Lecture 38.
Classical analogs for quantum resonance

(Ch. 4-6 of Unit 3   5.1.12)

Matrix-operator spectral decomposition, eigenvectors, and eigenvalues

Review of Lecture 37

Spectral decomposition of 2D-HO and mixed mode dynamics (B-Symmetric case))
- *Algebraic approach*
- *Geometric phasor view*

Spectral decomposition of 2D-HO and mixed mode dynamics (Asymmetric case)
- *Algebraic approach*
- *Mode trajectories*

2-State Schrodinger quantum analogy with classical 2D-HO
- *ABCD Symmetry operator analysis and U(2) spinors*
- *ABCD Time evolution operator and U(2)~R(3) spin spaces*

Lecture 38 ends here

Matrix-operator spectral decomposition, eigenvectors, and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases} \qquad \mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$$

They're *Ortho-Normal* and satisfy *Completeness Relation* $\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of $\mathbf{M}$

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Lecture 37 ended here

Example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

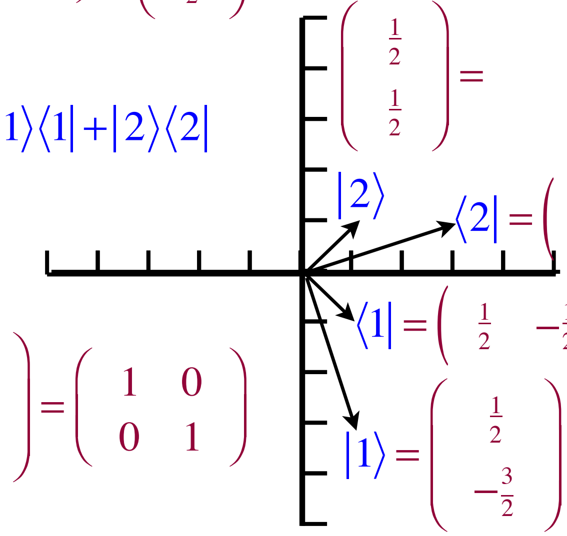
$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = |1\rangle\langle 1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} = |2\rangle\langle 2|$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|$$



$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|2\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

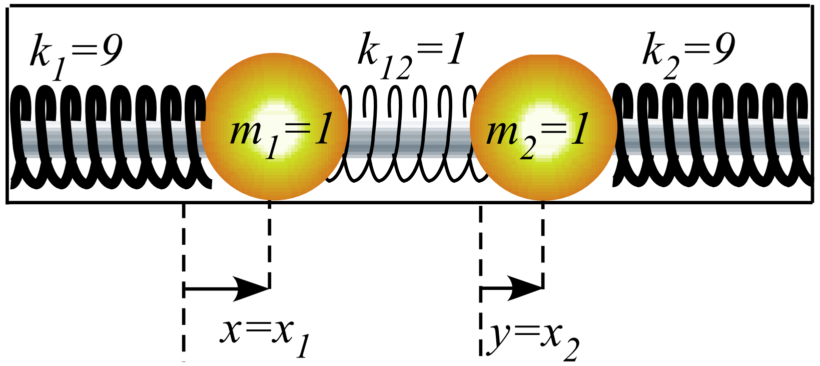
$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2|$$

$$= 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Examples with $\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm\sqrt{1}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm \sqrt{5}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics

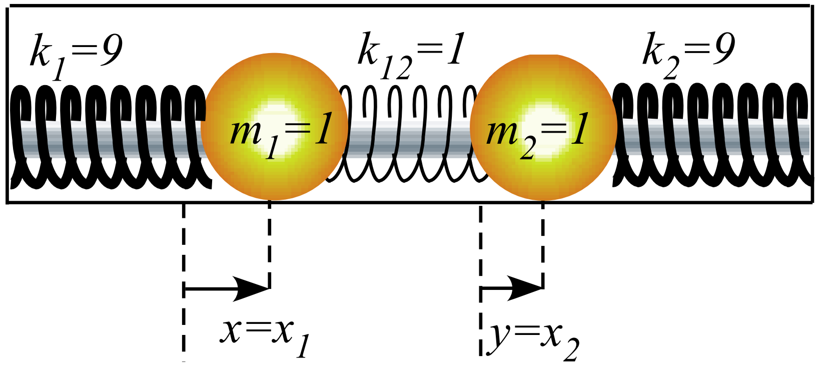


$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Spectral decomposition of 2D-HO mode dynamics



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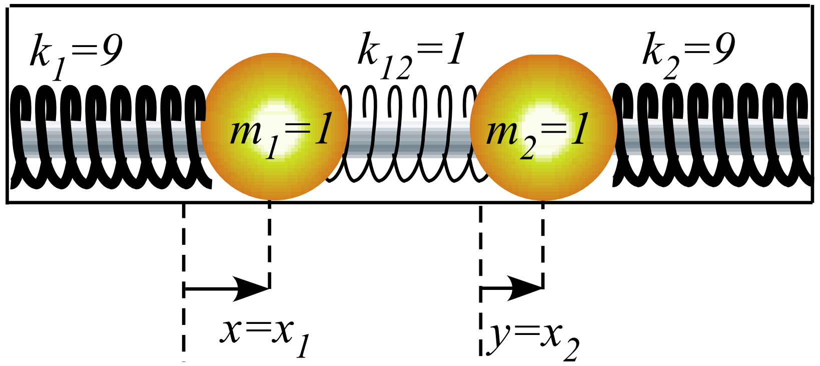
Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

Spectral decomposition of 2D-HO mode dynamics



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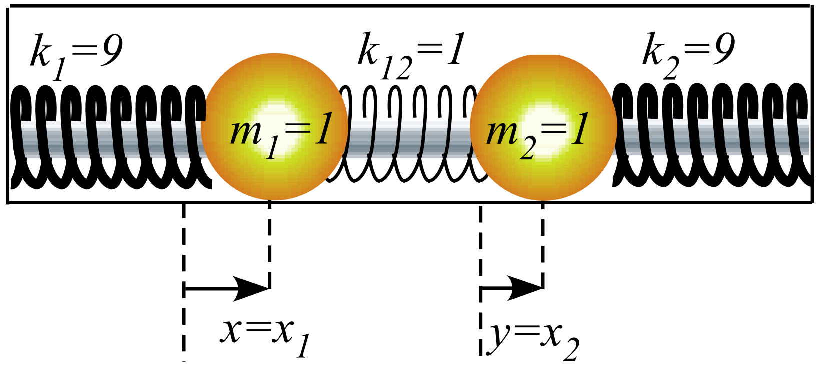
$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Spectral decomposition of 2D-HO mode dynamics



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

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$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

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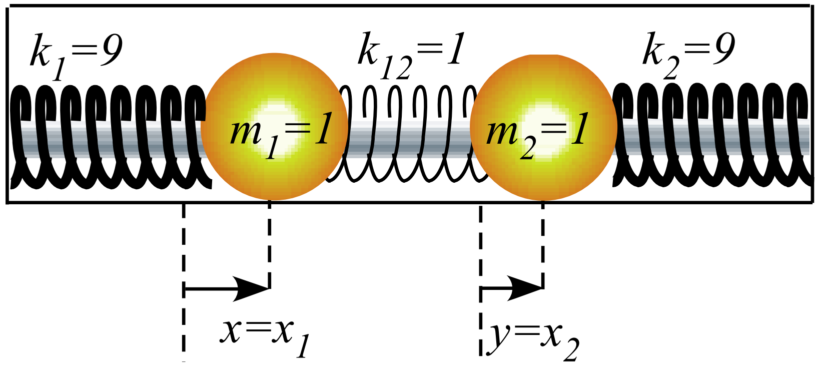
Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

Spectral decomposition of 2D-HO mode dynamics



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$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

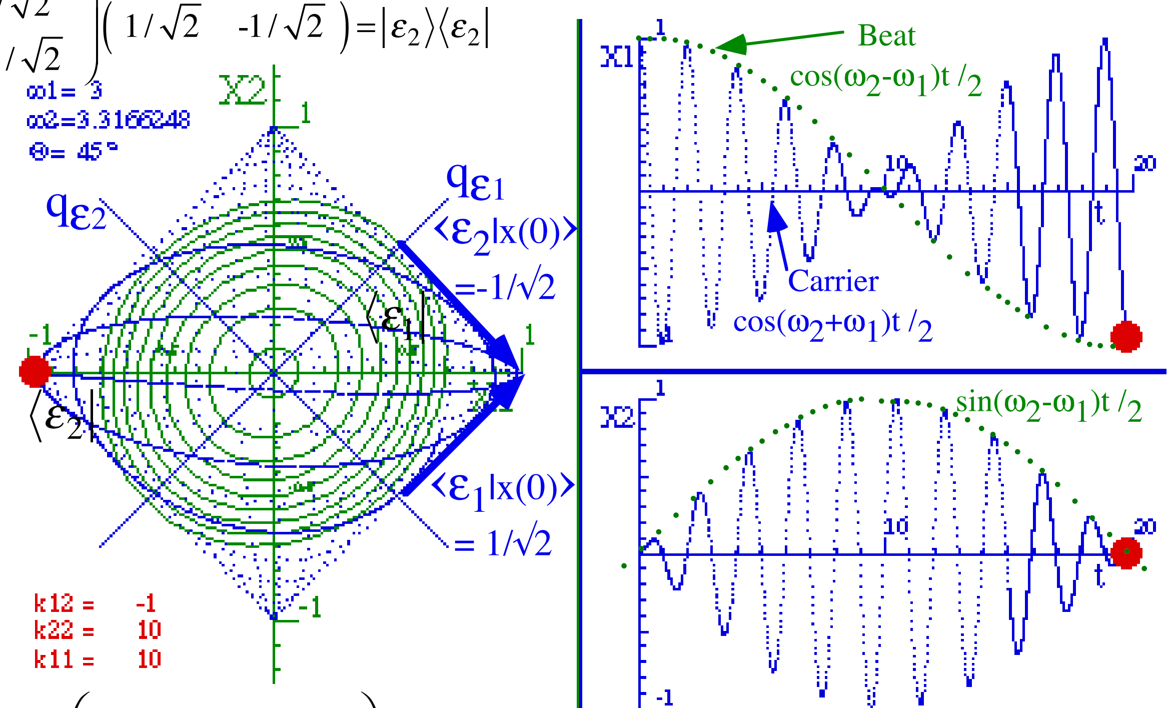
Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

