Lecture 14

Revised 12.22.12 from 10.09.2012

Complex Variables, Series, and Field Coordinates I.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
How good are those power series?
Taylor-Maclaurin series, imaginary interest, and complex exponentials
2. What good are complex exponentials?
Easy trig
Easy 2D vector analysis
Easy oscillator phase analysis
Easy rotation and "dot" or "cross" products
3. Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
Easy 2D vector field-potential theory
4. Riemann-Cauchy relations (What's analytic? What's not?)
Easy 2D curvilinear coordinate discovery
Easy 2D circulation and flux integrals
Easy 2D monopole, dipole, and 2^n-pole analysis
Easy 2^n-multipole field and potential expansion
Easy stereo-projection visualization
5. Non-analytic 2D source field analysis

Lecture 14 Tue. 10.09 starts here

1. *Complex numbers provide "automatic trigonometry"*
2. *Complex numbers add like vectors.*
3. *Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.*
4. *Complex products provide 2D rotation operations.*
5. *Complex products provide 2D "dot"(•) and "cross"(x) products.*
6. *Complex derivative contains "divergence"($\nabla \cdot \mathbf{F}$) and "curl"($\nabla \times \mathbf{F}$) of 2D vector field*
7. *Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]*
8. *Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla \times \mathbf{A}$) potentials*

   The half-n'-half results: (Riemann-Cauchy Derivative Relations)
9. *Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field*
10. *Complex integrals $\int f(z)dz$ count 2D "circulation"($\int \mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int \mathbf{F}\times d\mathbf{r}$)*
11. *Complex integrals define 2D monopole fields and potentials*
12. *Complex derivatives give 2D dipole fields*

Lecture 15 Thur. 10.11 starts here

13. *More derivatives give 2D 2^N-pole fields…*
14. *...and 2^N-pole multipole expansions of fields and potentials...*
15. *...and Laurent Series...*
16. *...and non-analytic source analysis.*

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Simple *interest* at some rate r based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time t later they would pay you $p(t)=(1+r\cdot t)p(0)$.

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$$p^{\frac{1}{2}}(t) = (1+r\cdot\tfrac{t}{2})p(\tfrac{t}{2}) = (1+r\cdot\tfrac{t}{2})\cdot(1+r\cdot\tfrac{t}{2})p(0) = \tfrac{3}{2}\cdot\tfrac{3}{2}\cdot 1 = \tfrac{9}{4} = 2.25$$

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Trimester compounded interest gives $p(\frac{t}{3})=(1+r\cdot\frac{t}{3})p(0)$ at the *1/3rd*-period $\frac{t}{3}$ or 1st trimester and then use that to figure the 2nd trimester and so on. Now $1.00 at rate $r=1$ earns $2.37.

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})p(2\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(0)=\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot 1=\tfrac{64}{27}=2.37$$

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So if you compound interest more and more frequently, do you approach **INFININTEREST***?*

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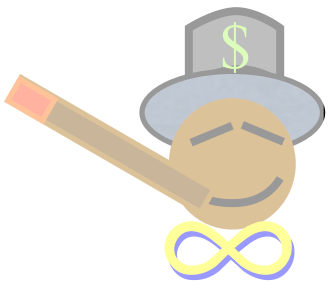

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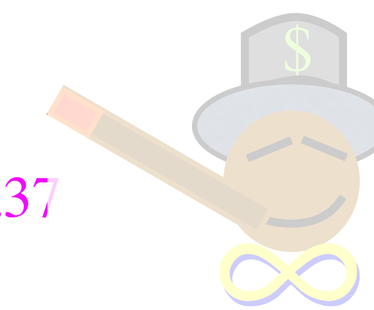

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+25¢

$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})^2 p(0)=\left(\tfrac{3}{2}\right)^2\cdot 1=\tfrac{9}{4}=2.25$$

+12¢

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})^3 p(0)=\left(\tfrac{4}{3}\right)^3\cdot 1=\tfrac{64}{27}=2.37$$

+7¢

$$p^{\frac{1}{4}}(t)=(1+r\cdot\tfrac{t}{4})^4 p(0)=\left(\tfrac{5}{4}\right)^4\cdot 1=\tfrac{625}{256}=2.44$$

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Monthly: $p^{\frac{1}{12}}(t)=(1+r\cdot\tfrac{t}{12})^{12}p(0)=\left(\tfrac{13}{12}\right)^{12}\cdot 1=2.613$

Weekly: $p^{\frac{1}{52}}(t)=(1+r\cdot\tfrac{t}{52})^{52}p(0)=\left(\tfrac{53}{52}\right)^{52}\cdot 1=2.693$

Daily: $p^{\frac{1}{365}}(t)=(1+r\cdot\tfrac{t}{365})^{365}p(0)=\left(\tfrac{366}{365}\right)^{365}\cdot 1=\mathbf{2.71}45$

Hrly: $p^{\frac{1}{8760}}(t)=(1+r\cdot\tfrac{t}{8760})^{8760}p(0)=\left(\tfrac{8761}{8760}\right)^{8760}\cdot 1=\mathbf{2.718}1$

Interest product formula is <u>really</u> inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

$$p^{1/m}(1) = (1+\tfrac{1}{m})^m \xrightarrow[m\to\infty]{} \mathbf{2.718281828459..} = e$$

Let: $m \cdot r \cdot t = n$

or: $1/m = r \cdot t/n$

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$p^{1/m}(1)$ = **2.71**69239322 *for m = 1,000*

$p^{1/m}(1)$ = **2.718**1459268 *for m = 10,000*

$p^{1/m}(1)$ = **2.7182**682372 *for m = 100,000*

$p^{1/m}(1)$ = **2.71828**04693 *for m = 1,000,000*

$p^{1/m}(1)$ = **2.718281**6925 *for m = 10,000,000*

$p^{1/m}(1)$ = **2.7182818**149 *for m = 100,000,000*

$p^{1/m}(1)$ = **2.71828182**71 *for m = 1,000,000,000*

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Can improve computational efficiency using binomial theorem:

$$(x+y)^n = x^n + n\cdot x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + n\cdot xy^{n-1} + y^n$$

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Define: *Factorials*(!):

$0!=1=1!$, $2!=1\cdot 2$, $3!=1\cdot 2\cdot 3, \ldots$

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$0! = 1 = 1!, \quad 2! = 1\cdot 2, \quad 3! = 1\cdot 2\cdot 3, \ldots$

As $n \to \infty$ let :

$$n(n-1) \to n^2,$$

$$n(n-1)(n-2) \to n^3, \text{ etc.}$$

$$e^{r\cdot t} = 1 + r\cdot t + \frac{1}{2!}(r\cdot t)^2 + \frac{1}{3!}(r\cdot t)^3 + \ldots = \sum_{p=0}^{o} \frac{(r\cdot t)^p}{p!}$$

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Precision order:

(o=1)-e-series = **2.**00000 =1+1
(o=2)-e-series = **2.**50000 =1+1+1/2
(o=3)-e-series = **2.**66667 =1+1+1/2+1/6
(o=4)-e-series = **2.7**0833 =1+1+1/2+1/6+1/24
(o=5)-e-series = **2.71**667 =1+1+1/2+1/6+1/24+1/120
(o=6)-e-series = **2.718**05 =1+1+1/2+1/6+1/24+1/120+1/720
(o=7)-e-series = **2.7182**5
(o=8)-e-series = **2.71828**

About 12 summed quotients for 6-figure precision (A lot better!)

Power Series Good! Need general power series development

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Rate of change of position $x(t)$ is *velocity $v(t)$*.

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$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

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Change of velocity $v(t)$ is *acceleration $a(t)$*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

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Rate of change of position $x(t)$ is *velocity $v(t)$*.   Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is *acceleration $a(t)$*.   Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is *jerk $j(t)$*. (*Jerk* is NASA term.)   Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

$$j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 t + 3\cdot 4\cdot 5c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is *inauguration $i(t)$*. (Be silly like NASA!)   Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.* Set $t=0$ to get $c_0 = x(0)$.

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Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +$$

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Good old UP I formula!

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Gives Maclaurin (or Taylor) power series

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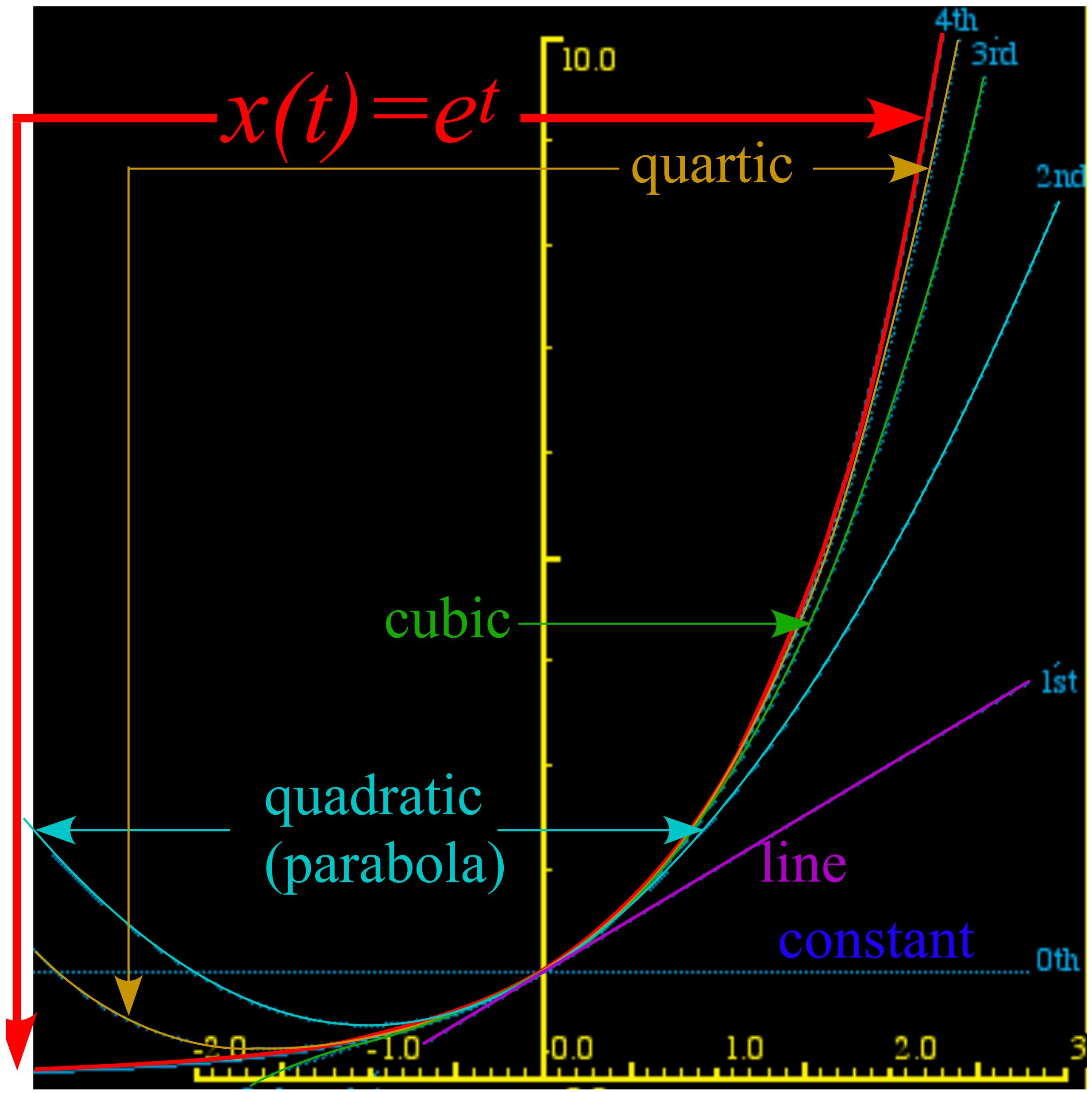
Good old UP I formula!

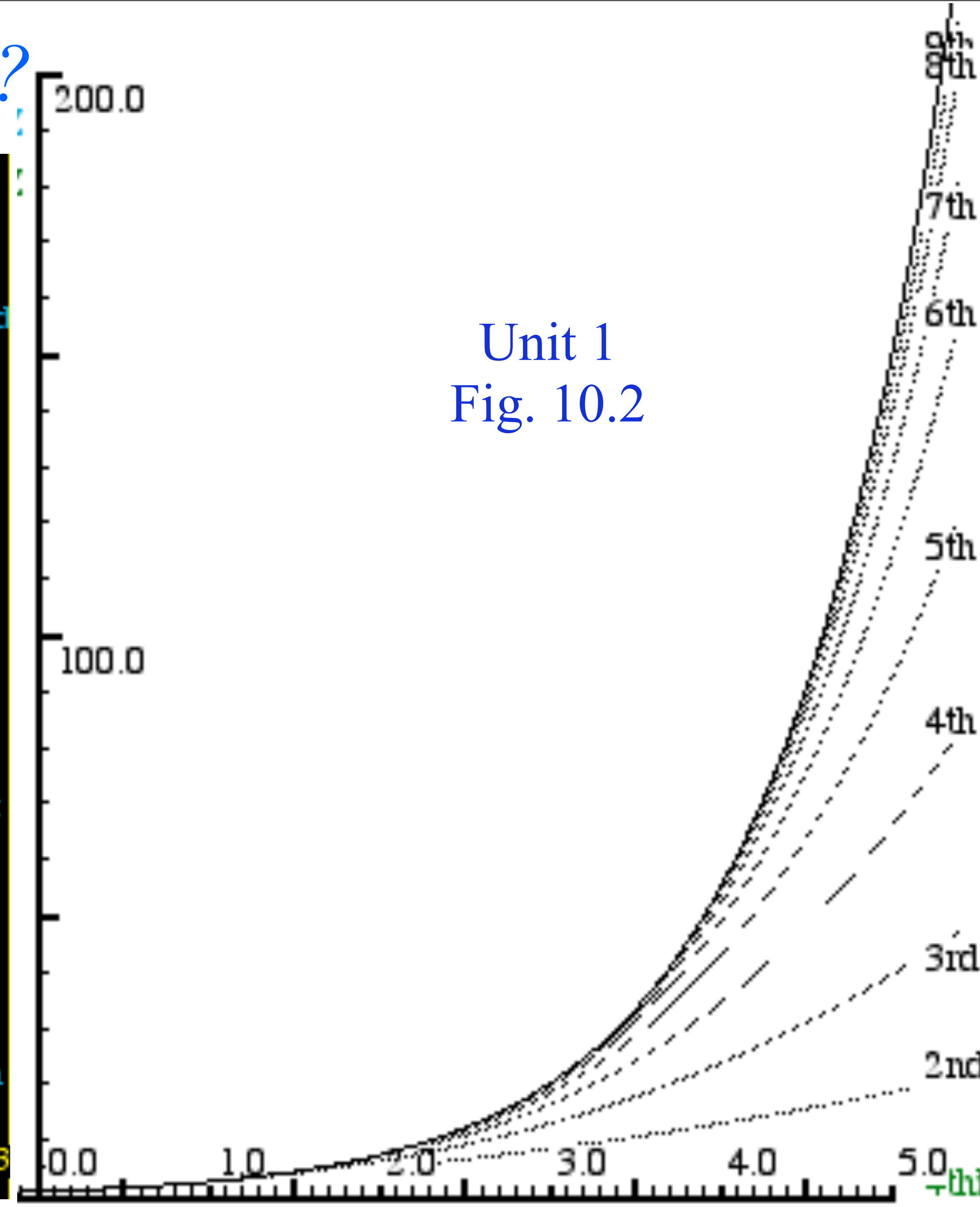
Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots$

gives *exponential:*

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \ldots + \frac{1}{n!}t^n +$$

But, how good are power series?





Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

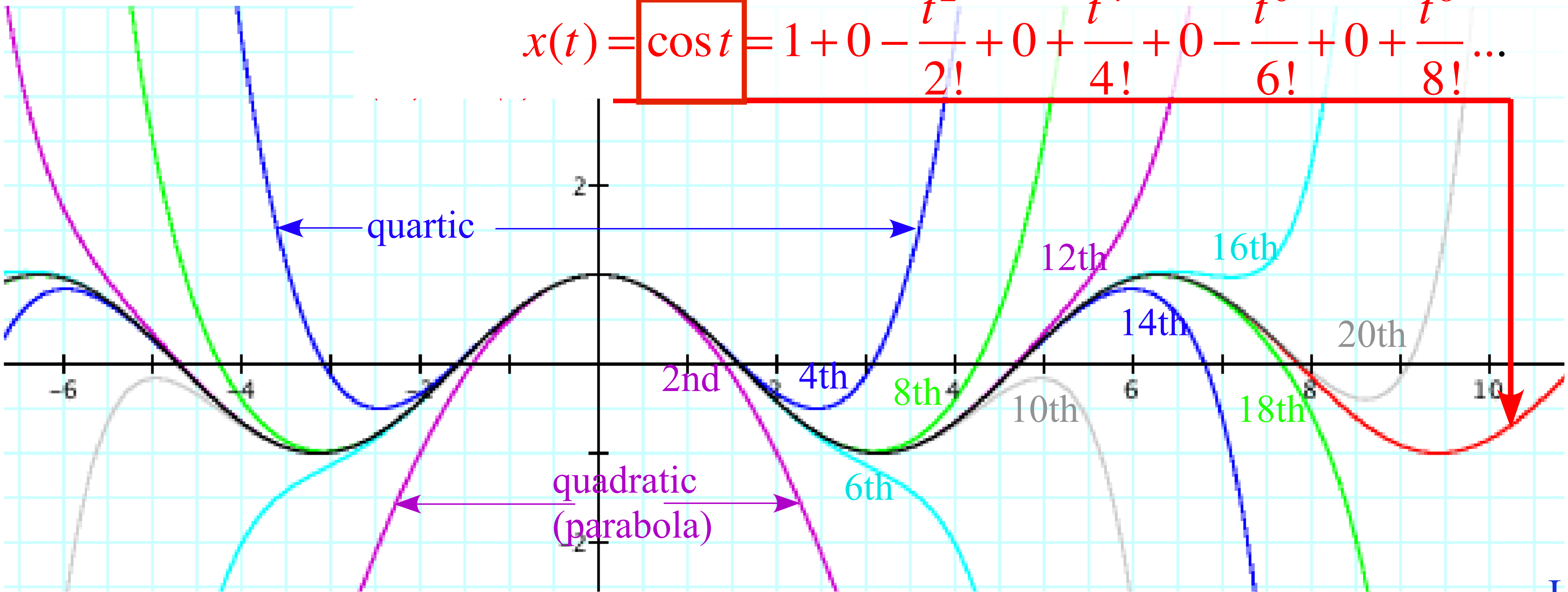
$$x(t) = x(0) + v(0)t + \tfrac{1}{2!}a(0)t^2 + \tfrac{1}{3!}j(0)t^3 + \tfrac{1}{4!}i(0)t^4 + \tfrac{1}{5!}r(0)t^5 + \ldots + \tfrac{1}{n!}x^{(n)}t^n +$$

Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots$

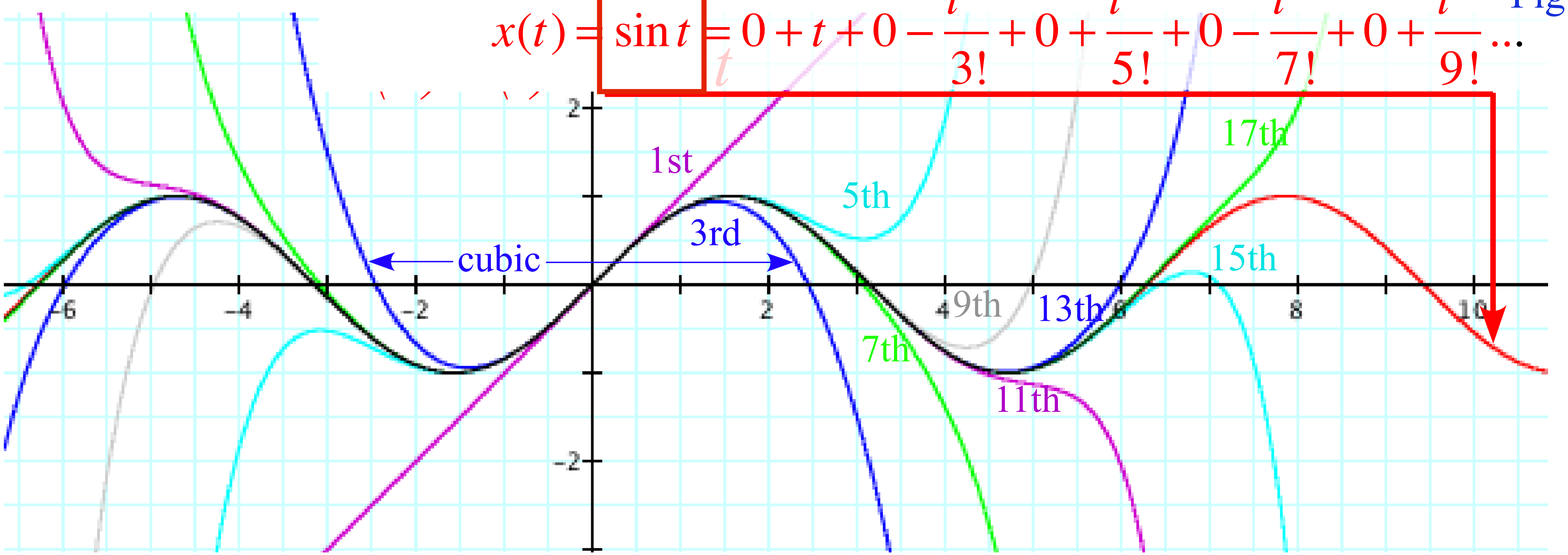
gives *exponential:* $$e^t = 1 + t + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3 + \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + \ldots + \tfrac{1}{n!}t^n +$$

How good are power series? Depends...

$$x(t) = \boxed{\cos t} = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \ldots$$



$$x(t) = \boxed{\sin t} = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \ldots$$



Unit 1
Fig. 10.3

1. The Story of e (A Tale of Great $Interest$)

How good are those power series?

