Lecture 16
Mon. 2.27.2012

Lagrangian and Hamiltonian dynamics: Living with duality in GCC cells and vectors,

(Ch. 12 of Unit 1)

1. GCC Cells and base vectors: Covariant $\mathbf{E}_m$ *vs.* Contravariant $\mathbf{E}^m$

Polar coordinate examples

2. Metric quadratic forms and tensors: Covariant g_{mn} *vs.* Invariant δ_m^n *vs.* Contravariant g^{mn}

Polar coordinate examples

3. Lagrange prefers Covariant g_{mn} *with* Contravariant *velocity* $\dot{q}^m$

Polar coordinate examples

How to finesse centrifugal and Coriolis "forces"

Lecture 16 ends here

4. Hamilton prefers Contravariant g^{mn} *with* Covariant *momentum* p_m

Deriving Hamilton's equations

How to finesse centrifugal and Coriolis energy and other things like phase space.

A dual set of *quasi-unit vectors* show up in Jacobian J and Kajobian K.

J-Columns are *covariant vectors* $\{\mathbf{E}_1=\mathbf{E}_r \;\; \mathbf{E}_2=\mathbf{E}_\phi\}$ K-Rows are *contravariant vectors* $\{\mathbf{E}^1=\mathbf{E}^r \;\; \mathbf{E}^2=\mathbf{E}^\phi\}$

$$\langle J\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} \\ \uparrow\mathbf{E}_1 \;\; \uparrow\mathbf{E}_2 \end{array} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial r}=\cos\phi & \frac{\partial x}{\partial \phi}=-r\sin\phi \\ \frac{\partial y}{\partial r}=\sin\phi & \frac{\partial y}{\partial \phi}=r\cos\phi \end{pmatrix} \\ \uparrow\mathbf{E}_r \qquad \uparrow\mathbf{E}_\phi \end{array}$$

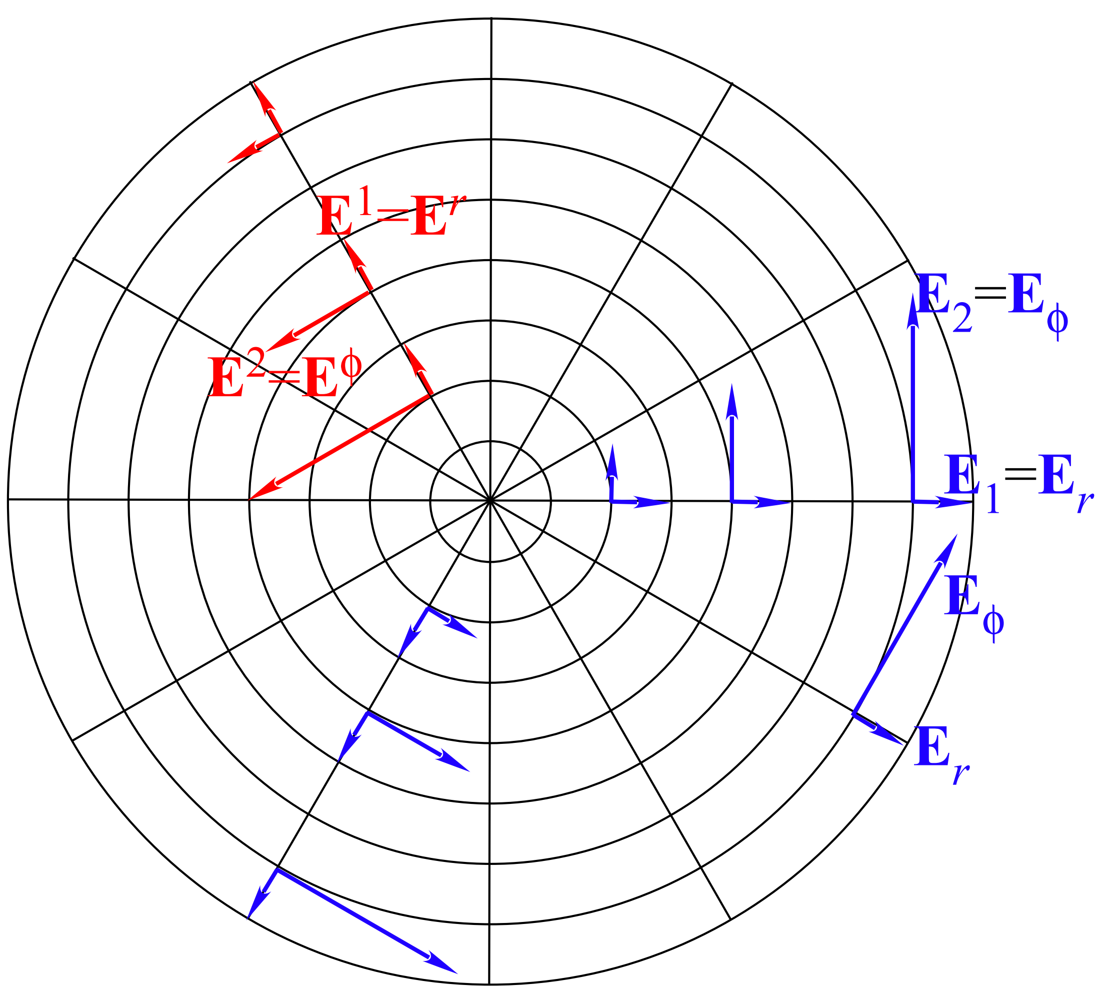
$$\langle K\rangle = \langle J^{-1}\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial r}{\partial x}=\cos\phi & \frac{\partial r}{\partial y}=\sin\phi \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{r} \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^r=\mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi=\mathbf{E}^2 \end{array} \end{array}$$