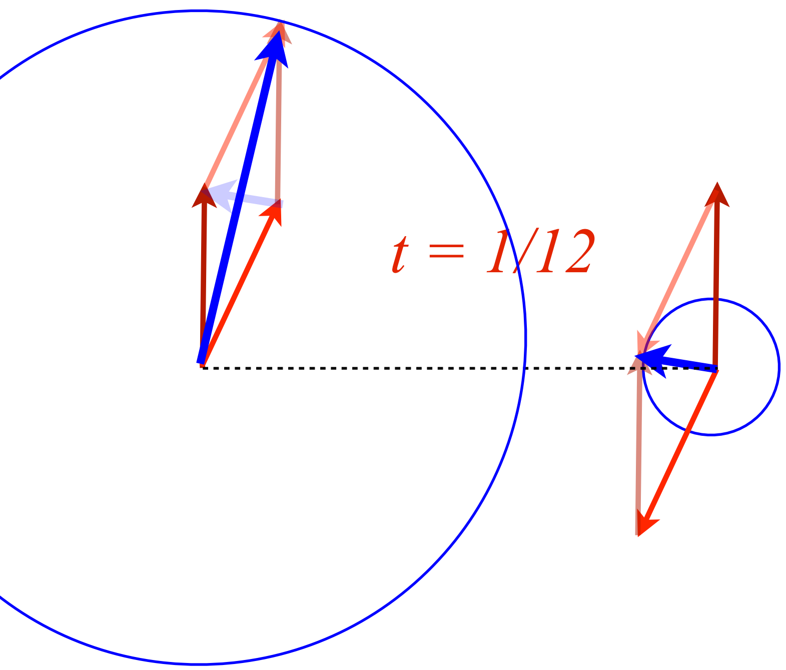
100% modulation (SWR=0)

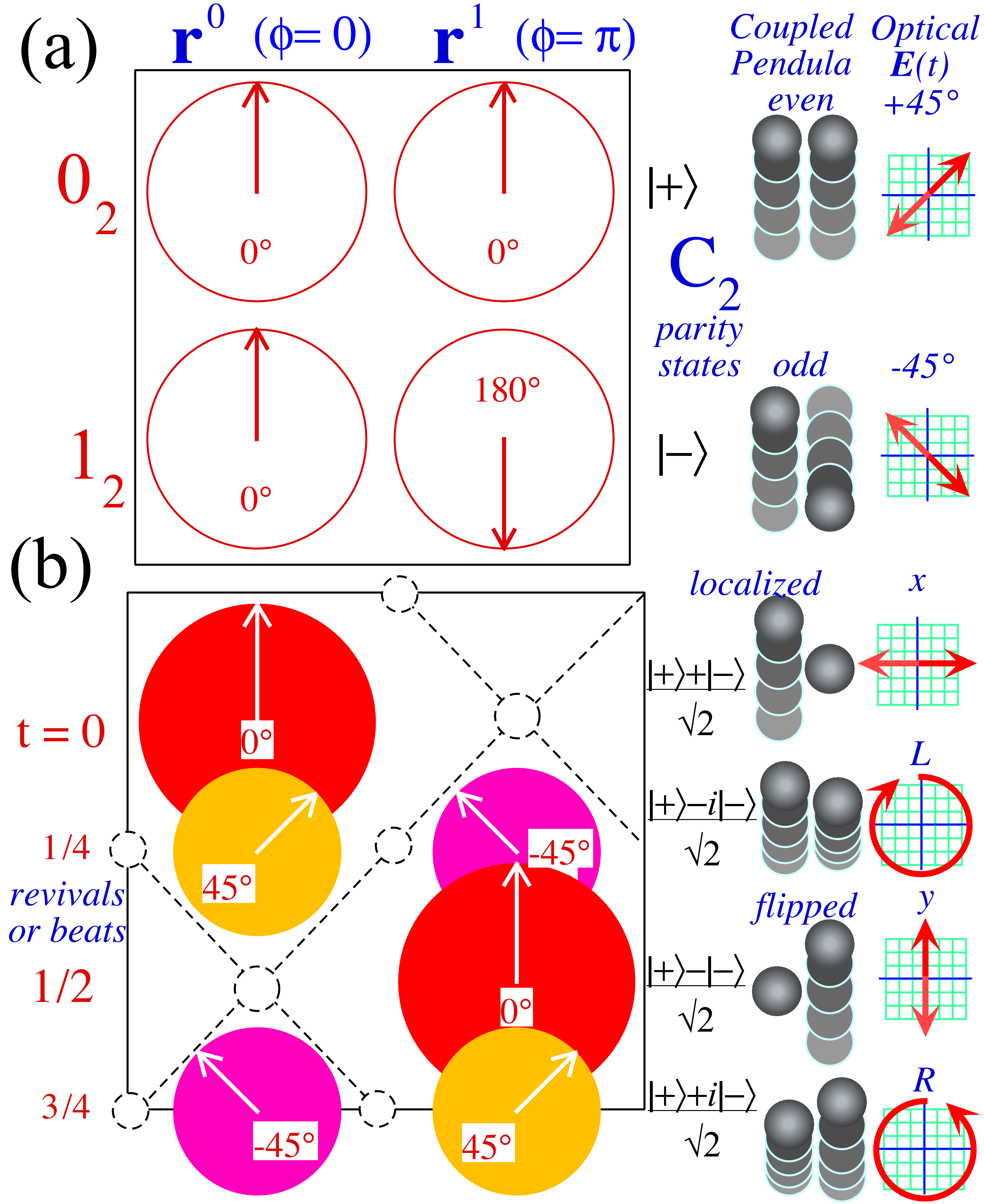


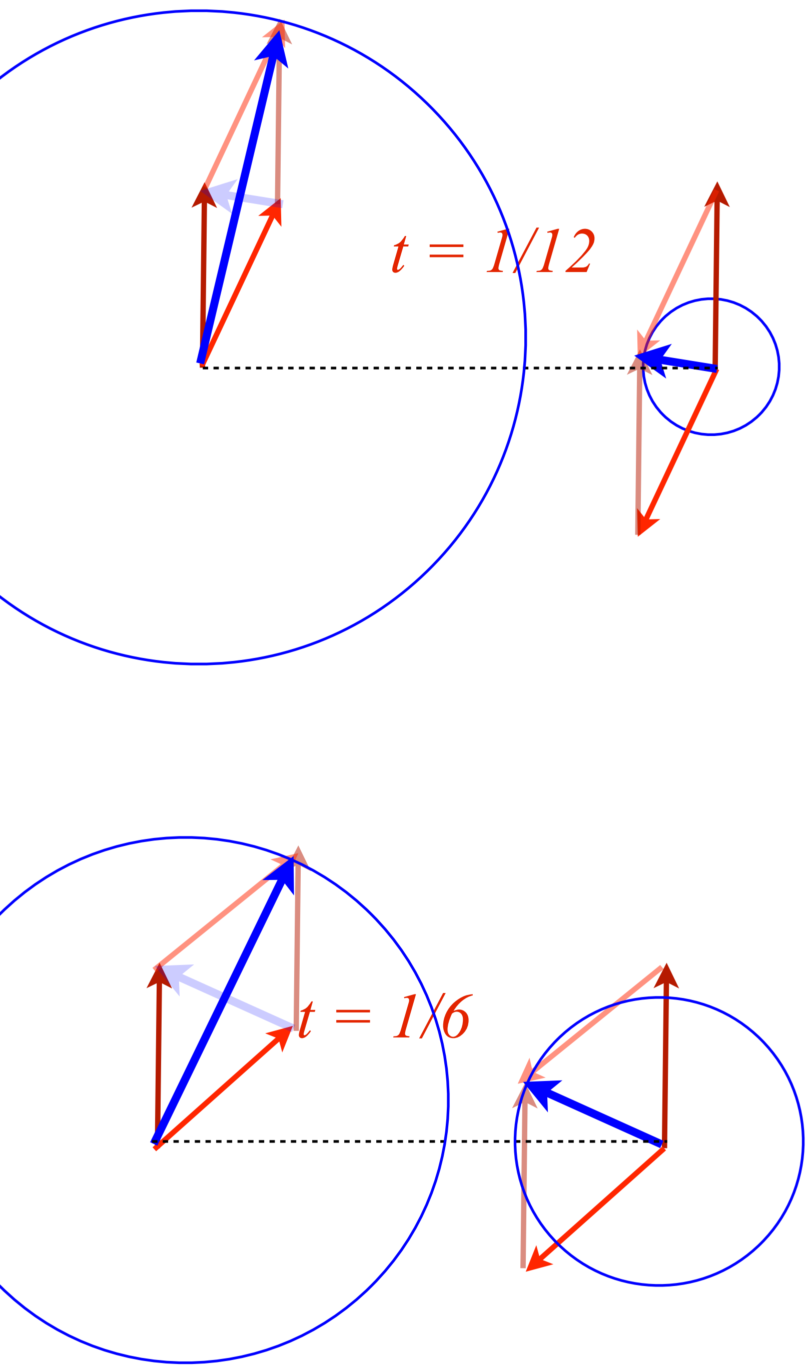
$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} \frac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\frac{(\omega_2-\omega_1)t}{2} \\ i\sin\frac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

