AMOP Lecture 11
Tue 3.13 2014

Based on Lectures 23-25
Group Theory in Quantum Mechanics

Quantum theory of harmonic oscillators U(1)⊂U(2)⊂U(3)...

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 20-22 , PSDS - Ch. 8)

EM Waves are made of (relativistic) oscillators?

1-D **a†a** *algebra of U(1) representations*
- *Creation-Destruction* **a†a** *algebra*
- *Eigenstate creationism (and destruction)*
  - *Vacuum state*
  - *1st excited state*
- *Normal ordering for matrix calculation*
  - *Commutator derivative identities*
  - *Binomial expansion identities*
- *Matrix* $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculations*
  - *Number operator and Hamiltonian operator*
  - *Expectation values of position, momentum, and uncertainty for eigenstate* $|n\rangle$
  - *Harmonic oscillator beat dynamics of mixed states*
- *Oscillator coherent states ("Shoved" and "kicked" states)*
  - *Translation operators vs. boost operators*
  - *Applying boost-translation combinations*
  - *Time evolution of coherent state*
  - *Properties of coherent state and "squeezed" states*

NEXT Lect 12:2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

EM Waves are made of (relativistic) oscillators?

(We add later the nonradiative or static field)

QTCA Unit 7 Ch. 22

$$\mathbf{E}_{non-rad} = -\nabla\Phi$$

Plane-wave solutions to Maxwell Equations

$$\mathbf{A} = \mathbf{e}_1 2|a| \sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \mathbf{e}_1 E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$$

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Electric E-polarization vector at zero phase

$$E_0\mathbf{e}_1 = 2|a|\omega\,\mathbf{e}_1$$

Magnetic B-polarization vector at zero phase

$$B_0\mathbf{b}_1 = B_0(\mathbf{k}\times\mathbf{e}_1) = \mathbf{e}_2 2|a|\omega/c \qquad (\text{Let: } k=\omega/c)$$

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Averaged EM field energy $\langle U\rangle V$ for a plane wave in volume V

(Use: $\langle \cos^2\omega t\rangle = \frac{1}{2}$)

$$\langle U\rangle V = \left\langle \frac{\varepsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right\rangle V = V\left(\frac{\varepsilon_0}{2}4|a|^2\omega^2 + 4\frac{|a|^2}{2\mu_0}k^2\right)\langle\cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\rangle$$

$$= 2\varepsilon_0\omega^2|a|^2 V = 2(k^2/\mu_0)|a|^2 V$$

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Einstein-Planck energy-frequency relation $(\langle U\rangle V = \hbar n\omega)$ for n=*1* photon.

$$|a| = \sqrt{\frac{\hbar\omega}{2\varepsilon_0\omega^2 V}} = \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} = A$$

Quantum field unit

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Sum every possible value of **k** and for each choice $\mathbf{e}_1$ or $\mathbf{e}_2$ of polarization orthogonal to **k**.

$$\mathbf{A} = \sum_{\mathbf{k}}\left[(a_{\mathbf{k}1}\mathbf{e}_1 + a_{\mathbf{k}2}\mathbf{e}_2)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{c.c.}\right] = \sum_{\mathbf{k}}\sum_{\alpha=1}^{2}\left[a_{\mathbf{k}\alpha}\mathbf{e}_\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + a^*_{\mathbf{k}\alpha}\mathbf{e}_\alpha e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

$$k_\beta = n_\beta\frac{2\pi}{L} \qquad (n_\beta = 1,2,\ldots j,\ \beta = x,y,z)$$

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A time derivative gives electric **E** field.

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The classical Hamiltonian is a volume V integral of energy density. Electric **E** field contribution is: $U_E V=\frac{\varepsilon_0}{2}\int d^3\mathbf{r}\ \mathbf{E}\cdot\mathbf{E}$,

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$$= \sum_{k'\alpha'} \sum_{k\alpha} \Big[-a_{\mathrm{k}'\alpha'} a_{\mathrm{k}\alpha} \omega'\omega\ \mathbf{e}_{\alpha'} \bullet \mathbf{e}_\alpha e^{i(\mathrm{k}'+\mathrm{k})\cdot\mathrm{r}-i(\omega'+\omega)t} - a^*_{\mathrm{k}'\alpha'} a^*_{\mathrm{k}\alpha} \omega'\omega\ \mathbf{e}_{\alpha'} \bullet \mathbf{e}_\alpha e^{i(\mathrm{k}'+\mathrm{k})\cdot\mathrm{r}+i(\omega'+\omega)t}$$

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simplified by normalization conditions:

$$\int d^3\mathrm{r}\ e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}} = \delta_{\mathrm{k}',-\mathrm{k}} V$$

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$$=\sum_{k'\alpha'}\sum_{k\alpha}\left[-a_{\mathrm{k}'\alpha'}a_{\mathrm{k}\alpha}\omega'\omega\ \mathbf{e}_{\alpha'}\bullet\mathbf{e}_\alpha e^{i(\mathrm{k}'+\mathrm{k})\cdot\mathrm{r}-i(\omega'+\omega)t}-a^*_{\mathrm{k}'\alpha'}a^*_{\mathrm{k}\alpha}\omega'\omega\ \mathbf{e}_{\alpha'}\bullet\mathbf{e}_\alpha e^{i(\mathrm{k}'+\mathrm{k})\cdot\mathrm{r}+i(\omega'+\omega)t}\right.$$

$$\left.+a^*_{\mathrm{k}'\alpha'}a_{\mathrm{k}\alpha}\omega'\omega\ \mathbf{e}_{\alpha'}\bullet\mathbf{e}_\alpha e^{i(\mathrm{k}'-\mathrm{k})\cdot\mathrm{r}-i(\omega'-\omega)t}+a_{\mathrm{k}'\alpha'}a^*_{\mathrm{k}\alpha}\omega'\omega\ \mathbf{e}_{\alpha'}\bullet\mathbf{e}_\alpha e^{i(\mathrm{k}'-\mathrm{k})\cdot\mathrm{r}+i(\omega'-\omega)t}\right]$$

simplified by normalization conditions:

$$\int d^3\mathrm{r}\ e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}}=\delta_{\mathrm{k}',-\mathrm{k}}V$$

$$\mathbf{e}_{\alpha'}\cdot\mathbf{e}_\alpha=\delta_{a'a}.$$

$$U_EV=\sum_{\mathrm{k}\alpha}\frac{\varepsilon_0V}{2}\left[2\left|a_{\mathrm{k}\alpha}\right|^2\omega^2-a^*_{-\mathrm{k}\alpha}a^*_{\mathrm{k}\alpha}\omega^2e^{-2i\omega t}-a_{-\mathrm{k}\alpha}a_{\mathrm{k}\alpha}\omega^2e^{-2i\omega t}\right].$$

