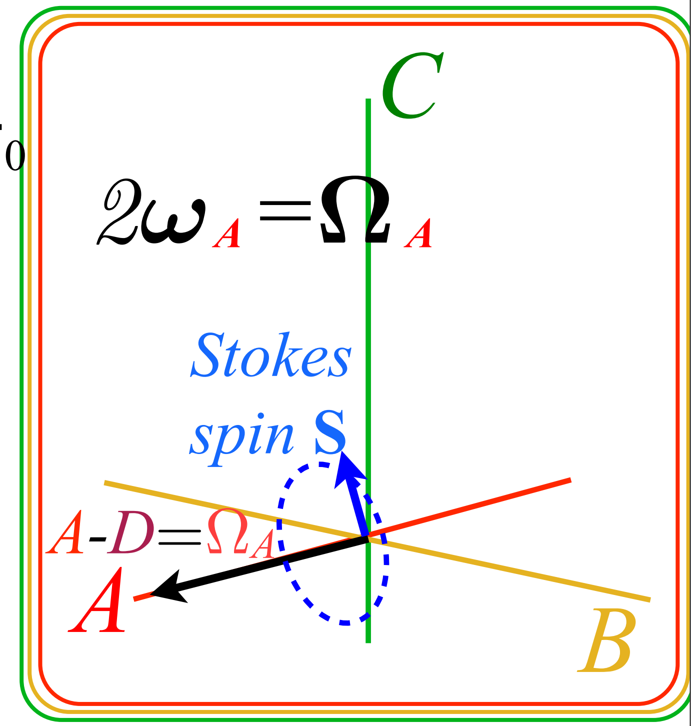
Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A^{\,2}}=\sqrt{\frac{(A-D)^2}{4}}$

$$\frac{\mathbf{H}}{\omega_{ABCD}}=\frac{A-D}{2\omega_{ABCD}}\boldsymbol{\sigma}_A+\frac{B}{\omega_{ABCD}}\boldsymbol{\sigma}_B+\frac{C}{\omega_{ABCD}}\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\boldsymbol{\sigma}_0$$

$$=\hat{\omega}_A\,\boldsymbol{\sigma}_A+\hat{\omega}_B\,\boldsymbol{\sigma}_B+\hat{\omega}_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

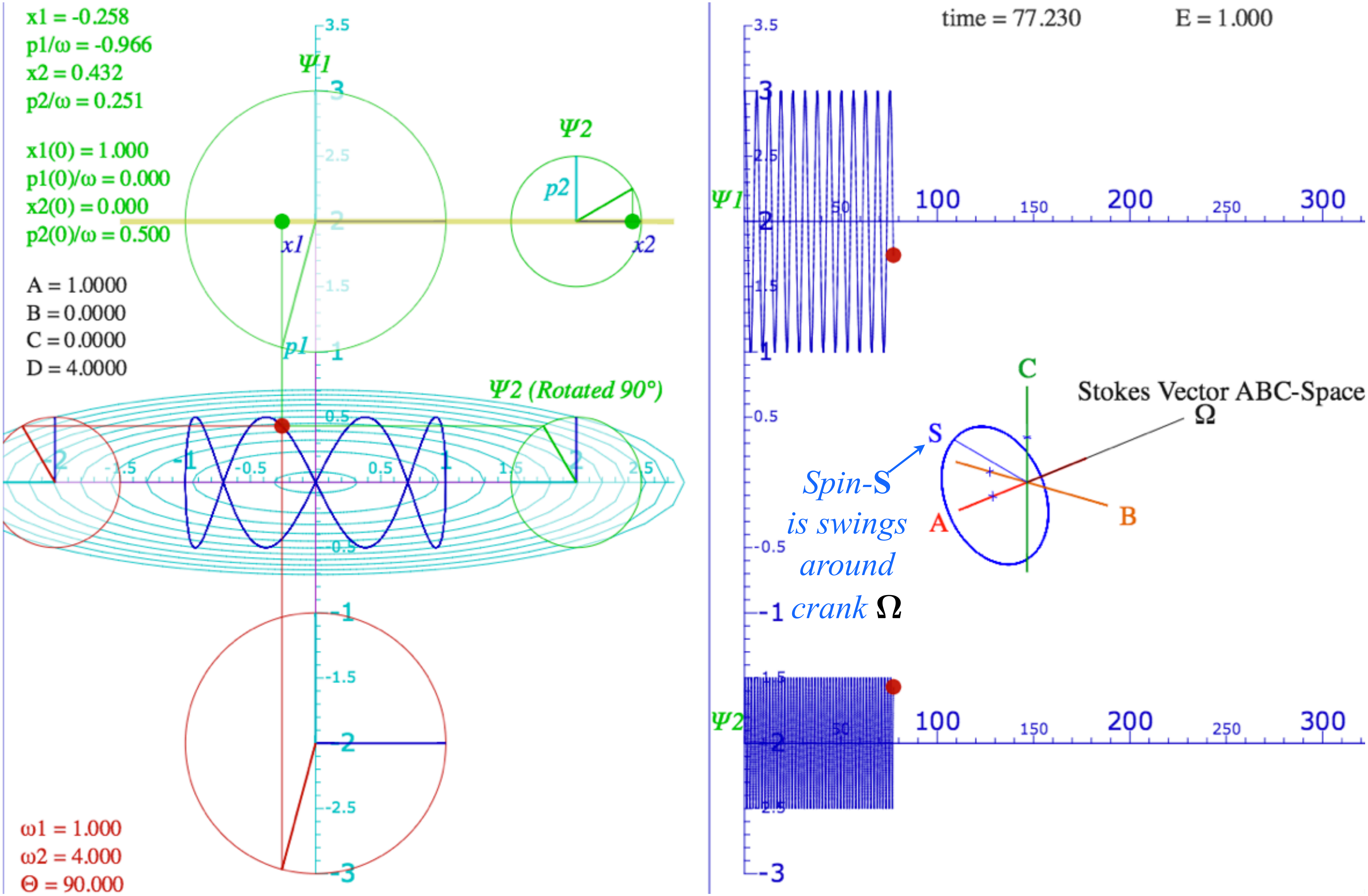
$$=\hat{\omega}_A\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\hat{\omega}_B\begin{pmatrix}0&1\\1&0\end{pmatrix}+\hat{\omega}_C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\underbrace{\frac{A+D}{2\omega_{ABCD}}}\mathbf{1}$$

$$=\begin{pmatrix}\hat{\omega}_{A=+1}&0\\0&-\hat{\omega}_{A=-1}\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\sigma_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\hat{\omega}_0\mathbf{1}$$



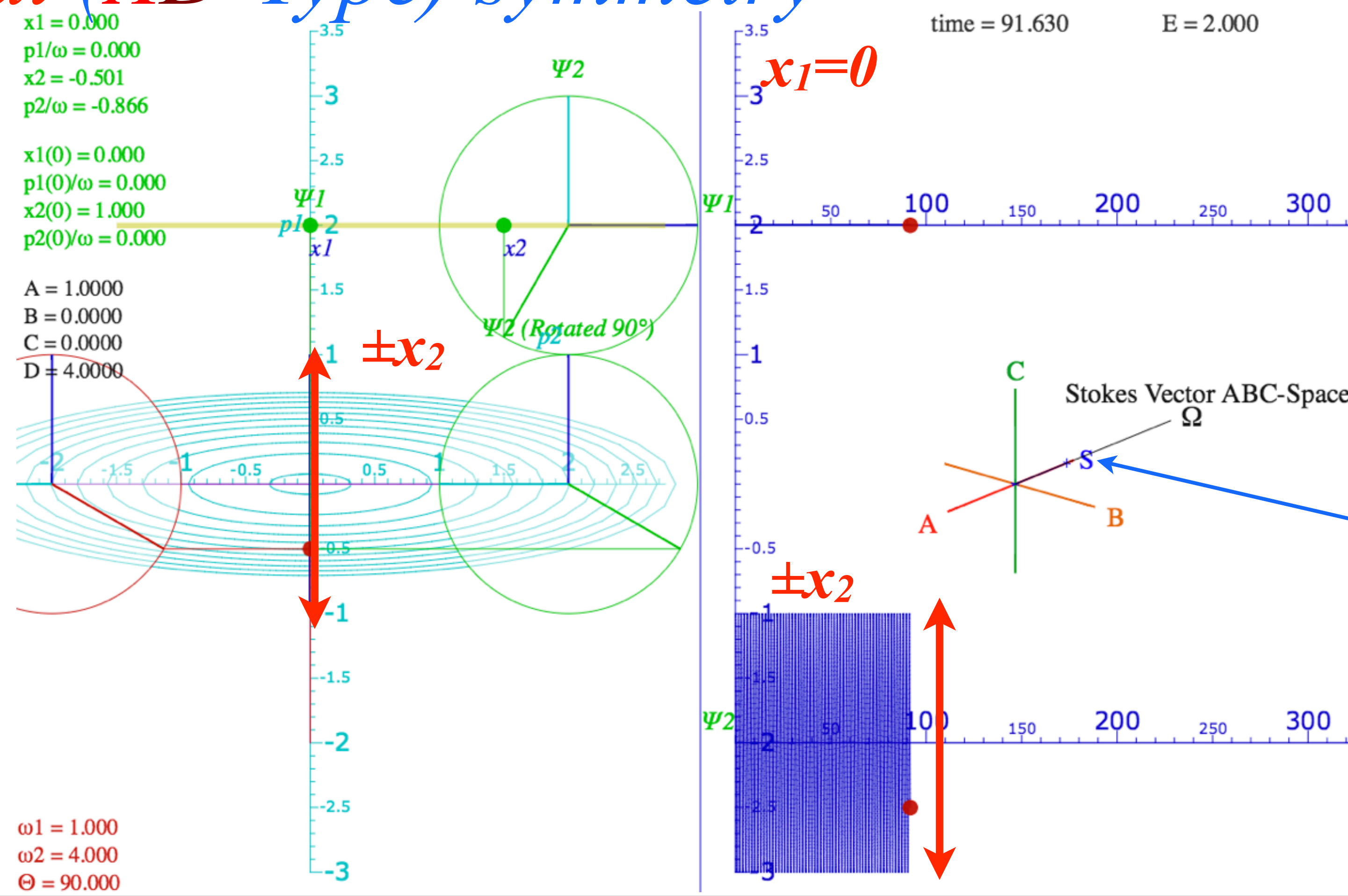
BoxIt Web Simulation
A-Type motion

Asymmetric-diagonal (AD-Type) symmetry



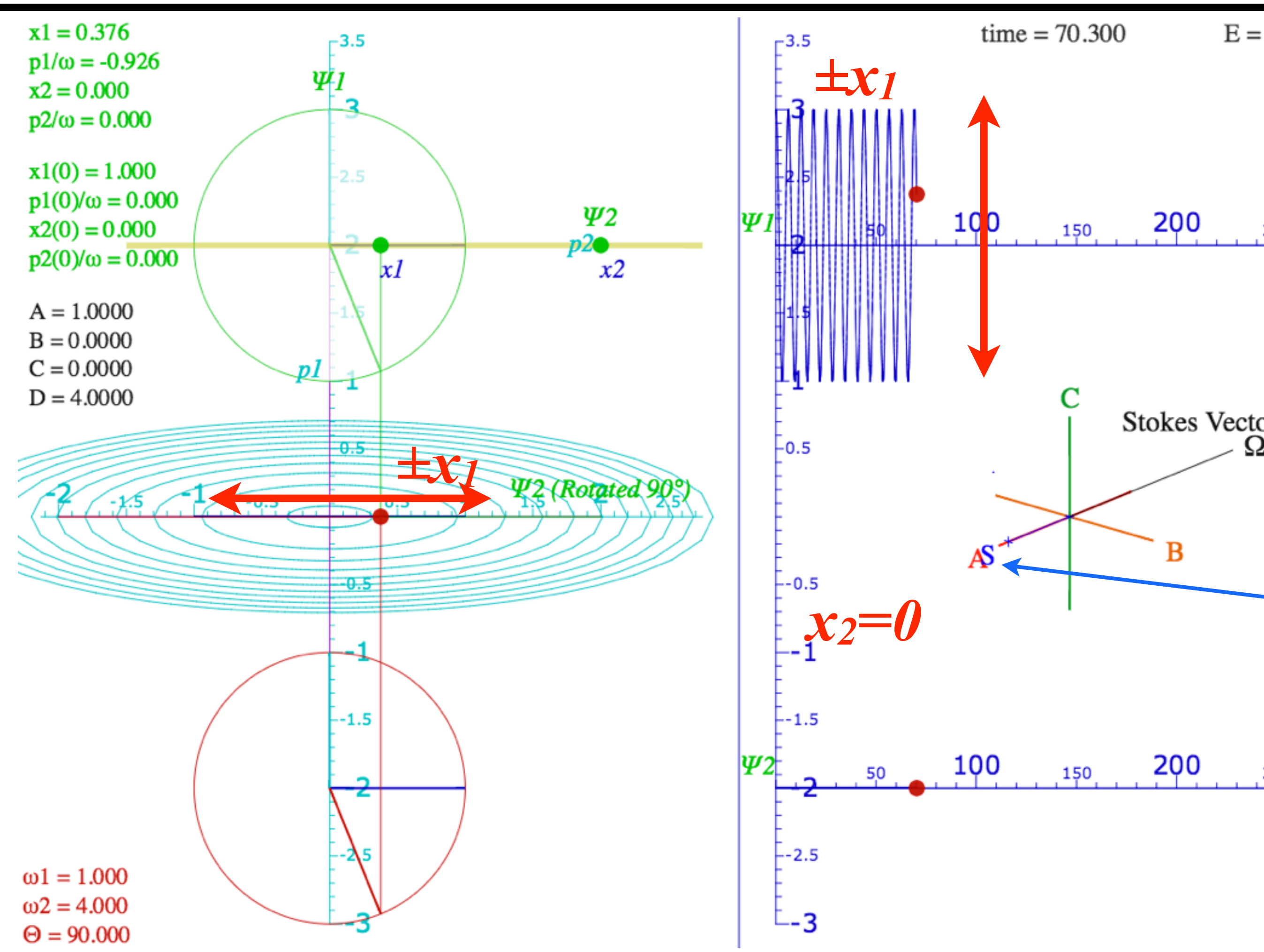
Asymmetric-diagonal (AD-Type) symmetry

High frequency A mode
x_2-linear polarized



Spin-S
is fixed up
crank Ω

Low frequency A mode
x_1-linear polarized



Spin-S
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crank Ω

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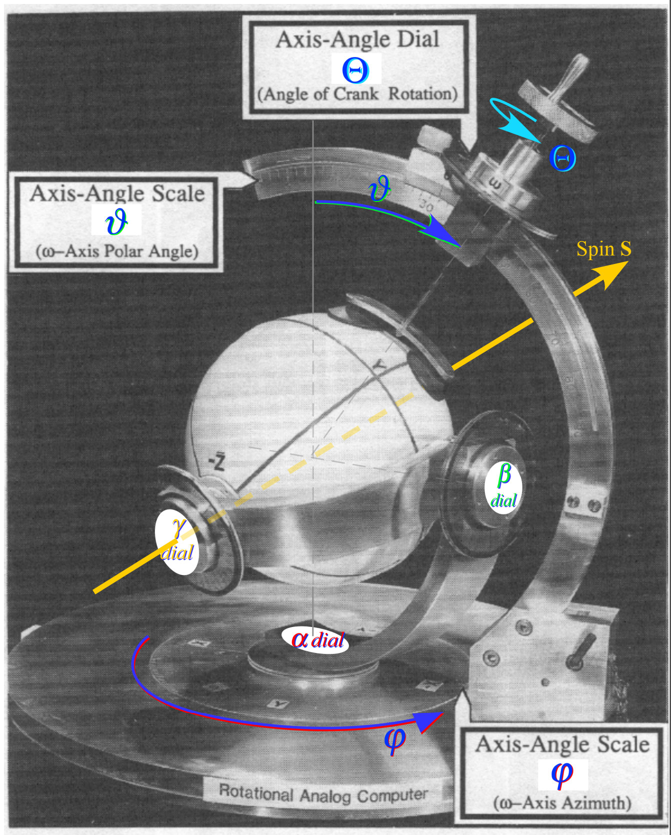

U(2)⊃C_2 ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

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→ Bilateral-balanced (B-Type) symmetry ←

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Bilateral-balanced (B-Type) symmetry

B-Type-matrix $\mathbf{H}=\boldsymbol{\sigma}_B+\omega_0\mathbf{1}$

$$\mathbf{H}=\frac{A-D}{2}\boldsymbol{\sigma}_A+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\boldsymbol{\sigma}_0$$

$$=\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\omega_A\,\boldsymbol{\sigma}_A+\omega_B\,\boldsymbol{\sigma}_B+\omega_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\mathbf{1}$$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD+}=\tfrac{1}{2}\begin{pmatrix}+1\\\hat{\omega}_B=1\end{pmatrix}$$

high eigenfrequency $\hat{\omega}_0+\omega_{ABCD}$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD-}=\tfrac{1}{2}\begin{pmatrix}-1\\\hat{\omega}_B=1\end{pmatrix}$$

low eigenfrequency $\hat{\omega}_0-\omega_{ABCD}$

Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_B{}^2}=\sqrt{B^2}$

$$\frac{\mathbf{H}}{\omega_{ABCD}}=\frac{A-D}{2\omega_{ABCD}}\boldsymbol{\sigma}_A+\frac{B}{\omega_{ABCD}}\boldsymbol{\sigma}_B+\frac{C}{\omega_{ABCD}}\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\boldsymbol{\sigma}_0$$

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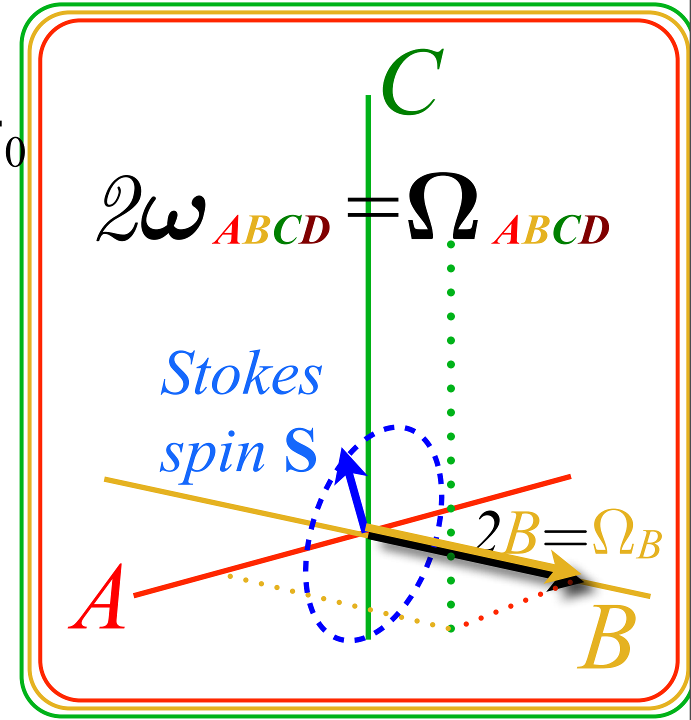
$$=\hat{\omega}_A\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\hat{\omega}_B\begin{pmatrix}0&1\\1&0\end{pmatrix}+\hat{\omega}_C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

$$=\begin{pmatrix}0&\hat{\omega}_B=1\\\hat{\omega}_B=1&0\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\hat{\omega}_0\mathbf{1}$$

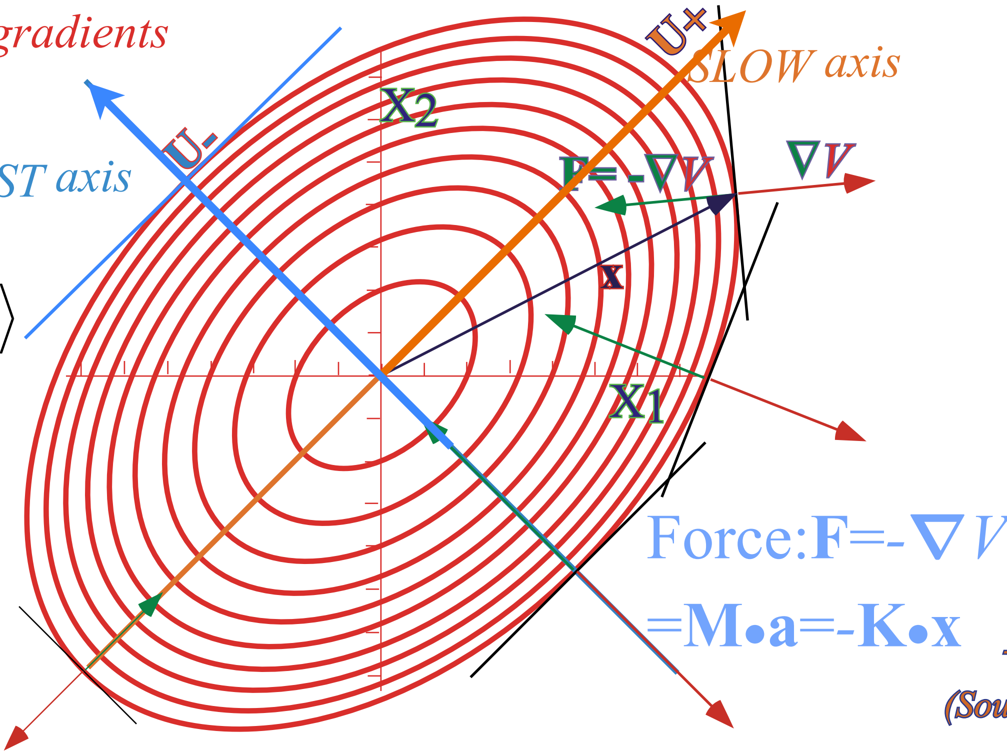


BoxIt Web Simulation
B-Type motion

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

Not most directions!