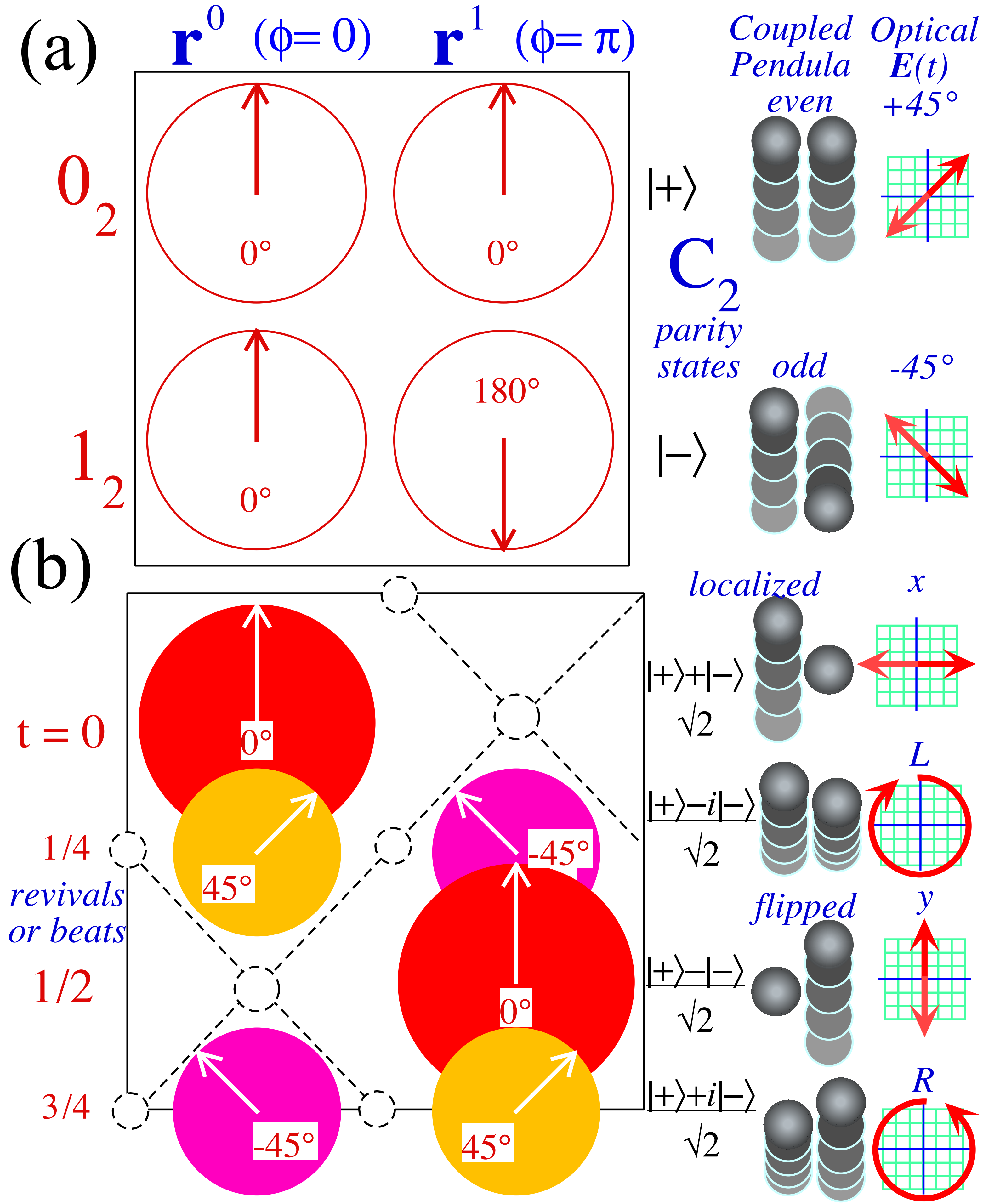
Note the i *phase*

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

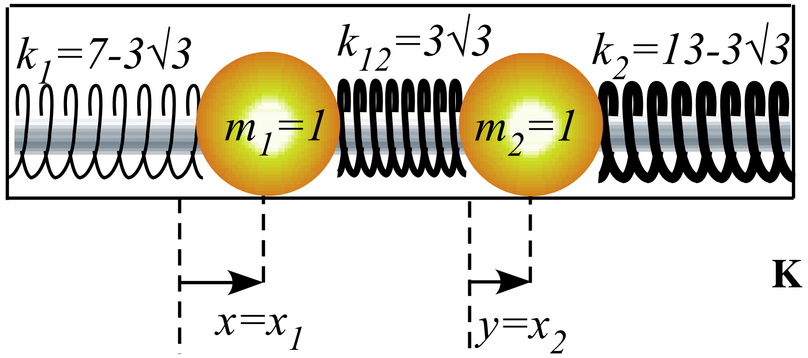

t = 1/12


(a)
r^0 (φ= 0)
r^1 (φ= π)
Coupled Pendula
Optical E(t)
even
+45°
0_2
1_2
0°
0°
0°
180°
|+⟩
|−⟩
C_2 parity states
odd
-45°
(b)
localized
x
t = 0
(|+⟩+|−⟩)/√2
L
(|+⟩−i|−⟩)/√2
1/4
revivals or beats
1/2
3/4
45°
-45°
flipped
y
(|+⟩−|−⟩)/√2
R
(|+⟩+i|−⟩)/√2


t = 1/12
t = 1/6


(a)
r0 (ϕ= 0)
r1 (ϕ= π)
0_2
1_2
0°
0°
0°
180°
|+⟩
|−⟩
C_2
parity
states
Coupled
Pendula
even
odd
Optical
E(t)
+45°
-45°
(b)
t = 0
1/4
revivals
or beats
1/2
3/4
0°
45°
-45°
0°
-45°
45°
localized
x
(|+⟩+|−⟩)/√2
L
(|+⟩−i|−⟩)/√2
flipped
y
(|+⟩−|−⟩)/√2
R
(|+⟩+i|−⟩)/√2

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



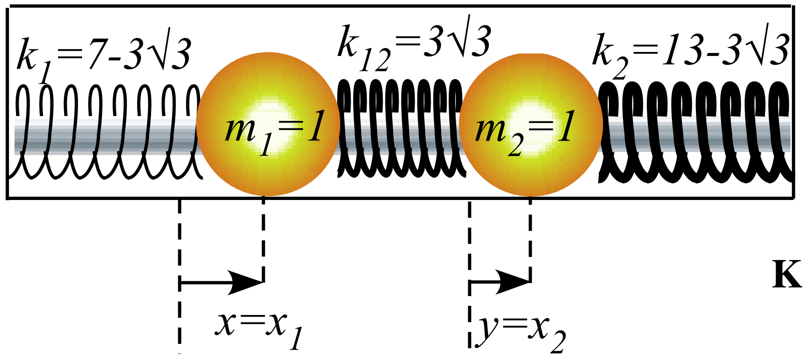
$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

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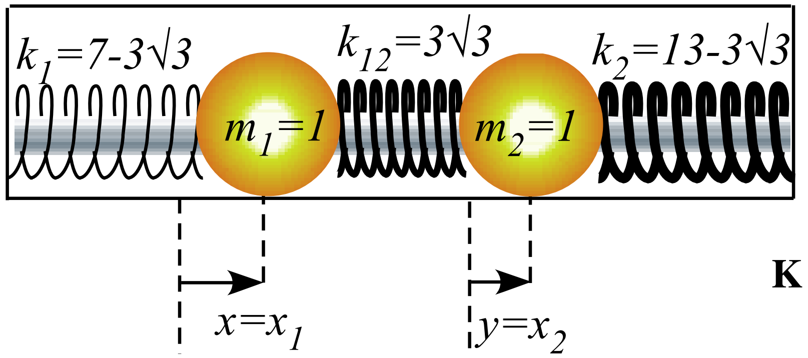
Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$
$$=\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$
$$=\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}=\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$=\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

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Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

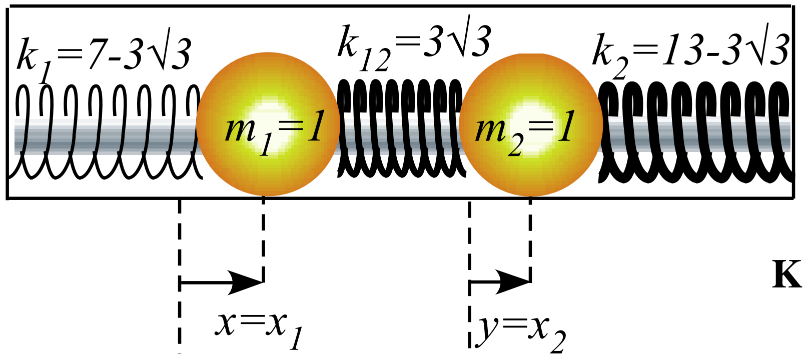
Spectral decomposition of initial state ***x****(0)=(1,0):*