Magnetic **B** energy $U_BV=\int d^3r\,\mathrm{B}\cdot\mathrm{B}/2u_0$ is like above with substitutions:

$$\mathbf{E}\to\mathbf{B},\quad \frac{\varepsilon_0}{2}\to\frac{1}{2\mu_0},\quad \omega e_\alpha\to k\mathbf{b}_\alpha\equiv\mathbf{k}\times\mathbf{e}_\alpha$$

$$U_BV=\sum_{\mathrm{k}\alpha}\frac{V}{2\mu_0}\left[2\left|a_{\mathrm{k}\alpha}\right|^2k^2+a^*_{-\mathrm{k}\alpha}a^*_{\mathrm{k}\alpha}k^2e^{2i\omega t}+a_{-\mathrm{k}\alpha}a_{\mathrm{k}\alpha}k^2e^{-2i\omega t}\right]$$

$$=\sum_{\mathrm{k}\alpha}\frac{\varepsilon_0V}{2}\left[2\left|a_{\mathrm{k}\alpha}\right|^2\omega^2+a^*_{-\mathrm{k}\alpha}a^*_{\mathrm{k}\alpha}\omega^2e^{2i\omega t}+a_{-\mathrm{k}\alpha}a_{\mathrm{k}\alpha}\omega^2e^{-2i\omega t}\right].$$

$$\omega^2=c^2k^2=k^2/(\mu_0\varepsilon_0)$$

EM Waves ***are*** *made of (relativistic) oscillators***!**

$$U_E V = \sum_{k\alpha} \frac{\varepsilon_0 V}{2}\left[2\left|a_{k\alpha}\right|^2 \omega^2 - a^*_{-k\alpha} a^*_{k\alpha} \omega^2 e^{-2i\omega t} - a_{-k\alpha} a_{k\alpha} \omega^2 e^{-2i\omega t} \right].$$

$$+ \quad U_B V = \sum_{k\alpha} \frac{\varepsilon_0 V}{2}\left[2\left|a_{k\alpha}\right|^2 \omega^2 + a^*_{-k\alpha} a^*_{k\alpha} \omega^2 e^{2i\omega t} + a_{-k\alpha} a_{k\alpha} \omega^2 e^{-2i\omega t} \right].$$

$$UV = \left(U_E + U_B\right)V = \sum_{k\alpha} 2\varepsilon_0 \omega^2 \left|a_{k\alpha}\right|^2 V \, .$$

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...to be continued after review of 1D-quantum oscillator mechanics...

1-D $a^\dagger a$ algebra of U(1) representations

Creation-Destruction $a^\dagger a$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle a^n a^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $a^\dagger a$ algebra of U(2) representations and R(3) angular momentum operators

1-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

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$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \mathbf{P}^2+\mathbf{X}^2 = (\mathbf{X}-i\mathbf{P})(\mathbf{X}+i\mathbf{P})/2 + (\mathbf{X}+i\mathbf{P})(\mathbf{X}-i\mathbf{P})/2$$ ...so make symmetric factors.

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp}-\mathbf{px} = \hbar i\,\mathbf{1}$

Proof: *QED:*

$$\langle x|\mathbf{xp}-\mathbf{px}|\psi\rangle = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - \frac{\partial}{\partial x}x\psi(x)\right) = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - x\frac{\partial}{\partial x}\psi(x) - \frac{\partial x}{\partial x}\psi(x)\right) = -\frac{\hbar}{i}\psi(x) = \hbar i\psi(x) = \langle x|[\mathbf{x},\mathbf{p}]|\psi\rangle$$

Q: How to convert *classical* HO Hamiltonian to *quantum* HO Hamiltonian?

$$E = H(x,p) = \frac{1}{2M}p^2 + \frac{1}{2}M\omega^2 x^2$$

A:Rewrite *classical H(x,p)* with a **thick** pen!

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \mathbf{p}^2/2M + V(\mathbf{x}) = \mathbf{p}^2/2M + M\omega^2\mathbf{x}^2/2 \qquad \text{with: } \mathbf{p} = \hbar\mathbf{k} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Q: How to *solve* a quantum HO Hamiltonian **H**(**x**,**p**)?

A:Rewrite **H**(**x**,**p**) with a **THICKER** pen! (Shows **x**⇄**p** symmetry)

$$\mathbf{H} = \mathbf{P}^2 + \mathbf{X}^2 \text{ where: } \mathbf{X} \equiv \sqrt{M\omega}\,\mathbf{x}/\sqrt{2} \text{ and } \mathbf{P} \equiv \mathbf{p}/\sqrt{(2M)}$$

Q: OK! HaHa! But, *really* how *do* you solve a quantum HO Hamiltonian **H**(**x**,**p**)?

A:Factor $\mathbf{P}^2+\mathbf{X}^2$! But, recall that **X** and **P** don't quite commute... (Use **x**⇄**p** symmetry)

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \mathbf{P}^2+\mathbf{X}^2 = (\mathbf{X}-i\mathbf{P})(\mathbf{X}+i\mathbf{P})/2 + (\mathbf{X}+i\mathbf{P})(\mathbf{X}-i\mathbf{P})/2$$ ...so make symmetric factors.

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp}-\mathbf{px} = \hbar i\mathbf{1}$ (They only miss commuting by 0.0000000000000... !)

Proof: *QED:*

$$\langle x|\mathbf{xp}-\mathbf{px}|\psi\rangle = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - \frac{\partial}{\partial x}x\psi(x)\right) = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - x\frac{\partial}{\partial x}\psi(x) - \frac{\partial x}{\partial x}\psi(x)\right) = -\frac{\hbar}{i}\psi(x) = \hbar i\psi(x) = \langle x|[\mathbf{x},\mathbf{p}]|\psi\rangle$$

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*2-D **a†a** algebra of U(2) representations and R(3) angular momentum operators*

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define

$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X} \equiv \sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P} \equiv \mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

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Recall *commutator* $[\mathbf{x}\ ,\ \mathbf{p}]$ *relation*: $[\mathbf{x},\mathbf{p}]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$

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1D-HO Hamiltonian in terms of $\mathbf{a}^\dagger\mathbf{a}$ *operator*

Recall: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger)/2$

Recall *commutator* $[\mathbf{x}\ ,\ \mathbf{p}]$ *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp} - \mathbf{px} = \hbar i\,\mathbf{1}$

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$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger)/2 \quad = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a} + \mathbf{1})/2 \quad = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a} + \mathbf{1}\hbar\omega/2$$

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Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a}, \mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

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But, $\mathbf{a}^\dagger$ acts on ground ket to give $|1\rangle = \mathbf{a}^\dagger|0\rangle$ with $\mathbf{H}$ eigenvalue $E_1 = \hbar\omega + E_0$. ($|1\rangle = \mathbf{a}^\dagger|0\rangle$, $\langle 0|\mathbf{a} = \langle 1|$.)

Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle \qquad + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

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Proof:

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But, $\mathbf{a}^\dagger$ acts on ground ket to give $|1\rangle = \mathbf{a}^\dagger|0\rangle$ with $\mathbf{H}$ eigenvalue $E_1 = \hbar\omega + E_0$. ($|1\rangle = \mathbf{a}^\dagger|0\rangle$, $\langle 0|\mathbf{a} = \langle 1|$.)

Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle \qquad + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$= \hbar\omega\,\mathbf{a}^\dagger|0\rangle + \boldsymbol{0} \qquad + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega \qquad + \hbar\omega/2)\,|1\rangle = E_1|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

Eigenstate creationism (and destruction)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

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One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

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For kets, $\mathbf{a}^\dagger$ is *creation operator* while $\mathbf{a}$ is *destruction operator*.

$$\mathbf{a}|1\rangle = \mathbf{a}\mathbf{a}^\dagger|0\rangle = (\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle = |0\rangle$$

Eigenstate creationism (and destruction)

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For bras, $\mathbf{a}^\dagger$ is *destruction operator* while $\mathbf{a}$ is *creation operator*.

$$\langle 1|\mathbf{a}^\dagger = \langle 0|\mathbf{a}\mathbf{a}^\dagger = \langle 0|(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1}) = \langle 0|$$

1-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Eigenstate creationism (and destruction)

Vacuum state

1^{st} excited state

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2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

Wavefunction creationism (Vacuum state)

Coordinate representation of the "nothing" equation $\langle x|\mathbf{a}|0\rangle = \boldsymbol{0}$

$$\langle x|\mathbf{a}|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle + i\langle x|\mathbf{p}|0\rangle / \sqrt{M\omega}\right) = 0$$

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$$\int \frac{d\psi}{\psi} = \int \frac{M\omega}{\hbar}\, xdx\,, \quad \ln\psi + \ln const. = \frac{-M\omega}{\hbar}\frac{x^2}{2}, \quad \psi = \frac{e^{-M\omega x^2/2\hbar}}{const.}$$

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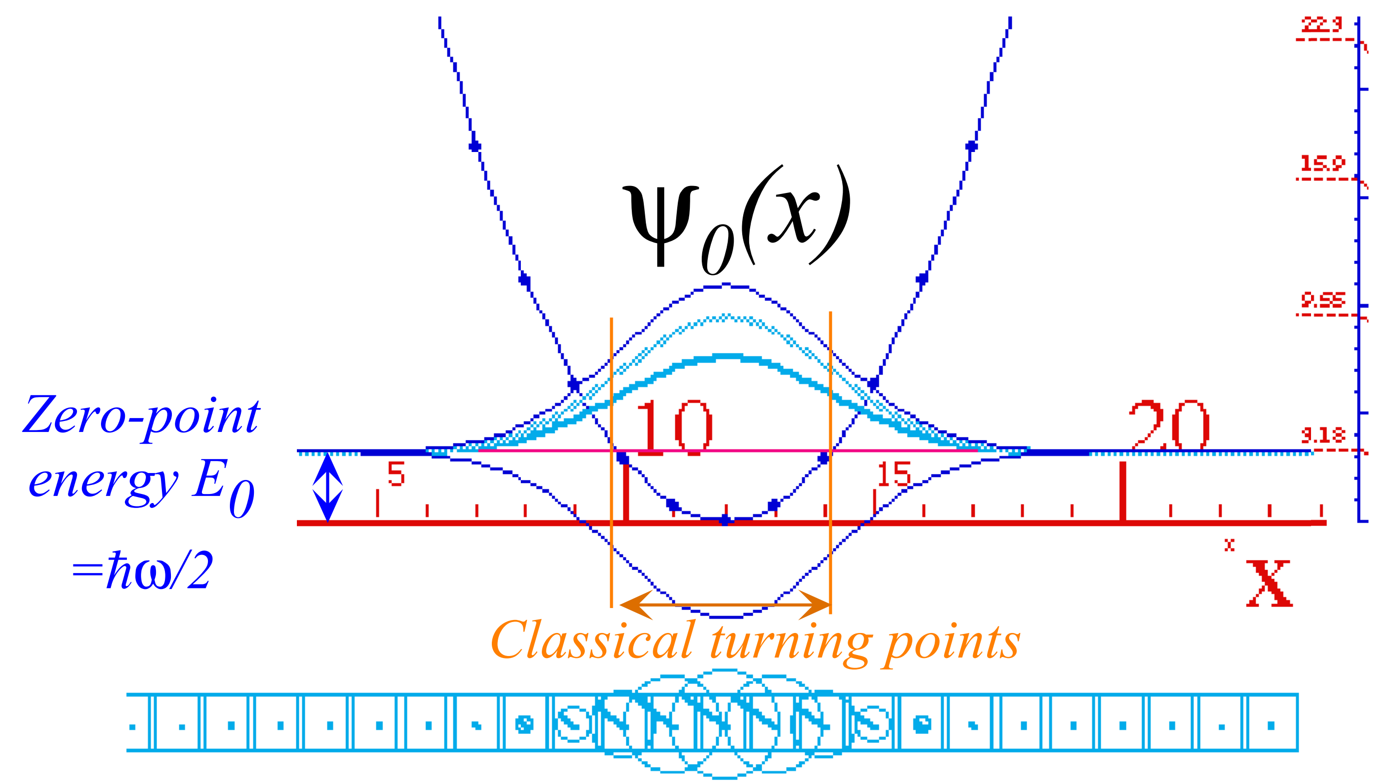