Taylor-Maclaurin series,

→ *imaginary interest, and complex exponentials*

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\ldots \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

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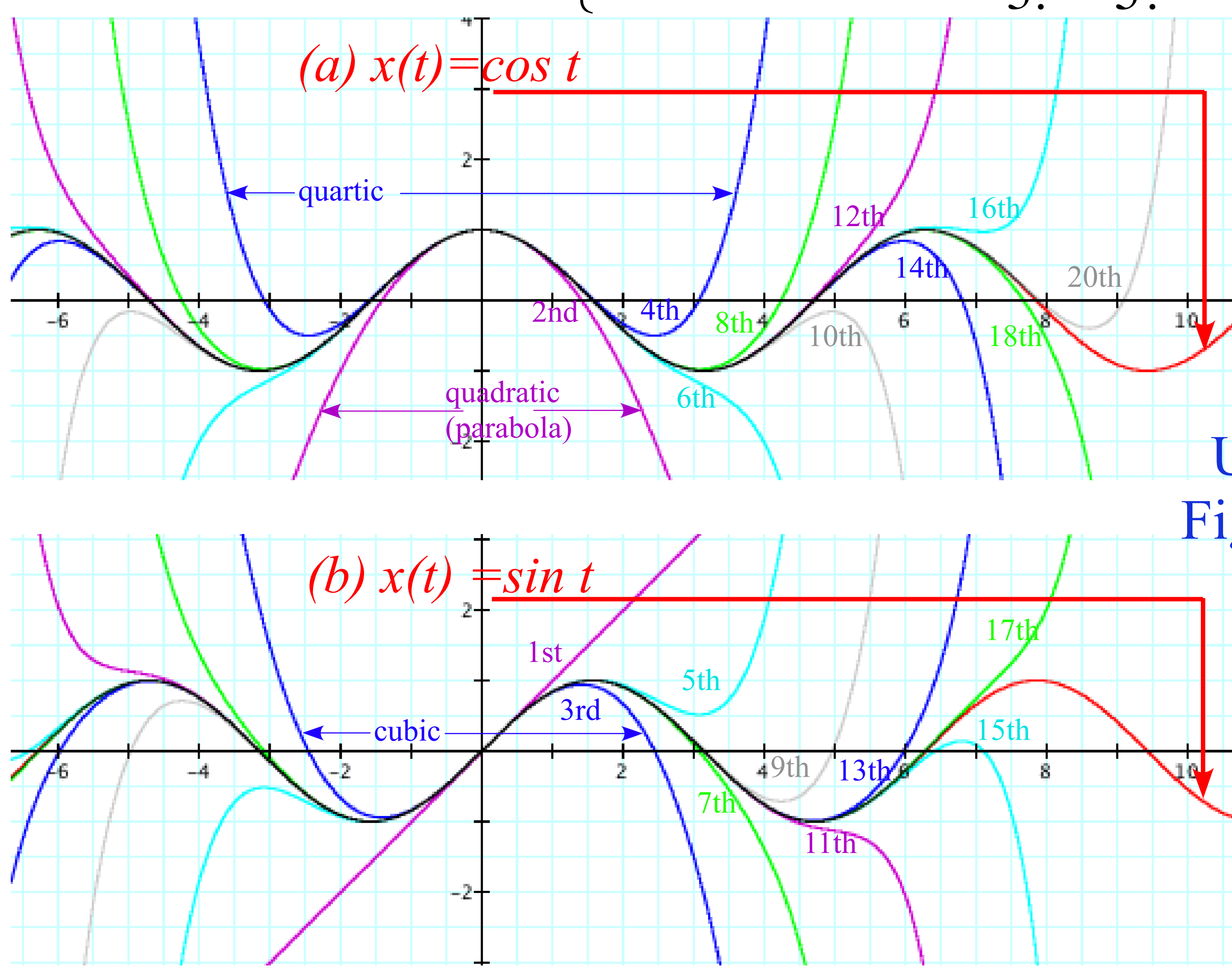
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

To match series for

$$\begin{cases} cosine: \cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ sine: \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

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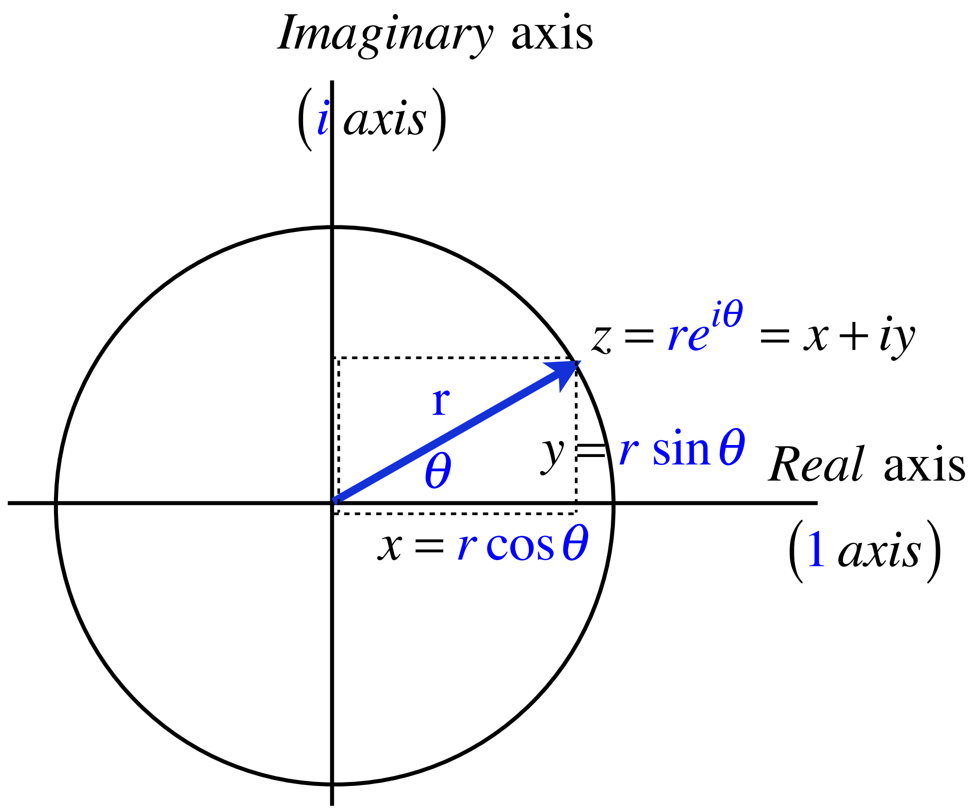
$$=1+i\theta-\frac{\theta^2}{2!}\ -i\frac{\theta^3}{3!}\ +\frac{\theta^4}{4!}\ +i\frac{\theta^5}{5!}\ -\ldots \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

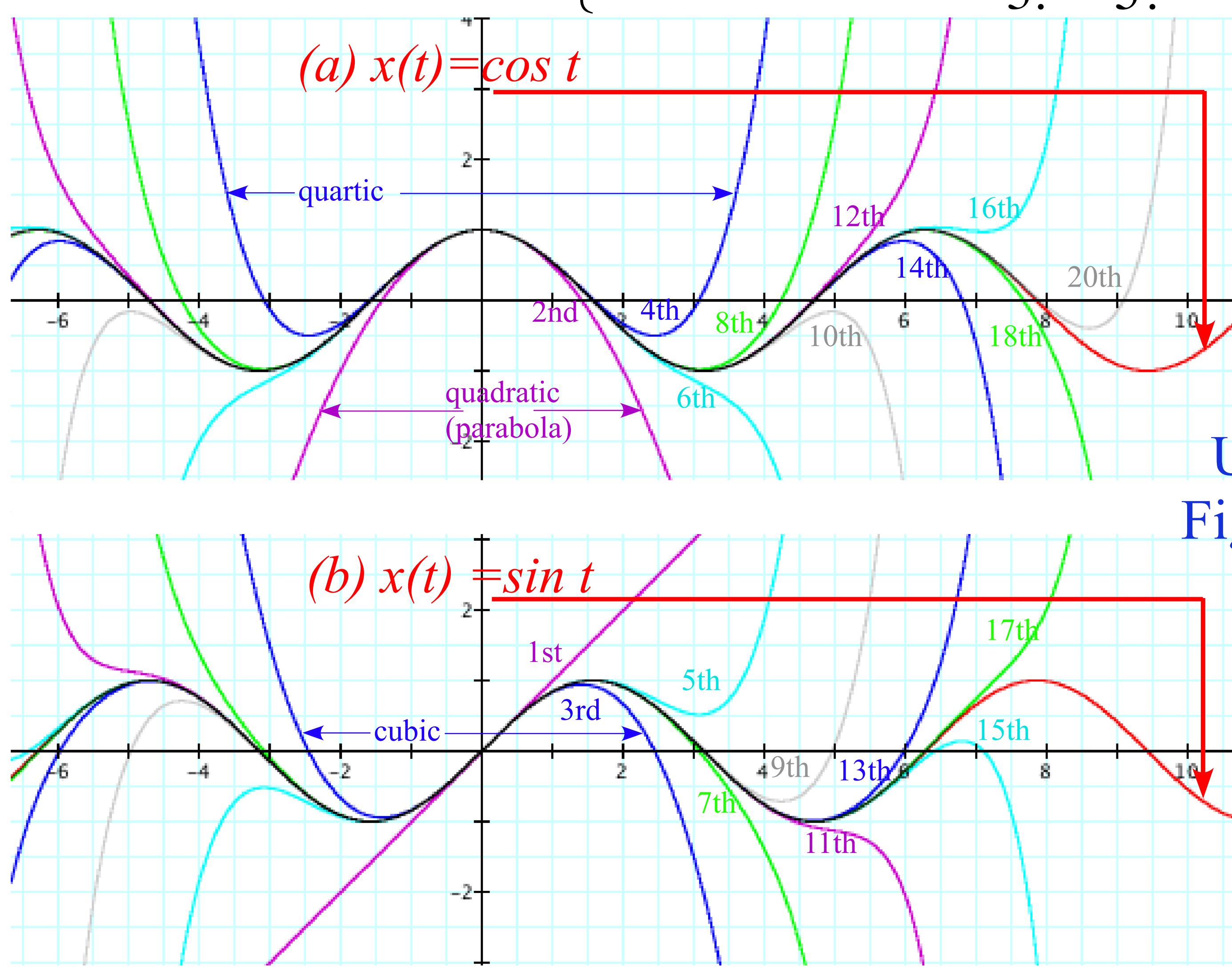
To match series for
$$\begin{cases} \textit{cosine}: \cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$e^{i\theta} \quad = \quad \cos\theta \quad + \quad i\sin\theta$$

Euler-DeMoivre Theorem



$$re^{i\theta} = r\cos\theta + i\sin\theta$$



Unit 1
Fig. 10.3

2. What Good Are Complex Exponentials?

Easy trig

→ *Easy 2D vector analysis*

→ *Easy oscillator phase analysis*

Easy rotation and "dot" or "cross" products

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ or sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$e^{i(a+b)} = e^{ia} \quad e^{ib}$$

$$\cos(a+b) + i\sin(a+b) = (\cos a + i\sin a)(\cos b + i\sin b)$$

$$\cos(a+b) + i\sin(a+b) = [\cos a\cos b - \sin a\sin b] + i[\sin a\cos b + \cos a\sin b]$$

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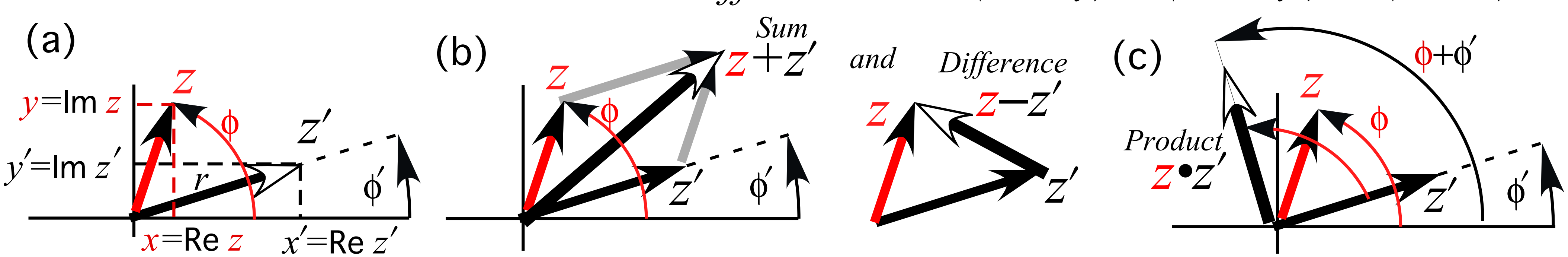
$$\cos(a+b) + i\sin(a+b) = [\cos a\cos b - \sin a\sin b] + i[\sin a\cos b + \cos a\sin b]$$

2. Complex numbers add like vectors.

$$z_{sum} = z + z' = (x+iy) + (x'+iy') = (x+x') + i(y+y')$$

$$z_{diff} = z - z' = (x+iy) - (x'+iy') = (x-x') + i(y-y')$$



Unit 1
Fig. 10.6

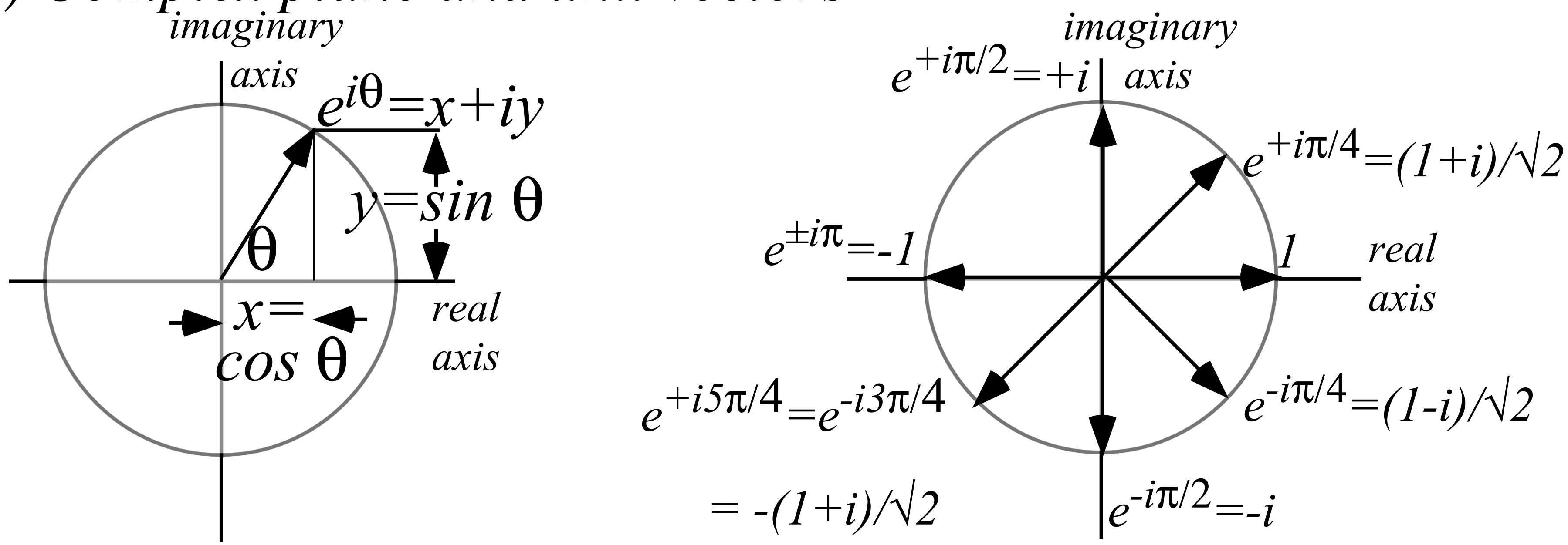
$$|z_{SUM}| = \sqrt{(z+z')^*(z+z')} = \sqrt{\left(re^{i\phi} + r'e^{i\phi'}\right)^*\left(re^{i\phi} + r'e^{i\phi'}\right)} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)}$$

$$= \sqrt{r^2 + r'^2 + rr'\left(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right)} = \sqrt{r^2 + r'^2 + 2rr'\cos(\phi-\phi')}$$

(quick derivation of Cosine Law)