Derived from polar definition: $x=r\cos\phi$ *and* $y=r\sin\phi$

Inverse polar definition:

$r^2=x^2+y^2$ *and* $\phi=\mathrm{atan2}(y,x)$

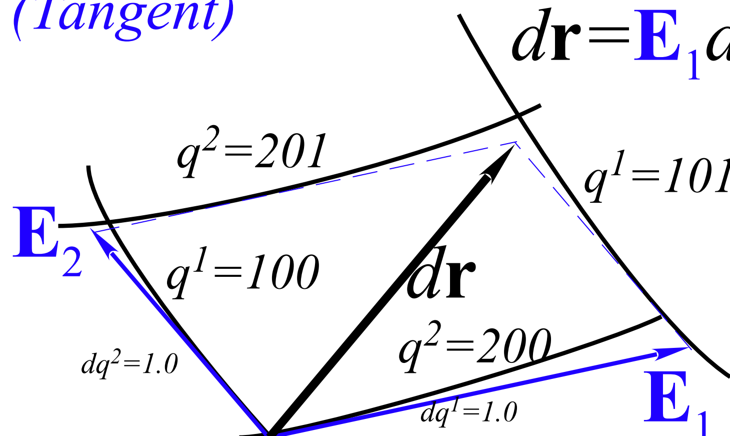
(a) Polar coordinate bases



(b) *Co*variant bases $\{\mathbf{E}_1\;\mathbf{E}_2\}$

(Tangent)

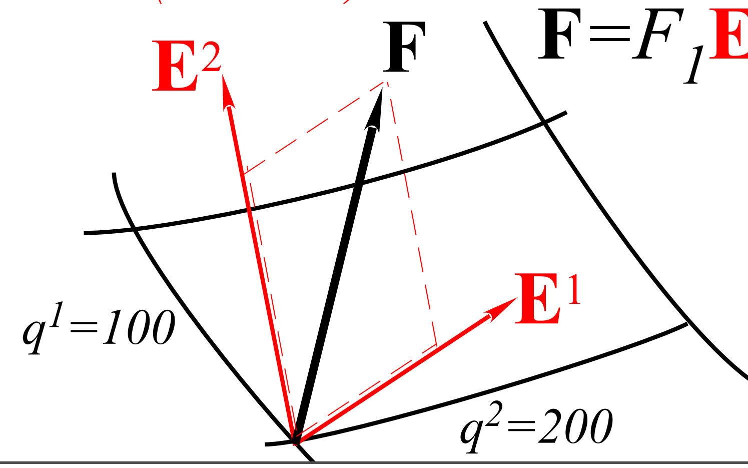
$$d\mathbf{r}=\mathbf{E}_1 dq^1+\mathbf{E}_2 dq^2$$



(c) *Contra*variant bases $\{\mathbf{E}^1\;\mathbf{E}^2\}$

(Normal)

$$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$$



Unit 1
Fig. 12.10

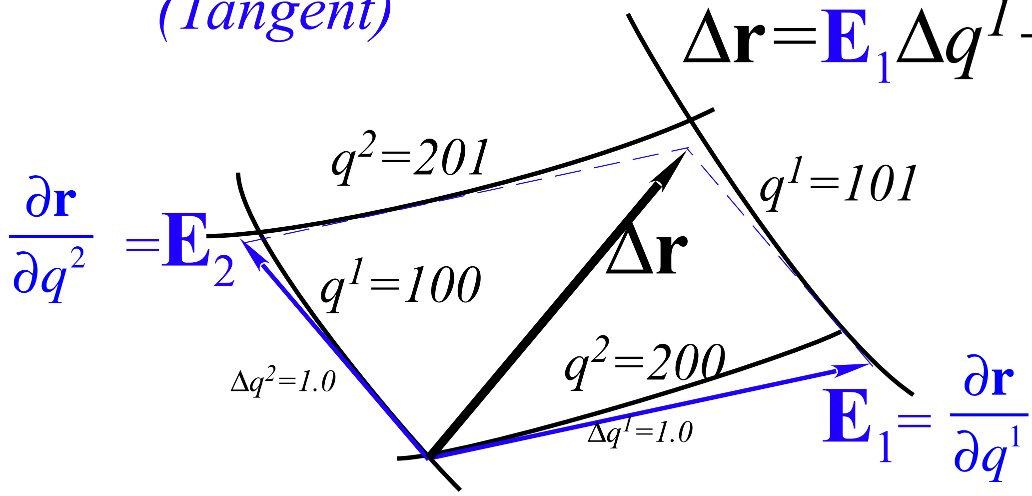
Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

geometric unit

*Co*variant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls

(Tangent)

$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$ *is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



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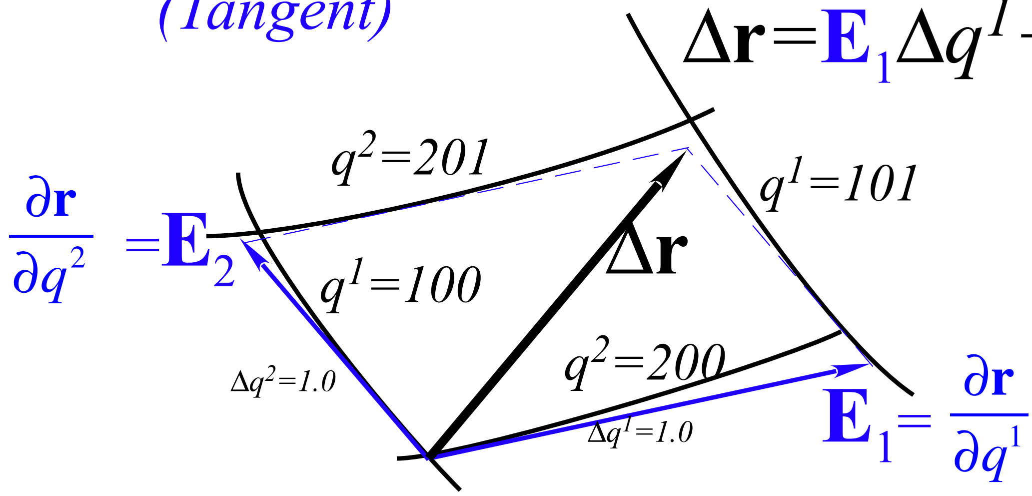


