Lecture 21 C_N Wave Modes
Tue. 3.29.2016

C_N-Symmetric Wave Modes

(Ch. 5 of Unit 4 3.29.15)

Wave resonance in cyclic C_n symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Projector analysis of 2D-HO modes and mixed mode dynamics*
  - *½-Sum-½-Diff-Identity for resonant beat analysis*
    - *Mode frequency ratios and continued fractions*
    - *Geometry of that 90°-phase lag (again)*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Deriving C_3 projectors*
    - *Deriving and labeling moving wave modes*
    - *Deriving dispersion functions and degenerate standing waves*
      - *Examples by* WaveIt *animation*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 moving waves and degenerate standing waves*
  - *C_6 dispersion functions for 1^{st}, 2^{nd}, and 3^{rd}-neighbor coupling*
  - *C_6 dispersion functions split by C-type* symmetry*(complex, chiral, ...)*

C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

Wave resonance in cyclic C_n symmetry

➜ *Harmonic oscillator with cyclic C_2 symmetry*

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C_6 dispersion functions split by C-type symmetry*(complex, chiral, ...)*

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Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry (*B*-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with *B*-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\sigma_B$$

$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\sigma_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

Wave resonance in cyclic C_n symmetry

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$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$=A\cdot\mathbf{1} \qquad +B\cdot\sigma_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
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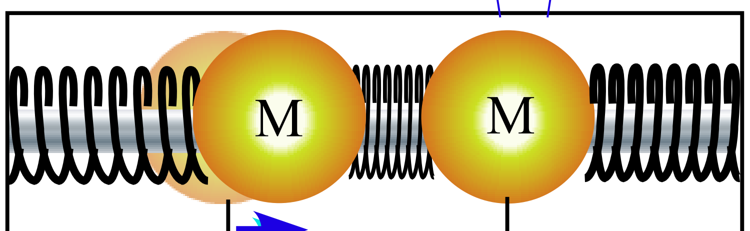
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

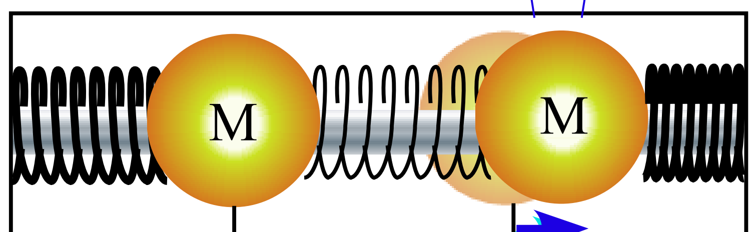
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad x_0=0$

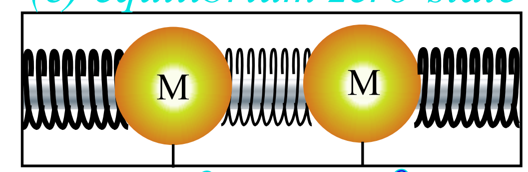
(b) unit base state $|\sigma_B\rangle=\sigma_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



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Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

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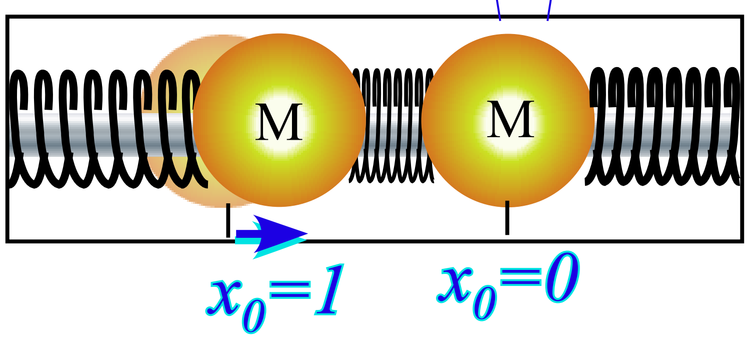

(b) unit base state $|\sigma_B\rangle = \sigma_B|\mathbf{1}\rangle$

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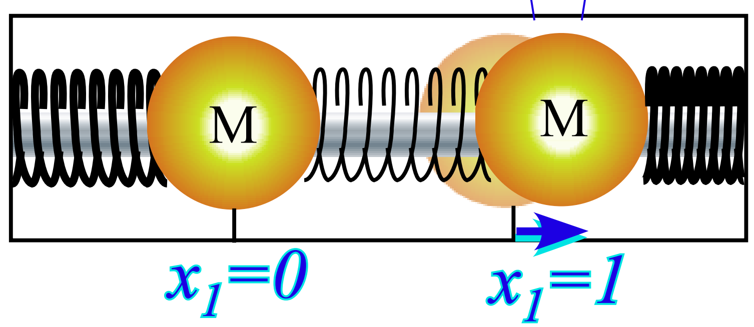

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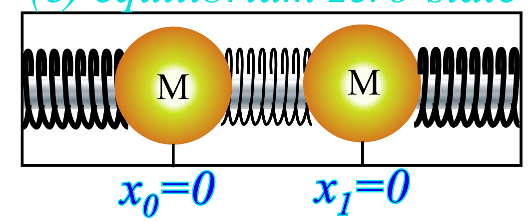

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

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Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

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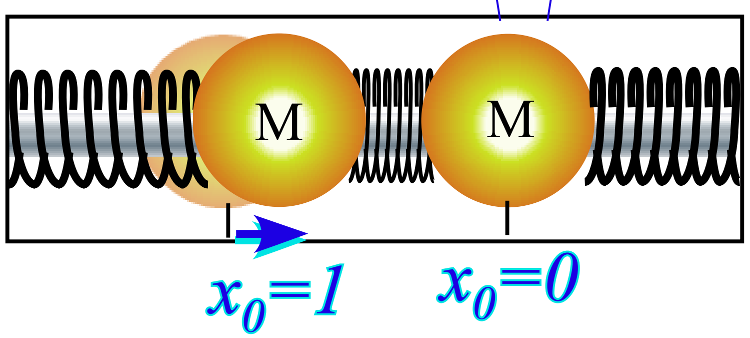

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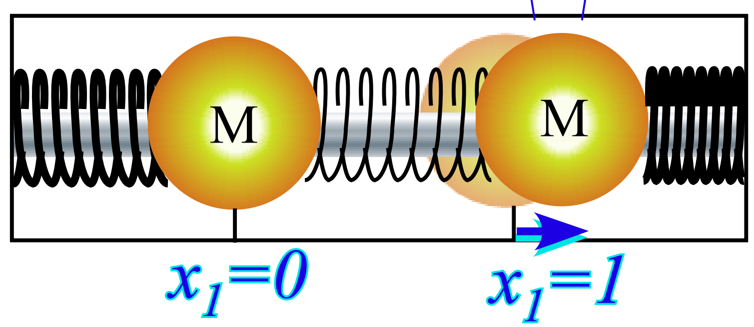

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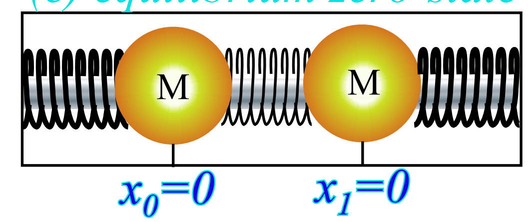

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Harmonic oscillator with cyclic C_2 symmetry

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C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

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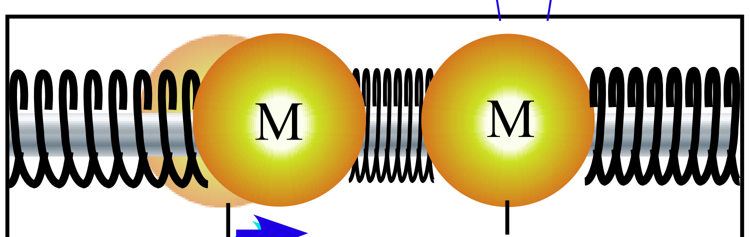

$x_0=1$ $\quad x_0=0$

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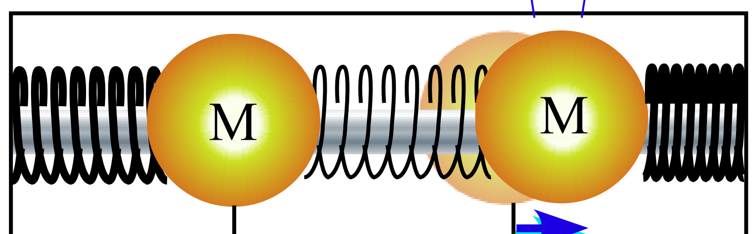

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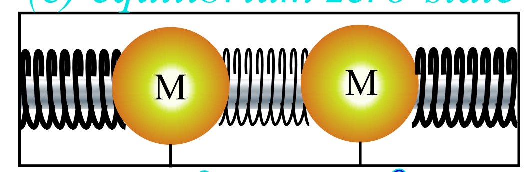

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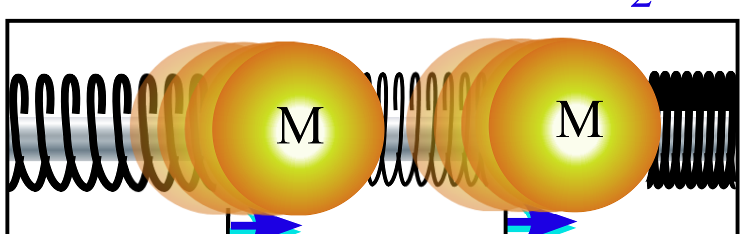
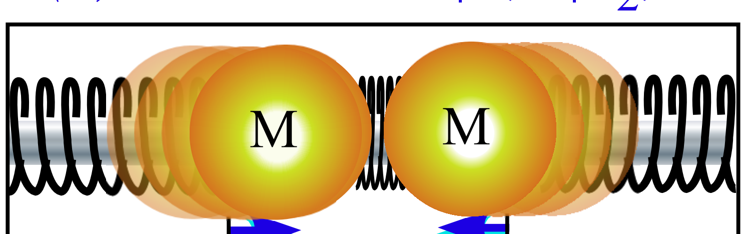
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



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C_2	$\mathbf{1}$	σ_B
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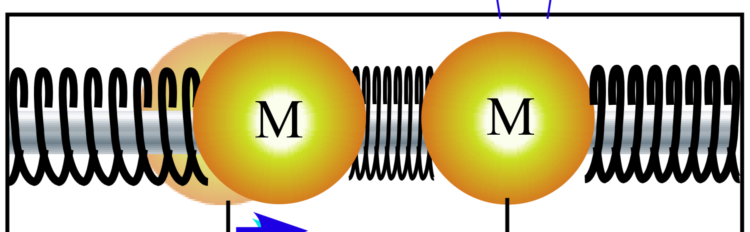

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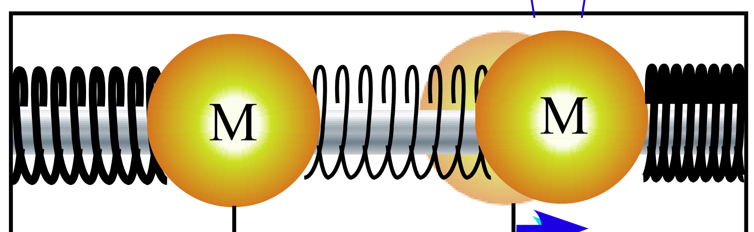

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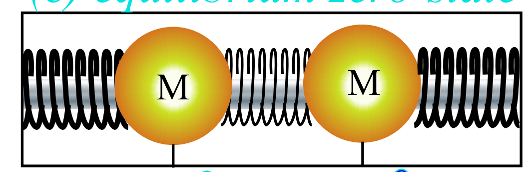
(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



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$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

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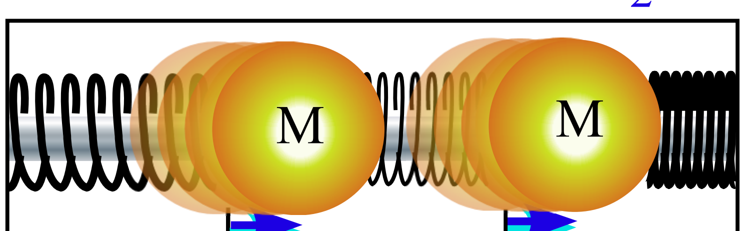
C_2 symmetry (B-type) modes

Mode state projection:

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



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$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$
$=(|0\rangle+|2\rangle)/\sqrt{2}$
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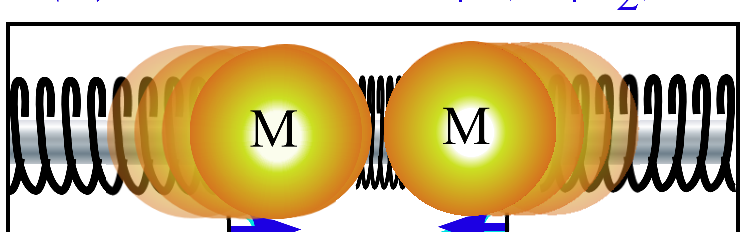

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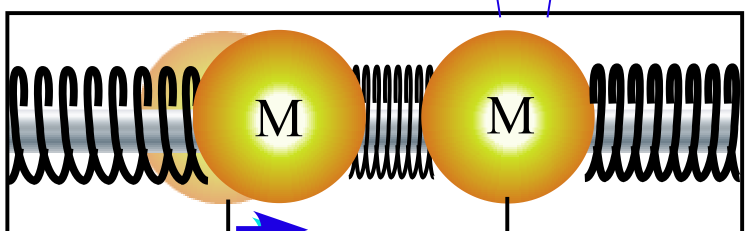

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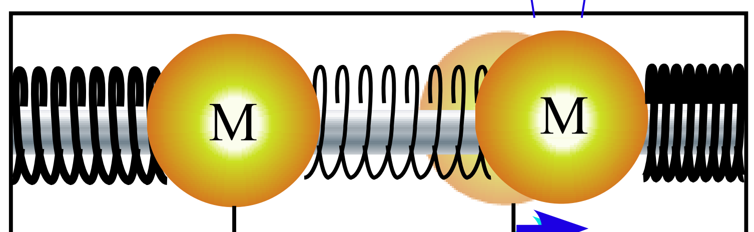

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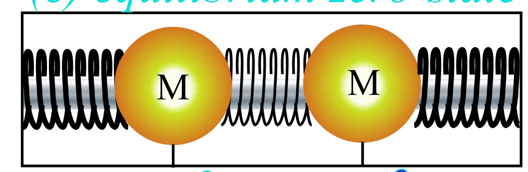

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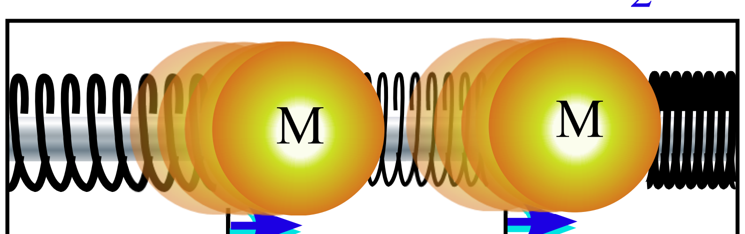

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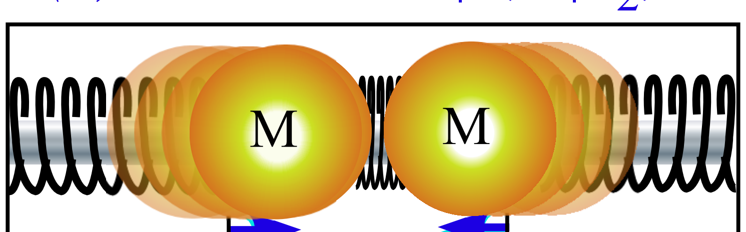

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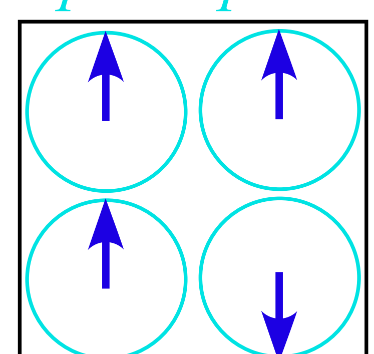
C_2 mode phase & character tables

p= position point (modulo-2)

p=0 p=1



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

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Hamiltonian matrix **H** or spring-constant matrix $\mathbf{K}=\mathbf{H}^2$ with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\sigma_B$$

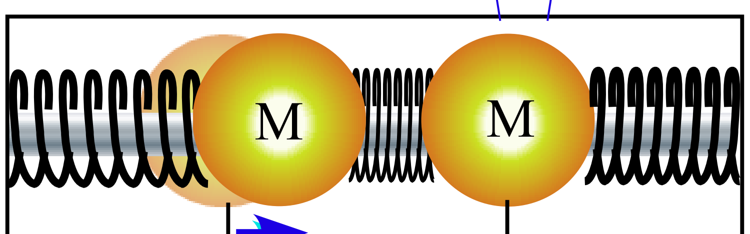
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\sigma_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

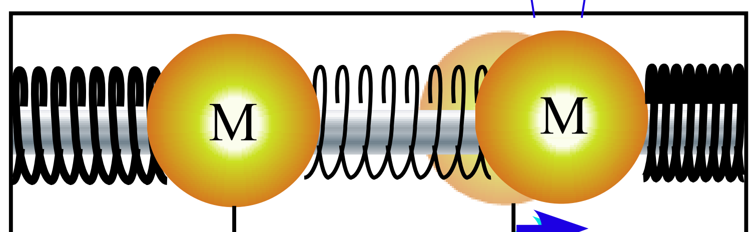
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $x_0=0$

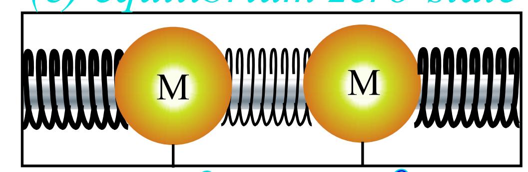
(b) unit base state $|\sigma_B\rangle=\sigma_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $x_1=0$

Note ½-sum-½-diff relations

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

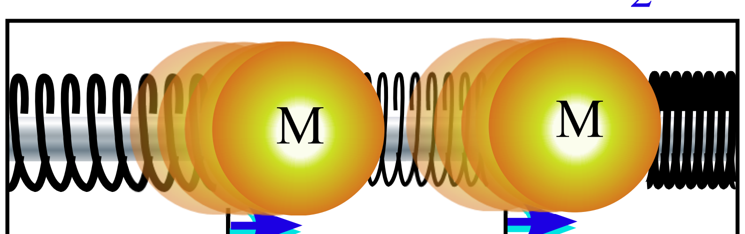
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\mathbf{1}+\sigma_B)/2$ and $\mathbf{P}^{(-)}=(\mathbf{1}-\sigma_B)/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

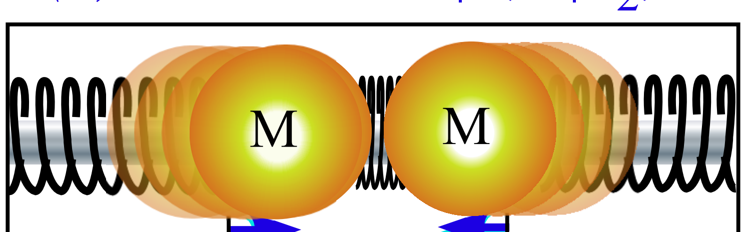
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $x_1=-1/\sqrt{2}$

Mode state projection:

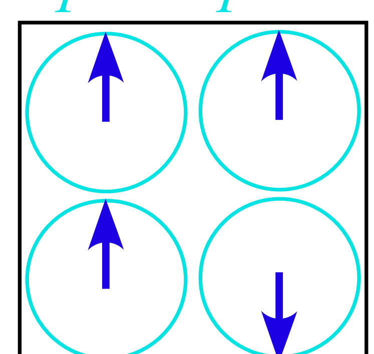
$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2} = (|0\rangle+|2\rangle)/\sqrt{2} = (|\mathbf{1}\rangle+|\sigma_B\rangle)/\sqrt{2}$$

$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2} = (|0\rangle-|2\rangle)/\sqrt{2} = (|\mathbf{1}\rangle-|\sigma_B\rangle)/\sqrt{2}$$

C_2 mode phase & character tables

p= position point (modulo-2)

p=0 *p=1*



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

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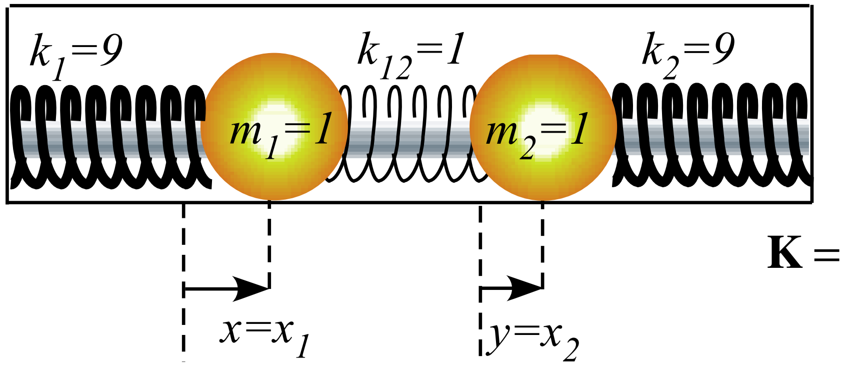

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$$Det(\mathbf{K})=10\cdot 10-1=99$$
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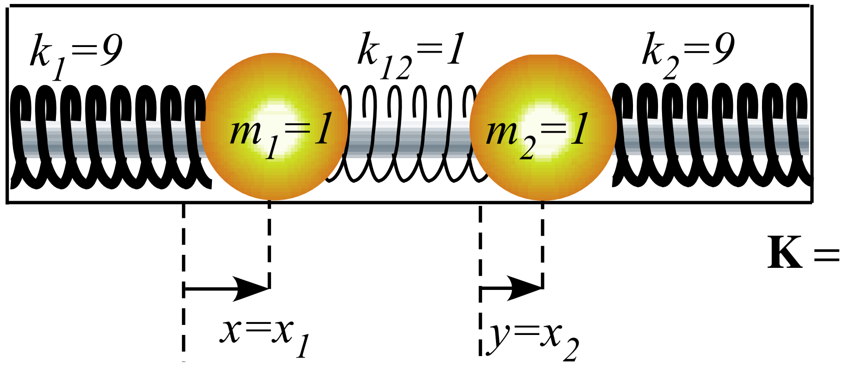
Projector analysis of 2D-HO modes and mixed mode dynamics



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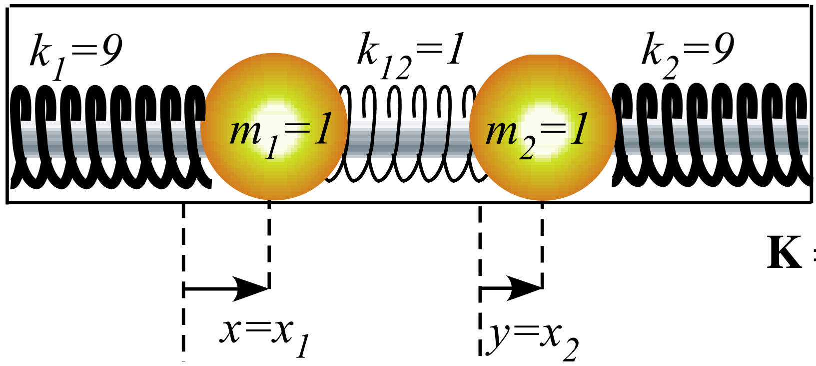

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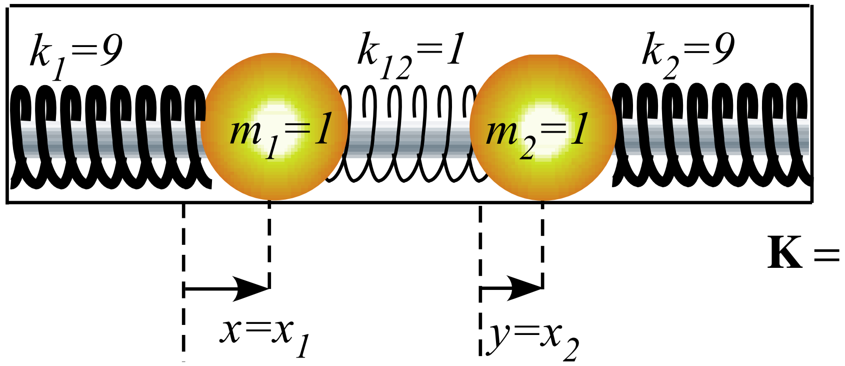

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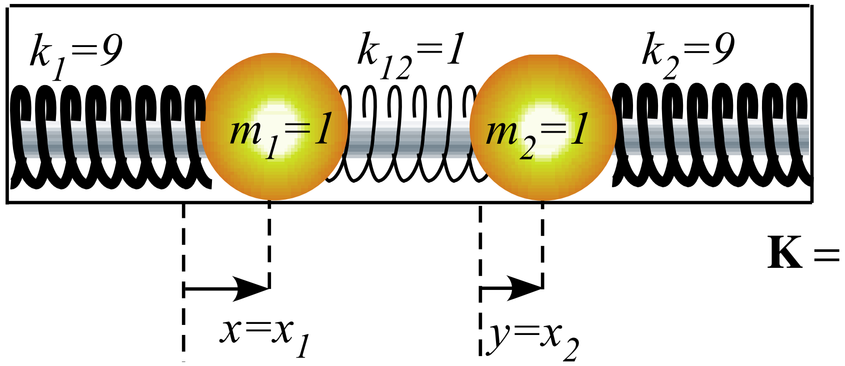

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...and eigen-ket-bras

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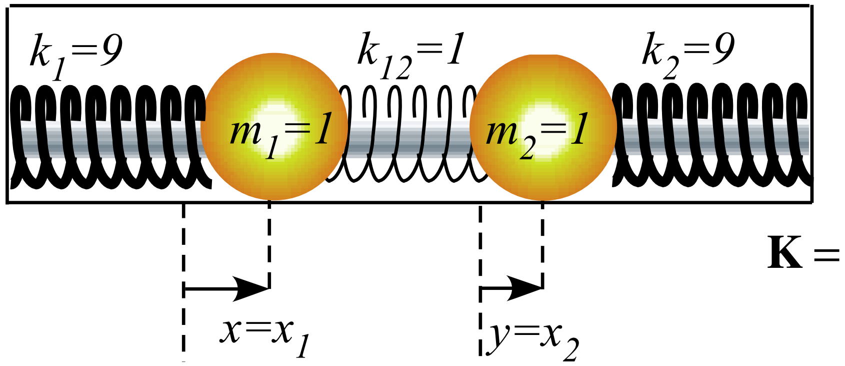
Projector analysis of 2D-HO modes and mixed mode dynamics



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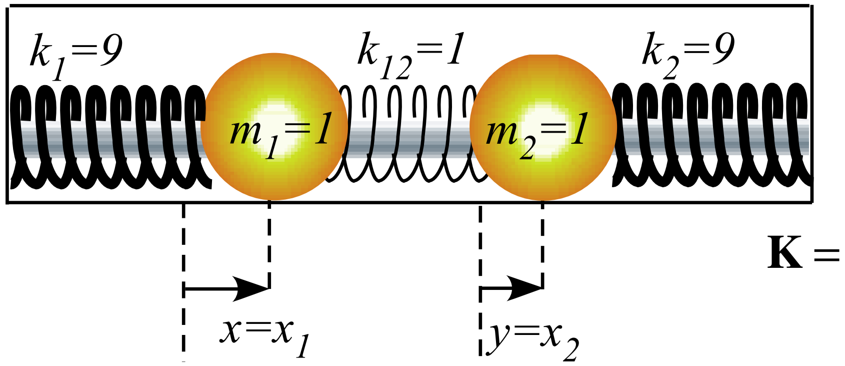

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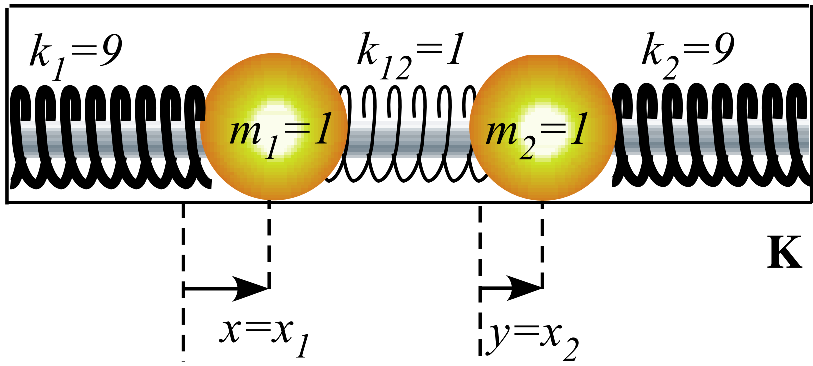

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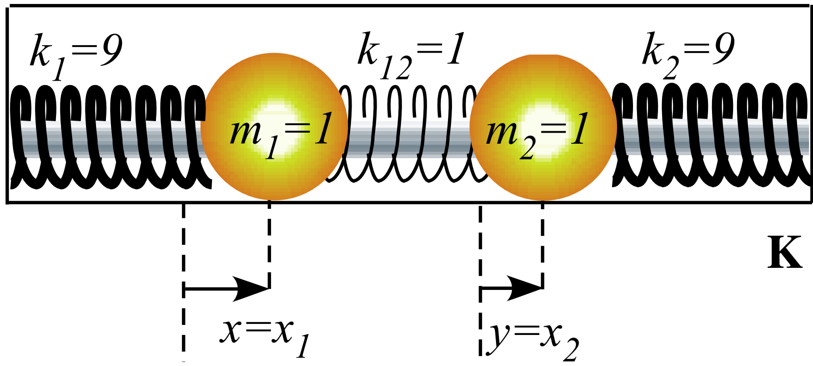
50-50 mix-mode dynamics results: $|\mathbf{x}(t)\rangle=\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{i\omega_1 t}+\tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{i\omega_2 t}=\tfrac{1}{\sqrt{2}}e^{i\omega_1 t}|\varepsilon_1\rangle+\tfrac{1}{\sqrt{2}}e^{i\omega_2 t}|\varepsilon_2\rangle$

Projector analysis of 2D-HO modes and mixed mode dynamics
½-Sum-½-Diff-Identity for resonant beat analysis



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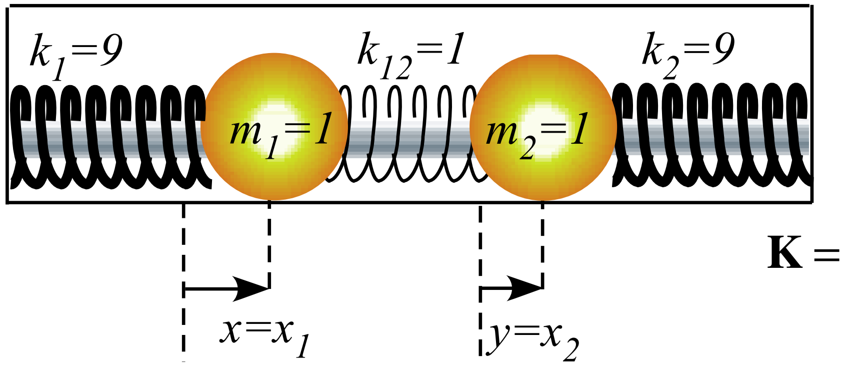

Projector analysis of 2D-HO modes and mixed mode dynamics
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...and eigen-ket-bras

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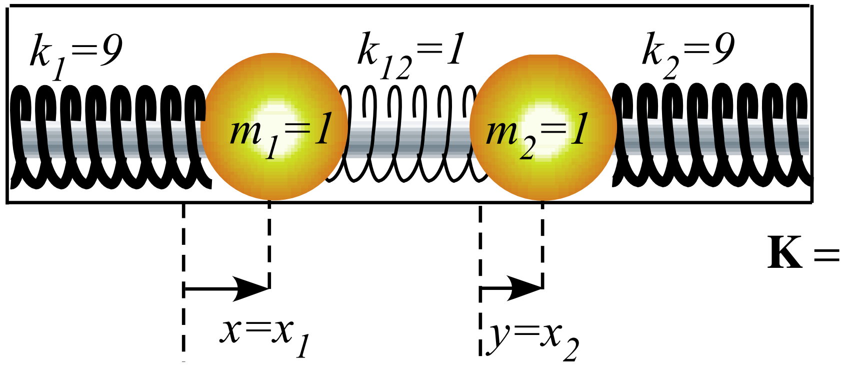

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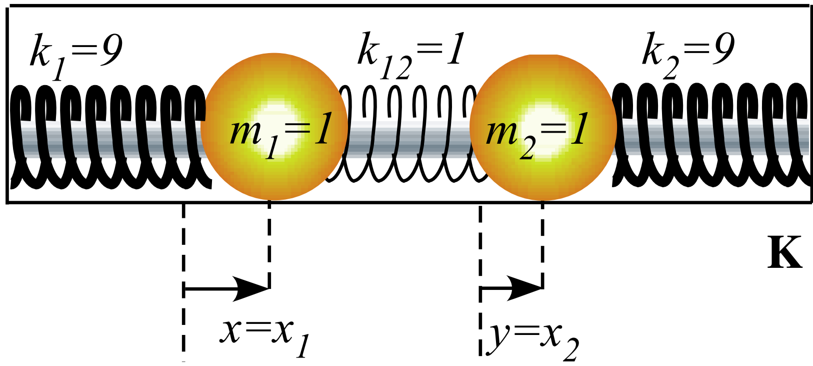

Projector analysis of 2D-HO modes and mixed mode dynamics
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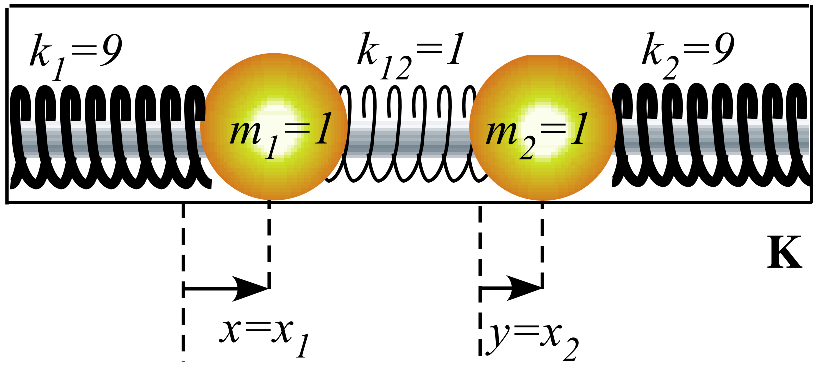
$$=ie^{\frac{i}{2}(a+b)}\sin\left(\tfrac{1}{2}(a-b)\right)$$

$$\cos\phi=\frac{e^{ix}+e^{-ix}}{2}$$
$$\sin\phi=\frac{e^{ix}+e^{-ix}}{2i}$$

Projector analysis of 2D-HO modes and mixed mode dynamics
½-Sum-½-Diff-Identity for resonant beat analysis



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...and eigen-ket-bras

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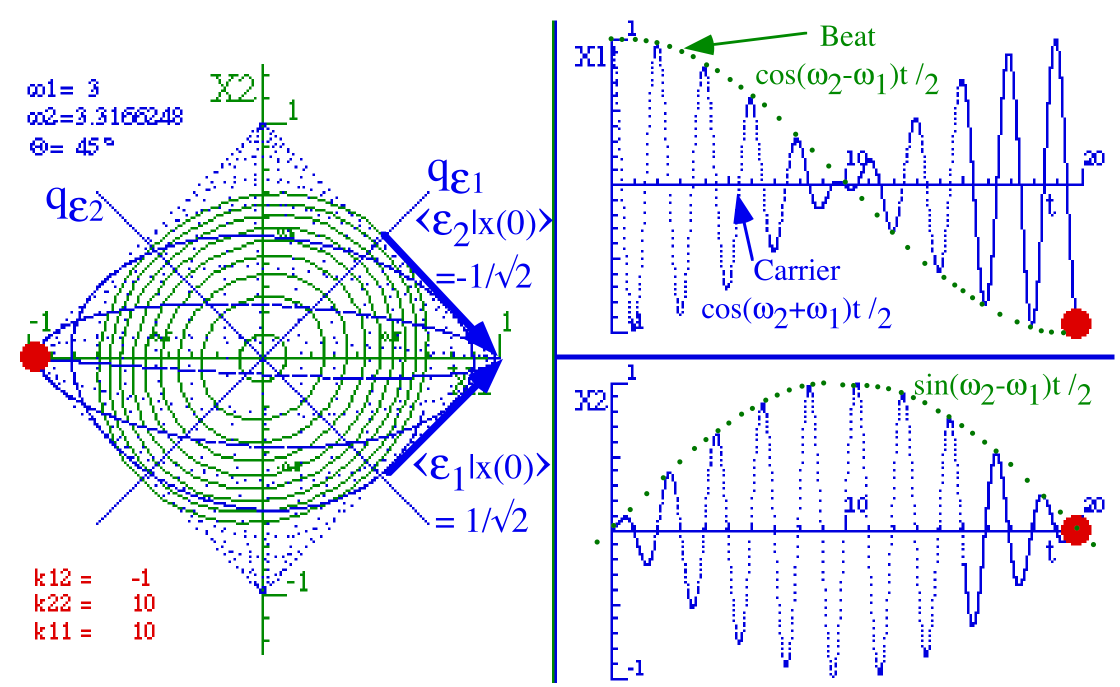
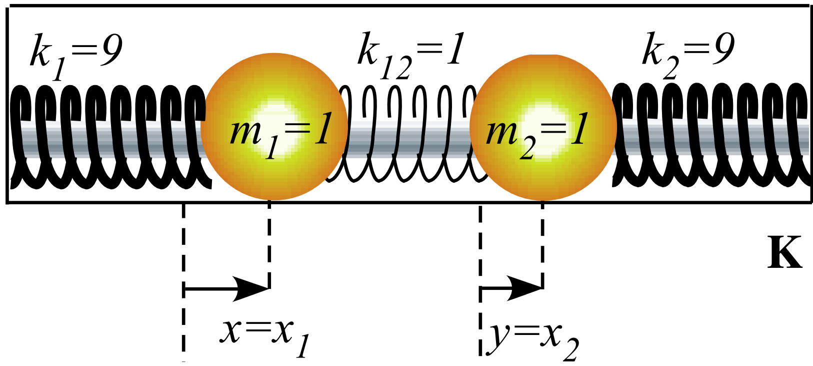
BoxIt Web Simulation