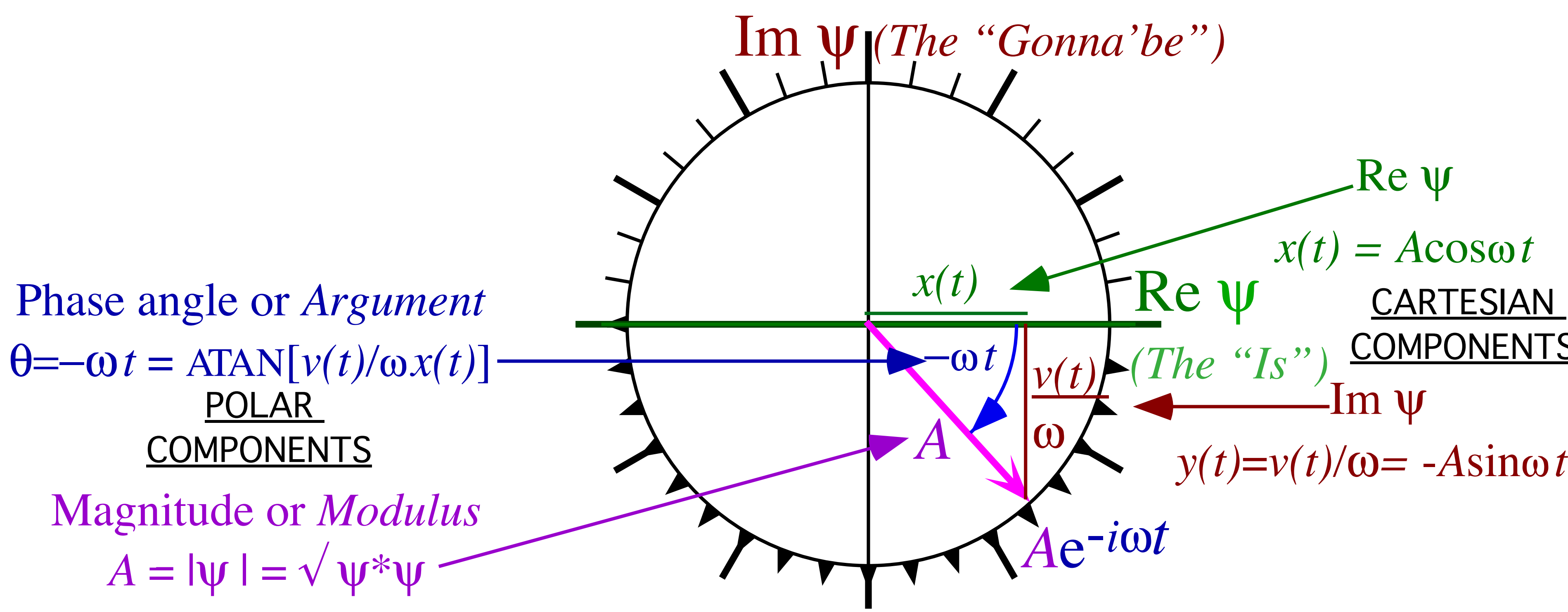
3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$

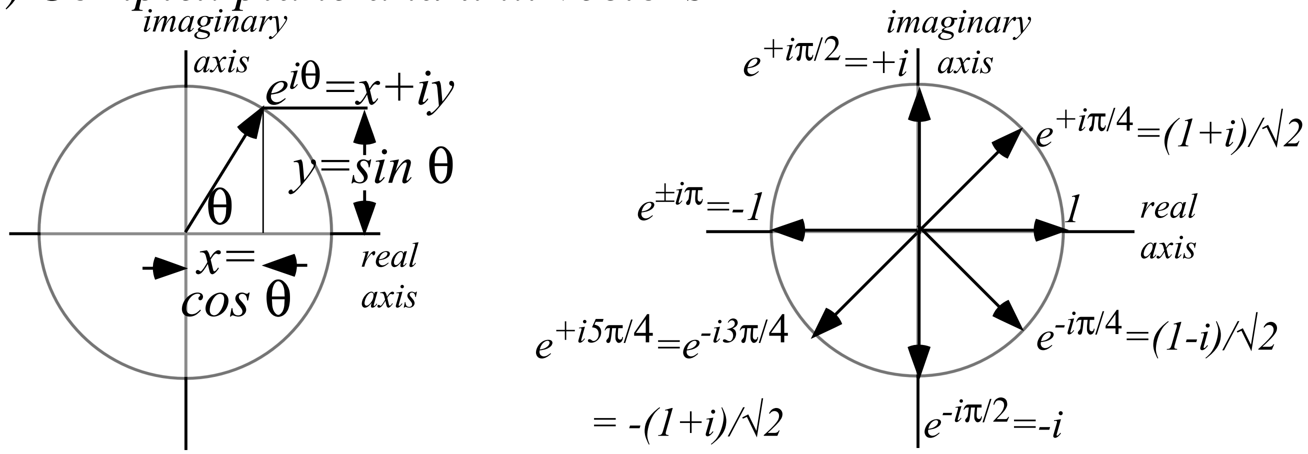
Unit 1
Fig. 10.5



What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors



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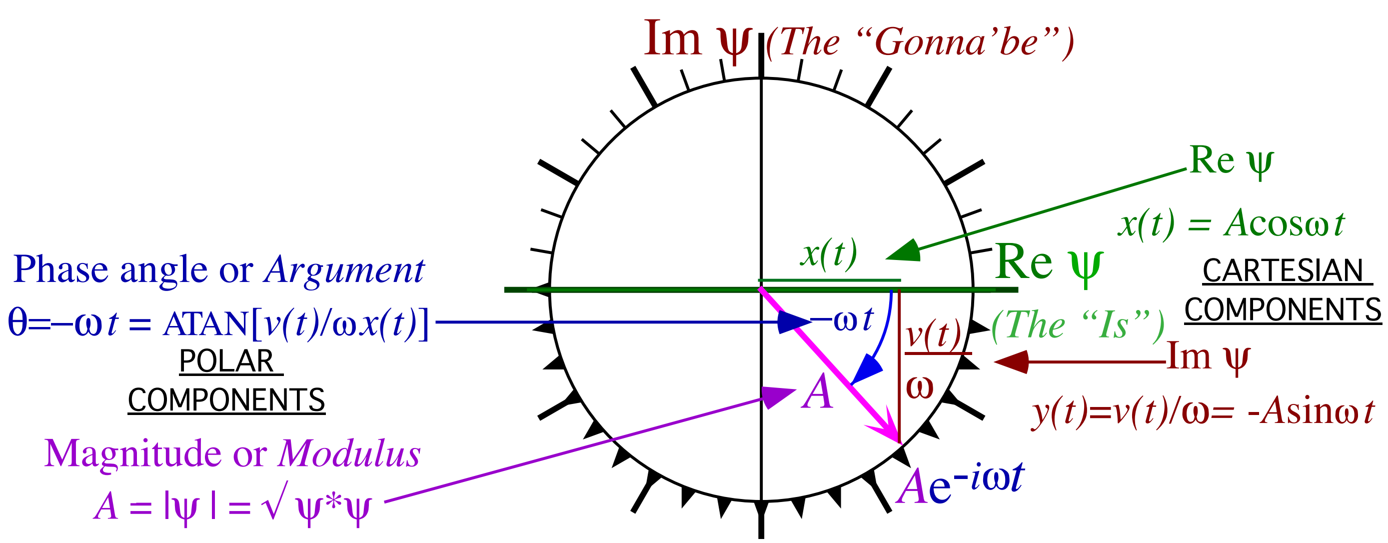

Unit 1
Fig. 10.5

Some Rect-vs-Polar *relations worth remembering*

Cartesian (x,y) form

$$\psi_x = \mathrm{Re}\,\psi(t) = x(t) = A\cos\omega t = \frac{\psi+\psi^*}{2}$$

$$\psi_y = \mathrm{Im}\,\psi(t) = \frac{v(t)}{\omega} = -A\sin\omega t = \frac{\psi-\psi^*}{2i}$$

$$\psi = re^{+i\theta} = re^{-i\omega t} = r(\cos\omega t - i\sin\omega t)$$

$$\psi^* = re^{-i\theta} = re^{+i\omega t} = r(\cos\omega t + i\sin\omega t)$$

Polar (r,θ) form

$$r = A = |\psi| = \sqrt{\psi_x^{\,2}+\psi_y^{\,2}} = \sqrt{\psi^*\psi}$$

$$\theta = -\omega t = \arctan(\psi_y/\psi_x)$$

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta}+e^{-i\theta}) \qquad \mathrm{Re}\,\psi = \frac{\psi+\psi^*}{2}$$

$$\sin\theta = \tfrac{1}{2i}(e^{+i\theta}-e^{-i\theta}) \qquad \mathrm{Im}\,\psi = \frac{\psi-\psi^*}{2i}$$

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

→ *Easy rotation and "dot" or "cross" products*

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

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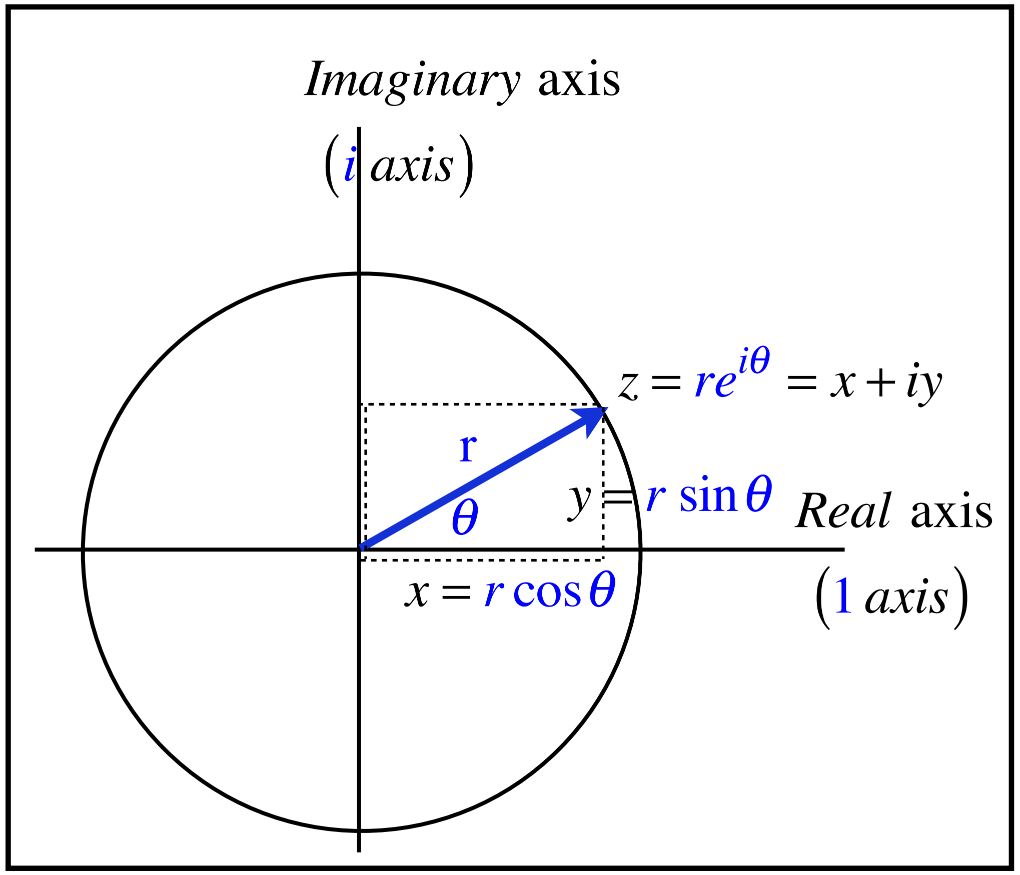
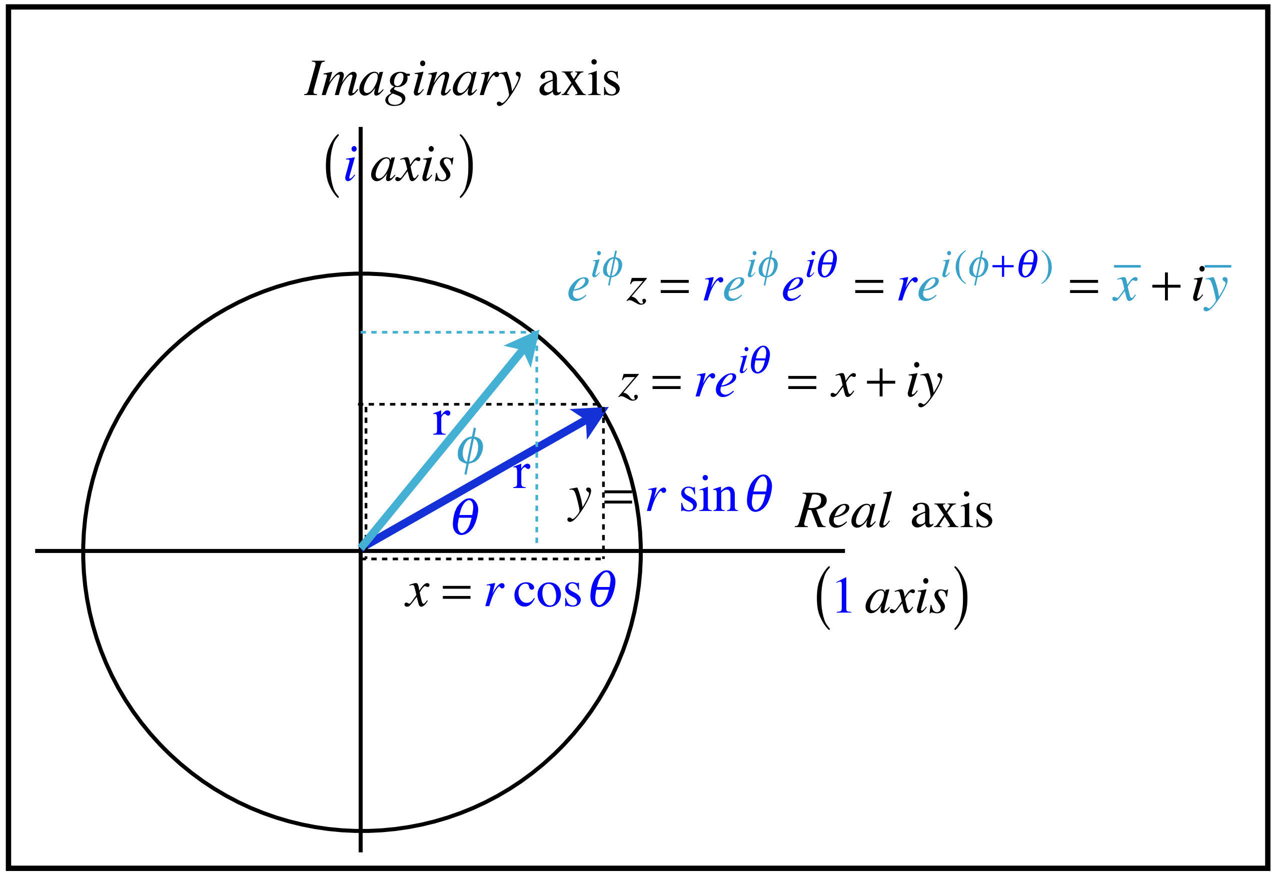
$e^{i\phi}$ acts on this: $z = re^{i\theta}$

to give this: $e^{i\phi}$ $e^{i\phi}z = re^{i\phi}e^{i\theta}$





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5. Complex products provide 2D “dot”(•) and “cross”(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their “star” (*)-product A^*B.

$$A^*B = (A_x + iA_y)^*(B_x + iB_y) = (A_x - iA_y)(B_x + iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

Real part is scalar or “dot”(•) product **A•B**.

Imaginary part is vector or “cross”(×) product, but just the Z-component *normal* to xy-plane.

Rewrite A^*B in polar form.

$$A^*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$

$$= |A||B|\cos(\theta_B - \theta_A) + i|A||B|\sin(\theta_B - \theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x+iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

5. Complex products provide 2D "dot"(•) and "cross"(x) products.

Two complex numbers $A = A_x + iA_y$ and $B = B_x + iB_y$ and their "star" (*)-product A^*B.

$$A^*B = (A_x + iA_y)^*(B_x + iB_y) = (A_x - iA_y)(B_x + iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

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$$\mathbf{A}\bullet\mathbf{B} = |A||B|\cos(\theta_B - \theta_A)$$
$$= |A|\cos\theta_A|B|\cos\theta_B + |A|\sin\theta_A|B|\sin\theta_B$$
$$= A_xB_x + A_yB_y$$

$$|\mathbf{A}\times\mathbf{B}| = |A||B|\sin(\theta_B - \theta_A)$$
$$= |A|\cos\theta_A|B|\sin\theta_B - |A|\sin\theta_A|B|\cos\theta_B$$
$$= A_xB_y - A_yB_x$$

What Good are complex variables?

Easy 2D vector calculus

→ *Easy 2D vector derivatives*

Easy 2D source-free field theory

Easy 2D vector field-potential theory

6. Complex derivative contains "divergence"($\nabla \cdot \mathbf{F}$) and "curl"($\nabla \times \mathbf{F}$) of 2D vector field

Relation of (z,z^*) to $(x=\mathrm{Re}z, y=\mathrm{Im}z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

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7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*. Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\frac{df^*}{dz} = 0$.

6. Complex derivative contains "divergence"(∇•F) and "curl"(∇xF) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

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For example: if *f(z)=a·z* then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ has *zero divergence*: $\nabla\bullet\mathbf{F}=0$ and has *zero curl*: $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]

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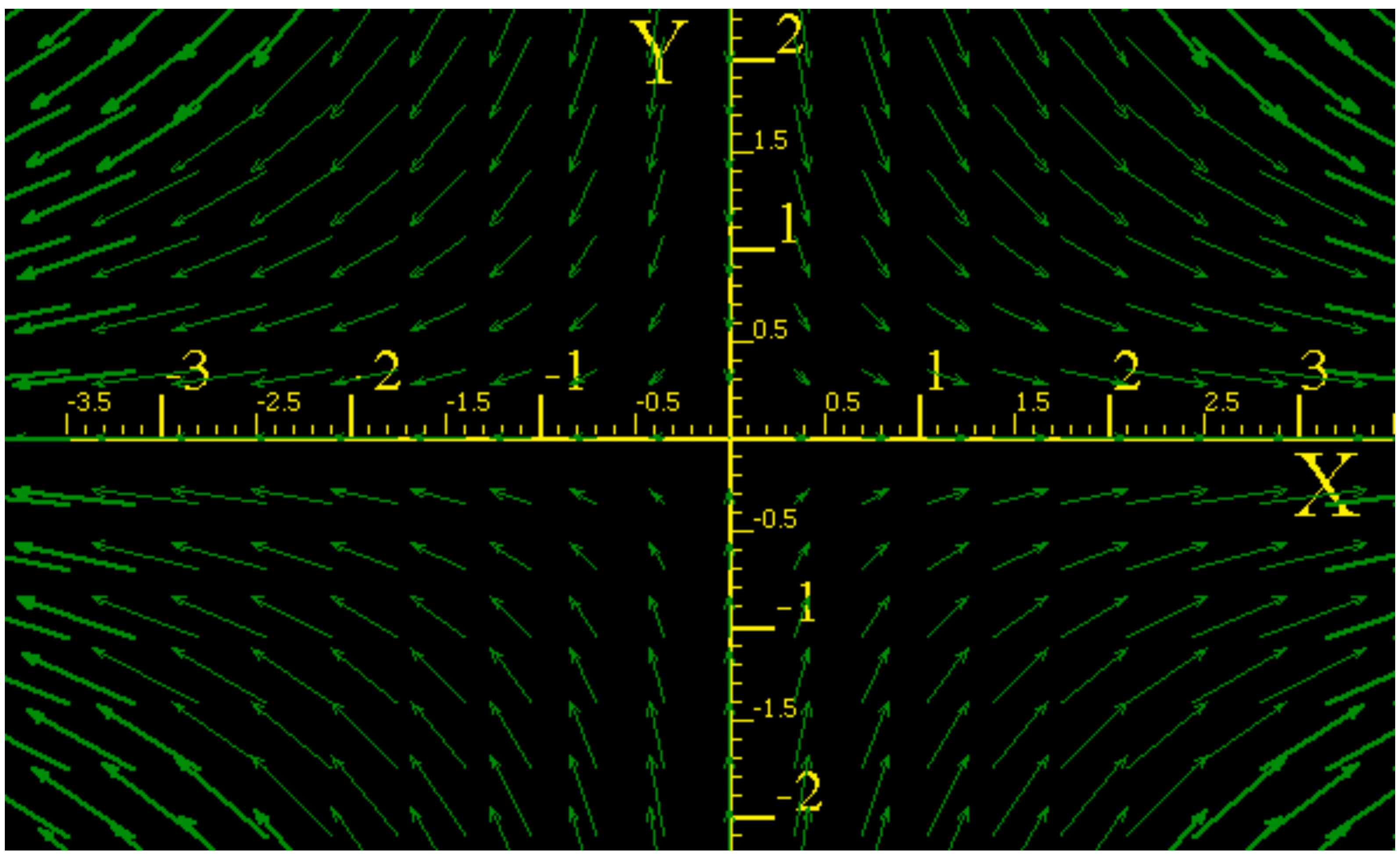

precursor to Unit 1 Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a \cdot x,-a \cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

8. Complex potential ϕ contains "scalar"(**F**=$\nabla\Phi$) and "vector"(**F**=∇x**A**) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* $\boldsymbol{\Phi}$ or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\boldsymbol{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\boldsymbol{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\boldsymbol{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\text{Re}\,\phi=\mathbf{\Phi}$) and imaginary ($\text{Im}\,\phi=\mathbf{A}$) parts.

