Lecture 12

Revised 12.22.12 from 10.2.2012

Hamiltonian vs. Lagrange mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 9-11 procedures:

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

*Hamilton prefers Contravariant g^{mn} with Covariant **momentum** p_m*

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m

Polar-coordinate example of Hamilton's equations

Hamilton's equations in Runga-Kutta (computer solution) form

Examples of Hamiltonian mechanics in effective potentials

Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)

Coulomb orbits in polar coordinates and effective potential (Simulation)

Parabolic and 2D-IHO orbital envelopes

Clues for take-home assignment 7 (Simulation)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)

1D-HO phase-space control (Simulation)

Quick Review of Lagrange Relations in Lectures 9-11

→ *0th and 1st equations of Lagrange and Hamilton and their geometric relations*

Quick Review of Lagrange Relations in Lectures 9-11

0^{th} and 1^{st} equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian and **Estrangian** have no explicit dependence on **momentum p**

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian and **Estrangian** have no explicit dependence on **velocity v**

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian and **Hamiltonian** have no explicit dependence on **speedinum V**

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2} = \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1^{st} equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2} = \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

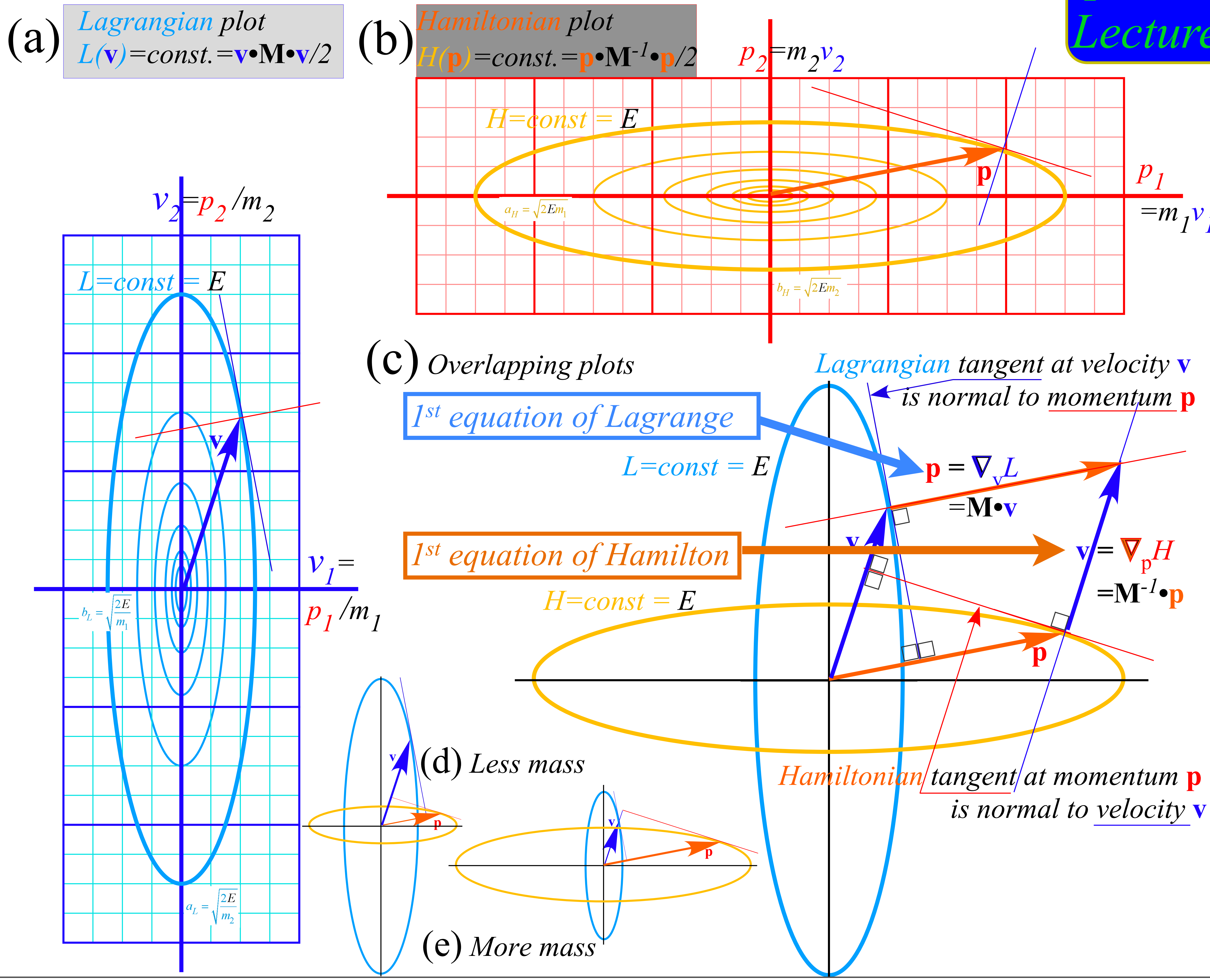
$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1^{st} equation(s)

$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

(Forget Estrangian for now)

Unit 1
Fig. 12.2



Review of Lagrange Equations in Lecture 11

Lagrange prefers Co*variant* g_{mn} *with* Contra*variant velocity* $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m *definition*

→ *GCC "canonical" force* F_m *definition*

Coriolis "fictitious" forces (... and weather effects)

Lagrange prefers Covariant g_{mn} *with* Contravariant *velocity*

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 11)**

$$L(\mathbf{v}) = \tfrac{1}{2} M \mathbf{v} \bullet \mathbf{v} - U = \tfrac{1}{2} M \dot{\mathbf{r}} \bullet \dot{\mathbf{r}} - U = \tfrac{1}{2} M (\mathbf{E}_m \dot{q}^m) \bullet (\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} *metric (1-page back)*

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

Lagrange prefers Covariant g_{mn} with Contravariant velocity

(Review of Lecture 11)

Lagrangian KE-U is supposed to be explicit function of velocity.

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GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\,g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

Lagrange prefers Covariant g_{mn} with Contravariant velocity

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2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

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Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

Centrifugal (center-fleeing) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity. **(Review of Lecture 11)**

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$$\dot{p}_r\equiv\frac{dp_r}{dt}=M\,\ddot{r}$$
$$=M\,r\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi\equiv\frac{dp_\phi}{dt}=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}$$
$$=0-\frac{\partial U}{\partial\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

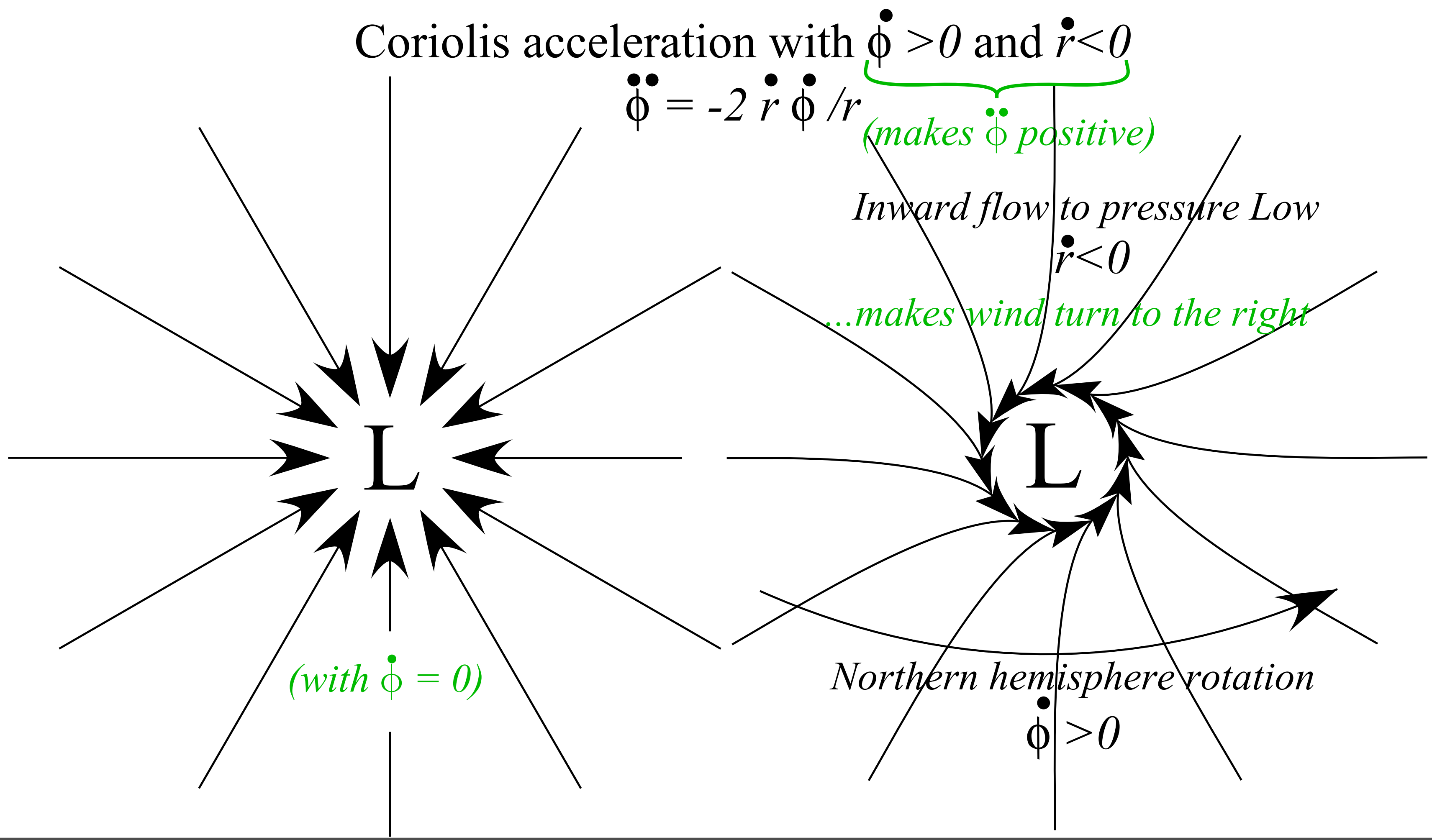
Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

radial force: $M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$ *angular force or torque:* $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\,\dot{\phi}^2$ *angular acceleration:* $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$



Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ *...*

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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...of coordinates and velocity and ***time****, too.* (*You can safely drop last chain-rule factor* [*1=dt/dt*])

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}\frac{dt}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

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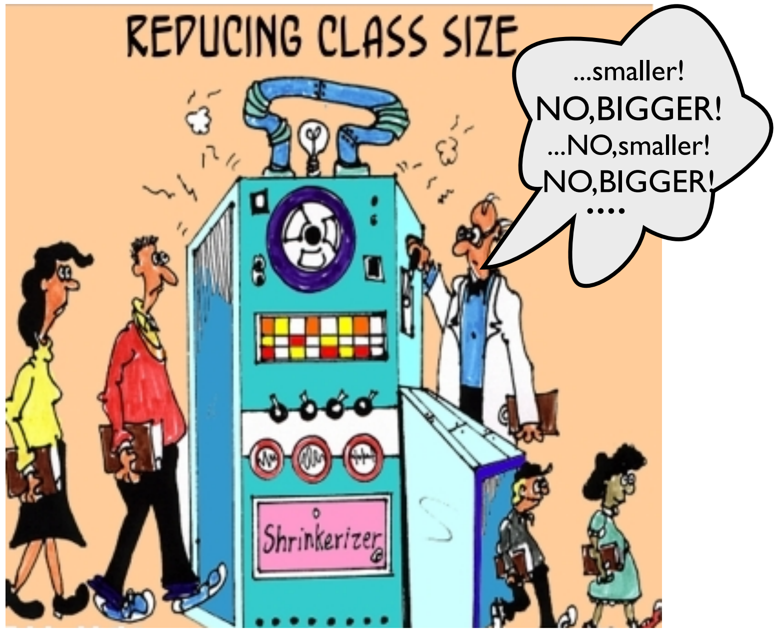
...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U(t)-dial.)

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



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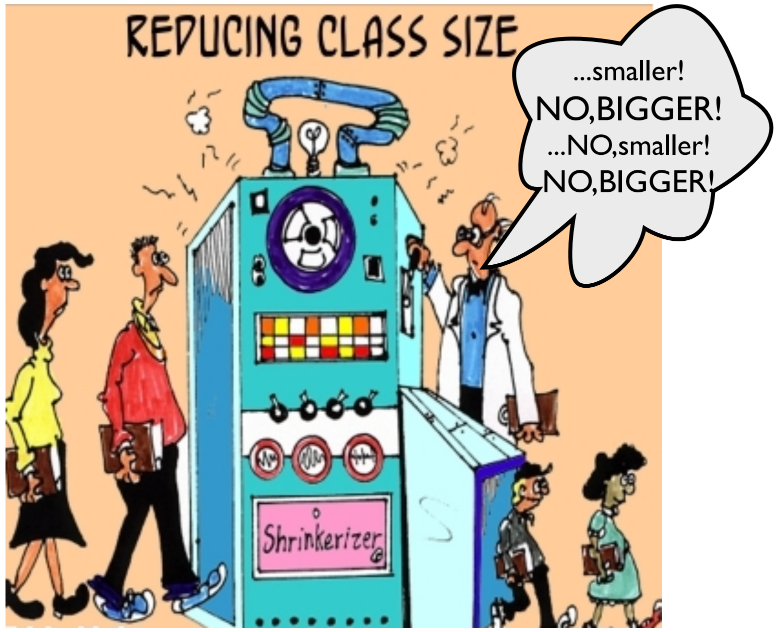

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Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

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Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
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$$\dot{L}(q,\dot{q},\ )=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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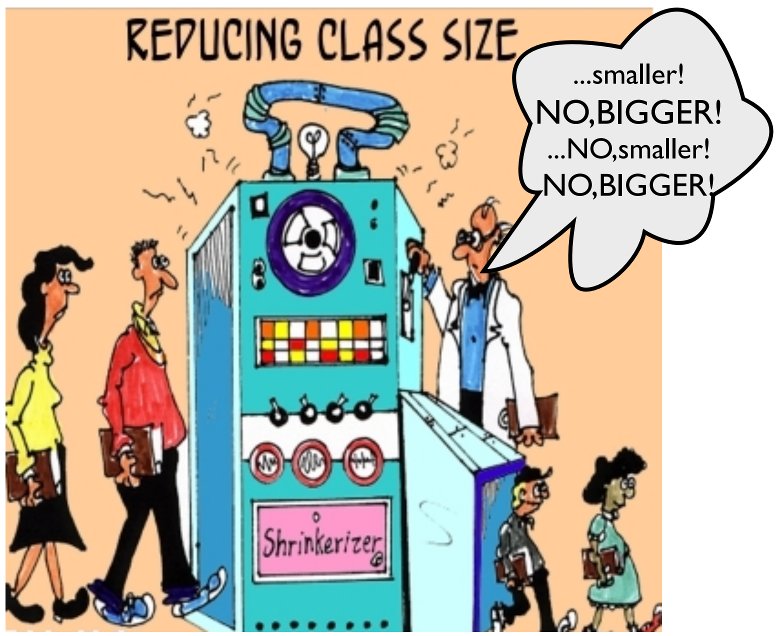

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$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:
$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$*...*

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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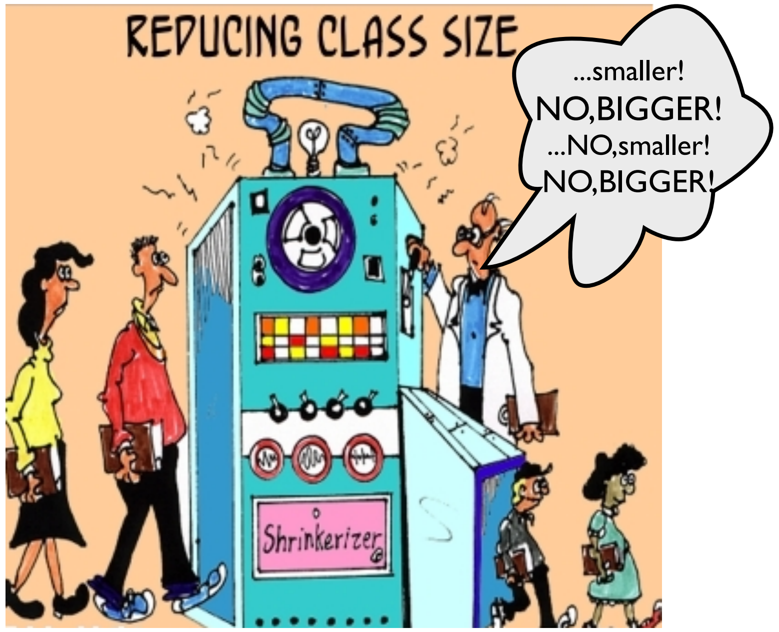

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Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

and switch the dL/dt and $\partial L/\partial t$ *to define the Hamiltonian function* $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}\qquad where: H=p_m\dot{q}^m-L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},\ )=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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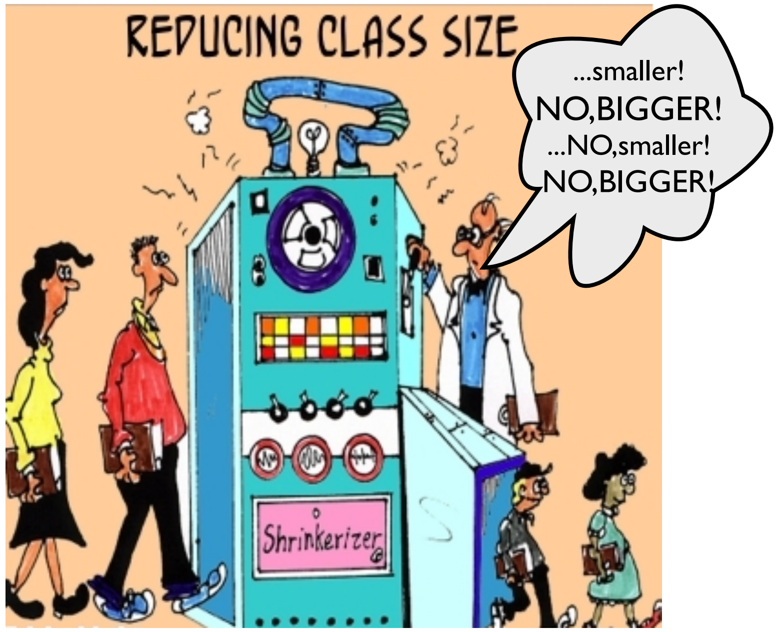

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$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$ *(That's the old Legendre transform)*

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}$$

where : $H=p_m\dot{q}^m-L$ *(Recall* $\frac{\partial L}{\partial p_m}\equiv 0$ *)*

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and <u>time</u>, too. (Imagine Mad Scientist turning U-dial.)