Only extremal axes work. (major or minor axes)

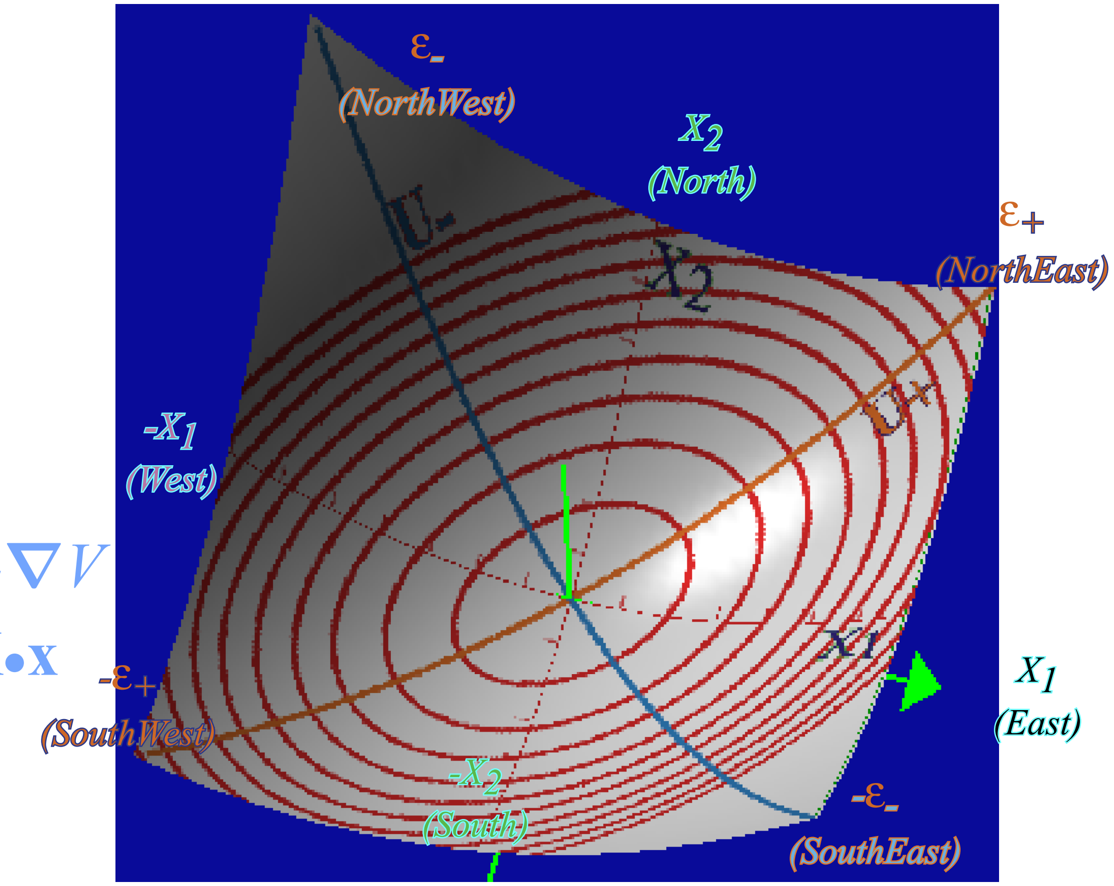
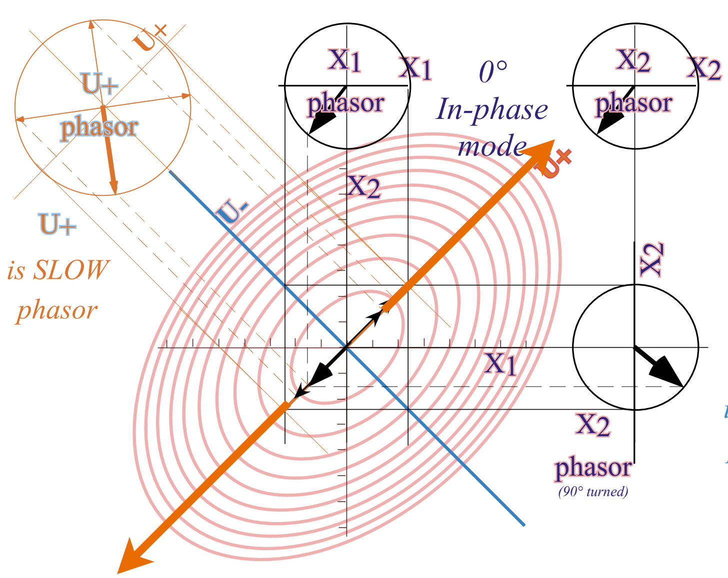
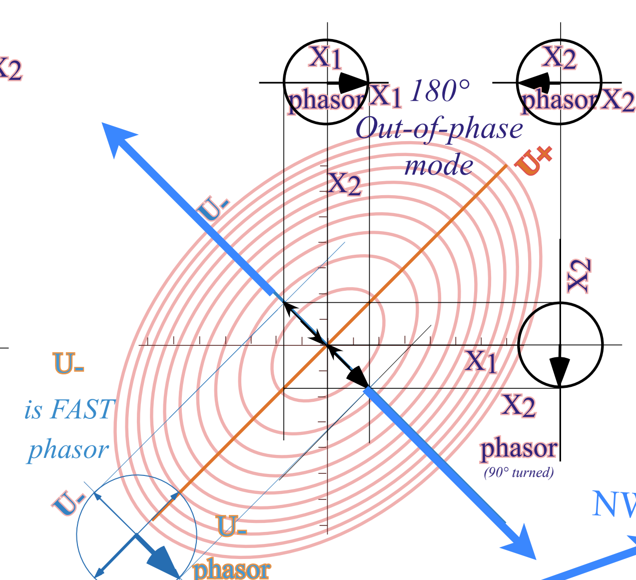


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

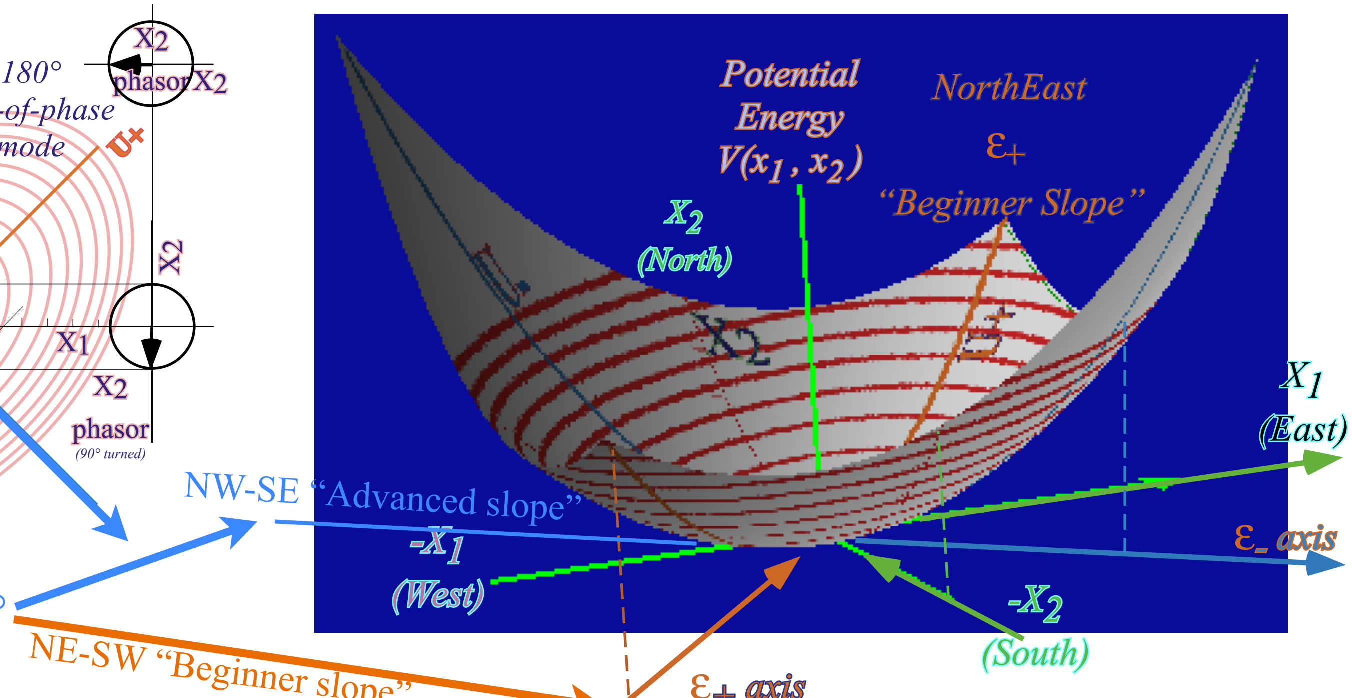
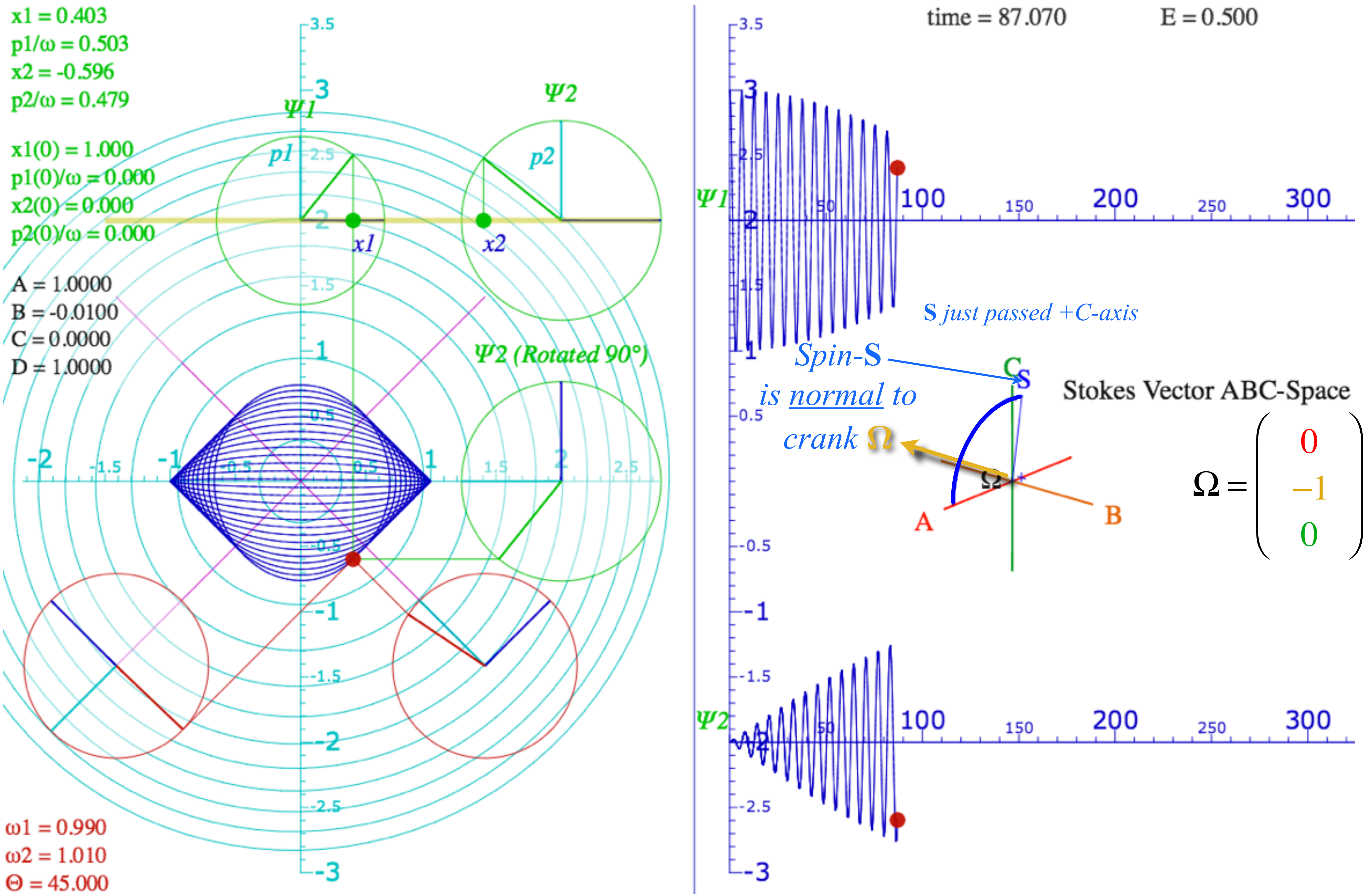
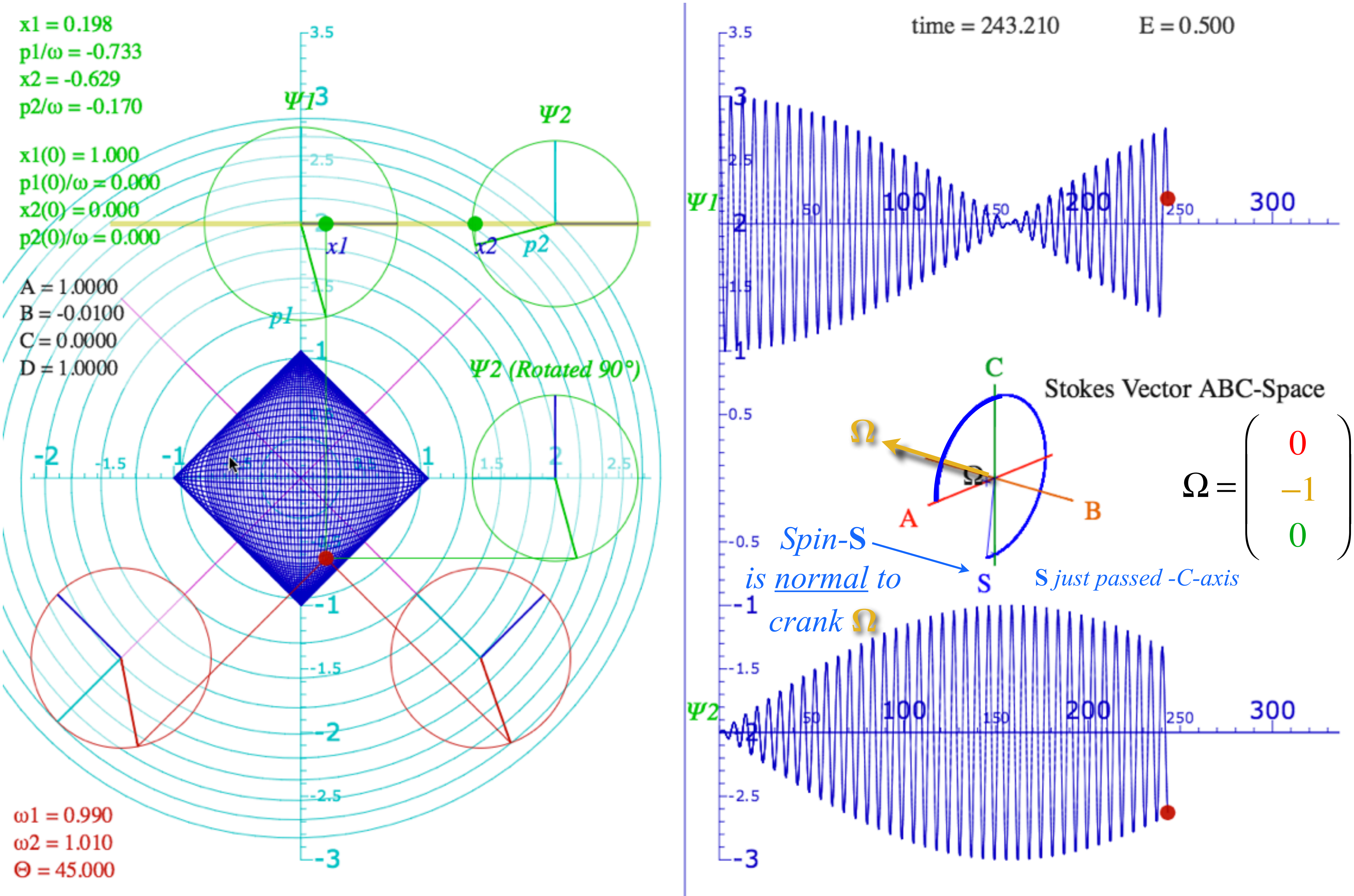


Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

Bilateral-balanced (B-Type) symmetry



Bilateral-balanced (B-Type) symmetry



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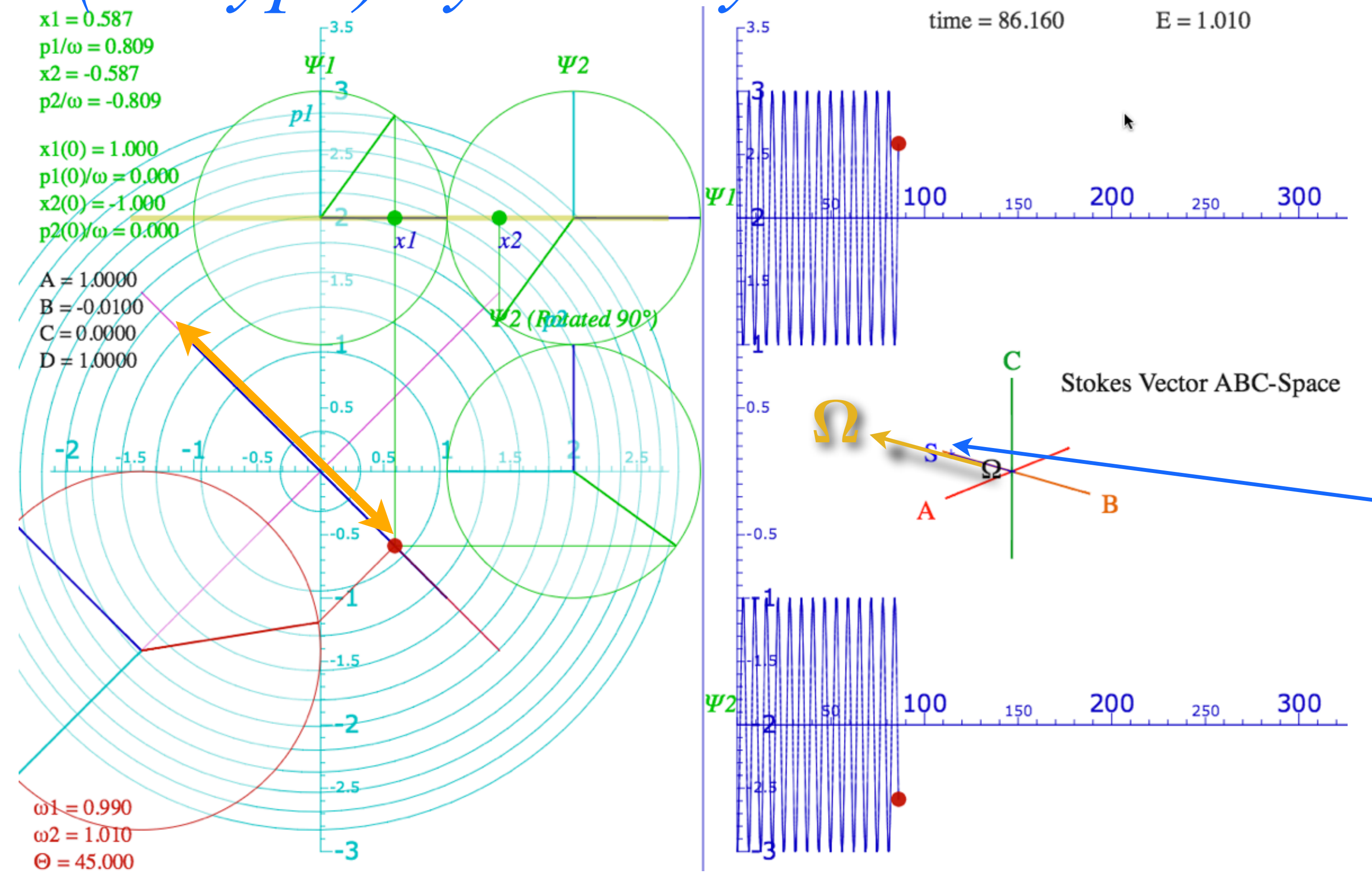
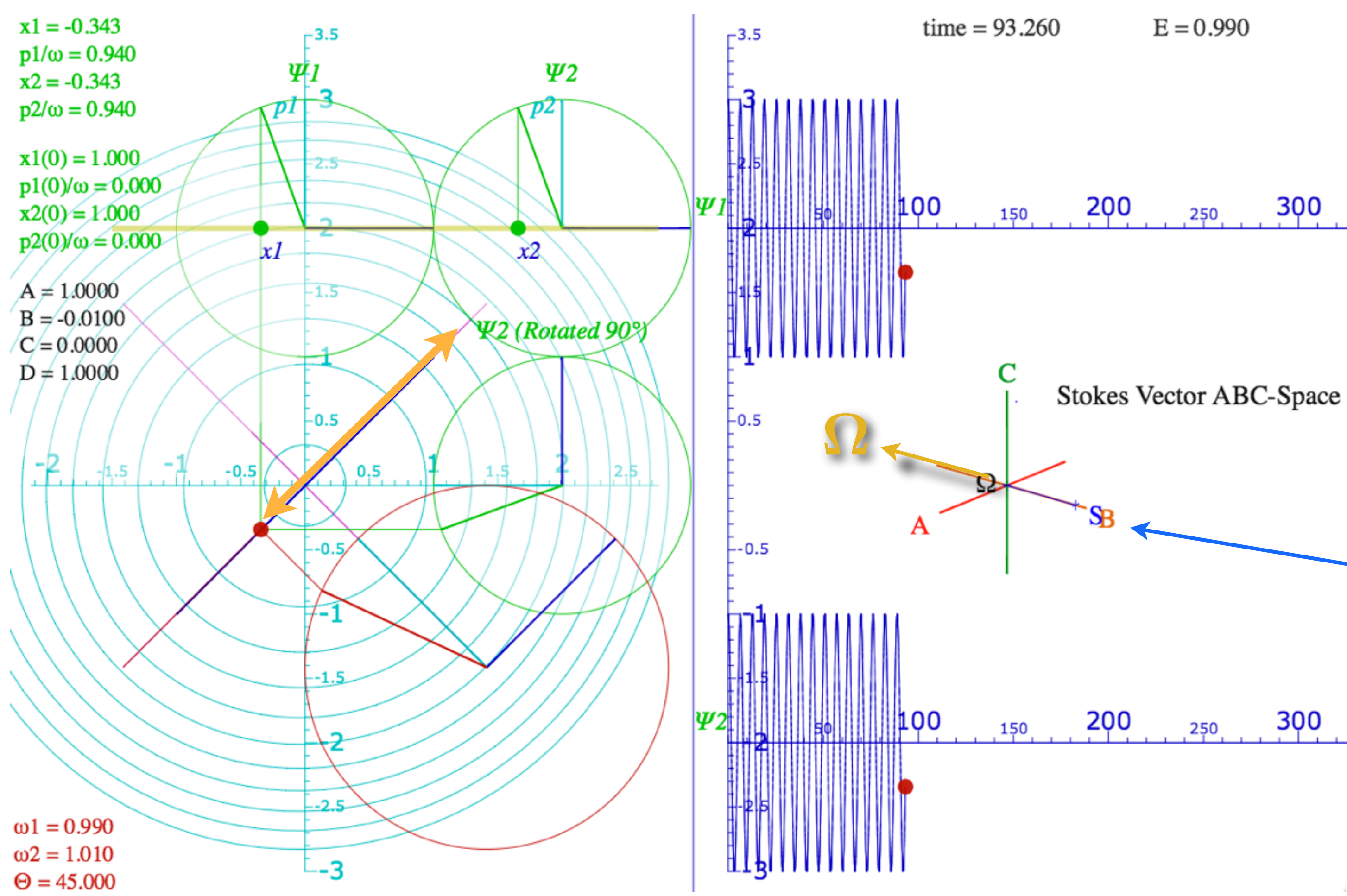
Bilateral-balanced (B-Type) symmetry

High frequency B mode
-45°-linear polarized



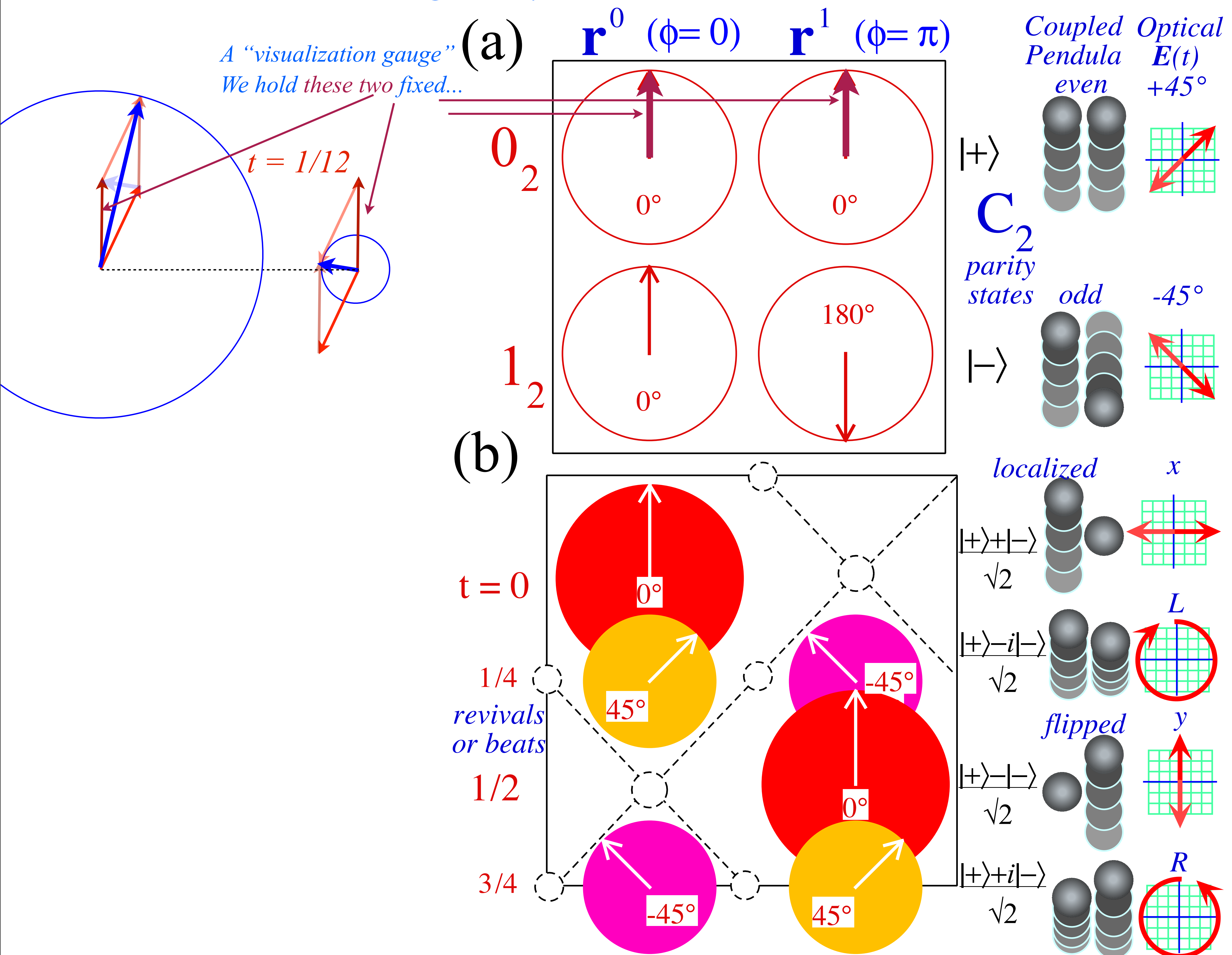
*Spin-**S***
is fixed up
crank Ω

Low frequency B mode
+45°-linear polarized

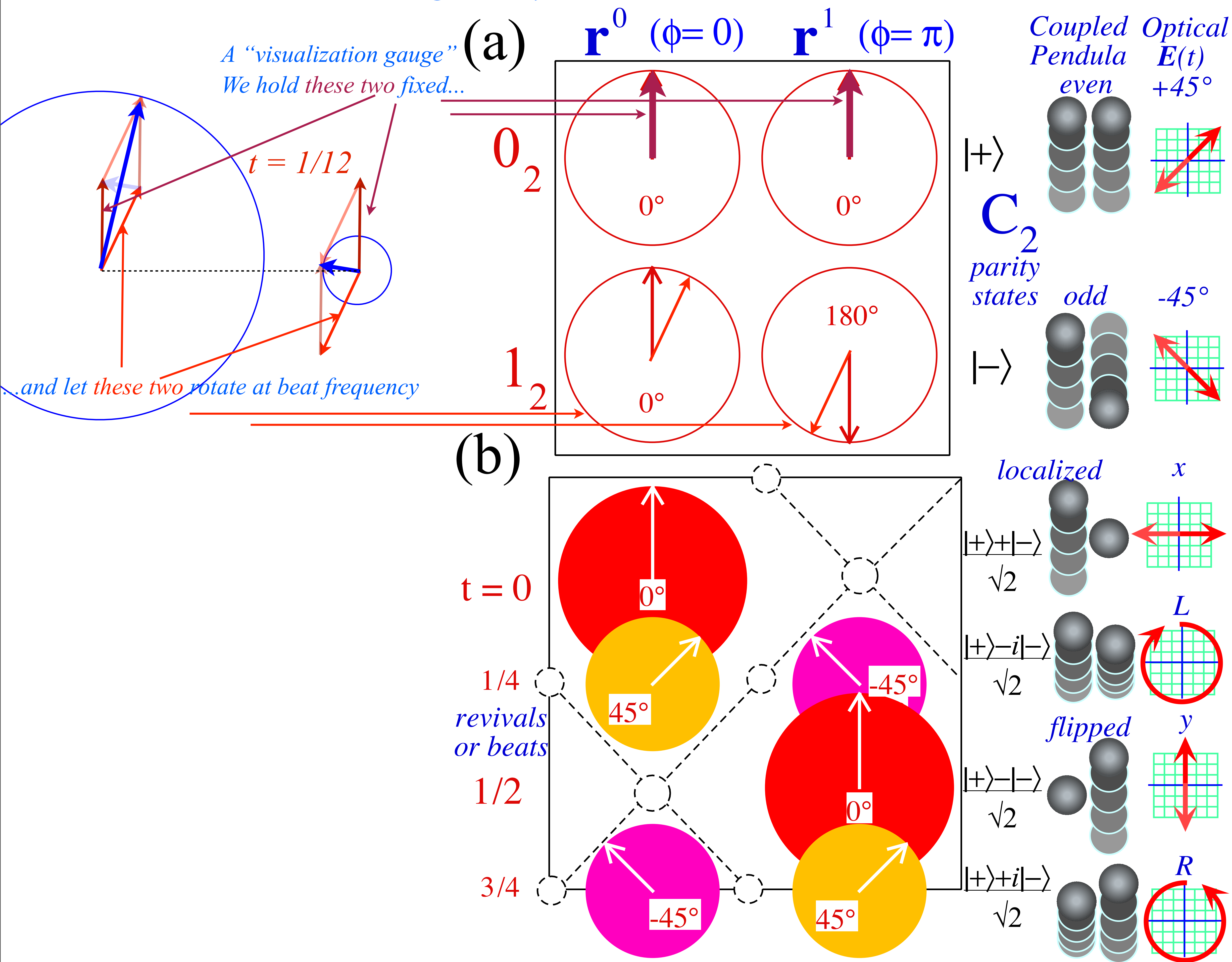


*Spin-**S***
is fixed down
crank Ω

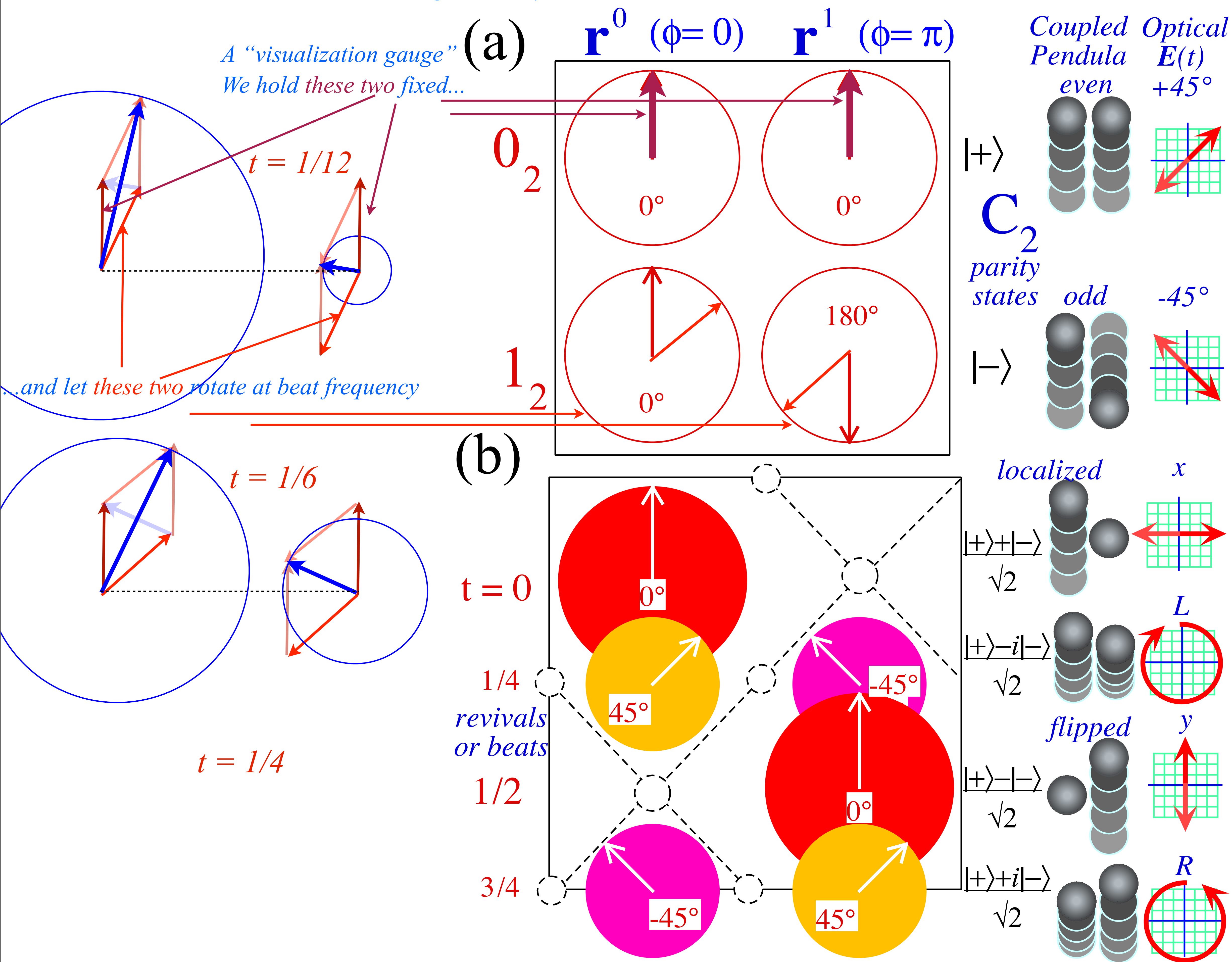
2D-HO beats and mixed mode geometry



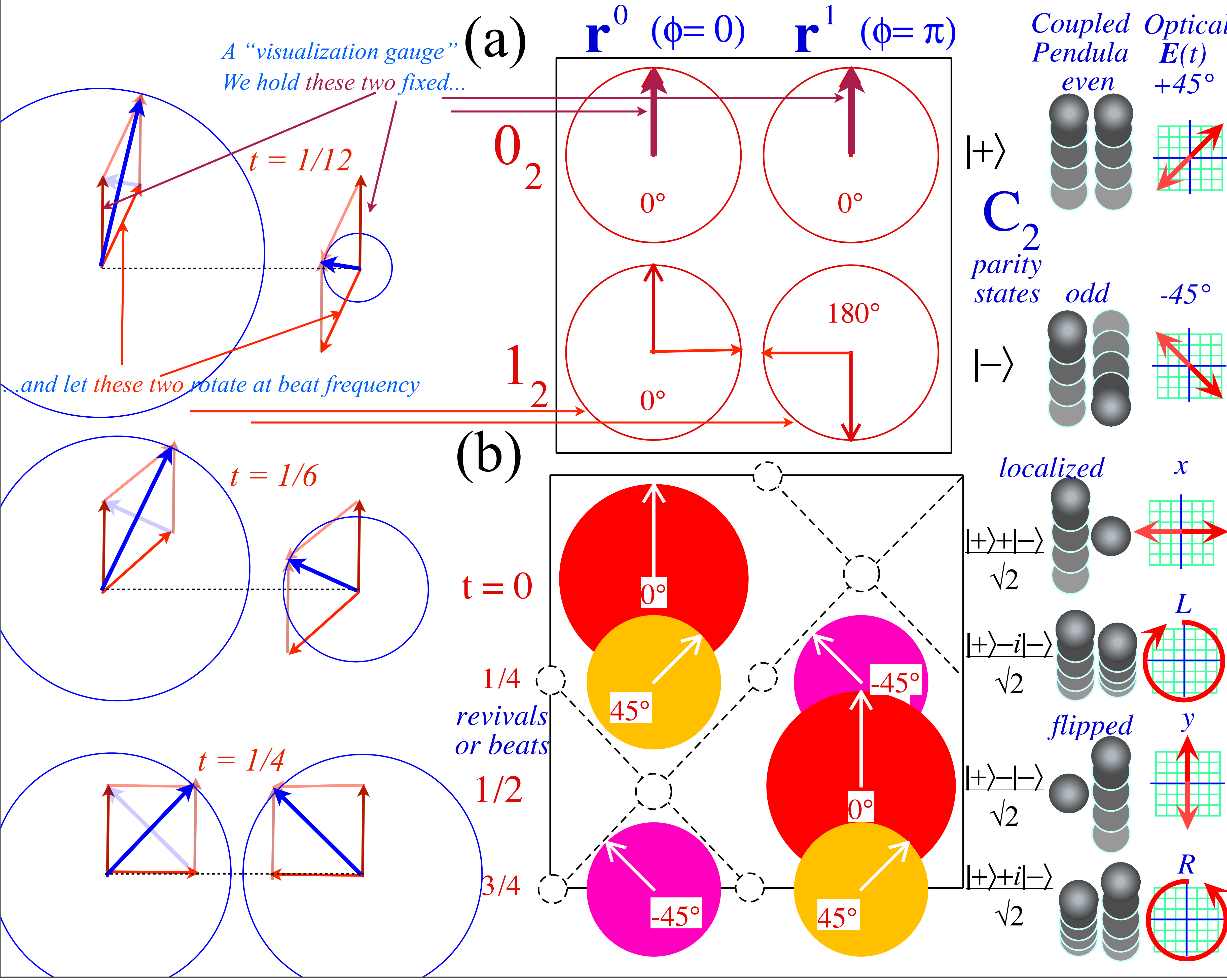
2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