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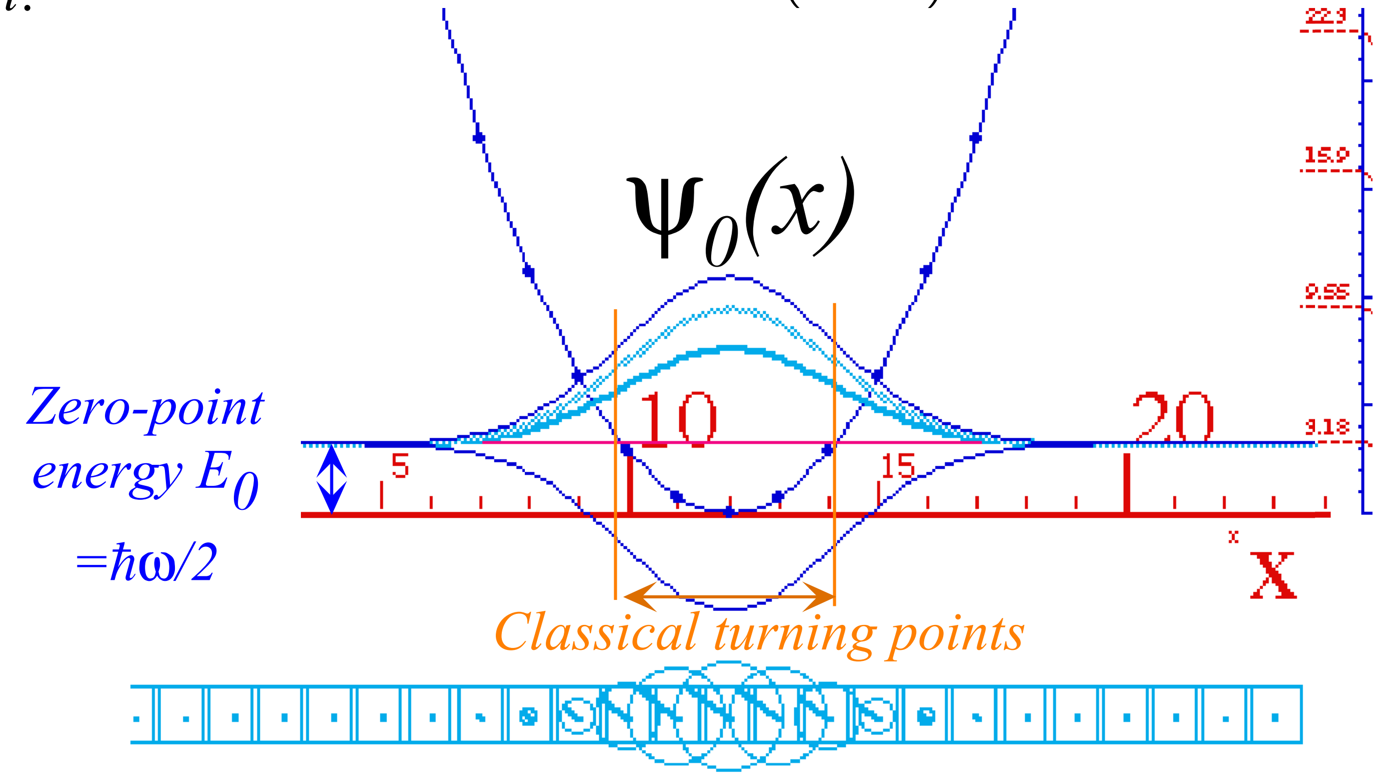
The normalization *const.* is evaluated using a standard Gaussian integral: $\int_{-\infty}^{\infty} dx\; e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\langle\psi_0|\psi_0\rangle = 1 = \int_{-\infty}^{\infty} dx\,\frac{e^{-M\omega x^2 2/2\hbar}}{const.^2} = \sqrt{\frac{\pi\hbar}{M\omega}}/const.^2 \Rightarrow const. = \left(\frac{\pi\hbar}{M\omega}\right)^{1/4}$$



1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

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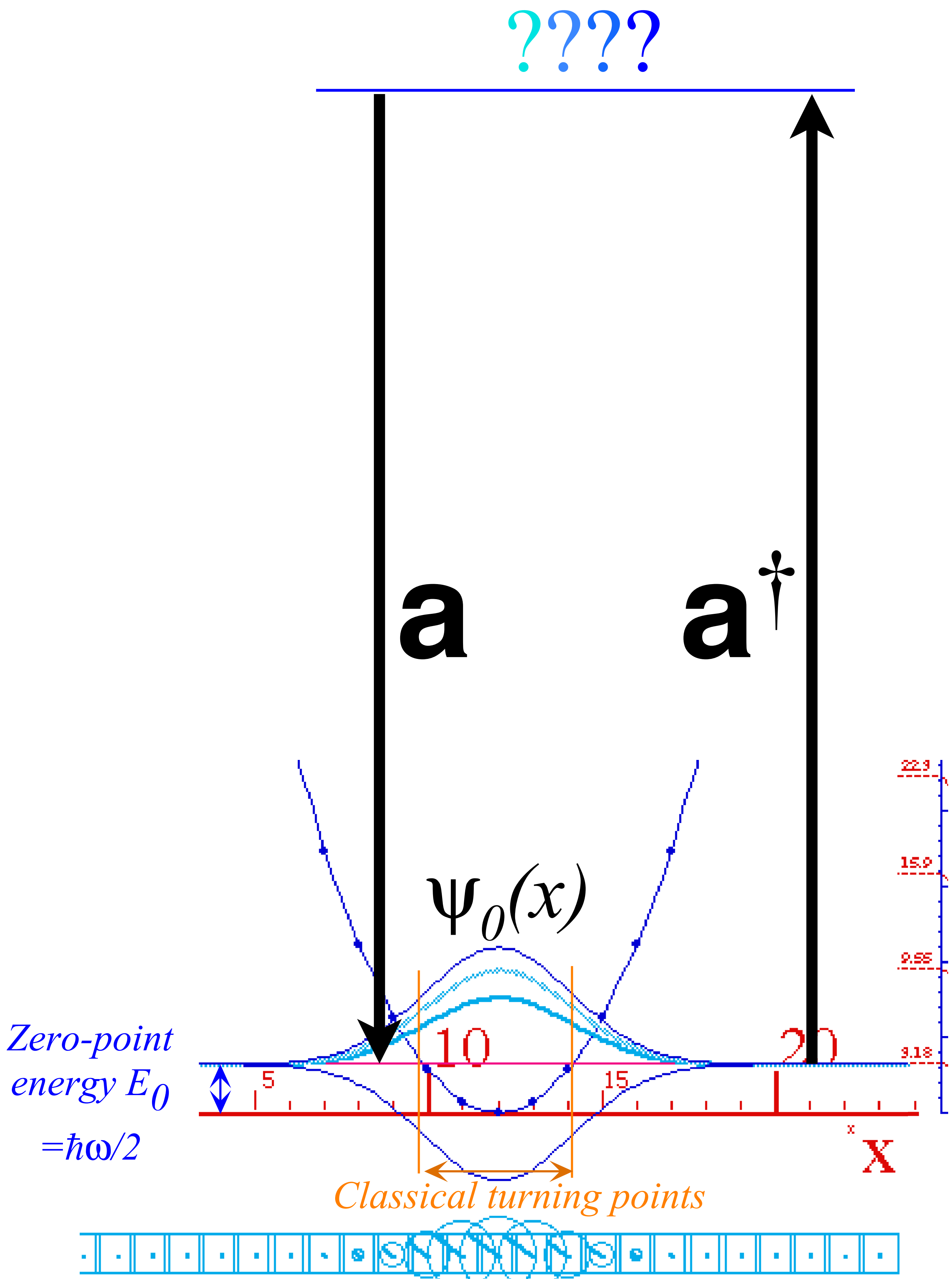
2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Wavefunction creationism (1st Excited state)