Coupled Oscillators $K_{11}=10$, $K_{12}=-1$





Projector analysis of 2D-HO modes and mixed mode dynamics
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...and eigen-ket-bras

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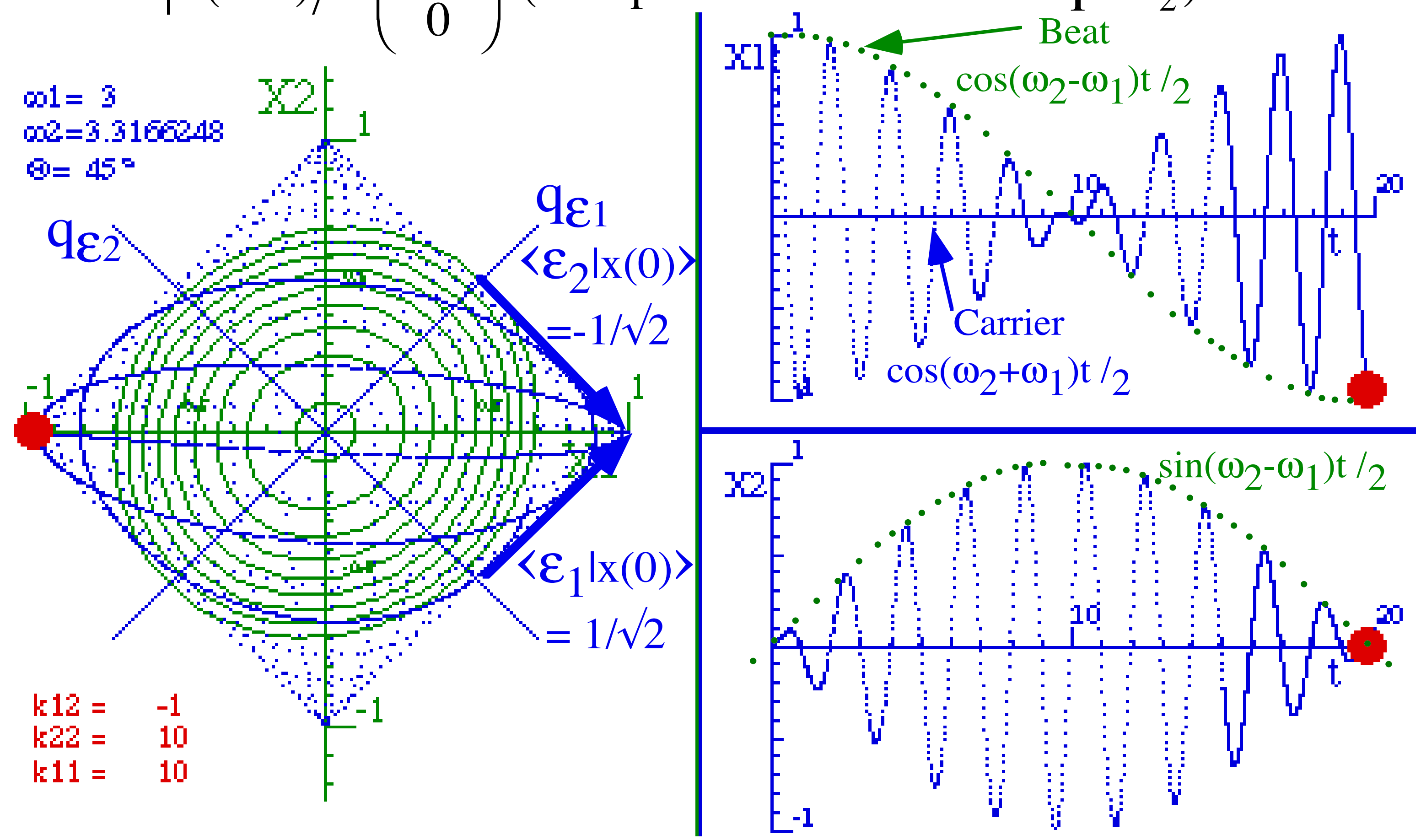
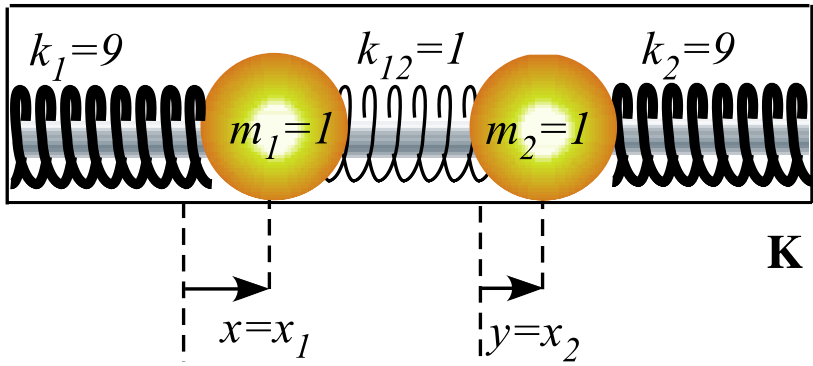


Fig. 2.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

Projector analysis of 2D-HO modes and mixed mode dynamics
½-Sum-½-Diff-Identity for resonant beat analysis



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$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2}, 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

...and eigen-ket-bras

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2}, \text{-}1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Apply projector sum $\mathbf{1}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$ to initial state $|\mathbf{x}(t{=}0)\rangle=\begin{pmatrix}1\\0\end{pmatrix}$ (Completeness Relation $\mathbf{1{=}P_1}+\mathbf{P}_2$)

$$\mathbf{1}|\mathbf{x}(0)\rangle=\left[\tfrac{1}{2}\begin{pmatrix} 1 & +1 \\ +1 & 1\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix}$$

50-50 mix-mode dynamics results:

$$\begin{pmatrix} x_1(t) \\ x_2(2)\end{pmatrix}=e^{\frac{i}{2}(\omega_1+\omega_2)t}\,\frac{1}{2}\begin{pmatrix} \cos\frac{1}{2}(\omega_1-\omega_2)t \\ i\sin\frac{1}{2}(\omega_1-\omega_2)t\end{pmatrix}$$

Note the ***i*** *phase*

Using ½-sum-½-diff-identity:

$$\tfrac{1}{2}(e^{ia}-e^{ib})=e^{\frac{i}{2}(a+b)}\tfrac{1}{2}(e^{\frac{i}{2}(a-b)}-e^{-\frac{i}{2}(a-b)})$$

$$=ie^{\frac{i}{2}(a+b)}\sin\left(\tfrac{1}{2}(a-b)\right)$$

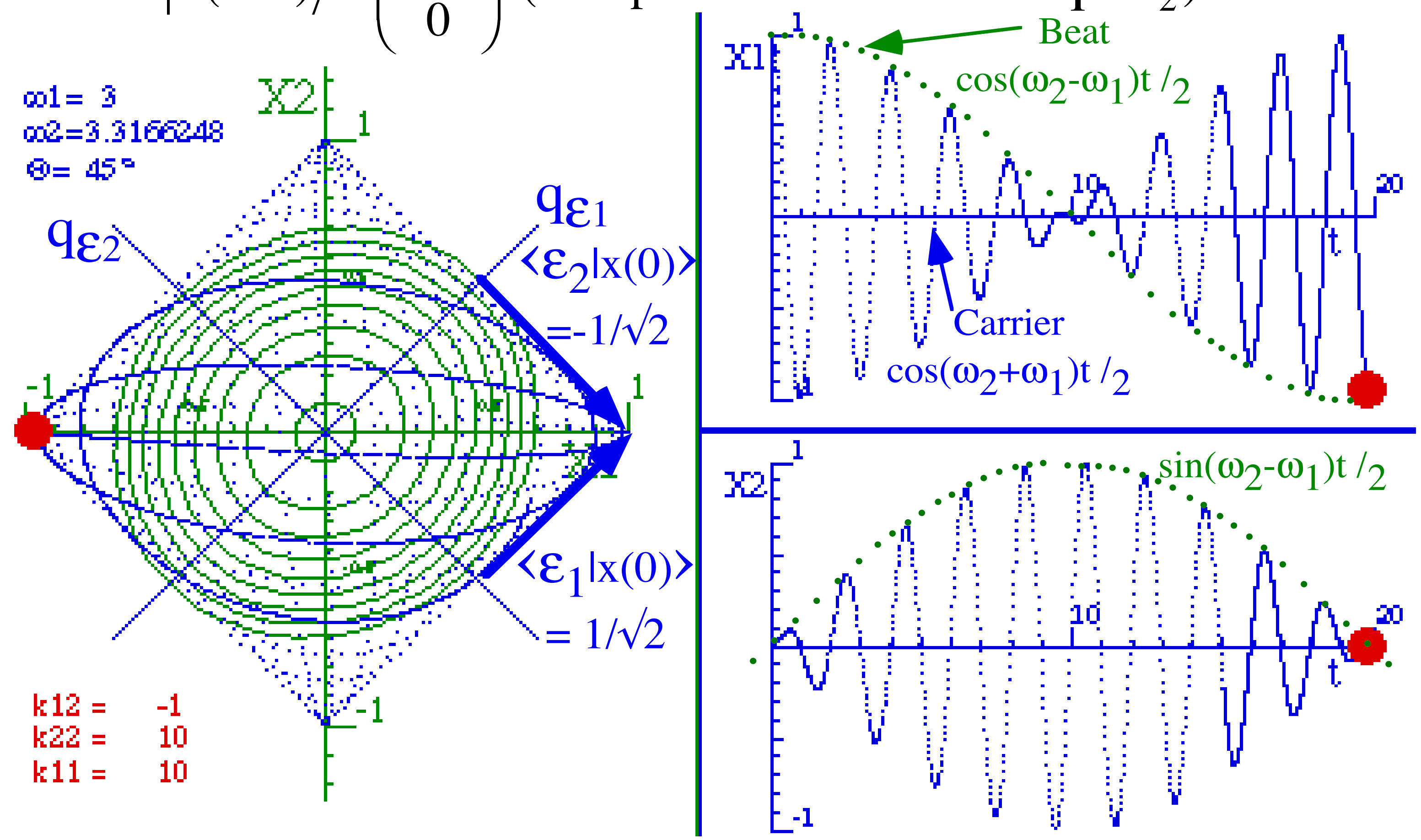
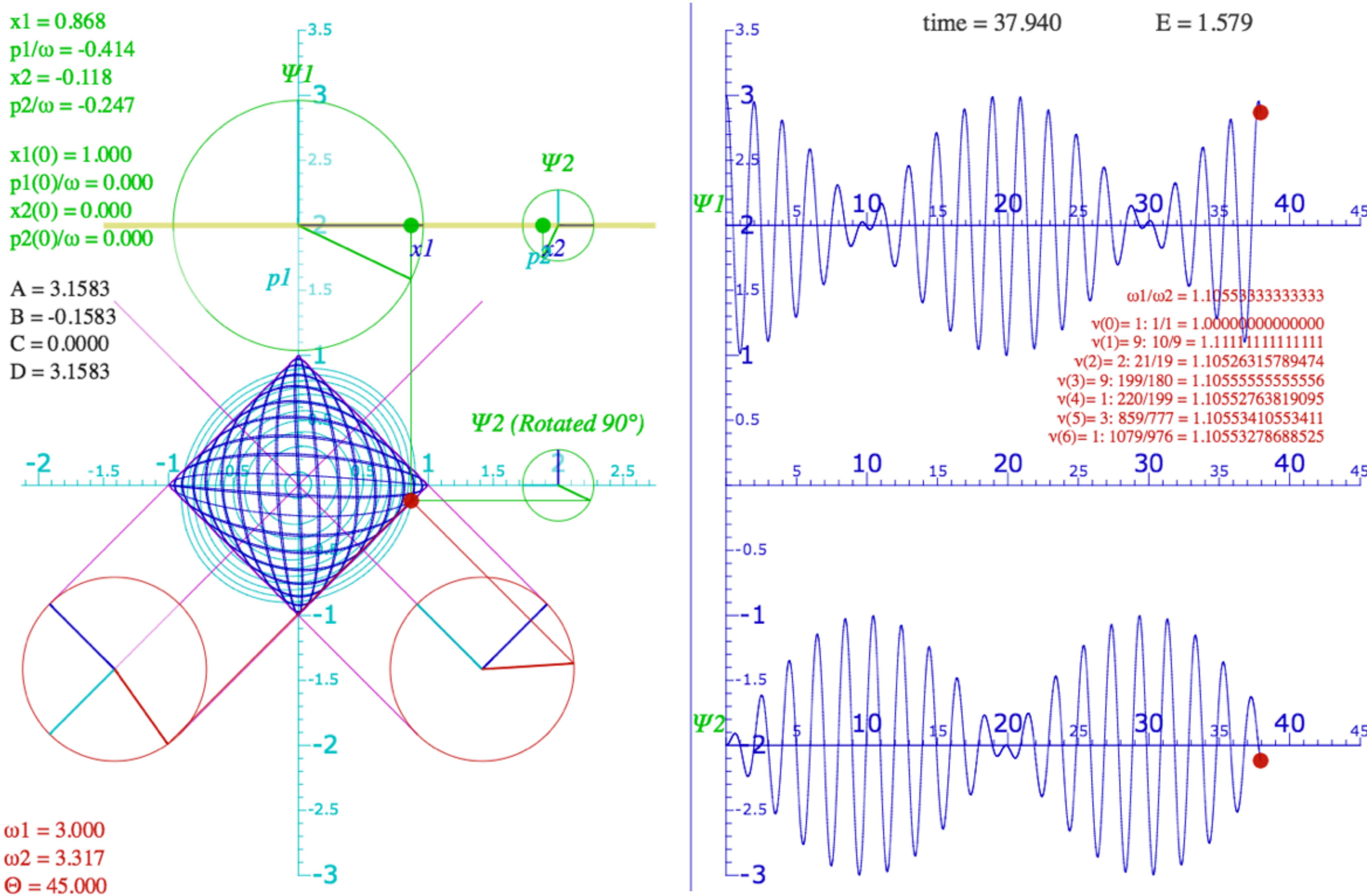


Fig. 2.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

Projector analysis of 2D-HO modes and mixed mode dynamics
½-Sum-½-Diff-Identity for resonant beat analysis



[BoxIt Web Simulation - *Coupled Oscillators $K_{11}=10$, $K_{12}=-1$*]

Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

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➔ *Mode frequency ratios and continued fractions*

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Mode frequency ratios and continued fractions

Recipe for continued fraction approximation of π

$$A_0 = \alpha = 3.14159265\ldots \qquad n_0 = INT(A_0) = 3 \qquad \pi \cong \qquad = 3.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06\ldots \qquad n_1 = INT(A_1) = 7 \qquad \pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99\ldots \qquad n_2 = INT(A_2) = 15 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003\ldots \qquad n_3 = INT(A_3) = 1 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15 + 1}} = \frac{355}{113} = 3.14159292$$

Recipe for continued fraction approximation of the Golden Mean $G=(1+\sqrt{5})/2=1.618$...

$$A_0 = G = 1.618033989\ldots \qquad n_0 = INT(A_0) = 1 \qquad G \cong \qquad = 1.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 1.6180\ldots \qquad n_1 = INT(A_1) = 1 \qquad G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180\ldots \qquad n_2 = INT(A_2) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = 1.500$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.6180\ldots \qquad n_3 = INT(A_3) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}} = \frac{5}{3} = 1.666\ldots$$

Continued fraction approximation of mode frequency ratio (√11)/3=1.1055416

$$A_0 = \alpha = 1.1055416...$$

$$A_1 = \frac{1}{A_0 - n_0} = \frac{1}{0.1055416} = 9.474937...$$

$$A_2 = \frac{1}{A_1 - n_1} = \frac{1}{0.474937} = 2.1055416...$$

$$A_3 = \frac{1}{A_2 - n_2} = \frac{1}{0.1055416} = 9.474936...$$

$$n_0 = INT(A_0) = 1$$

$$n_1 = INT(A_1) = 9$$

$$n_2 = INT(A_2) = 2$$

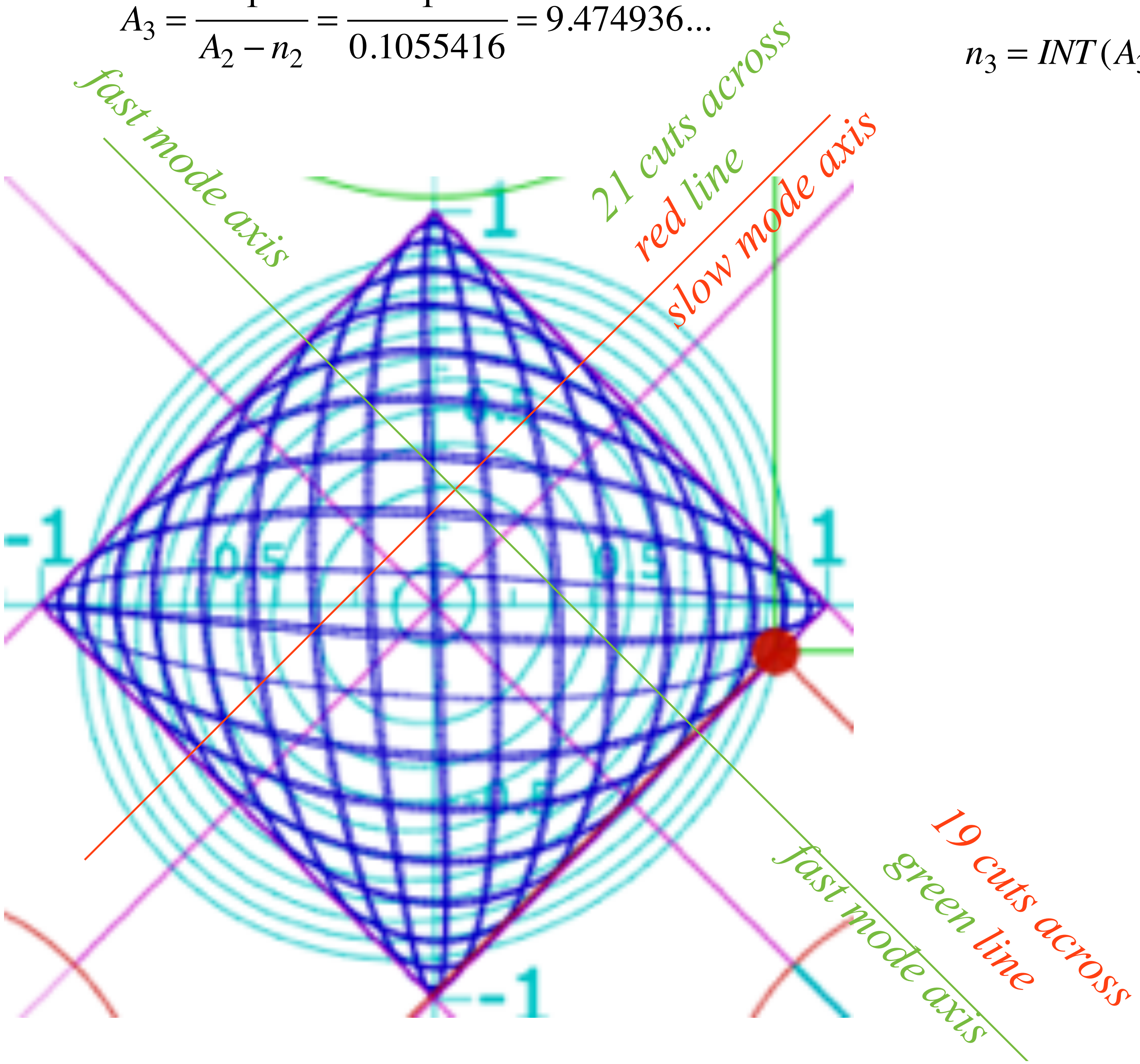
$$n_3 = INT(A_3) = 9$$

$$r \cong \quad = 1.000..$$

$$r \cong 1 + \frac{1}{9} = \frac{10}{9} = 1.111111$$

$$r \cong 1 + \cfrac{1}{9 + \cfrac{1}{2}} = \frac{21}{19} = 1.10526$$

$$r \cong 1 + \cfrac{1}{9 + \cfrac{1}{2 + \cfrac{1}{9}}} = \frac{199}{180} = 1.055556$$



ω1/ω2 = 1.10553333333333

ν(0)= 1: 1/1 = 1.00000000000000
ν(1)= 9: 10/9 = 1.11111111111111
ν(2)= 2: 21/19 = 1.10526315789474
ν(3)= 9: 199/180 = 1.10555555555556
ν(4)= 1: 220/199 = 1.10552763819095
ν(5)= 3: 859/777 = 1.10553410553411

Wave resonance in cyclic C_n symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Projector analysis of 2D-HO modes and mixed mode dynamics*
  - *½-Sum-½-Diff-Identity for resonant beat analysis*
    - *Mode frequency ratios and continued fractions*
    - *Geometry of that 90°-phase lag (again)*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Deriving C_3 projectors*
    - *Deriving and labeling moving wave modes*
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      - *Examples by* WaveIt *animation*
- *C_6 symmetric mode model:Distant neighbor coupling*
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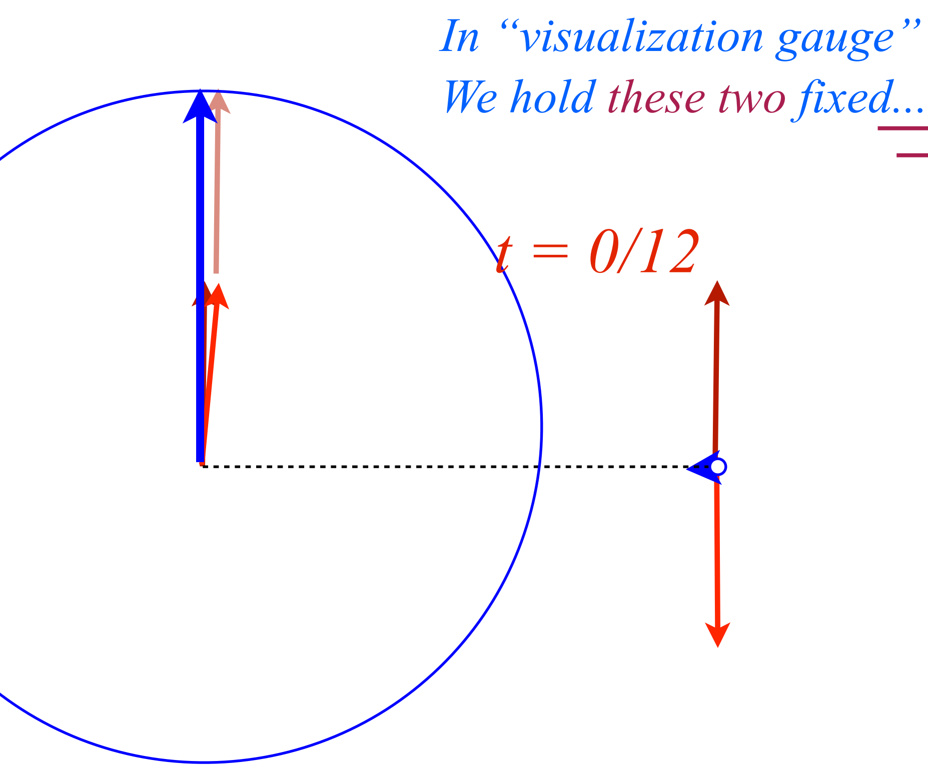
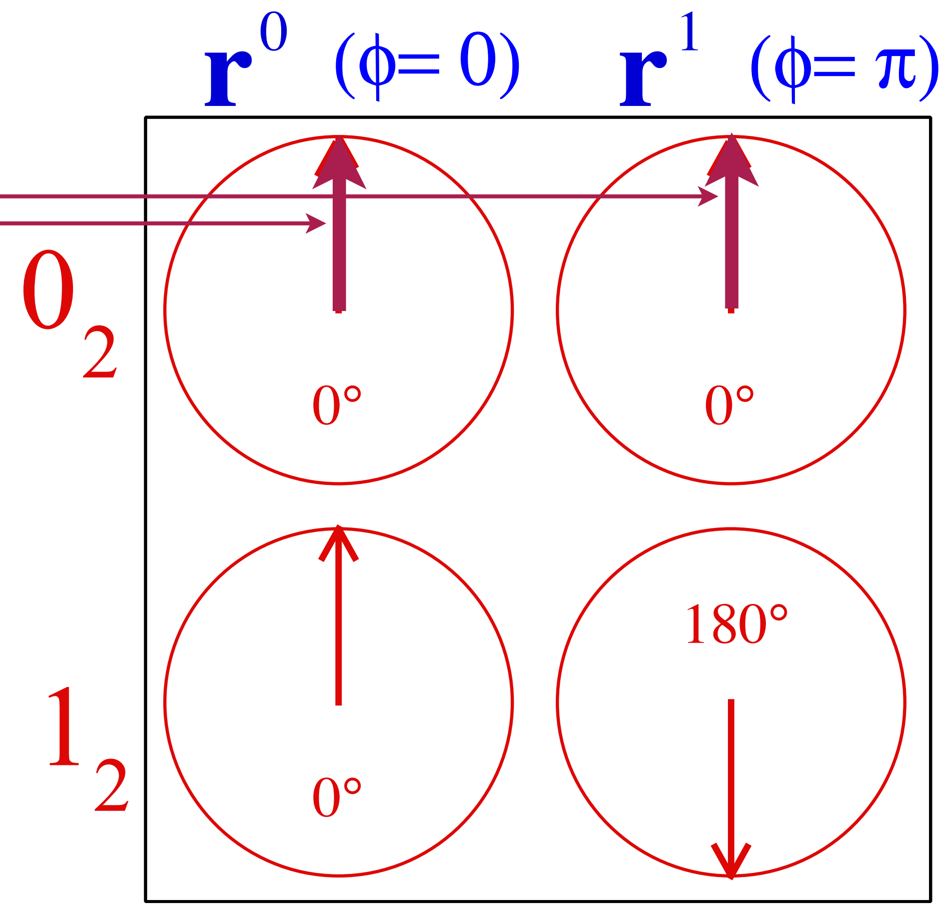
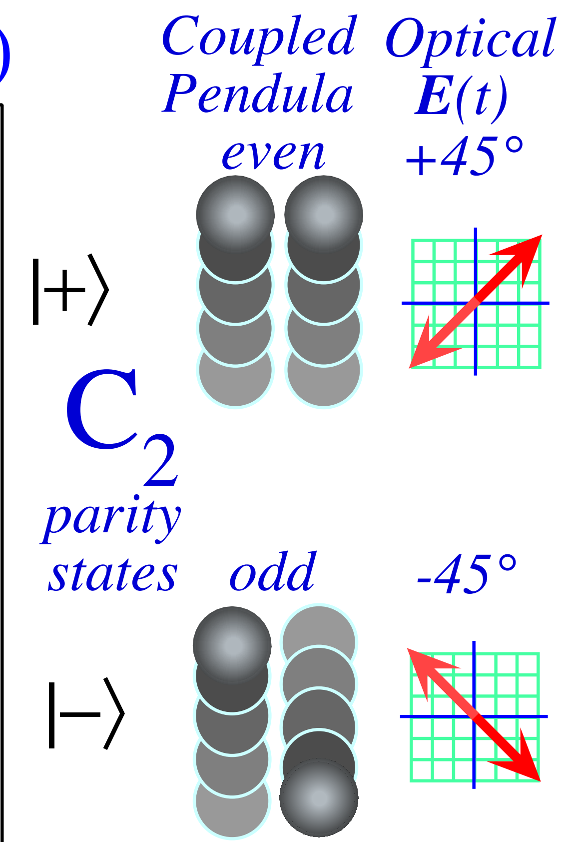
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*

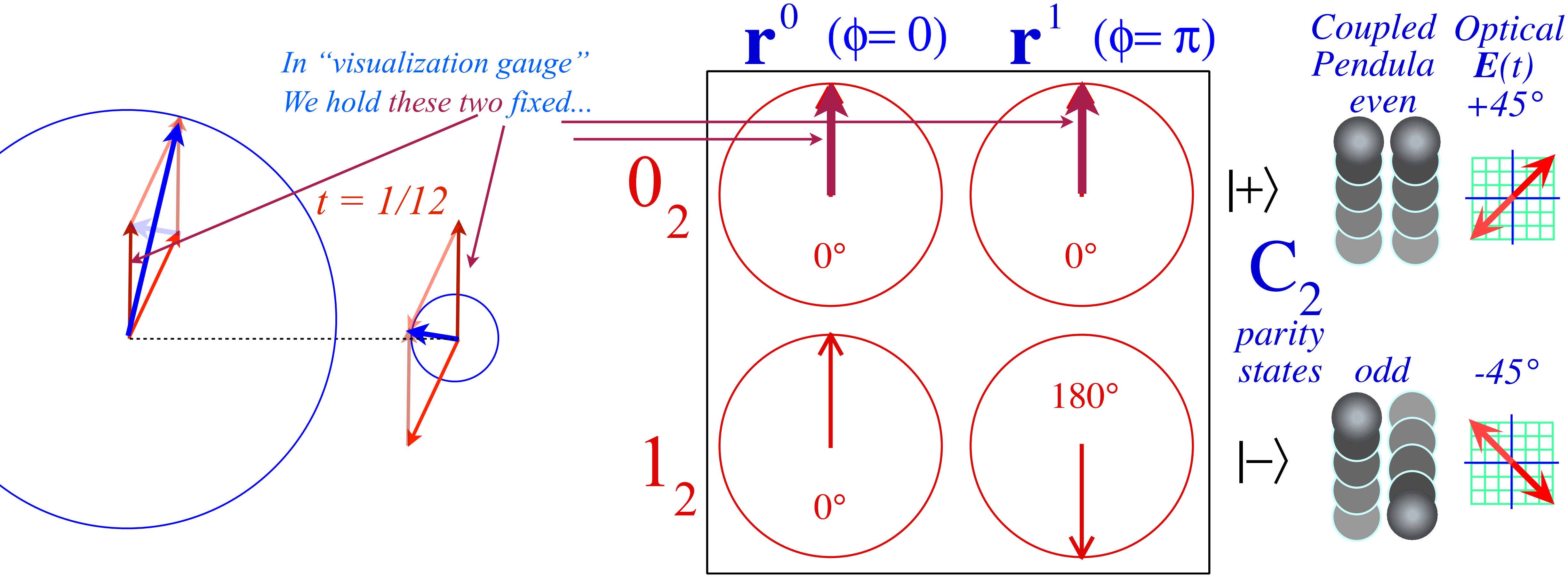
½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)

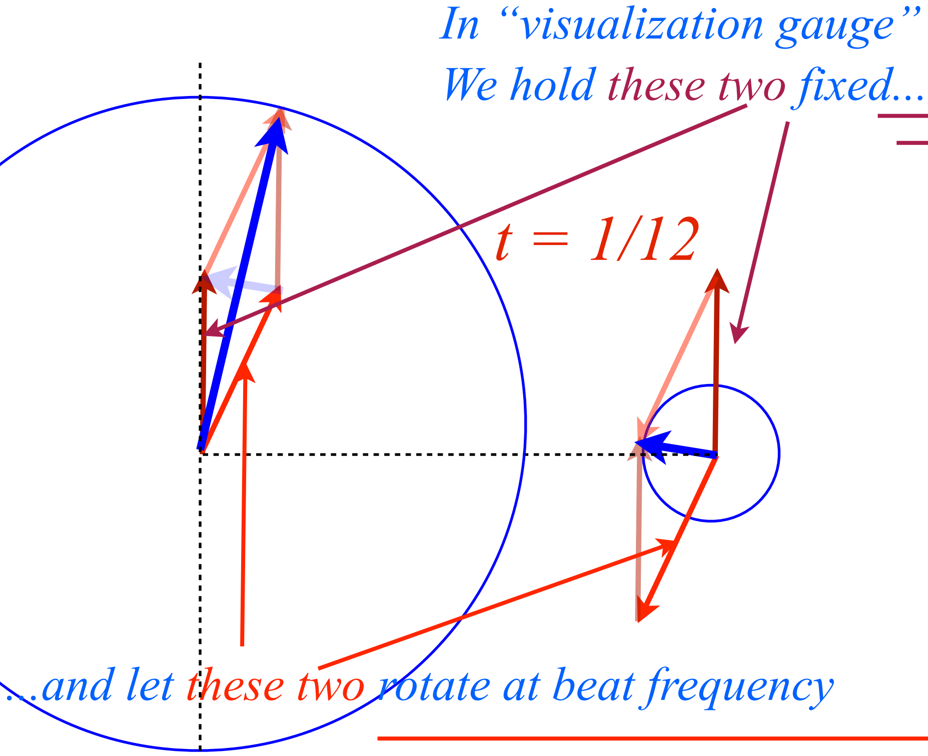




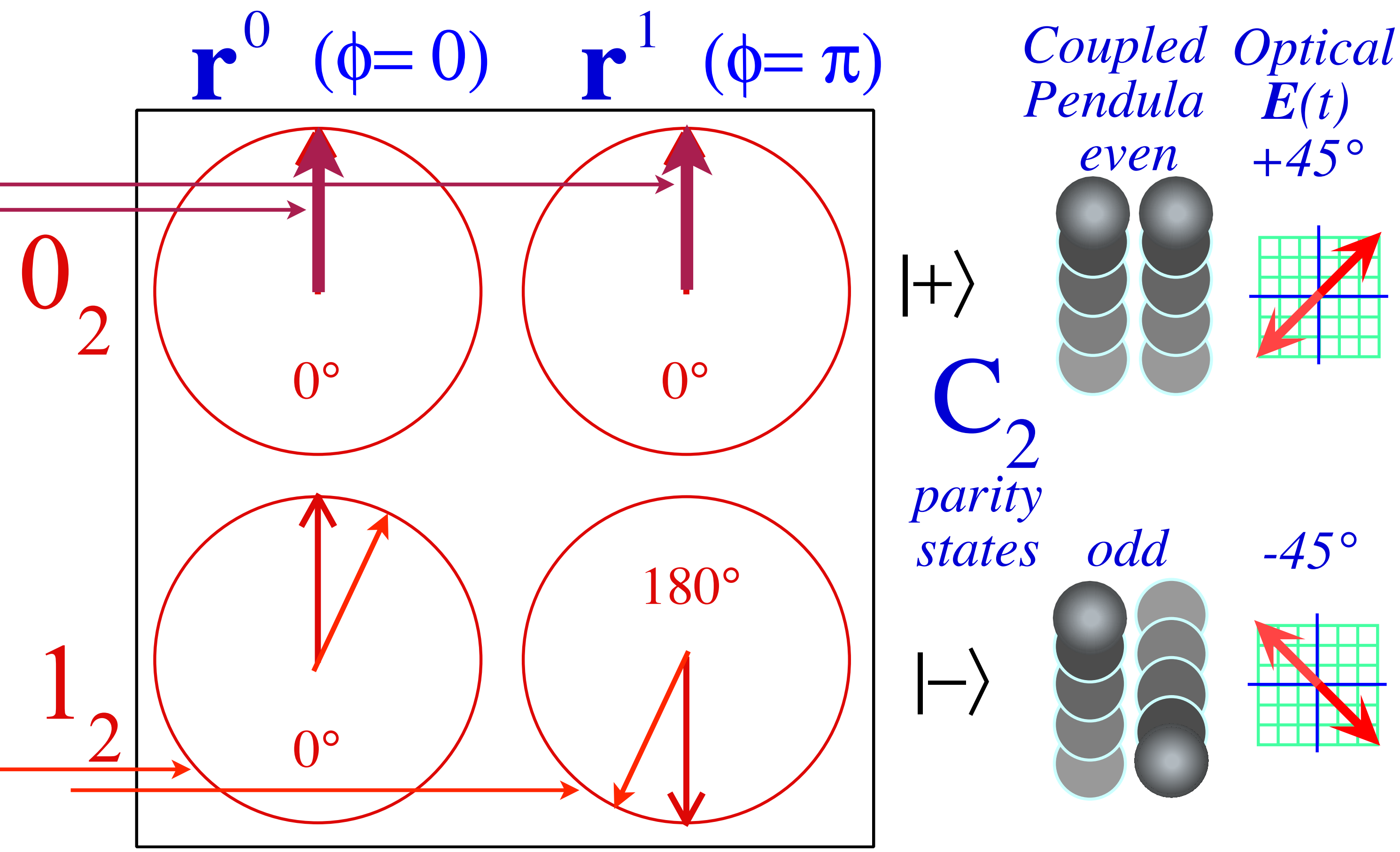


½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)