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Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

➔ Circular-chiral-cyclotron (C-Type) symmetry ⟵

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Lagrange functional interpolation formula

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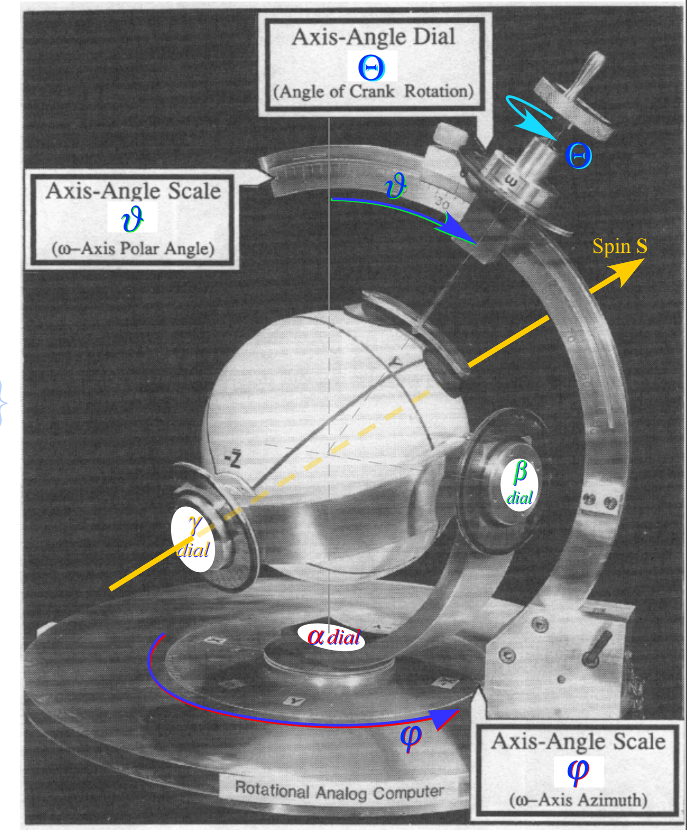


Circular-chiral-cyclotron (C-Type) symmetry

$$\left(\begin{array}{c} x_1+ip_1 \\ x_2+ip_2 \end{array}\right)^{ABCD+} = \frac{1}{2}\left(\begin{array}{c} +1 \\ +i\hat{\omega}_C \end{array}\right) = \left(\begin{array}{c} \frac{1}{2} \\ \frac{i}{2} \end{array}\right)$$

high eigenfrequency $\hat{\omega}_0 + \omega_{ABCD}$

$$\left(\begin{array}{c} x_1+ip_1 \\ x_2+ip_2 \end{array}\right)^{ABCD-} = \frac{1}{2}\left(\begin{array}{c} -1 \\ +i\hat{\omega}_C \end{array}\right) = \left(\begin{array}{c} \frac{-1}{2} \\ \frac{i}{2} \end{array}\right)$$

low eigenfrequency $\hat{\omega}_0 - \omega_{ABCD}$

C-Type H-matrix $\mathbf{H} = \boldsymbol{\sigma}_C + \omega_0 \mathbf{1}$

$$\mathbf{H} = +C\ \boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

$$= +C\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right) + \frac{A+D}{2}\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$$

$$= +\omega_C\ \boldsymbol{\sigma}_C + \frac{A+D}{2}\mathbf{1}$$

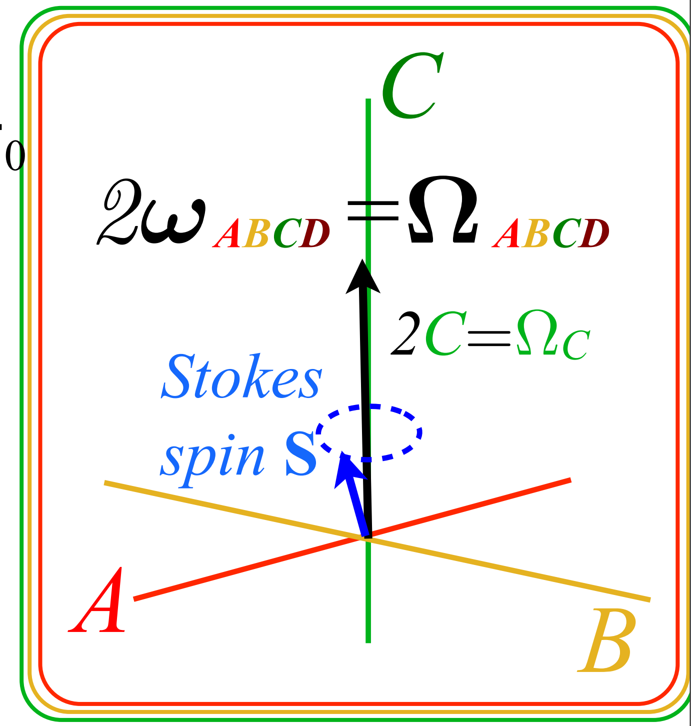
Divide H by beat frequency $\omega_{ABCD} = \sqrt{\omega_C{}^2} = \sqrt{C^2}$

$$\frac{\mathbf{H}}{\omega_{ABCD}} = +\frac{C}{\omega_{ABCD}}\boldsymbol{\sigma}_C + \frac{A+D}{2\omega_{ABCD}}\boldsymbol{\sigma}_0$$

$$= +\ \hat{\omega}_C\ \boldsymbol{\sigma}_C + \frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

$$= +\ \hat{\omega}_C\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right) + \frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

$$= \left(\begin{array}{cc} 0 & -i\hat{\omega}_C \\ +i\hat{\omega}_C & 0 \end{array}\right) + \hat{\omega}_0\mathbf{1} = \sigma_{\hat{\omega}} + \hat{\omega}_0\mathbf{1} = \boldsymbol{\sigma}\bullet\hat{\omega} + \hat{\omega}_0\mathbf{1}$$



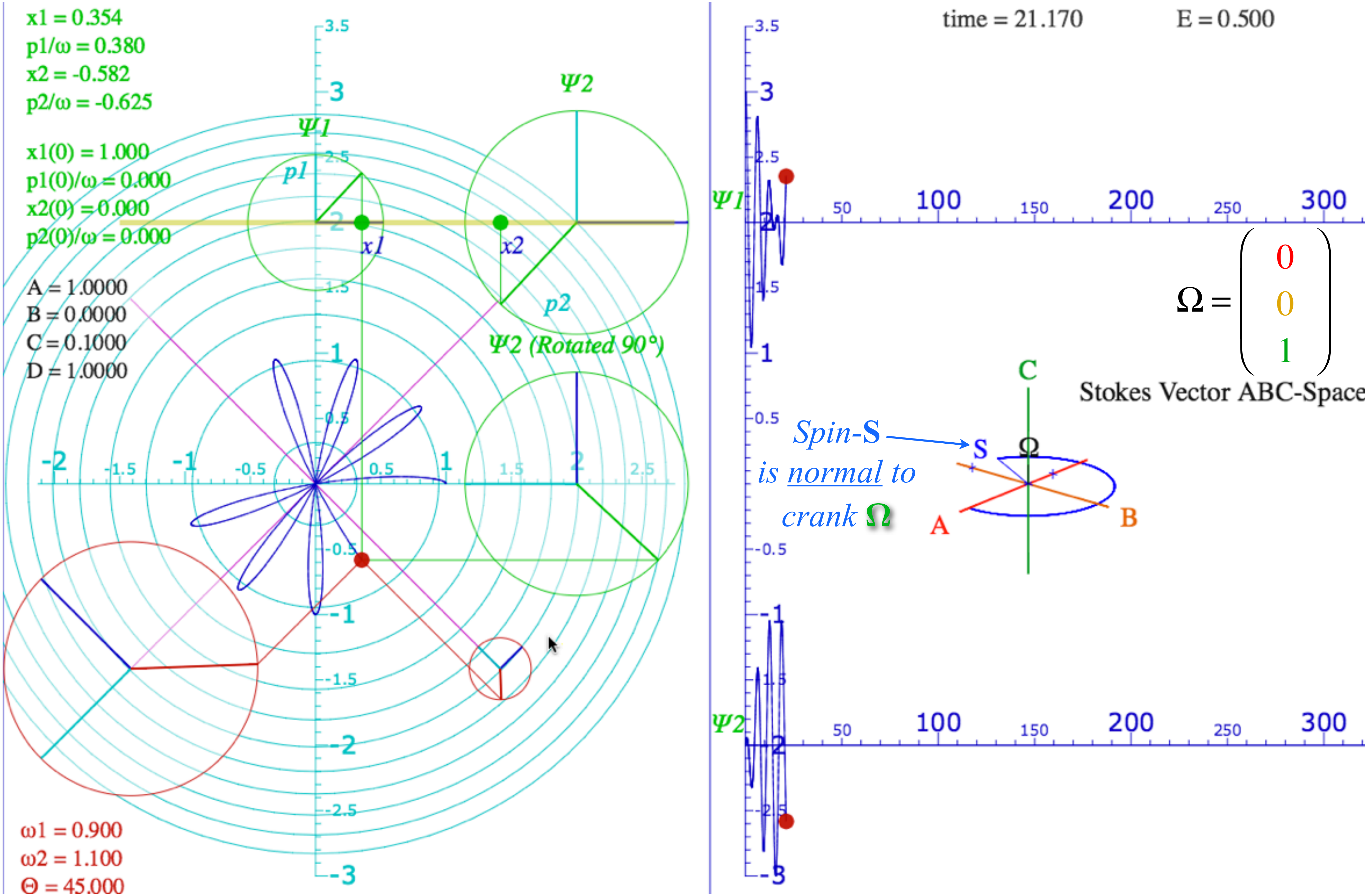
BoxIt Web Simulation

C-Type Hamiltonian - Foucault pendulum

Circular-chiral-cyclotron (C-Type) symmetry

Foucault pendulum motion due to 50-50 mix of left and right polarization eigenstates

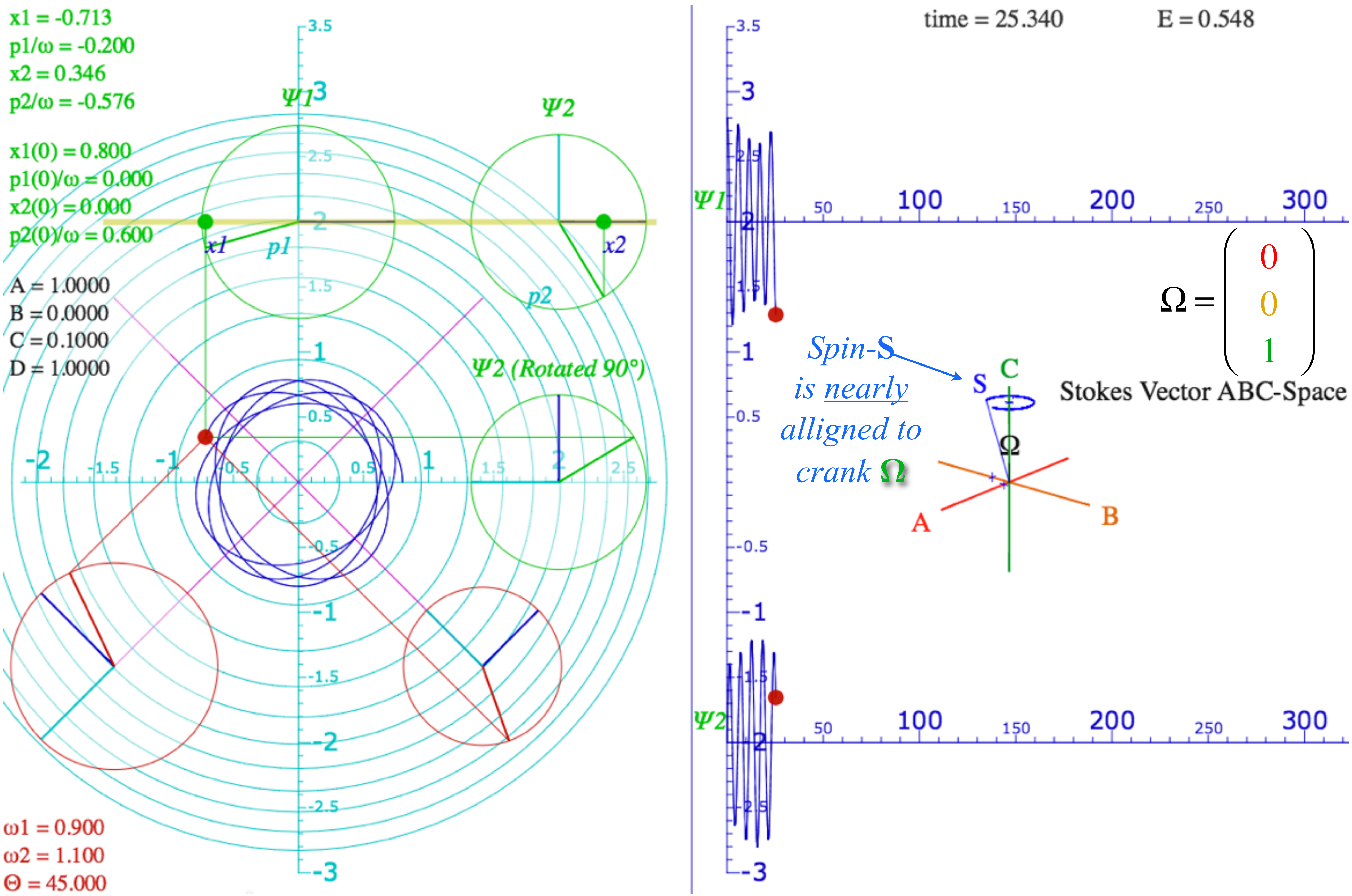
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$$



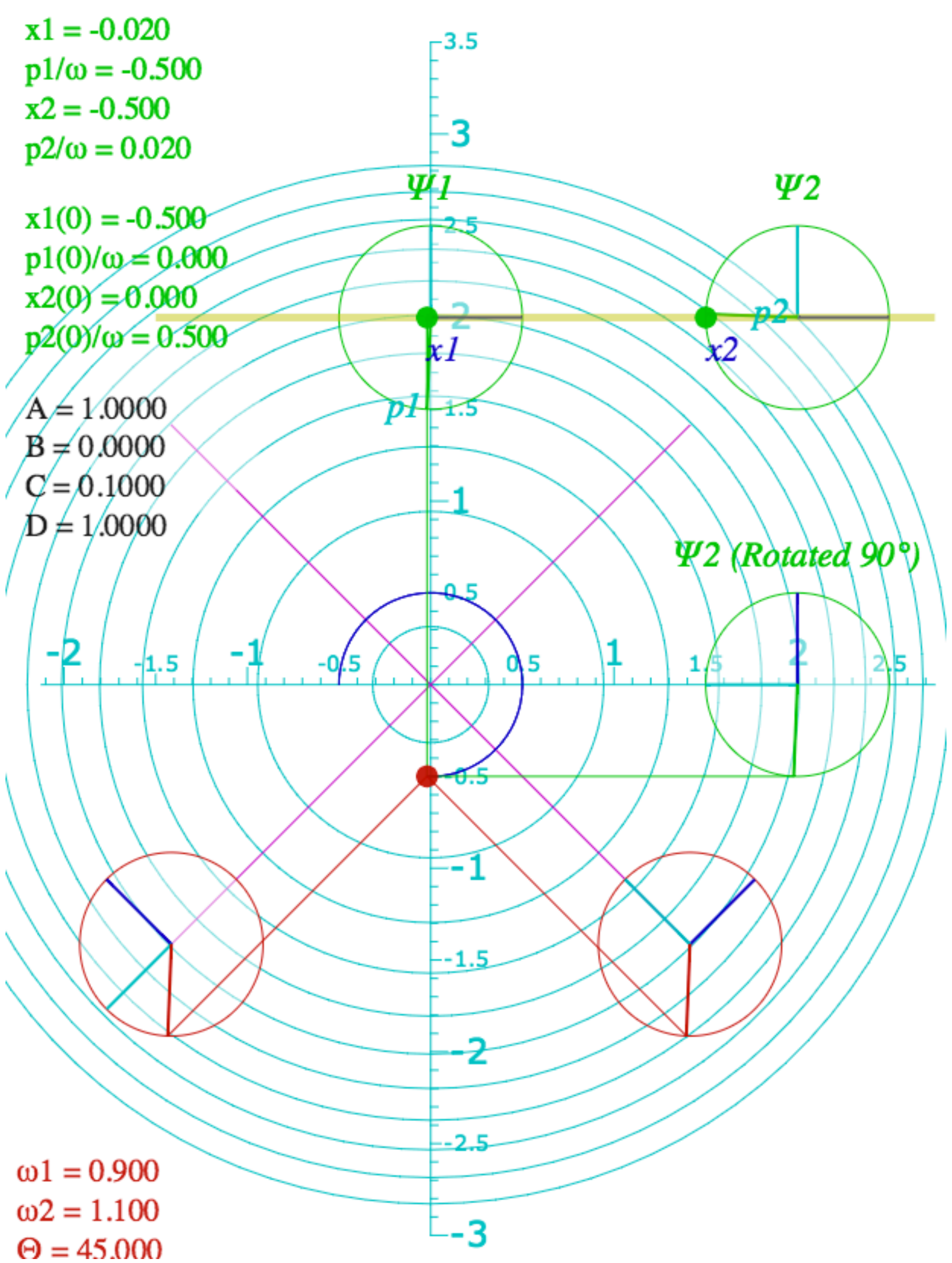
Circular-chiral-cyclotron (C-Type) symmetry

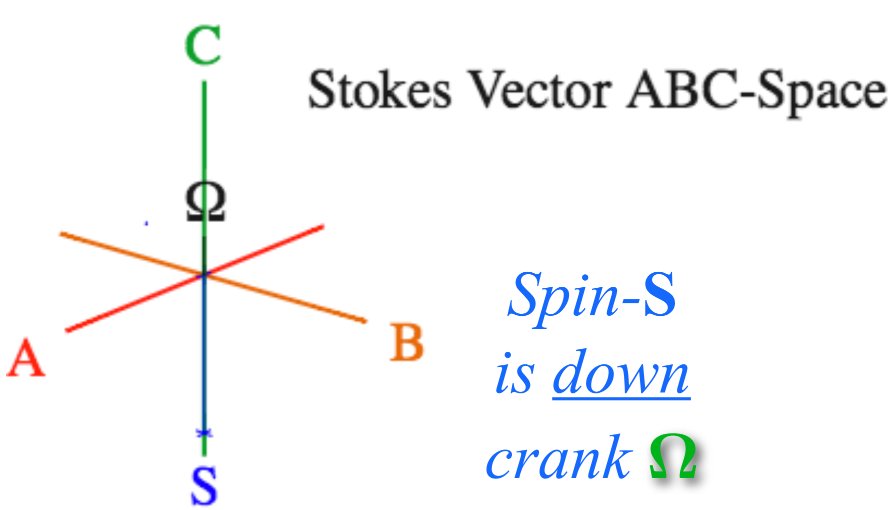
Foucault pendulum motion due to 30-70 mix of left and right polarization eigenstates