1st excited state wavefunction $\psi_1(x)=\langle x\,|1\rangle$

$$\langle x\,|\mathbf{a}^\dagger|0\rangle = \langle x\,|1\rangle = \psi_1(x)$$



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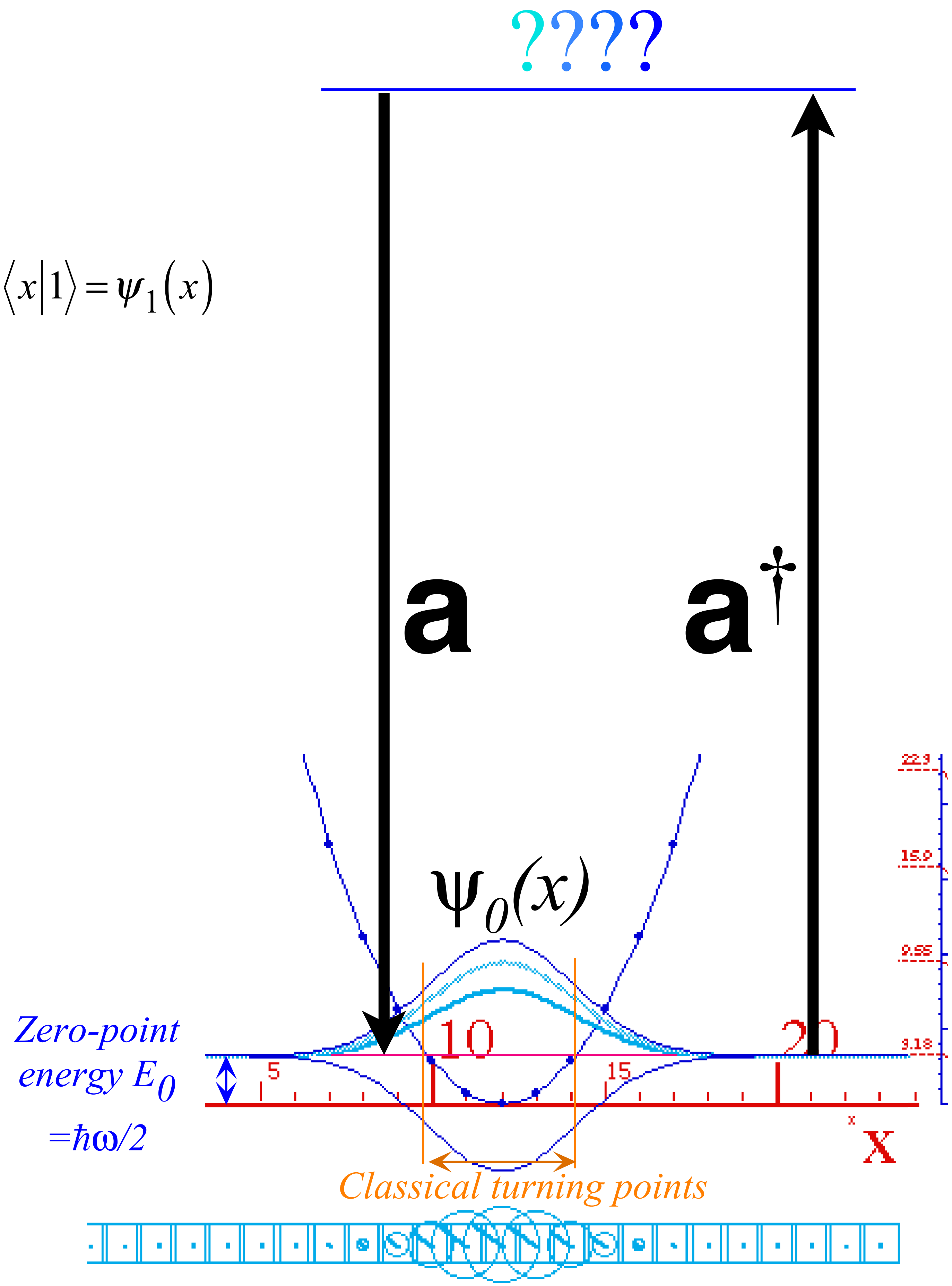
Expanding the creation operator

$$\langle x|\mathbf{a}^\dagger|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle - i\langle x|\mathbf{p}|0\rangle/\sqrt{M\omega}\right) = \langle x|1\rangle = \psi_1(x)$$



Wavefunction creationism (1st Excited state)

1st excited state wavefunction $\psi_1(x) = \langle x|1\rangle$

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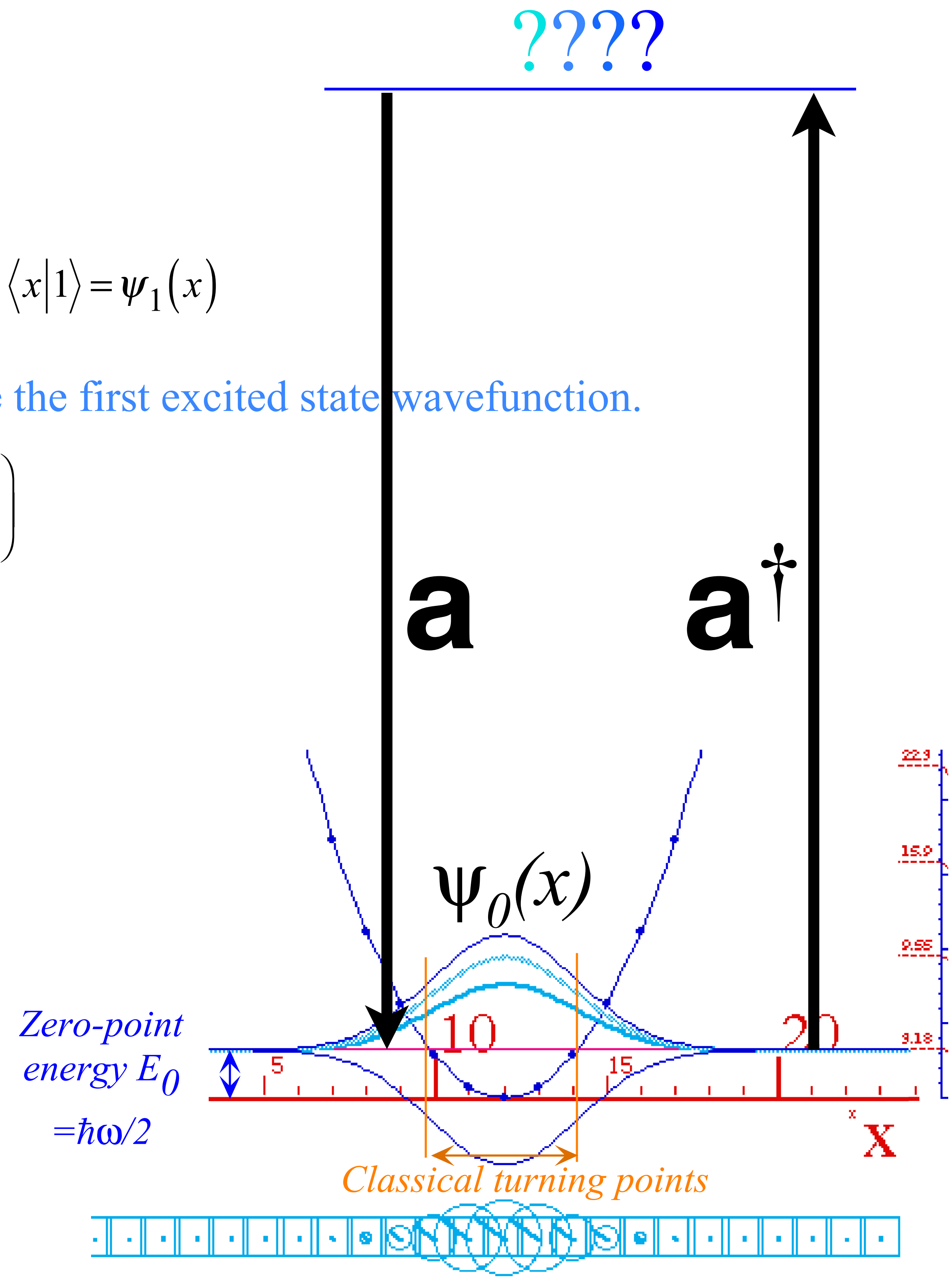
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x|1\rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\psi_0(x) - i\frac{\hbar}{i}\frac{\partial\psi_0(x)}{\partial x}/\sqrt{M\omega}\right)$$