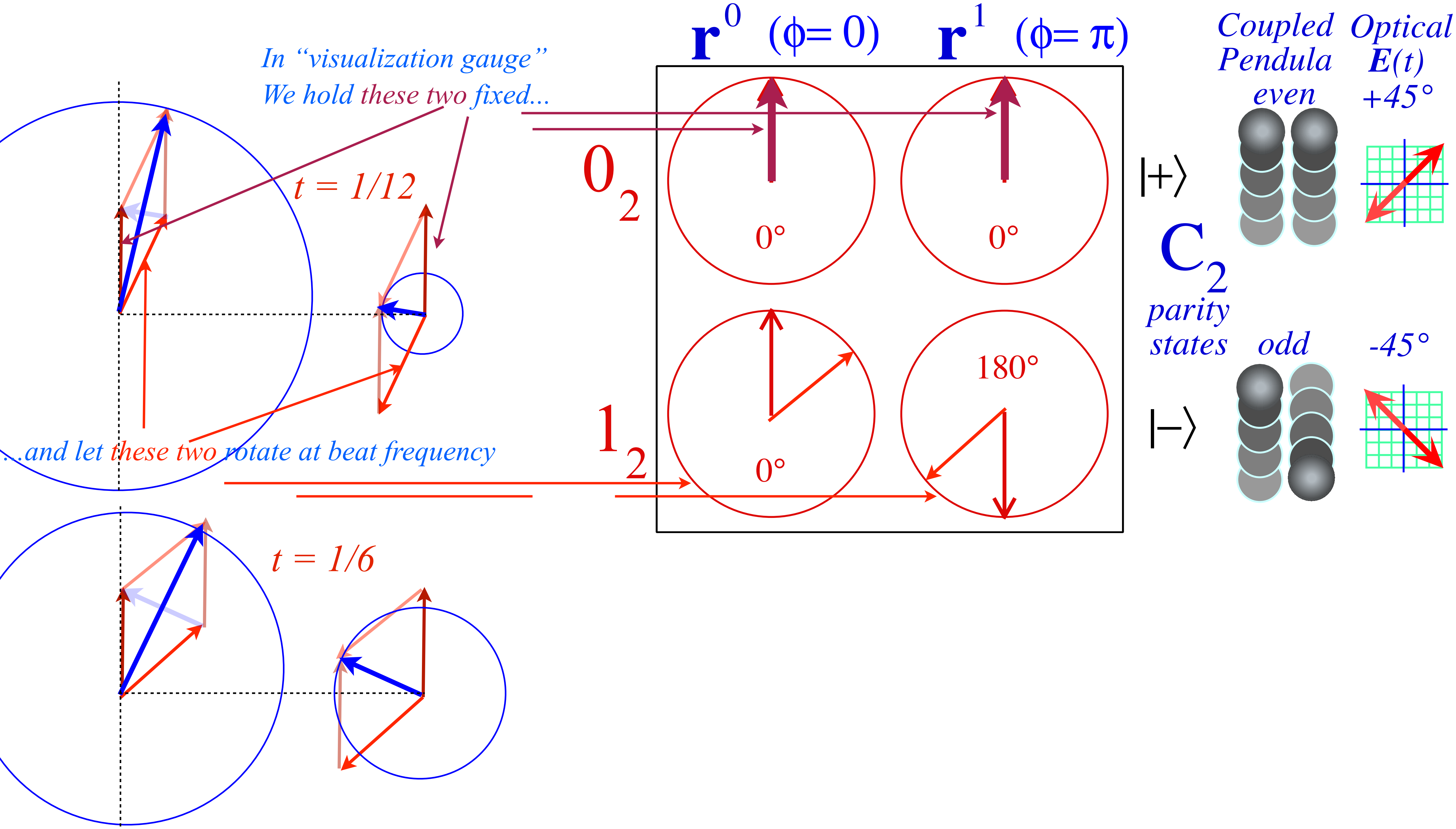


In "visualization gauge"
We hold these two fixed...
t = 1/12
...and let these two rotate at beat frequency

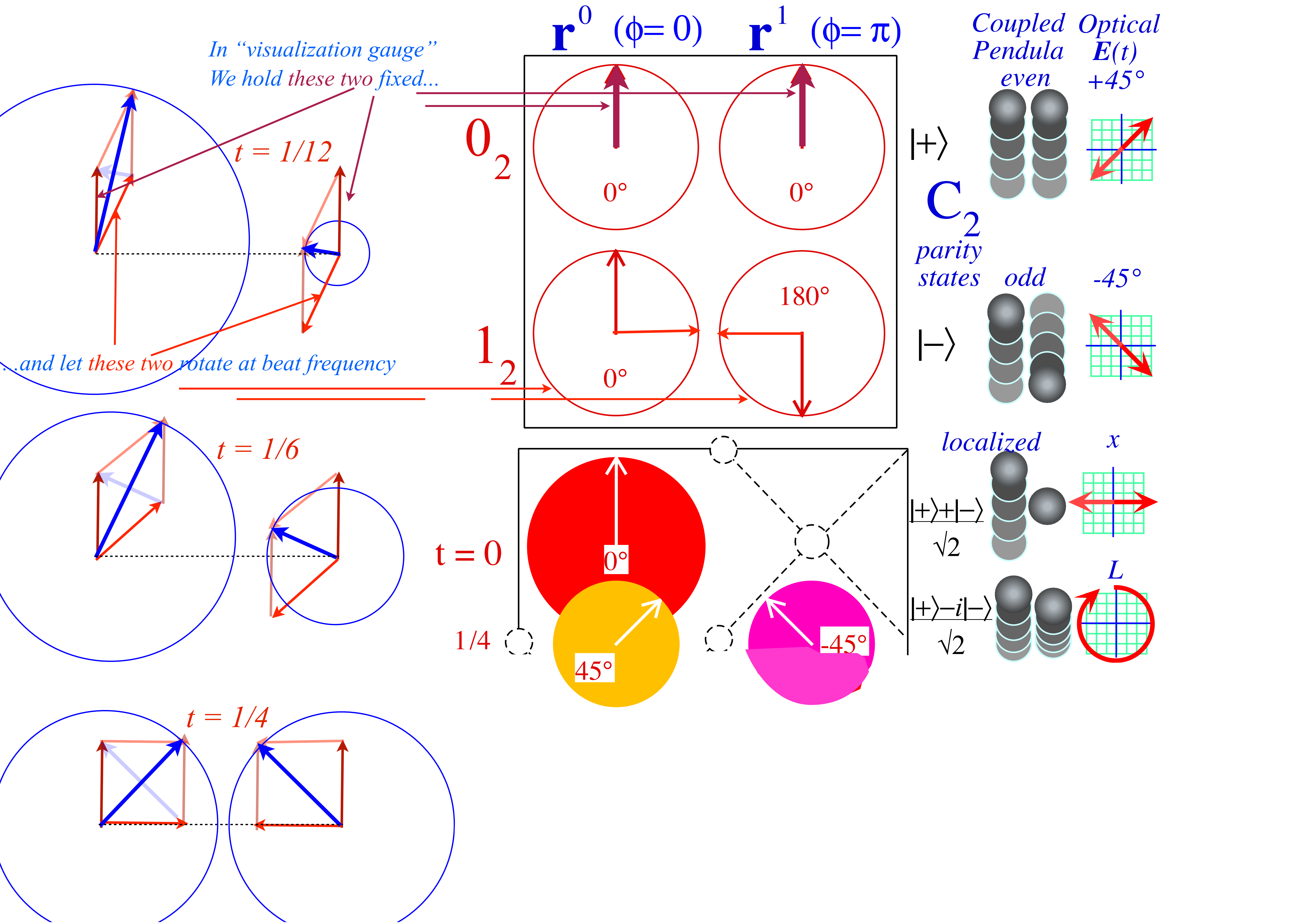


r^0 (ϕ= 0)
r^1 (ϕ= π)
0_2
1_2
0°
0°
0°
180°
|+⟩
|−⟩
C_2
parity
states
Coupled
Pendula
even
odd
Optical
E(t)
+45°
-45°

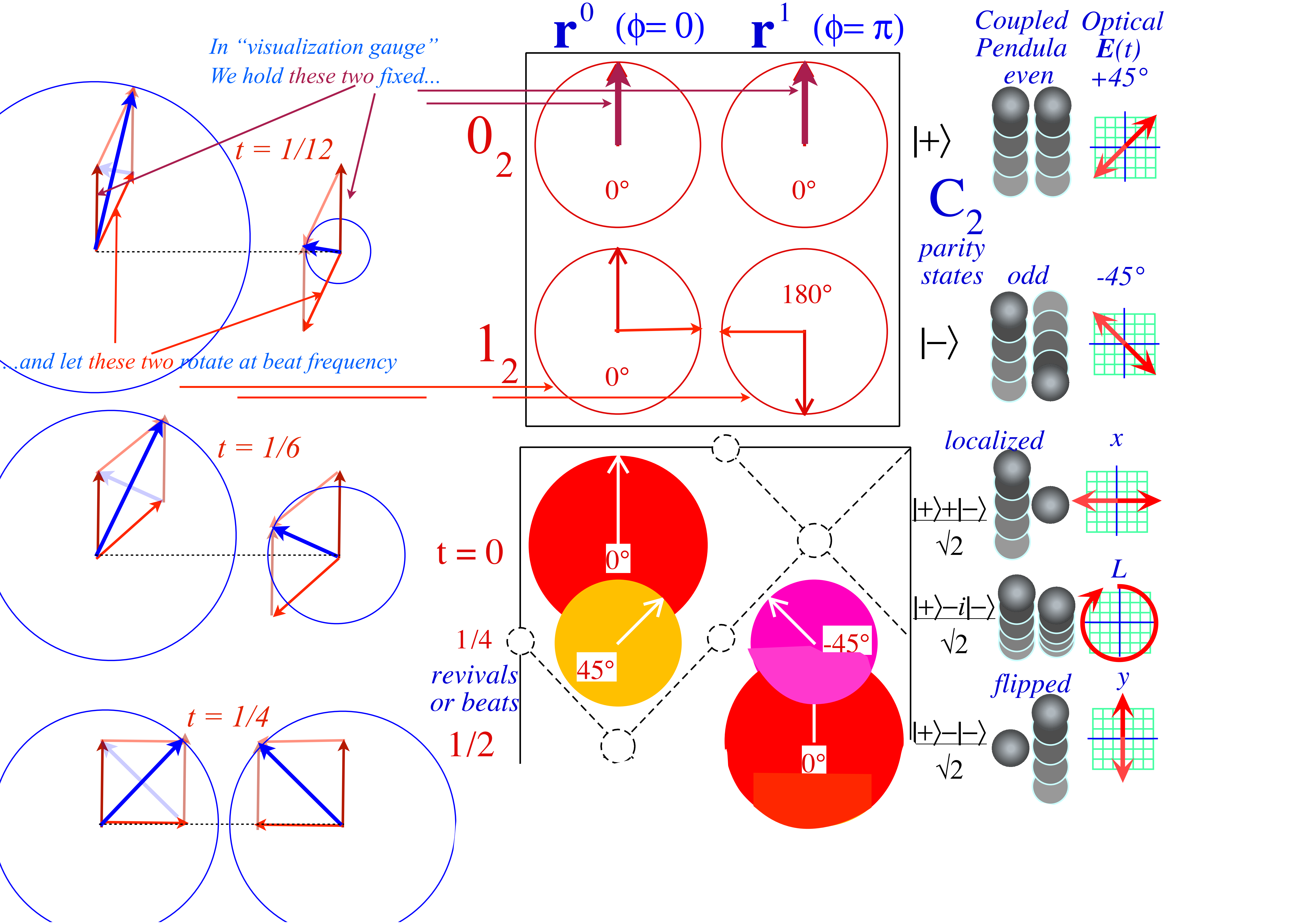
½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)



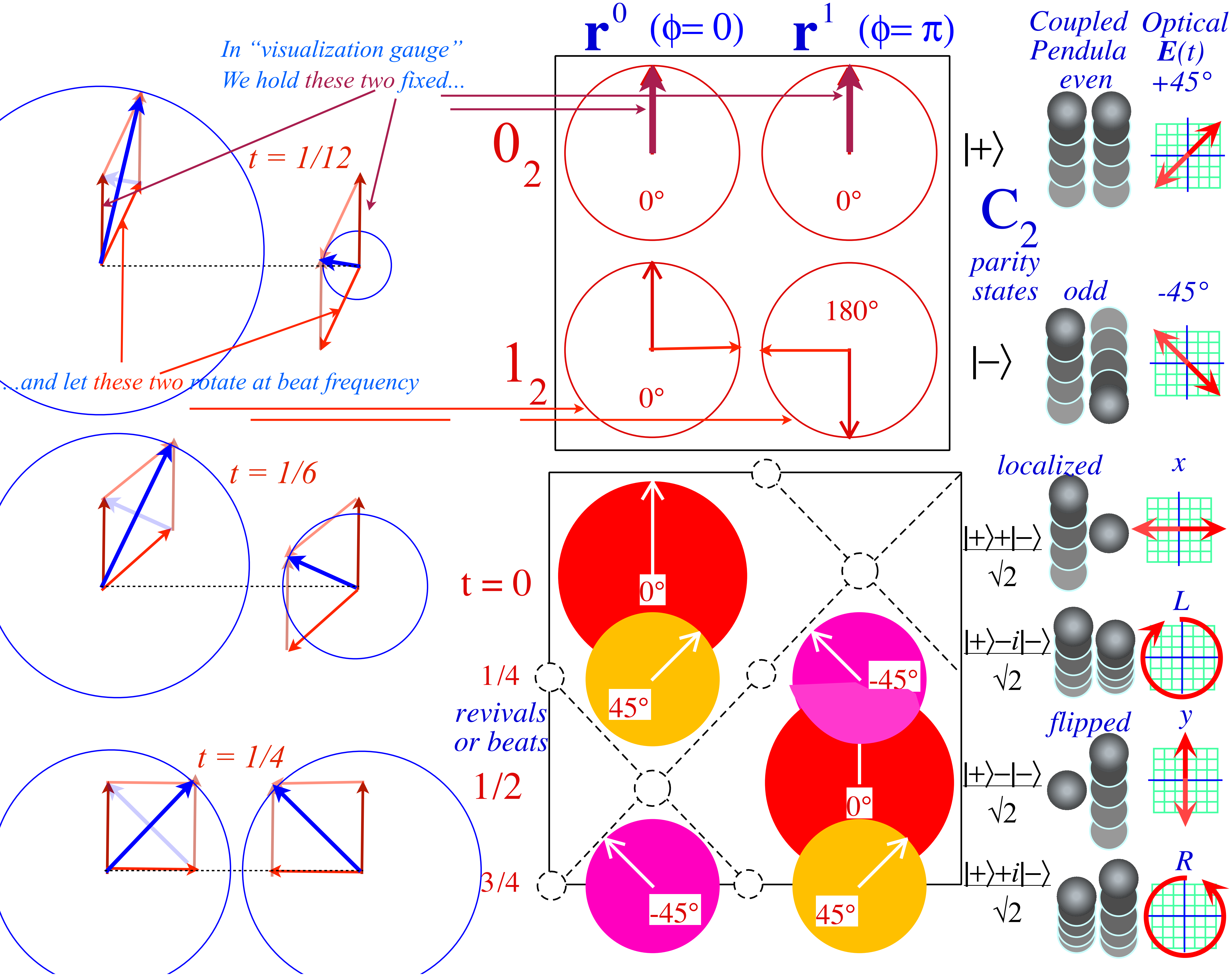
½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)



½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)



½-Sum-½-Diff-phasors - Geometry of that 90°-phase lag (again)



Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Projector analysis of 2D-HO modes and mixed mode dynamics

½-Sum-½-Diff-Identity for resonant beat analysis

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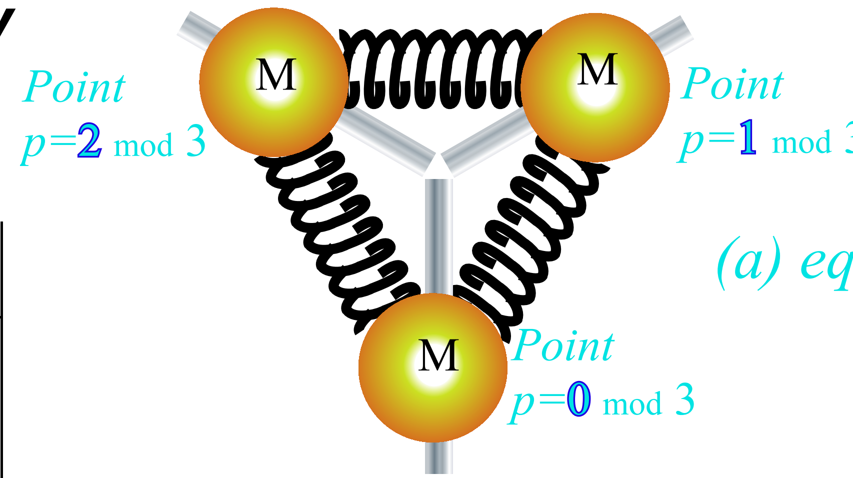
3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

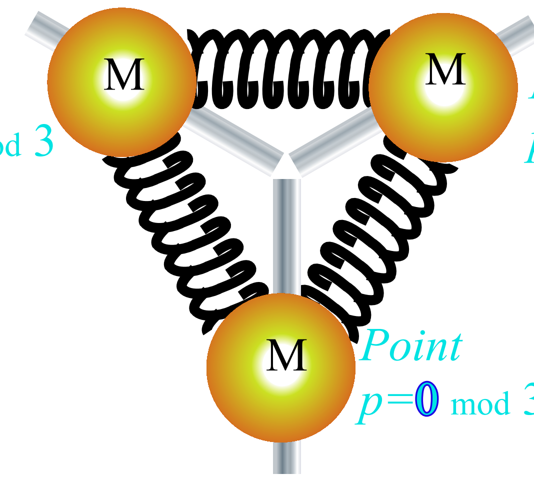
Orig. Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$

g$=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

C_3 unit base states

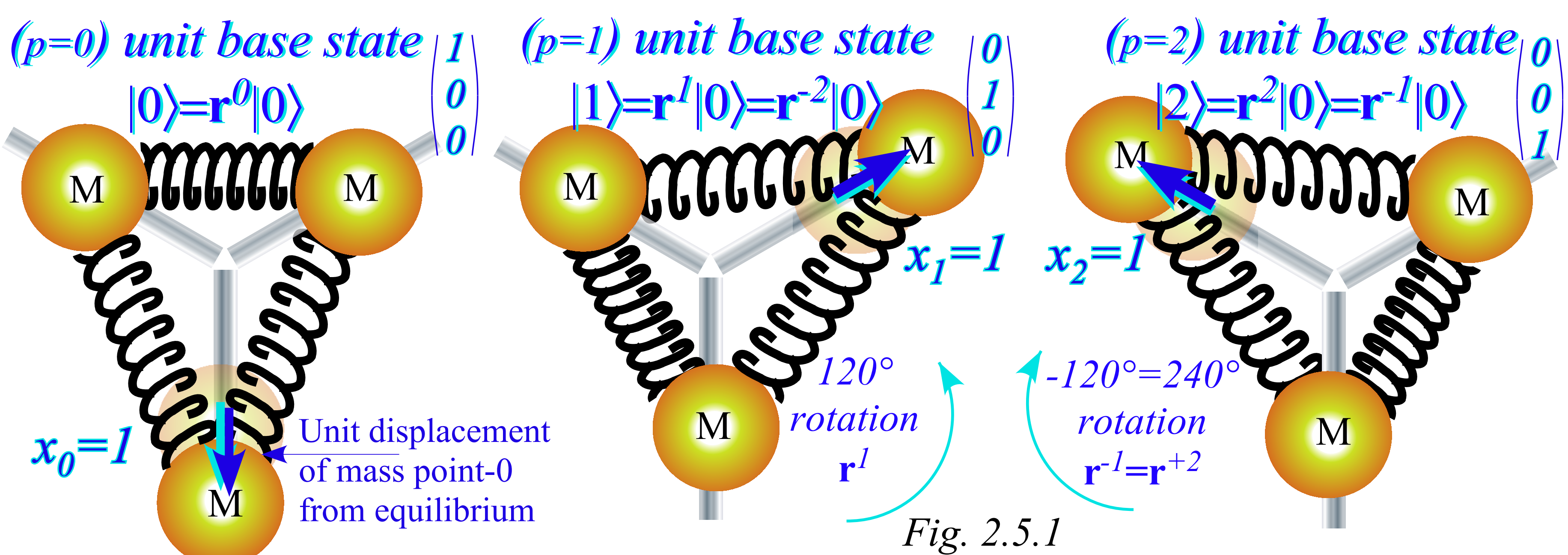


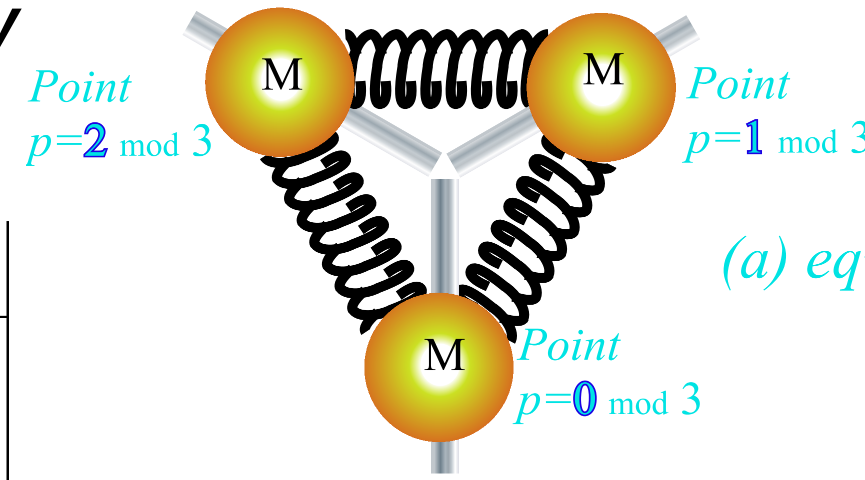
Fig. 2.5.1
Unit 2
Honors Physics

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state
$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row
then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0 \cdot \mathbf{1} + r_1 \cdot \mathbf{r}^1 + r_2 \cdot \mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

C_3 unit base states

(p=0) unit base state $|0\rangle=\mathbf{r}^0|0\rangle$ $\begin{pmatrix}1\\0\\0\end{pmatrix}$

(p=1) unit base state $|1\rangle=\mathbf{r}^1|0\rangle=\mathbf{r}^{-2}|0\rangle$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$

(p=2) unit base state $|2\rangle=\mathbf{r}^2|0\rangle=\mathbf{r}^{-1}|0\rangle$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

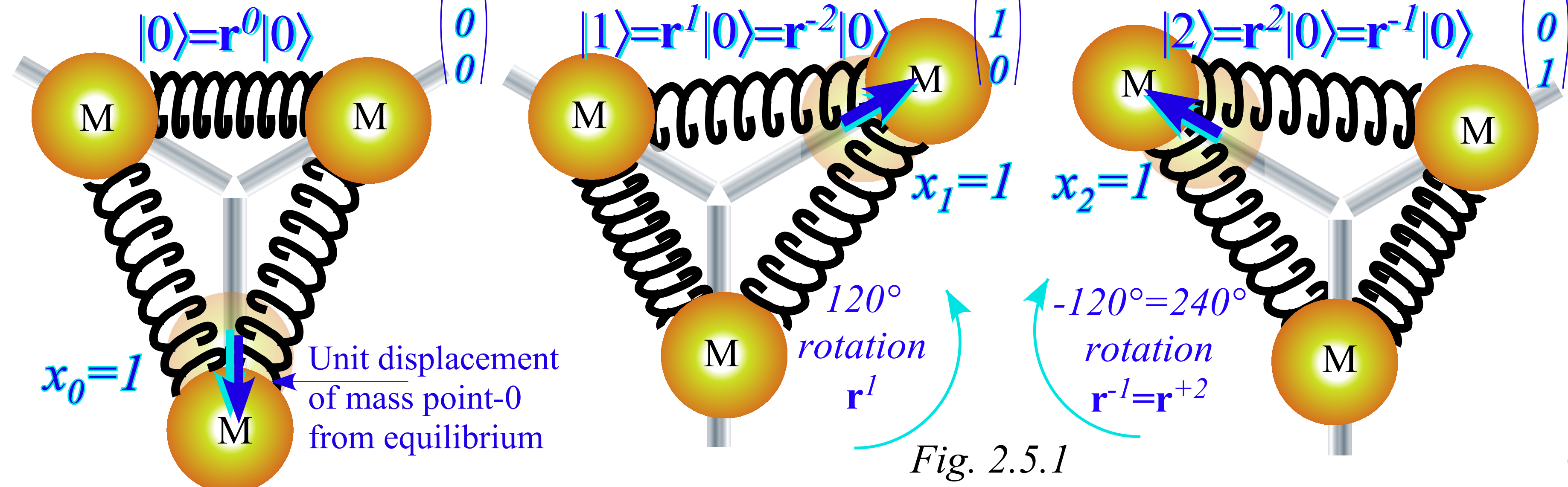
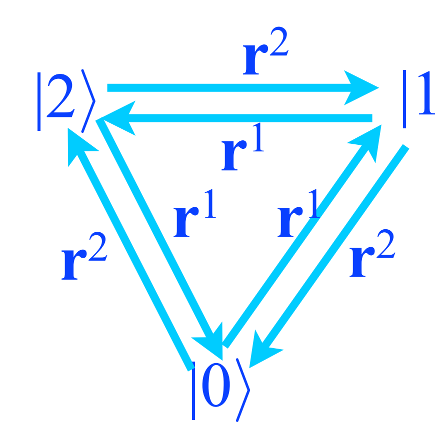


Fig. 2.5.1
Unit 2
Honors Physics



$\mathbf{r}^0=\mathbf{1}$
$=\mathbf{r}^3$

Usually assume <u>Real</u> $r_1=r=r_2$
Stability only requires $(r_1)^*=r_2$

Each **H**-matrix coupling constant $r_p=\{r_0, r_1, r_2\}$ is amplitude of its operator power $\mathbf{r}^p=\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$

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C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

C_3 Spectral resolution: 3^{rd} roots of unity

For any integer m : $e^{2\pi i m}=\cos(2\pi m)+i\sin(2\pi m)=1$

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

$$\sqrt[3]{1}=(1)^{1/3}$$
$$=(e^{2\pi i m})^{1/3}$$
$$=e^{2\pi i m/3}$$

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3\mathbf{-1=0}$ and resolves to factors of *3^{rd} roots of unity* $\rho_m=e^{im2\pi/3}$.

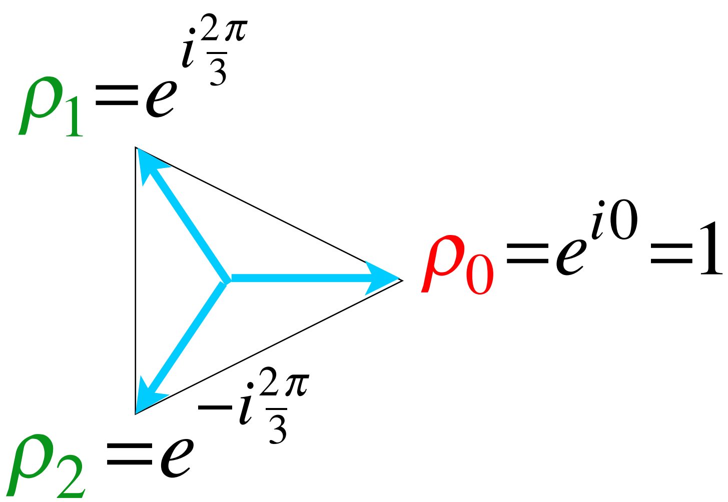
C_3 Spectral resolution: 3^{rd} roots of unity

For any integer m : $e^{2\pi i m} = \cos(2\pi m) + i\sin(2\pi m) = 1$

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p \mathbf{r}^p$ of powers $\mathbf{r}^p$.

$$\sqrt[3]{1} = (1)^{1/3} = (e^{2\pi i m})^{1/3} = e^{2\pi i m/3}$$

r-symmetry is cubic $\mathbf{r}^3 = \mathbf{1}$, or $\mathbf{r}^3 - \mathbf{1} = \mathbf{0}$ and resolves to factors of *3^{rd} roots of unity* $\rho_m = e^{im2\pi/3}$.



C_3 Spectral resolution: 3^{rd} roots of unity

For any integer m : $e^{2\pi i m} = \cos(2\pi m) + i\sin(2\pi m) = 1$

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p \mathbf{r}^p$ of powers $\mathbf{r}^p$.

$$\sqrt[3]{1} = (1)^{1/3} = (e^{2\pi i m})^{1/3} = e^{2\pi i m/3}$$

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$$\mathbf{1} = \mathbf{r}^3 \;\; implies: \;\; \mathbf{0} = \mathbf{r}^3 - \mathbf{1} = (\mathbf{r} - \rho_0 \mathbf{1})(\mathbf{r} - \rho_1 \mathbf{1})(\mathbf{r} - \rho_2 \mathbf{1}) \;\; where: \;\; \rho_m = e^{im\frac{2\pi}{3}}$$

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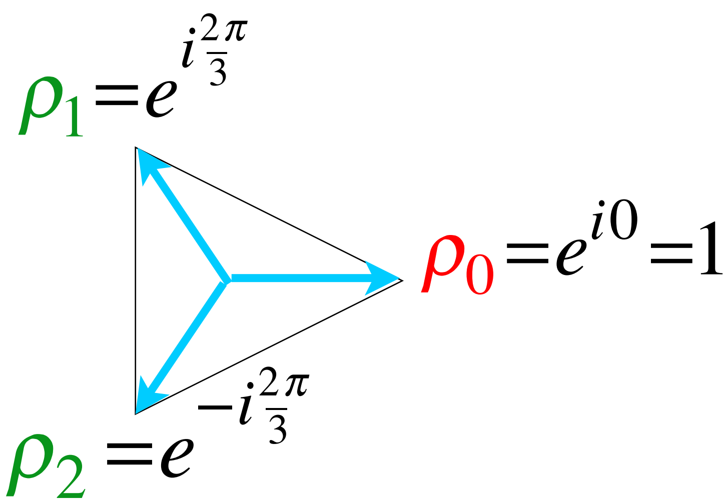

C_3 Spectral resolution: 3^{rd} roots of unity

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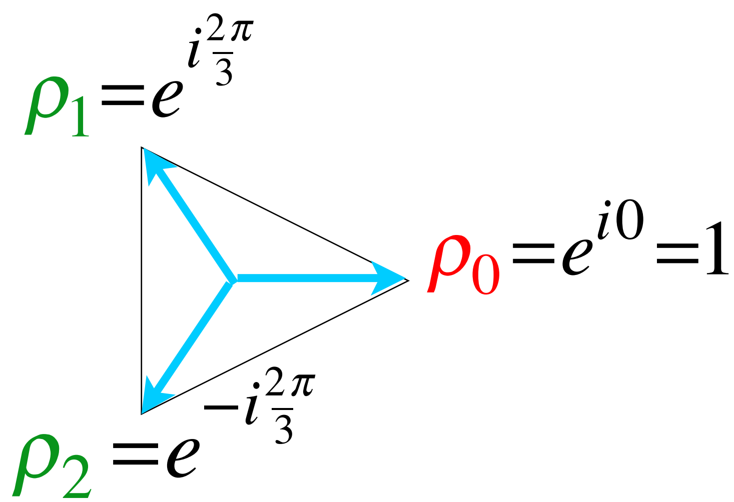

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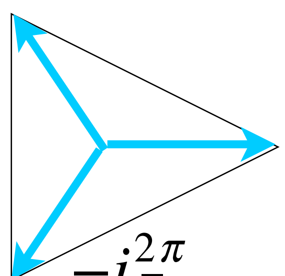

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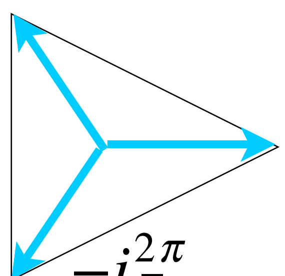

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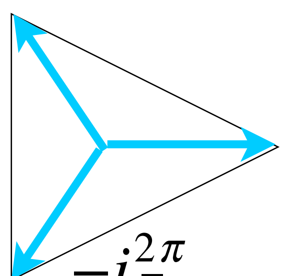

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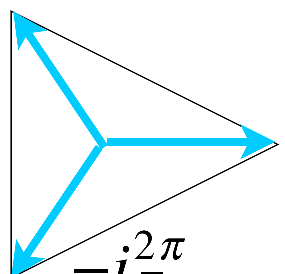

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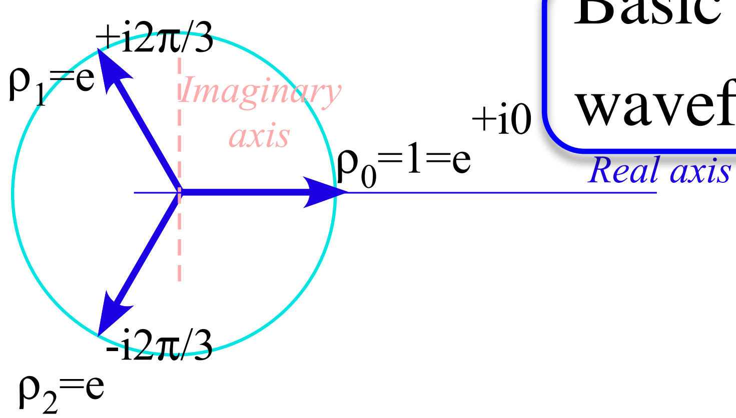

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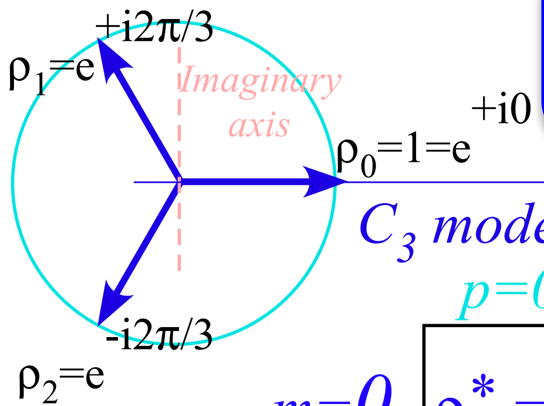

C_3 mode phase character tables

wave-number m= "momentum"

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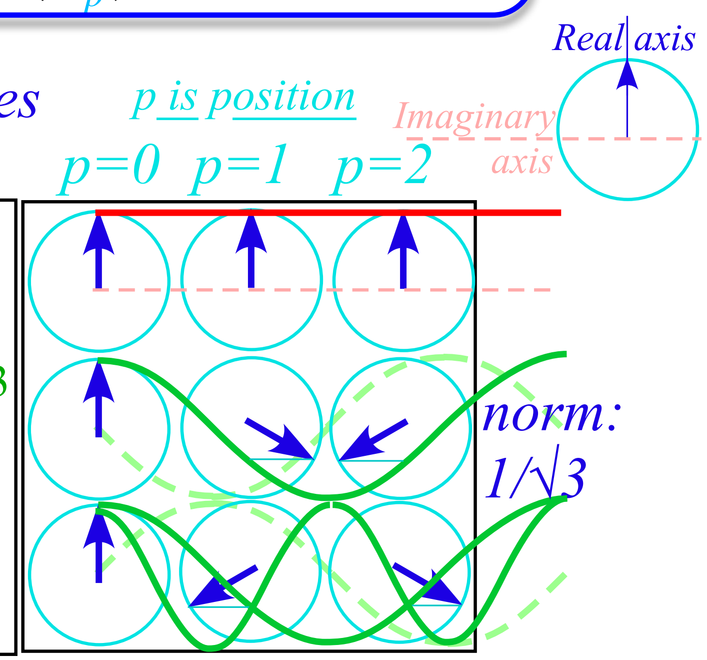

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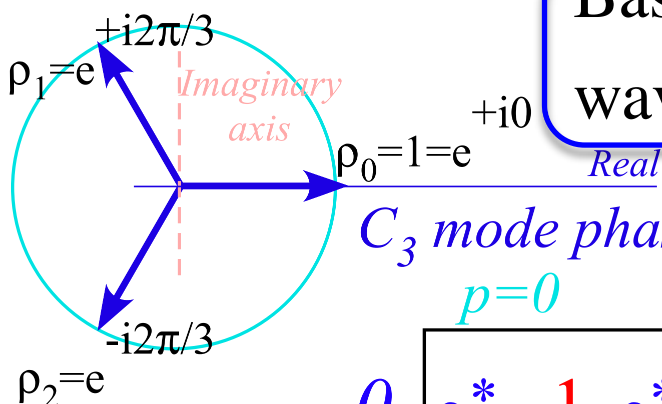

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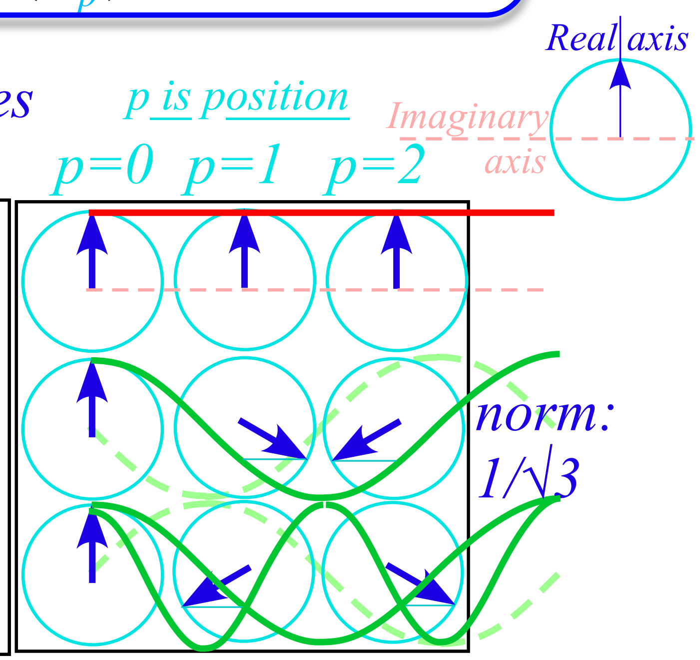