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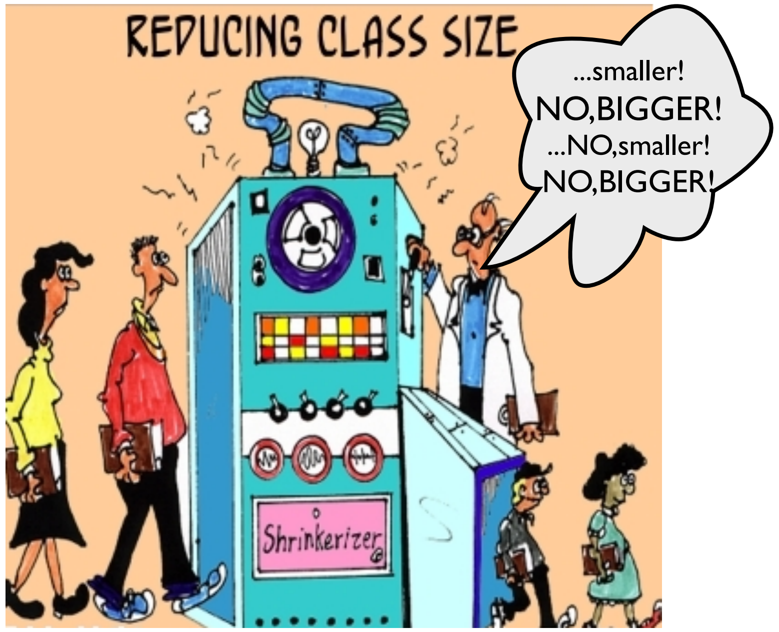
Recall Lagrange equations: $\boxed{\dot{p}_m = \frac{\partial L}{\partial q^m}}$ $\boxed{p_m = \frac{\partial L}{\partial \dot{q}^m}}$



$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \dot{p}_m\frac{dq^m}{dt} + p_m\frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule:
$\dot{u}\frac{dv}{dt} + u\frac{d\dot{v}}{dt} = \frac{d}{dt}(u\dot{v})$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot{q}^m\right) + \frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p}\bullet\mathbf{v} - L(\mathbf{v})$ (That's the old Legendre transform)

$$\frac{d}{dt}\left(p_m\dot{q}^m - L\right) = -\frac{\partial L}{\partial t} = \frac{dH}{dt} \quad where: H = p_m\dot{q}^m - L$$

(Recall: $\frac{\partial L}{\partial p_m} \equiv 0$
and: $\frac{\partial H}{\partial \dot{q}^m} \equiv 0$)

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m} = \dot{q}^m$$

$$\frac{\partial H}{\partial q^m} = 0\cdot 0 - \frac{\partial L}{\partial q^m} = -\dot{p}_m$$

Hamilton's 2nd GCC equation

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$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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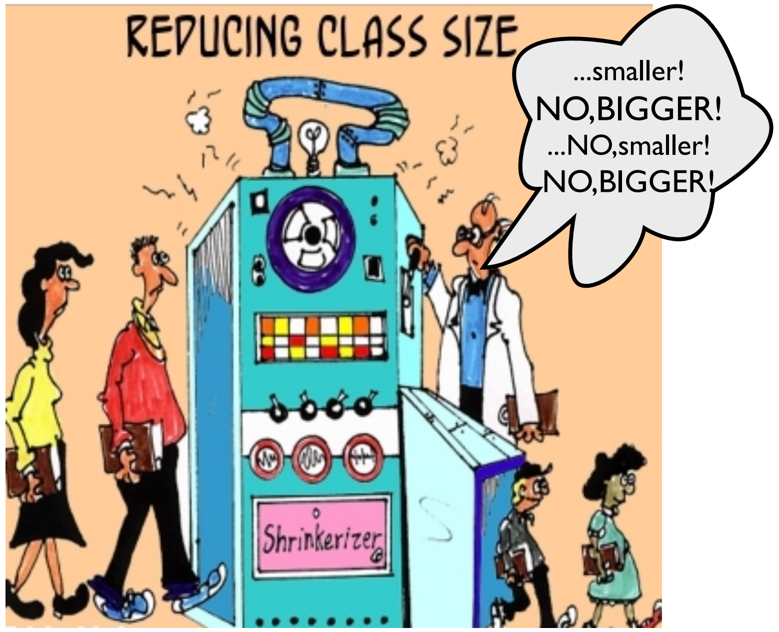

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a most **peculiar** relation
involving partial vs total

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

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Hamilton prefers Contra*variant* g^{mn} *with* Co*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

→ *Expressing Hamiltonian* $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations

Hamilton's equations in Runga-Kutta (computer solution) form

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

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This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity $\dot{q}^m$*.)*

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

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We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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$$\boxed{H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E}$$

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Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r,\phi) = \frac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r,\phi)$$

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Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

→ *Polar-coordinate example of Hamilton's equations*

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Hamiltonian $H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + U(r,\phi)$ *in 2D-polar coordinates satifies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m} = \dot{q}^m$ *Hamilton's 2nd equations:* $\frac{\partial H}{\partial q^m} = -\dot{p}_m$

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$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

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Polar coordinate example of Hamilton's equations

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$$\frac{\partial H}{\partial r}=-\dot{p}_r \qquad \frac{\partial H}{\partial r}=-2\frac{p_\phi^2}{2Mr^3}+\frac{\partial U(r,\phi)}{\partial r} \qquad \dot{p}_r=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi \qquad \frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

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$$\frac{\partial H}{\partial \phi} = -\dot{p}_\phi$$

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$$\dot{p}_\phi = -\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ *in 2D-polar coordinates satifies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$p_r=M\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2}$$

$$p_\phi=Mr^2\dot{\phi}$$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial r}=-2\frac{p_\phi^2}{2Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

$$\dot{p}_r=M\ddot{r}=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

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$$\frac{\partial H}{\partial p_r}=\dot{r} \qquad \frac{\partial H}{\partial p_r}=\frac{p_r}{M} \qquad p_r=M\dot{r}$$

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$$\dot{p}_r=M\ddot{r}=\frac{p_\phi^2}{Mr^3}-\frac{\partial U(r,\phi)}{\partial r}$$

$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi \qquad \frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi} \qquad \dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

Polar coordinate example of Hamilton's equations

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$$p_\phi=Mr^2\dot{\phi}$$

$$\dot{p}_\phi=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}=-\partial_\phi U(r,\phi)$$

Hamilton prefers <u>Contra</u>*variant* g^{mn} *with* <u>Co</u>*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations

→ *Hamilton's equations in Runga-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r} \qquad \dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$$p_\phi = Mr^2\dot{\phi} \qquad \dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi$$

Runga-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) \quad = \frac{p_r}{M}$$

$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi^2}{Mr^3} - \partial_r U(r,\phi)$$

$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi}{Mr^2}$$

$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) \quad = -\partial_\phi U(r,\phi)$$

Examples of Hamiltonian mechanics in effective potentials

→ *Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)*

Coulomb orbits in polar coordinates and effective potential (Simulation)

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r{}^2 + g^{\phi\phi} p_\phi{}^2) + k \cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2} \cdot p_\phi{}^2) + \frac{k \cdot r^2}{2} = E = const.$$

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Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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Same applies to any radial potential U(r)

$$E = \frac{p_r^{\ 2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{U(r)}^{\text{“real” PE}}}^{\text{“effective” PE}}$$

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Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2} \cdot r^2$

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}$$

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Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M}\cdot r^2}}$

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Called the “quadrature” or 1/4-cycle solution if $r_< = 0$ and $r_> =$ max amplitude

Radial KE is: $\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2} \cdot r^2$

Radial velocity:

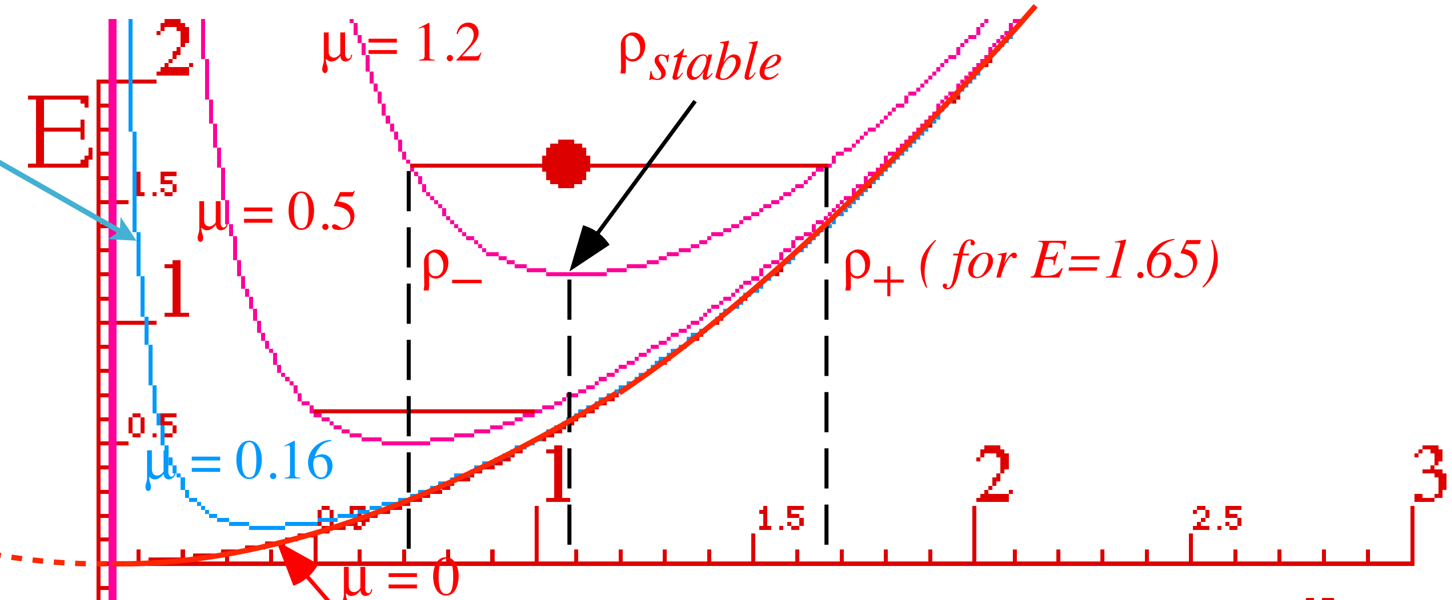
$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}$$