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$$f(z)=\frac{d\phi}{dz} \quad\Rightarrow\quad \phi=\quad \Phi \quad +i\ \mathbf{A}=\int f\cdot dz=\int az\cdot dz=\frac{1}{2}az^2$$

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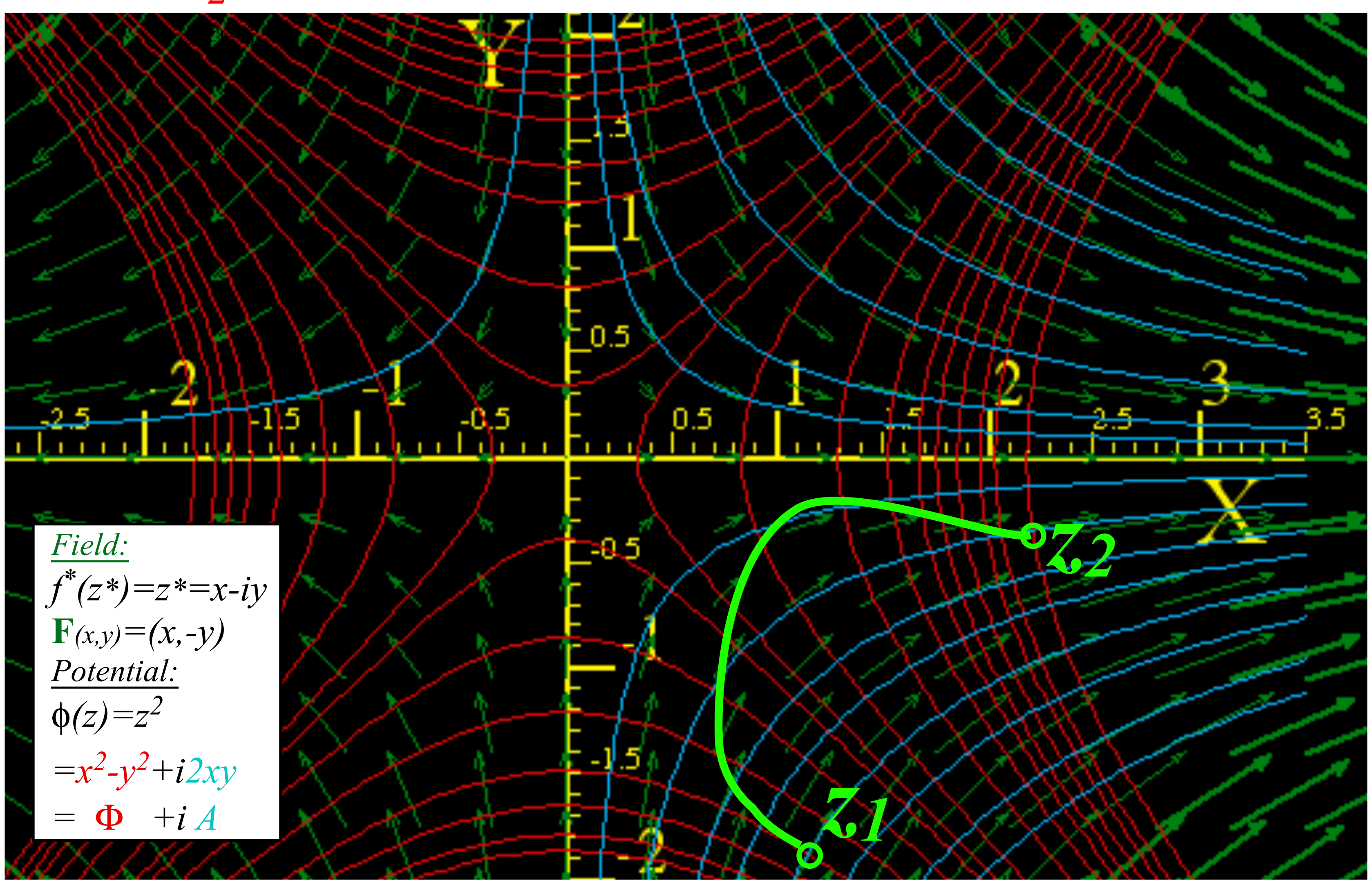
Unit 1
Fig. 10.7



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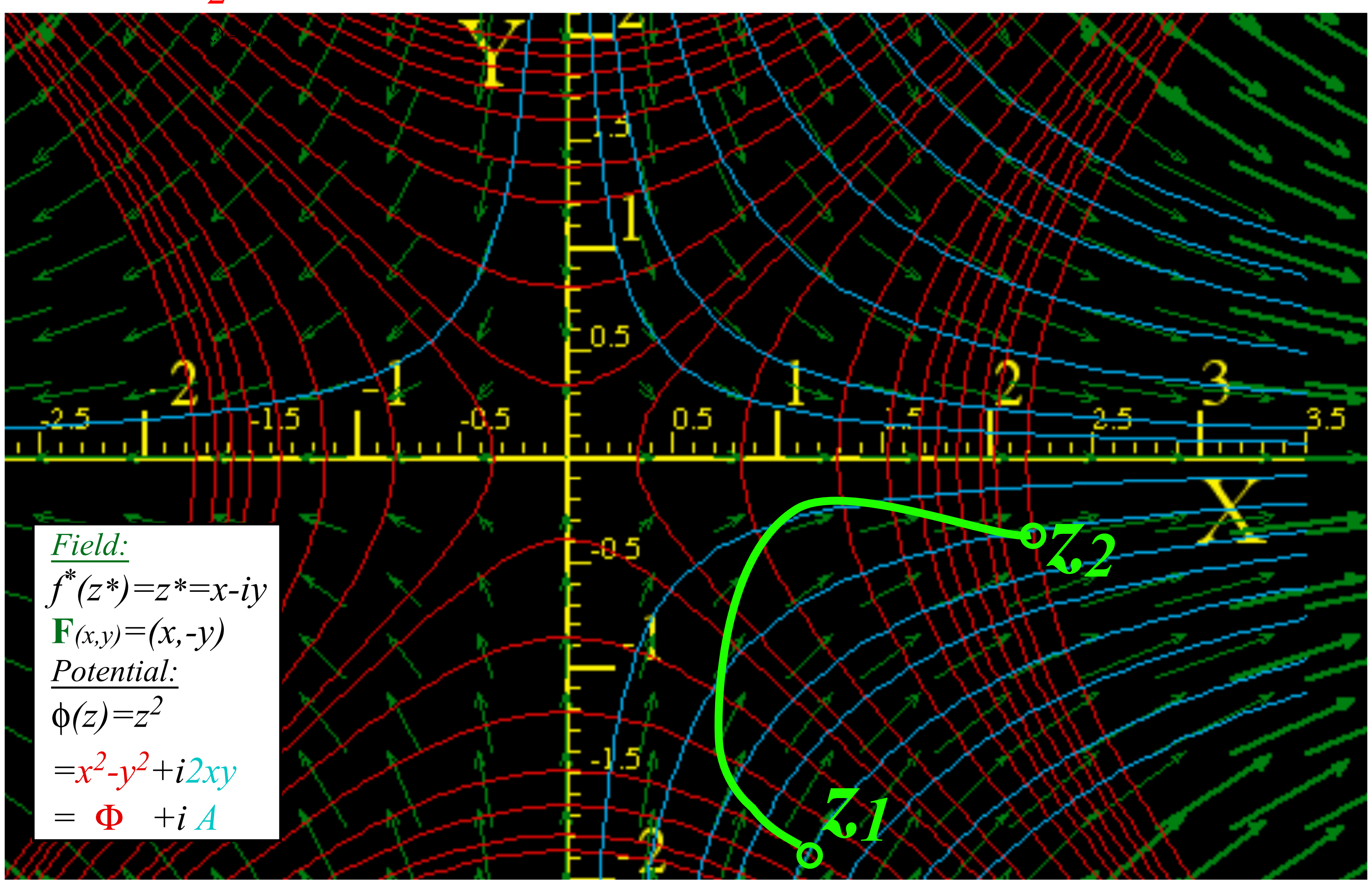
$$=\frac{1}{2}a(x^2-y^2)+i\,axy$$

BONUS! Get a free coordinate system!



Unit 1 Fig. 10.7

The (Φ,**A**) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=$ *const.*

$q^2=\mathbf{A}=(xy)=$ *const.*

*Actually it's OCC.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

→ *The half-n'-half results: (Riemann-Cauchy Derivative Relations)*

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**$\nabla\times$**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial y} \end{pmatrix}$ of vector **A** (*and they're <u>equal</u>!*)

$f(z) = \frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right) = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

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Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

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$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

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Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

Given ϕ:

$$\phi = \Phi + i\,A = \tfrac{1}{2}a(x^2-y^2) + i\,axy$$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

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$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\tfrac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\tfrac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\tfrac{1}{2}\left(\frac{\partial A}{\partial y}-i\frac{\partial A}{\partial x}\right)=\tfrac{1}{2}\nabla\Phi+\tfrac{1}{2}\nabla\times\mathbf{A}$$

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The half-n'-half result

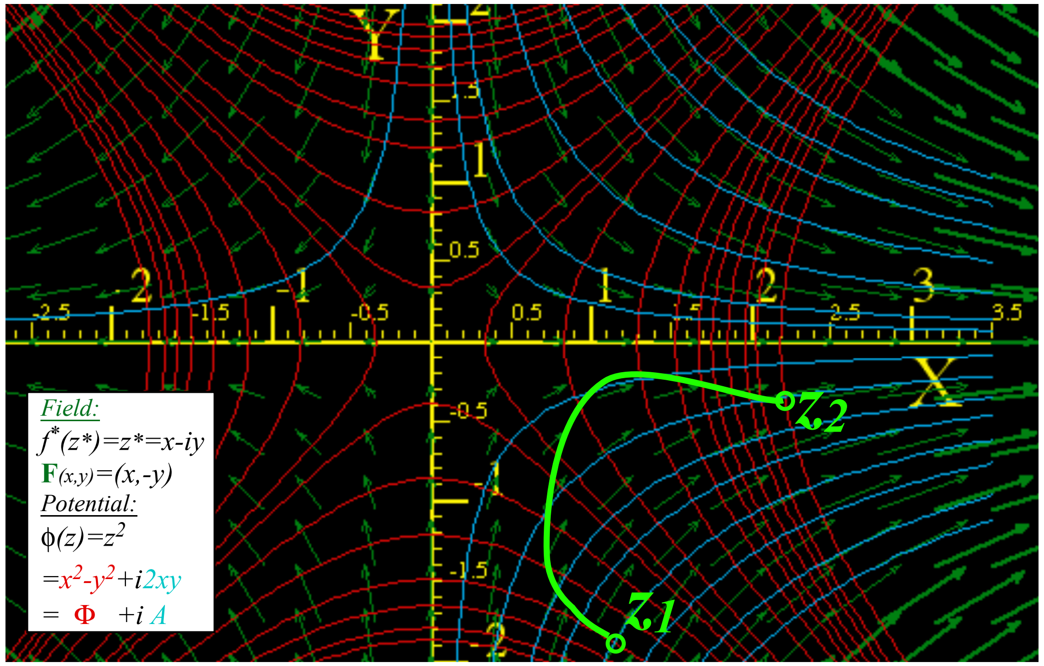
find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial A}{\partial y}\\ -\frac{\partial A}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial \mathbf{A}}{\partial y}\\ -\frac{\partial \mathbf{A}}{\partial x}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

Given ϕ: $\phi=\Phi+i\mathbf{A}=\frac{1}{2}a(x^2-y^2)+i\,axy$

The half-n'-half result

find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

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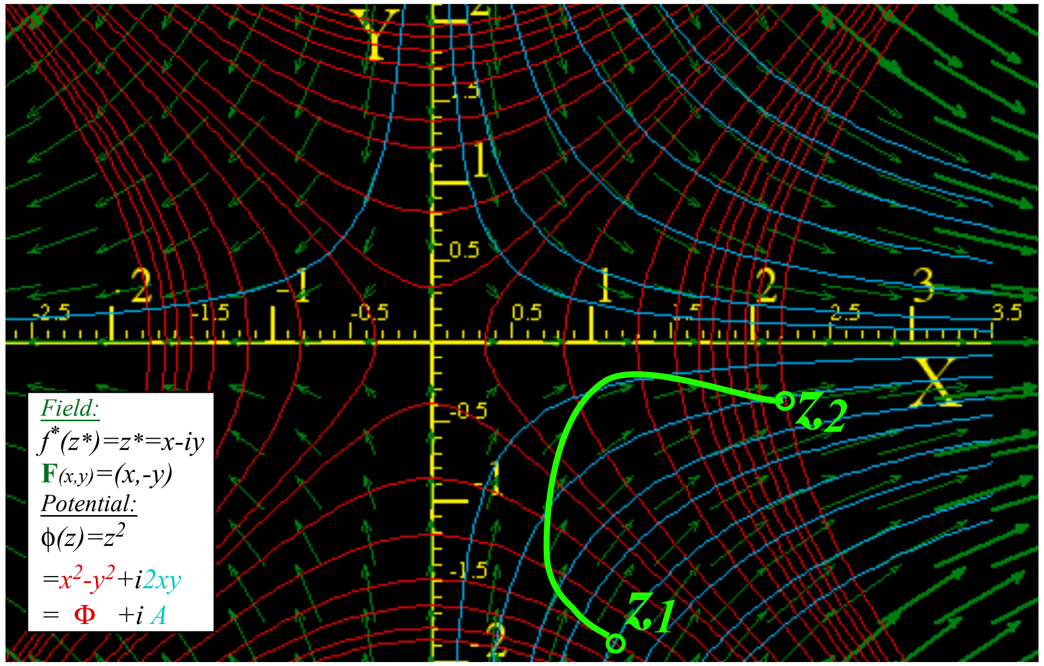

The half-n'-half results are called Riemann-Cauchy Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

→ *4. Riemann-Cauchy conditions* *What's analytic? (...and what's <u>not</u>?)*

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>not</u> be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies: } \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \text{ and: } \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

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First, f(z) must <u>not</u> be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies : } \frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \text{ and : } \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's <u>not</u>?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z = z + iy$?

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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$$= \quad z+z^* \quad + \quad (2z-2z^*)$$

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$$
\begin{aligned}
f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
&= z+z^* + (2z-2z^*) \\
&= 3z-z^*
\end{aligned}
$$

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$$= \quad 3z-z^*$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=z+iy$?

A: *NO!* *$r(xy)=z^*z$ is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

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Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=z+iy$?

A: *YES!* *$s(xy)=(x+iy)^2 =z^2$ is analytic function of z.* (Yay!)

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

→ *Easy 2D circulation and flux integrals*

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F•dr**) and "flux"(∫**Fxdr**)*

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2, y_2) - \mathbf{\Phi}(x_1, y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

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$$\int f(z)dz = \int\left(f^*(z^*)\right)^* dz = \int\left(f^*(z^*)\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* + i\,f_y^*\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int(f_x^* dx + f_y^* dy) + i\int(f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

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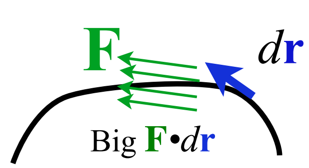
$$= \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

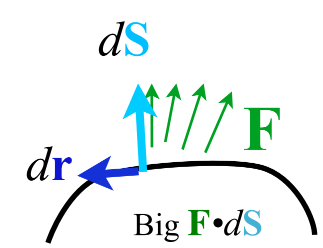
$$= \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \bullet d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

$$= \boxed{\int \mathbf{F} \bullet d\mathbf{r}} \quad + i \boxed{\int \mathbf{F} \bullet d\mathbf{S}}$$

where: $d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z$



Real part $\int_1^2 \mathbf{F} \bullet d\mathbf{r} = \Delta\mathbf{\Phi}$
sums **F** projections *along* path
*d***r** that is, *circulation* on path
to get $\Delta\mathbf{\Phi}$.