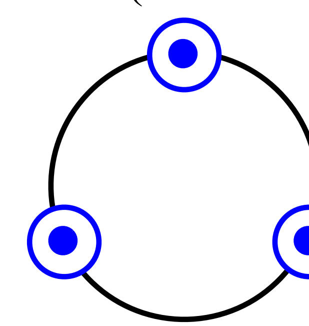

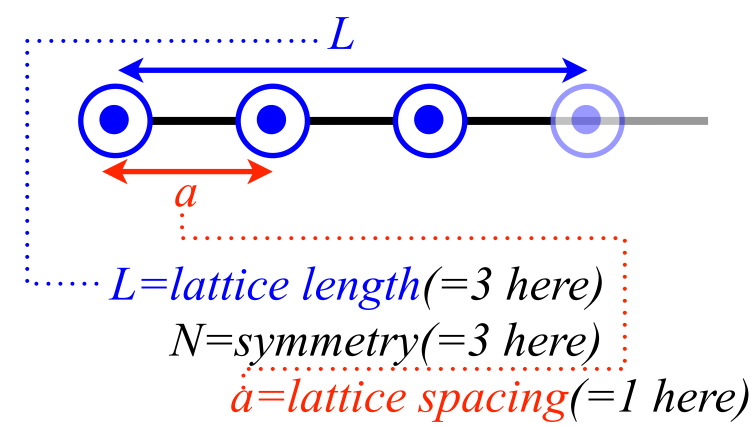

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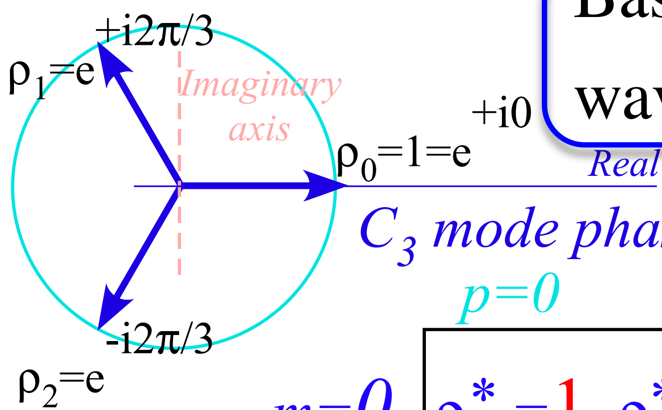

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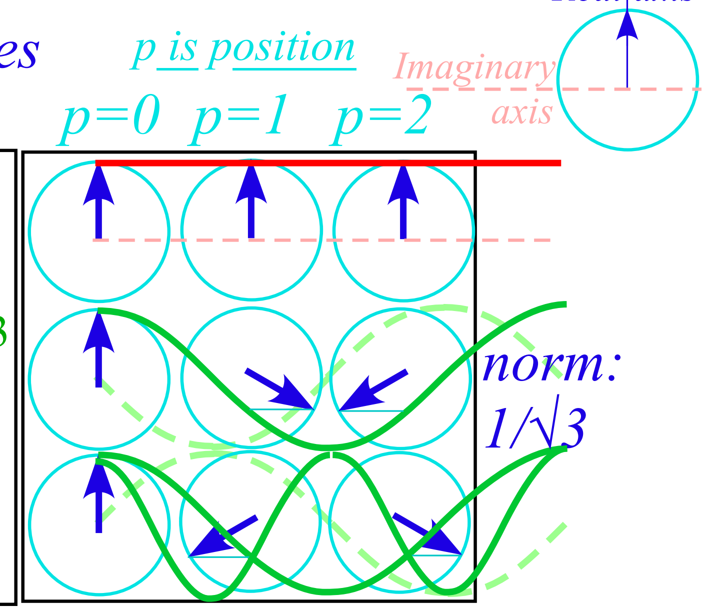

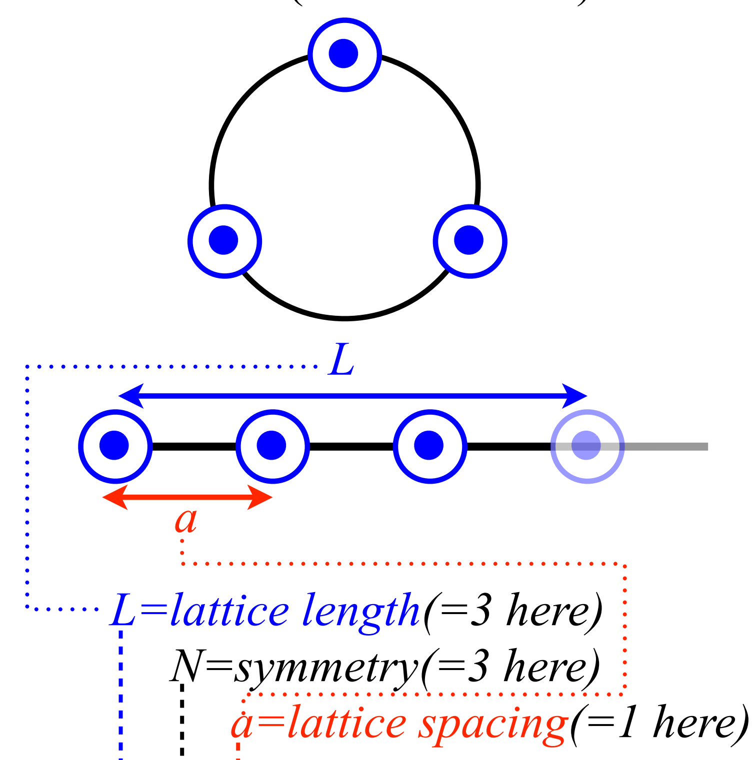

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(Sample *WaveIt* animation follows)

Wave resonance in cyclic C_n symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Projector analysis of 2D-HO modes and mixed mode dynamics*
  - *½-Sum-½-Diff-Identity for resonant beat analysis*
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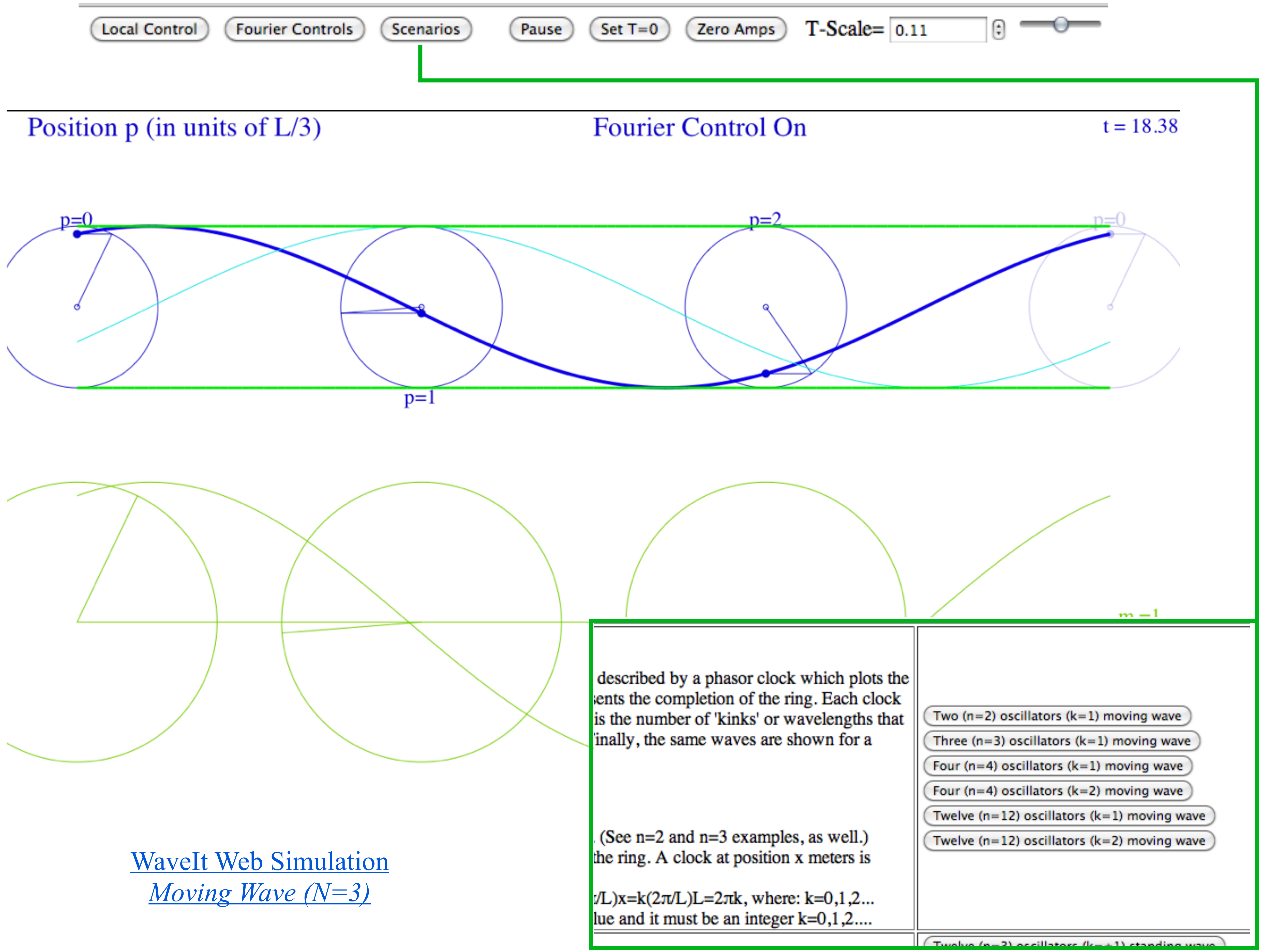
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
- *½-Sum-½-Diff-theory of 2-CW group and phase velocity*



WaveIt Web Simulation

Moving Wave (N=3)

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)} = \tfrac{1}{3}(\mathbf{r}^0 + \mathbf{r}^1 + \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2)$$

$$\mathbf{P}^{(1)} = \tfrac{1}{3}(\mathbf{r}^0 + \rho_1^* \mathbf{r}^1 + \rho_2^* \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{-i2\pi/3}\mathbf{r}^1 + e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)} = \tfrac{1}{3}(\mathbf{r}^0 + \rho_2^* \mathbf{r}^1 + \rho_1^* \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{+i2\pi/3}\mathbf{r}^1 + e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle (0_3)| = \langle 0|\mathbf{P}^{(0)}\sqrt{3} = \sqrt{\tfrac{1}{3}}\,(\;1 \quad 1 \quad 1\;)$$

$$\langle (1_3)| = \langle 0|\mathbf{P}^{(1)}\sqrt{3} = \sqrt{\tfrac{1}{3}}\,(\;1 \quad e^{-i2\pi/3} \quad e^{+i2\pi/3}\;)$$

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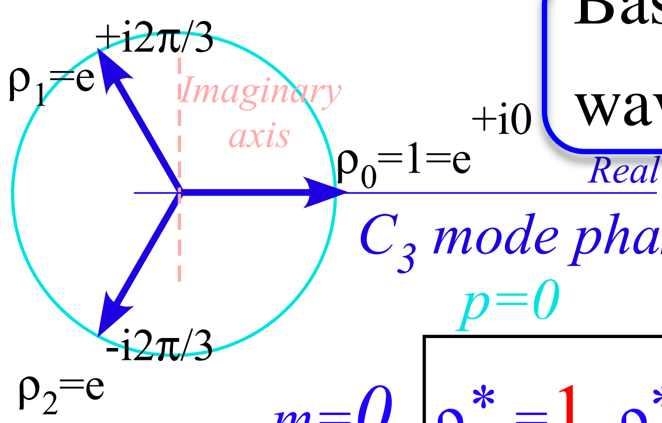


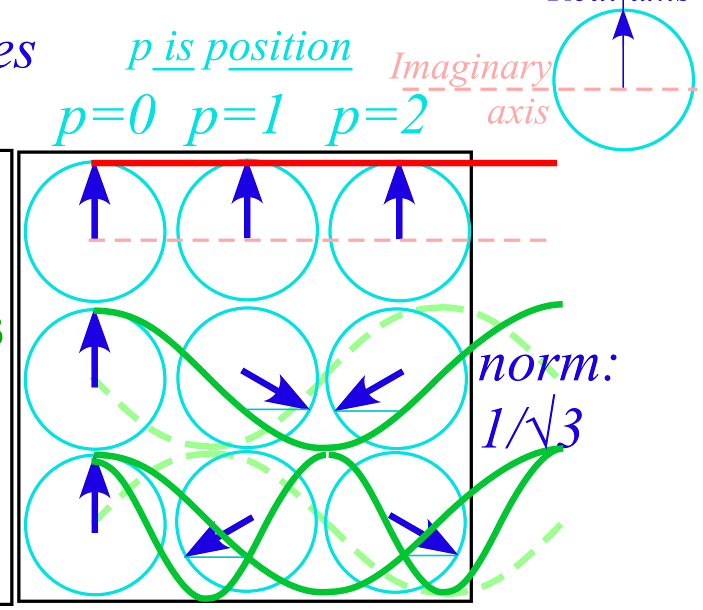
Basic "trinary" wavefunction: $\rho_{mp} = \psi^{k_m}(x_p) = e^{ik_m x_p} = e^{i\frac{2\pi mp}{3}}$

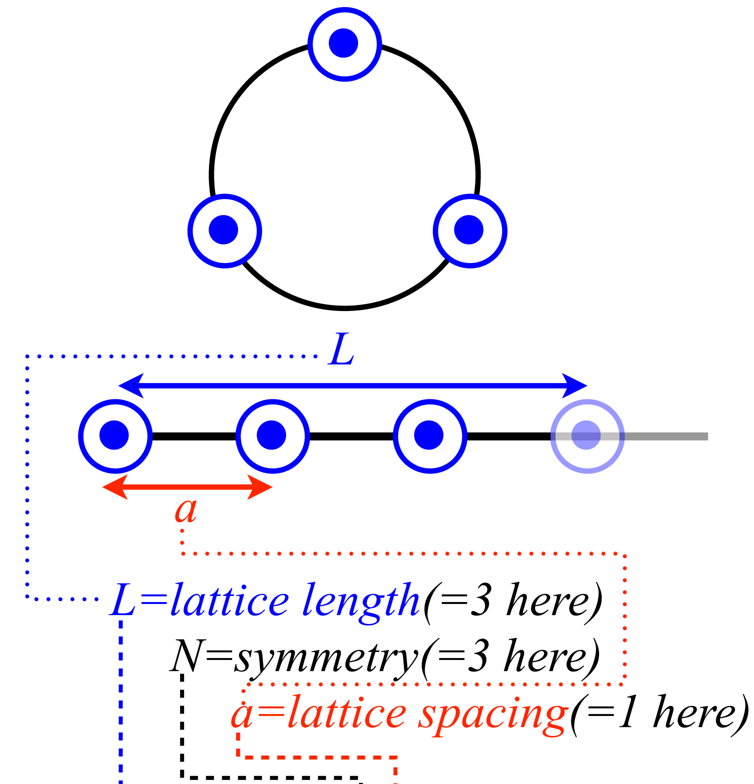
(m_3) means: *m-modulo-3* (Details follow)

C_3 mode phase character tables

wave-number m= "momentum"

	$p=0$	$p=1$	$p=2$
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





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Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always $0,1,$or $2,$ or else $-1,0,$or $1,$ or else $-2,-1,$or $0,$ *etc.,* depending on choice of origin.

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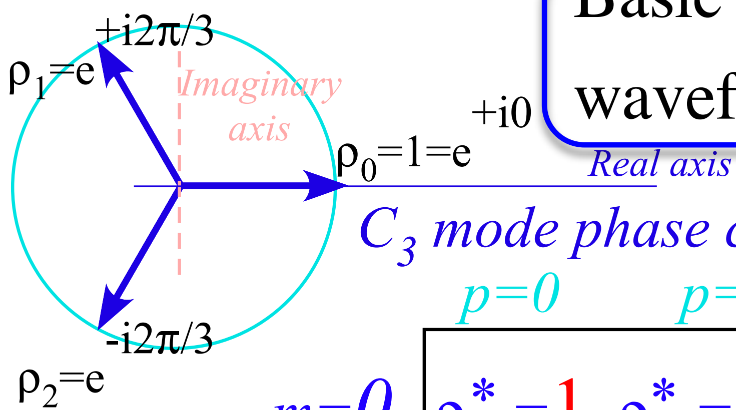

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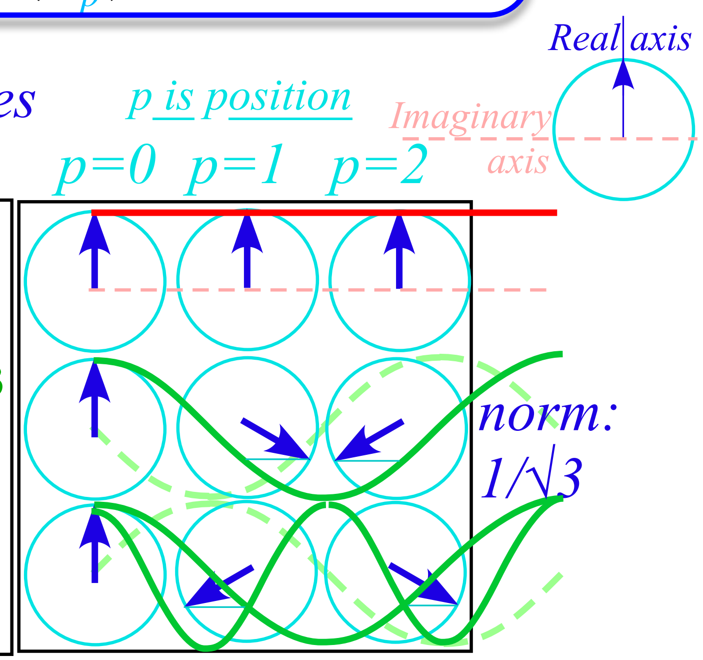

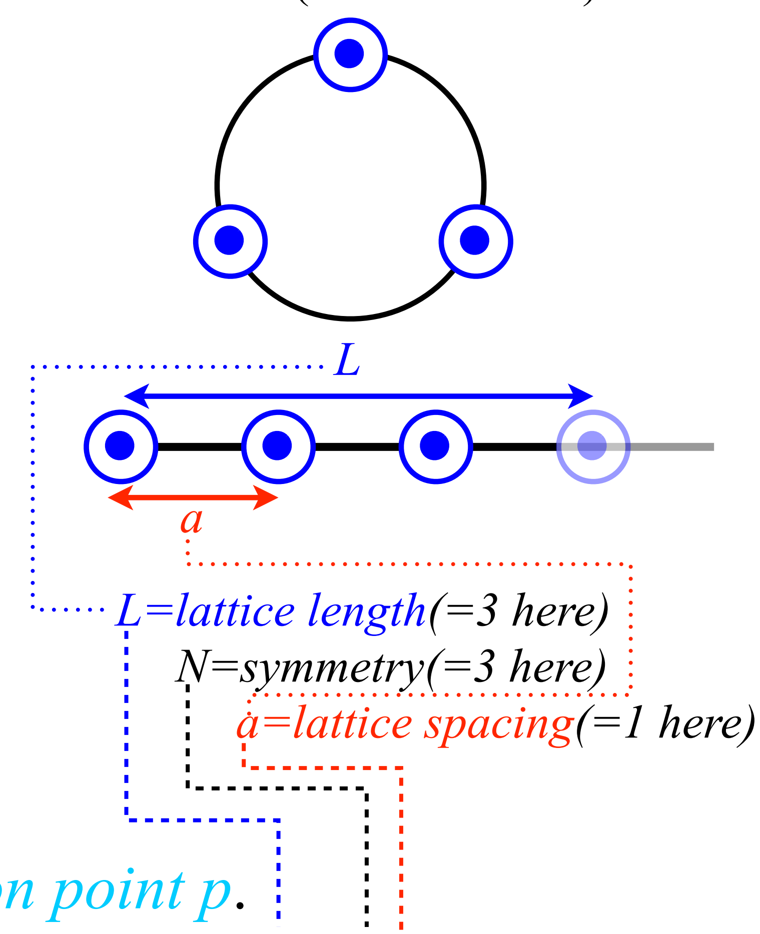

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For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}=e^{i4\cdot 2\pi/3}=e^{i1\cdot 2\pi/3}\,e^{i2\pi}=e^{i2\pi/3}=\rho_1$.
That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3=1*, the remainder of *4* divided by *3*.)

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H-eigenvalues:

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

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K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

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Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

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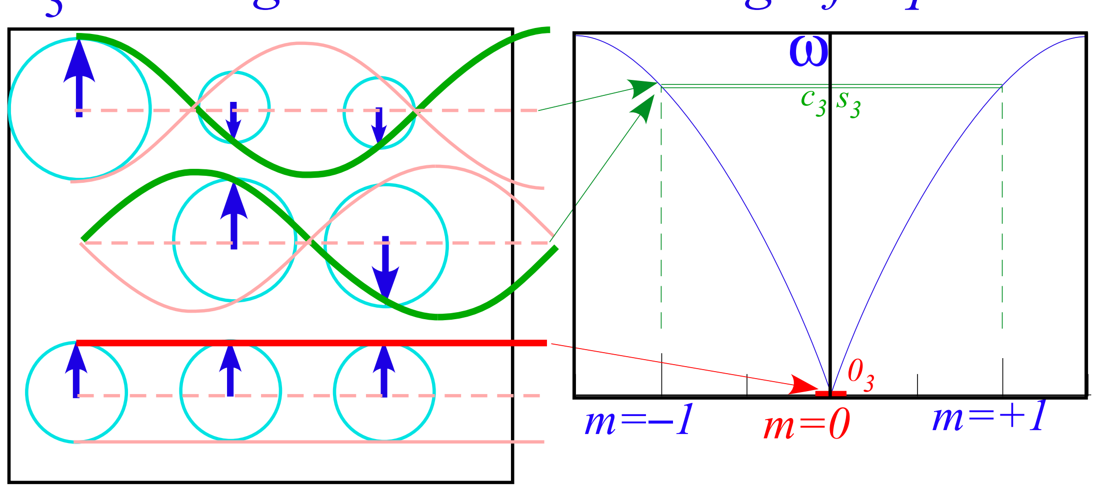
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

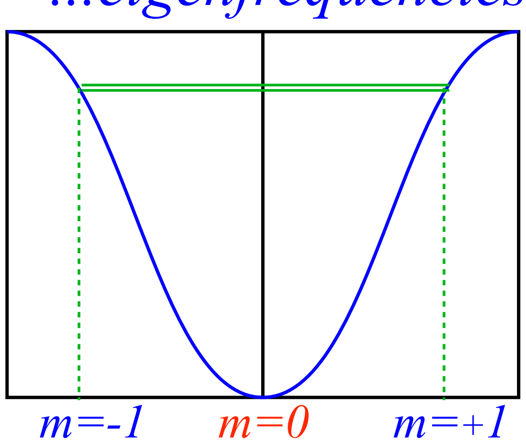
Animation screen shots (2 pages ahead)

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
$m=0_3$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

C_3 standing wave modes and eigenfrequencies of **K**



...eigenfrequencies of **H**



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

Here we assume Real $r_1=r=r_2$

Stability only requires $(r_1)^*=r_2$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m=0) \\ r_0 - r & (\text{for } m=\pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

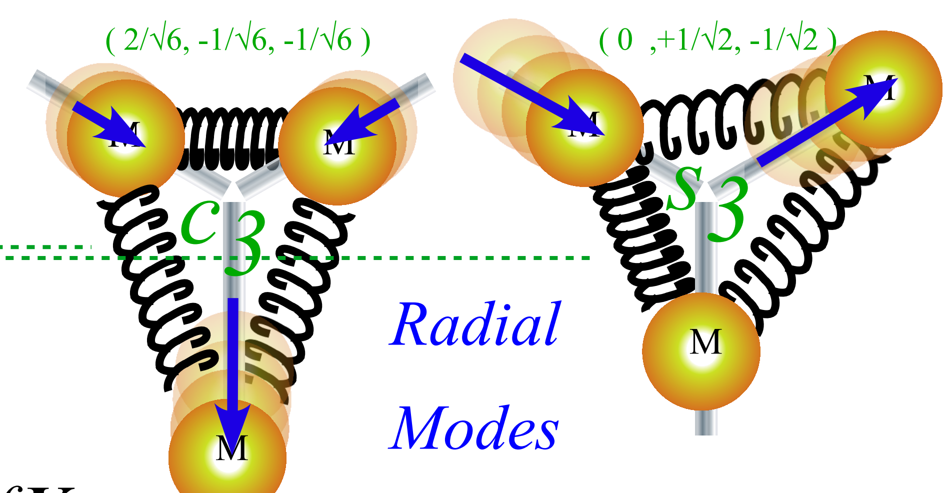
K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Animation screen shots (1 page ahead)

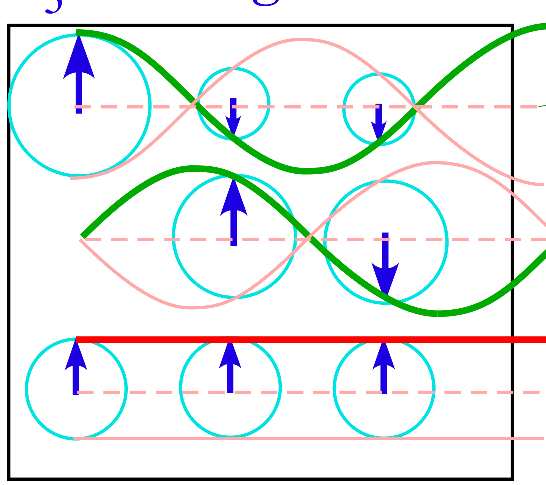
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}$ *– eigenfrequencies*	$\mathbf{K}$ *– eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

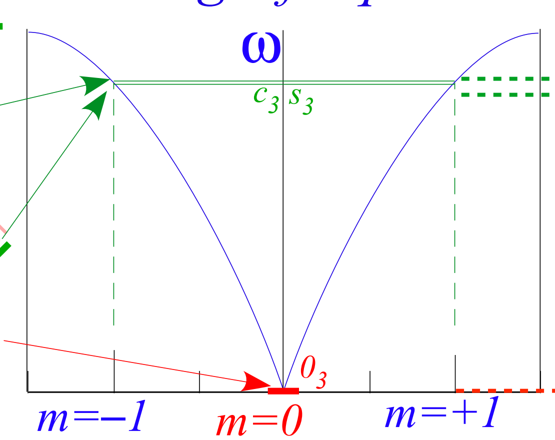
Transverse (to k) Waves

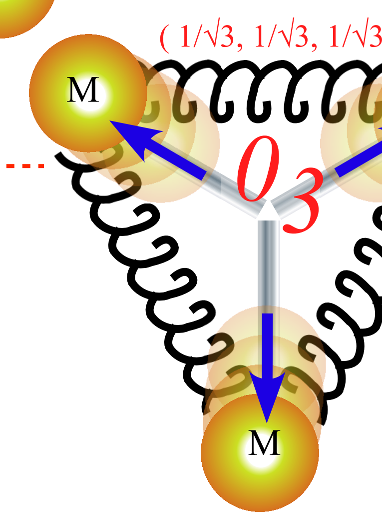


C_3 standing wave modes and eigenfrequencies of $\mathbf{K}$

	p=0	*p=1*	*p=2*
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3



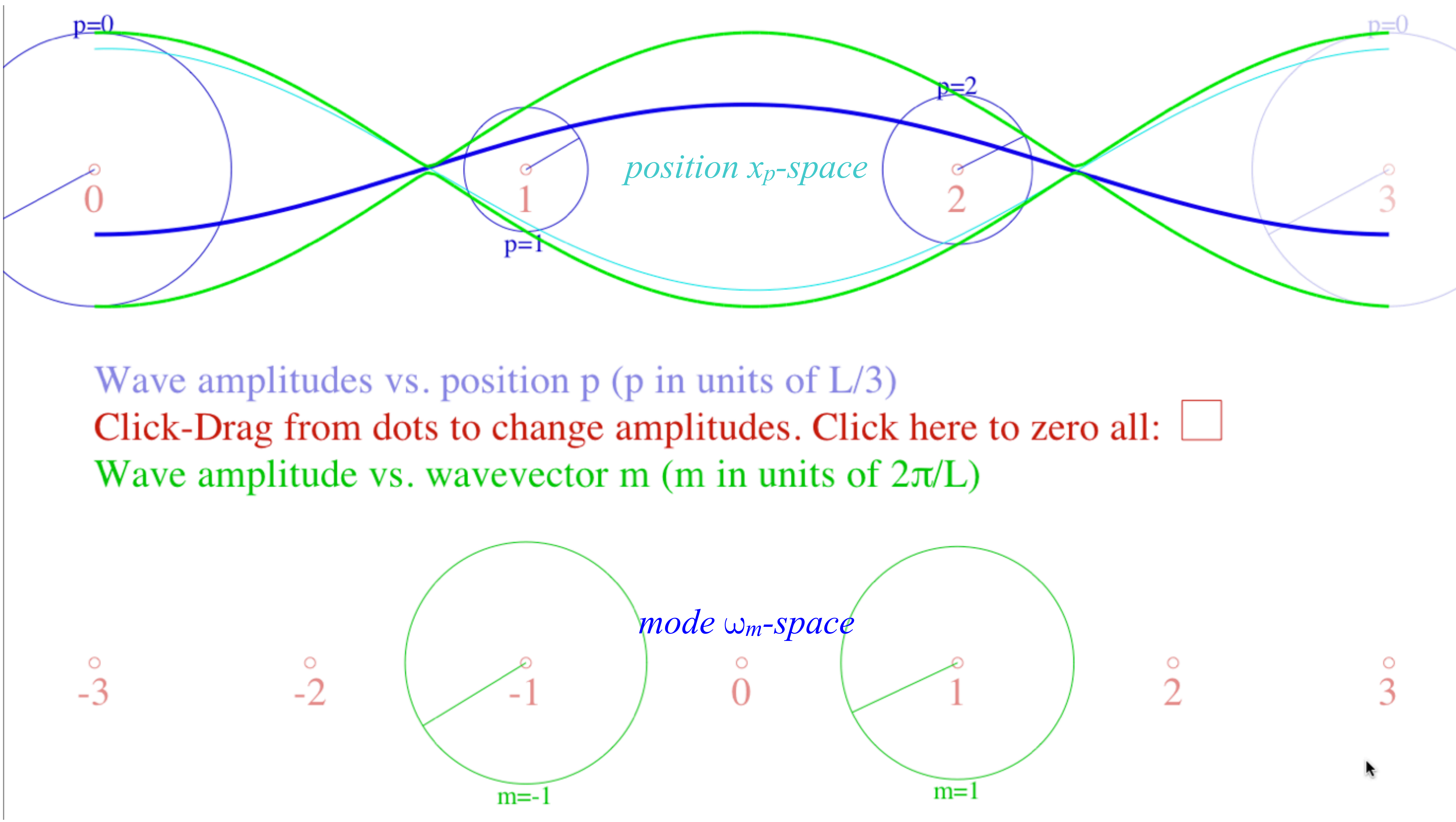




Unit 2 Fig. 2.5.3 Honors Physics CPRelawavity

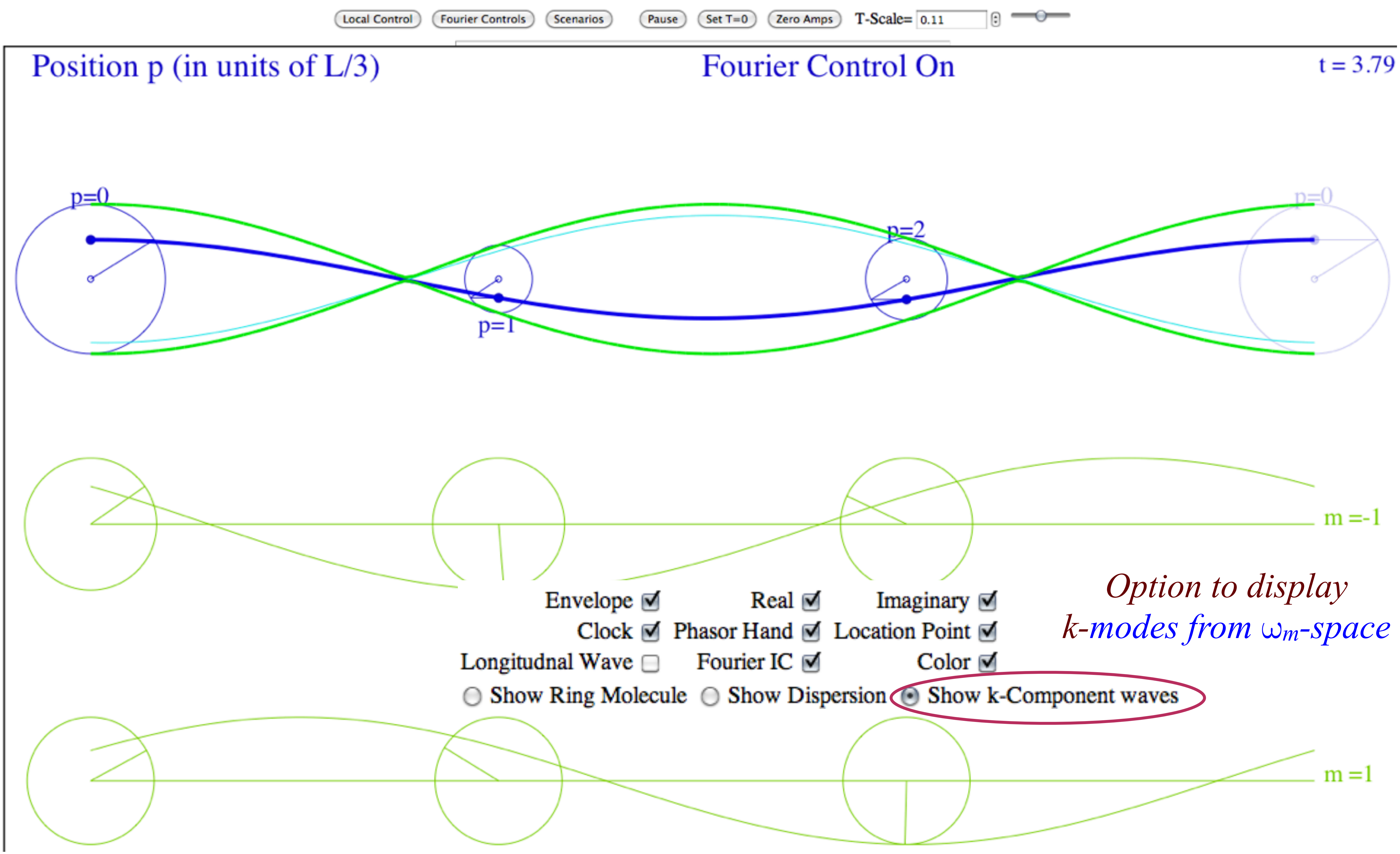
Orig. Fig. 4.5.3 CMwBang

Fourier Controls let you set modes in position x_p-space or in mode ω_m-space



[WaveIt Web Simulation](#)
[*Standing Wave (N-3)*](#)

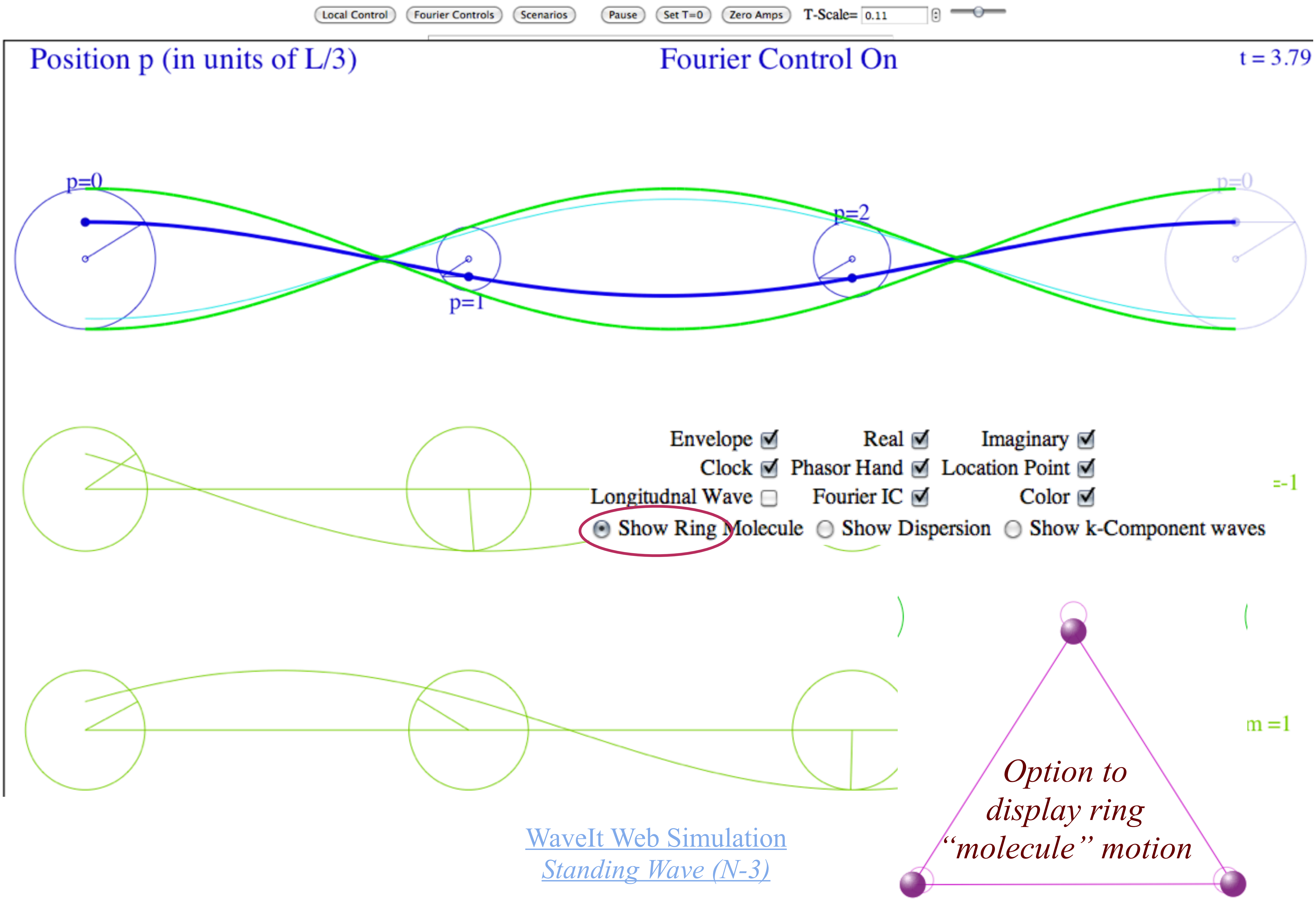
Controls in position x_p-space can display Fourier component modes from ω_m-space