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$$



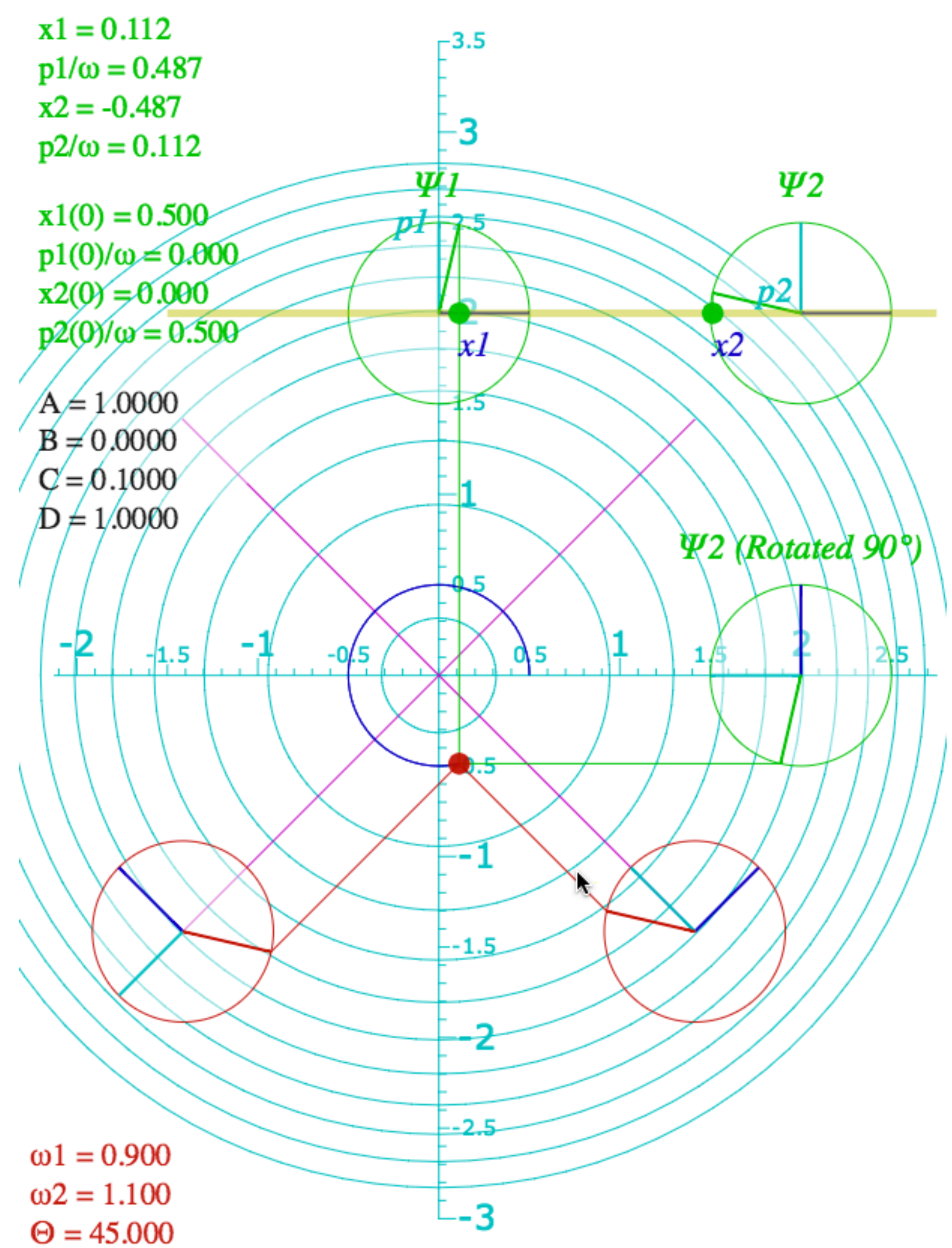
Left handed polarization eigenstate





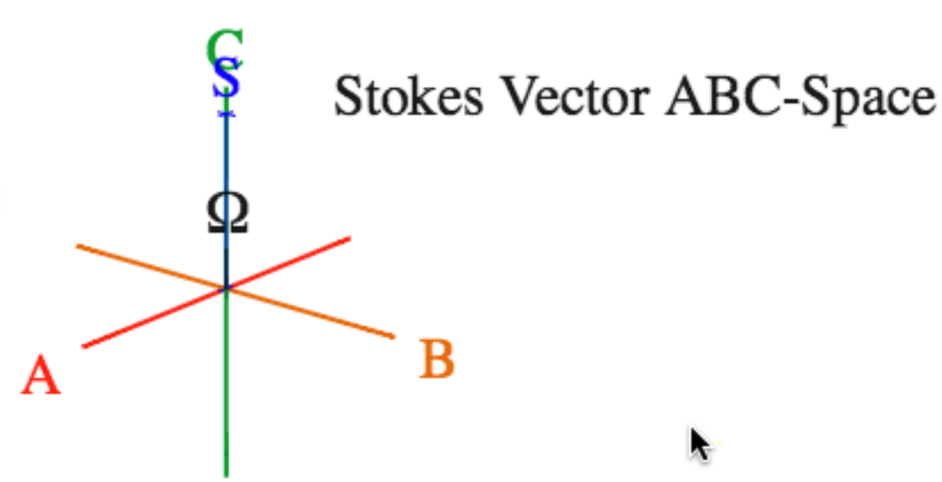
*Spin-***S**
is down
crank **Ω**

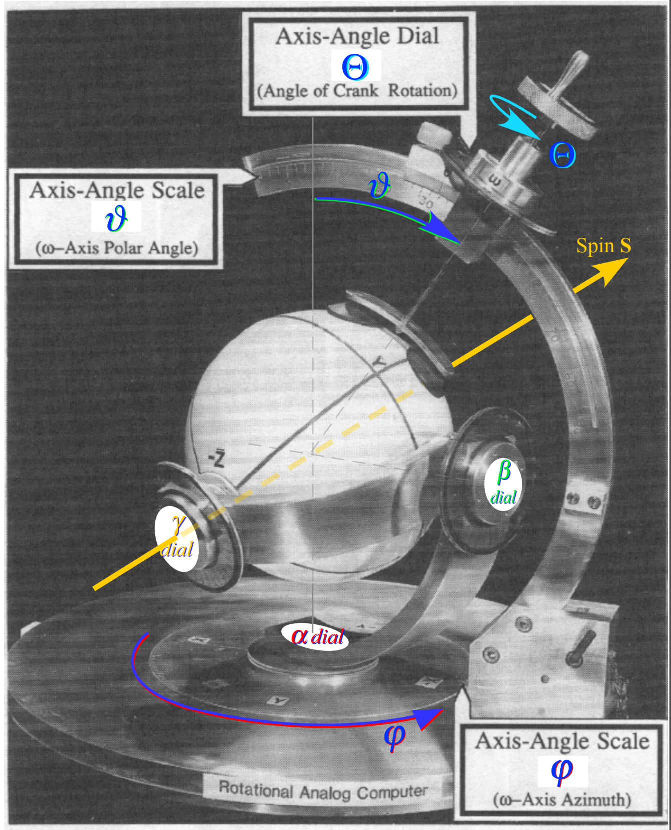
Right handed polarization eigenstate



*Spin-***S**
is up
crank **Ω**

$$\Omega = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$





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➡ Mixed ABCD symmetry examples ⬅

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Lagrange functional interpolation formula

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Mixed ABCD symmetry example

(A=3, B=1, C=1, D=1) *H-matrix* $\mathbf{H}=\boldsymbol{\sigma}_A+\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_C+\omega_0\mathbf{1}$

$$\mathbf{H}=\frac{A-D}{2}\boldsymbol{\sigma}_A+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A+D}{2}\boldsymbol{\sigma}_0$$

$$=\frac{3-1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+1\begin{pmatrix}0&1\\1&0\end{pmatrix}+1\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{3+1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\omega_A\,\boldsymbol{\sigma}_A+\omega_B\,\boldsymbol{\sigma}_B+\omega_C\,\boldsymbol{\sigma}_C+2\,\mathbf{1}$$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD+}=\frac{1}{2}\begin{pmatrix}1\text{-}1/\sqrt{3}\\-(1+i)/\sqrt{3}\end{pmatrix}$$

high eigenfrequency $\hat{\omega}_0+\omega_{ABCD}$

$$\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}^{ABCD-}=\frac{1}{2}\begin{pmatrix}1+1/\sqrt{3}\\(1+i)/\sqrt{3}\end{pmatrix}$$

low eigenfrequency $\hat{\omega}_0-\omega_{ABCD}$

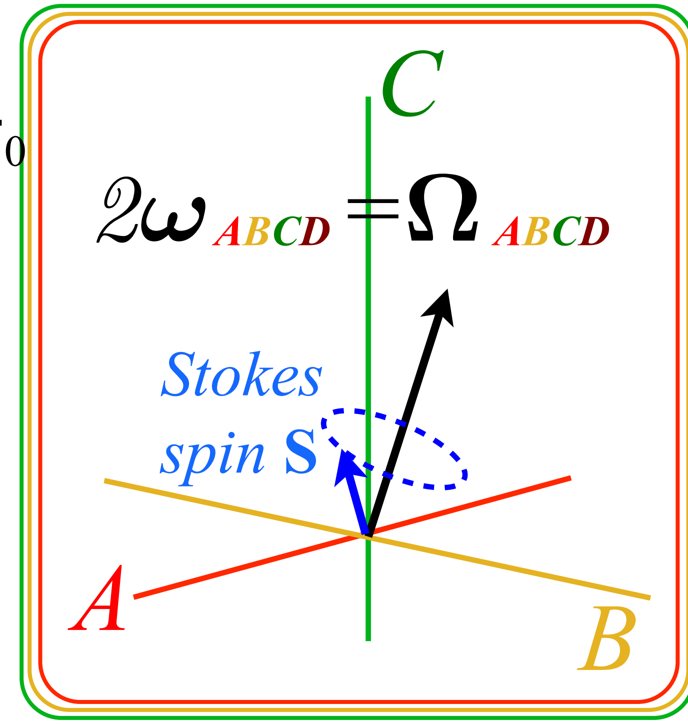
Divide H by beat frequency $\omega_{ABCD}=\sqrt{\omega_A{}^2+\omega_B{}^2+\omega_C{}^2}=\sqrt{\frac{(3-1)^2}{4}+1^2+1^2}=\sqrt{3}$

$$\frac{\mathbf{H}}{\omega_{ABCD}}=\frac{3-1}{2\omega_{ABCD}}\boldsymbol{\sigma}_A+\frac{1}{\omega_{ABCD}}\boldsymbol{\sigma}_B+\frac{1}{\omega_{ABCD}}\boldsymbol{\sigma}_C+\frac{3+1}{2\omega_{ABCD}}\boldsymbol{\sigma}_0$$

$$=\hat{\omega}_A\,\boldsymbol{\sigma}_A+\hat{\omega}_B\,\boldsymbol{\sigma}_B+\hat{\omega}_C\,\boldsymbol{\sigma}_C+\frac{A+D}{2\omega_{ABCD}}\mathbf{1}$$

$$=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{1}{\sqrt{3}}\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{\sqrt{3}}\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{4}{2\sqrt{3}}\mathbf{1}$$

$$=\mathbf{h}+\hat{\omega}_0\mathbf{1}=\begin{pmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}-i\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}+i\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{3}}\end{pmatrix}+\hat{\omega}_0\mathbf{1}=\sigma_{\hat{\omega}}+\hat{\omega}_0\mathbf{1}=\boldsymbol{\sigma}\bullet\hat{\omega}+\frac{2}{\sqrt{3}}\mathbf{1}$$



High ω Projector: $\mathbf{P}^{ABCD+}=\frac{\mathbf{1}-\mathbf{h}}{2}=\frac{1}{2}\begin{pmatrix}1\text{-}\frac{1}{\sqrt{3}}&\frac{-1+i}{\sqrt{3}}\\\frac{-1-i}{\sqrt{3}}&1\text{-}\frac{1}{\sqrt{3}}\end{pmatrix}$

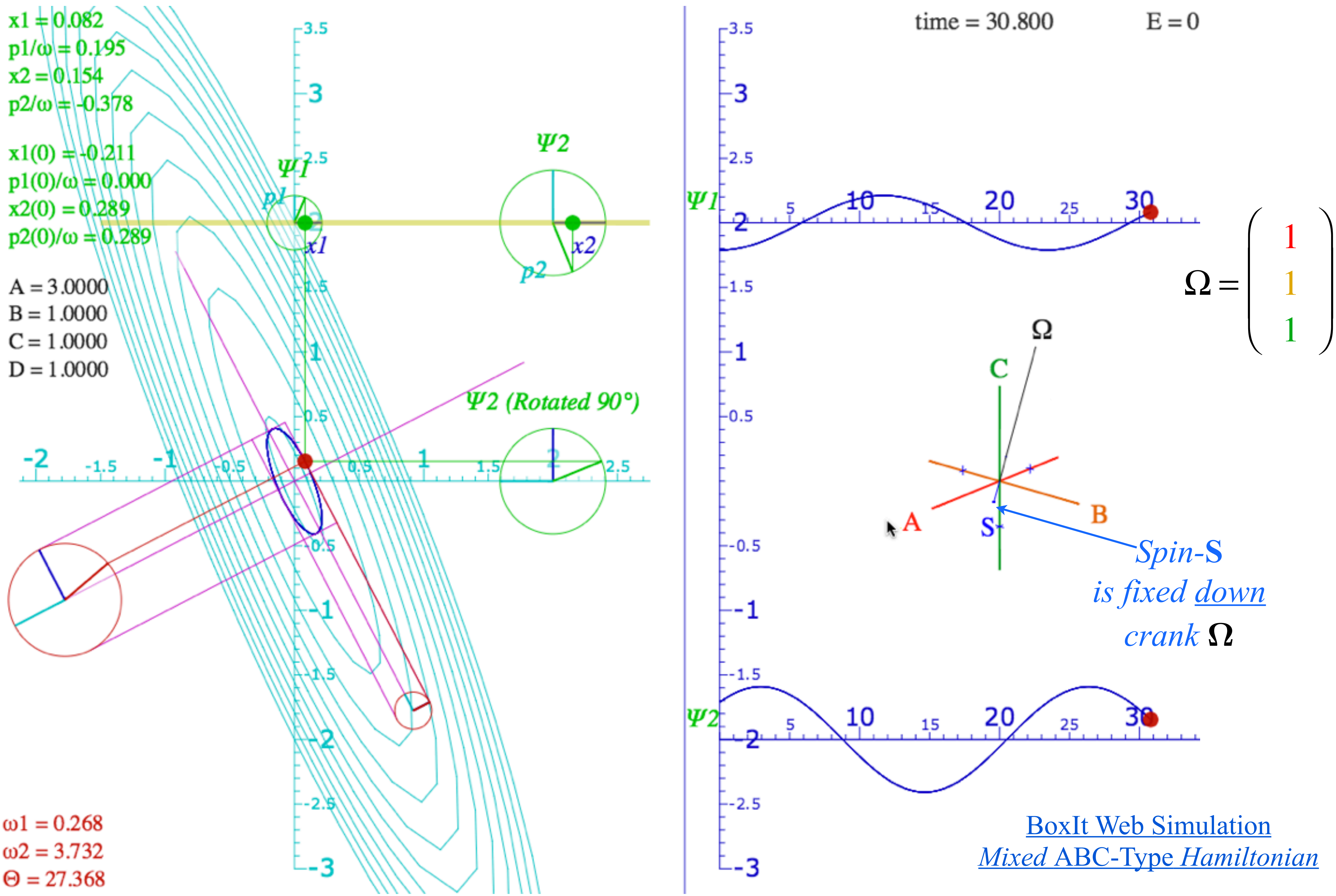
$$\mathbf{P}^{ABCD+}=\frac{\mathbf{1}-\mathbf{h}}{2}$$

$$\mathbf{P}^{ABCD-}=\frac{\mathbf{1}+\mathbf{h}}{2}$$

Low ω Projector: $\mathbf{P}^{ABCD-}=\frac{\mathbf{1}+\mathbf{h}}{2}=\frac{1}{2}\begin{pmatrix}1+\frac{1}{\sqrt{3}}&\frac{1-i}{\sqrt{3}}\\\frac{1+i}{\sqrt{3}}&1+\frac{-1}{\sqrt{3}}\end{pmatrix}$

Mixed ABCD symmetry example   *High eigenmode*

(A=3, B=1, C=1, D=1) *H-matrix* $\mathbf{H}=\boldsymbol{\sigma}_A+\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_C+\omega_0\mathbf{1}$