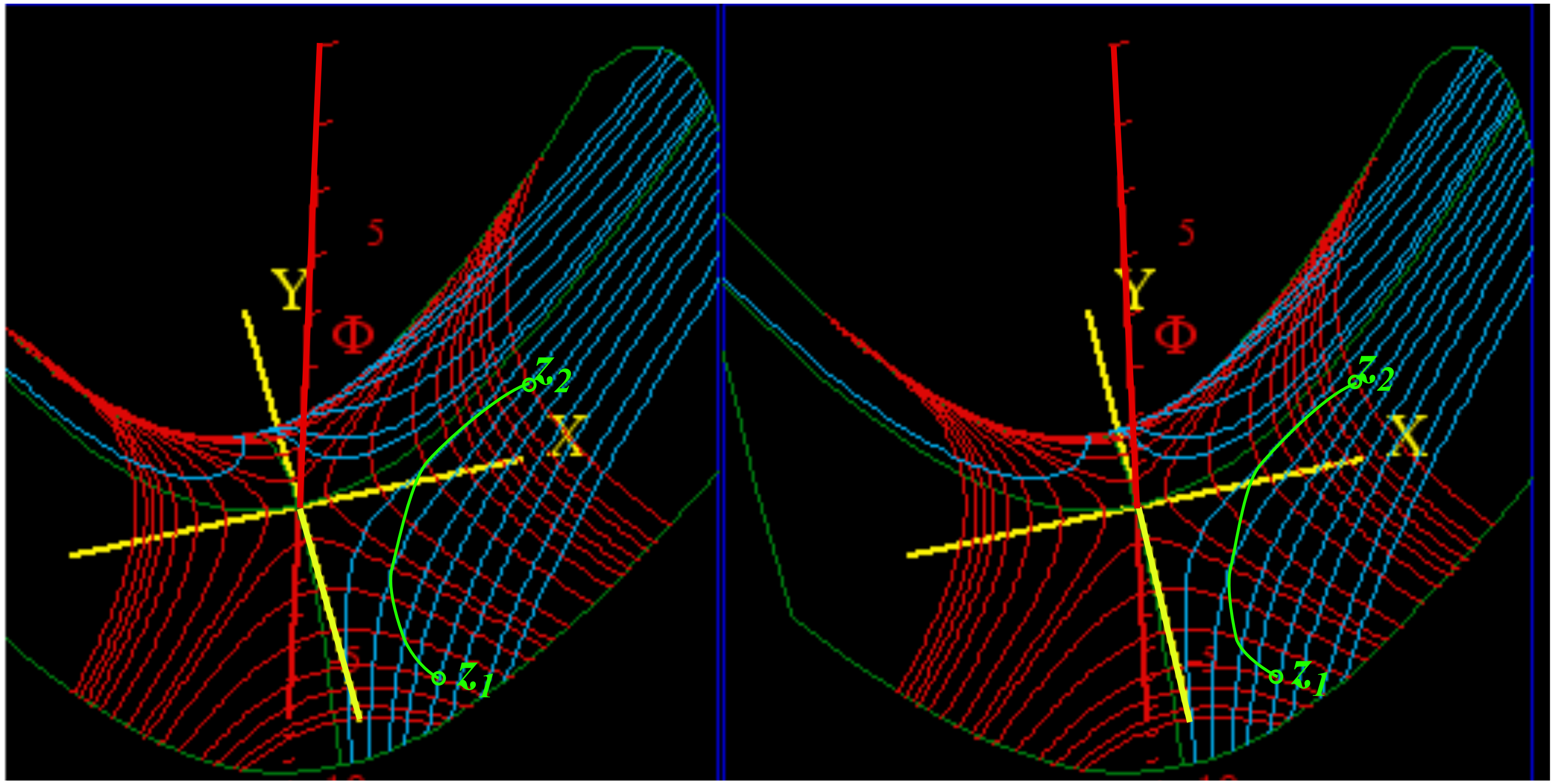


Imaginary part $\int_1^2 \mathbf{F} \bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path *d***r**
that is, *flux* thru surface
elements $d\mathbf{S} = d\mathbf{r} \times \mathbf{e}_Z$ normal to *d***r**
to get $\Delta\mathbf{A}$.

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



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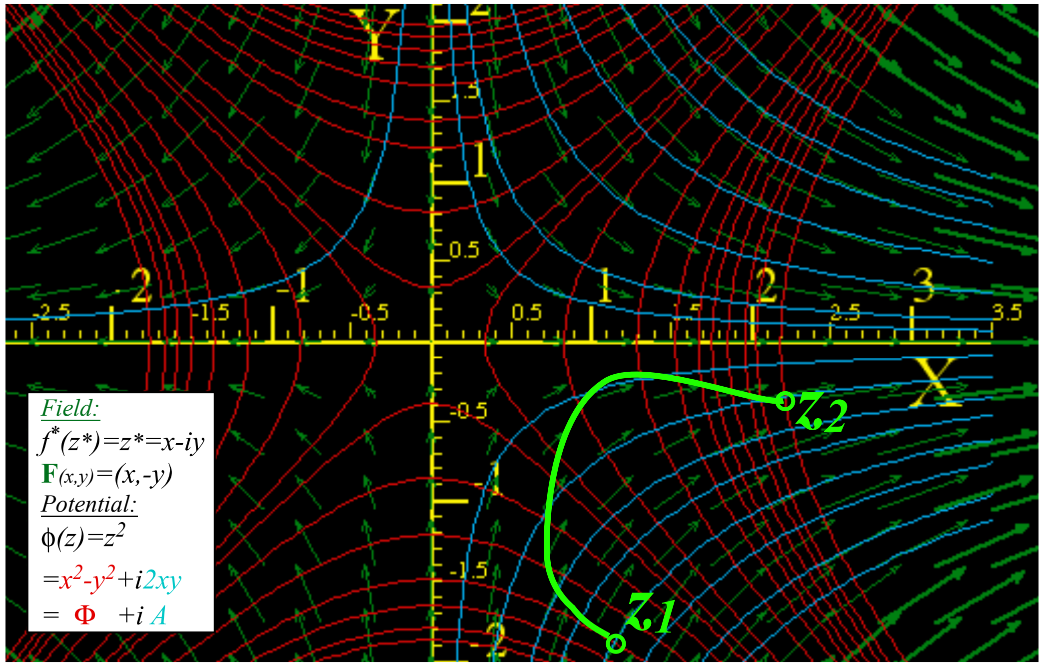
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2+y^2$$

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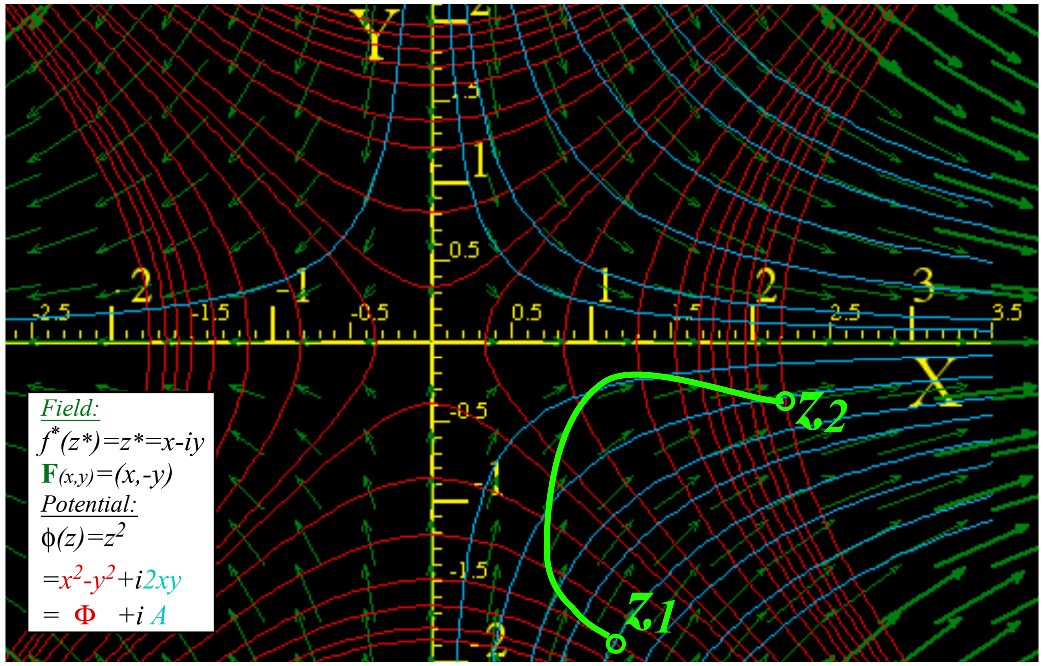


$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

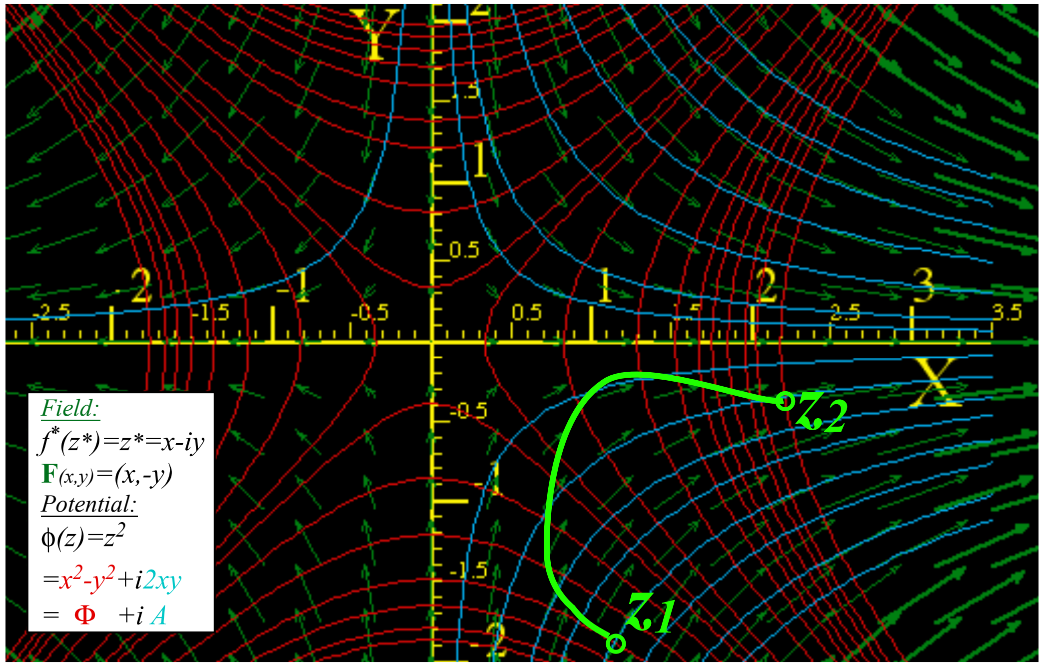
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



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or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$

and so does **A**

potential Φ *obeys Laplace equation*

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}= z^{-1}$ $f(z)=\frac{a}{z}= az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z)= \quad \Phi \quad + \quad i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z)$$

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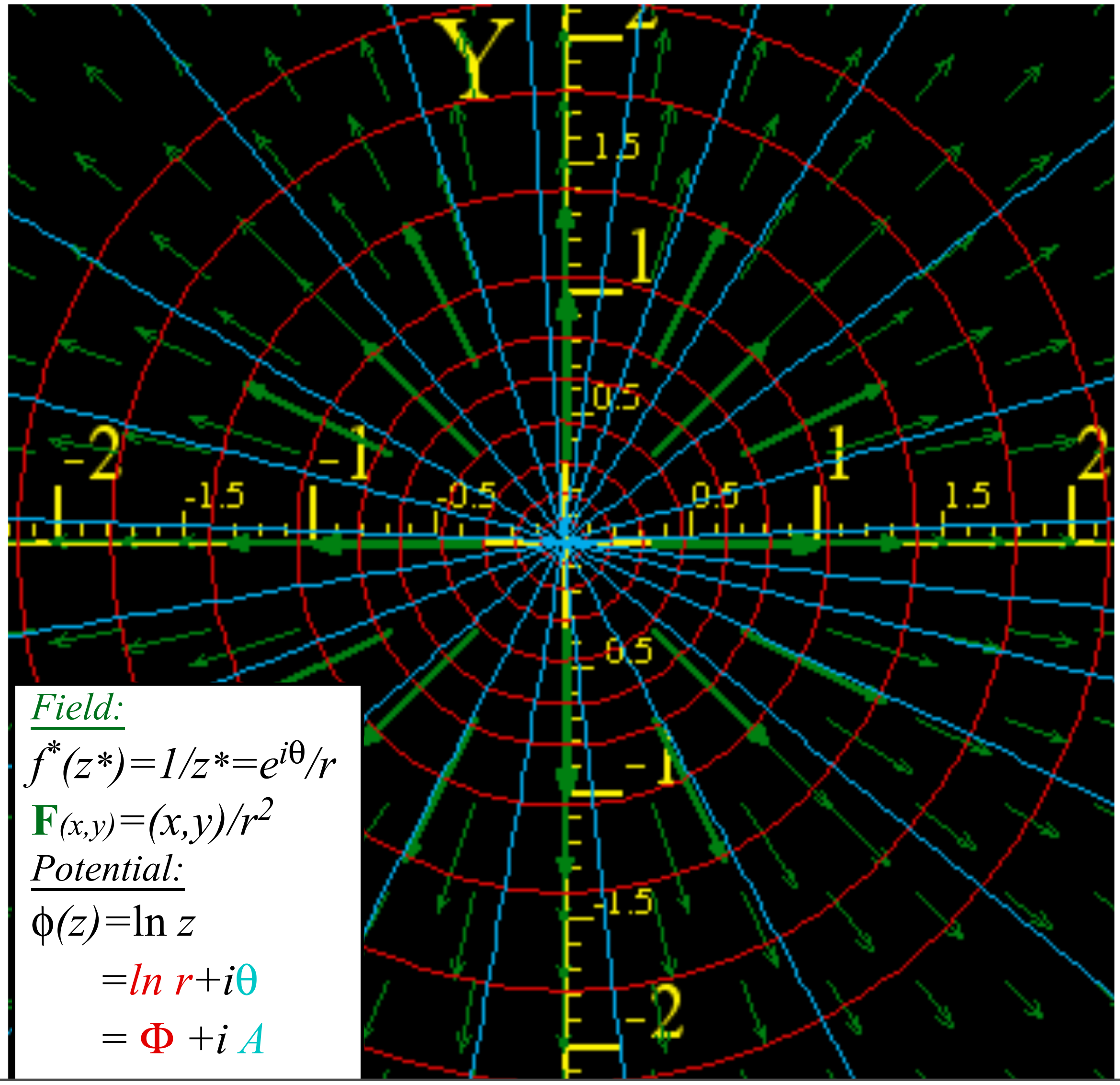
$$=\overbrace{a\ln(r)}+i\overbrace{a\theta}$$

(a) Unit Z-line-flux field $f(z)=1/z$



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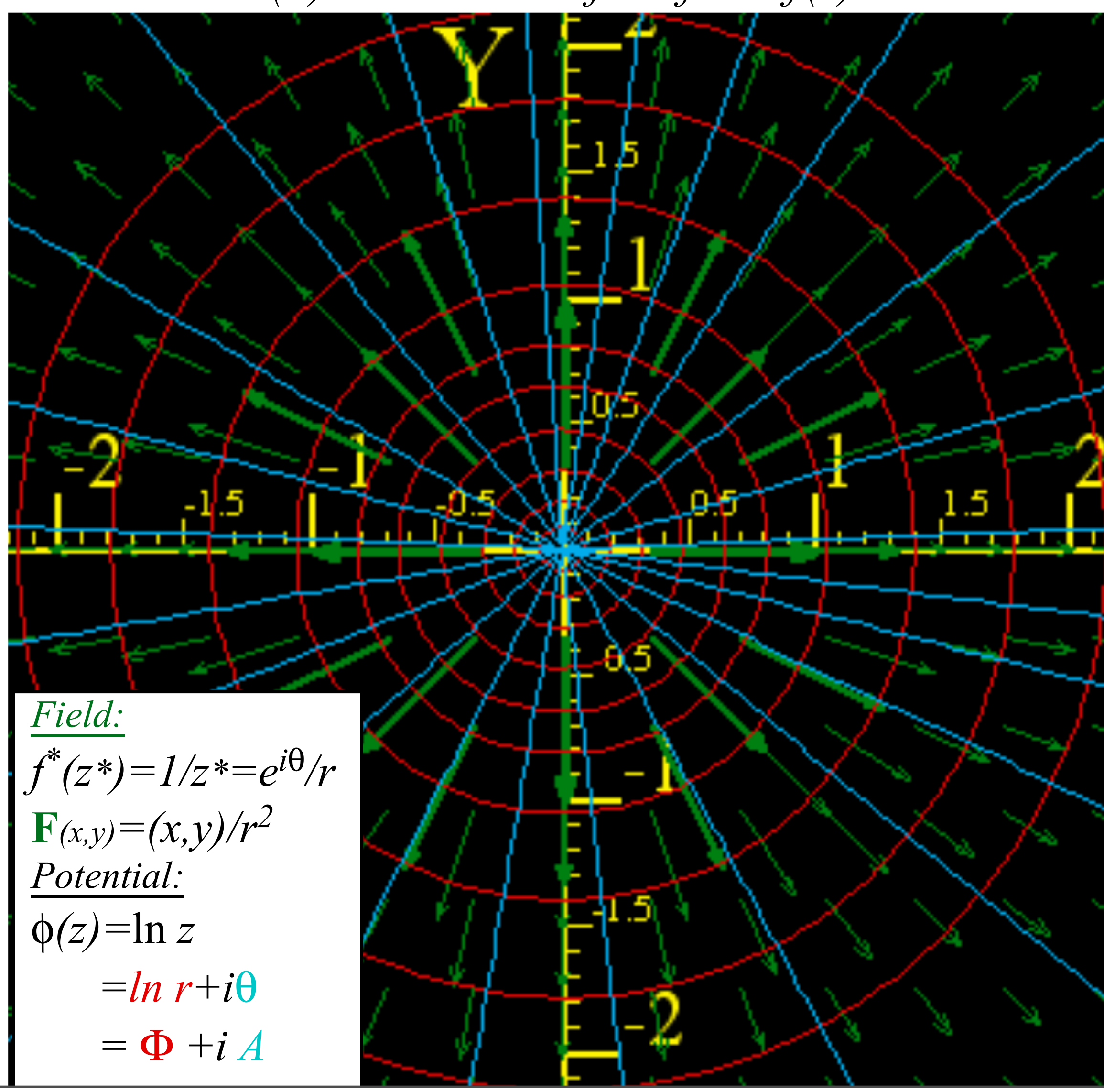
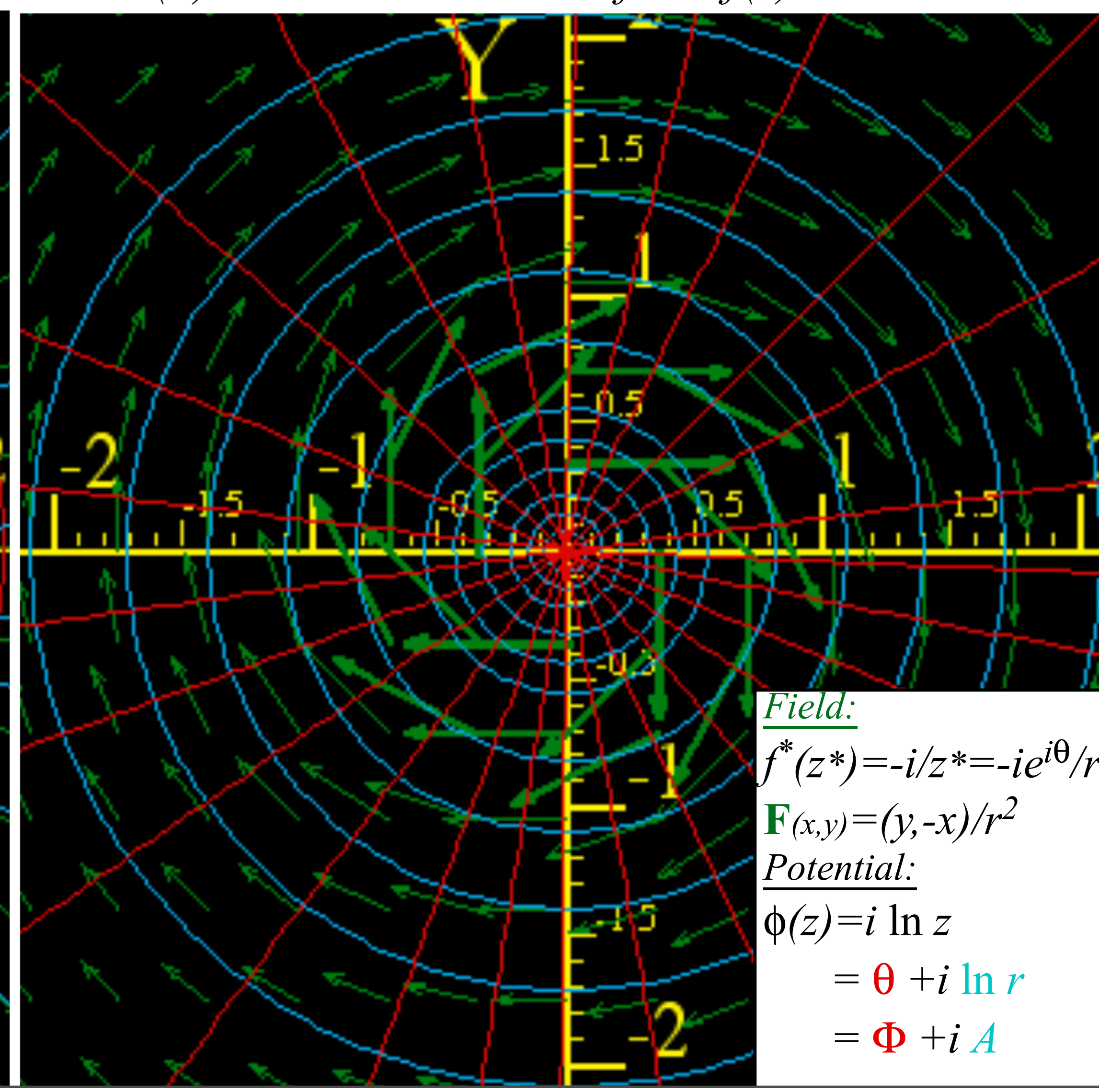


(b) Unit Z-line-vortex field f(z)=i/z



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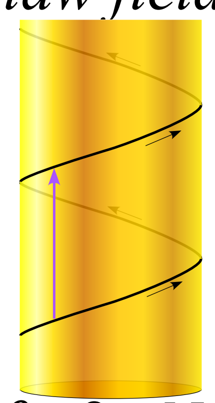
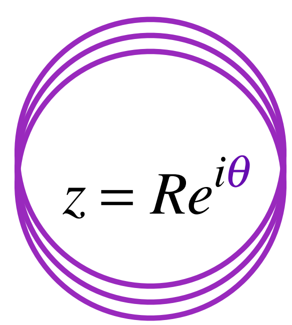
A monopole field is the only power-law field whose integral (potential) depends on path of integration.