Option to display k-modes from ω_m-space

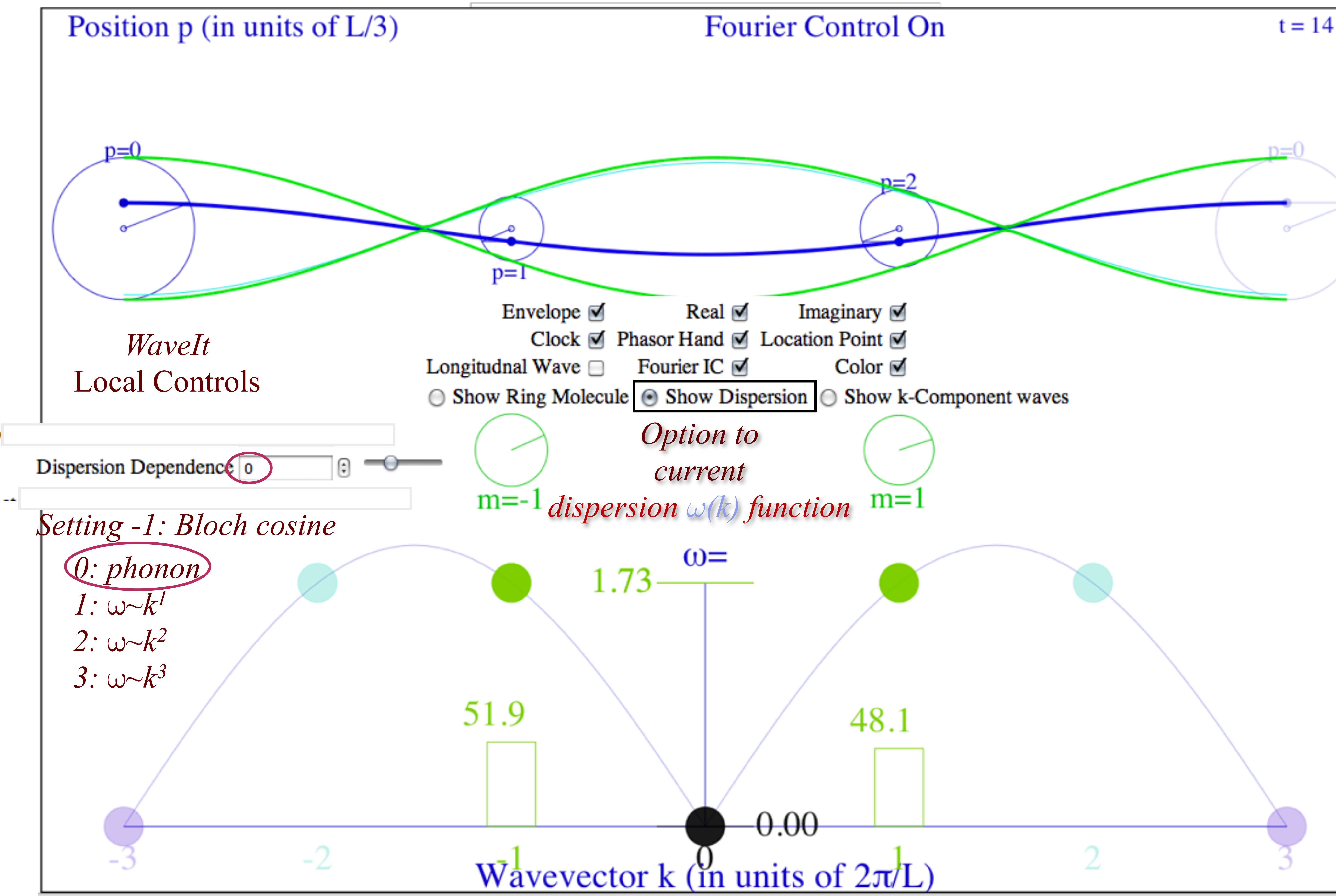
[WaveIt Web Simulation](#)
[Standing Wave (N-3)](#)

Controls in position x_p-space can display Fourier component modes from ω_m-space



Option to display ring "molecule" motion

[WaveIt Web Simulation](#)
[*Standing Wave (N-3)*](#)


Position p (in units of L/3)
Fourier Control On
t = 14
p=0
p=1
p=2
p=0
Envelope
Real
Imaginary
Clock
Phasor Hand
Location Point
Longitudnal Wave
Fourier IC
Color
Show Ring Molecule
Show Dispersion
Show k-Component waves
WaveIt
Local Controls
Dispersion Dependence
0
Option to
current
dispersion ω(k) function
m=-1
m=1
Setting -1: Bloch cosine
0: phonon
1: ω~k^1
2: ω~k^2
3: ω~k^3
ω=
1.73
51.9
48.1
0.00
-3
-2
-1
0
1
2
3
Wavevector k (in units of 2π/L)

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

Here we assume Real $r_1=r=r_2$
Stability only requires $(r_1)^*=r_2$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th} \textit{ Eigenvalue of } \mathbf{r}^p \\ \langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3} \end{array}}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r \ (\text{for } m = 0) \\ r_0 - r \ \ (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

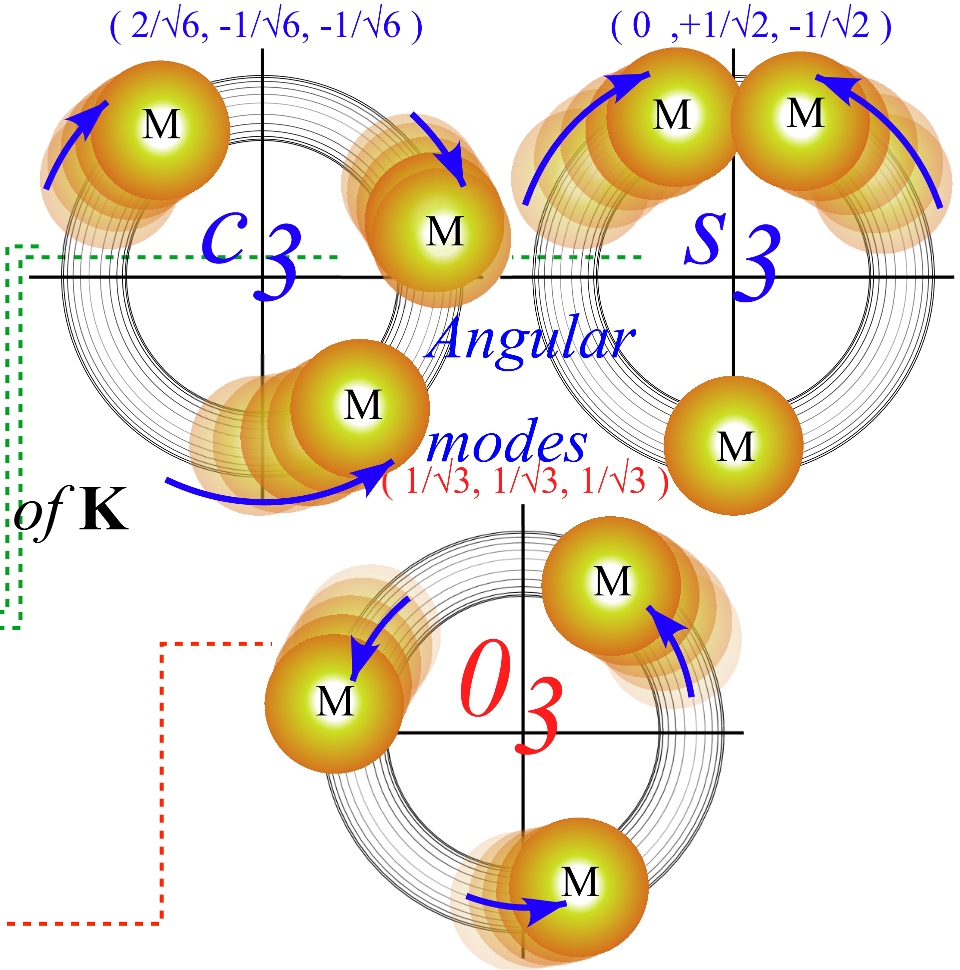
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

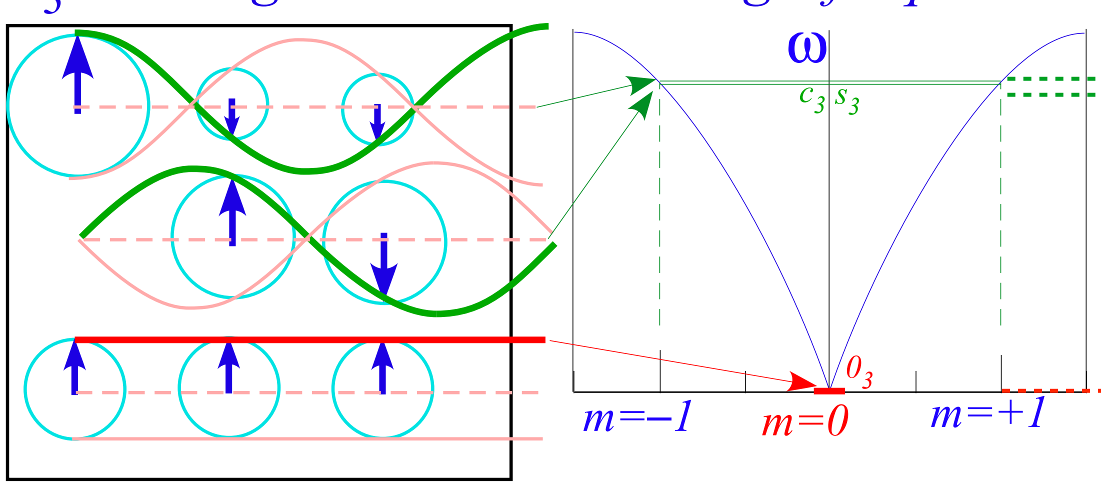
Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\vert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\vert c_3\rangle = \frac{\vert(+1)_3\rangle + \vert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\vert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\vert s_3\rangle = \frac{\vert(+1)_3\rangle - \vert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\vert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

Longitudinal (to k) Waves



C_3 standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3



Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Projector analysis of 2D-HO modes and mixed mode dynamics

½-Sum-½-Diff-Identity for resonant beat analysis

Mode frequency ratios and continued fractions

Geometry of that 90°-phase lag (again)

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C_3 symmetric spectral decomposition by 3rd roots of unity

Deriving C_3 projectors

Deriving and labeling moving wave modes

Deriving dispersion functions and degenerate standing waves

Examples by WaveIt *animation*

C_6 symmetric mode model:Distant neighbor coupling

C_6 moving waves and degenerate standing waves

C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling

C_6 dispersion functions split by C-type symmetry*(complex, chiral, ...)*

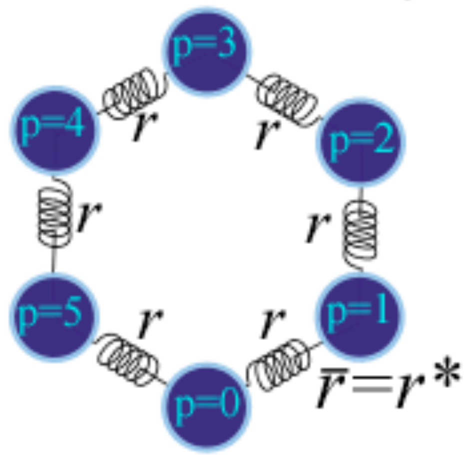
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity

½-Sum-½-Diff-theory of 2-CW group and phase velocity

C_6 Symmetric Mode Model: 1^{st} neighbor coupling

We usually assume <u>Real</u> $r=\bar{r}$
Stability only requires $(r)^*=\bar{r}$

(a) 1^{st} Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c} \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} & \\ \begin{pmatrix} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

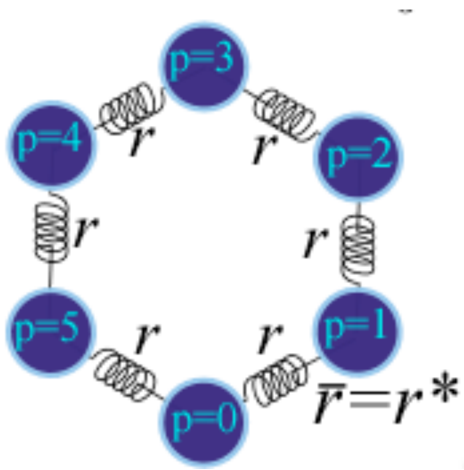
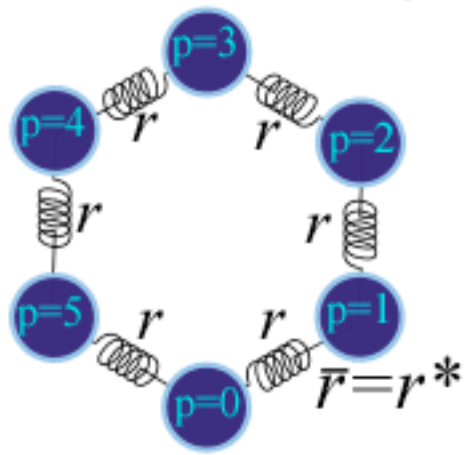


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Symmetric Mode Model: 1st and 2nd neighbor coupling

We usually assume Real $r=\bar{r}$
Stability only requires $(r)^*=\bar{r}$

(a) 1^{st} Neighbor C_6

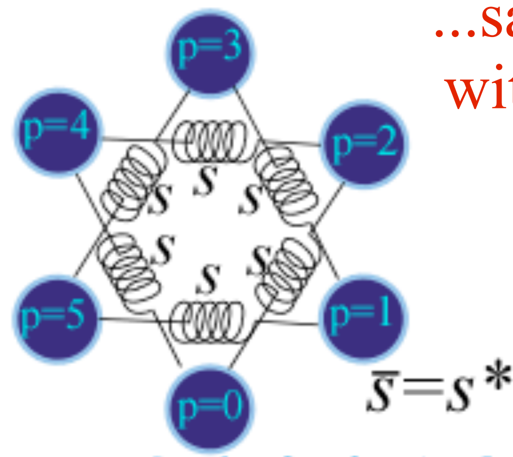


$$\mathbf{H}^{B1(6)}= \begin{array}{c} p{=}0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{pmatrix} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

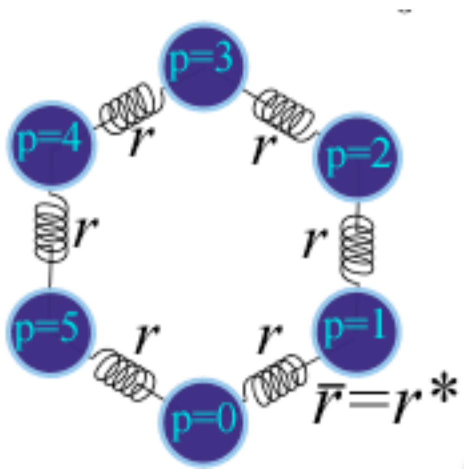
(b) 2^{nd} Neighbor C_6

...same with s



$$\mathbf{H}^{B2(6)}= \begin{array}{c} p{=}0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{pmatrix} H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot \\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s} \\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot \\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s \\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot \\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$



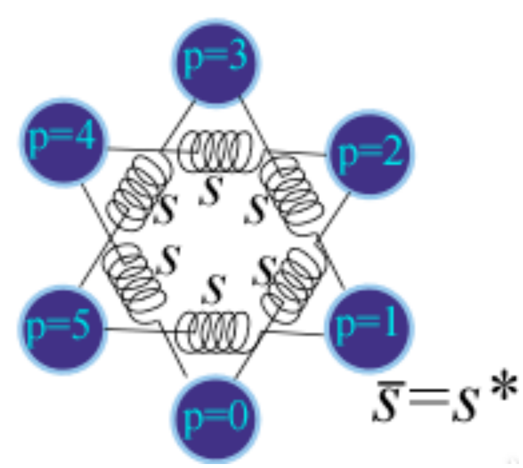
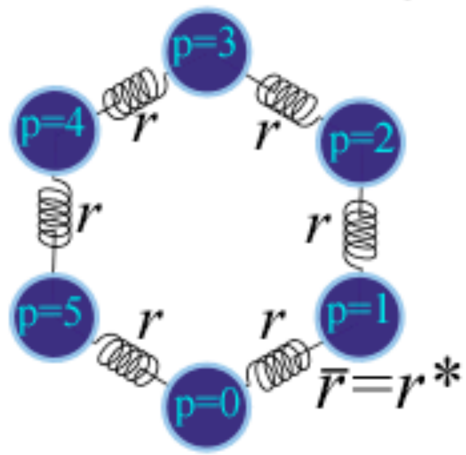


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Symmetric Mode Model: 1st , 2nd and 3rd neighbor coupling

(a) 1st Neighbor C_6

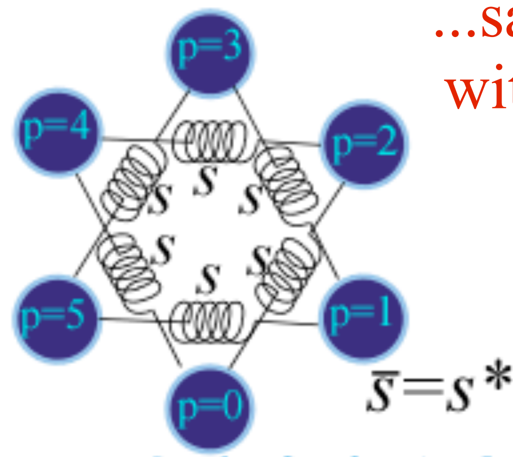


$$
\mathbf{H}^{B1(6)}=
\begin{array}{c}
\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left(\begin{array}{cccccc}
H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\
-\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\
\cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\
\cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\
\cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\
-r & \cdot & \cdot & \cdot & -\bar{r} & H_1
\end{array}\right)
\end{array}
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

(b) 2nd Neighbor C_6

...same with s

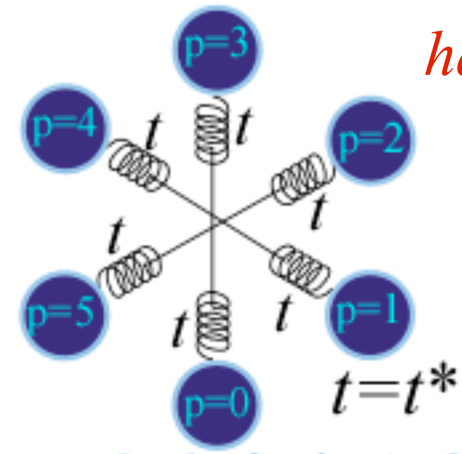


$$
\mathbf{H}^{B2(6)}=
\begin{array}{c}
\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left(\begin{array}{cccccc}
H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot \\
\cdot & H_2 & \cdot & -s & \cdot & -\bar{s} \\
-\bar{s} & \cdot & H_2 & \cdot & -s & \cdot \\
\cdot & -\bar{s} & \cdot & H_2 & \cdot & -s \\
-s & \cdot & -\bar{s} & \cdot & H_2 & \cdot \\
\cdot & -s & \cdot & -\bar{s} & \cdot & H_2
\end{array}\right)
\end{array}
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$

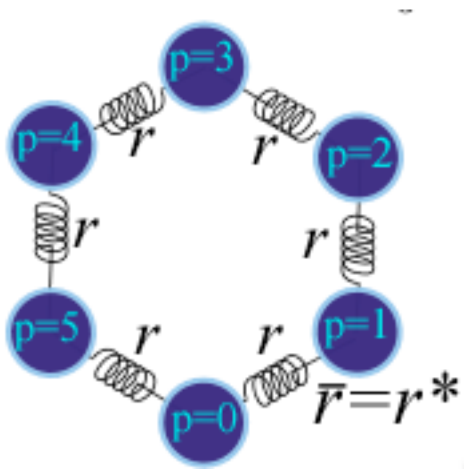
(c) 3rd Neighbor C_6

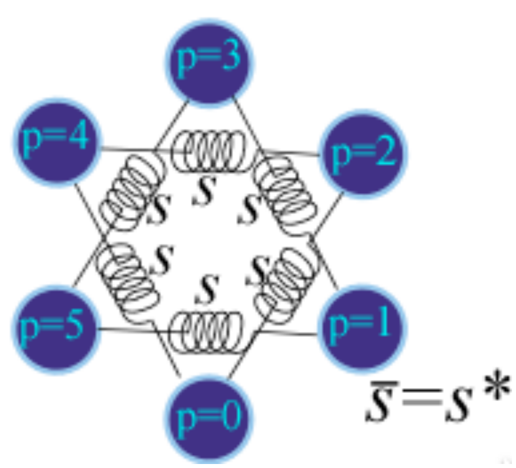
...but t *has* to be real



$$
\mathbf{H}^{B3(6)}=
\begin{array}{c}
\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left(\begin{array}{cccccc}
H_3 & \cdot & \cdot & -t & \cdot & \cdot \\
\cdot & H_3 & \cdot & \cdot & -t & \cdot \\
\cdot & \cdot & H_3 & \cdot & \cdot & -t \\
-t & \cdot & \cdot & H_3 & \cdot & \cdot \\
\cdot & -t & \cdot & \cdot & H_3 & \cdot \\
\cdot & \cdot & -t & \cdot & \cdot & H_3
\end{array}\right)
\end{array}
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$





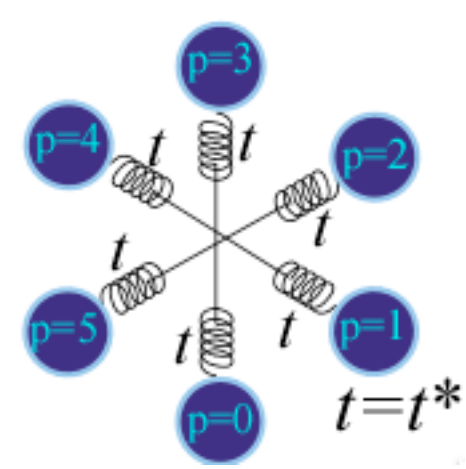


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

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C_6 symmetric mode model:Distant neighbor coupling

C_6 moving waves and degenerate standing waves

C_6 dispersion functions for 1^{st}, 2^{nd}, and 3^{rd}-neighbor coupling

C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)

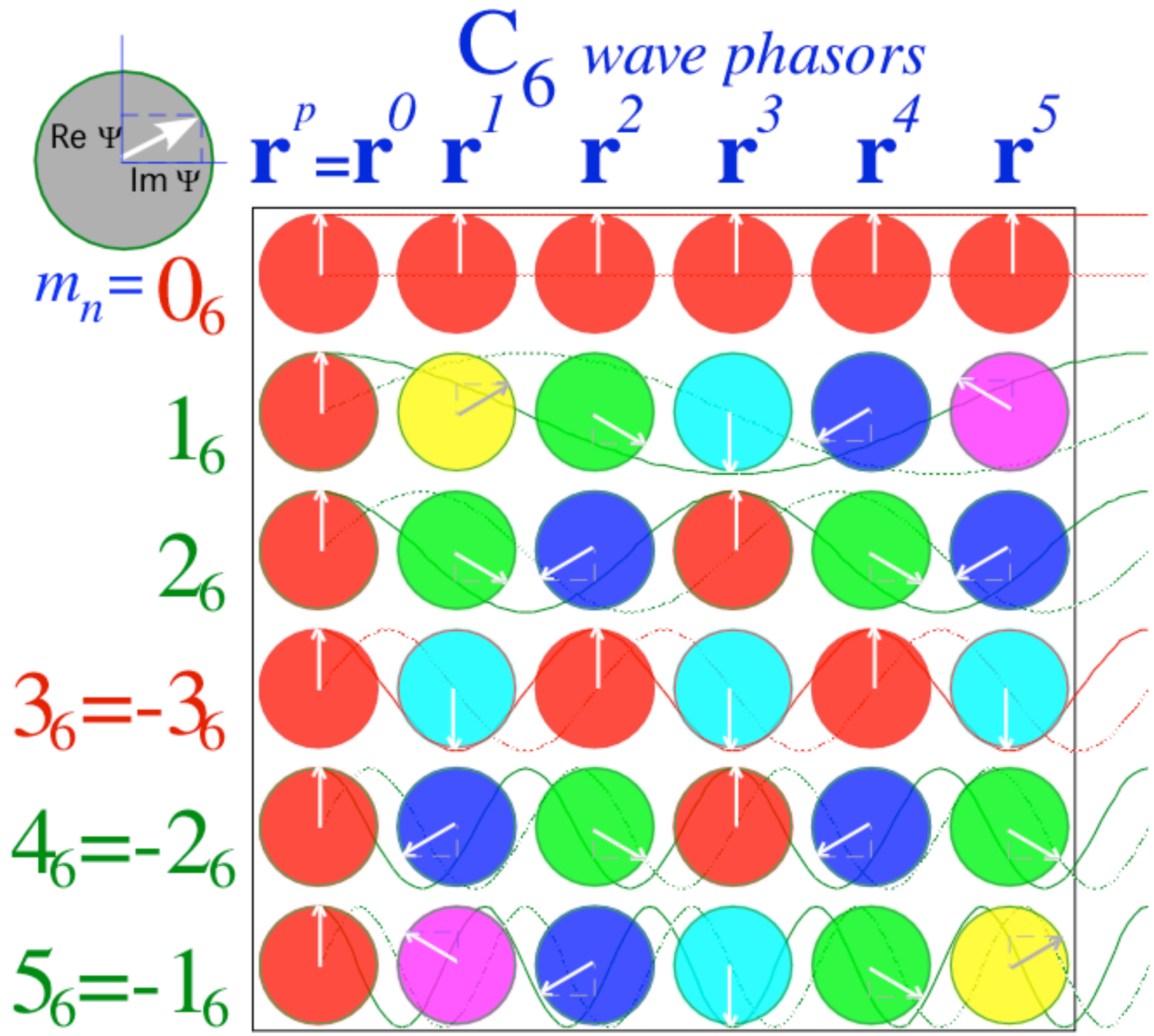
C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity

½-Sum-½-Diff-theory of 2-CW group and phase velocity

C_6 Spectral resolution: 6^{th} roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

$$Wavefunction: \ \Psi^m(x_p) = \chi_p^{m*} = D^{m*}(\mathbf{r}^p)$$



$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi i m p}{6}}$$

WaveIt C_6 Character Phasors Web Simulation

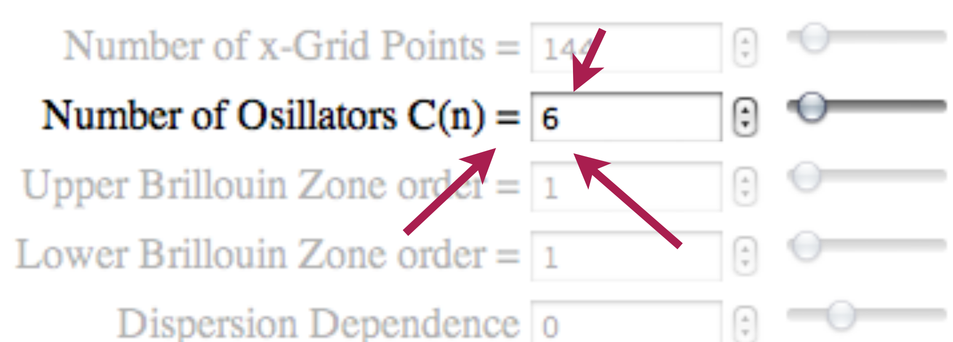
Fig. 13 International Journal of Molecular Science 14, 752 (2013)

C_6 Spectral resolution: 6^{th} roots of unity

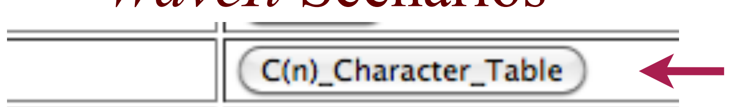
$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

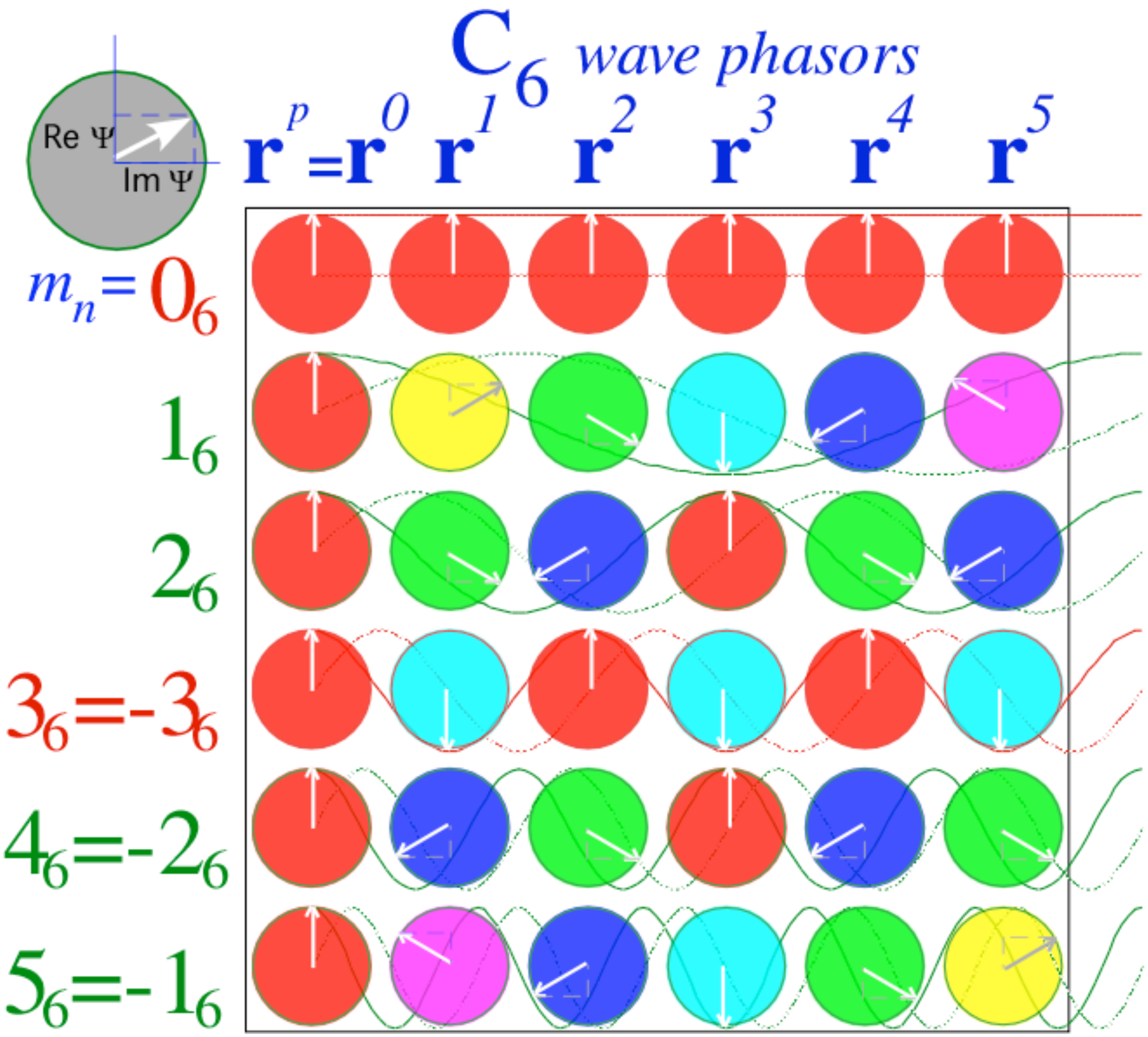
$$Wavefunction: \ \Psi^m(x_p) = \chi_p^{m*} = D^{m*}(\mathbf{r}^p)$$

WaveIt
Local Controls



WaveIt Scenarios

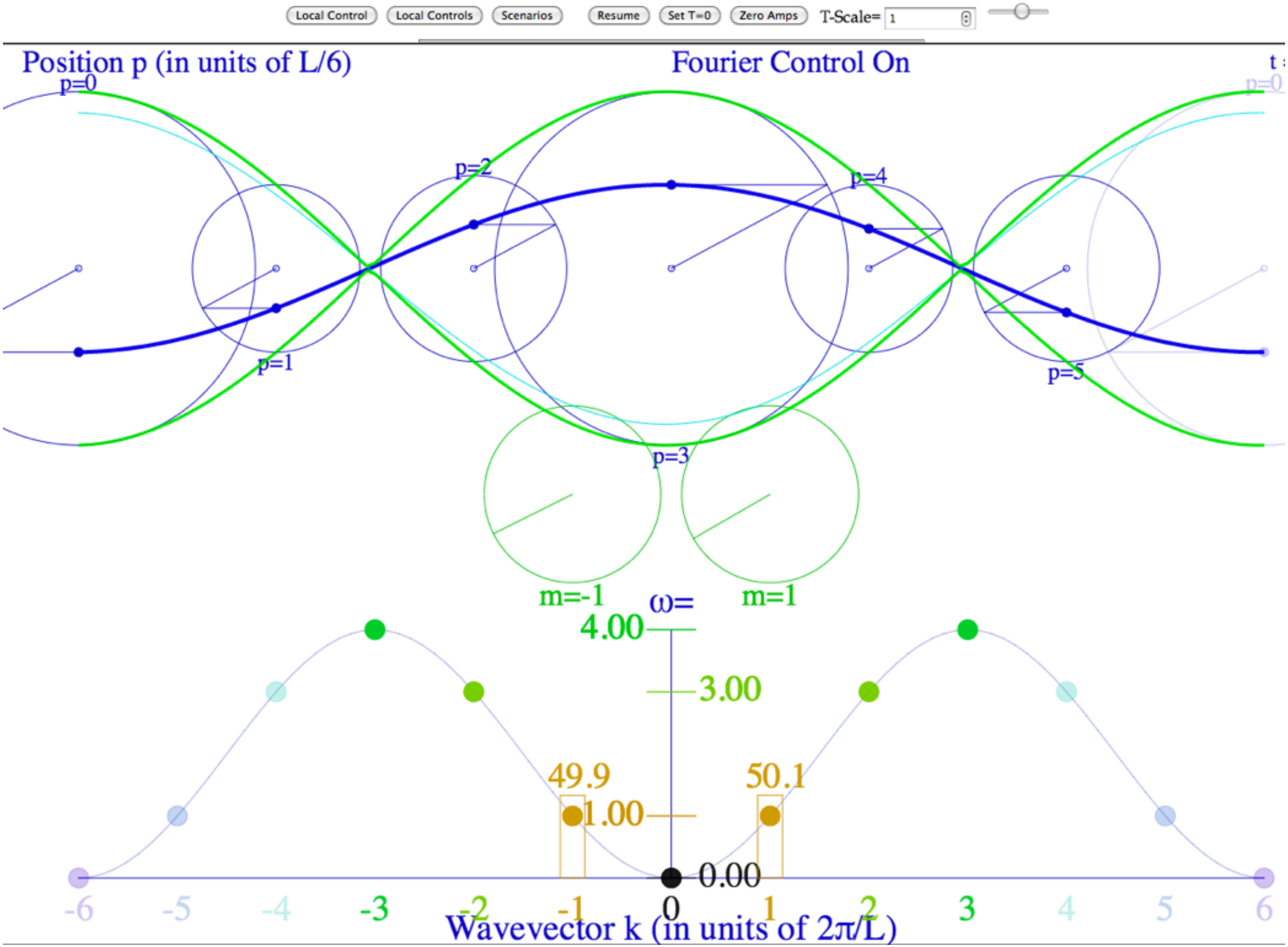




$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi i m p}{6}}$$

[WaveIt C_6 Character Phasors Web Simulation]

Fig. 13 International Journal of Molecular Science 14, 752 (2013)



[WaveIt Web Simulation - *Standing Wave (N=6)*]

[WaveIt Web Simulation - *Galloping Wave (N=6)*]