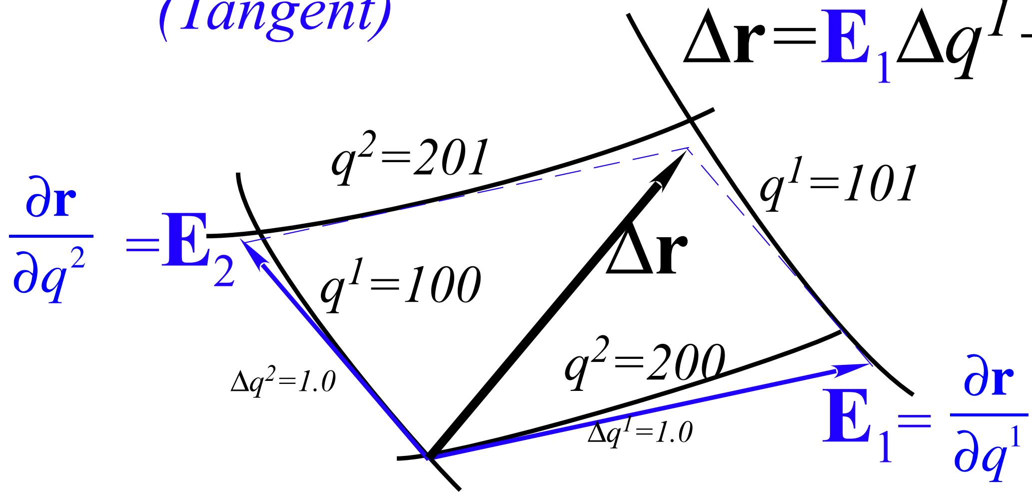
$\mathbf{E}_1$ *follows tangent to* q^2*=const. ...*
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... *remain constant*

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match geometric unit cell walls

(Tangent)

$\Delta\mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2$ *is based on chain rule:* $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2=*const.* ...
since only q^1 *varies in* $\frac{\partial \mathbf{r}}{\partial q^1}$
while q^2, q^3,... *remain constant*

$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ vs. Contravariant $\mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m$

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(Tangent)

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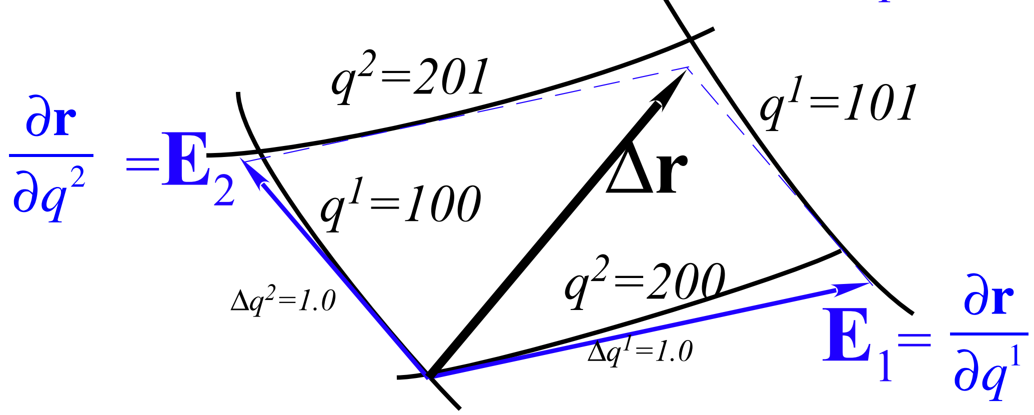

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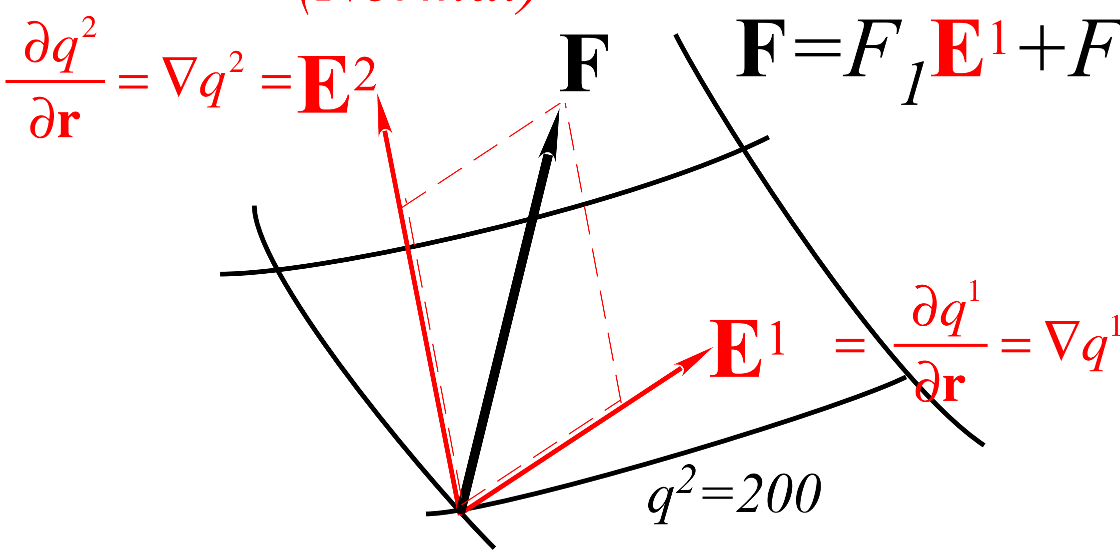
$$\mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