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}\left(\frac{\sqrt{3}}{2}\right)+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}\left(-\frac{1}{2}\right)$$

(Note projection onto eigen-axes)

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$ $\qquad Det(\mathbf{K})=7\cdot 13-27=91-27=64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\qquad K_1=\omega_0^2(\varepsilon_1)=4, \quad K_2=\omega_0^2(\varepsilon_2)=16,$

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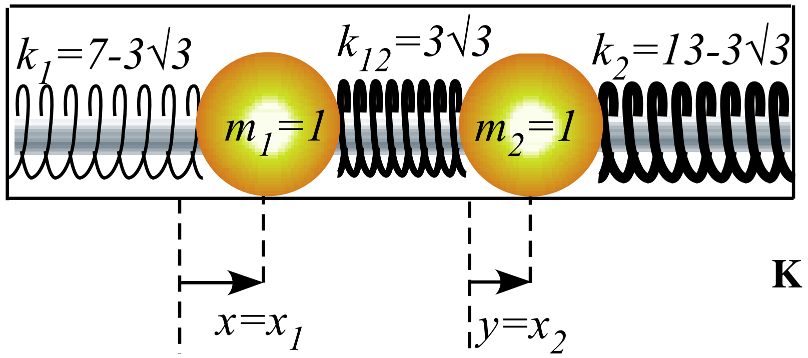
(Note projection onto eigen-axes)

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t)=-\frac{1}{2}\cos 4t \right)$$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}q_1^2+\frac{1}{2}$$

Spectral decomposition of 2D-HO mode dynamics for lower symmetry



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$$=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}\left(\tfrac{\sqrt{3}}{2}\right)+\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\tfrac{1}{2}\right)$$

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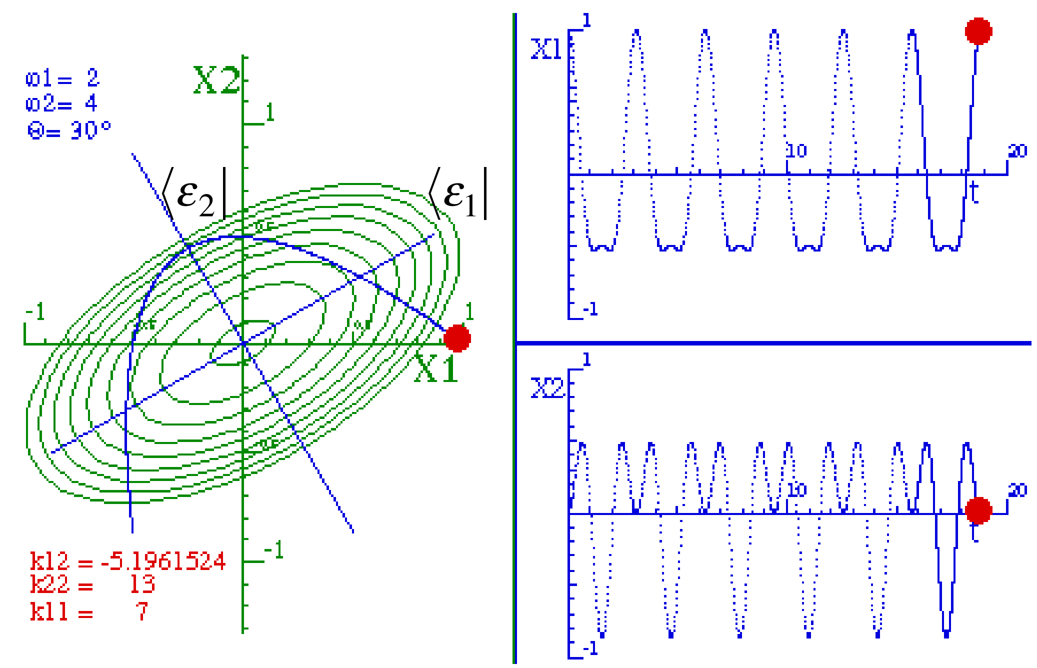

Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

2-State Schrodinger equations $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ are analogous to classical 2D coupled oscillator equations.

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

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$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

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$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

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2-State Schrodinger equations $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ are analogous to classical 2D coupled oscillator equations.

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

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QM vs. Classical Equations are identical

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$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

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Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

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Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

The $\{\sigma_I, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_I=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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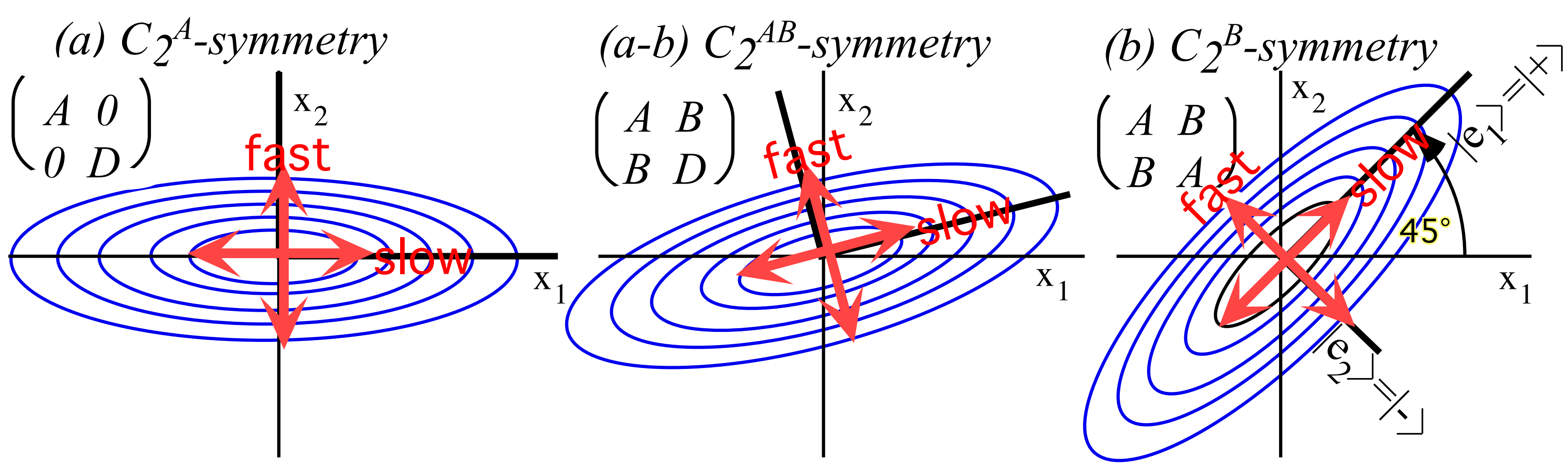

Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

$$= e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\Omega}\cdot t/2} e^{-i\Omega_0\cdot t} \text{ where: } \boldsymbol{\Theta} = \boldsymbol{\Omega}\cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t \text{ and: } \Omega_0 = \frac{A+D}{2}$$

ABCD Time evolution operator

Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

Each σ_X squares to one (unit matrix $\mathbf{1} = \sigma_X \cdot \sigma_X$) and each quaternion squares to minus-one ($-\mathbf{1} = \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j}$, *etc.*) just like $i = \sqrt{-1}$.

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$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

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Each σ_X squares to one (unit matrix $\mathbf{1} = \sigma_X\cdot\sigma_X$) and each quaternion squares to minus-one (**–1=i·i=j·j**, *etc.*) just like $i = \sqrt{-1}$.

This is true for spinor components based on *any* unit vector $\hat{\mathbf{a}} = (a_X, a_Y, a_Z)$ for which $\hat{\mathbf{a}}\bullet\hat{\mathbf{a}} = 1 = a_X^2 + a_Y^2 + a_Z^2$.

To see this just try it out on any $\hat{\mathbf{a}}$-component: $\sigma_a = \sigma\bullet\hat{\mathbf{a}} = a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z$.

$$\sigma_a^2 = (\sigma\bullet\hat{\mathbf{a}})(\sigma\bullet\hat{\mathbf{a}}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)$$

$$\begin{aligned} & a_X\sigma_X a_X\sigma_X \quad +a_X\sigma_X a_Y\sigma_Y \quad +a_X\sigma_X a_Z\sigma_Z \qquad a_X a_X\sigma_X\sigma_X \quad +a_X a_Y\sigma_X\sigma_Y \quad +a_X a_Z\sigma_X\sigma_Z \\ = & +a_Y\sigma_Y a_X\sigma_X \quad +a_Y\sigma_Y a_Y\sigma_Y \quad +a_Y\sigma_Y a_Z\sigma_Z \; = \; +a_Y a_X\sigma_Y\sigma_X \quad +a_Y a_Y\sigma_Y\sigma_Y \quad +a_Y a_Z\sigma_Y\sigma_Z \\ & +a_Z\sigma_Z a_X\sigma_X \quad +a_Z\sigma_Z a_Y\sigma_Y \quad +a_Z\sigma_Z a_Z\sigma_Z \qquad +a_Z a_X\sigma_Z\sigma_X \quad +a_Z a_Y\sigma_Z\sigma_Y \quad +a_Z a_Z\sigma_Z\sigma_Z \end{aligned}$$

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ABCD Time evolution operator

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

$$= e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\Omega}\cdot t/2} e^{-i\Omega_0\cdot t} \text{ where: } \boldsymbol{\Theta} = \boldsymbol{\Omega}\cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t \text{ and: } \Omega_0 = \frac{A+D}{2}$$

Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

Each σ_X squares to one (unit matrix $\mathbf{1} = \sigma_X\cdot\sigma_X$) and each quaternion squares to minus-one (**–1=i·i=j·j**, *etc.*) just like $i = \sqrt{-1}$.