C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion

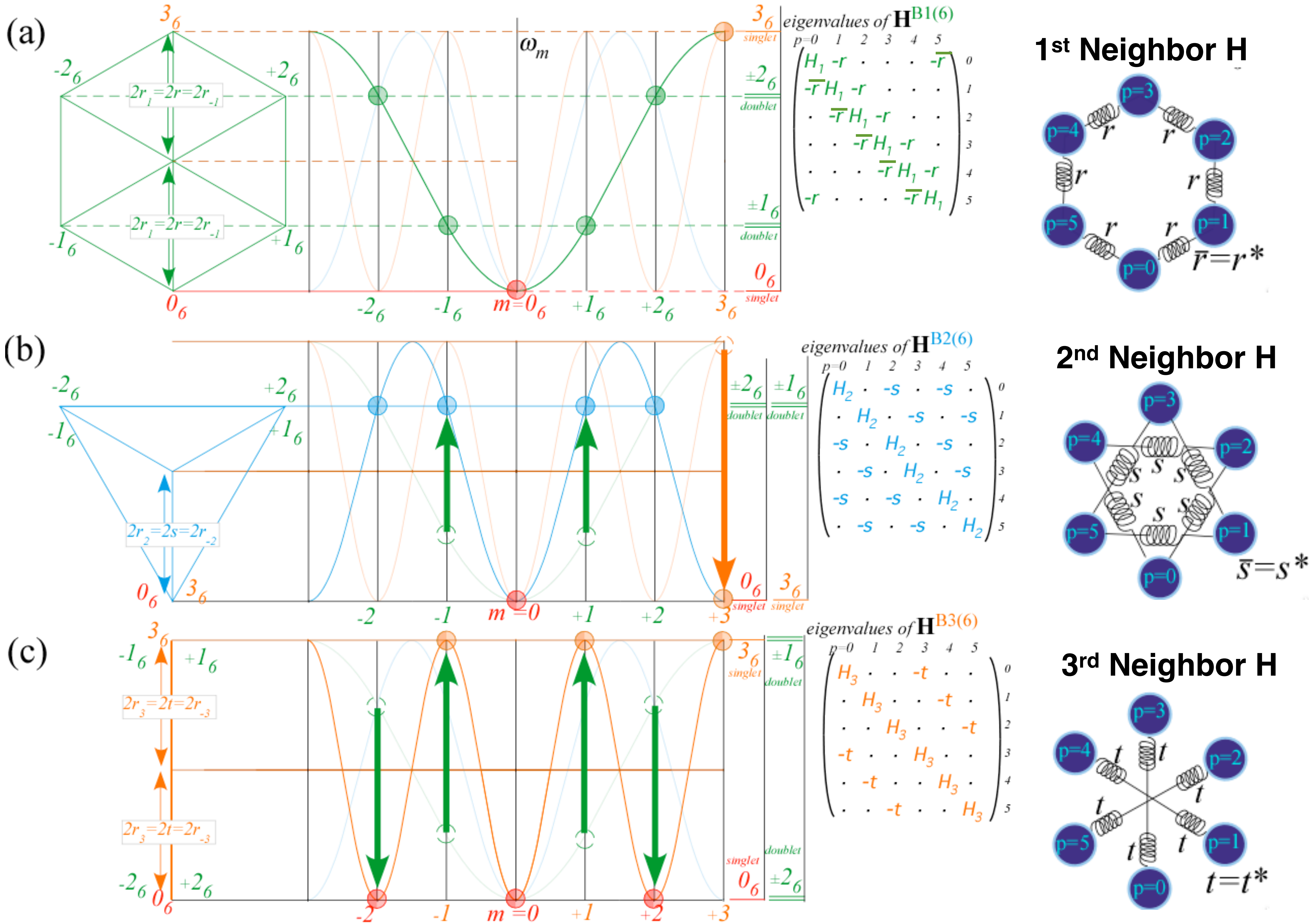
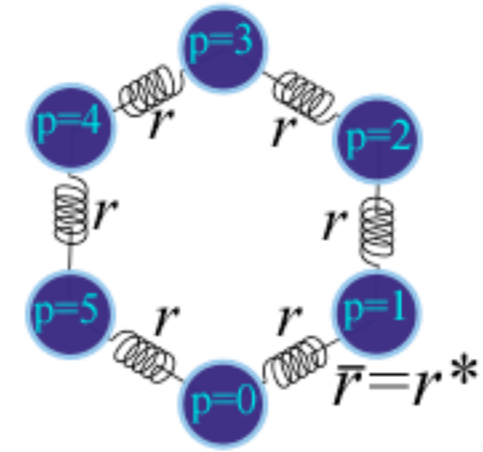
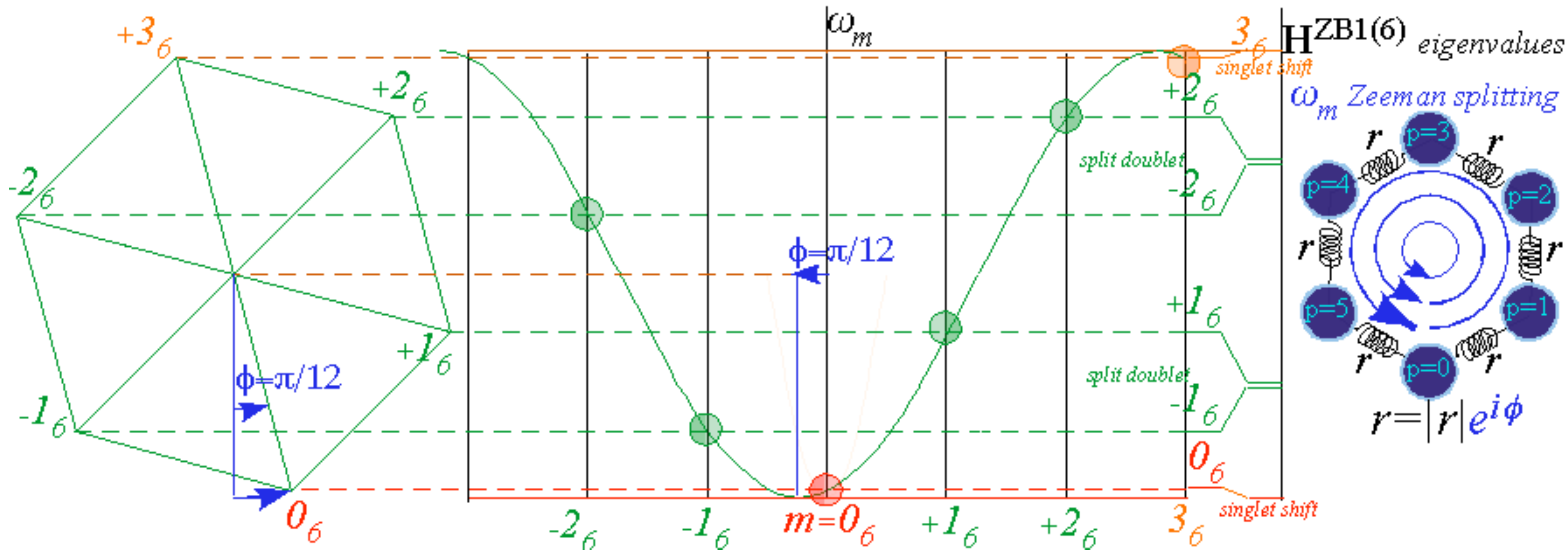


Fig. 14 International Journal of Molecular Science 14, 754 (2013)

C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,

1st Neighbor H





Standing wave combinations like $\cos kx=(e^{+ikx}+e^{-ikx})/2$
are not eigenmodes unless $\phi=0$.

Fig. 15 International Journal of Molecular Science 14, 755 (2013)

Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Projector analysis of 2D-HO modes and mixed mode dynamics

½-Sum-½-Diff-Identity for resonant beat analysis

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➔ *C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity*

½-Sum-½-Diff-theory of 2-CW group and phase velocity

C_N Symmetric Mode Models:

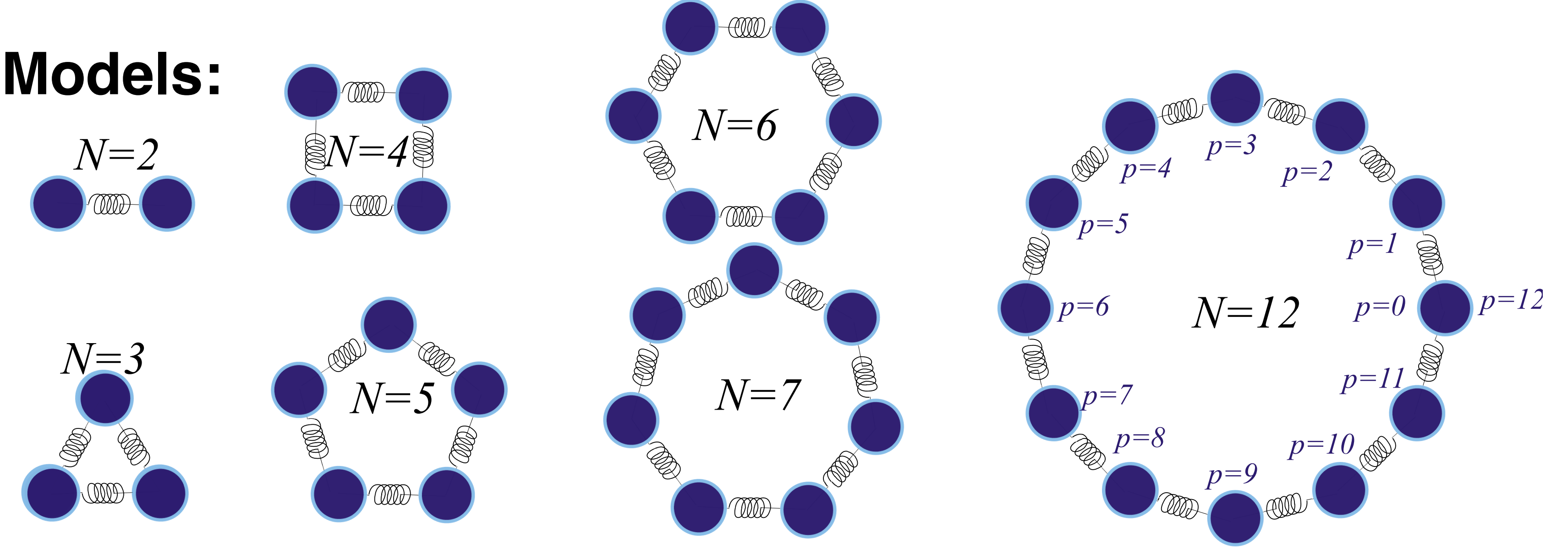


Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

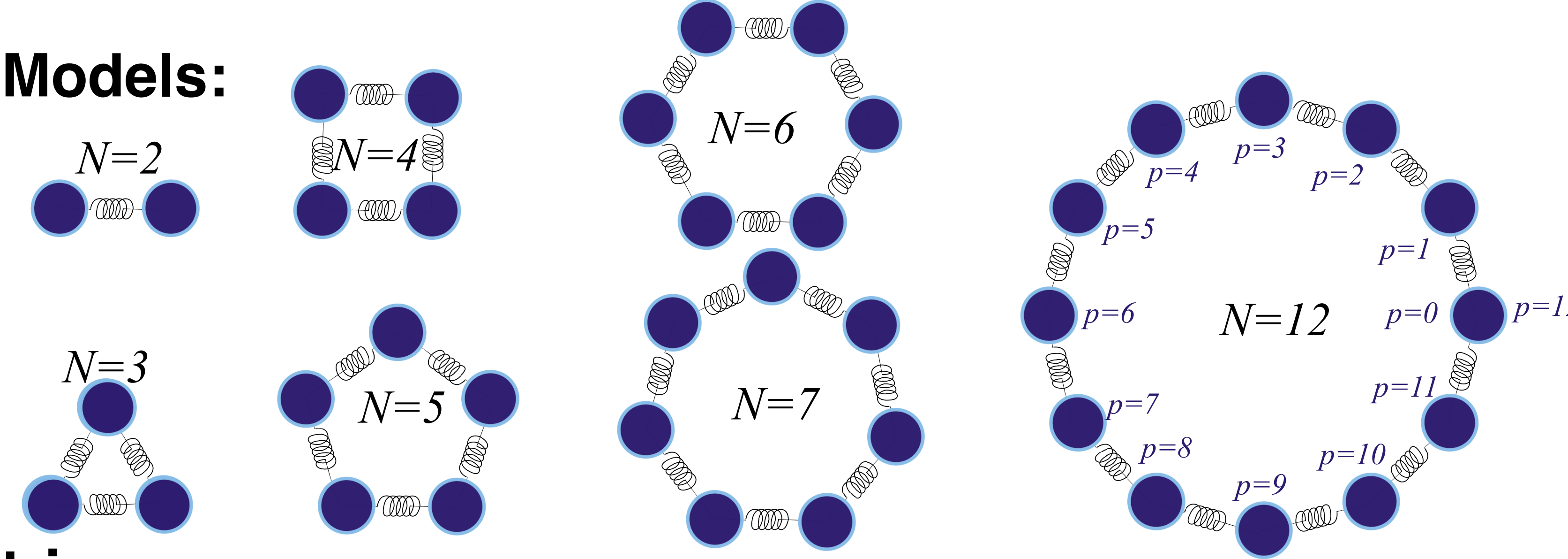


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$
-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix}
=
\begin{pmatrix}
K & -k_{12} & . & . & . & \cdots & -k_{12} \\
-k_{12} & K & -k_{12} & . & . & \cdots & . \\
. & -k_{12} & K & -k_{12} & . & \cdots & . \\
. & . & -k_{12} & K & -k_{12} & \cdots & . \\
. & . & . & -k_{12} & K & \cdots & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\
-k_{12} & . & . & . & . & -k_{12} & K
\end{pmatrix}
\bullet
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}
\quad \text{where:} \quad
\begin{matrix} K = k + 2k_{12} \\ k = \dfrac{Mg}{\ell} \\ (\cdot) = 0 \end{matrix}
$$

C_N Symmetric Mode Models:

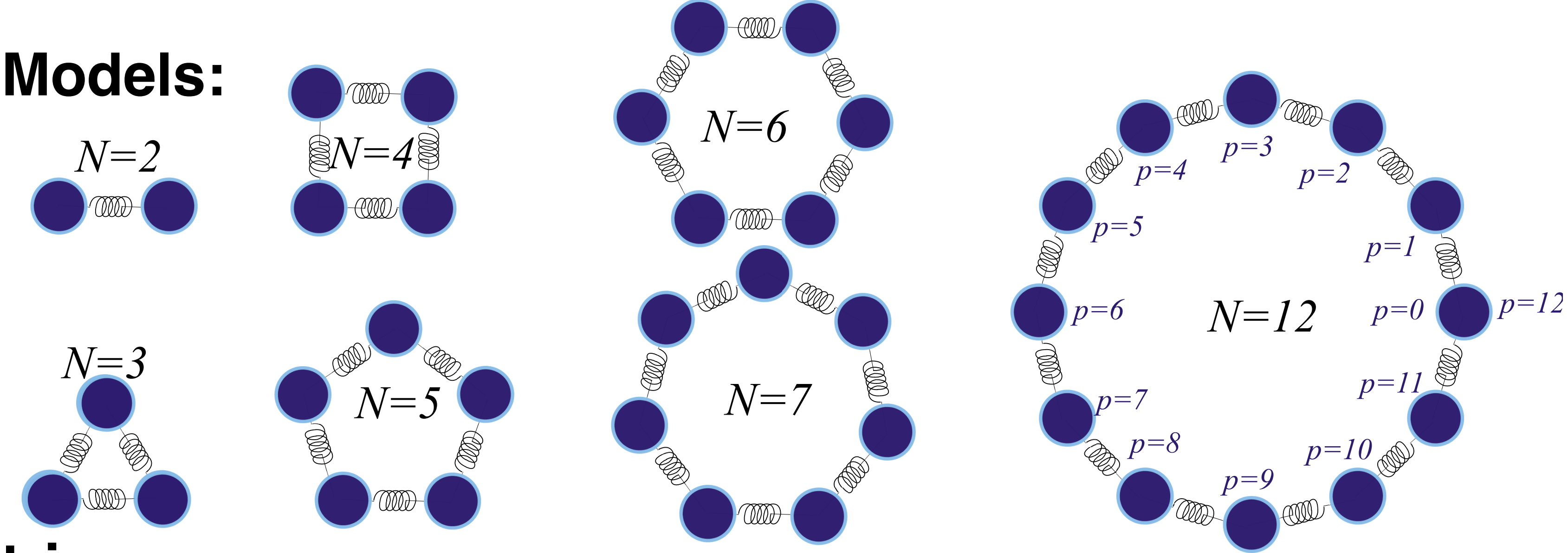


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix}F_0\\F_1\\F_2\\F_3\\F_4\\\vdots\\F_{N-1}\end{pmatrix}=\begin{pmatrix}K & -k_{12} & \cdot & \cdot & \cdot & \cdots & -k_{12}\\ -k_{12} & K & -k_{12} & \cdot & \cdot & \cdots & \cdot\\ \cdot & -k_{12} & K & -k_{12} & \cdot & \cdots & \cdot\\ \cdot & \cdot & -k_{12} & K & -k_{12} & \cdots & \cdot\\ \cdot & \cdot & \cdot & -k_{12} & K & \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12}\\ -k_{12} & \cdot & \cdot & \cdot & \cdot & -k_{12} & K\end{pmatrix}\bullet\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\x_4\\\vdots\\x_{N-1}\end{pmatrix}$$

where: $K = k + 2k_{12}$, $k = \dfrac{Mg}{\ell}$, $(\cdot) = 0$

Nth roots of 1 $e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

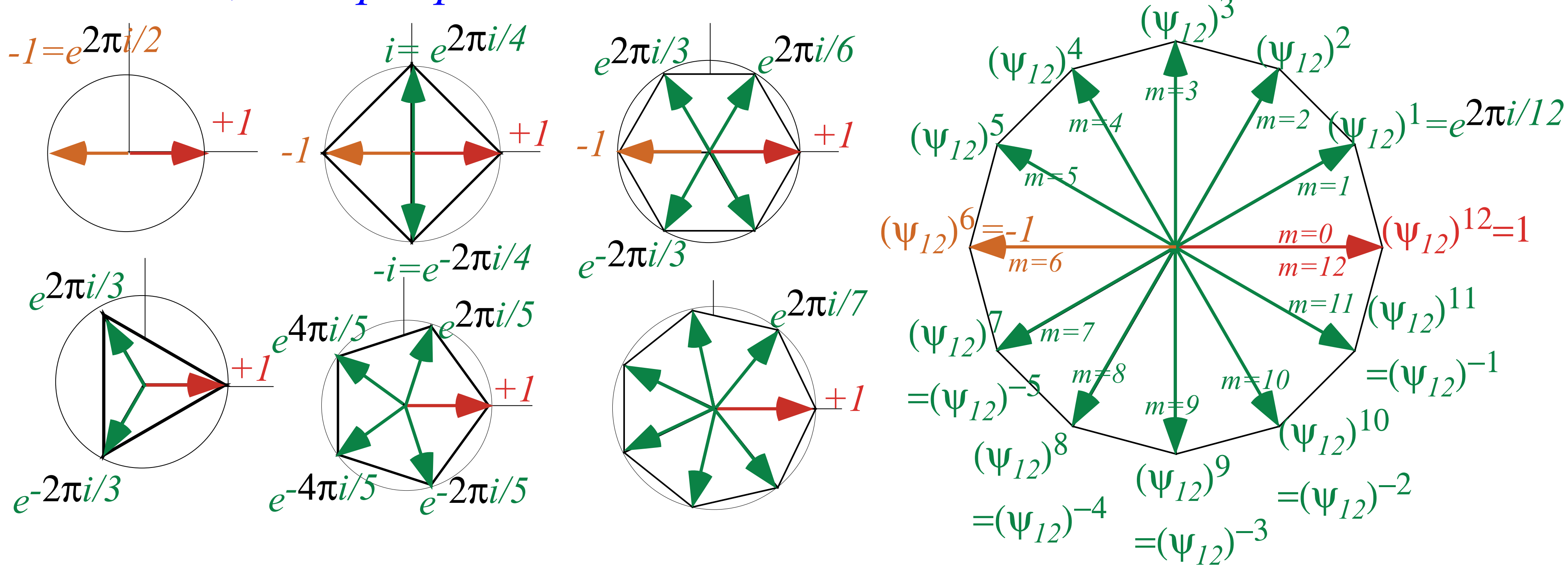
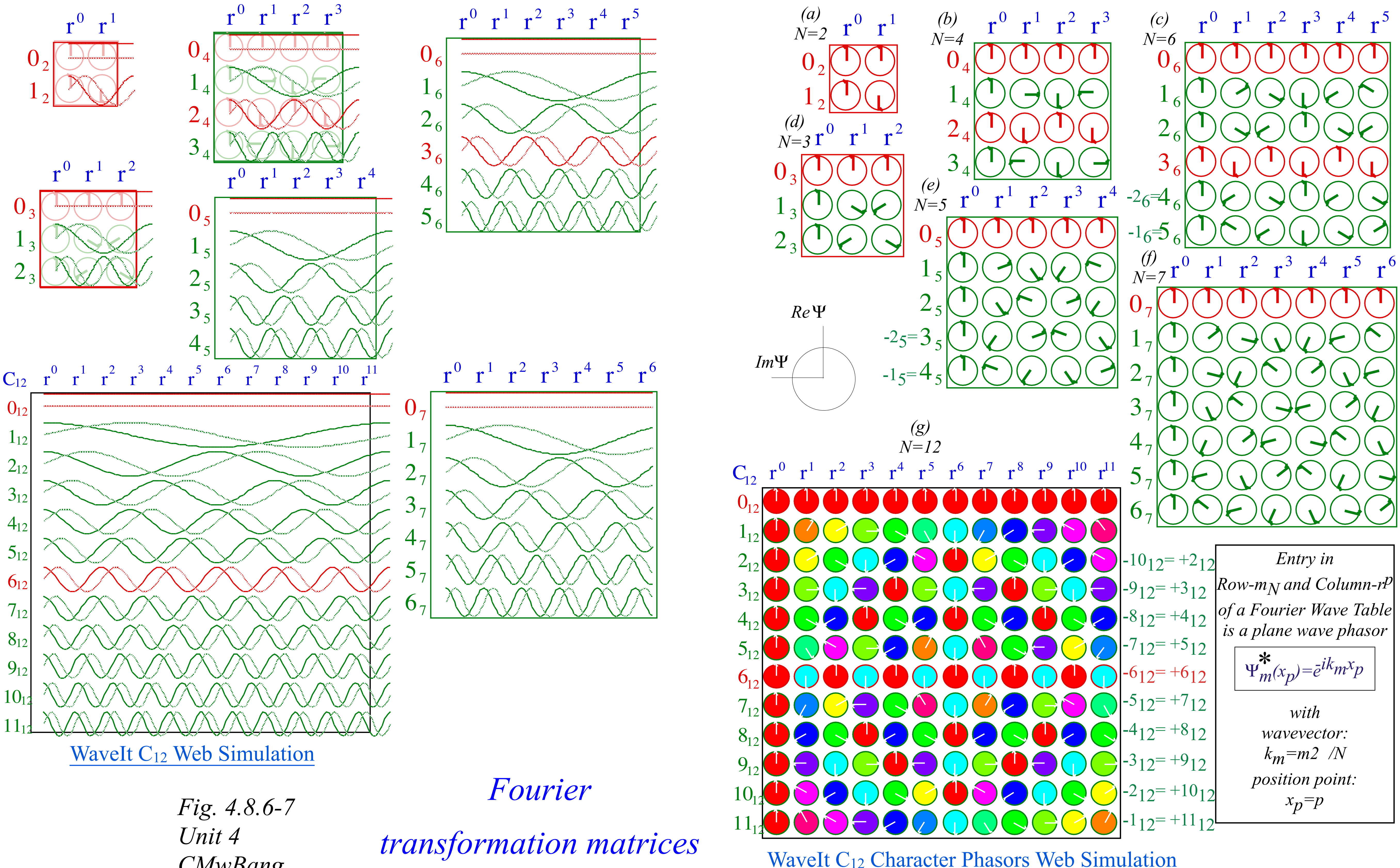


Fig. 4.8.5
Unit 4
CMwBang

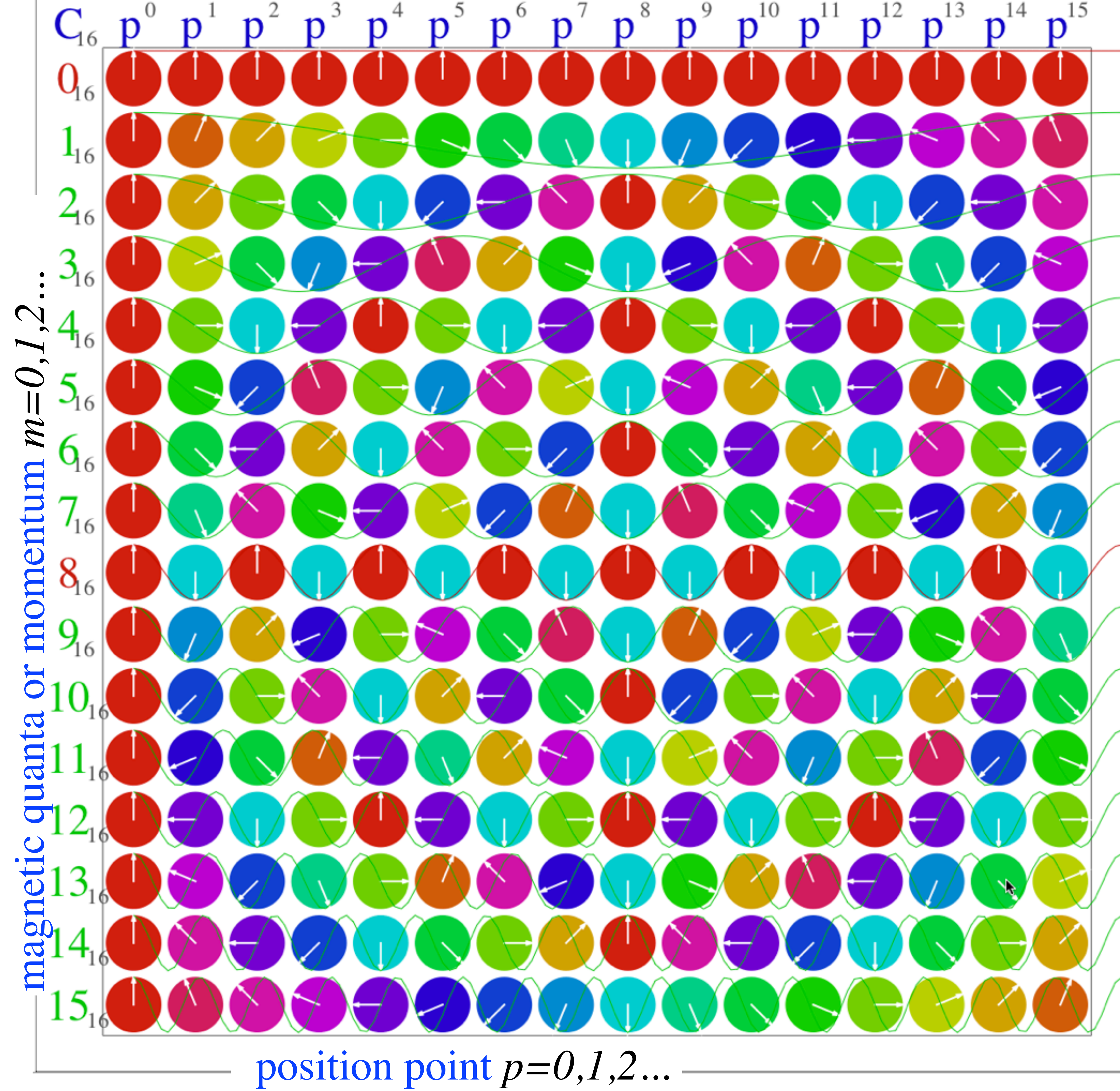
C_N Symmetric Mode Models:

N^{th} roots of 1 $e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*



Fourier transformation matrices

Fig. 4.8.6-7
Unit 4
CMwBang

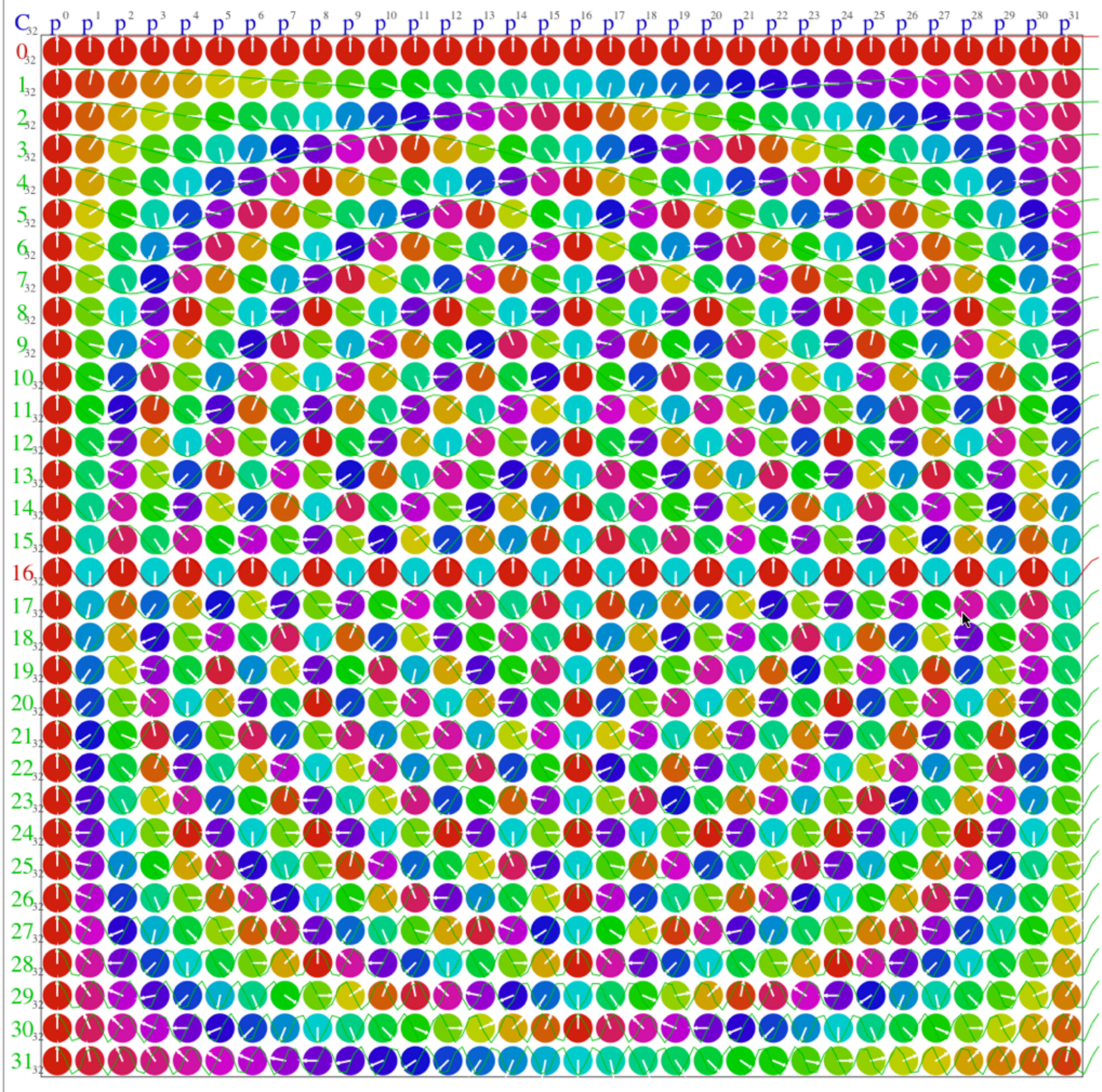


C16

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{16}}$$

[WaveIt C_{16} Character Phasors Web Simulation] magnetic quanta or momentum $m=0,1,2\ldots$



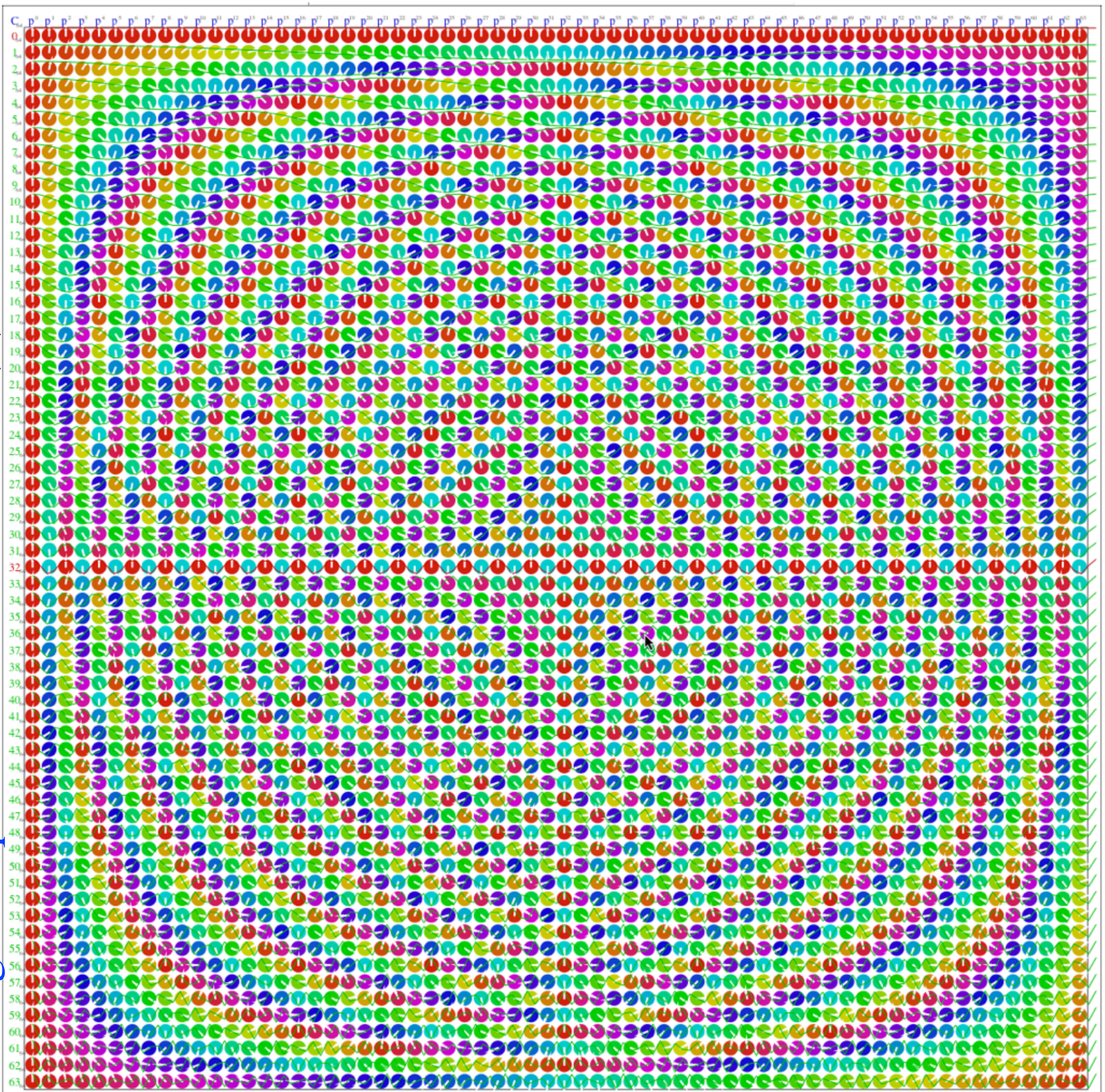
position point $p=0,1,2\ldots$

C_{32}

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi i m p}{32}}$$

[WaveIt C_{32} Character Phasors Web Simulation]



magnetic quanta or momentum *m=0,1,2…*

position point *p=0,1,2…*

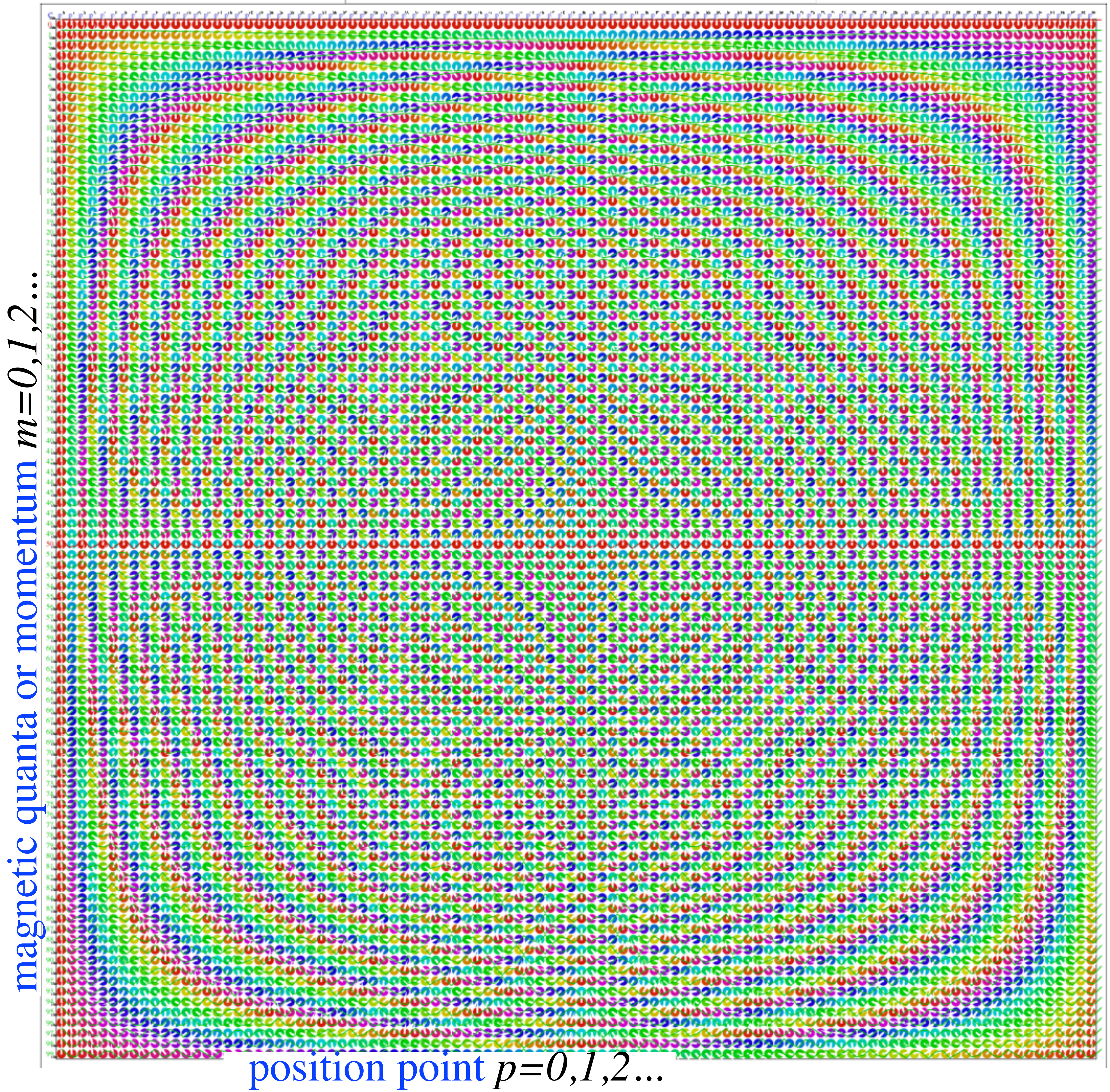
C_{64}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{64}}$$

Invariant phase
“Uncertainty”
hyperbolas:
$m \cdot p = const.$



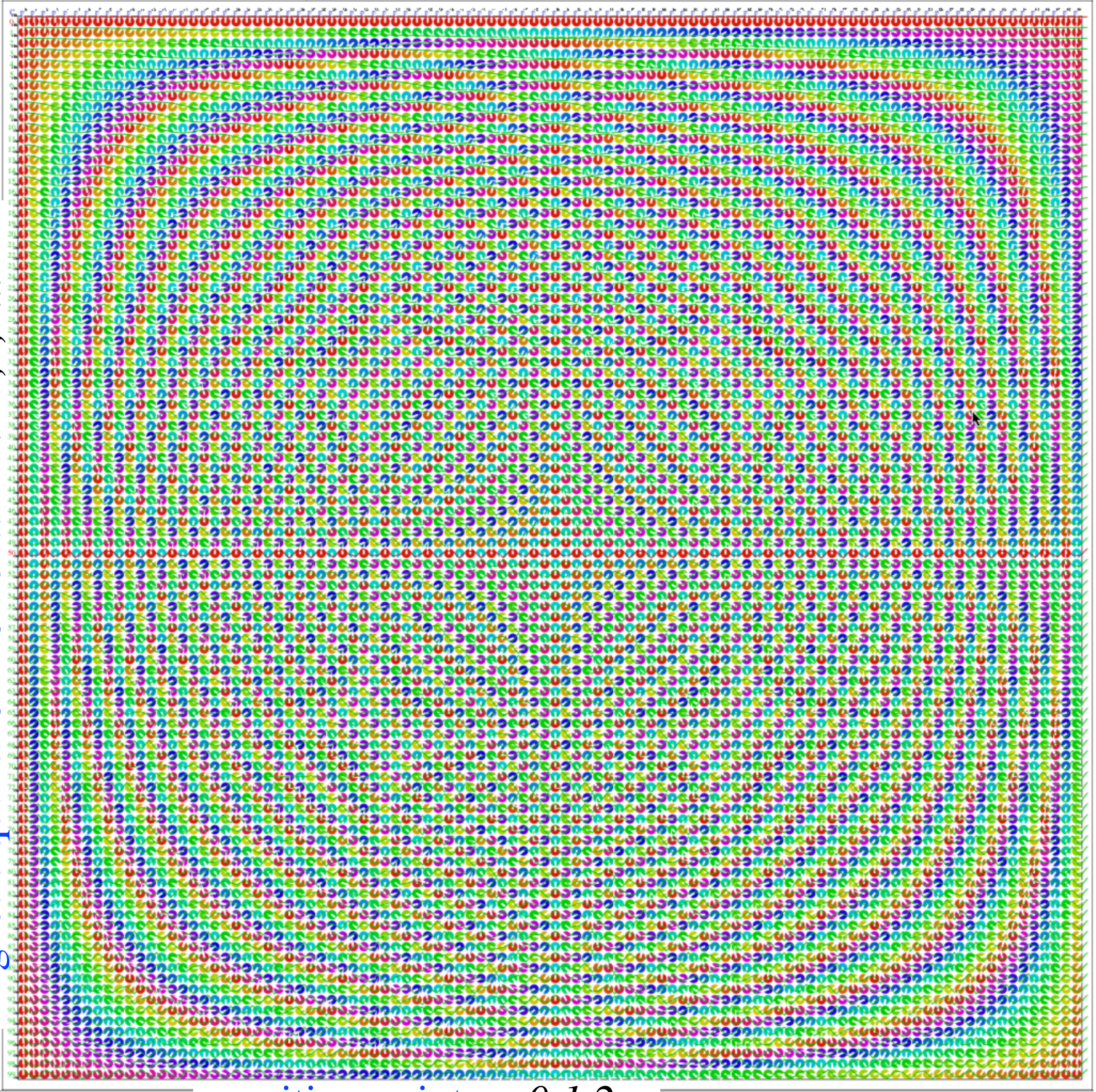
C_{100}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$



magnetic quanta or momentum m=0,1,2...

position point p=0,1,2...