Contravariant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$



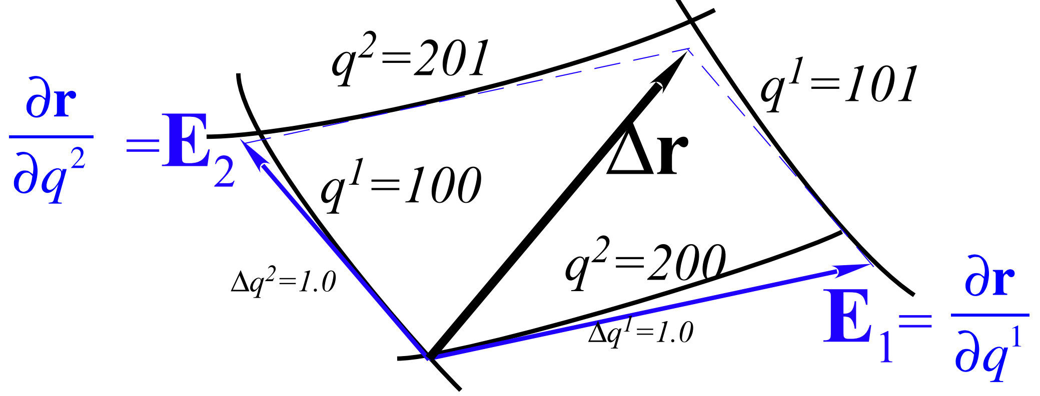
$\mathbf{E}^1$ *is normal to* q^1=const. *since*
gradient *of* q^1 *is vector sum* $\nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix}$
of all its partial derivatives

Comparison: Covariant $\mathbf{E}_m=\frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^m=\frac{\partial q^m}{\partial \mathbf{r}}=\nabla q^m$

geometric unit

Covariant bases $\{\mathbf{E}_1\ \mathbf{E}_2\}$ match cell walls
(Tangent)

$\Delta\mathbf{r}=\mathbf{E}_1\Delta q^1+\mathbf{E}_2\Delta q^2$ *is based on chain rule:* $d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial q^1}dq^1+\frac{\partial \mathbf{r}}{\partial q^2}dq^2=\mathbf{E}_1 dq^1+\mathbf{E}_2 dq^2$



$\mathbf{E}_1$ *follows tangent to* q^2=*const. ...*
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$\mathbf{E}_m$ are convenient bases for *ex*tensive quantities like distance and velocity.

$$\mathbf{V}=V^1\mathbf{E}_1+V^2\mathbf{E}_2=V^1\frac{\partial \mathbf{r}}{\partial q^1}+V^2\frac{\partial \mathbf{r}}{\partial q^2}$$

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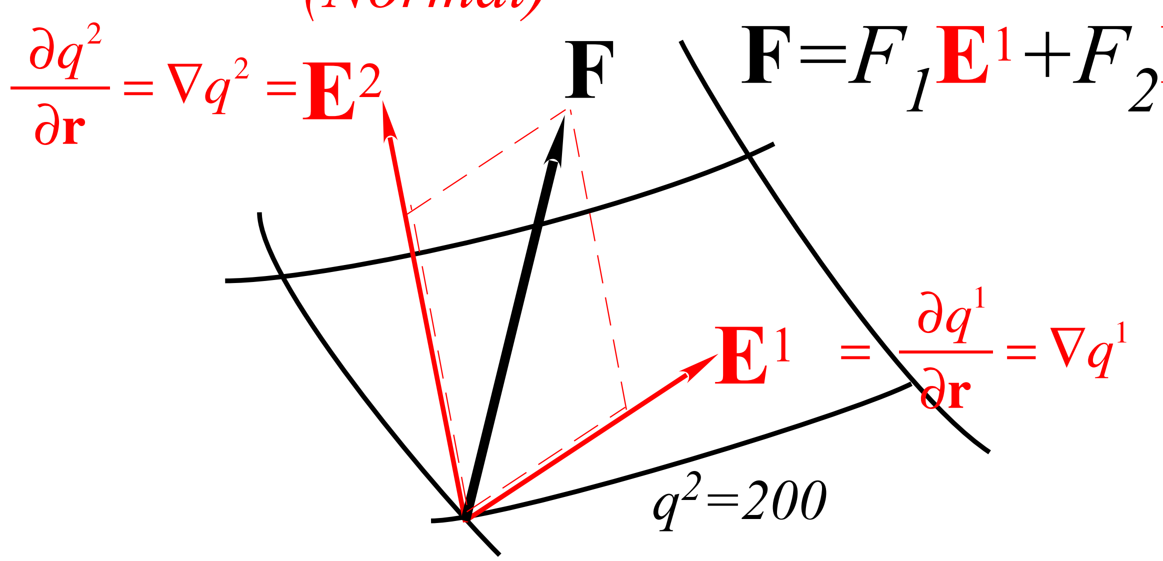
$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2$



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gradient *of* q^1 *is vector sum* $\nabla q^1=\begin{pmatrix}\frac{\partial q^1}{\partial x}\\ \frac{\partial q^1}{\partial y}\end{pmatrix}$
of all its partial derivatives

$\mathbf{E}^m$ are convenient bases for *in*tensive quantities like force and momentum.

$$\mathbf{F}=F_1\mathbf{E}^1+F_2\mathbf{E}^2=F_1\frac{\partial q^1}{\partial \mathbf{r}}+F_2\frac{\partial q^2}{\partial \mathbf{r}}=F_1\nabla q^1+F_2\nabla q^2$$