This is true for spinor components based on *any* unit vector $\hat{\mathbf{a}} = (a_X, a_Y, a_Z)$ for which $\hat{\mathbf{a}}\bullet\hat{\mathbf{a}} = 1 = a_X^2 + a_Y^2 + a_Z^2$.

To see this just try it out on any $\hat{\mathbf{a}}$-component: $\sigma_a = \boldsymbol{\sigma}\bullet\hat{\mathbf{a}} = a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z$.

$$\sigma_a^2 = (\boldsymbol{\sigma}\bullet\hat{\mathbf{a}})(\boldsymbol{\sigma}\bullet\hat{\mathbf{a}}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)$$

$$= \begin{matrix} a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z \\ +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z \\ +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z \end{matrix} = \begin{matrix} a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

$$\sigma_Z\cdot\sigma_X: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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To finish we need another symmetry property called *anti-commutation*: $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$, *etc.*

$$\sigma_a^2 = (\boldsymbol{\sigma}\bullet\hat{\mathbf{a}})(\boldsymbol{\sigma}\bullet\hat{\mathbf{a}}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)$$

$$= \begin{matrix} a_X^2\mathbf{1} & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ -a_X a_Y\sigma_X\sigma_Y & +a_Y^2\mathbf{1} & +a_Y a_Z\sigma_Y\sigma_Z \\ -a_X a_Z\sigma_X\sigma_Z & -a_Y a_Z\sigma_Y\sigma_Z & +a_Z^2\mathbf{1} \end{matrix} = (a_X^2 + a_Y^2 + a_Z^2)\mathbf{1} = \mathbf{1}$$

So: $\sigma_a^2 = \mathbf{1}$

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$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(b_X\sigma_X + b_Y\sigma_Y + b_Z\sigma_Z)$$

$$= \begin{matrix} a_Xb_X\mathbf{1} & +a_Xb_Y\sigma_X\sigma_Y & -a_Xb_Z\sigma_Z\sigma_X \\ -a_Yb_X\sigma_X\sigma_Y & +a_Yb_Y\mathbf{1} & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & -a_Zb_X\sigma_Y\sigma_Z & +a_Zb_Z\mathbf{1} \end{matrix} = (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} \begin{matrix} +i(a_Yb_Z - a_Zb_Y)\sigma_X \\ +i(a_Zb_X - a_Xb_Z)\sigma_Y \\ +i(a_Xb_Y - a_Yb_X)\sigma_Z \end{matrix}$$

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Write the product in Gibbs notation. (Where do you think Gibbs *got* his {**i,j,k**} notation!)

$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\sigma$$

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$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\sigma$$

(Recall (1.10.29). in complex variable unit.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
$$= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) = (\mathbf{A}\bullet\mathbf{B}) + i(\mathbf{A}\times\mathbf{B})_Z$$

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$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$

$$- i(\varphi + \frac{1}{3!}\varphi^3 \cdots) \quad - i(\sin\varphi)$$

Note even powers of *(-i)* are ±*1* and odd powers of *(-i)* are ±*i*.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

Lecture 38 ends here

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$$(-i\sigma_a)^0 = +\mathbf{1},\ (-i\sigma_a)^1 = -i\sigma_a,\ (-i\sigma_a)^2 = -\mathbf{1},\ (-i\sigma_a)^3 = +i\sigma_a,\ (-i\sigma_a)^4 = +\mathbf{1},\ (-i\sigma_a)^5 = -i\sigma_a,\ etc.$$

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This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\boldsymbol{\sigma}_a\varphi}$ for any $\sigma_a = (\boldsymbol{\sigma}\bullet\mathbf{a}) = a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z$.

$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$ *generalizes to:* $e^{-i\boldsymbol{\sigma}_a\varphi} = \mathbf{1}\cos\varphi \ - \ i\ \boldsymbol{\sigma}_a \sin\varphi$

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ABCD Time evolution operator

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$= e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\Omega}\cdot t/2} e^{-i\Omega_0\cdot t} \text{ where: } \boldsymbol{\Theta} = \boldsymbol{\Omega}\cdot t = \begin{pmatrix}\Omega_A \\ \Omega_B \\ \Omega_C\end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C\end{pmatrix}\cdot t \text{ and: } \Omega_0 = \frac{A+D}{2}$$

Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall (1.10.17).)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = \left[1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots\right] = [\cos\varphi]$$

$$-i\left(\varphi + \frac{1}{3!}\varphi^3 \cdots\right) \qquad -i(\sin\varphi)$$

Note even powers of *(-i)* are ± 1 and odd powers of *(-i)* are $\pm i$.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

Hamilton replaces *(-i)* with $-i\sigma_a$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

$$(-i\sigma_a)^0 = +\mathbf{1},\ (-i\sigma_a)^1 = -i\sigma_a,\ (-i\sigma_a)^2 = -\mathbf{1},\ (-i\sigma_a)^3 = +i\sigma_a,\ (-i\sigma_a)^4 = +\mathbf{1},\ (-i\sigma_a)^5 = -i\sigma_a,\ etc.$$

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$$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$$

generalizes to:

$$e^{-i\boldsymbol{\sigma}_a\varphi} = \mathbf{1}\cos\varphi - i\ \boldsymbol{\sigma}_a\sin\varphi$$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t}$$

$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

$$= e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\Omega}\cdot t/2} e^{-i\Omega_0\cdot t} \text{ where: } \boldsymbol{\Theta} = \boldsymbol{\Omega}\cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t \text{ and: } \Omega_0 = \frac{A+D}{2}$$

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Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

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$$- i\left(\varphi + \frac{1}{3!}\varphi^3 \cdots\right) \quad - i(\sin\varphi)$$

Note even powers of *(-i)* are ± 1 and odd powers of *(-i)* are $\pm i$.: $(-i)^0 = +1$, $(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = +i$, $(-i)^4 = +1$, $(-i)^5 = -i$, *etc.*

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This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\boldsymbol{\sigma}_a\varphi}$ for any $\sigma_a = (\boldsymbol{\sigma}\bullet\mathbf{a}) = a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z$.

$$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$$ *generalizes to:* $$e^{-i\boldsymbol{\sigma}_a\varphi} = \mathbf{1}\cos\varphi - i\ \boldsymbol{\sigma}_a\sin\varphi$$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2: C or Y rotation

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$= e^{-i\boldsymbol{\sigma}\bullet\boldsymbol{\Omega}\cdot t/2} e^{-i\Omega_0\cdot t} \text{ where: } \boldsymbol{\Theta} = \boldsymbol{\Omega}\cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t \text{ and: } \Omega_0 = \frac{A+D}{2}$$

ABCD Time evolution operator

Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

3D crank vector $\vec{\Theta} = \vec{\Omega}\cdot t$ and *spin operator* $\mathbf{S}$ defines *3D ABC*-rotation with ratio $\frac{1}{2}$ or 2 between Θ_a and $\varphi_a = \frac{1}{2}\Theta_a$ or between $\mathbf{S}$ and $\boldsymbol{\sigma} = 2\mathbf{S}$.

$$e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}} = e^{-i\boldsymbol{\sigma}\bullet\vec{\Theta}/2} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = \mathbf{1}\cos\frac{\Theta}{2} - i\,(\boldsymbol{\sigma}\bullet\hat{\Theta})\sin\frac{\Theta}{2} = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_A\sin\frac{\Theta}{2} & (-i\hat{\Theta}_B - \hat{\Theta}_C)\sin\frac{\Theta}{2} \\ (-i\hat{\Theta}_B + \hat{\Theta}_C)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + i\hat{\Theta}_A\sin\frac{\Theta}{2} \end{pmatrix}$$

Example 3:
Any $\boldsymbol{\Theta} = \boldsymbol{\Omega}t$*-axial rotation*

2D angle : $\varphi = \frac{1}{2}\Theta$ $\qquad$ *3D Crank vector* : $\vec{\Theta} = \Theta\hat{\Theta} = 2\varphi_a\hat{\mathbf{a}} = 2\vec{\varphi}$ $\qquad$ *2D spin matrix* : $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$