C_{256}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{256}}$$

Invariant phase "Uncertainty" hyperbolas: $m \cdot p = const.$

WaveIt C_{256} Character Phasors Web Simulation

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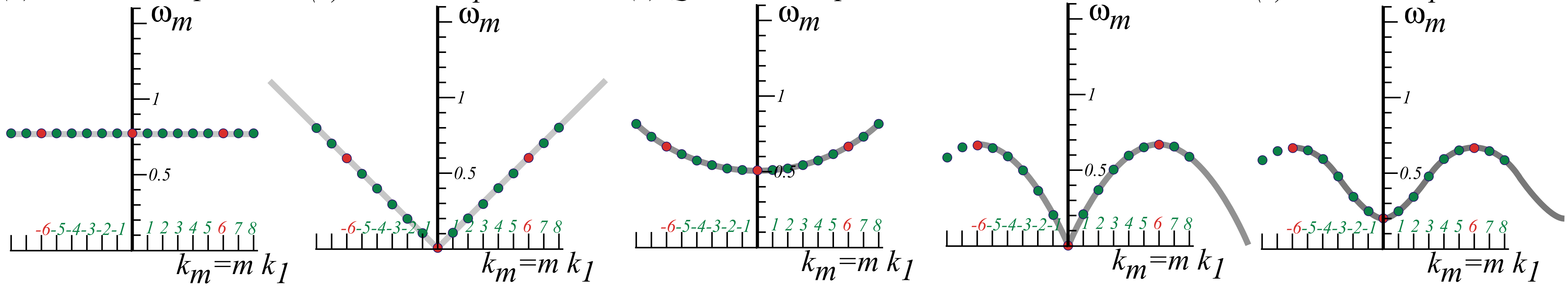
Archetypical Examples of C_{12} Dispersion Functions

(a) Constant dispersion *(b) Linear dispersion* *(c) Quadratic dispersion* *(d) Phonon dispersion* *(e) Exciton dispersion*



Applications:

(a) Uncoupled pendulums

Movie marquis
Xmas lights

(b) Weakly coupled pendu-lums (No gravity)

Light in vacuum (Exactly)
Sound (Approximately)

(c) Weakly coupled pendu-lums (With gravity)

Light in fiber (Approx)
Non-relativistic
Schrodinger matter wave

(d) Strongly coupled pendu-lums (No gravity)

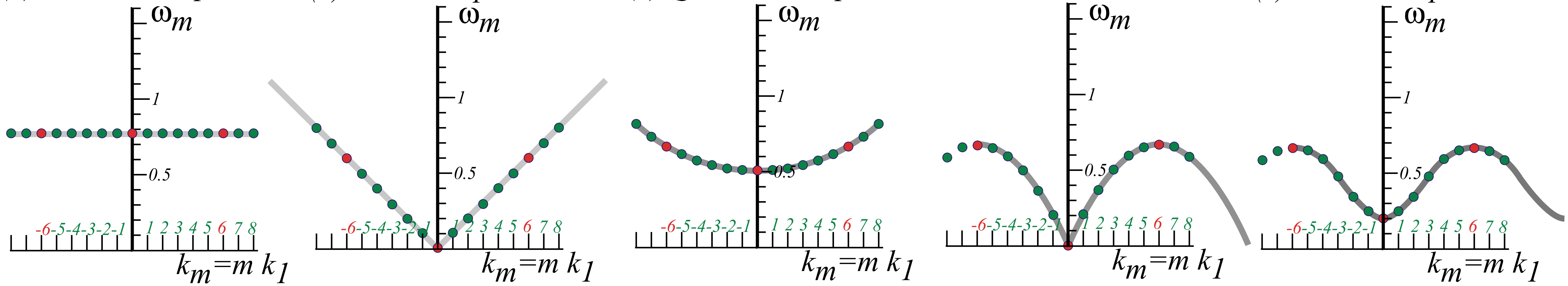
Acoustic mode in solids

(e) Strongly coupled pendu-lums (With gravity)

Optical mode in solids
Relativistic matter
(If exact hyperbola)

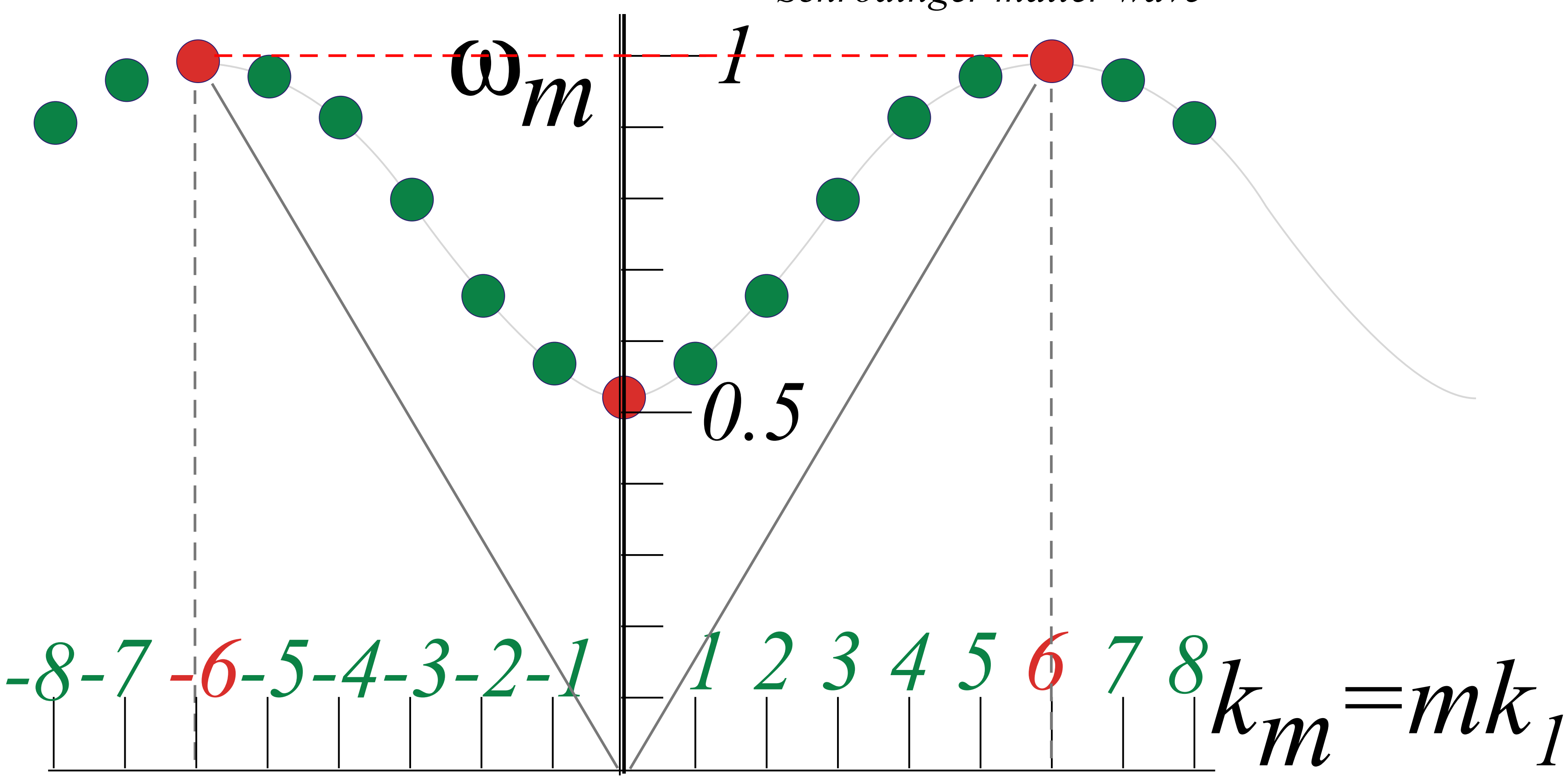
Archetypical Examples of C_{12} Dispersion Functions

(a) Constant dispersion (b) Linear dispersion (c) Quadratic dispersion (d) Phonon dispersion (e) Exciton dispersion



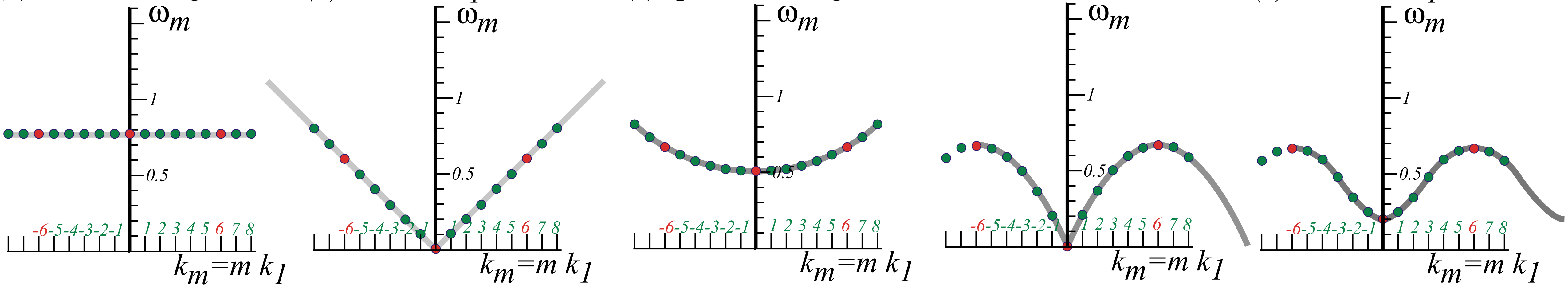
Applications:

(a)	(b)	(c)	(d)	(e)
Uncoupled pendulums	Weakly coupled pendulums (No gravity)	Weakly coupled pendulums (With gravity)	Strongly coupled pendulums (No gravity)	Strongly coupled pendulums (With gravity)
Movie marquis Xmas lights	Light in vacuum (<u>Exactly</u>) Sound (Approximately)	Light in fiber (Approx) Non-relativistic Schrodinger matter wave	Acoustic mode in solids	Optical mode in solids Relativistic matter (If exact hyperbola)



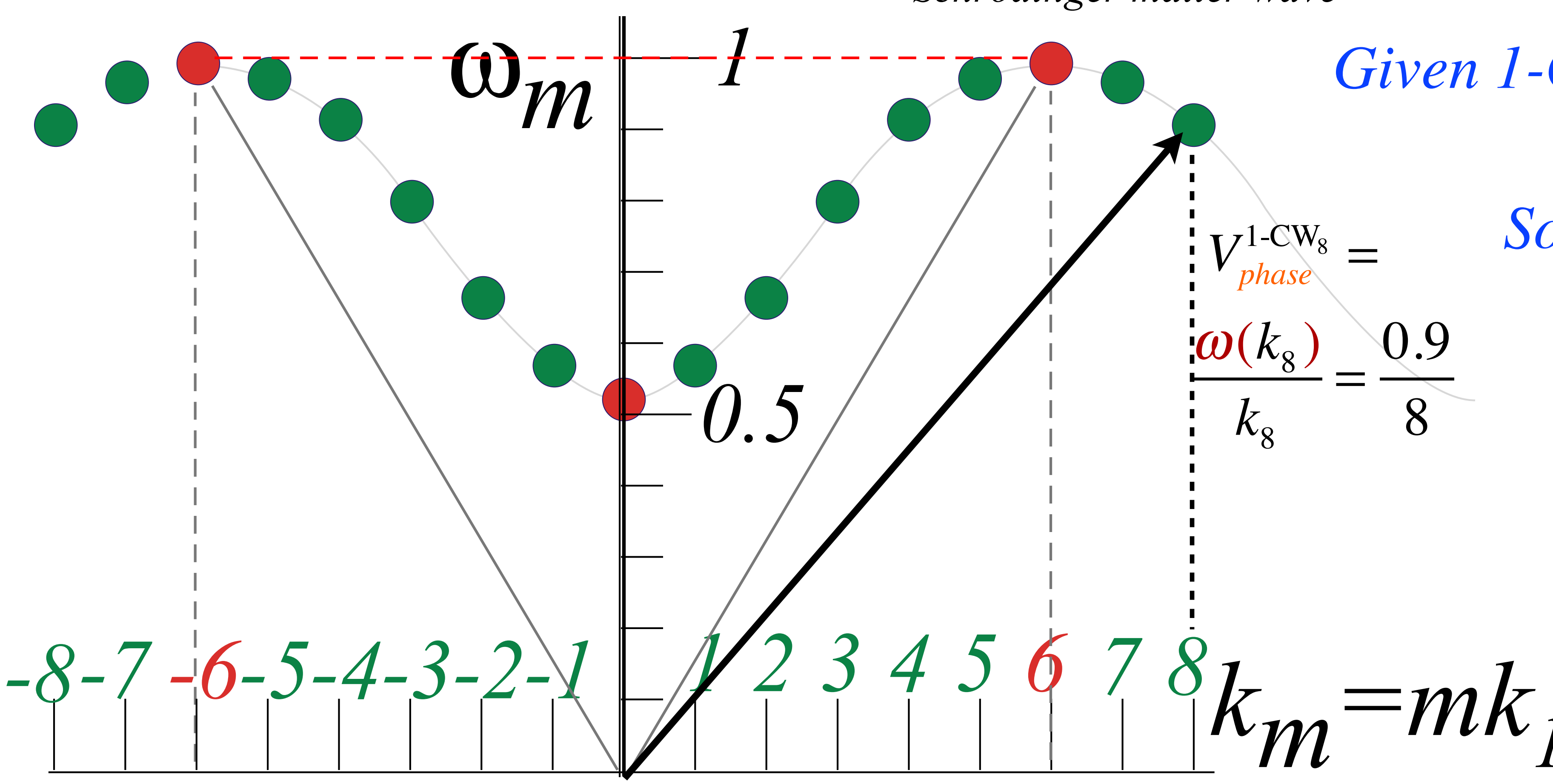
Archetypical Examples of C_{12} Dispersion Functions

(a) Constant dispersion | *(b) Linear dispersion* | *(c) Quadratic dispersion* | *(d) Phonon dispersion* | *(e) Exciton dispersion*



Applications:

(a)	(b)	(c)	(d)	(e)
Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis *Xmas lights*	*Light in vacuum (Exactly)* *Sound (Approximately)*	*Light in fiber (Approx)* *Non-relativistic Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids* *Relativistic matter* *(If exact hyperbola)*



$$V^{1\text{-}CW_8}_{phase} = \frac{\omega(k_8)}{k_8} = \frac{0.9}{8}$$

Given 1-CW phase of wave $e^{\overbrace{i(kx-\omega t)}}$:

$$a = \underbrace{k\cdot x - \omega\cdot t}$$

Solve for 1-CW $\underbrace{\textit{phase velocity}}$

$$x = \frac{\omega}{k}\cdot t + \frac{a}{k}$$

Wave velocities depend on
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

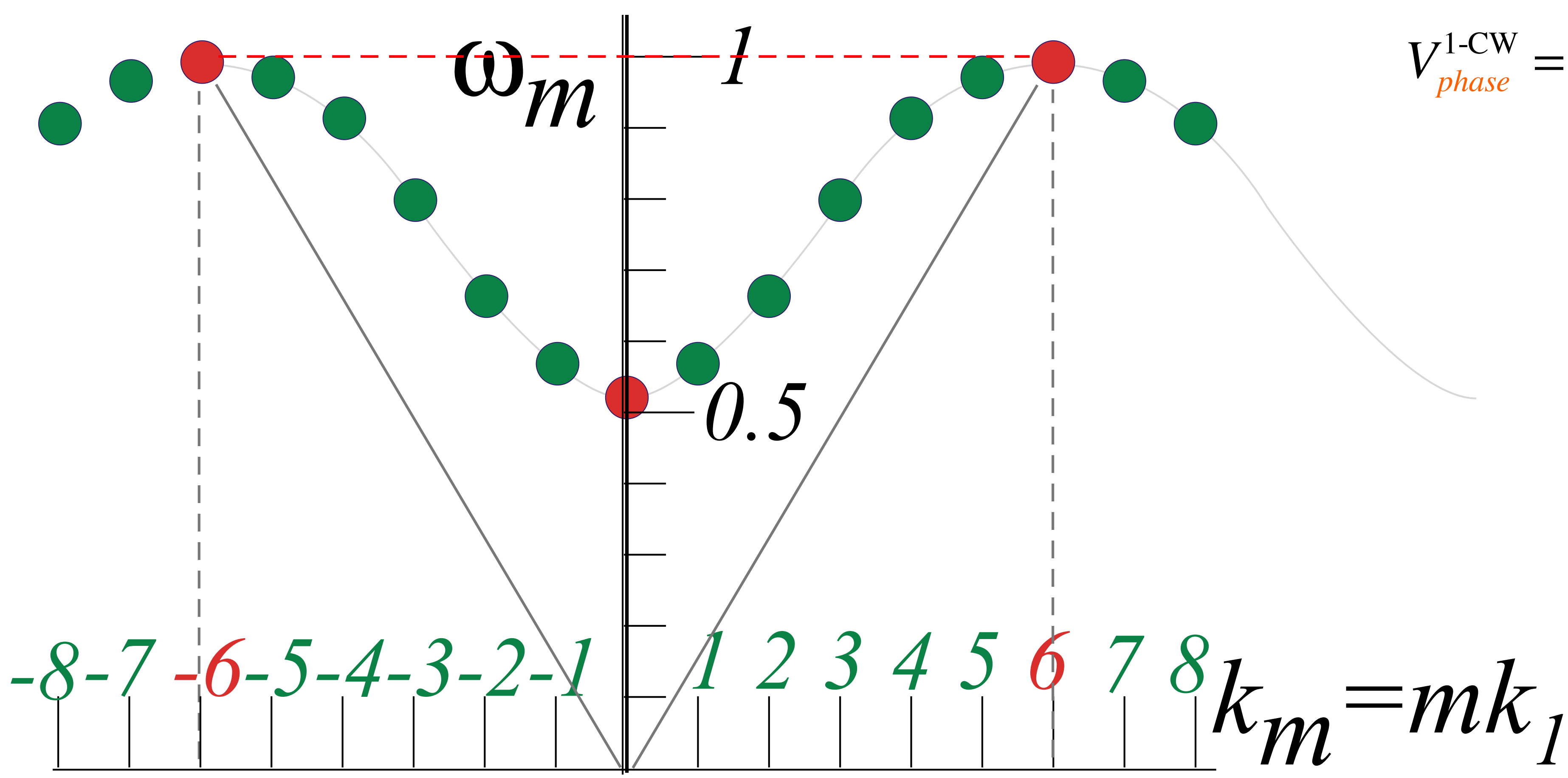
$$V^{1\text{-}CW}_{phase} = \frac{\omega(k)}{k}$$

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

...find 2-CW phase velocity $V_{phase}^{2\text{-CW}}$ *and group velocity* $V_{group}^{2\text{-CW}}$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V_{phase}^{1\text{-CW}} = \frac{\omega(k)}{k}$$

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: ...*find 2-CW phase velocity* $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right) = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

Velocities depend upon
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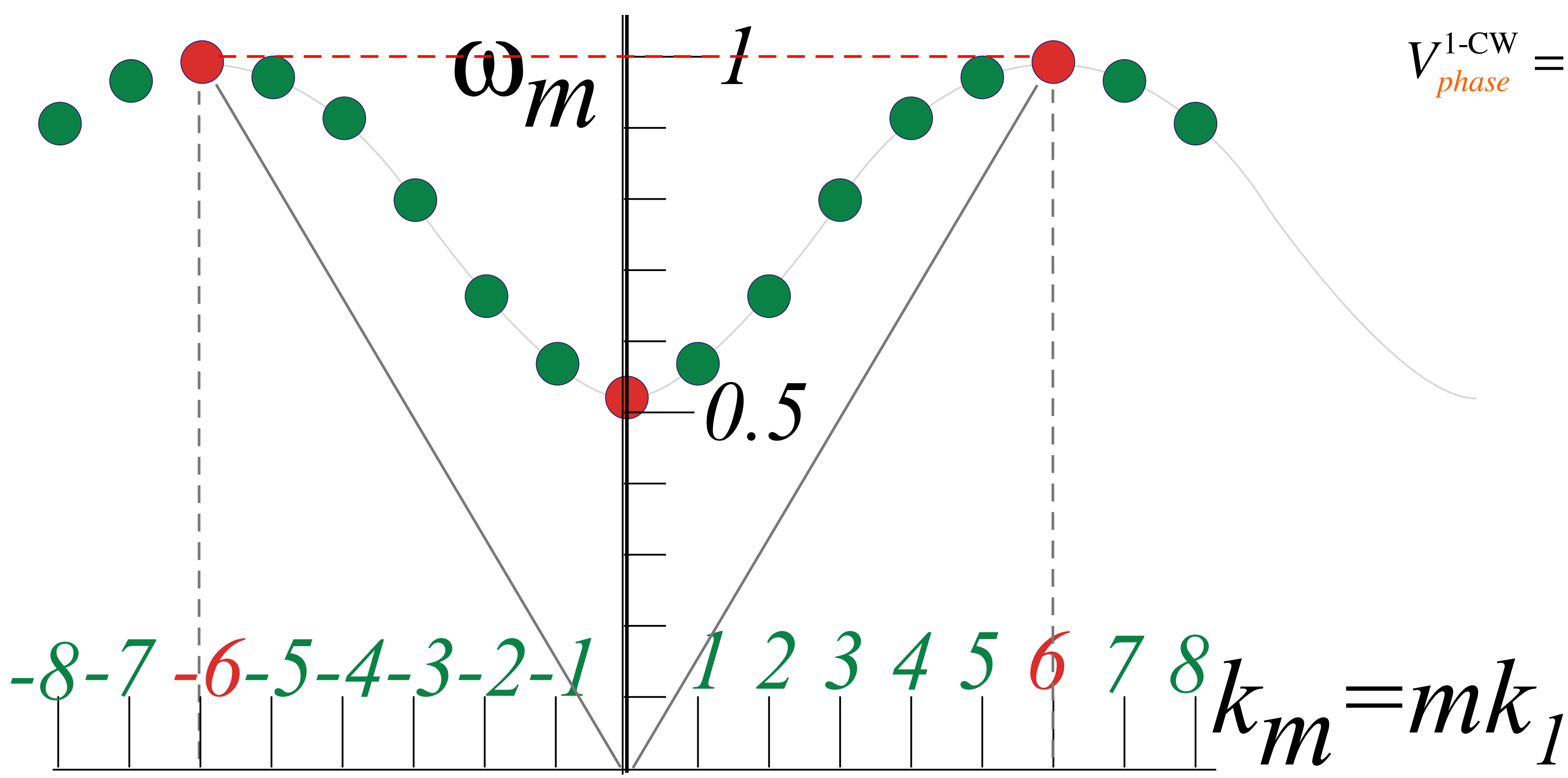
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: ...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ and group velocity $V^{2\text{-CW}}_{group}$

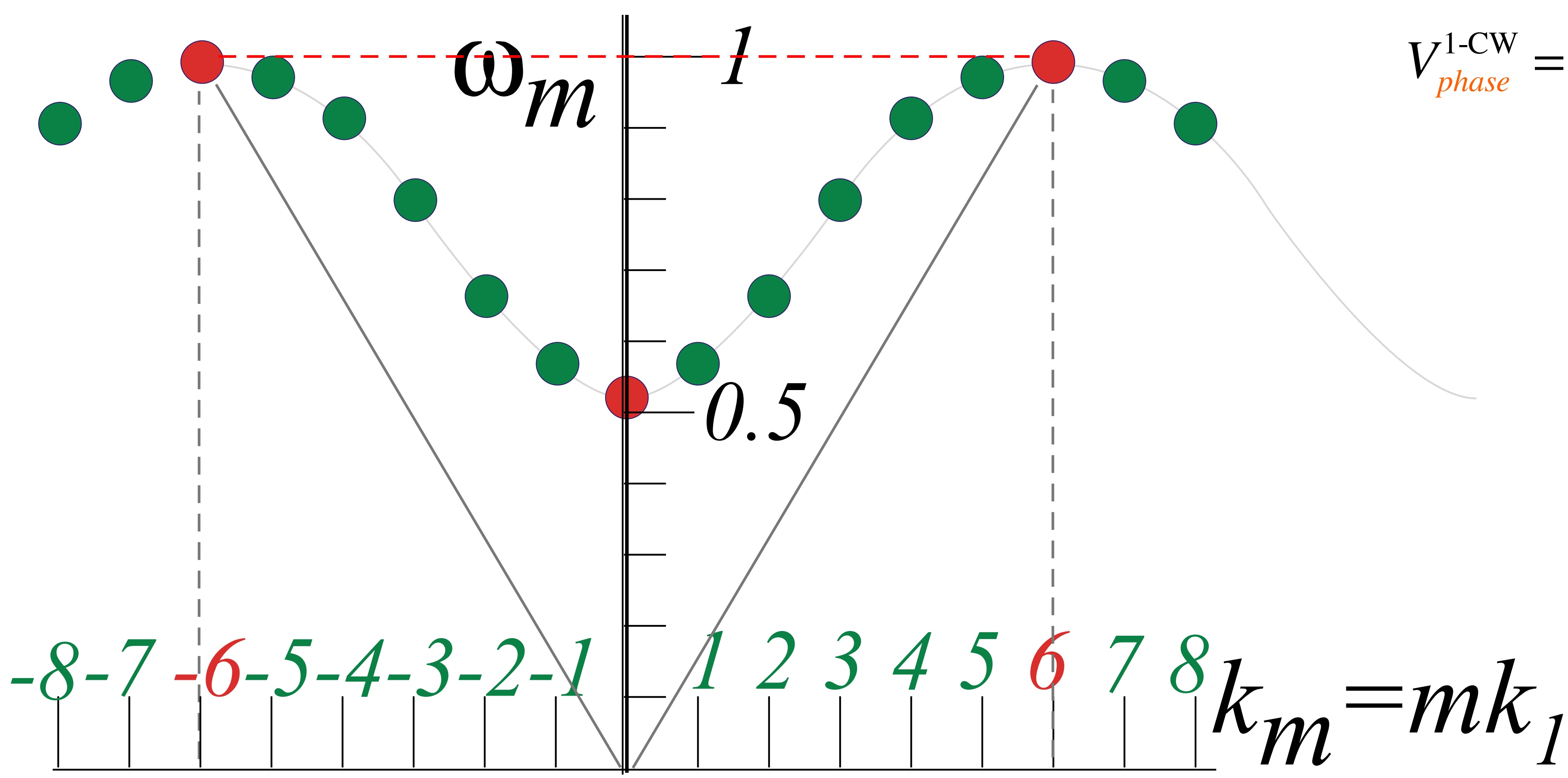
$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right) = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x - \frac{(\omega_a+\omega_b)}{2}t} \cos\left(\frac{(k_a - k_b)}{2} x - \frac{(\omega_a - \omega_b)}{2} t \right)$$

Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:
$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: ...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ and group velocity $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

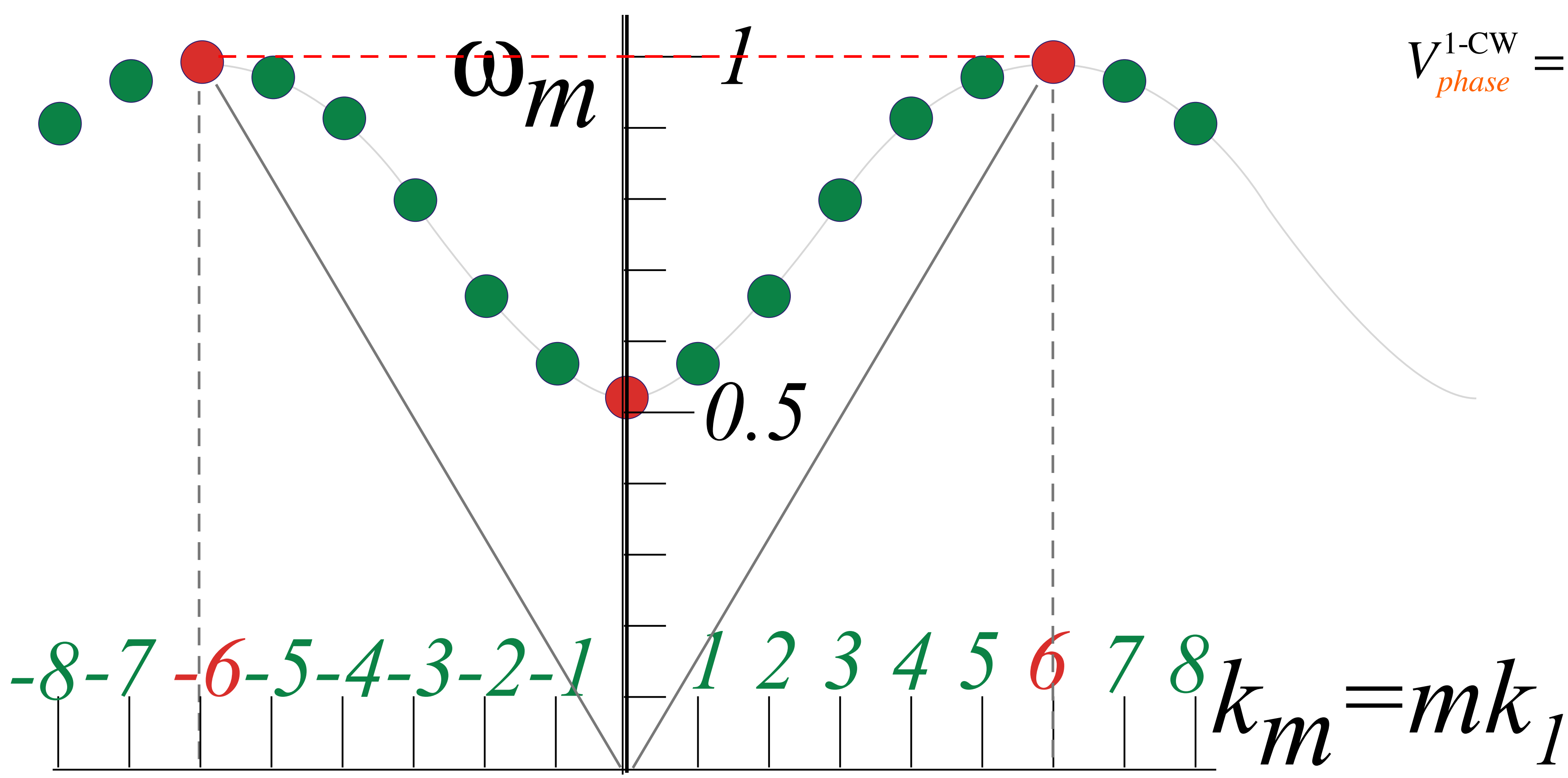
$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_a+\omega_b)}{(k_a+k_b)} \qquad V^{2\text{-CW}}_{group} = \frac{(\omega_a-\omega_b)}{(k_a-k_b)}$$

Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

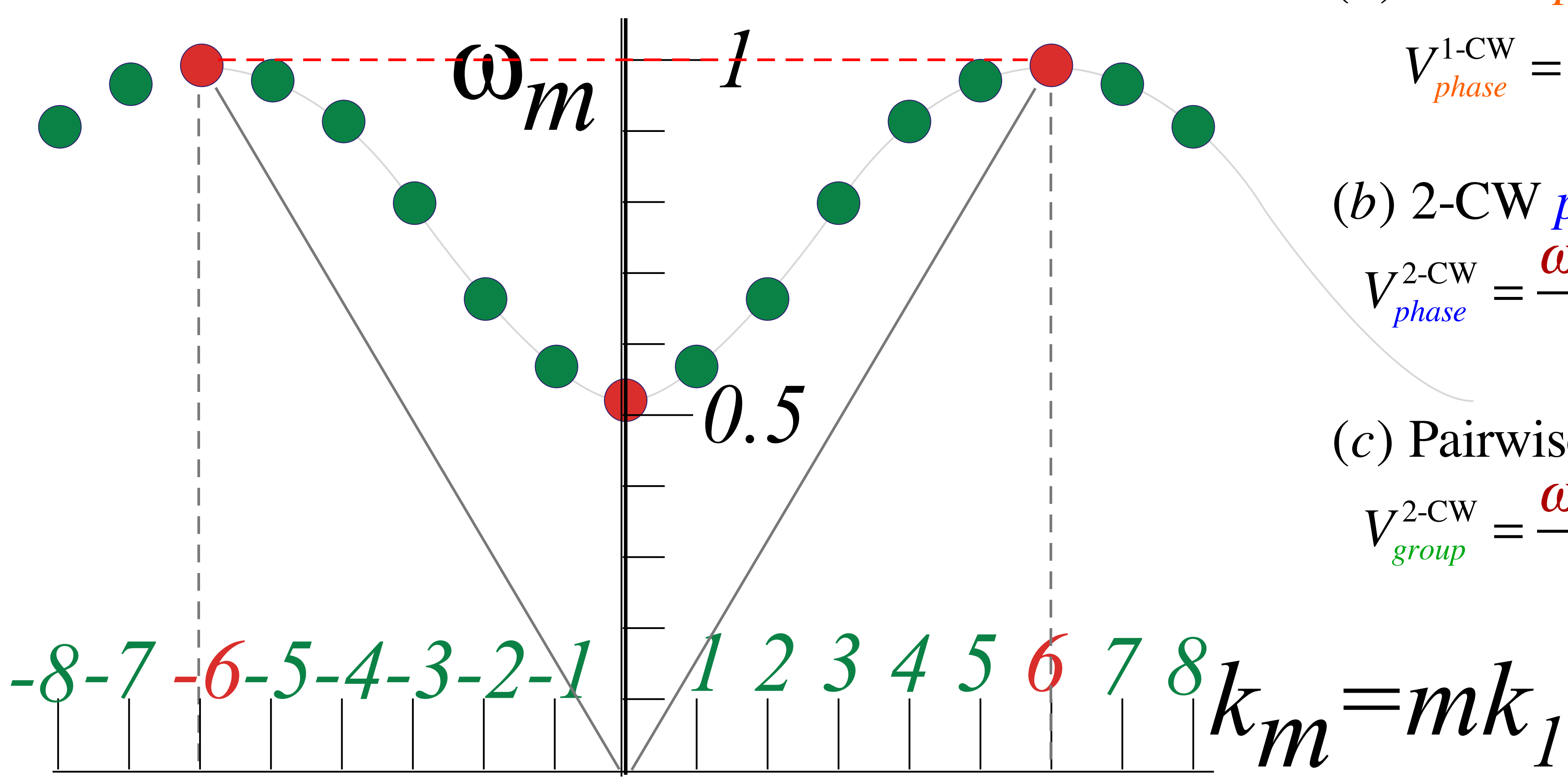
Given 2-CW phases: ...*find 2-CW phase velocity* $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x - \frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x - \frac{(\omega_a-\omega_b)}{2}t\right)$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_a+\omega_b)}{(k_a+k_b)} \qquad V^{2\text{-CW}}_{group} = \frac{(\omega_a-\omega_b)}{(k_a-k_b)}$$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