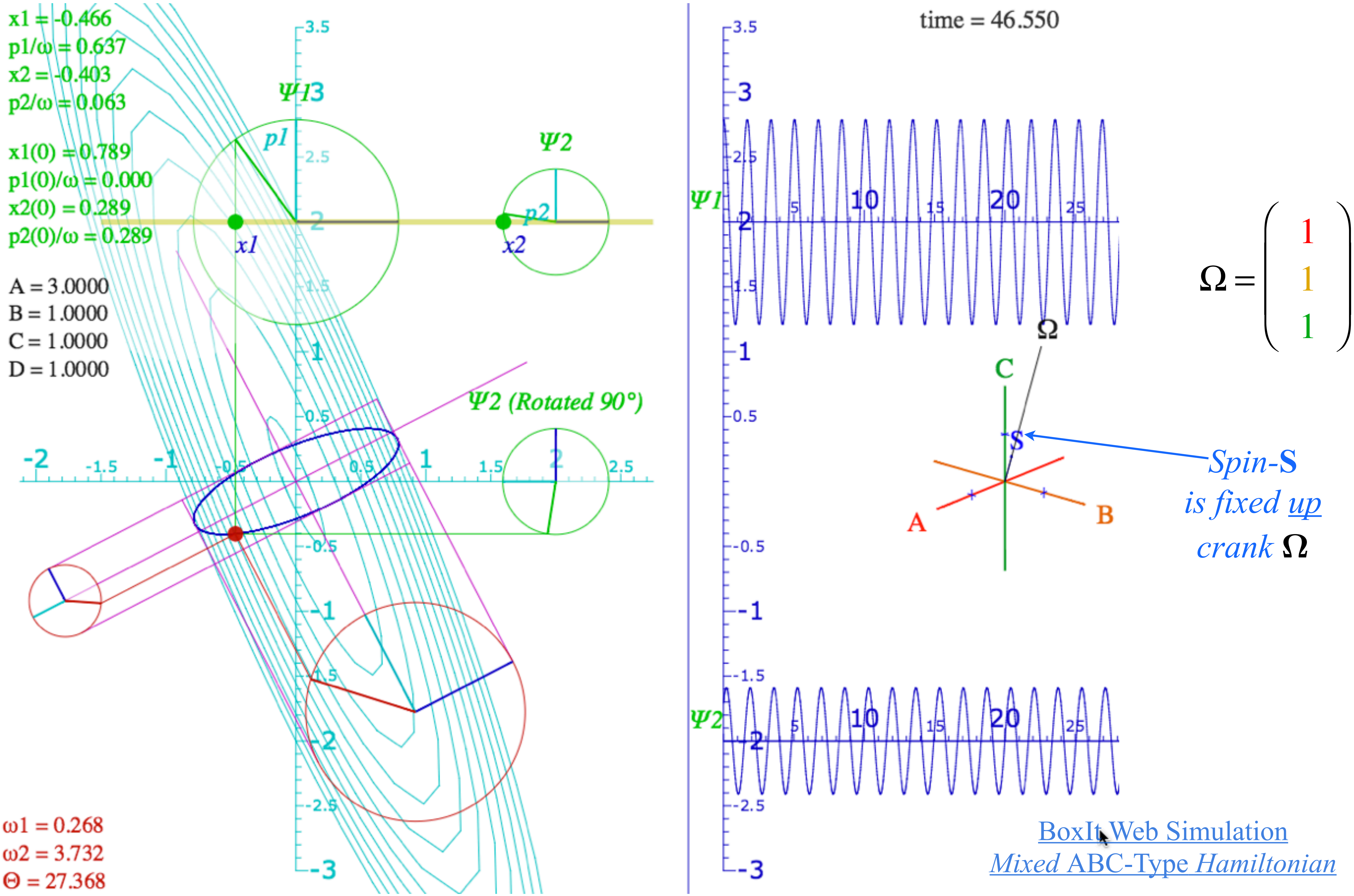


BoxIt Web Simulation
Mixed ABC-Type *Hamiltonian*

Mixed ABCD symmetry example

Low eigenmode

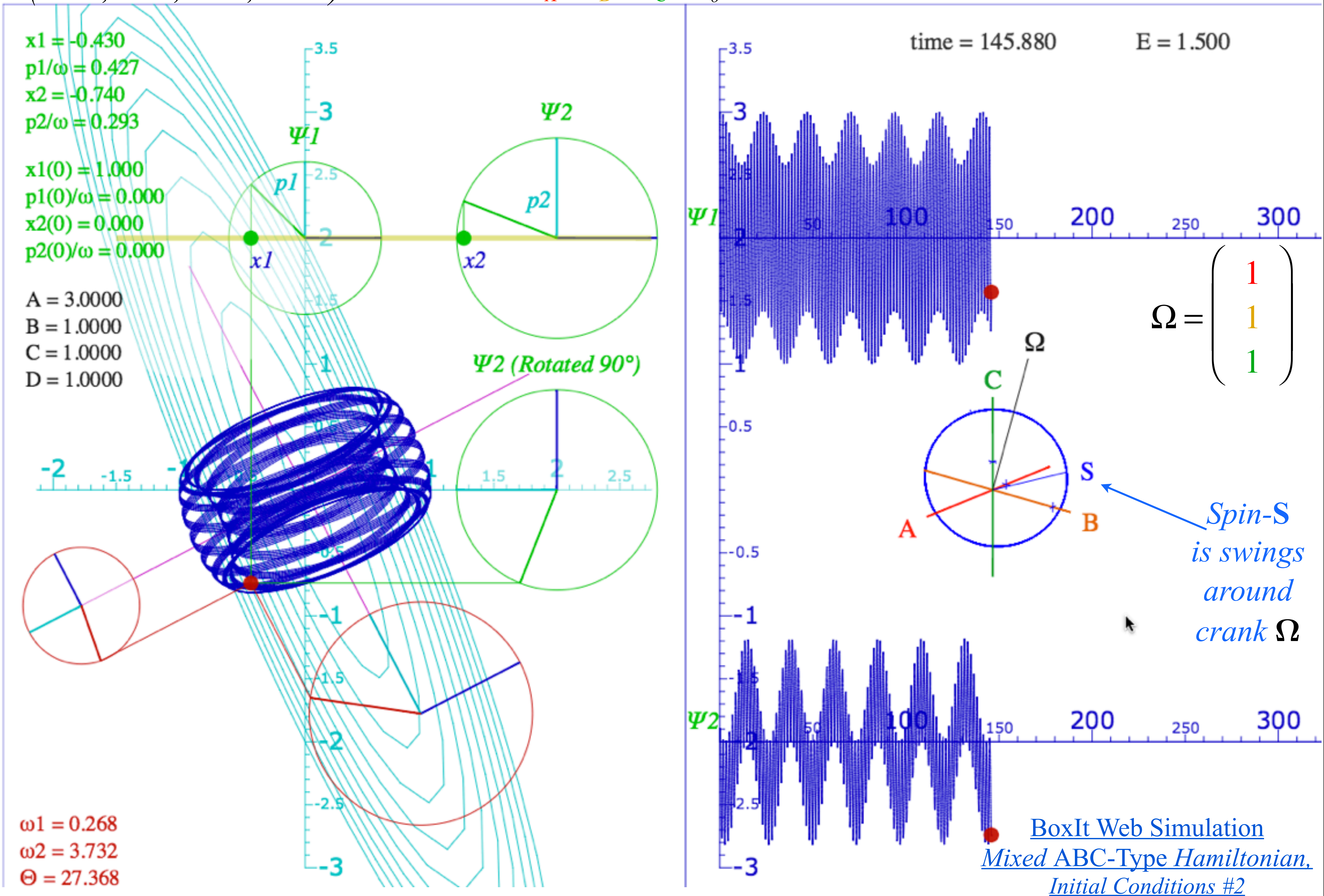
(A=3, B=1, C=1, D=1) *H-matrix* $\mathbf{H}=\boldsymbol{\sigma}_A+\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_C+\omega_0\mathbf{1}$



Mixed ABCD symmetry example

Mixed High and Low eigenmodes

($A=3$, $B=1$, $C=1$, $D=1$) *H-matrix* $\mathbf{H}=\boldsymbol{\sigma}_A+\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_C+\omega_0\mathbf{1}$



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➔ *Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State*

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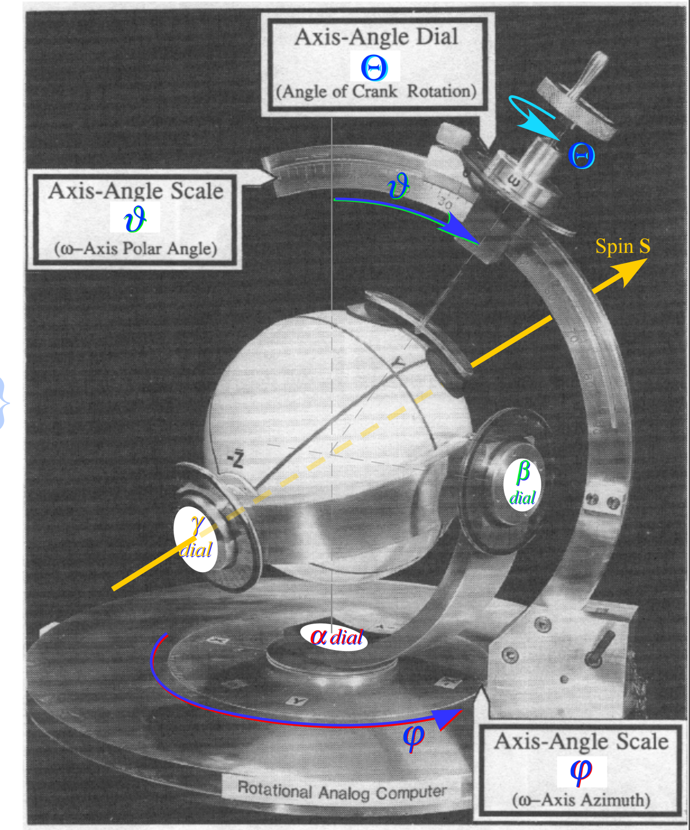


Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$ *implies:* $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

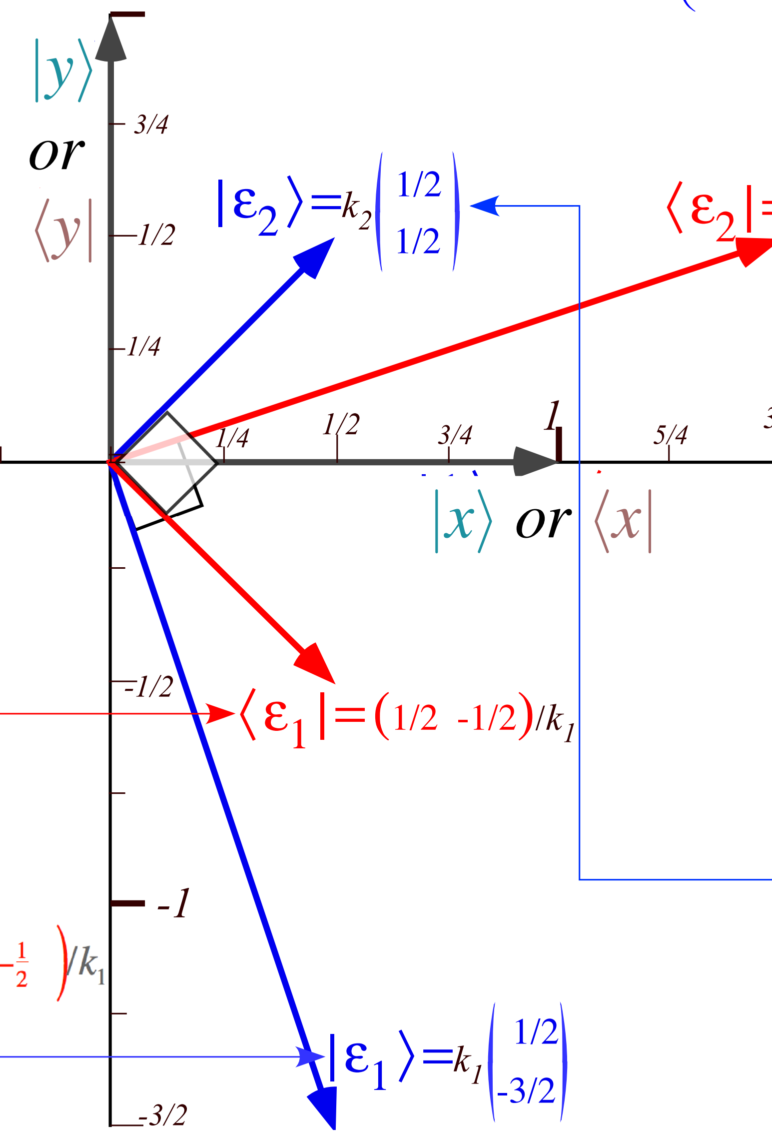
"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Eigen-bra-ket or projectors of matrix: $\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$



$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$
$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle,|y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j \neq k \\ \mathbf{P}_k & if: j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

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$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

*Operator expressions for orthonormality appear quite **different** from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

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*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

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Dirac δ-function

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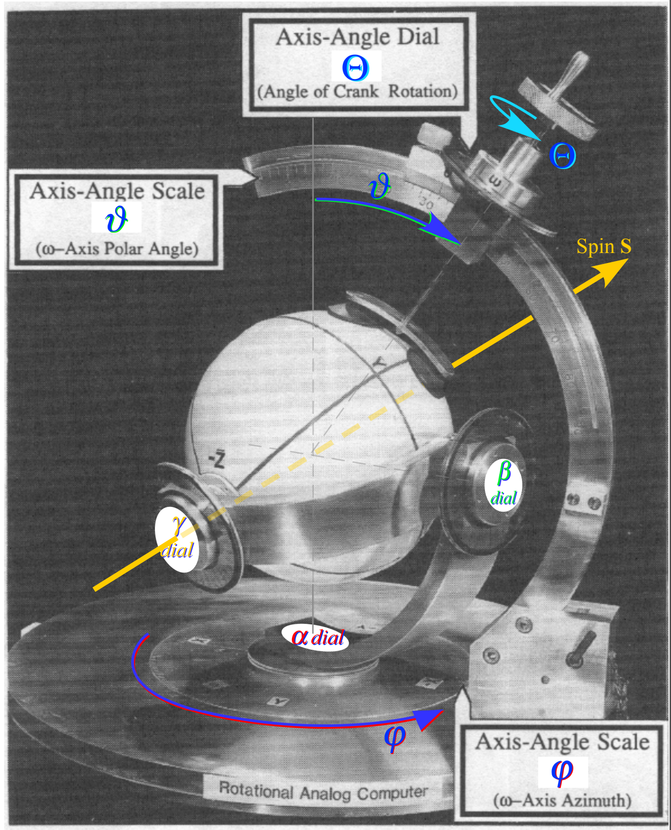
$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \qquad \ldots + \psi_i^*(x)\psi_j(x) + \psi_2(y)\psi_2^*(y) + \ldots . \rightarrow \int dx\, \psi_i^*(x)\psi_j(x)$$

However Schrodinger wavefunction notation $\psi(x) = \langle x|\psi\rangle$ shows quite a difference…

…particularly in the orthonormality integral.



(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

σ_A*-products 3D vector analysis and "Crazy-Thing-Theorem"*

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

*Idempotent (**P·P=P**) projectors (how eigenvalues⇒eigenvectors)*

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

U(2)⊃C₂ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

Circular-chiral-cycloton (C-Type) symmetry

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

➔ *Lagrange functional interpolation formula*

Diagonalizing Transformations (D-Ttran) from projectors

A Proof of Projector Completeness *(Truer-than-true by Lagrange interpolation)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

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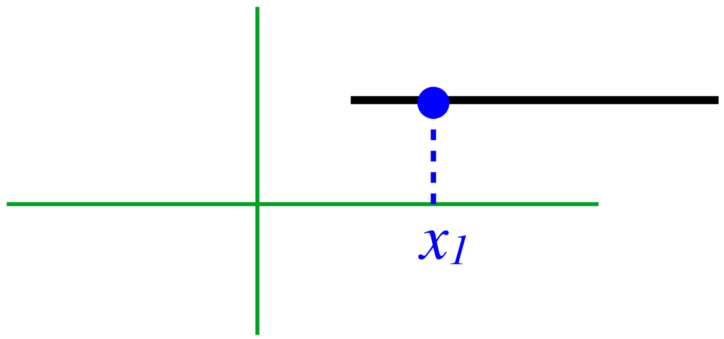

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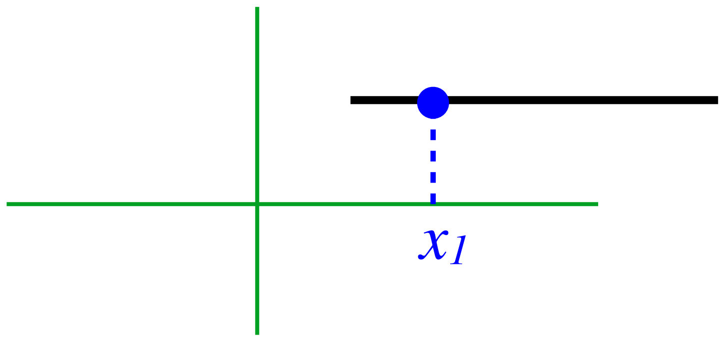

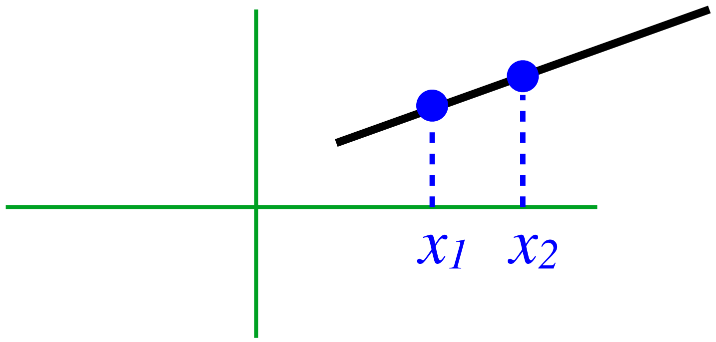

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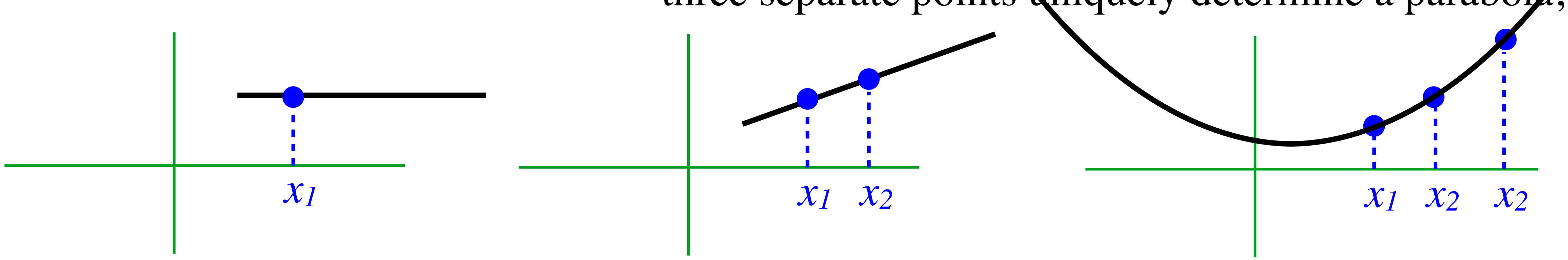

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All distinct values $\varepsilon_1 \neq \varepsilon_2 \neq \ldots \neq \varepsilon_N$ satisfy $\Sigma \mathbf{P}_k = \mathbf{1}$.

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All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_1-\varepsilon_j\right)}+\frac{\prod_{j\neq1}\left(\mathbf{M}-\varepsilon_j\mathbf{1}\right)}{\prod_{j\neq1}\left(\varepsilon_2-\varepsilon_j\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}+\frac{\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_2-\varepsilon_1\right)}=\frac{\left(\mathbf{M}-\varepsilon_2\mathbf{1}\right)-\left(\mathbf{M}-\varepsilon_1\mathbf{1}\right)}{\left(\varepsilon_1-\varepsilon_2\right)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{\left(\varepsilon_1-\varepsilon_2\right)}=\mathbf{1}\ (\text{for } all\ \varepsilon_j)$$

A Proof of Projector Completeness (Truer-than-true)

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}\qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at N points $x_1, x_2, \ldots x_N$.

$$L\big(f(x)\big)=\sum_{k=1}^{N} f(x_k)\,P_k(x)\qquad\text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}\left(x-x_j\right)}{\prod_{j\neq k}^{N}\left(x_k-x_j\right)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m(x)\qquad x=\sum_{m=1}^{N}x_m P_m(x)\qquad x^2=\sum_{m=1}^{N}x_m^2 P_m(x)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula→*Completeness formula* as $x\rightarrow\mathbf{M}$ and as $x_k\rightarrow\varepsilon_k$ and as $P_k(x_k)\rightarrow\mathbf{P}_k$

All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

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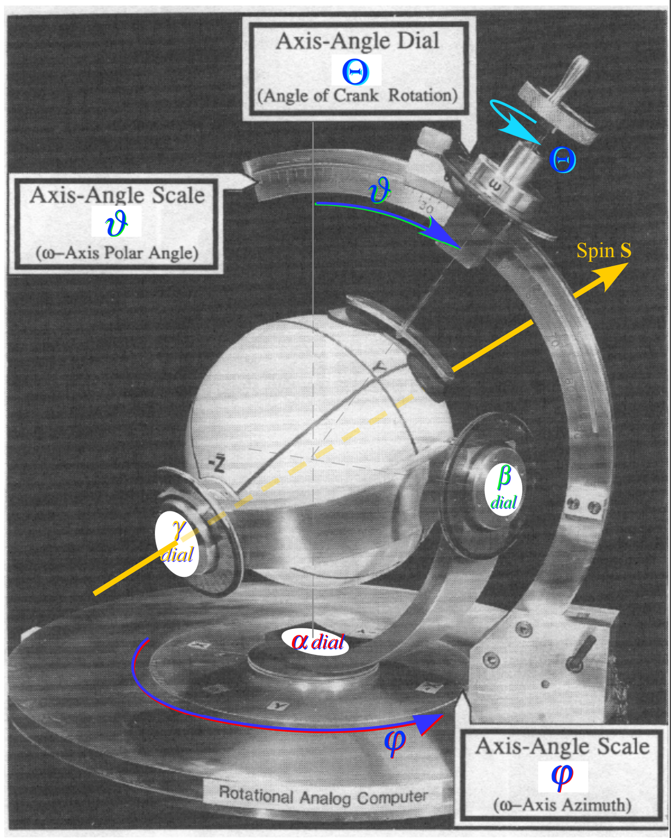
However, only *select* values ε_k work for eigen-forms $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ or orthonormality $\mathbf{P}_j\mathbf{P}_k=\delta_{jk}\mathbf{P}_k$.



(REVIEW) 2D classical HO compared to U(2) quantum 2-state system

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Algebra of Hamilton/Pauli hypercomplex operators $\{\sigma_A,\sigma_B,\sigma_C\}=\{\sigma_Z,\sigma_X,\sigma_Y\}$

σ_A*-products 3D vector analysis and "Crazy-Thing-Theorem"*

Eigensolutions by general matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent (**P·P=P**) *projectors (how eigenvalues⇒eigenvectors)*

Eigenvector orthonormality and completeness

Spectral Decompositions

Functional spectral decomposition

U(2)⊃C₂ ABCD group theory method to find 2D-HO eigenmodes and eigenvalues

Asymmetric-diagonal (AD-Type) symmetry

Bilateral-balanced (B-Type) symmetry

Circular-chiral-cycloton (C-Type) symmetry

Mixed ABCD symmetry examples

More theory of matrix diagonalization

Discussion of orthogonality vs. completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

➔ *Diagonalizing Transformations (D-Ttran) from projectors*

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Diagonalizing Transformations (D-Ttran) from projectors

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$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into inverse d-tran columns.