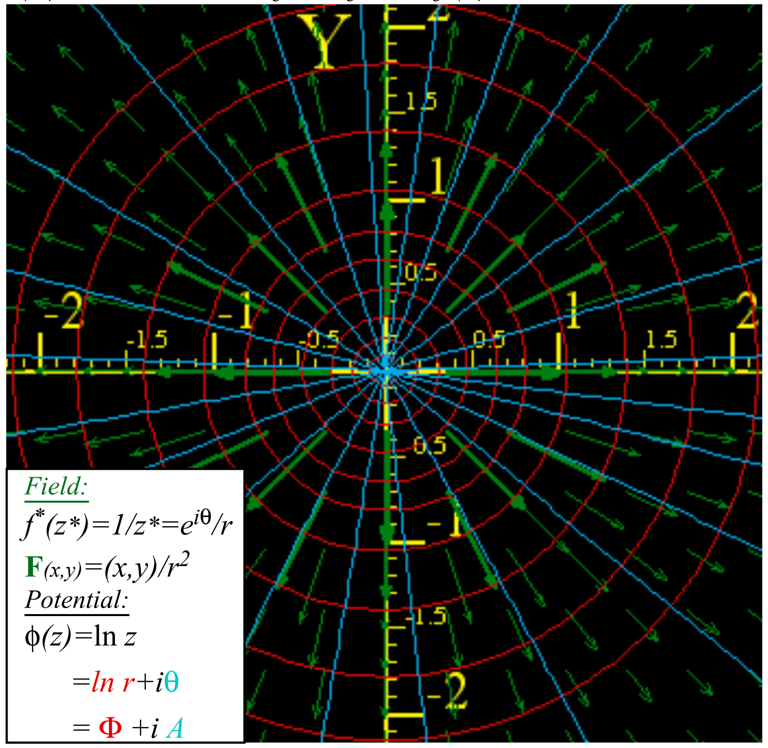




path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}id\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

(a) Unit Z-line-flux field f(z)=1/z



$$\phi(z) = \underbrace{\Phi}_{\ln(r)} + i\underbrace{\mathbf{A}}_{\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta})$$

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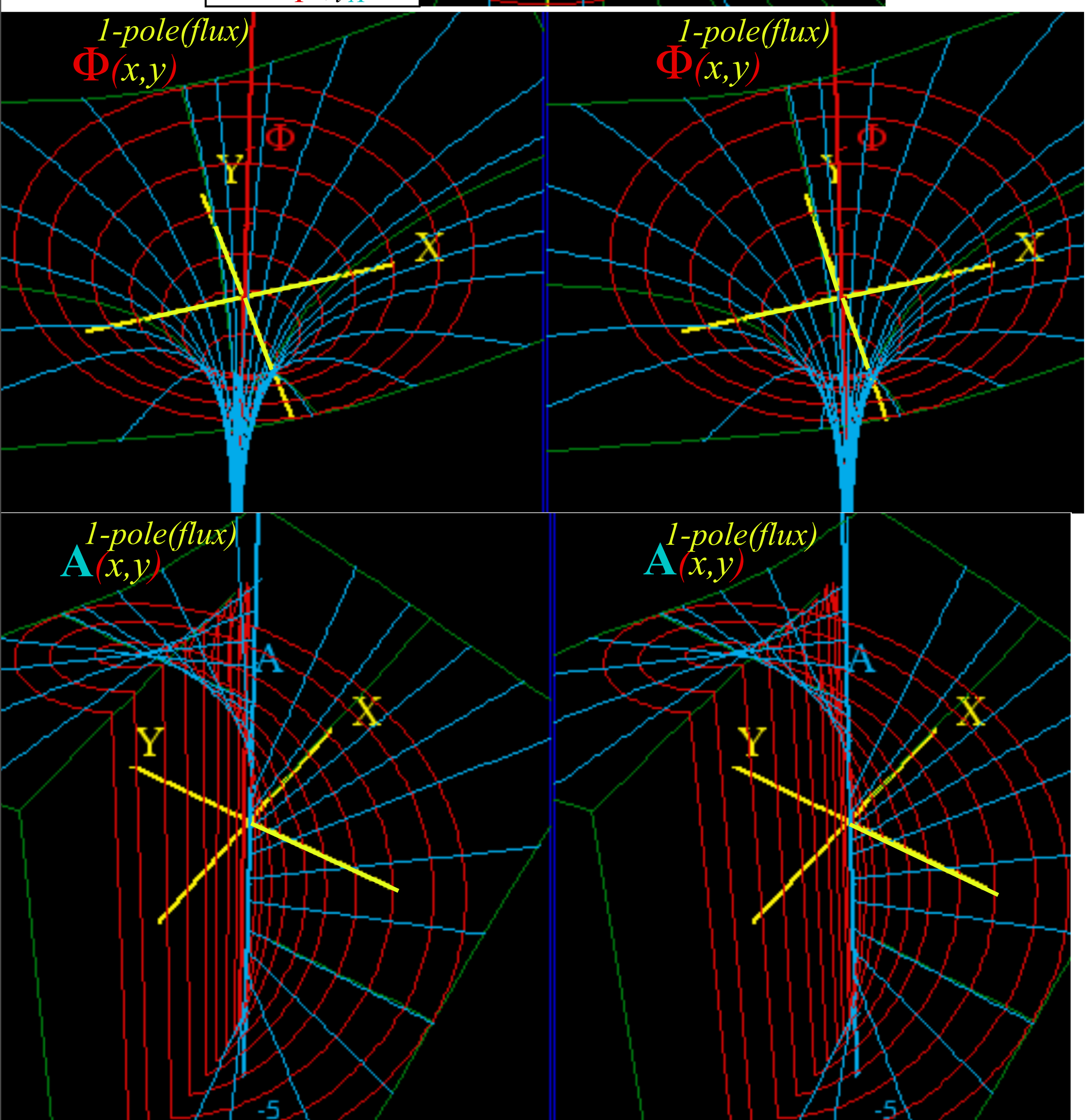
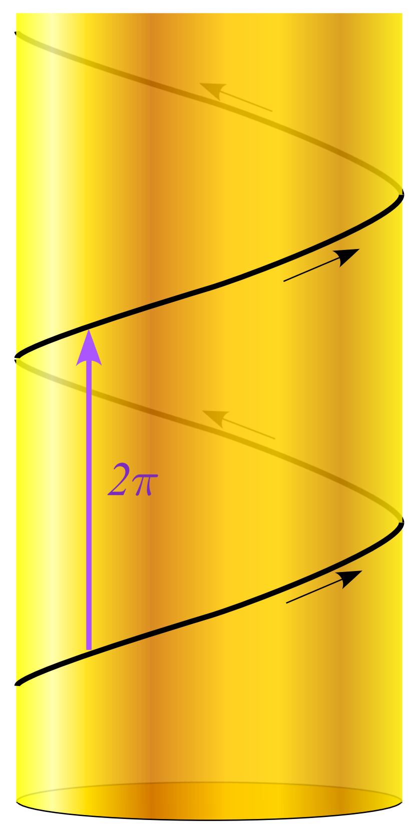


*Each turn around origin adds 2πi to vector potential i***A**



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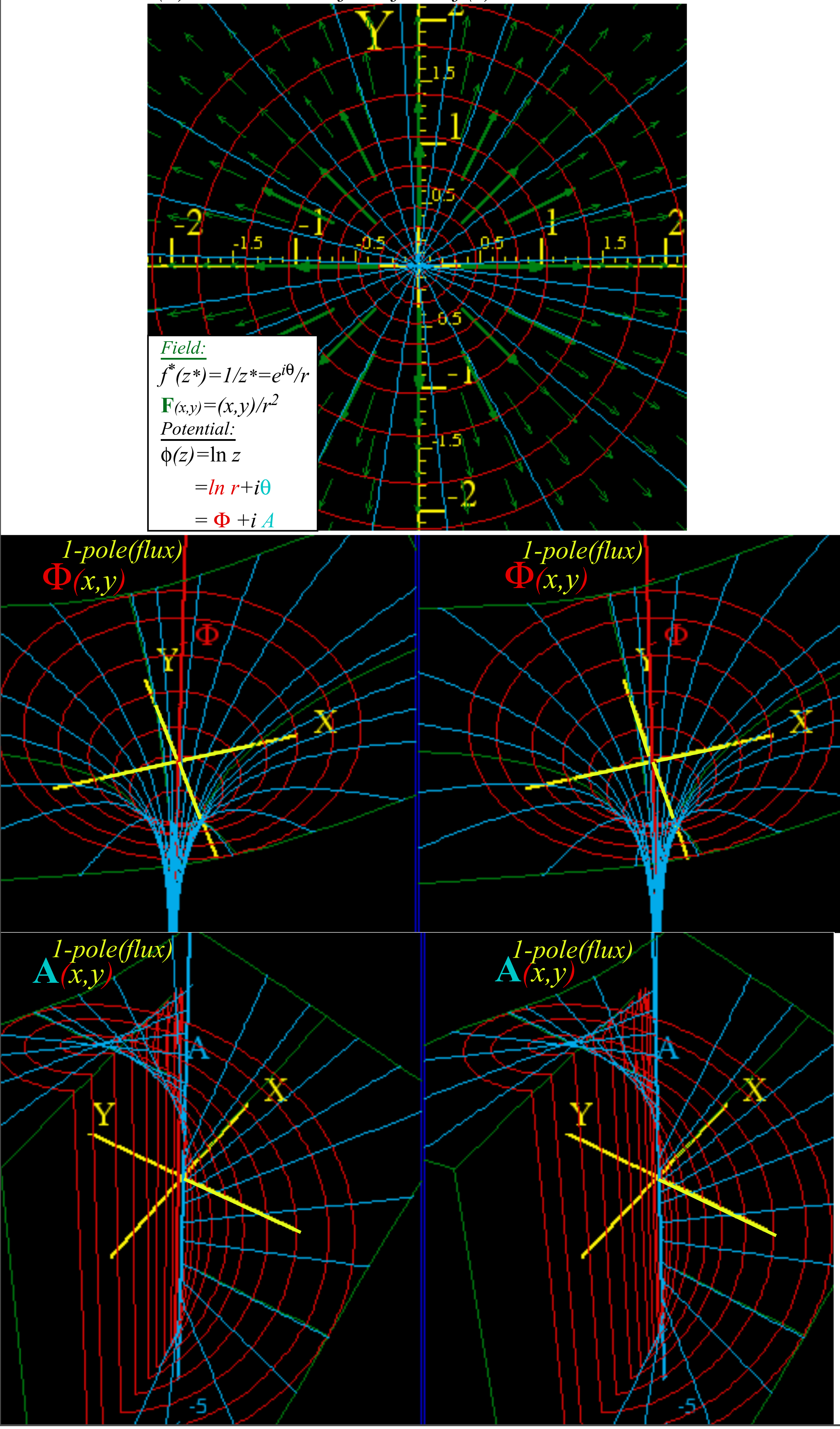

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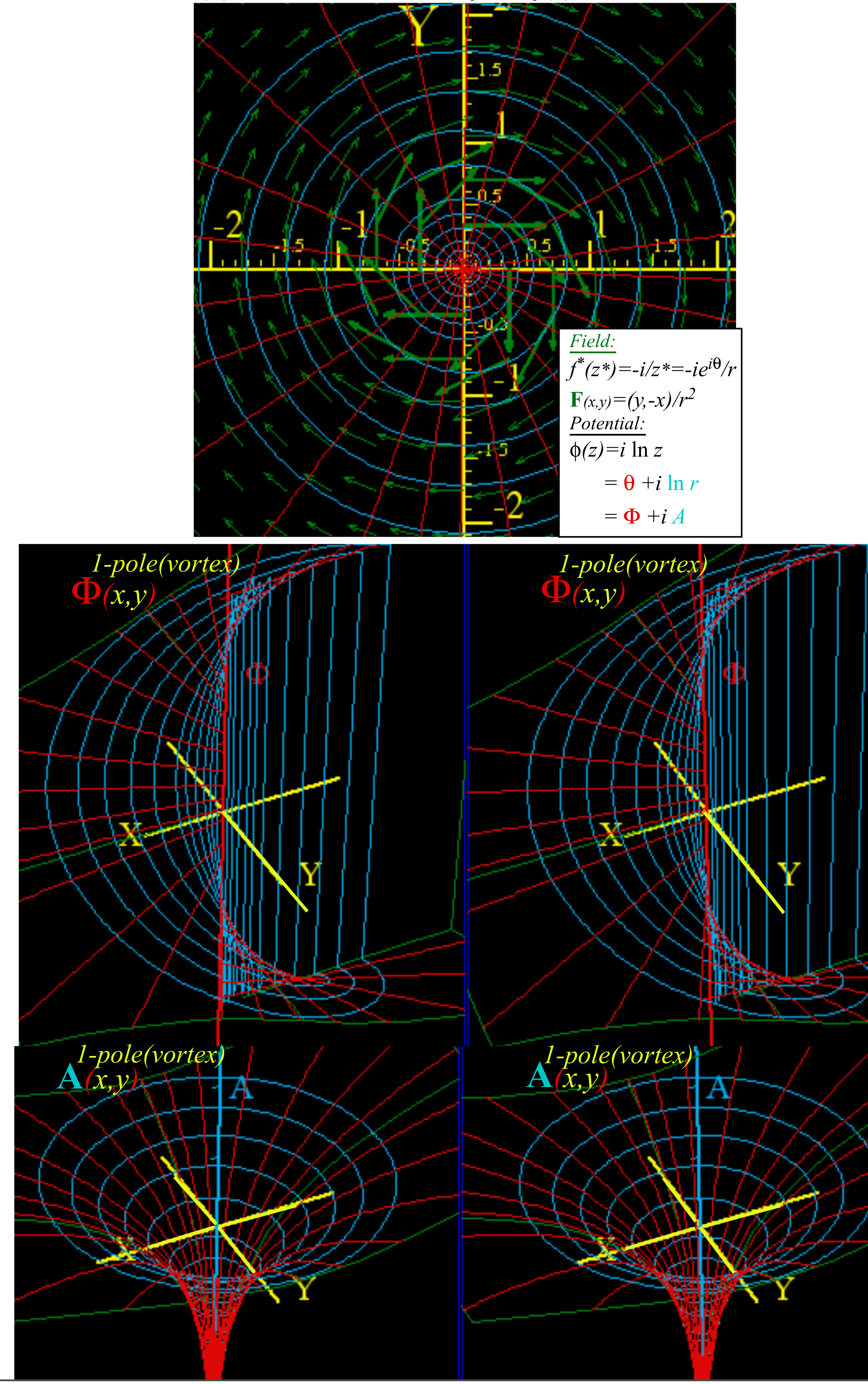
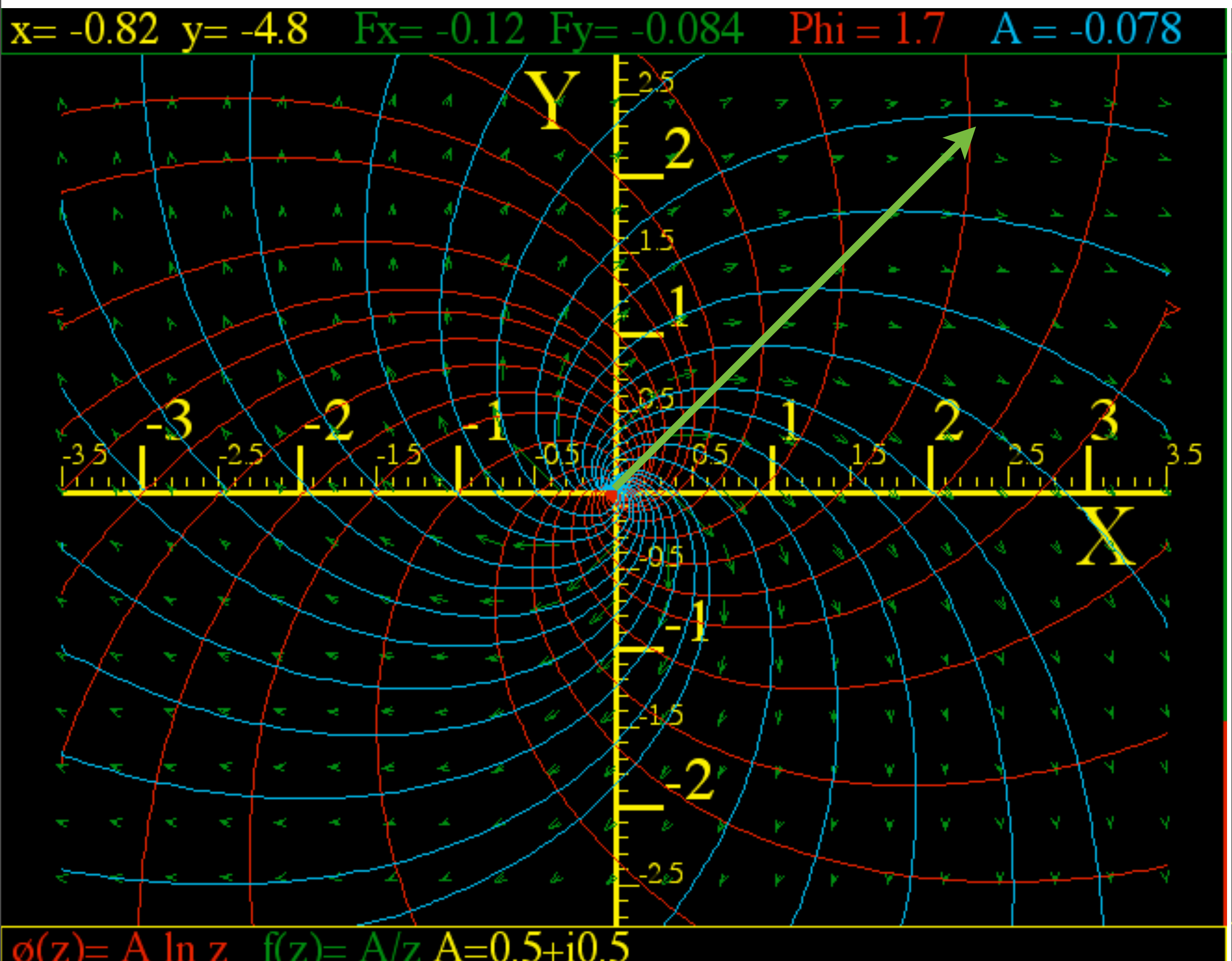
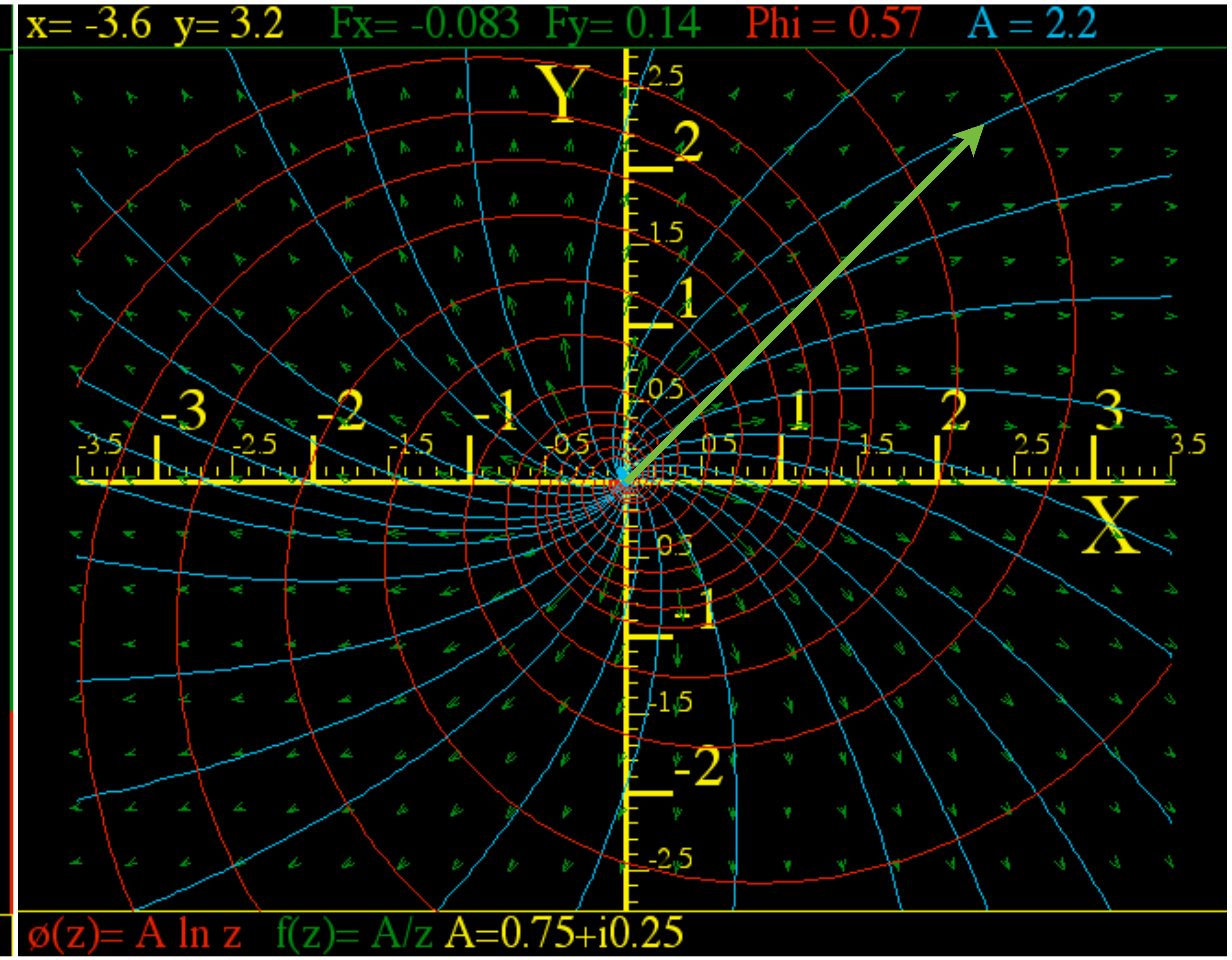