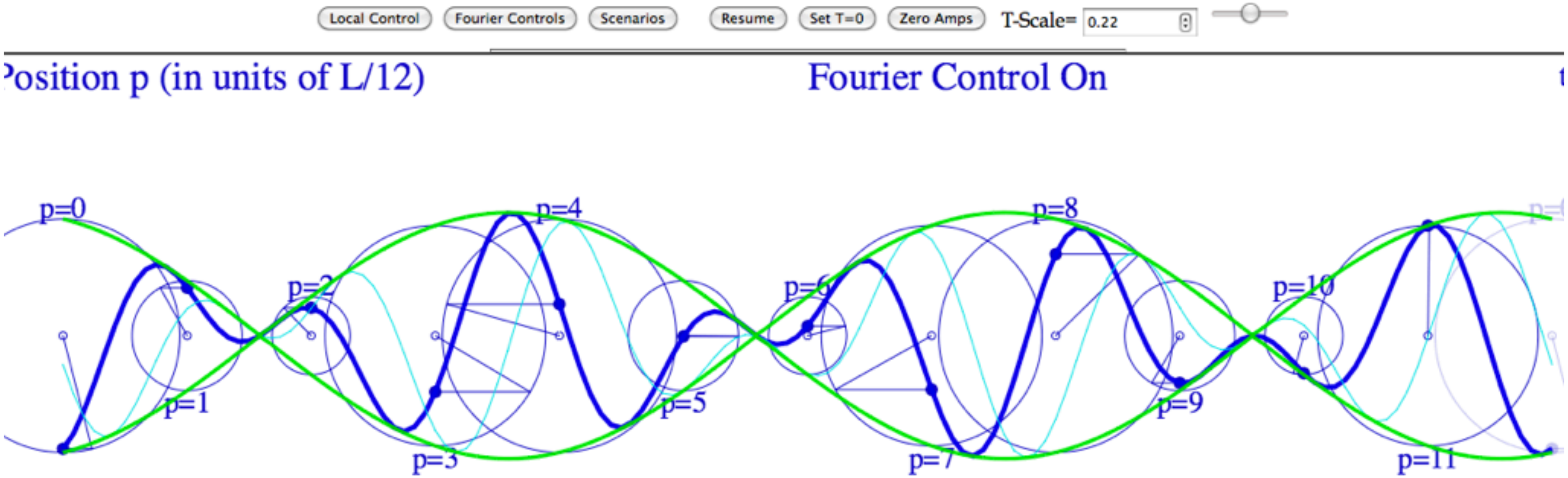
$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$

(b) 2-CW *phase* velocity:

$$V^{2\text{-CW}}_{phase} = \frac{\omega(k_1)+\omega(k_2)}{k_1+k_2}$$

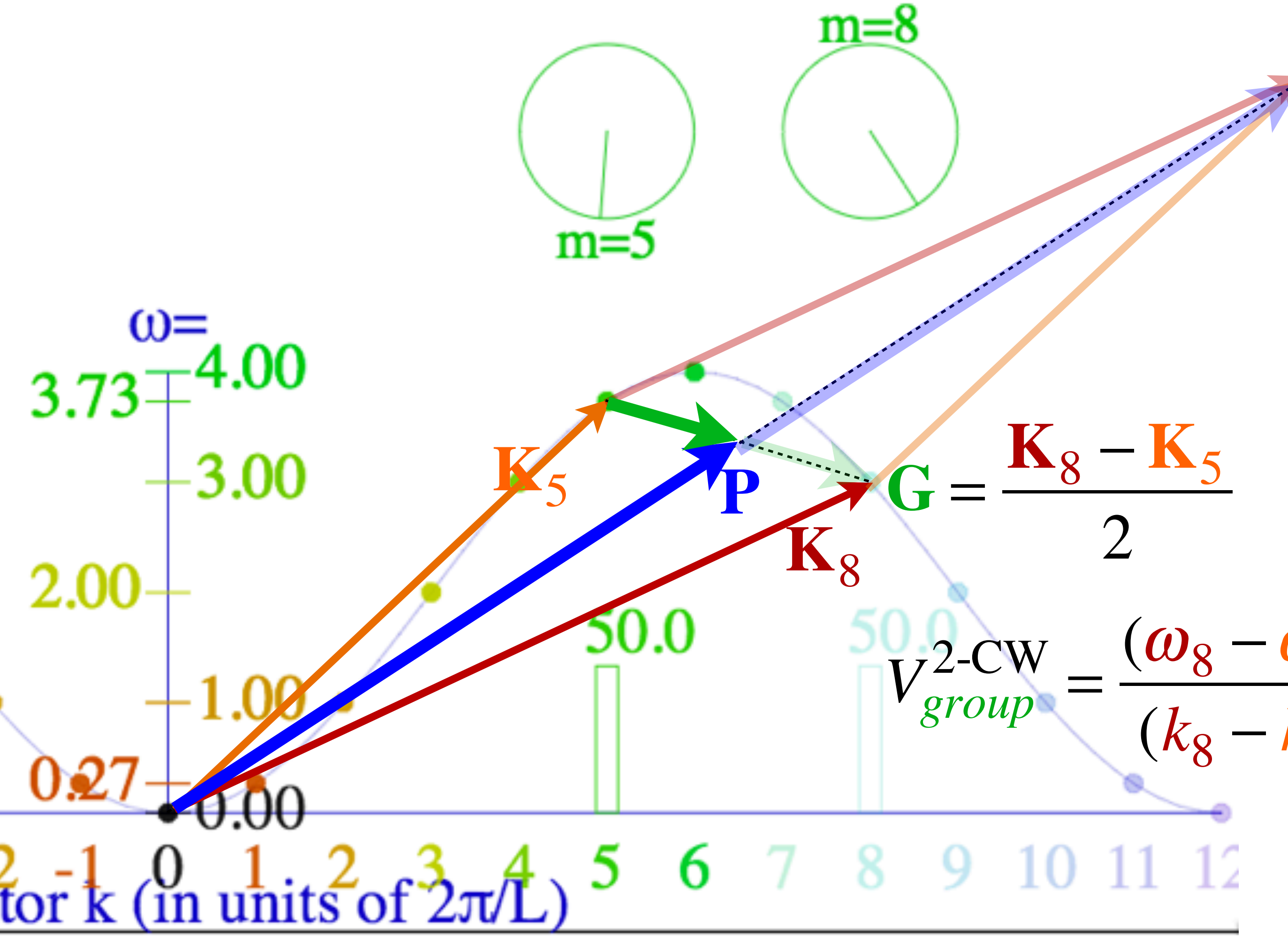
(c) Pairwise *group* velocity:

$$V^{2\text{-CW}}_{group} = \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}$$



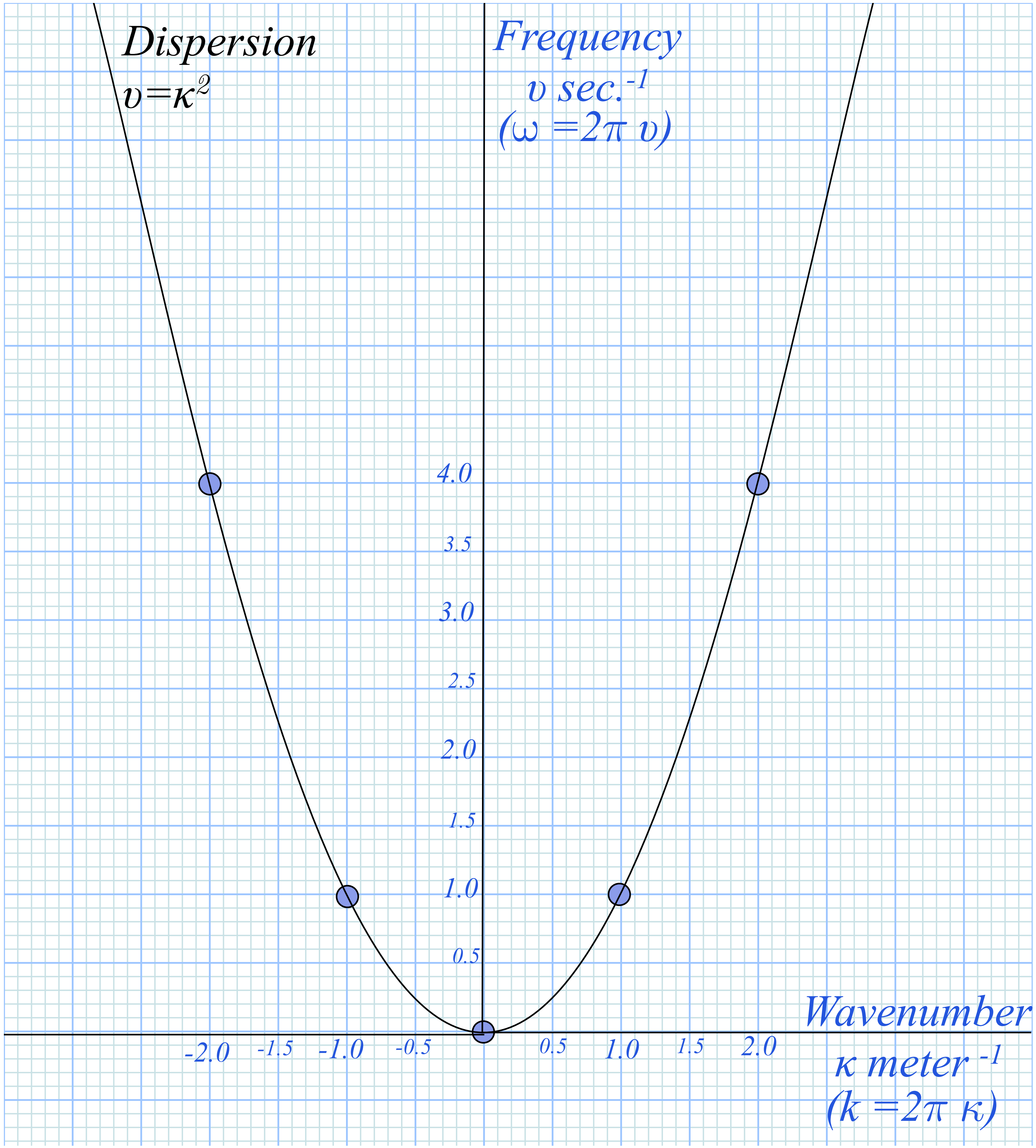
$$\mathbf{P} = \frac{\mathbf{K}_8 + \mathbf{K}_5}{2} = \frac{1}{2}\begin{pmatrix} k_8 \\ \omega_8 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} k_5 \\ \omega_5 \end{pmatrix}$$

$$V_{phase}^{2\text{-CW}} = \frac{(\omega_8 + \omega_5)}{(k_8 + k_5)}$$

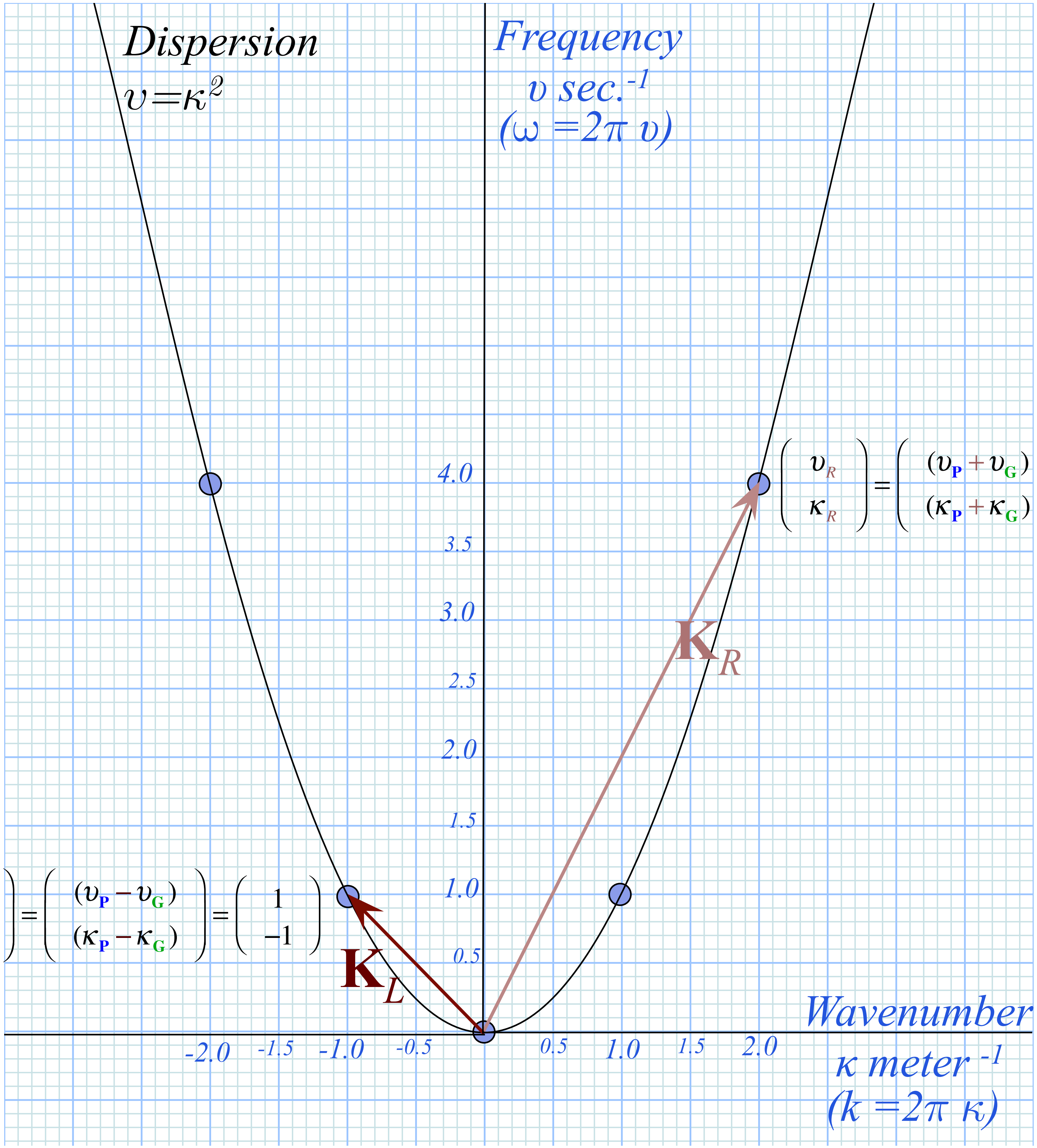


$$\mathbf{G} = \frac{\mathbf{K}_8 - \mathbf{K}_5}{2}$$

$$V_{group}^{2\text{-CW}} = \frac{(\omega_8 - \omega_5)}{(k_8 - k_5)}$$

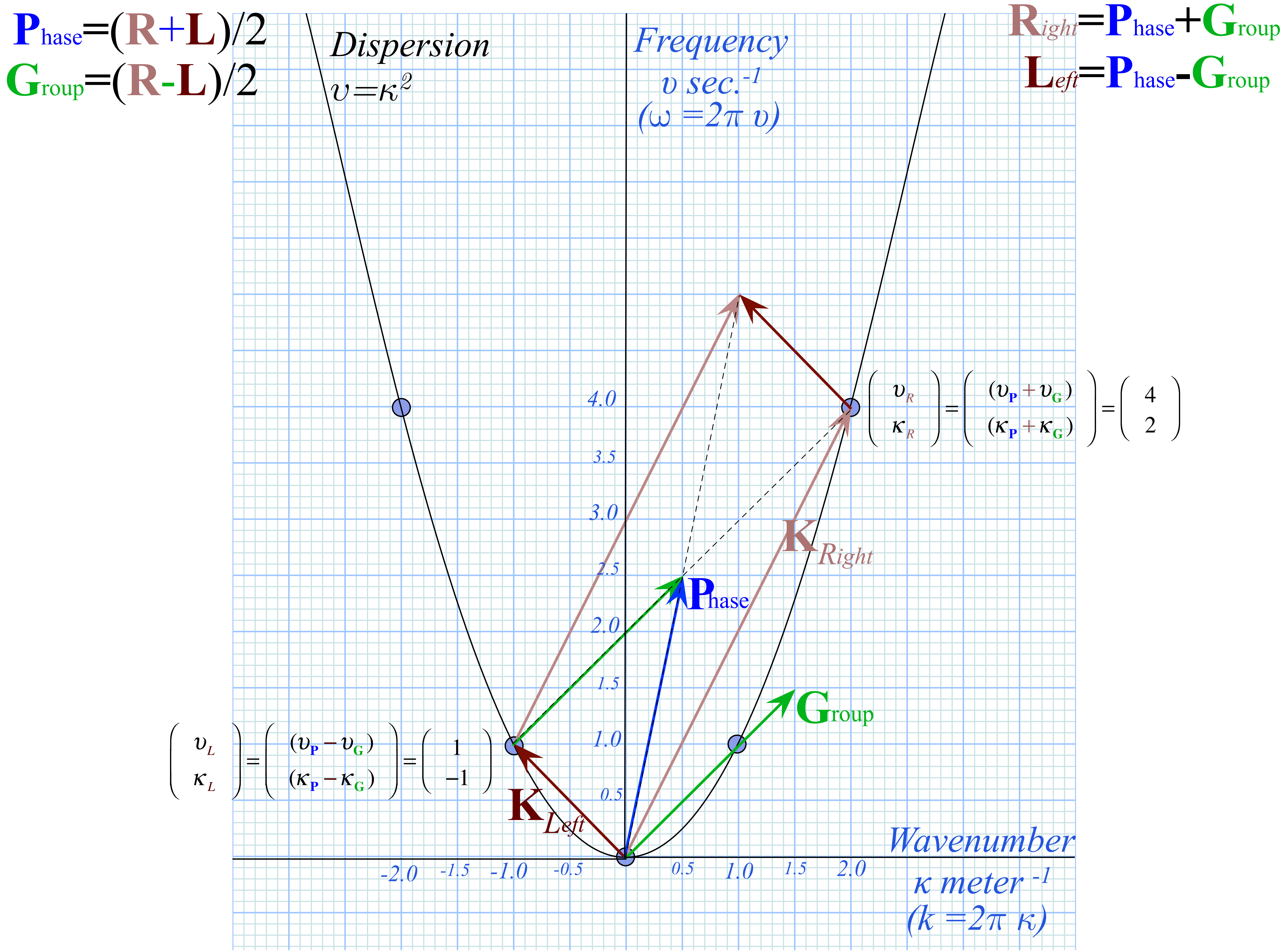


Dispersion
υ=κ²
Frequency
υ sec.⁻¹
(ω =2π υ)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
-2.0
-1.5
-1.0
-0.5
0.5
1.0
1.5
2.0
Wavenumber
κ meter ⁻¹
(k =2π κ)



$$\begin{pmatrix} \upsilon_R \\ \kappa_R \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} + \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} + \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_L \\ \kappa_L \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} - \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} - \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

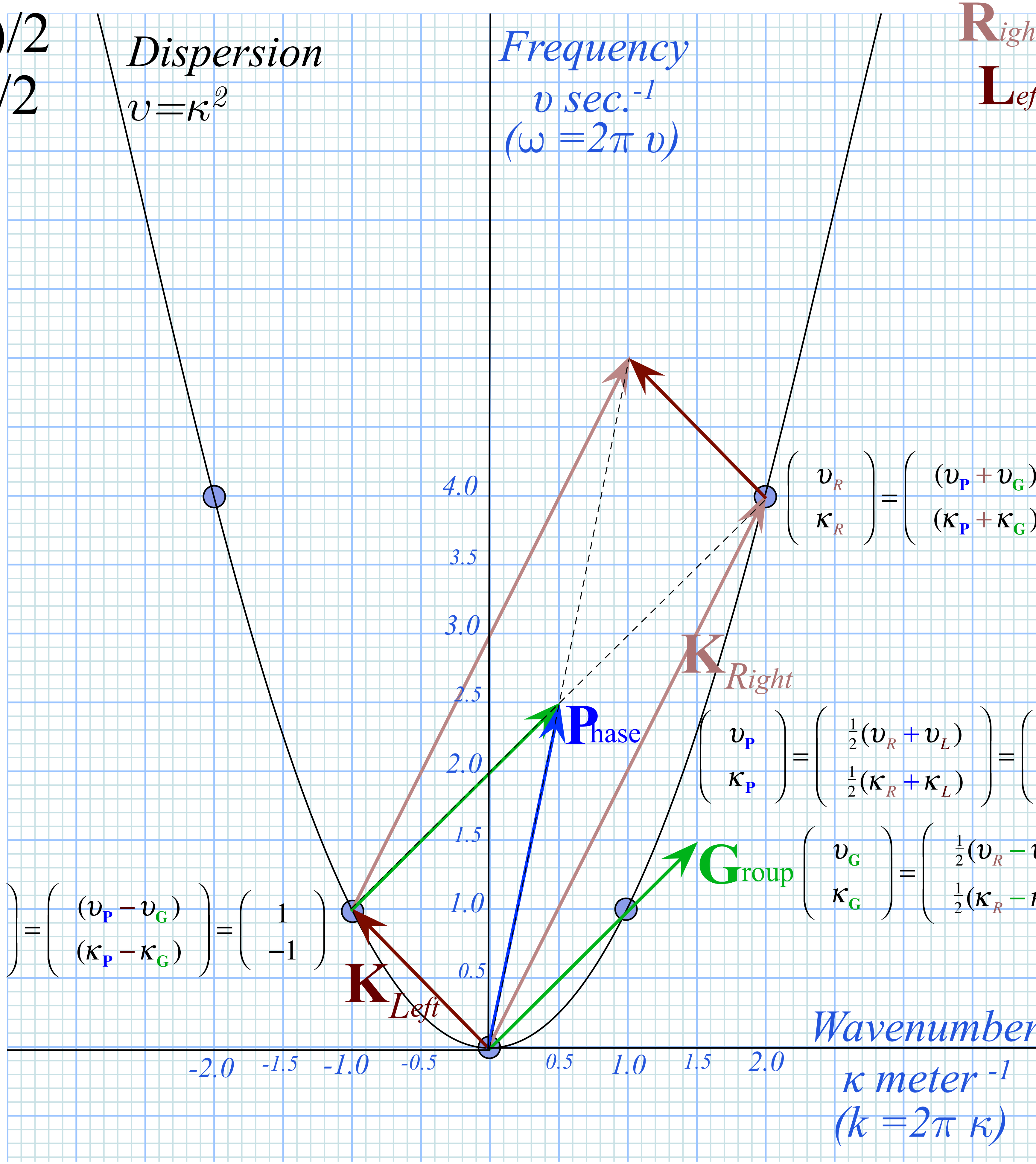

Phase=(R+L)/2
Group=(R-L)/2
Dispersion
$\upsilon=\kappa^2$
Frequency
υ sec.$^{-1}$
($\omega = 2\pi\,\upsilon$)
Right=Phase+Group
Left=Phase-Group
$\begin{pmatrix} \upsilon_R \\ \kappa_R \end{pmatrix} = \begin{pmatrix} (\upsilon_P + \upsilon_G) \\ (\kappa_P + \kappa_G) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
K_{Right}
Phase
Group
$\begin{pmatrix} \upsilon_L \\ \kappa_L \end{pmatrix} = \begin{pmatrix} (\upsilon_P - \upsilon_G) \\ (\kappa_P - \kappa_G) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
K_{Left}
Wavenumber
-2.0
-1.5
-1.0
-0.5
0.5
1.0
1.5
2.0
κ meter $^{-1}$
($k = 2\pi\,\kappa$)

$\mathbf{P}_{\text{hase}} = (\mathbf{R}+\mathbf{L})/2$

$\mathbf{G}_{\text{roup}} = (\mathbf{R}-\mathbf{L})/2$

$\mathbf{R}_{ight} = \mathbf{P}_{\text{hase}} + \mathbf{G}_{\text{roup}}$

$\mathbf{L}_{eft} = \mathbf{P}_{\text{hase}} - \mathbf{G}_{\text{roup}}$



$$\begin{pmatrix} \upsilon_R \\ \kappa_R \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} + \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} + \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_{\mathbf{P}} \\ \kappa_{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\upsilon_R + \upsilon_L) \\ \frac{1}{2}(\kappa_R + \kappa_L) \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_{\mathbf{G}} \\ \kappa_{\mathbf{G}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\upsilon_R - \upsilon_L) \\ \frac{1}{2}(\kappa_R - \kappa_L) \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_L \\ \kappa_L \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} - \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} - \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

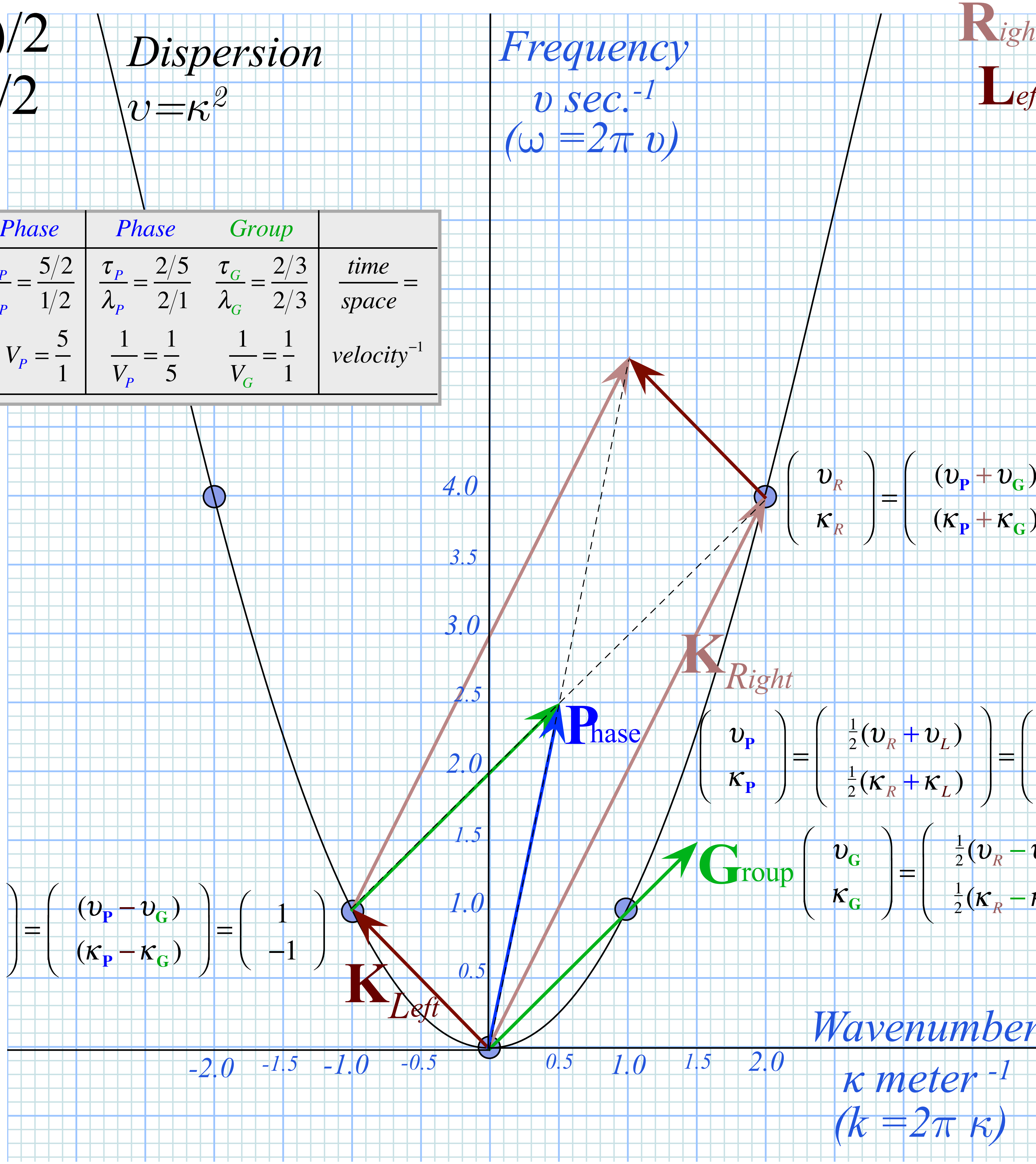
Phase=(**R**+**L**)/2

Group=(**R**-**L**)/2

Right=**Phase**+**Group**

Left=**Phase**-**Group**

	Group	*Phase*	*Phase*	*Group*	
$\frac{\textit{per-time}}{\textit{per-space}}$	$\frac{\upsilon_G}{\kappa_G}=\frac{3/2}{3/2}$	$\frac{\upsilon_P}{\kappa_P}=\frac{5/2}{1/2}$	$\frac{\tau_P}{\lambda_P}=\frac{2/5}{2/1}$	$\frac{\tau_G}{\lambda_G}=\frac{2/3}{2/3}$	$\frac{\textit{time}}{\textit{space}}=$
$=$ *velocity*	$=V_G=\frac{1}{1}$	$=V_P=\frac{5}{1}$	$\frac{1}{V_P}=\frac{1}{5}$	$\frac{1}{V_G}=\frac{1}{1}$	*velocity*$^{-1}$



$$\begin{pmatrix}\upsilon_R\\ \kappa_R\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}+\upsilon_{\mathbf{G}})\\ (\kappa_{\mathbf{P}}+\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}4\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{P}}\\ \kappa_{\mathbf{P}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R+\upsilon_L)\\ \frac{1}{2}(\kappa_R+\kappa_L)\end{pmatrix}=\begin{pmatrix}5/2\\ 1/2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_{\mathbf{G}}\\ \kappa_{\mathbf{G}}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(\upsilon_R-\upsilon_L)\\ \frac{1}{2}(\kappa_R-\kappa_L)\end{pmatrix}=\begin{pmatrix}3/2\\ 3/2\end{pmatrix}$$

$$\begin{pmatrix}\upsilon_L\\ \kappa_L\end{pmatrix}=\begin{pmatrix}(\upsilon_{\mathbf{P}}-\upsilon_{\mathbf{G}})\\ (\kappa_{\mathbf{P}}-\kappa_{\mathbf{G}})\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

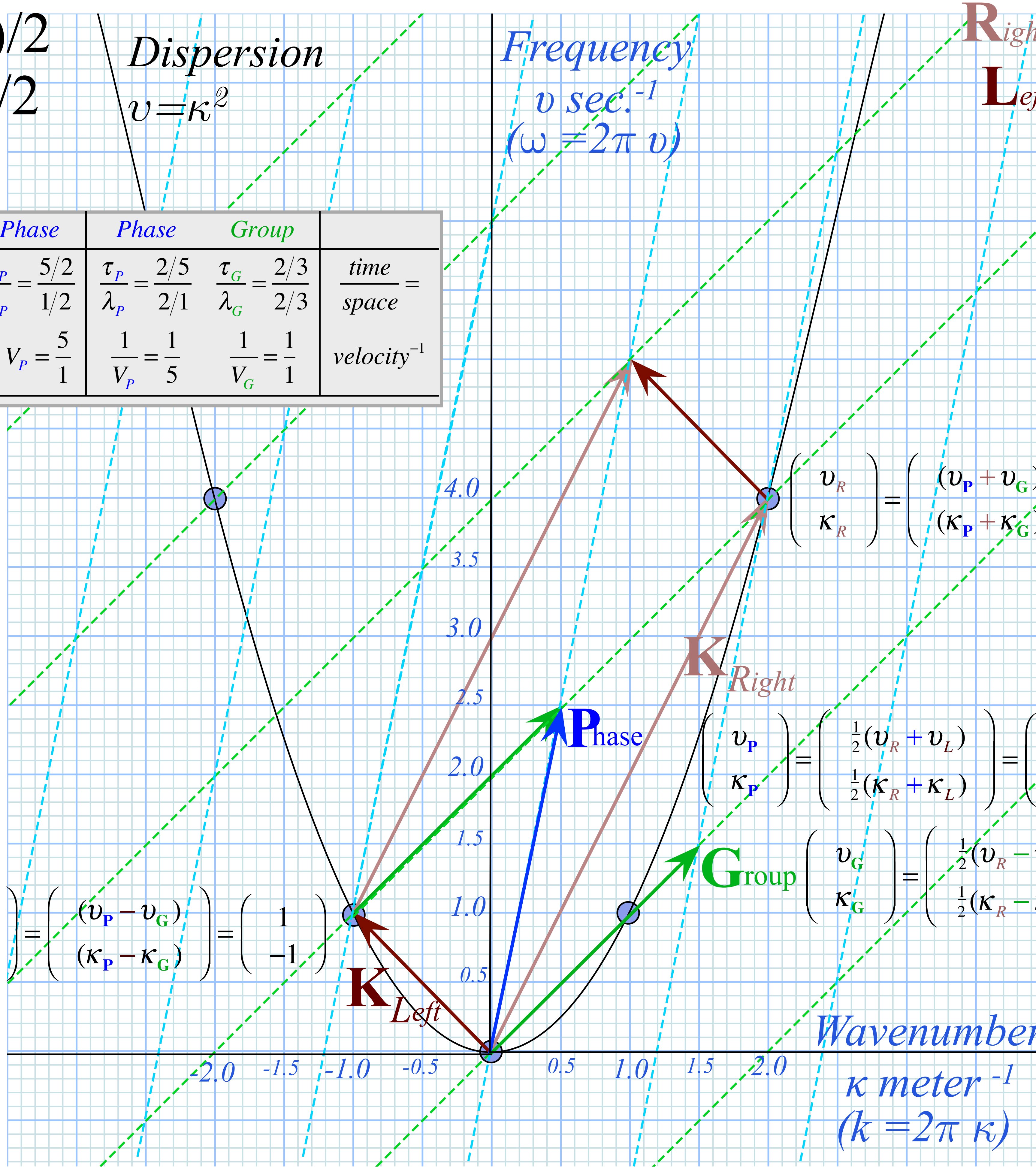
$\mathbf{P}_{\text{hase}} = (\mathbf{R}+\mathbf{L})/2$

$\mathbf{G}_{\text{roup}} = (\mathbf{R}-\mathbf{L})/2$

$\mathbf{R}_{\textit{ight}} = \mathbf{P}_{\text{hase}} + \mathbf{G}_{\text{roup}}$

$\mathbf{L}_{\textit{eft}} = \mathbf{P}_{\text{hase}} - \mathbf{G}_{\text{roup}}$

	Group	*Phase*	*Phase*	*Group*	
$\dfrac{\textit{per-time}}{\textit{per-space}}$	$\dfrac{\upsilon_G}{\kappa_G} = \dfrac{3/2}{3/2}$	$\dfrac{\upsilon_P}{\kappa_P} = \dfrac{5/2}{1/2}$	$\dfrac{\tau_P}{\lambda_P} = \dfrac{2/5}{2/1}$	$\dfrac{\tau_G}{\lambda_G} = \dfrac{2/3}{2/3}$	$\dfrac{\textit{time}}{\textit{space}} =$
$= \textit{velocity}$	$= V_G = \dfrac{1}{1}$	$= V_P = \dfrac{5}{1}$	$\dfrac{1}{V_P} = \dfrac{1}{5}$	$\dfrac{1}{V_G} = \dfrac{1}{1}$	$\textit{velocity}^{-1}$



$$\begin{pmatrix} \upsilon_R \\ \kappa_R \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} + \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} + \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_{\mathbf{P}} \\ \kappa_{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\upsilon_R + \upsilon_L) \\ \frac{1}{2}(\kappa_R + \kappa_L) \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_{\mathbf{G}} \\ \kappa_{\mathbf{G}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\upsilon_R - \upsilon_L) \\ \frac{1}{2}(\kappa_R - \kappa_L) \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

$$\begin{pmatrix} \upsilon_L \\ \kappa_L \end{pmatrix} = \begin{pmatrix} (\upsilon_{\mathbf{P}} - \upsilon_{\mathbf{G}}) \\ (\kappa_{\mathbf{P}} - \kappa_{\mathbf{G}}) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\bar{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad \bar{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\bar{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \bar{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$