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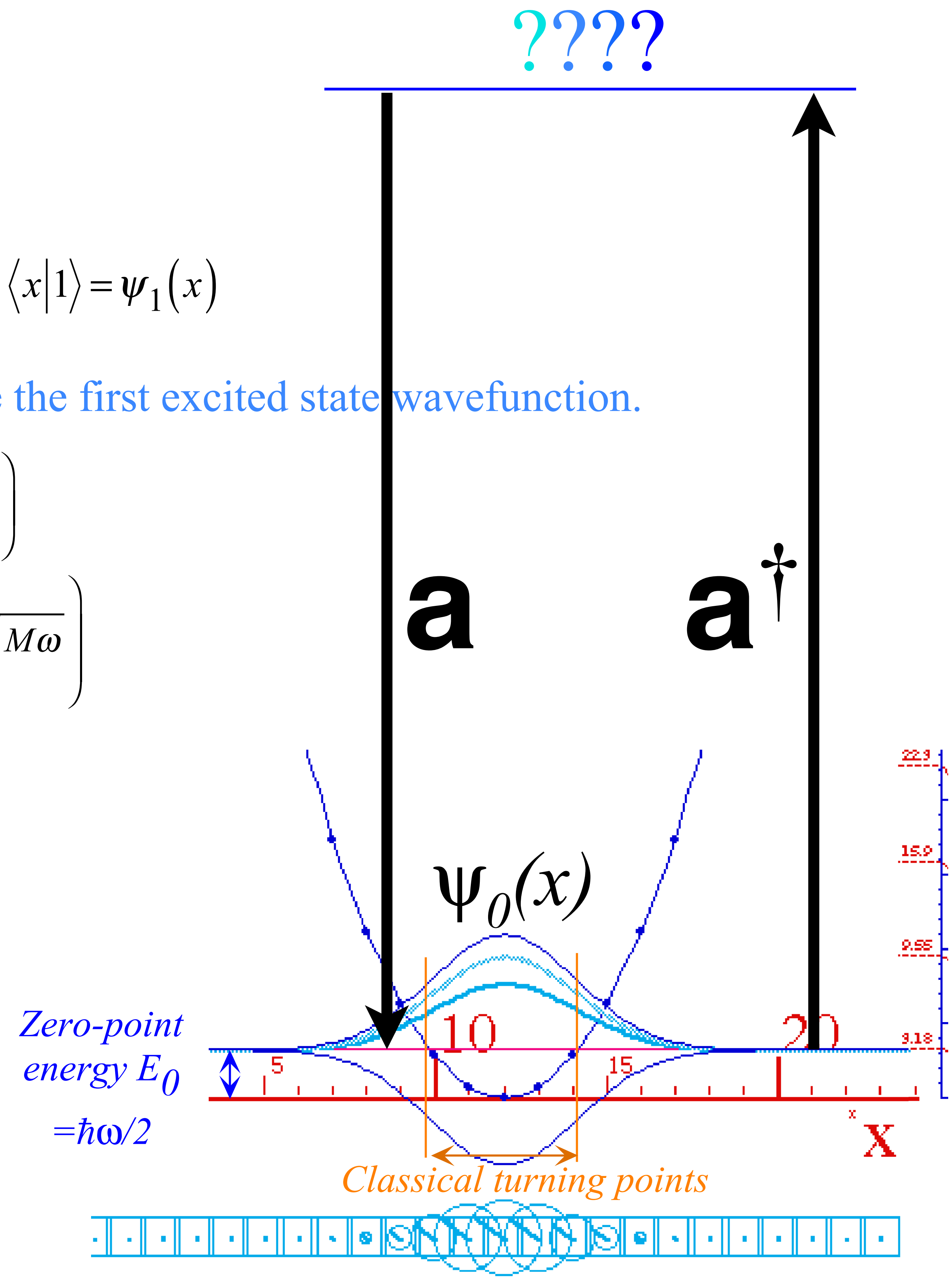