Time vs r: $t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}}$

Time vs r for any radial U(r):

$$t = \int_{r<}^{r>} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{2U(r)}{M}}}$$

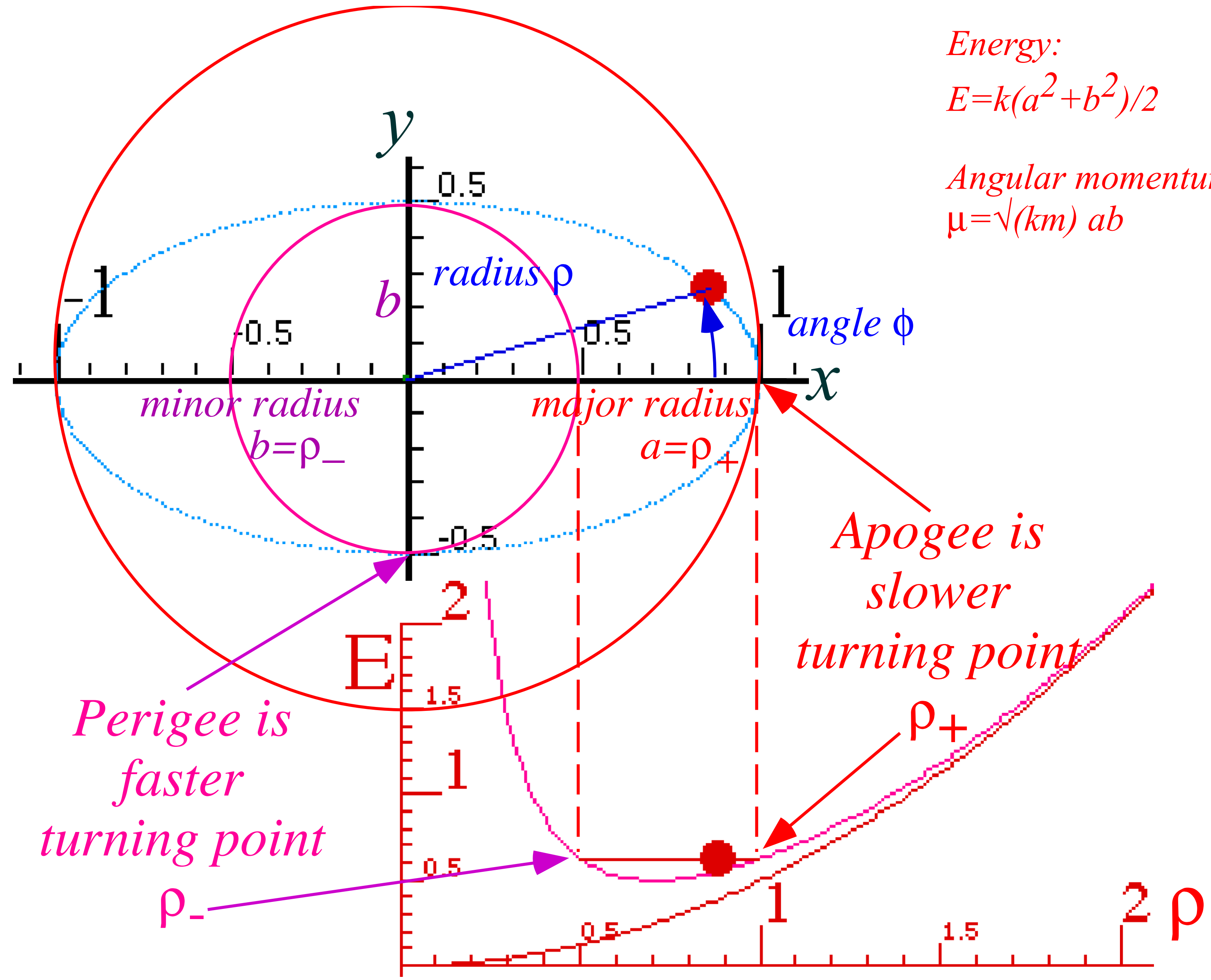
$$E = \frac{p_r^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$

Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$



Examples of Hamiltonian mechanics in effective potentials

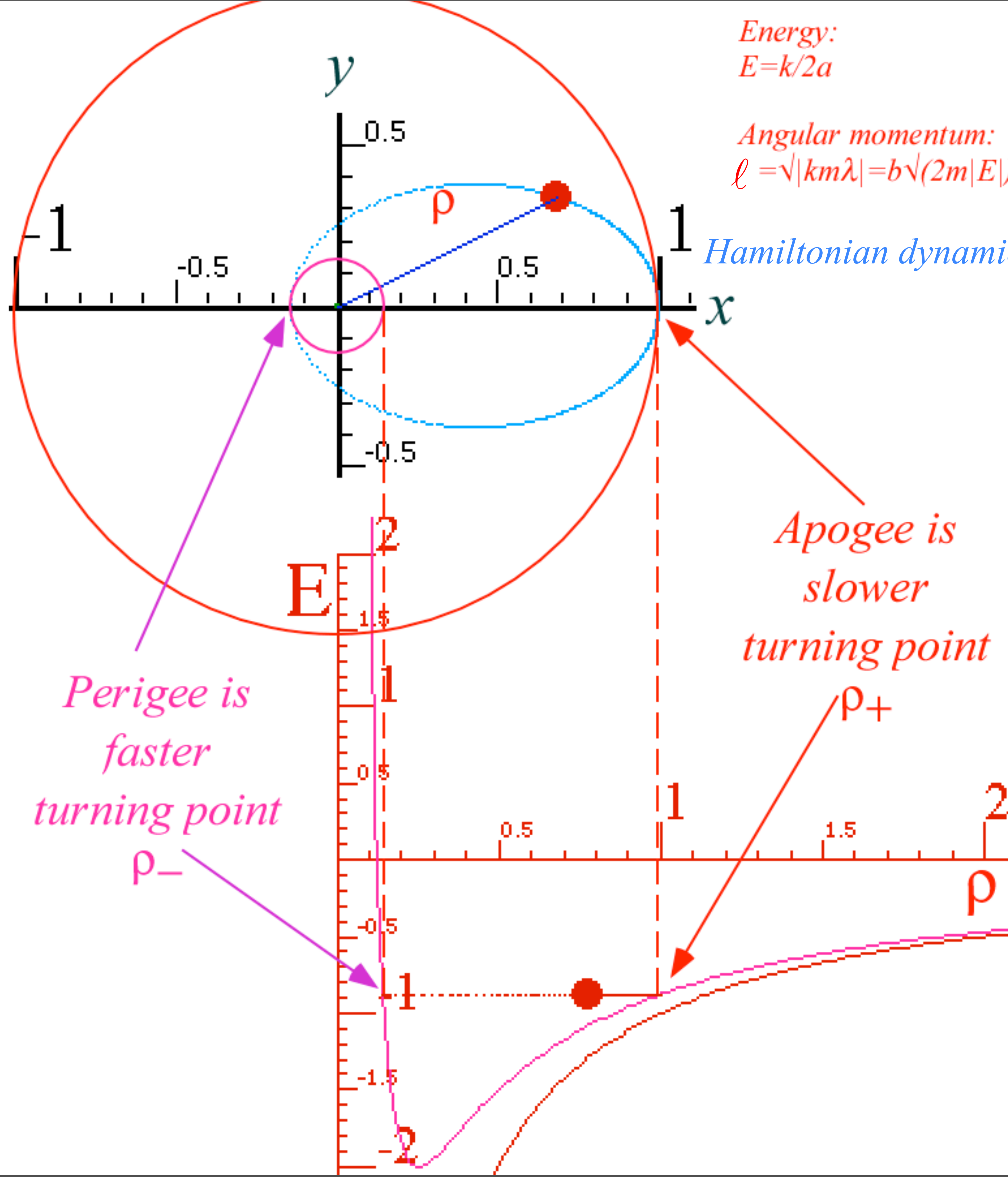
Isotropic Harmonic Oscillator in polar coordinates and effective potential (Simulation)

Coulomb orbits in polar coordinates and effective potential (Simulation)