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$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

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$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

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Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

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Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

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Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are "easy"

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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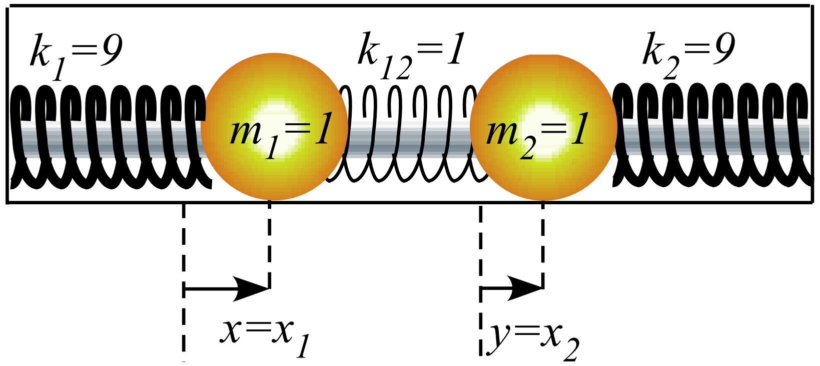
2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

➔ *2D-HO eigensolution example with bilateral (B-Type) symmetry* ←
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)



Analyzing 2D-HO beats and mixed mode eigen-solutions

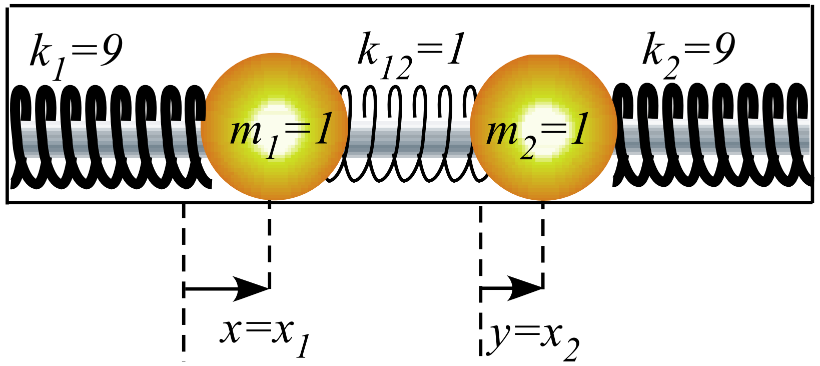
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$Det(\mathbf{K}) = 10 \cdot 10 - 1 = 99$

$Trace(\mathbf{K}) = 10 + 10 = 20$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-K_1)(K-K_2)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$



Analyzing 2D-HO beats and mixed mode eigen-solutions

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$$Det(\mathbf{K})=10\cdot 10-1=99$$
$$Trace(\mathbf{K})=10+10=20$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)=(K-K_1)(K-K_2)$

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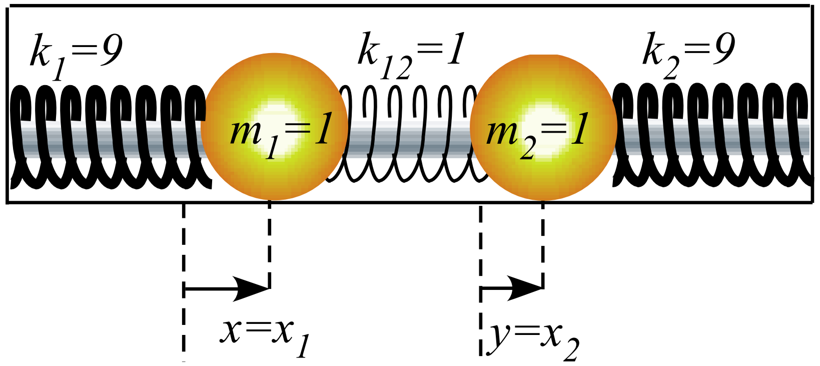

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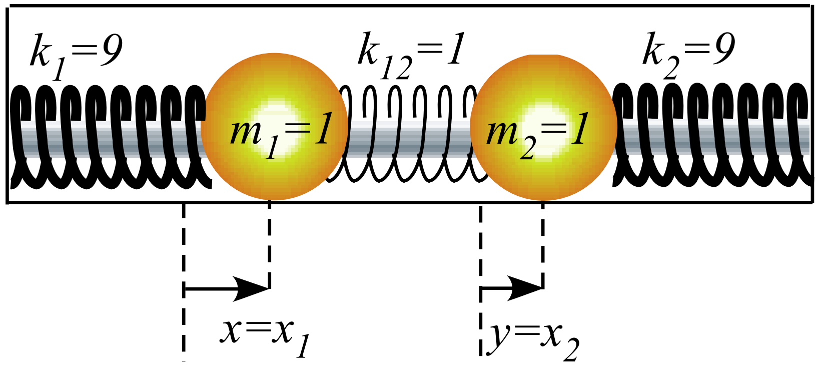

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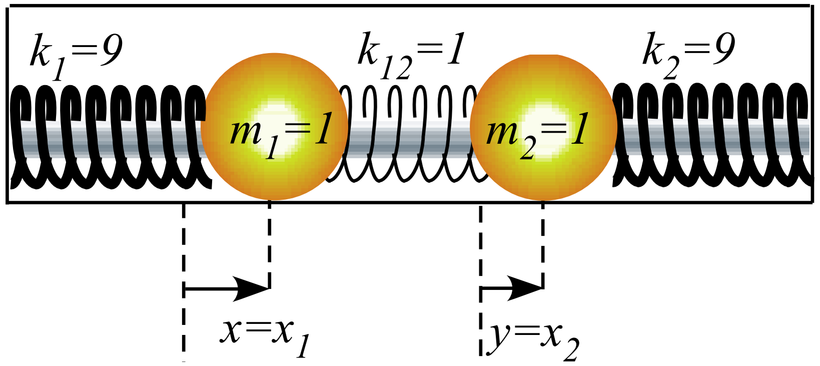
Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\ \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\ |\varepsilon_2\rangle\ \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



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Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

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$$|x(t)\rangle = |\varepsilon_1\rangle \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

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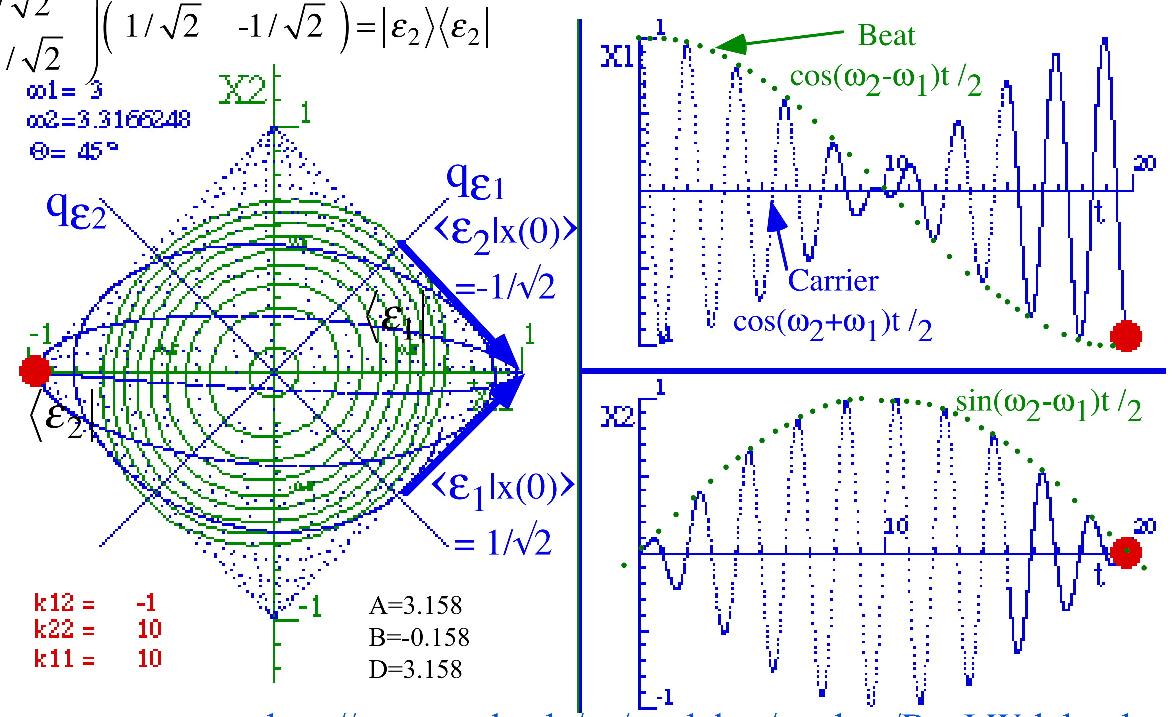
100% modulation (SWR=0) $\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}} \frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2}$

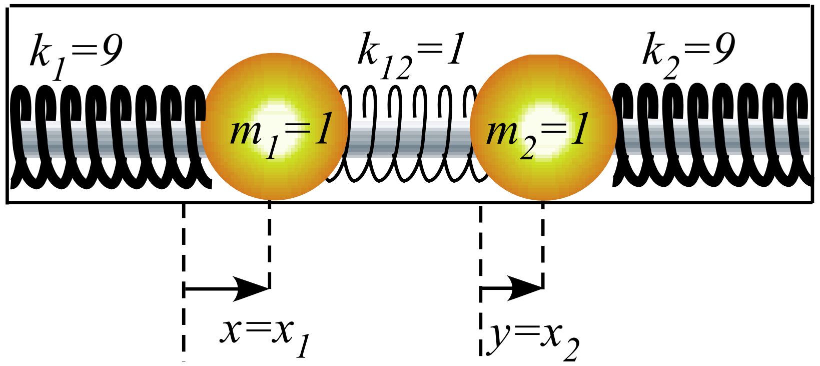
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{e^{-i\omega_1 t} + e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t} - e^{-i\omega_2 t}}{2} \end{pmatrix} = \frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2} \begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t} + e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t} - e^{i\frac{(\omega_1-\omega_2)}{2}t} \end{pmatrix}$$



http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

BoxIt (Beating) Simulation

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.



Analyzing 2D-HO beats and mixed mode eigen-solutions

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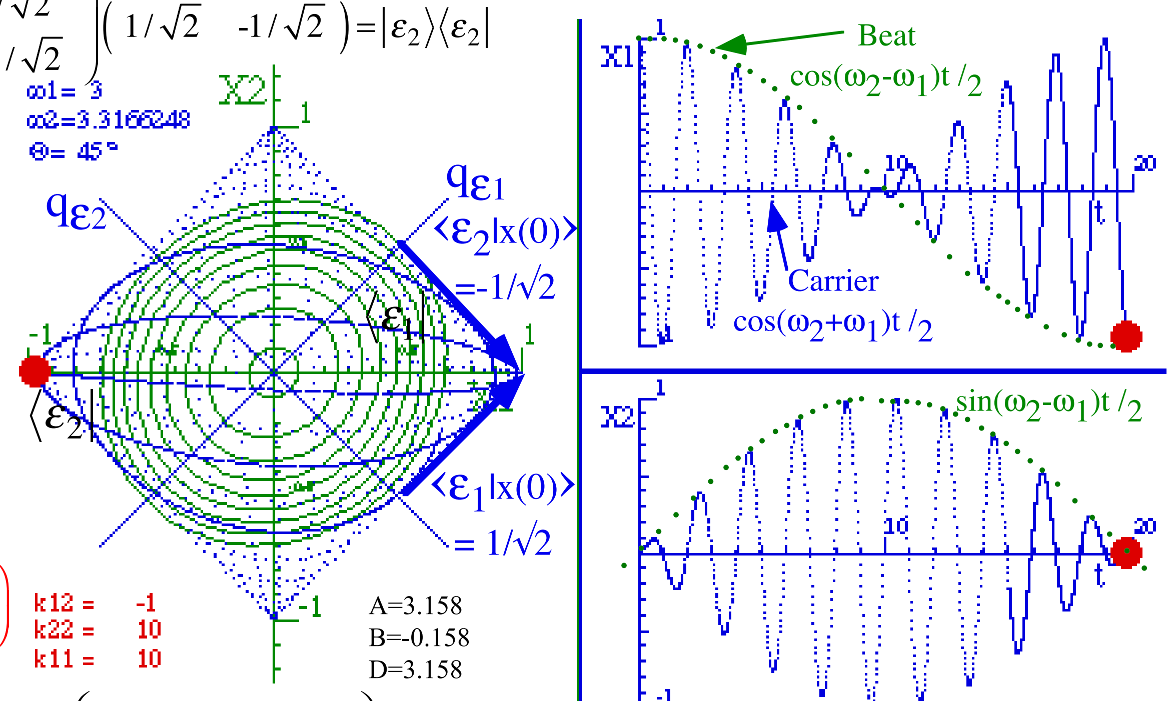

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$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\frac{(\omega_2-\omega_1)t}{2} \\ i\sin\frac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

Note the i phase

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2D harmonic oscillator equations

Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions

Geometric method

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry

Mixed mode beat dynamics and fixed π/2 phase

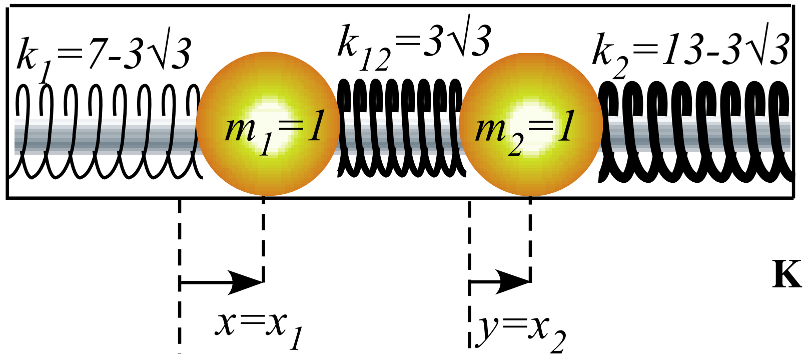
➡ *2D-HO eigensolution example with asymmetric (A-Type) symmetry* ⬅

Initial state projection, mixed mode beat dynamics with variable phase

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (ABCD-Types)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$$

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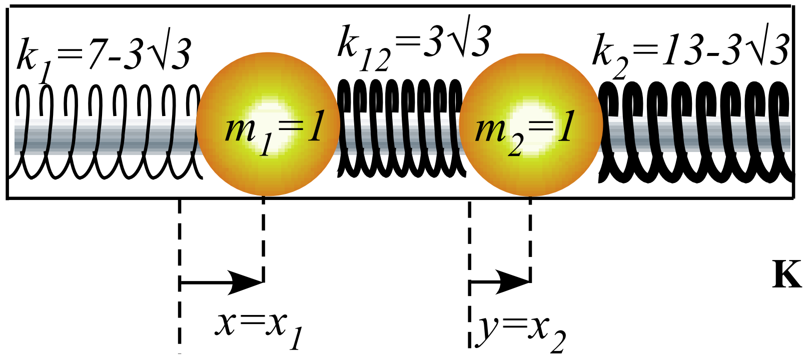

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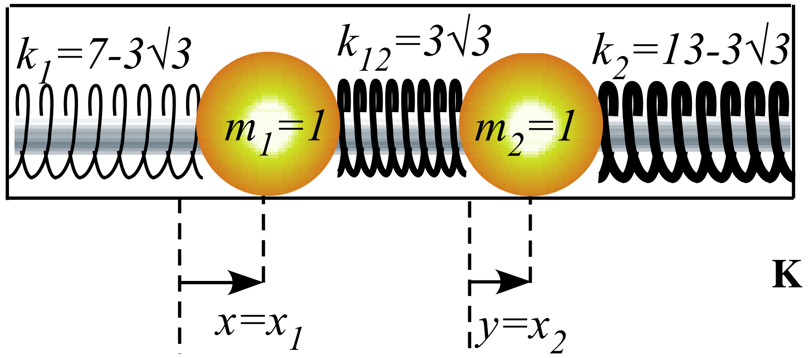

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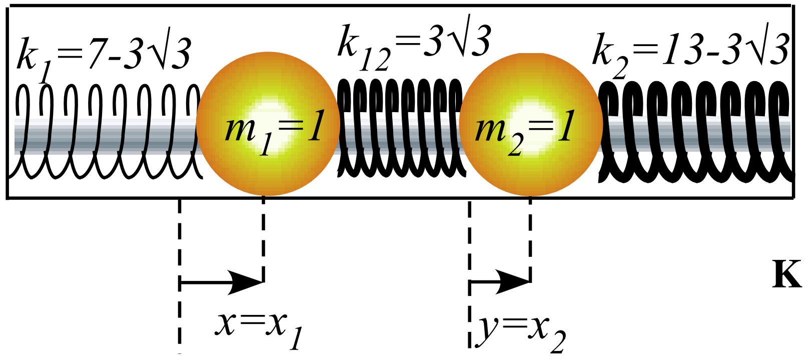

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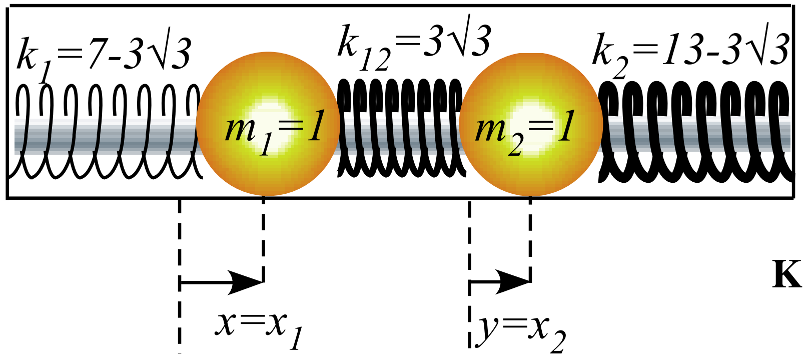

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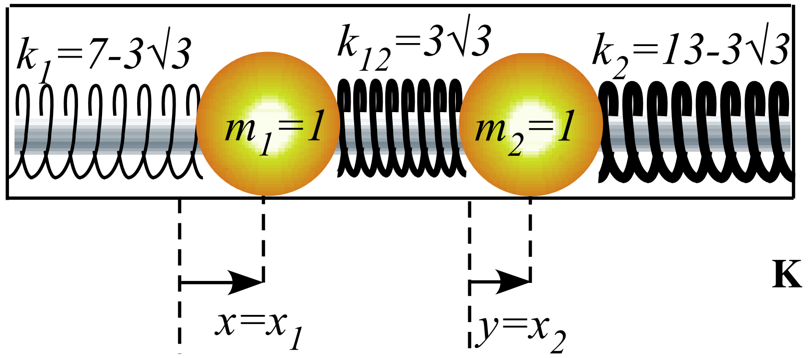
$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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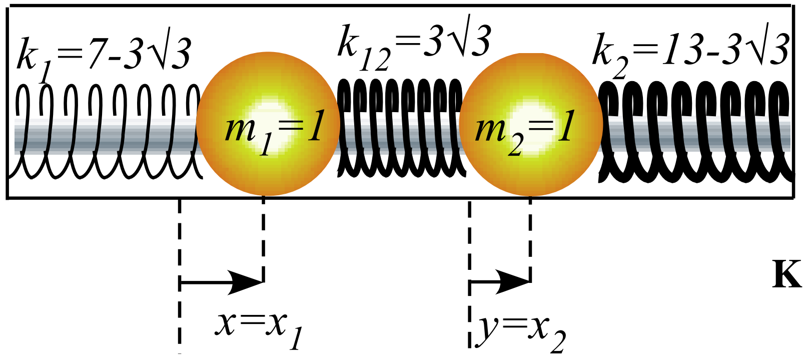
$$=\frac{\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix}\begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix}K_{11}-K_1 & K_{12}\\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix}7-4 & -3\sqrt{3}\\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix}3 & -3\sqrt{3}\\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix}-1/2\\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

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2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