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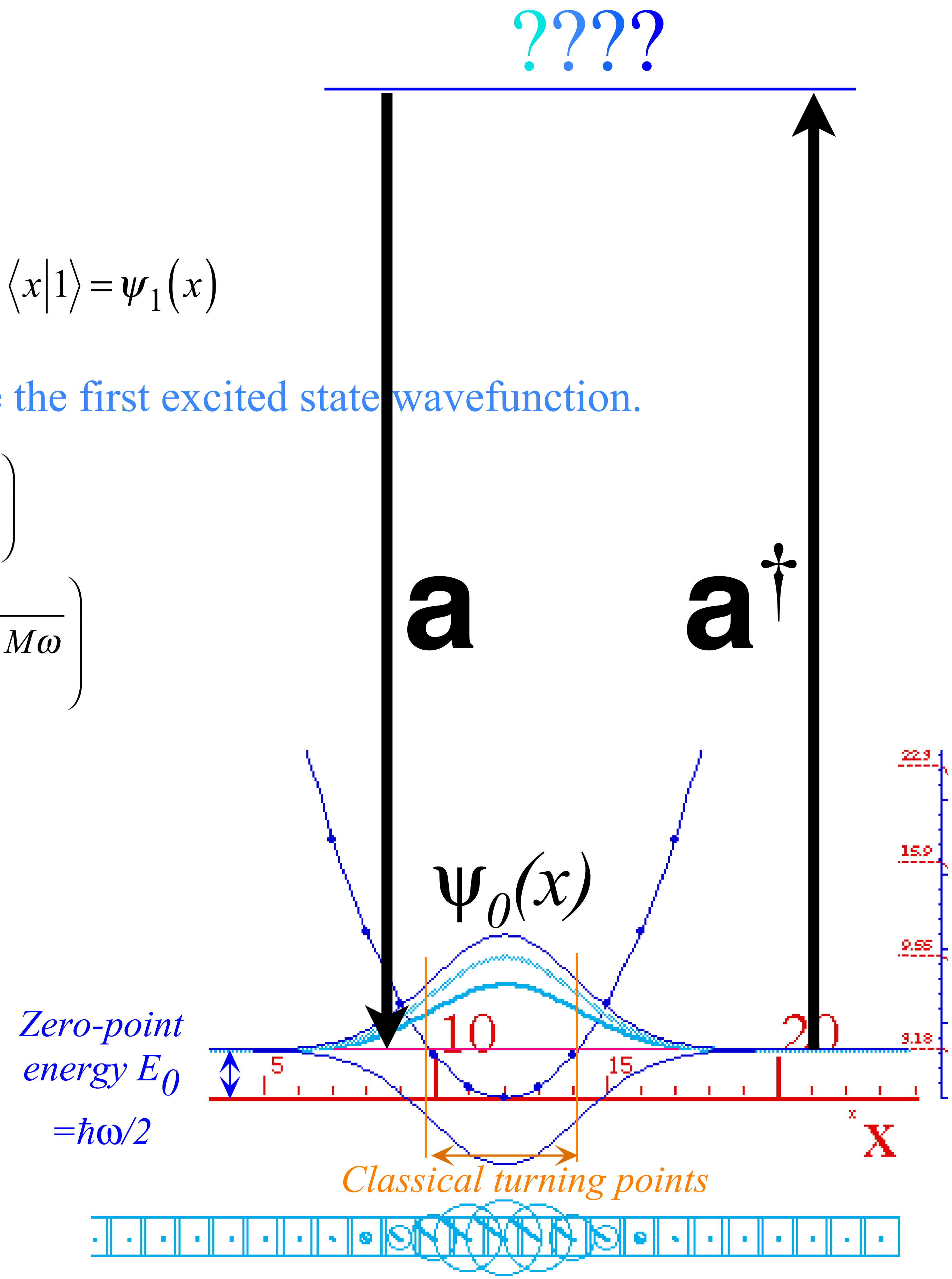

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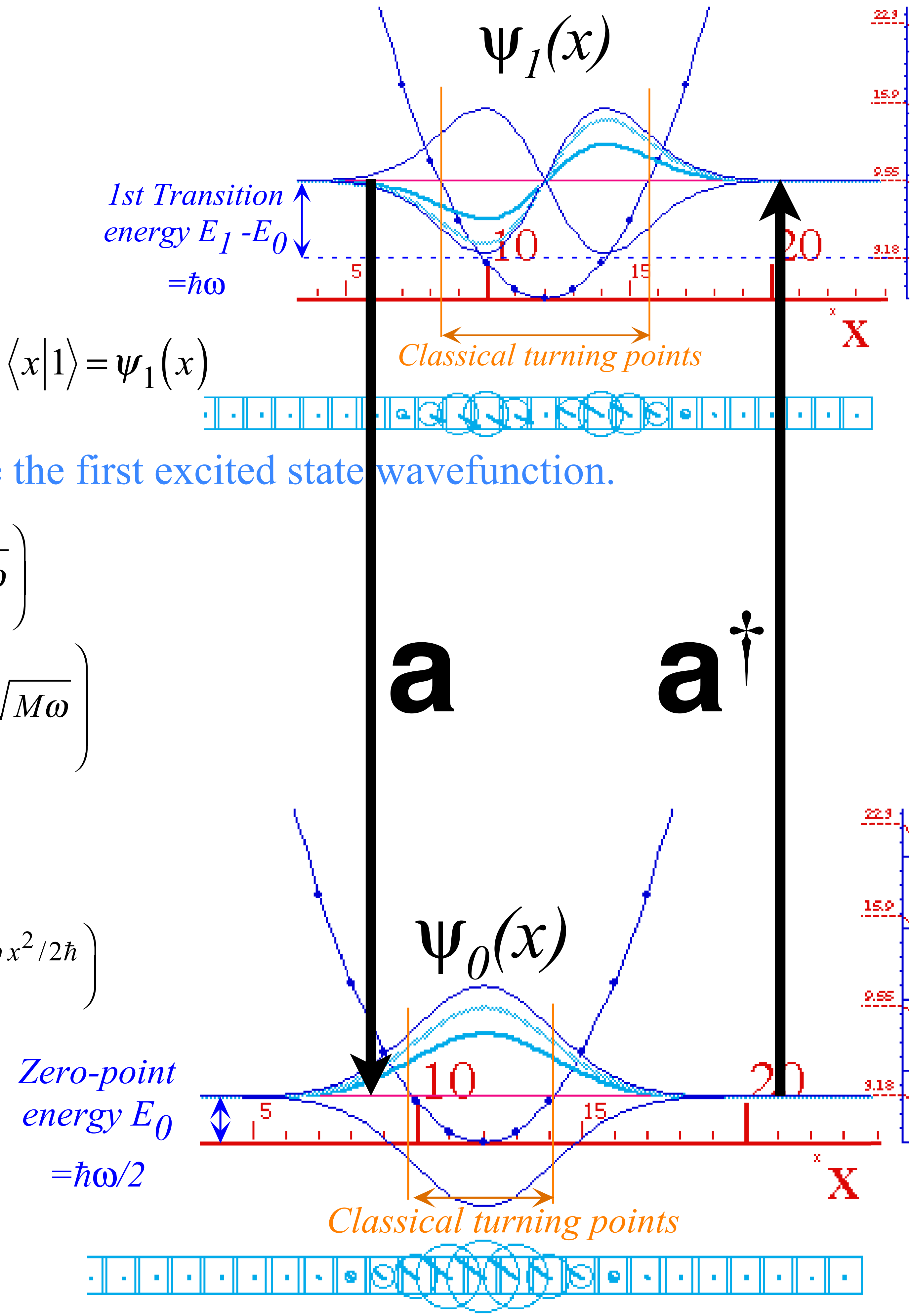
$$= \frac{1}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}\left(\sqrt{M\omega}\,x + i\frac{\hbar}{i}\frac{M\omega x}{\hbar}/\sqrt{M\omega}\right)$$