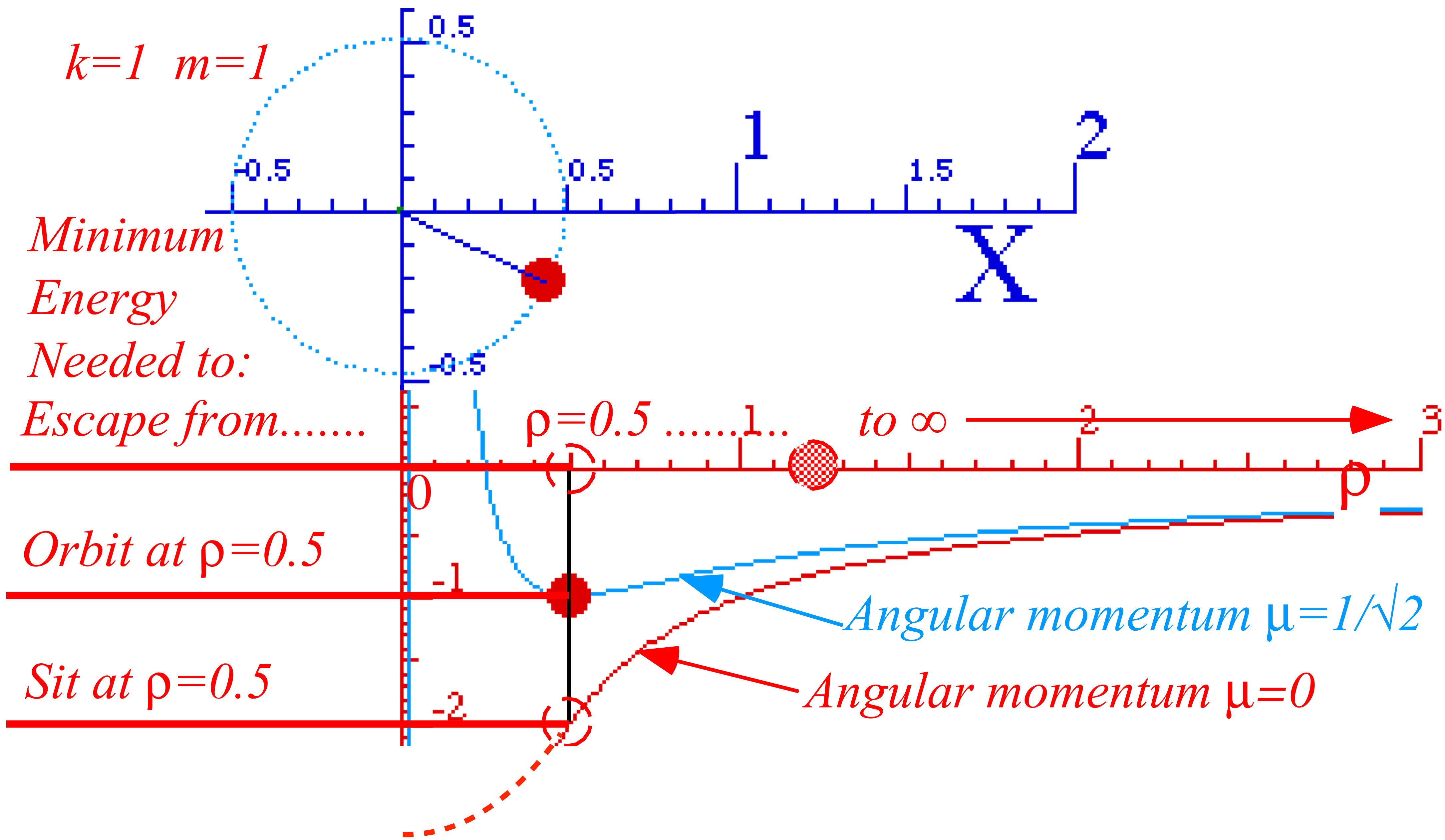
Energy:
$E=k/2a$

Angular momentum:
$\ell = \sqrt{|km\lambda|} = b\sqrt{(2m|E|)}$

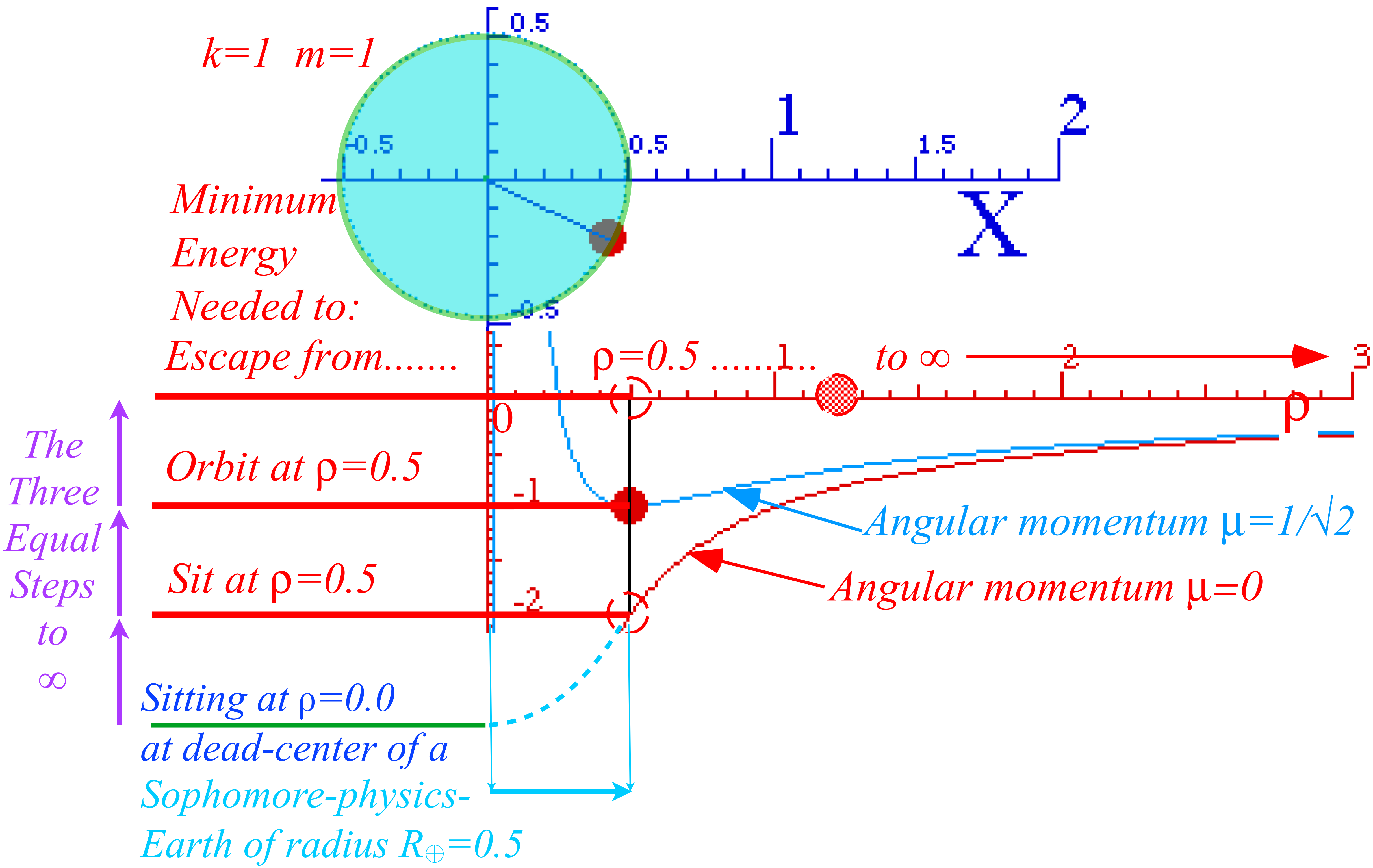
Hamiltonian dynamics for Coulomb potential $U(r) = -k/r$



$$E = \frac{p_r^{\,2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{"centifugal-barrier" PE}} - \overbrace{\frac{k}{r}}^{\text{"real" PE}}}^{\text{"effective" PE}}$$



k=1 m=1
0.5
-0.5
0.5
1
1.5
2
-0.5
X
Minimum
Energy
Needed to:
Escape from.......
ρ=0.5 ......1...
to ∞
2
3
0
ρ
Orbit at ρ=0.5
-1
Angular momentum μ=1/√2
Sit at ρ=0.5
-2
Angular momentum μ=0