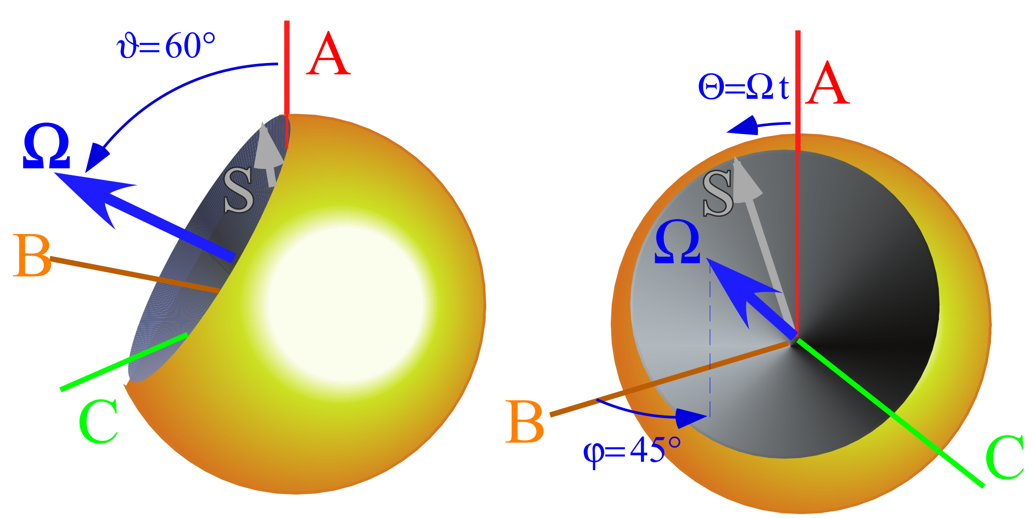


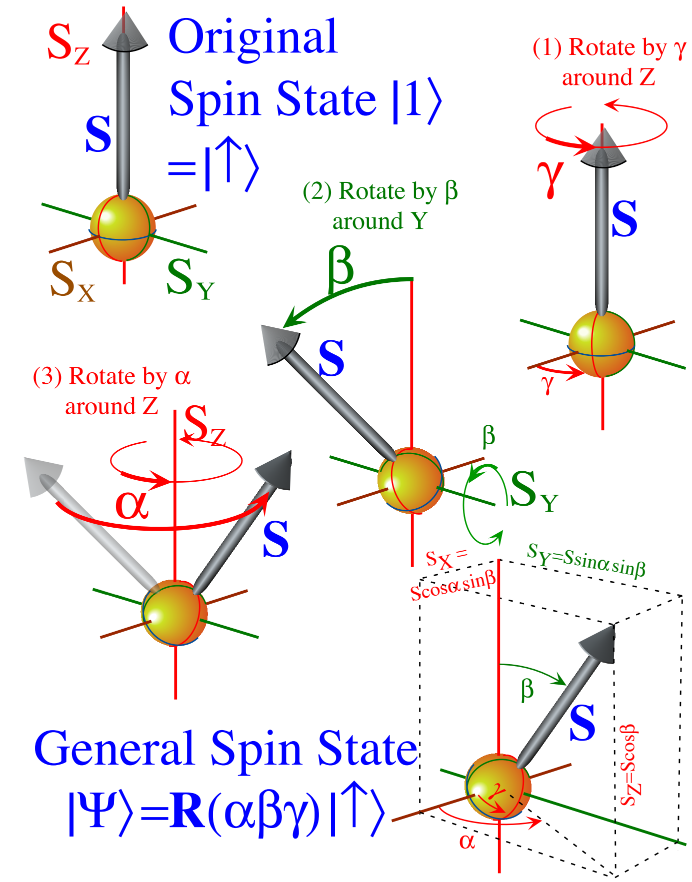
Fig. 3.4.2 Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* ***S*** *in ABC-space.*

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Vector space

3D vector $\hat{\mathbf{a}}$ defines a combination $\sigma_a = a_A\sigma_A + a_B\sigma_B + a_C\sigma_C$ of *operators* $\sigma_A, \sigma_B, \sigma_C$.
These may be rotated by 2-by-2 σ_a matrices acting *twice*, fore and aft^{-1}
The result is rotation by *twice* the *2D* angle φ_a.

$$\mathbf{R}(\varphi_C) \cdot \boldsymbol{\sigma}_A \cdot \mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin 2\varphi_C$$
$$= \boldsymbol{\sigma}_A \cos 2\varphi_C + \boldsymbol{\sigma}_B \sin 2\varphi_C$$

$$\mathbf{R}(\varphi_C) \cdot \boldsymbol{\sigma}_B \cdot \mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C \end{pmatrix}$$
$$= \begin{pmatrix} -2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C \\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\sin 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos 2\varphi_C$$
$$= -\boldsymbol{\sigma}_A \sin 2\varphi_C + \boldsymbol{\sigma}_B \cos 2\varphi_C$$



3D *Stokes vector components* S_a define polarization ellipses, *2D* HO orbits, and spin-$\frac{1}{2}$ states.

Each point $\{E_1, E_2\}$ in complex *2D* oscillator space or in analogous to Ψ-space given by 2D array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$
maps to real 3D spin vector (S_A, S_B, S_C) in that "points" to a particular state of polarization.

Asymmetry $S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$

Balance $S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$

Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$

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$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$=\begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$=\begin{pmatrix}\cos^2\varphi_C-\sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C-\cos^2\varphi_C\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos 2\varphi_C+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$

$$=\boldsymbol{\sigma}_A\cos 2\varphi_C+\boldsymbol{\sigma}_B\sin 2\varphi_C$$

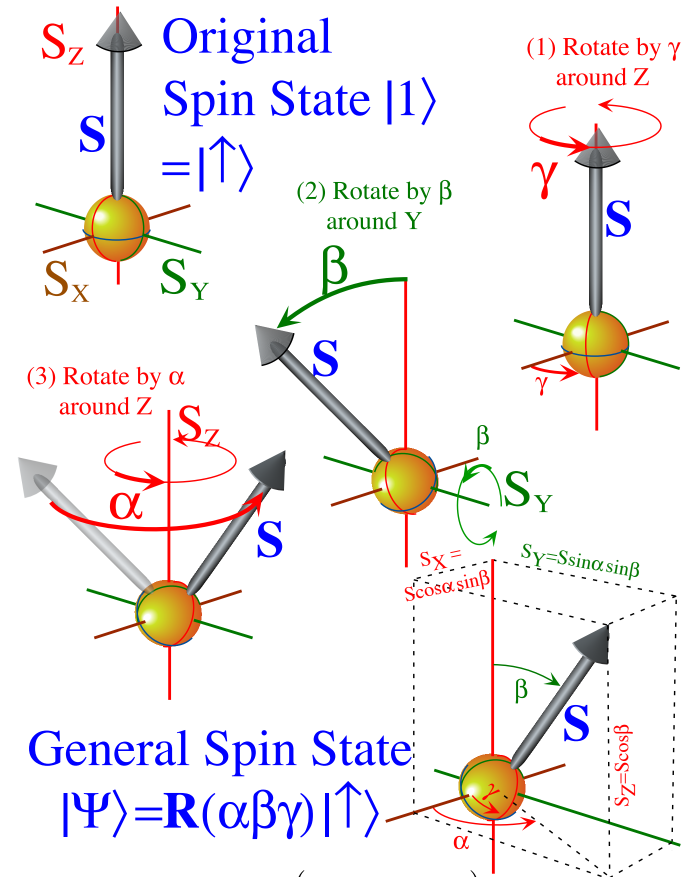
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$=\begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$=\begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C-\sin^2\varphi_C\\ \cos^2\varphi_C-\sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\sin 2\varphi_C+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cos 2\varphi_C$$

$$=-\boldsymbol{\sigma}_A\sin 2\varphi_C+\boldsymbol{\sigma}_B\cos 2\varphi_C$$



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$$\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=A\begin{pmatrix}e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$

Asymmetry $S_A=\frac{1}{2}\langle a|\sigma_A|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_1-a_2^*a_2\right]=\frac{1}{2}\left[x_1^2+p_1^2-x_2^2-p_2^2\right]=\frac{I}{2}[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}]=\frac{I}{2}\cos\beta$

Balance $S_B=\frac{1}{2}\langle a|\sigma_B|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\frac{1}{2}\left[a_1^*a_2+a_2^*a_1\right]=\left[p_1p_2+x_1x_2\right]=I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}+\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C=\frac{1}{2}\langle a|\sigma_C|a\rangle=\frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\frac{-i}{2}\left[a_1^*a_2-a_2^*a_1\right]=\left[x_1p_2-x_2p_1\right]=I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}-\cos\frac{\alpha-\gamma}{2}\cdot-\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{I}{2}\sin\alpha\sin\beta$

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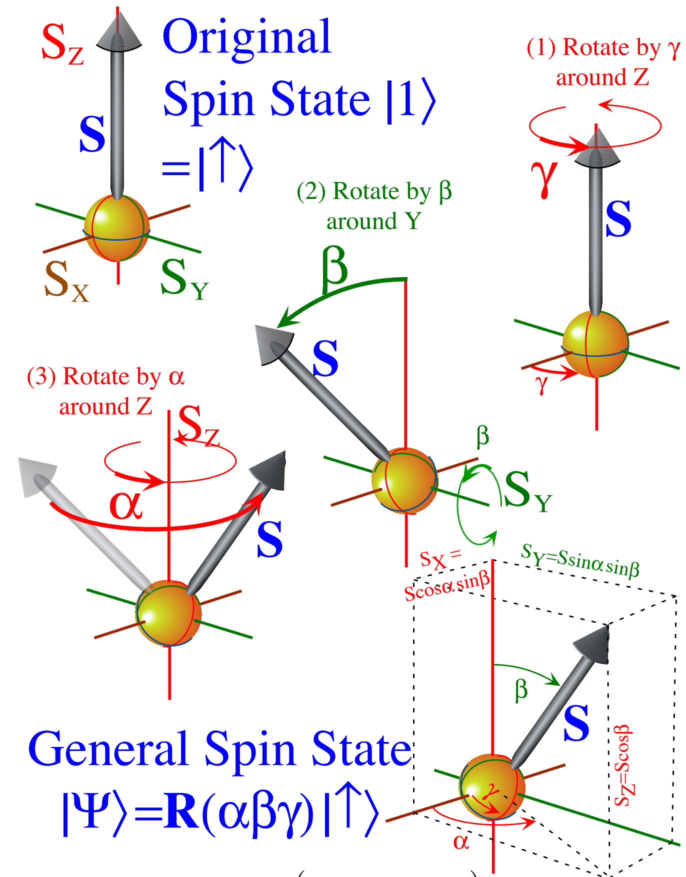