Comparison: Covariant $\mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m}$ *vs. Contravariant* $\mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n$

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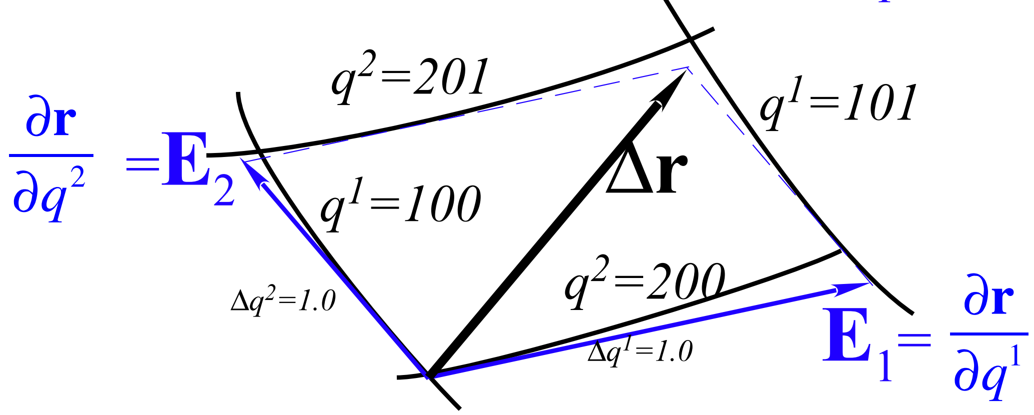

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$$\mathbf{V} = V^1\mathbf{E}_1 + V^2\mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

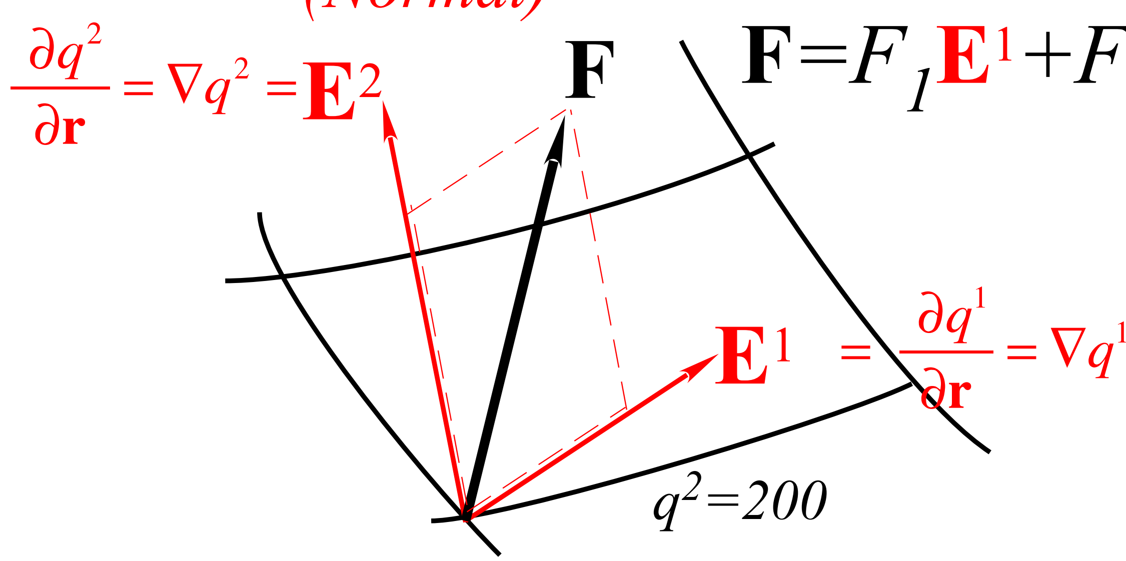
Co-Contra dot products $\mathbf{E}_m \cdot \mathbf{E}^n$ *are orthonormal!*

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta^n_m$$

*Contra*variant $\{\mathbf{E}^1\ \mathbf{E}^2\}$ match reciprocal cells

(Normal)

$$\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2$$



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2. Metric quadratic forms and tensors:

Co*variant* g_{mn} *vs.* In*variant* $\delta_m{}^n$ *vs.* Contra*variant* g^{mn}

Covariant g_{mn} vs. Invariant $\delta_m{}^n$ vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

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Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

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Contravariant
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Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad\qquad \uparrow \mathbf{E}_r \qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant g_{mn} vs. Invariant δ_m^n vs. Contravariant g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn}$$

$$\mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n$$

$$\mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
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$$\begin{array}{cccc} \uparrow \mathbf{E}_1 & \uparrow \mathbf{E}_2 & \uparrow \mathbf{E}_r & \uparrow \mathbf{E}_\phi \end{array}$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{array}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

3. Lagrange prefers <u>Co</u>*variant* g_{mn} *with* <u>Contra</u>*variant velocity* $\dot{q}^m$

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian L=KE-U is supposed to be explicit function of velocity.

$$L(\mathbf{v}) = \tfrac{1}{2} M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2} M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2} M(\mathbf{E}_m \dot{q}^m)\bullet(\mathbf{E}_n \dot{q}^n) - U = \tfrac{1}{2} M(g_{mn}\dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

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Use polar coordinate Covariant g_{mn} metric (1-page back)
$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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Use polar coordinate Covariant g_{mn} metric (1-page back)

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This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

Lagrange prefers Covariant g_{mn} with Contravariant velocity

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Use polar coordinate Covariant g_{mn} metric (1-page back)
$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2} M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2} M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising;
radial momentum p_r has the
usual linear $M\cdot v$ form

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

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Use polar coordinate Covariant g_{mn} metric (1-page back)

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GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