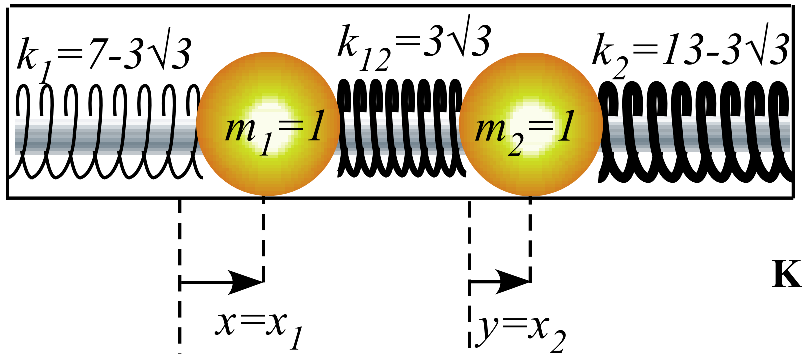
2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
➡ *Initial state projection, mixed mode beat dynamics with variable phase* ⬅

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
Hamilton-Pauli spinor symmetry (ABCD-Types)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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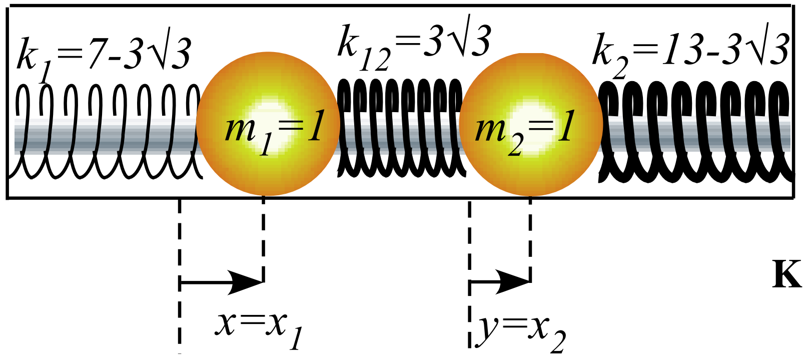
$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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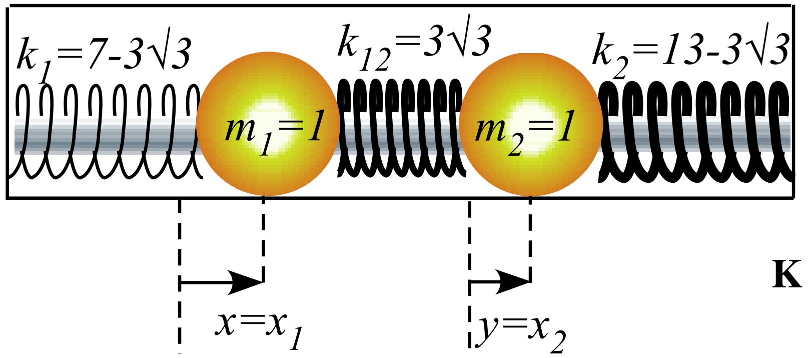
$$= \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\tfrac{\sqrt{3}}{2}\right) + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\tfrac{1}{2}\right)$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

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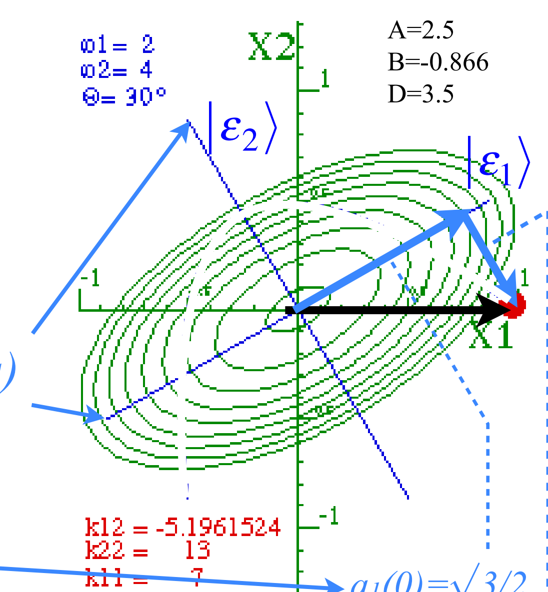
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \left(\frac{\sqrt{3}}{2}\right) + \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \left(-\frac{1}{2}\right)$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

$$q_1(t) = \frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t) = -\frac{1}{2}\cos 4t$$

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$q_1(0) = \sqrt{3}/2$

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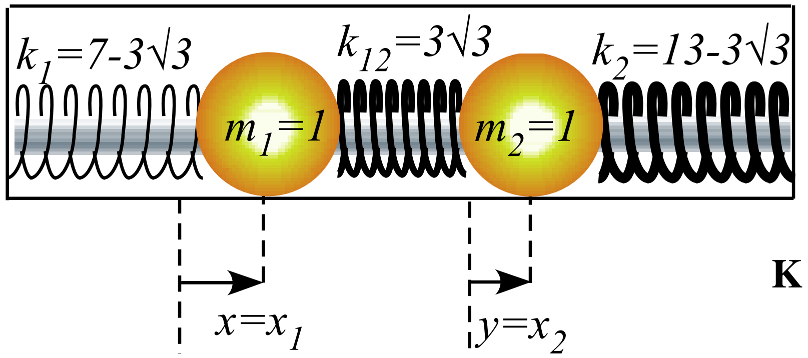
Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$= \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\frac{\sqrt{3}}{2}\right) + \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\frac{1}{2}\right)$$

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Using $\cos 4t = 2\cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2(t) = -\frac{1}{2}2\cos^2 2t + \frac{1}{2} = -\frac{4}{3}\left[q_1(t)\right]^2 + \frac{1}{2}$$

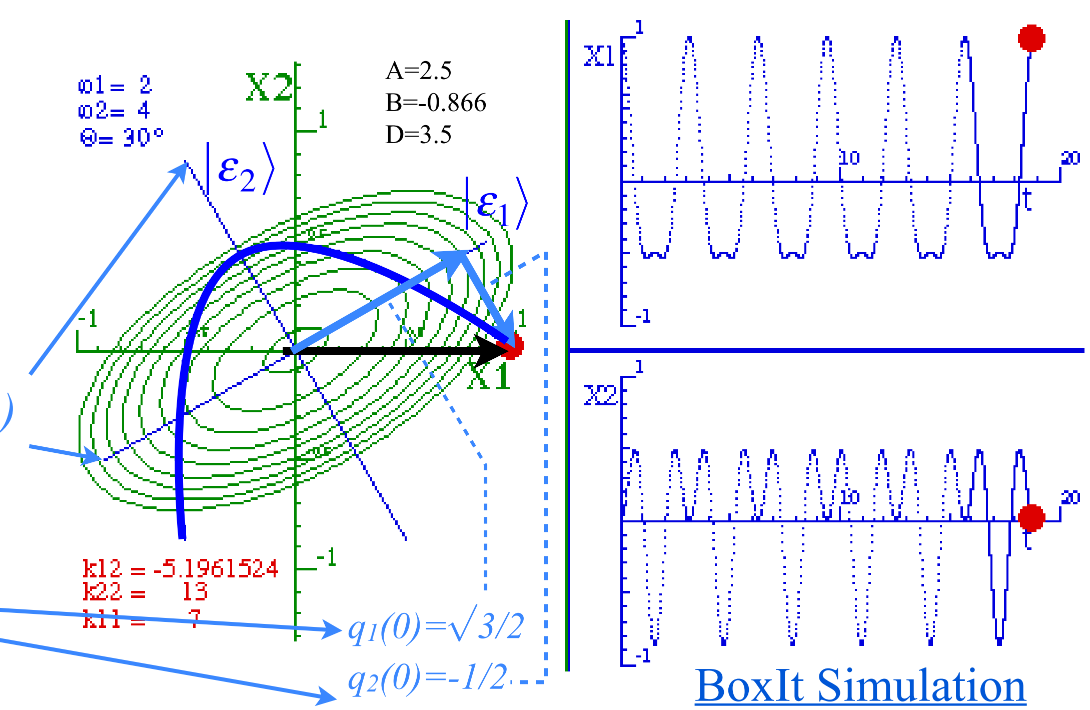
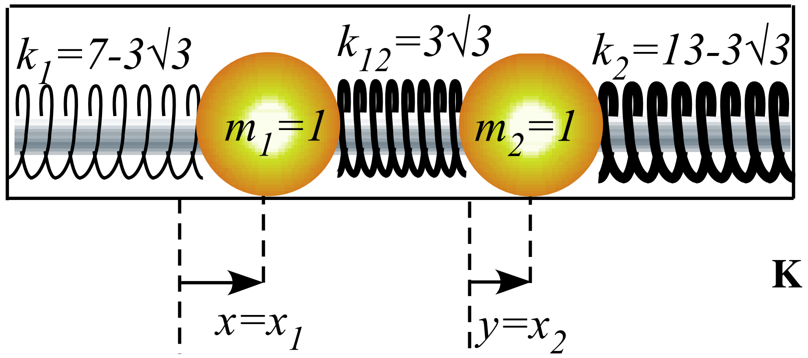


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.
http://www.uark.edu/ua/modphys/markup/BoxItWeb.html

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}(\tfrac{\sqrt{3}}{2})+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}(-\tfrac{1}{2})$$

(Note projection of $\mathbf{x}(0)$ *onto eigen-axes)*

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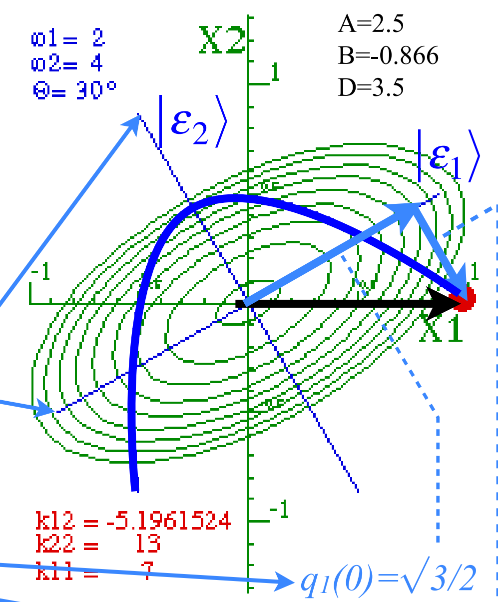
$$q_2(t)=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}[q_1(t)]^2+\frac{1}{2}$$

$$\mathbf{x}(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

$q_1(0)=\sqrt{3/2}$

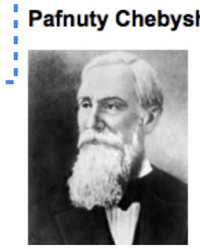
$q_2(0)=-1/2$



Example of a Tschebycheff Polynomial order 2

BoxIt Simulation

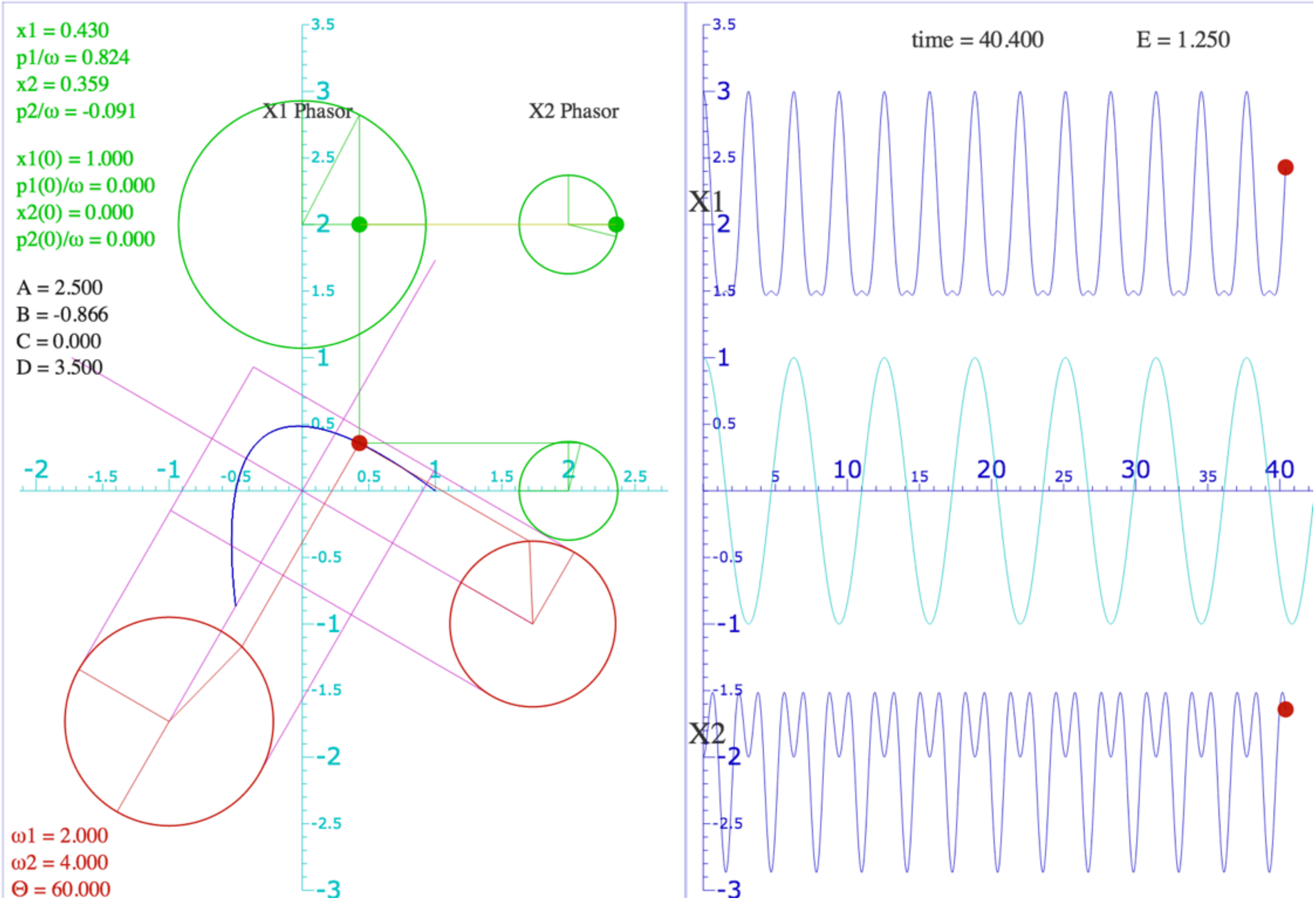
Pafnuty Chebyshev



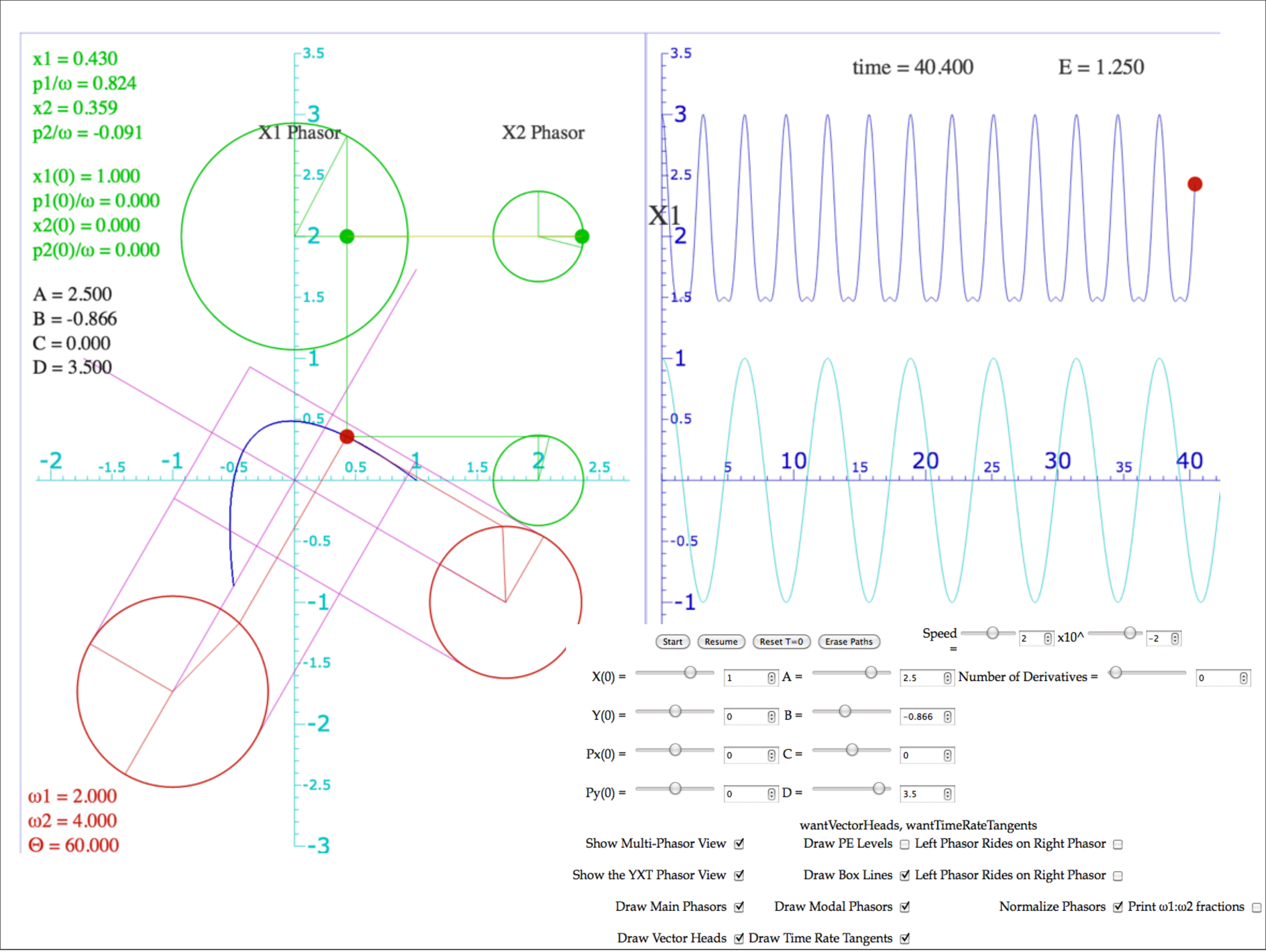
Pafnuty Lvovich Chebyshev was a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff. Wikipedia

Born: May 16, 1821, Borovsk

Died: December 8, 1894, Saint Petersburg



BoxIt Simulation



x1 = 0.430
p1/ω = 0.824
x2 = 0.359
p2/ω = -0.091
x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.000
A = 2.500
B = -0.866
C = 0.000
D = 3.500
ω1 = 2.000
ω2 = 4.000
Θ = 60.000
X1 Phasor
X2 Phasor
time = 40.400
E = 1.250
X1
Start
Resume
Reset T=0
Erase Paths
Speed =
x10^
X(0) =
A =
Number of Derivatives =
Y(0) =
B =
Px(0) =
C =
Py(0) =
D =
wantVectorHeads, wantTimeRateTangents
Show Multi-Phasor View
Draw PE Levels
Left Phasor Rides on Right Phasor
Show the YXT Phasor View
Draw Box Lines
Left Phasor Rides on Right Phasor
Draw Main Phasors
Draw Modal Phasors
Normalize Phasors
Print ω1:ω2 fractions
Draw Vector Heads
Draw Time Rate Tangents

2D harmonic oscillator equations
Lagrangian and matrix forms and Reciprocity symmetry

2D harmonic oscillator equation eigensolutions
Geometric method
Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $\mathbf{P}\cdot\mathbf{P}=\mathbf{P}$*)*
Spectral Decompositions
Functional spectral decomposition
Orthonormality vs. Completeness vis-a`-vis Operator vs. State
Lagrange functional interpolation formula
Diagonalizing Transformations (D-Ttran) from projectors

2D-HO eigensolution example with bilateral (B-Type) symmetry
Mixed mode beat dynamics and fixed π/2 phase

2D-HO eigensolution example with asymmetric (A-Type) symmetry
Initial state projection, mixed mode beat dynamics with variable phase

➤ *ANALOGY: 2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$ ➤
Hamilton-Pauli spinor symmetry (ABCD-Types)

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$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

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that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Both have 4 parameters *($2^2 = 2+2$)*

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

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For C=0
Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

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$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

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$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$
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QM vs. Classical Equations are identical

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Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

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$$= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2$$

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For C=0
Is form of 2D Hooke harmonic oscillator

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Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

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Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

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