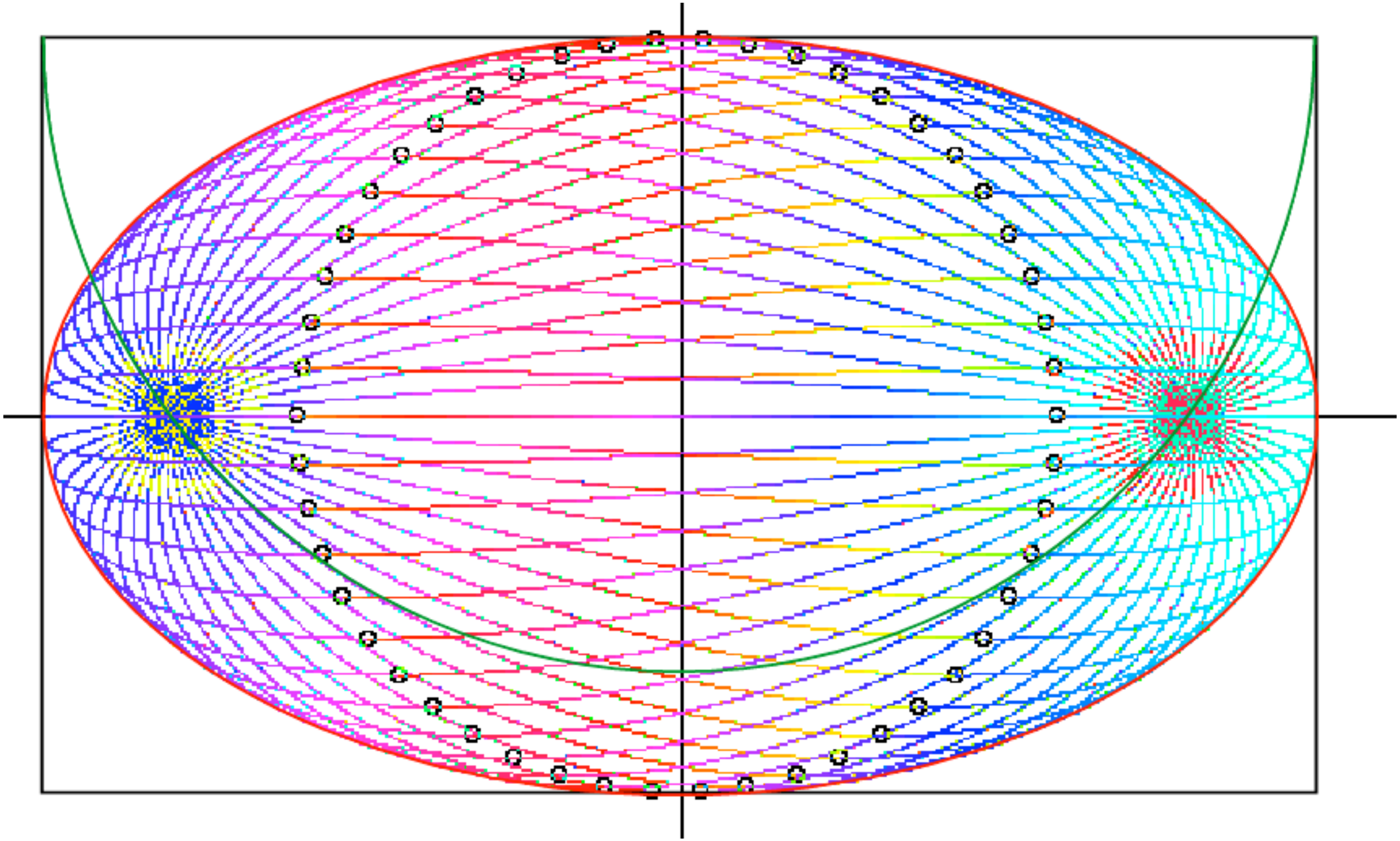


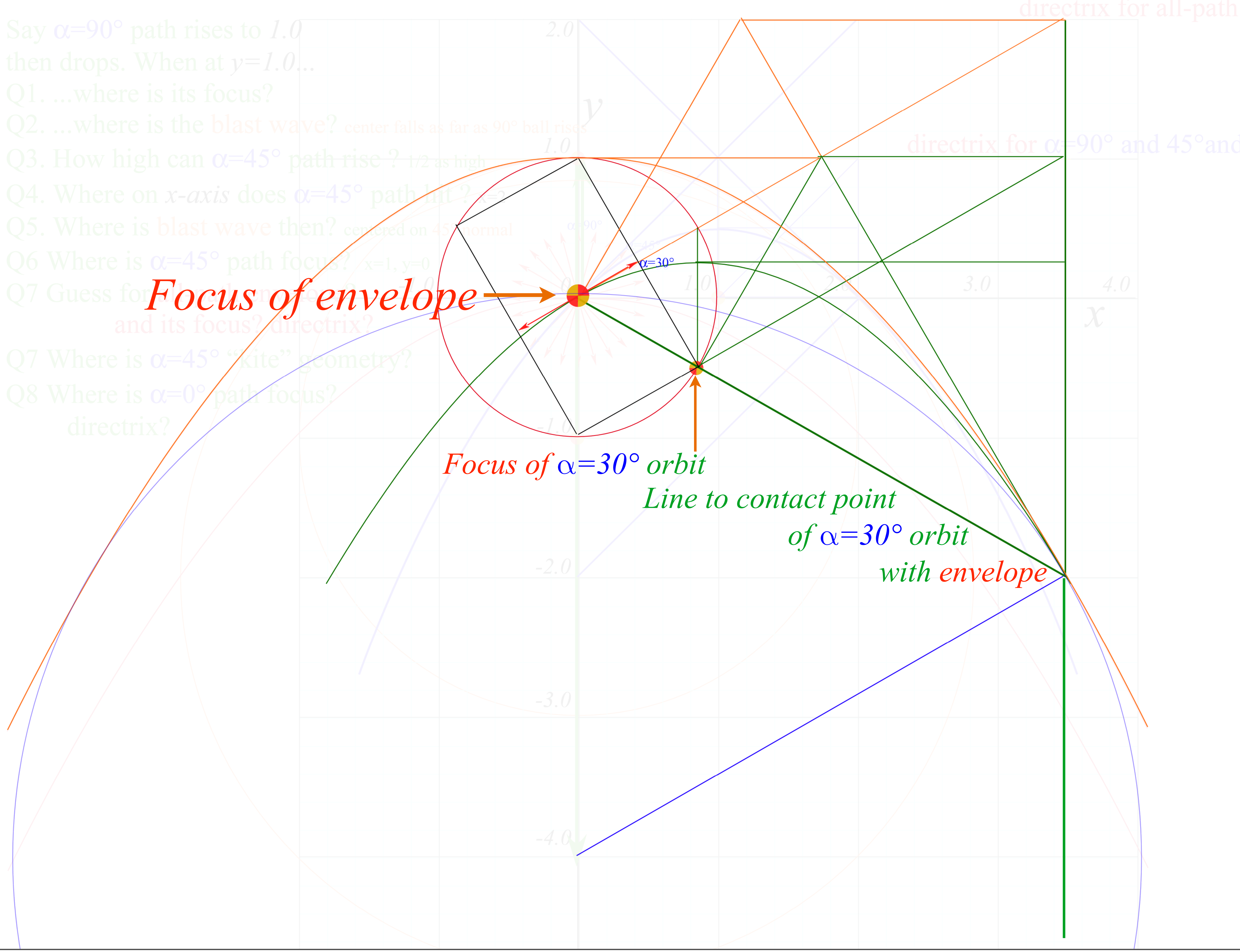
k=1 m=1
0.5
-0.5
0.5
1
1.5
2
X
-0.5
Minimum
Energy
Needed to:
Escape from.......
ρ=0.5 ......1..
to ∞
2
3
0
ρ
The
Three
Equal
Steps
to
∞
Orbit at ρ=0.5
-1
Angular momentum μ=1/√2
Sit at ρ=0.5
-2
Angular momentum μ=0
Sitting at ρ=0.0
at dead-center of a
Sophomore-physics-
Earth of radius R⊕=0.5

Parabolic and 2D-IHO elliptic orbital envelopes

Some clues for take-home assignment 7.2 (Simulation)

Exploding-starlet elliptical envelope and contacting elliptical trajectories





Focus of envelope
Focus of α=30° orbit
Line to contact point
of α=30° orbit
with envelope
α=30°
directrix for all-path

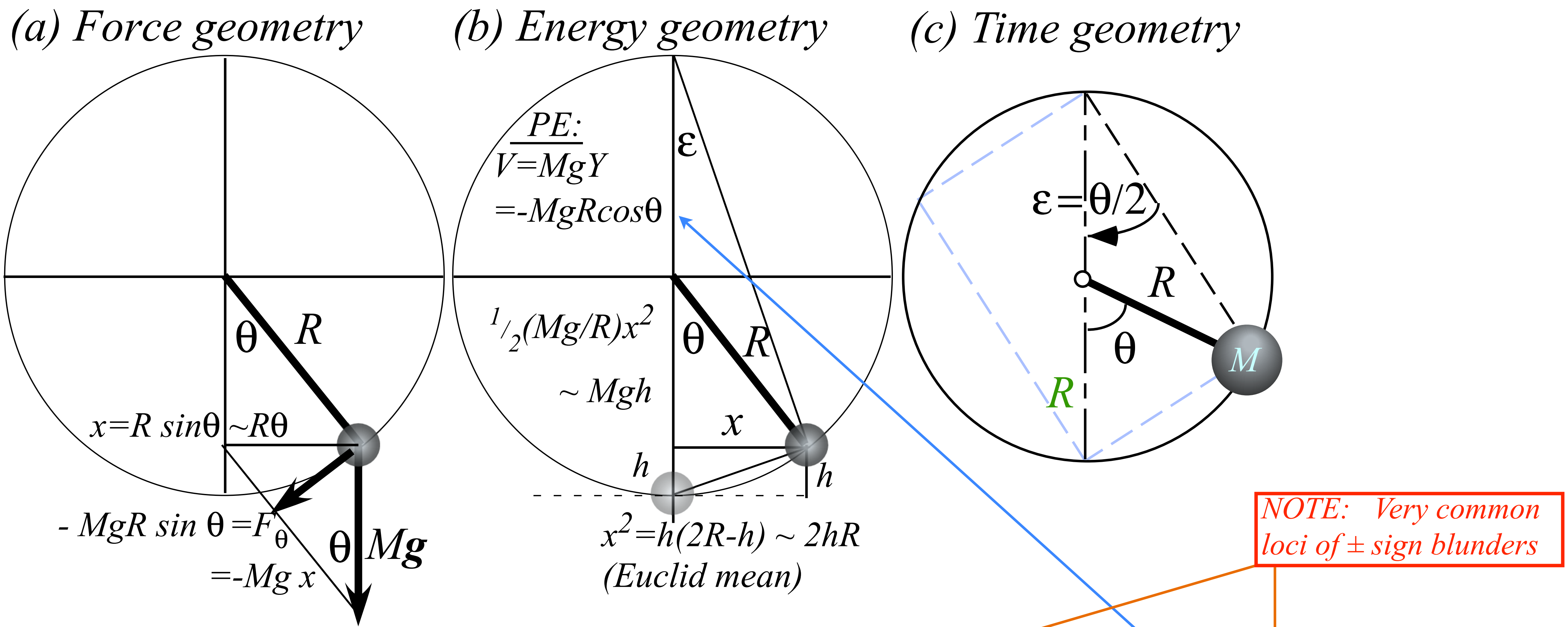
Lecture 12 ends here
Thur. 10.2.2012

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)

1D-HO phase-space control (Simulation)