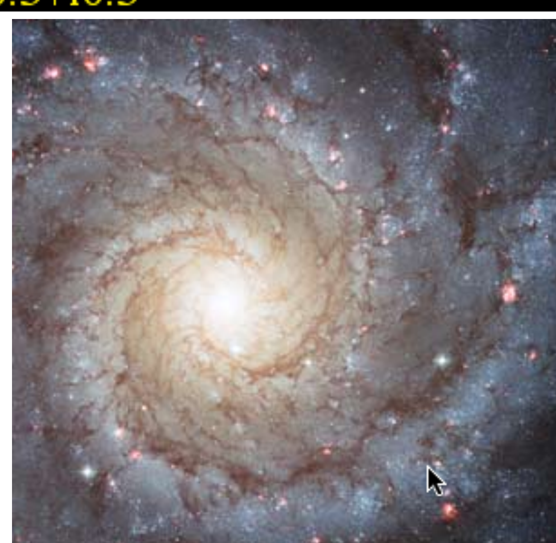
What Good Are Complex Exponentials? (contd.)

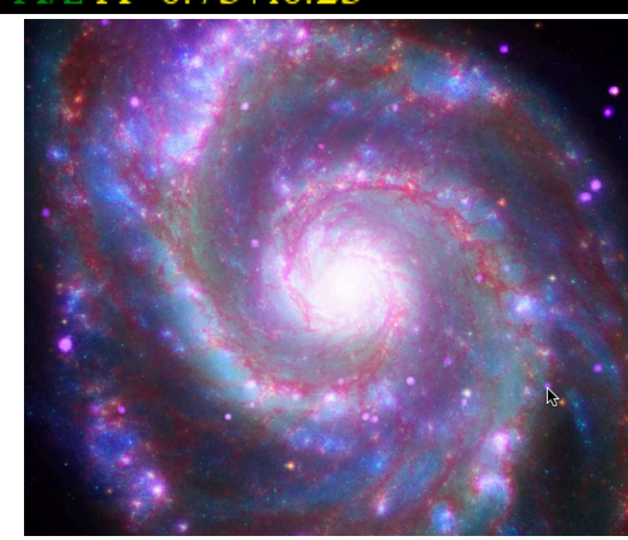
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$ $\qquad$ $f(z) = (0.75 + i0.25)/z = e^{i18^\circ}/z\sqrt{n}$

"Vortex" "Hurricane"









4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

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Easy stereo-projection visualization

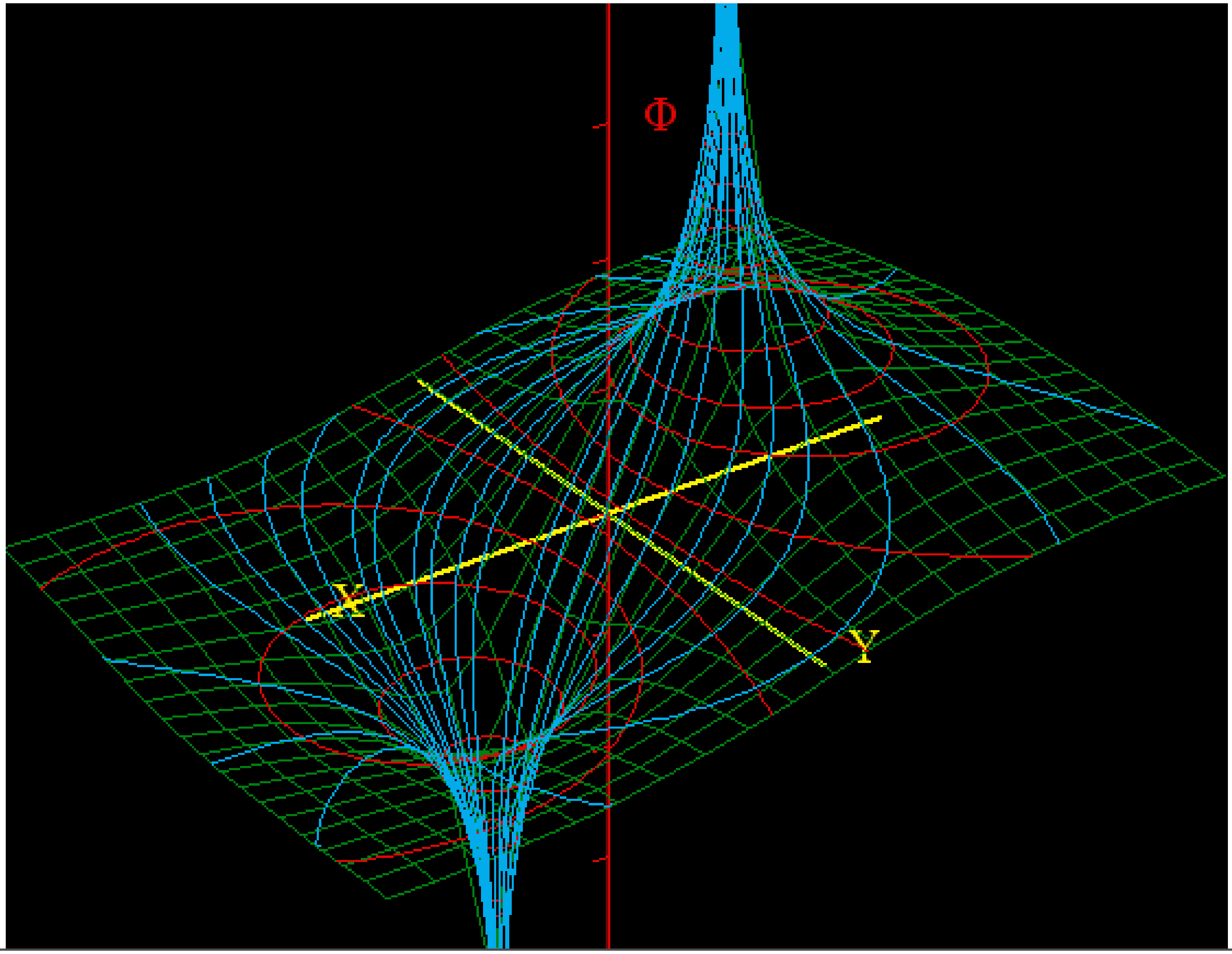
12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$



So-called "physical dipole" has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

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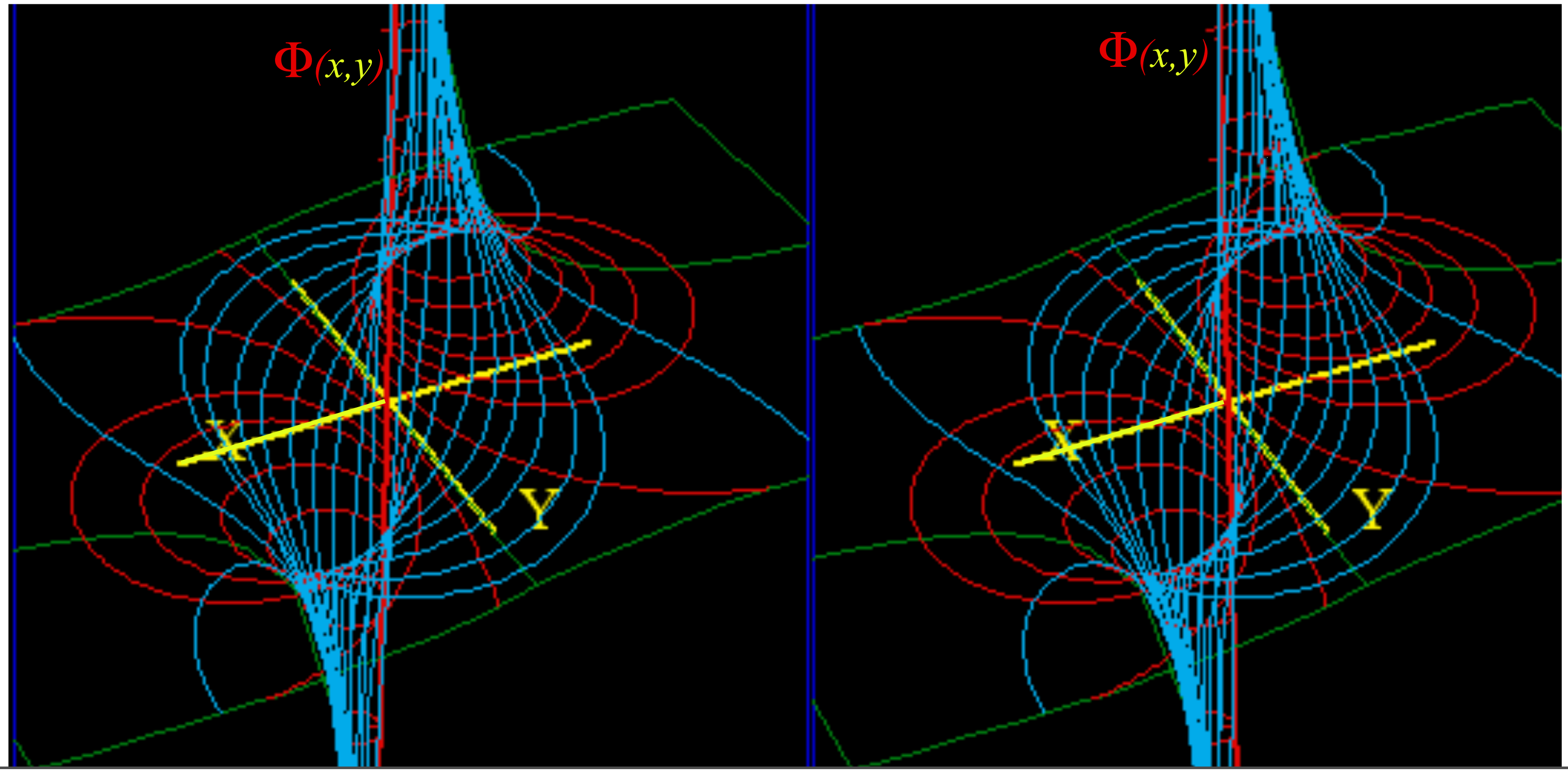

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

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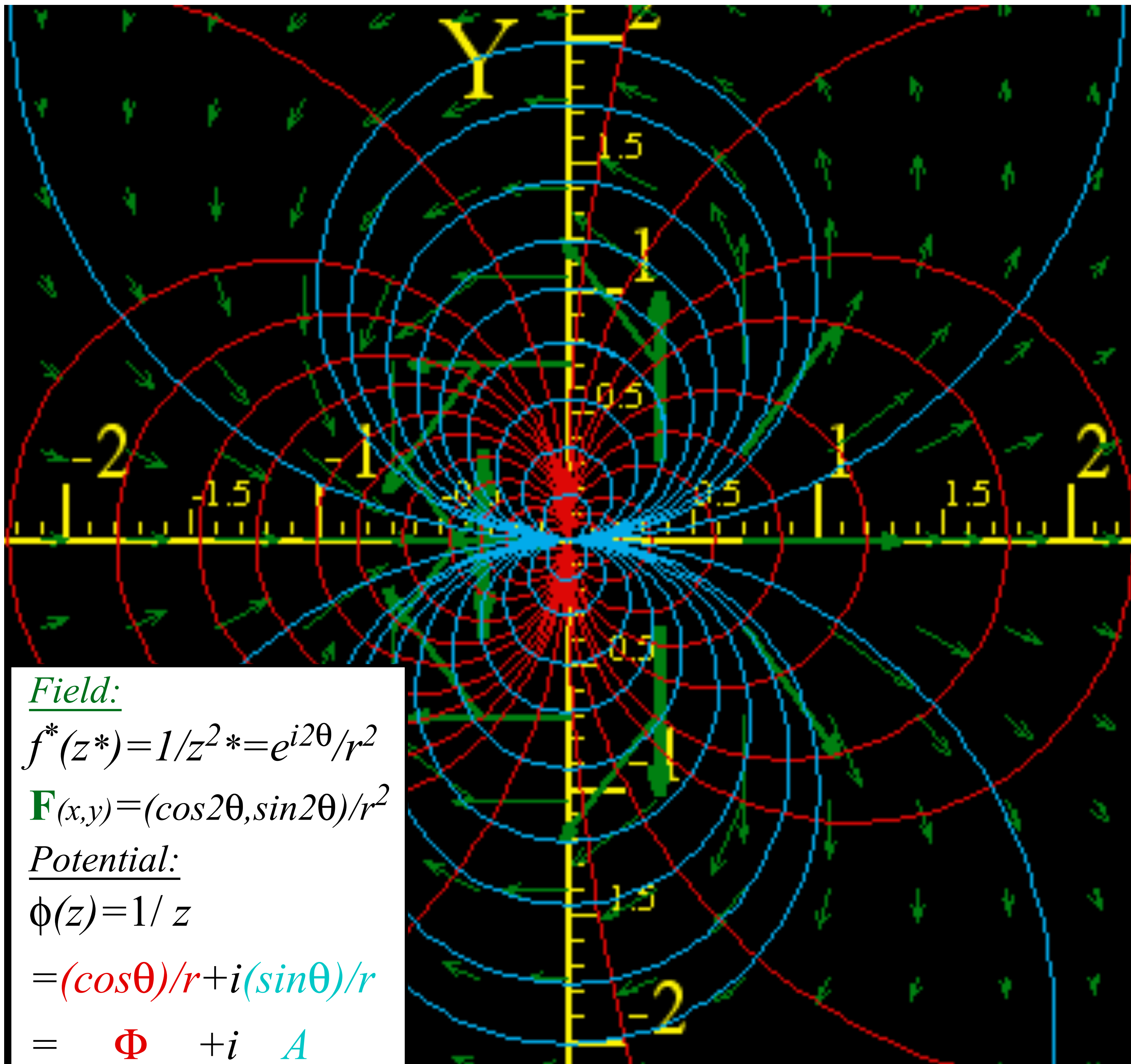
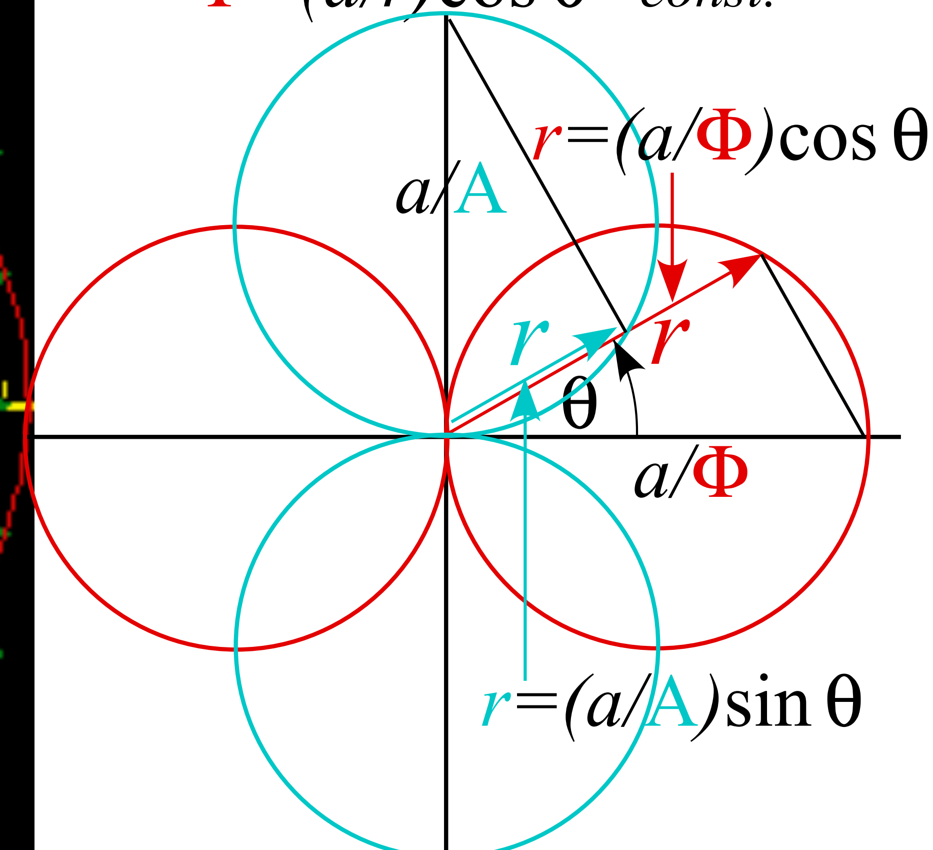


Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$A=(a/r)\sin\theta=const.$

2^n-pole analysis (quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, ..., pole dancer,

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{\text{4-pole}} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{\text{2-pole}}}{dz} = \frac{d\phi^{\text{4-pole}}}{dz}$$

$$\phi^{\text{4-pole}} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{\text{2-pole}}}{dz}$$

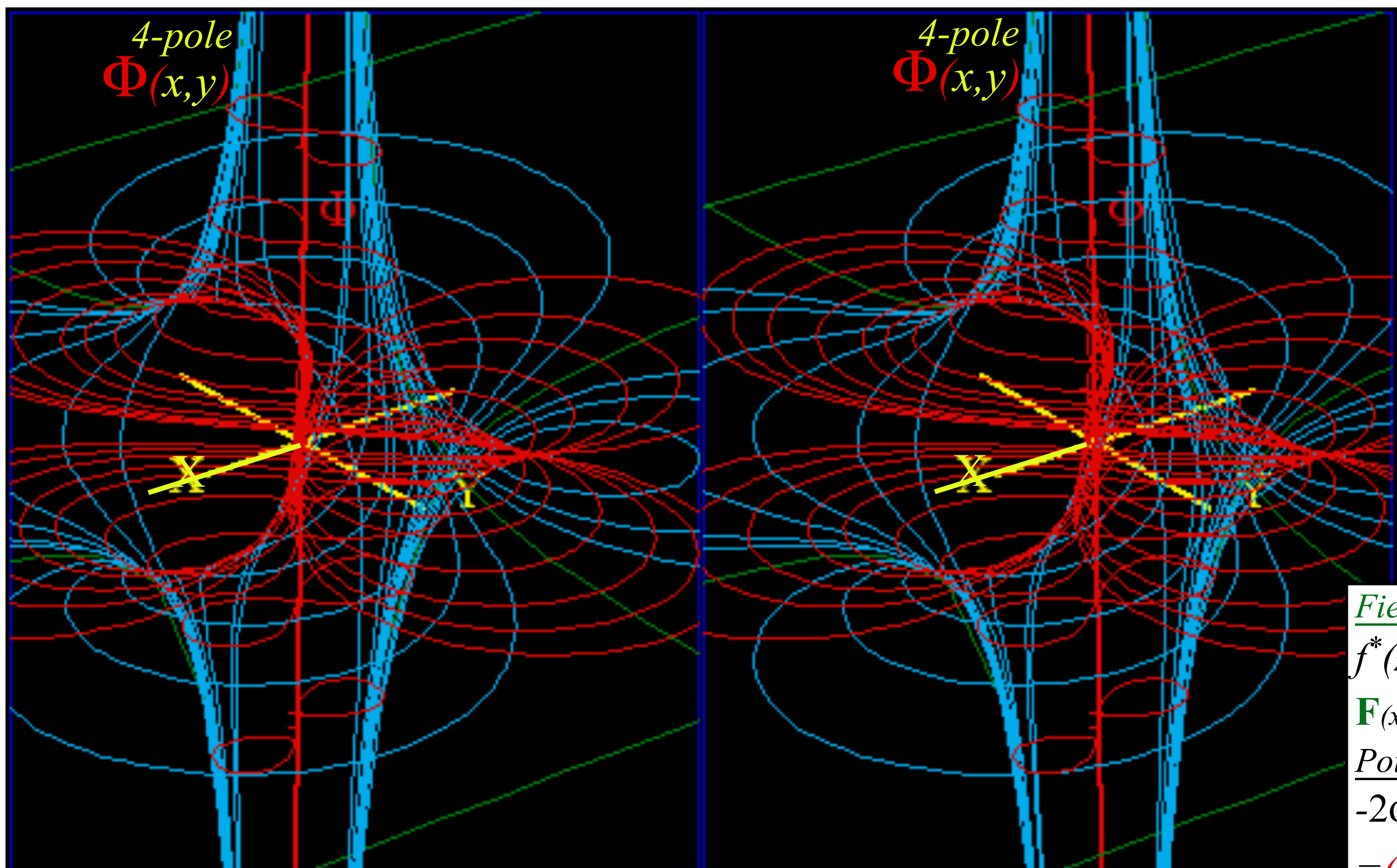
2^n-pole analysis *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

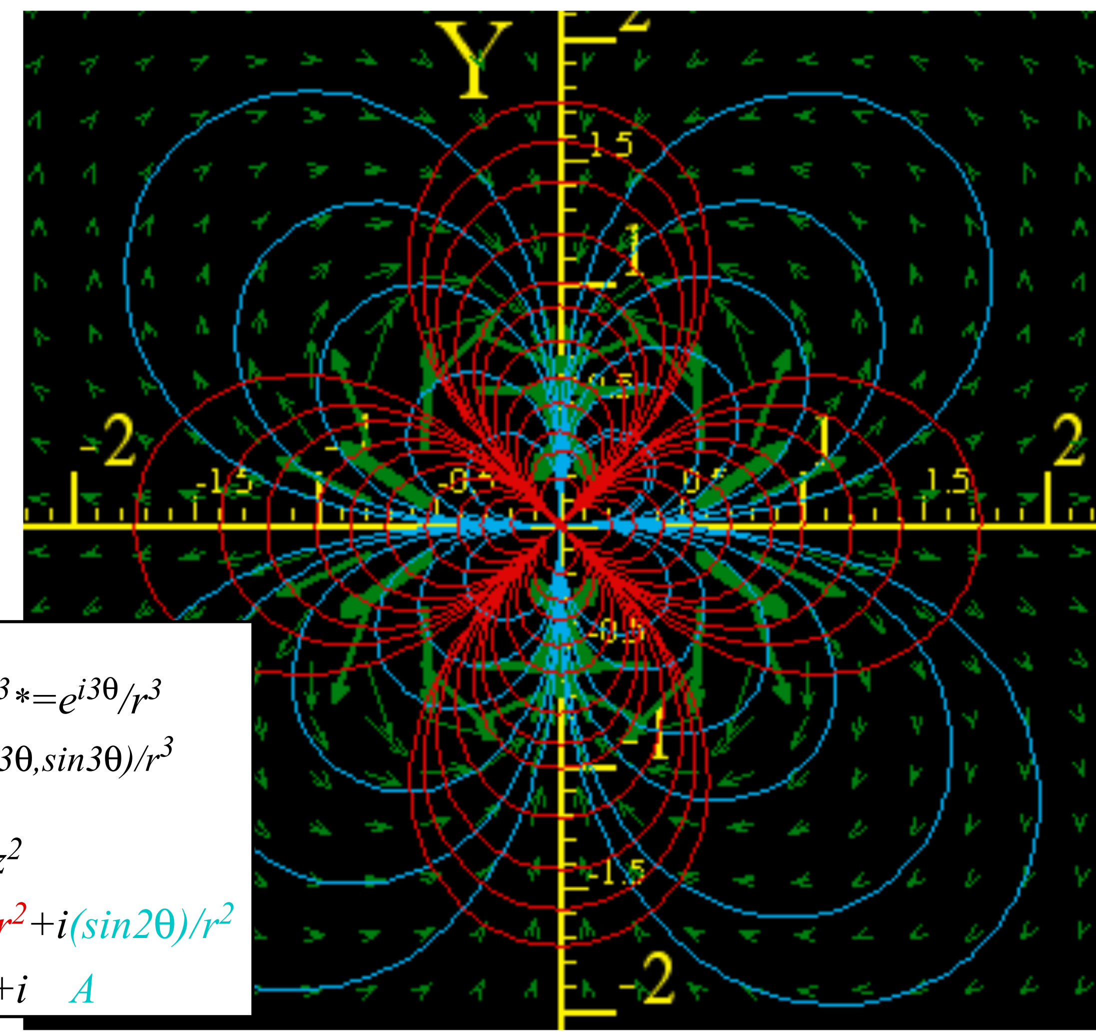
Well, the result is *4-pole* or *quadrupole* field $f^{\,4\text{-pole}}$ and potential $\phi^{4\text{-pole}}$.

Each a *z*-derivative of $f^{\,2\text{-pole}}$ and $\phi^{2\text{-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad\qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$



Field:
$f^*(z^*)=1/z^3{}^*=e^{i3\theta}/r^3$
$\mathbf{F}_{(x,y)}=(\cos 3\theta,\sin 3\theta)/r^3$
Potential:
$-2\phi(z)=1/z^2$
$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$
$=\quad \Phi \quad +i \quad A$



4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

→ *Easy 2^n-multipole field and potential expansion*

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\phi(z) = \ldots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

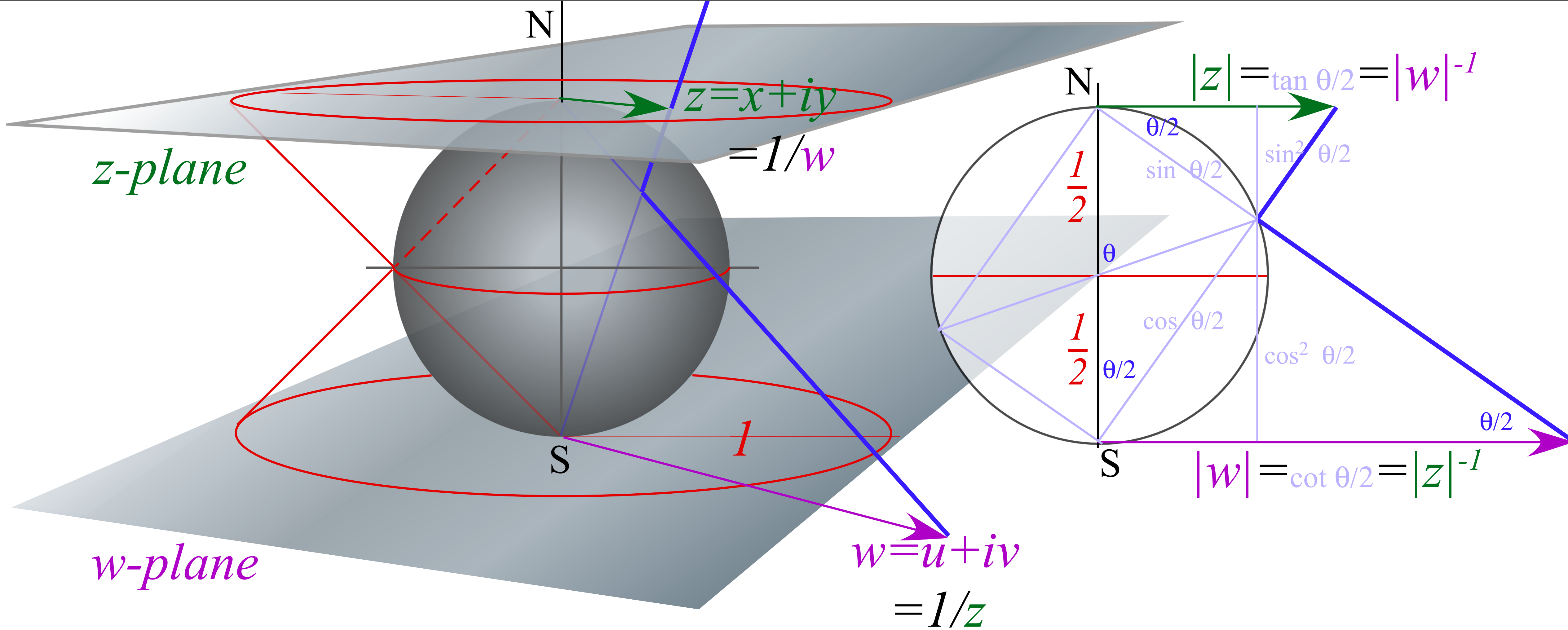
$$\phi(z) = \ldots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

$$\phi(w) = \ldots \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-2}}{-1}w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2}w^2 + \frac{a_2}{3}w^3 + \ldots$$

(*with* $z=w^{-1}$)

$$= \ldots \frac{a_2}{3}z^{-2} + \frac{a_1}{2}z^{-2} + a_0 z^{-1} - a_{-1}\ln z + \frac{a_{-2}}{-1}z + \frac{a_{-3}}{-2}z^2 + \frac{a_{-3}}{-2}z^3 + \ldots$$

(*with* $w=z^{-1}$)



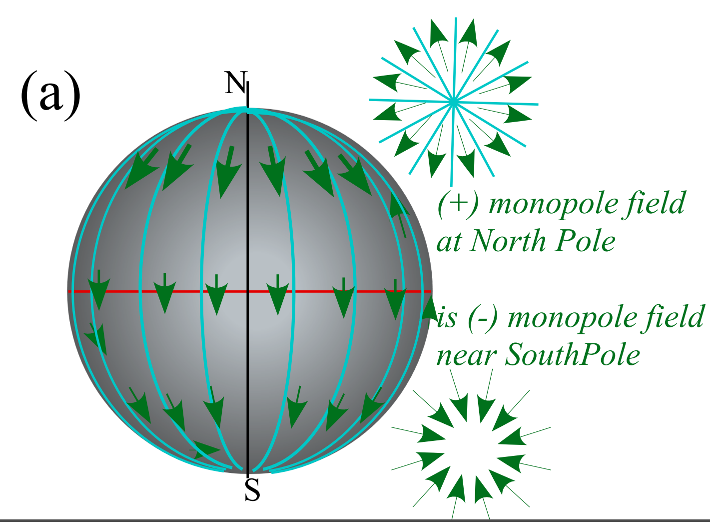
$$\phi(z) = \ldots \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + a_{-1} \ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \ldots$$

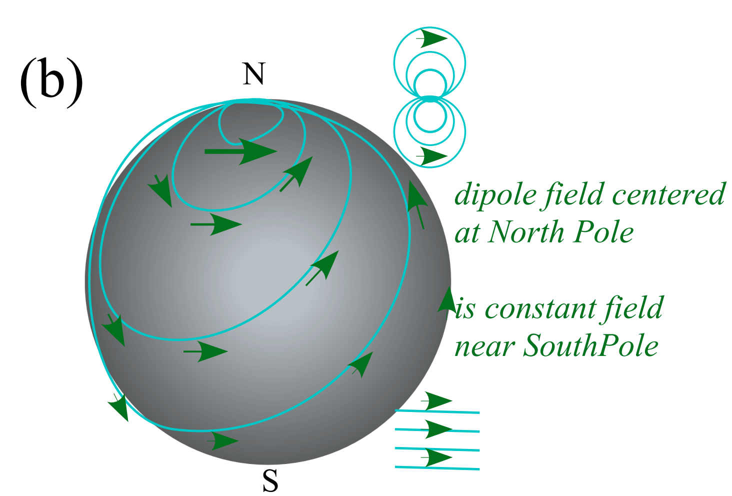
$$\phi(w) = \ldots \frac{a_{-3}}{-2} w^{-2} + \frac{a_{-3}}{-2} w^{-2} + \frac{a_{-2}}{-1} w^{-1} + a_{-1} \ln w + a_0 w + \frac{a_1}{2} w^2 + \frac{a_2}{3} w^3 + \ldots$$

(*with* $z=w^{-1}$)

$$= \ldots \frac{a_2}{3} z^{-2} + \frac{a_1}{2} z^{-2} + a_0 z^{-1} - a_{-1} \ln z + \frac{a_{-2}}{-1} z + \frac{a_{-3}}{-2} z^2 + \frac{a_{-3}}{-2} z^3 + \ldots$$

(*with* $w=z^{-1}$)





Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of Laurent coefficient expressions.

$$\cdots a_{-3} = \frac{1}{2\pi i}\oint z^2 f(z)dz \ , \ a_{-2} = \frac{1}{2\pi i}\oint z^1 f(z)dz \ , \ a_{-1} = \frac{1}{2\pi i}\oint f(z)dz \ , \ a_0 = \frac{1}{2\pi i}\oint \frac{f(z)}{z}dz \ , \ a_1 = \frac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \ ,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function *f(z)* is just *f(a)* on a tiny circle around point-a.

$$\oint \frac{f(z)}{z-a}dz = \oint \frac{f(a)}{z-a}dz = f(a)\oint \frac{1}{z-a}dz = 2\pi i f(a) \qquad f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

The *f(a)* result is called a *Cauchy integral*. Then repeated a-derivatives gives a sequence of them.

$$\frac{df(a)}{da} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz \ , \ \frac{d^2 f(a)}{da^2} = \frac{2}{2\pi i}\oint \frac{f(z)}{(z-a)^3}dz \ , \ \frac{d^3 f(a)}{da^3} = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-a)^4}dz, \ \cdots, \frac{d^n f(a)}{da^n} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz$$

This leads to a general *Taylor-Laurent* power series expansion of function *f(z)* around point-a.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad where: \ a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \left(= \frac{1}{n!}\frac{d^n f(a)}{da^n} \quad for: \ n \geq 0 \right)$$