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Centrifugal force $Mr\omega^2$

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Torque relates to two distinct parts: Coriolis and angular acceleration

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Rewriting GCC Lagrange equations :

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Conventional forms

radial force: $M\ddot{r} = Mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$

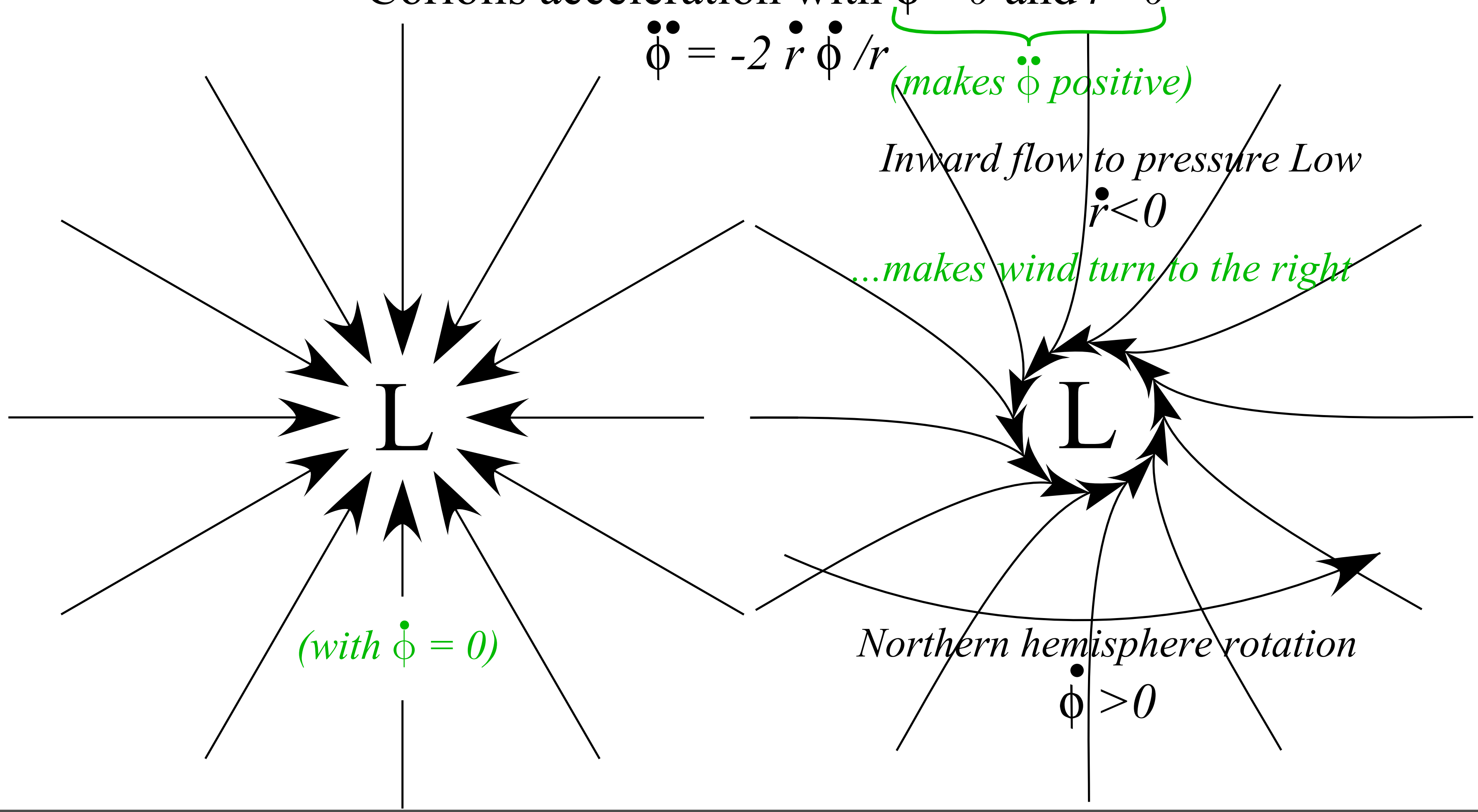
angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$

Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}_{\text{(makes } \ddot{\phi} \text{ positive)}}$



Effect on Northern Hemisphere local weather

Cyclonic flow around lows

*4. Hamilton prefers Contravariant g^{mn} with Covariant **momentum** p_m*

Deriving Hamilton's Equations

How to finesse centrifugal and Coriolis energy and other things like phase space.

Deriving Hamilton's equations

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ *...*

$$\dot{L}\left(q,\dot{q},t\right)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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...of coordinates and velocity and <u>time</u>, too. (*You can safely drop last chain-rule factor* [*1=dt/dt*])

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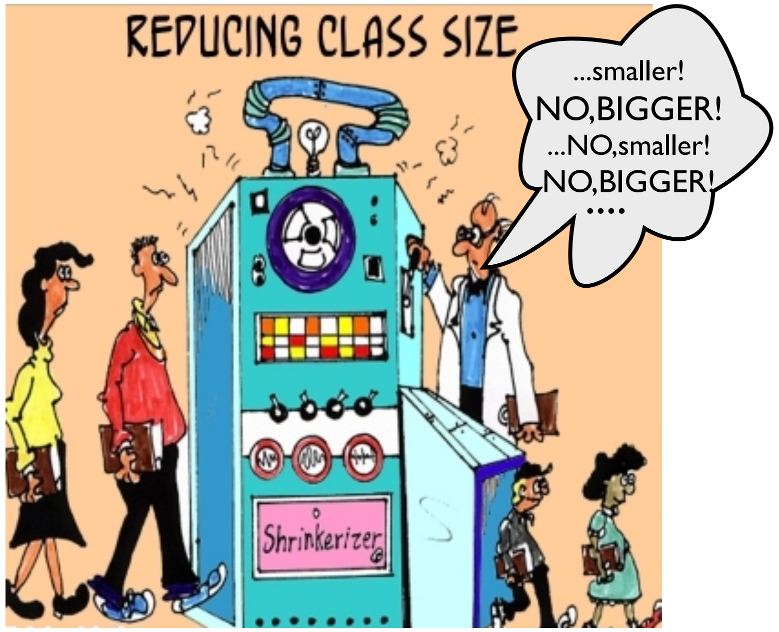