1D Pendulum and phase plot

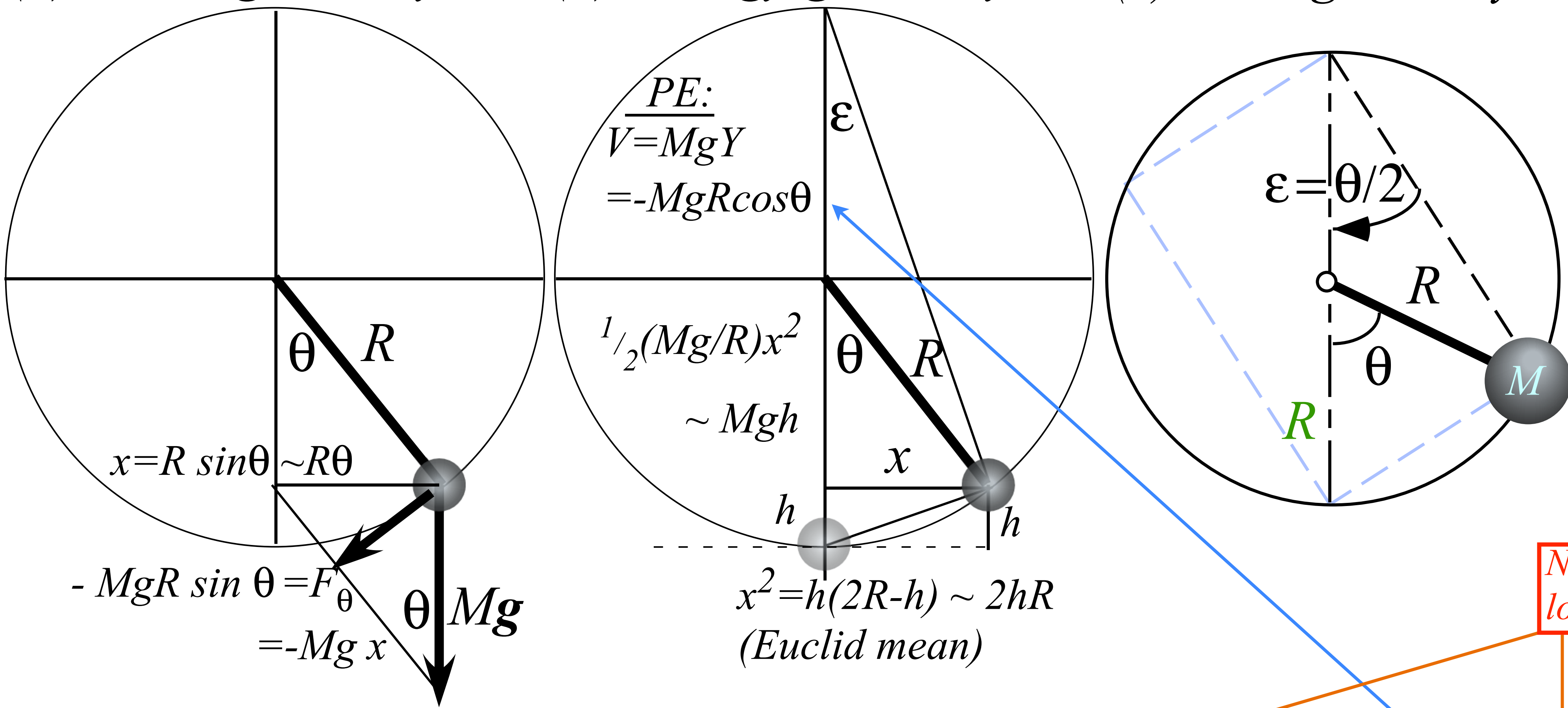


Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2} I\dot{\theta}^2 - U(\theta) = \frac{1}{2} I\dot{\theta}^2 + MgR\cos\theta$$

1D Pendulum and phase plot

(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



NOTE: Very common loci of ± sign blunders

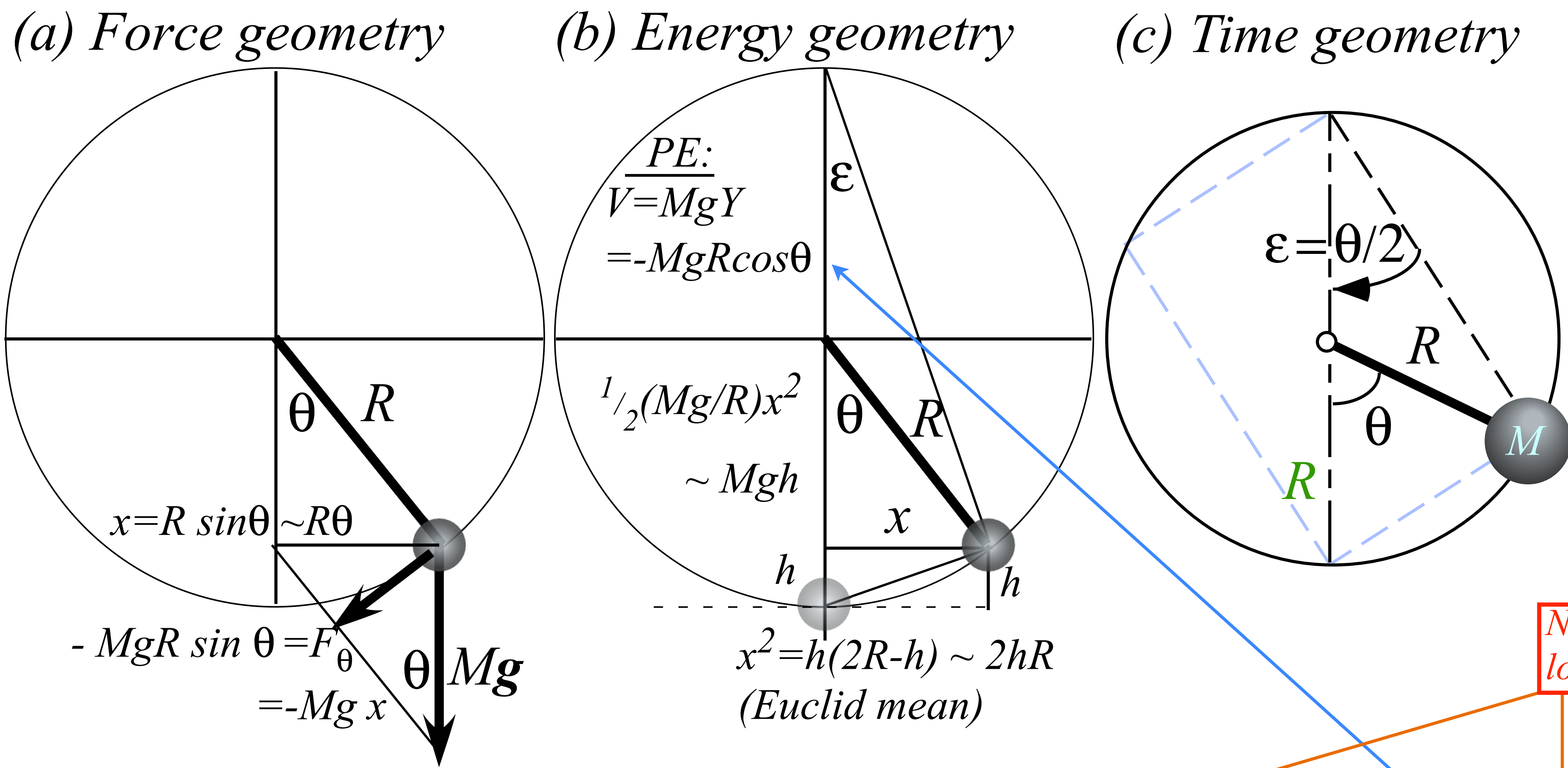
Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot\theta,\theta) = \frac{1}{2}I\dot\theta^2 - U(\theta) = \frac{1}{2}I\dot\theta^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;= E \quad =\text{const.}$$

1D Pendulum and phase plot



NOTE: Very common loci of ± sign blunders

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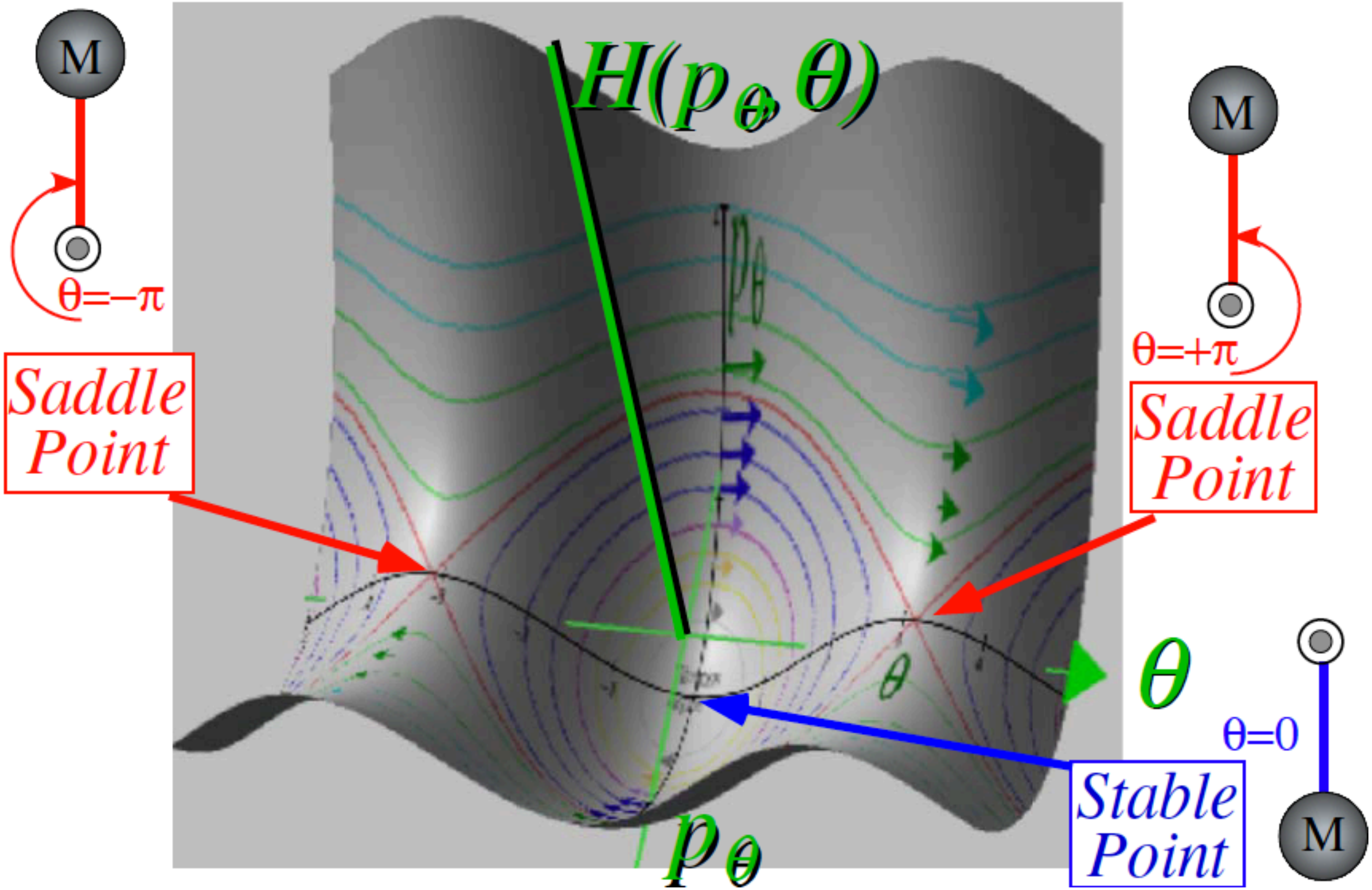
Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