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Asymmetry $S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{1}{2}\left[a_1^*a_1 - a_2^*a_2\right] = \frac{1}{2}\left[x_1^2+p_1^2-x_2^2-p_2^2\right] = \frac{I}{2}[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}] = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{1}{2}\left[a_1^*a_2 + a_2^*a_1\right] = \left[p_1p_2 + x_1x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\ \cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix}a_1^* & a_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{-i}{2}\left[a_1^*a_2 - a_2^*a_1\right] = \left[x_1p_2 - x_2p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

Euler's definition of state $|a\rangle$ *of rotation using spin-1/2 matrix* **R**(α/2) *and* **R**(β/2))

$$|a\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle$$
$$= \mathbf{R}[\alpha\ \text{about}\ Z]\cdot\mathbf{R}[\beta\ \text{about}\ Y]\cdot\mathbf{R}[\gamma\ \text{about}\ Z]|\uparrow\rangle$$
$$= \begin{pmatrix}e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}\begin{pmatrix}A\\ 0\end{pmatrix} = \begin{pmatrix}e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}A\\ 0\end{pmatrix} = A\begin{pmatrix}e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}} = \begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}$$

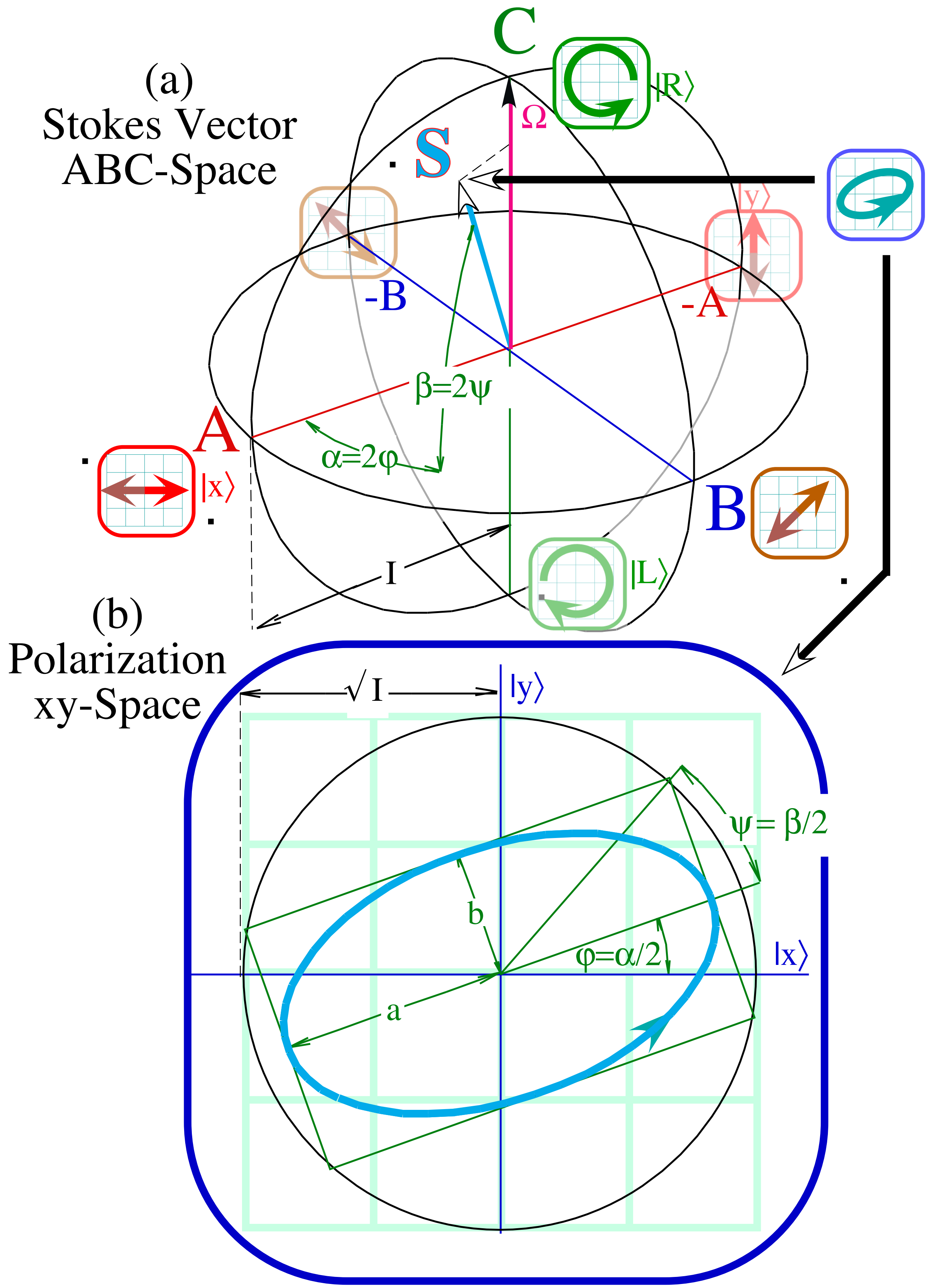

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

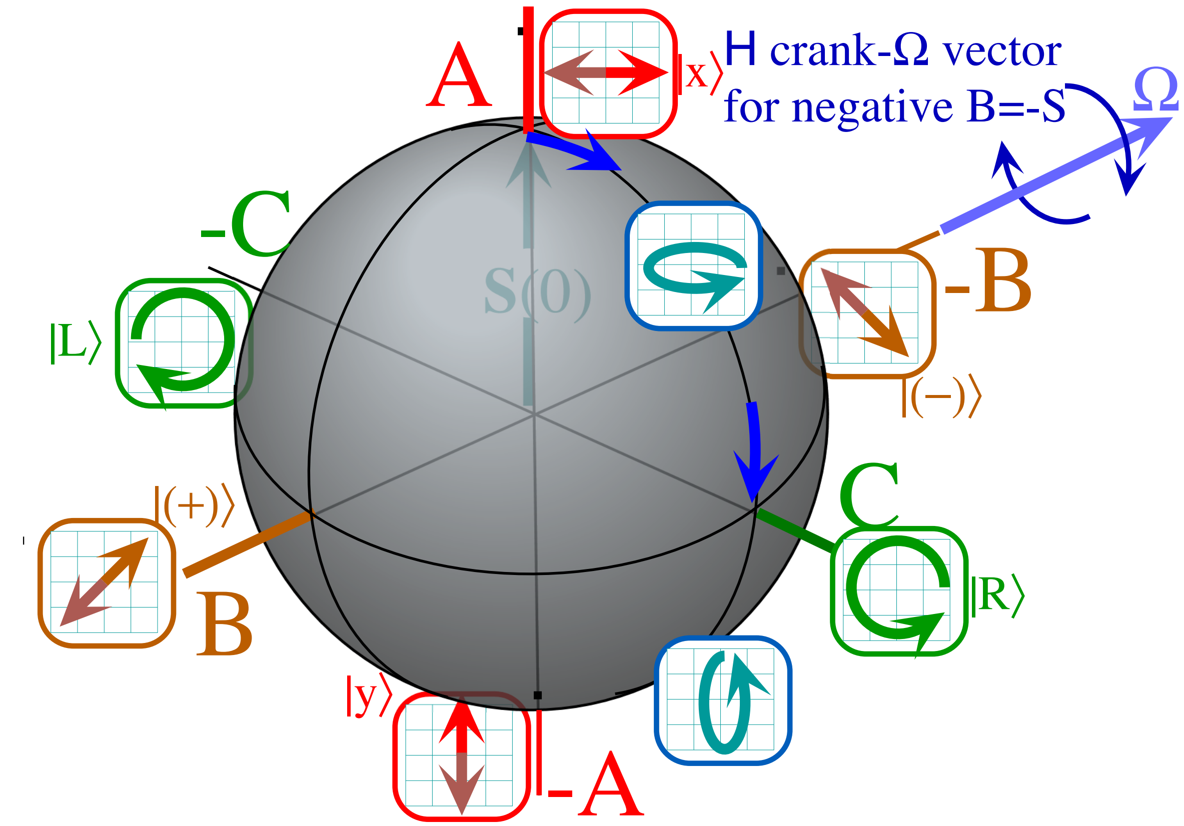

Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

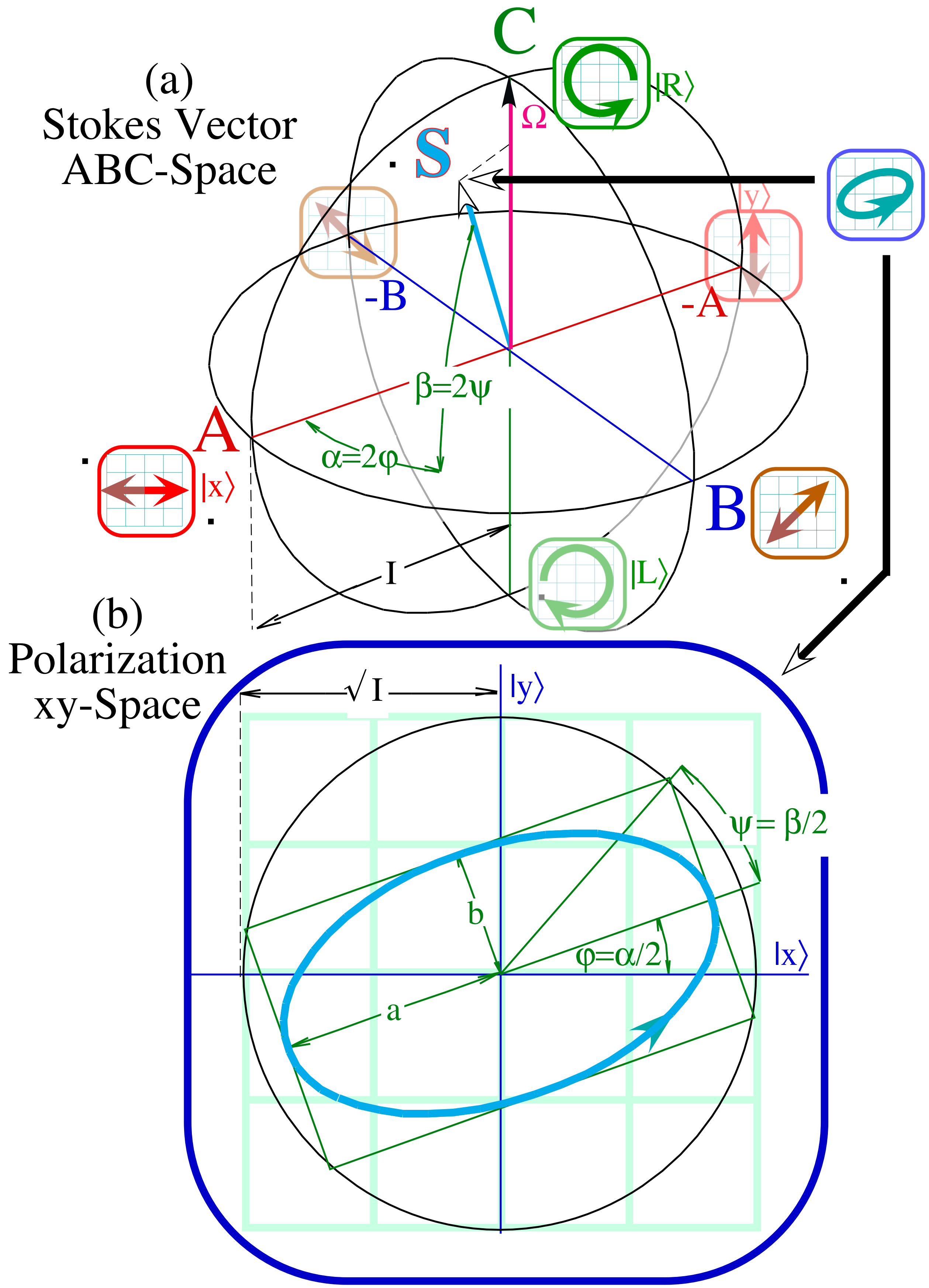

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

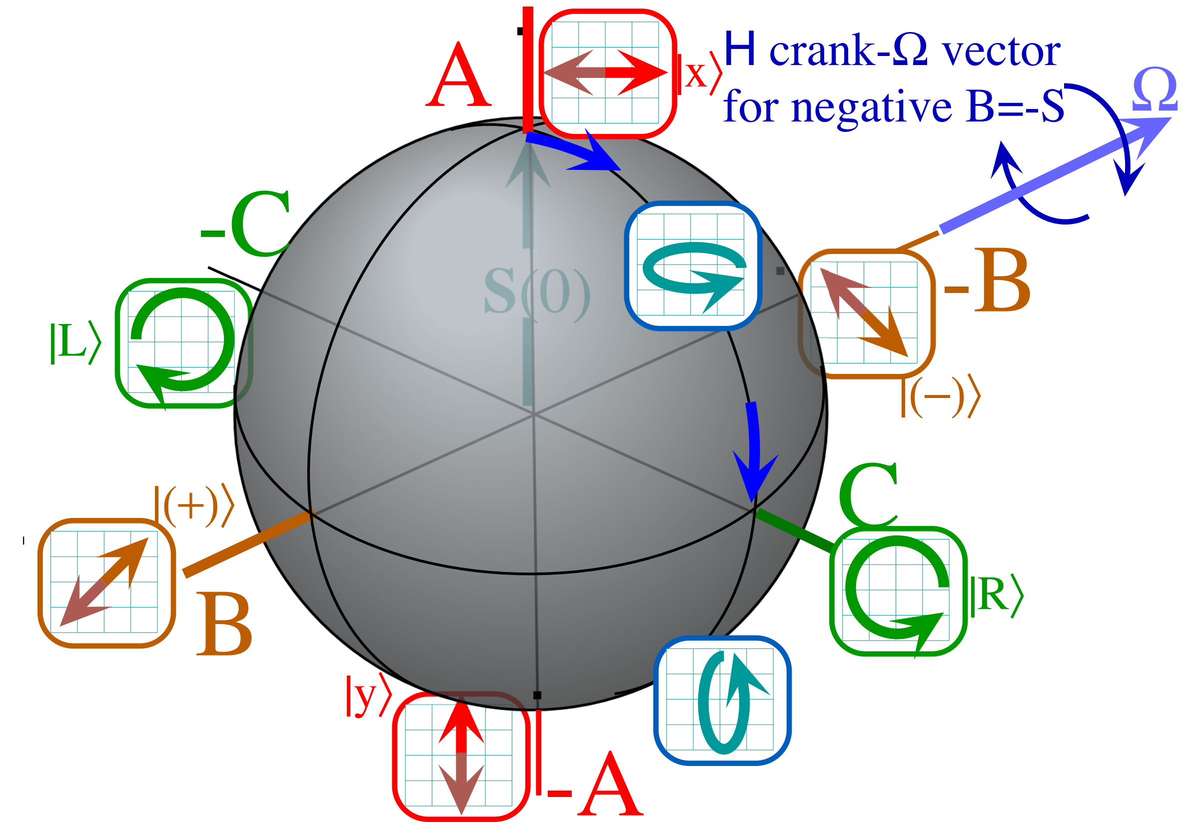

Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

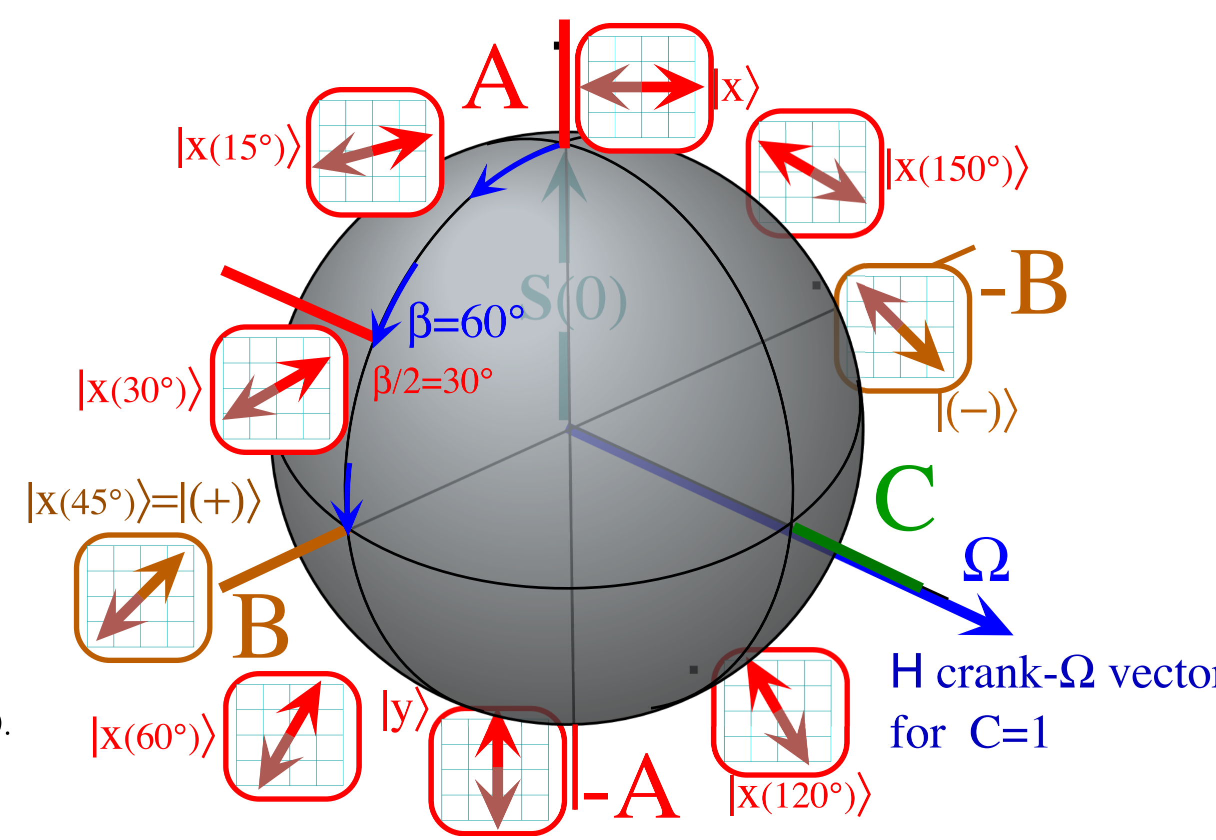

Fig. 3.4.7 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*