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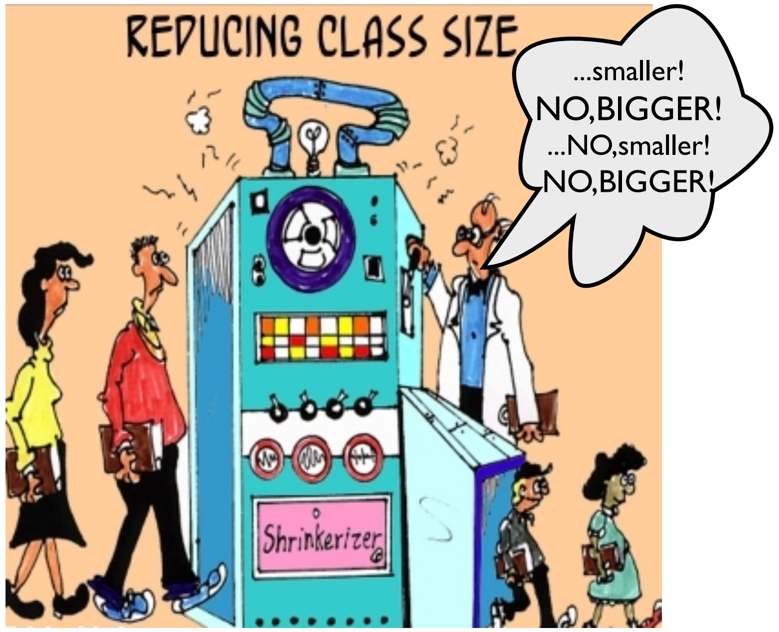

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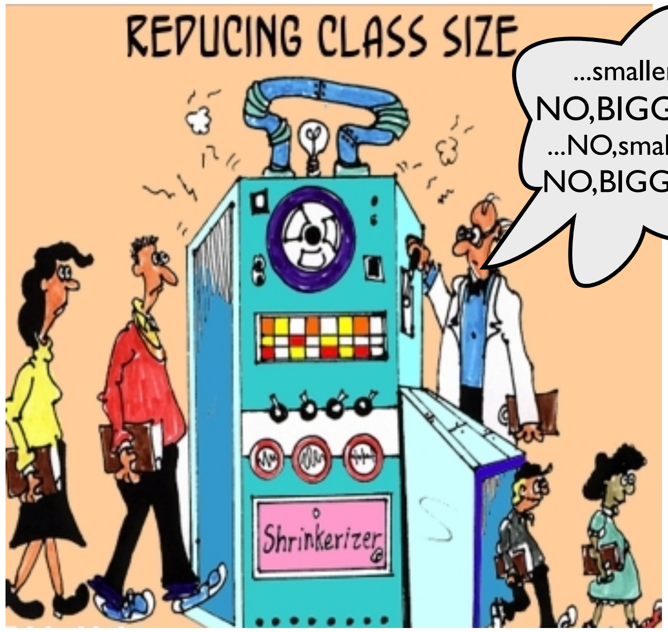

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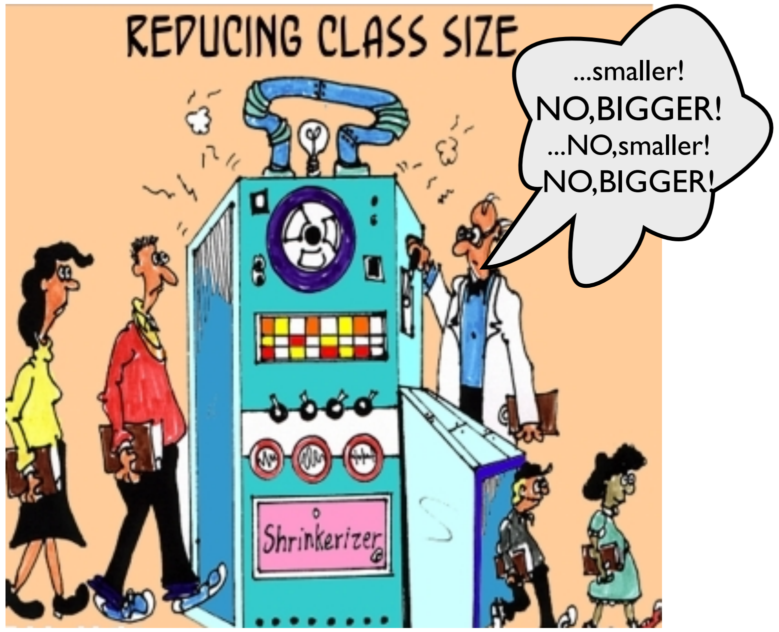

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$$-\frac{\partial L}{\partial t} = \frac{d}{dt}\left(p_m\dot{q}^m\right) - \frac{dL}{dt}$$

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p}\bullet\mathbf{v} - L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m\dot{q}^m - L\right) = -\frac{\partial L}{\partial t} = \frac{dH}{dt} \quad \text{where}: H = p_m\dot{q}^m - L$$