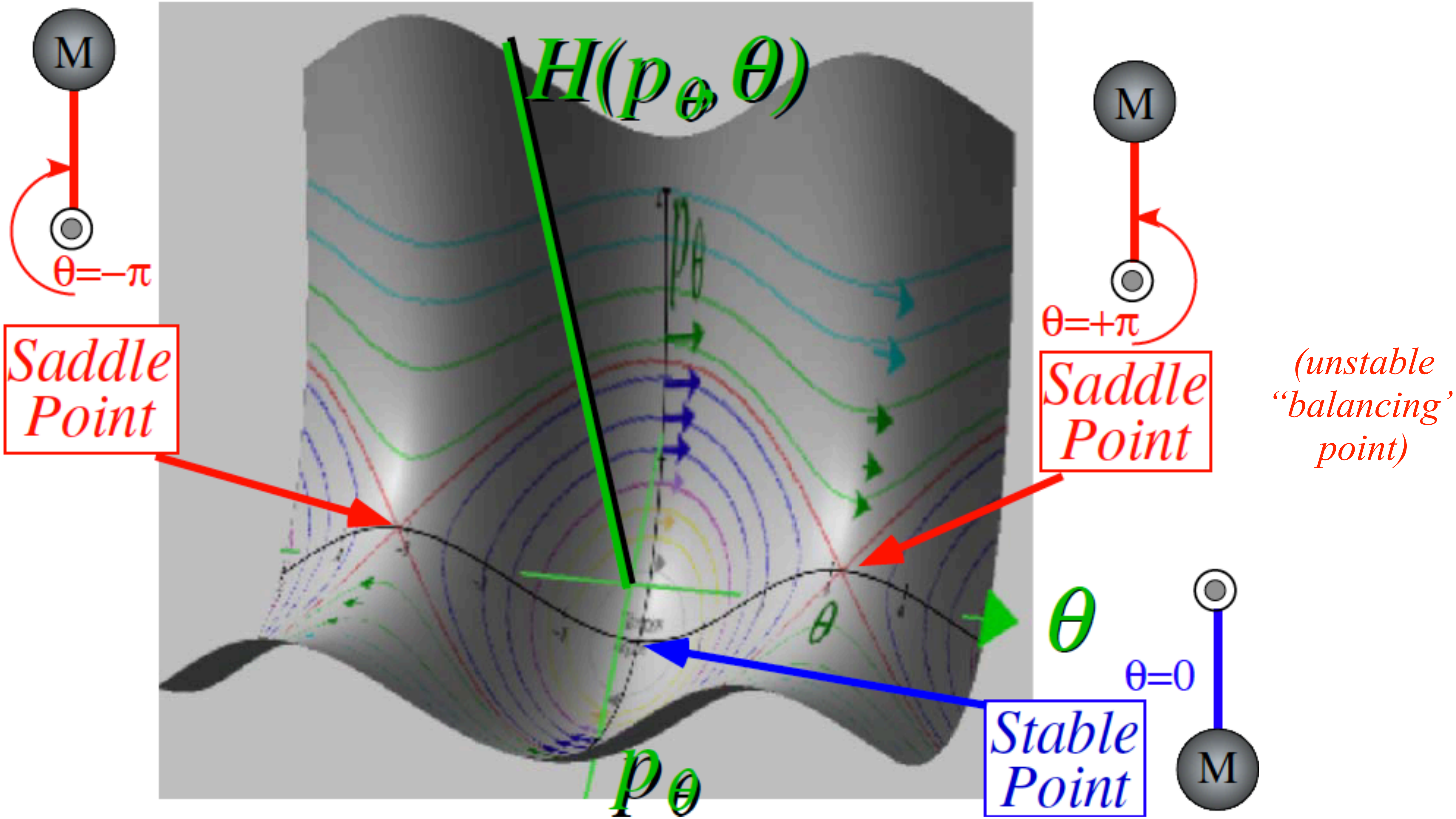
$$H(p_\theta,\theta) = \frac{1}{2I} p_\theta{}^2 + U(\theta) = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \;=E \quad =const.$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;, \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

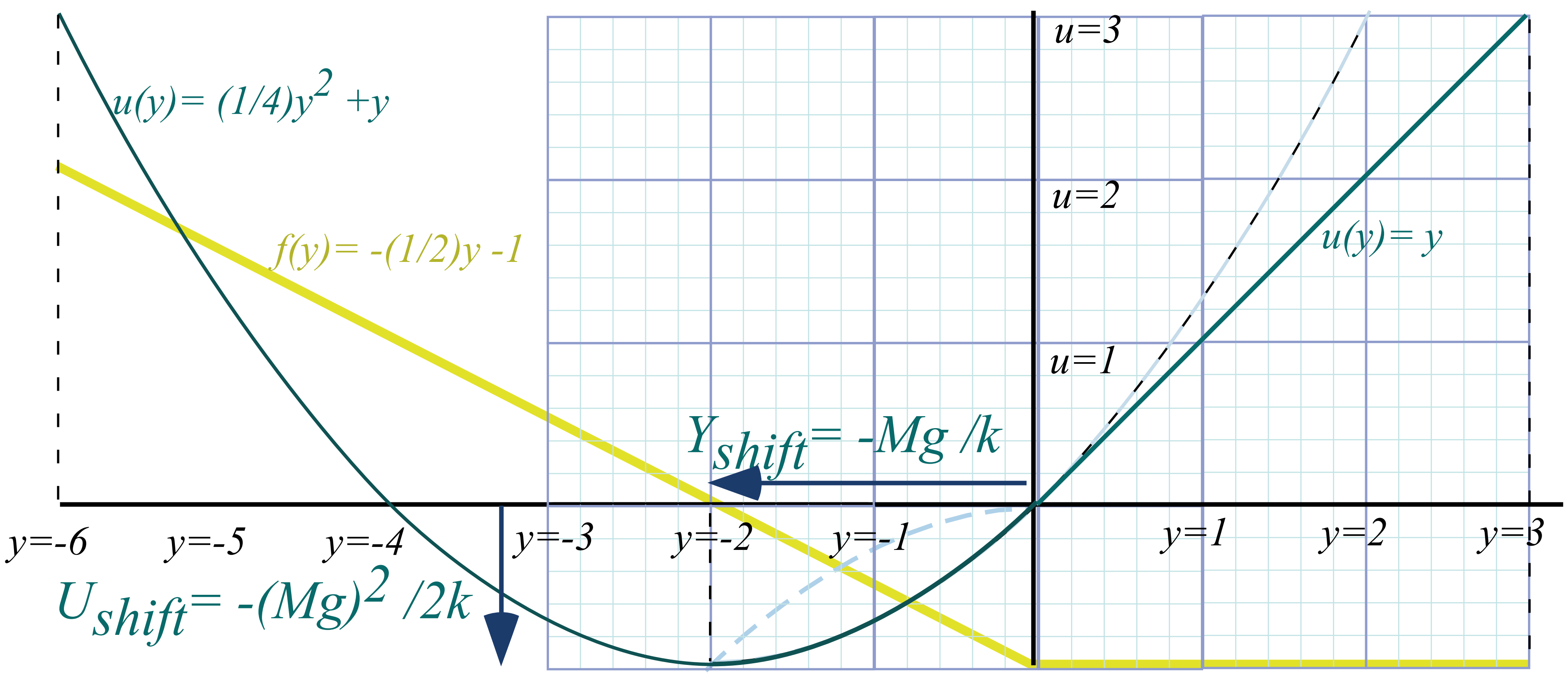
$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e_H} \times \left(-\nabla H\right) = (\text{H-axis}) \times (\text{fall line}), \text{ where:} \begin{cases} (\text{H-axis}) = \mathbf{e_H} = \mathbf{e_q} \times \mathbf{e_p} \\ (\text{fall line}) = -\nabla H \end{cases}$$

2. Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)

Phase control *(Simulation)*

$F(Y)=-kY-Mg$ $\qquad$ $U(Y)=(1/2)kY^2+Mg\ Y$



Unit 1
Fig. 7.4

Simulation of atomic classical (*or semi-classical*) *dynamics using varying phase control*