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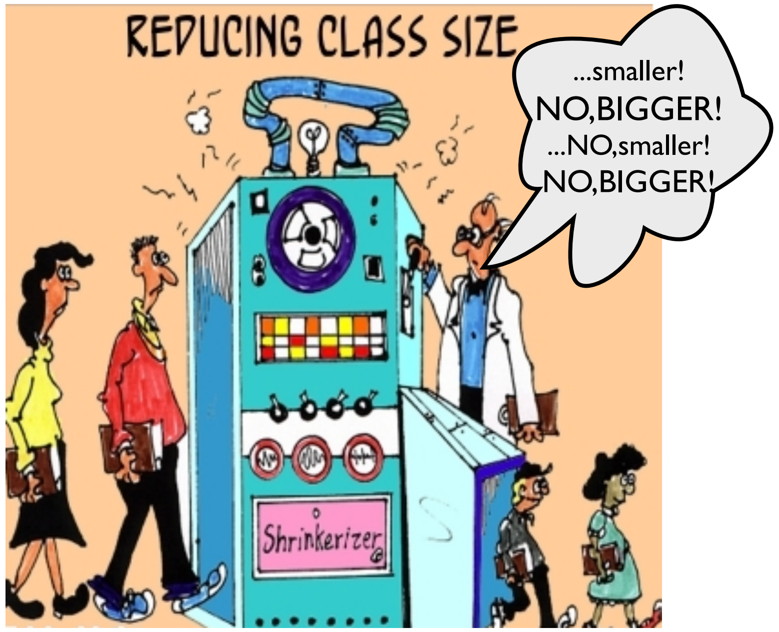

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Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

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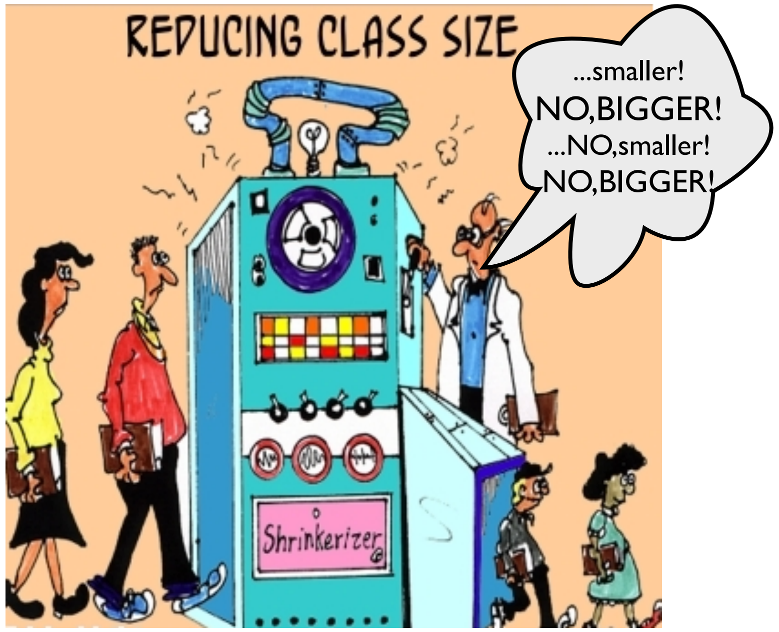

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(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$ *and:* $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$ *)*

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

$$\frac{\partial H}{\partial q^m}=0\cdot 0-\frac{\partial L}{\partial q^m}=-\dot{p}_m$$

Hamilton's 2nd GCC equation

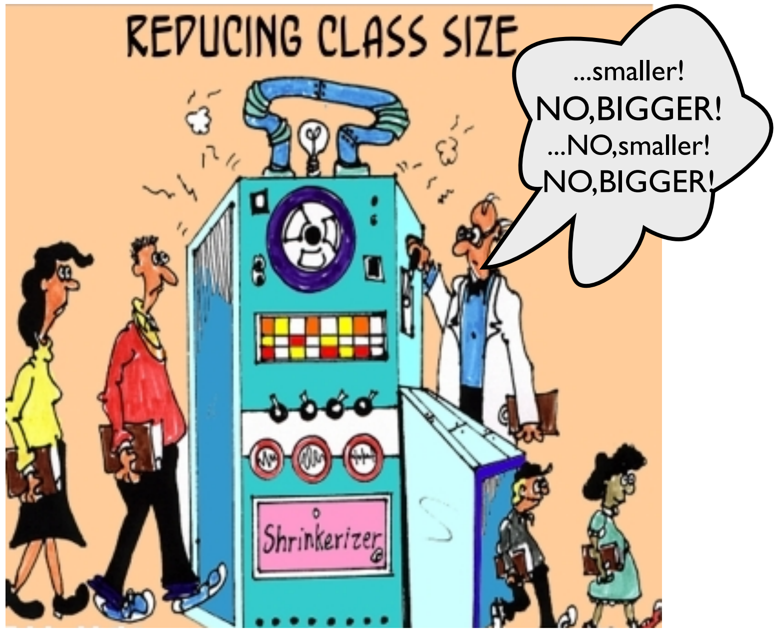
$$\frac{\partial H}{\partial q^m}=-\dot{p}_m$$

Deriving Hamilton's equations

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and ***time****, too. (Imagine Mad Scientist turning U-dial.)*



$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:
$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$-\frac{\partial L}{\partial t}=\frac{d}{dt}\left(p_m\dot{q}^m\right)-\frac{dL}{dt}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$ *(That's the old Legendre transform)*

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=\boxed{-\frac{\partial L}{\partial t}=\frac{dH}{dt}}\quad where: H=p_m\dot{q}^m-L$$

(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$
and: $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$ *)*

a most ***peculiar*** *relation involving partial vs total*

Hamilton's 1st GCC equation

$$\frac{\partial H}{\partial p_m}=\dot{q}^m$$

Hamilton's 2nd GCC equation

$$\frac{\partial H}{\partial q^m}=-\dot{p}_m$$

End of this Lecture