$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x\,e^{-M\omega x^2/2\hbar}\right)$$



1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

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➡ *Normal ordering for matrix calculation* ⬅

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Normal ordering for matrix calculation

Normal ordering: move destructive $\mathbf{a}$ operators to the right of creation $\mathbf{a}^\dagger$ to zero out on vacuum $|0\rangle$.

$$f(\mathbf{a})g(\mathbf{a}^\dagger)|0\rangle = [f(\mathbf{a}), g(\mathbf{a}^\dagger)]\,|0\rangle + g(\mathbf{a}^\dagger)f(\mathbf{a})\,|0\rangle$$

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$$\left[\mathbf{a},\mathbf{a}^{\dagger 2}\right] = 2\mathbf{a}^\dagger \ , \quad \left[\mathbf{a},\mathbf{a}^{\dagger 3}\right] = 3\mathbf{a}^{2\dagger}, \ \cdots \ , \quad , \quad \left[\mathbf{a},\mathbf{a}^{\dagger n}\right] = n\mathbf{a}^{\dagger n-1}$$

(Power-law derivative-like relations)

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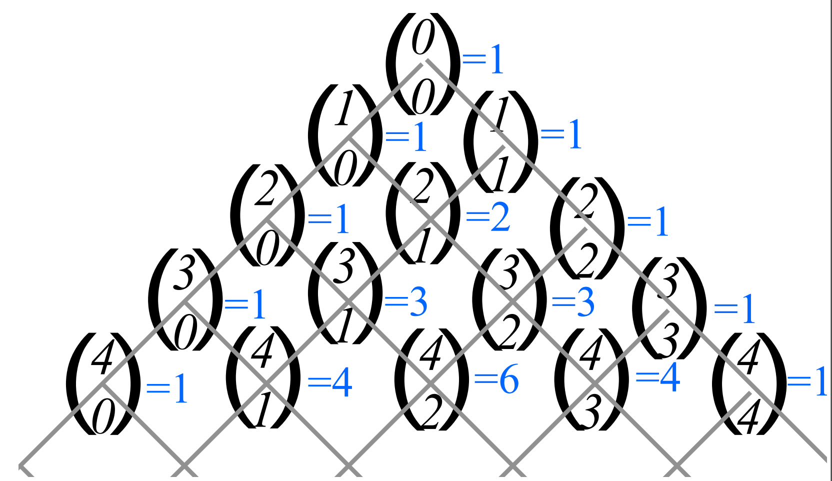

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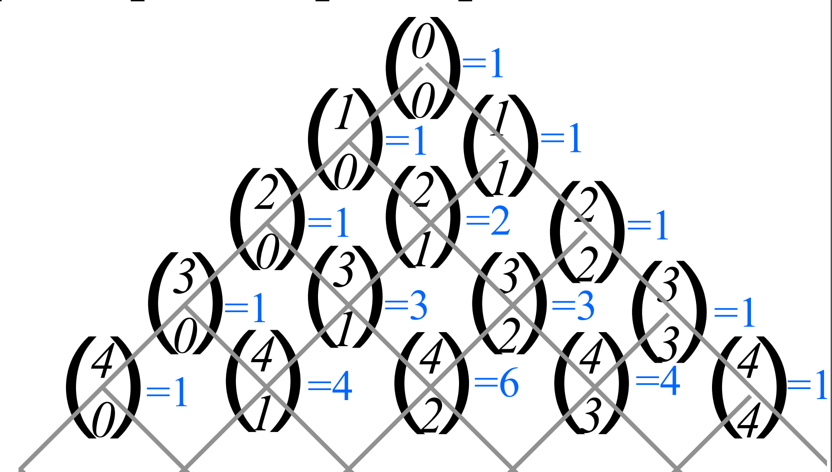

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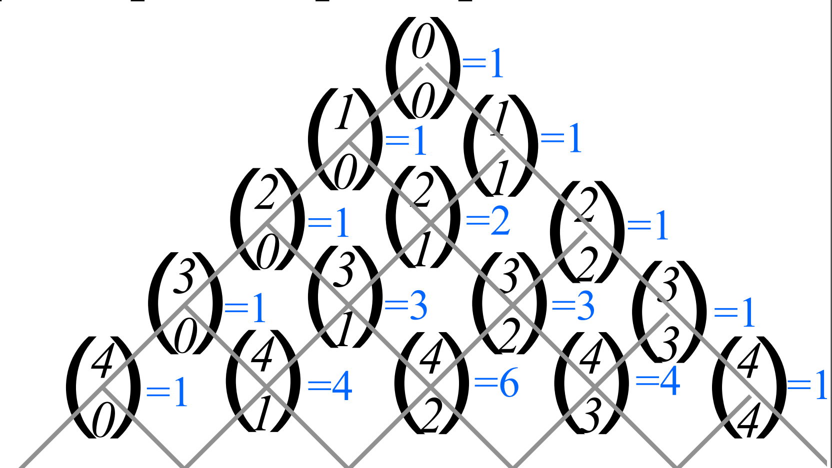

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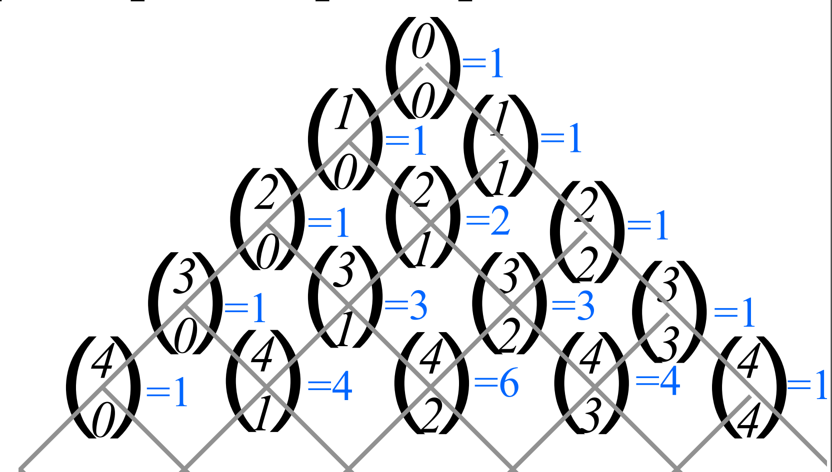

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$\mathbf{a}^n\mathbf{a}^{\dagger n}$ *to* $\mathbf{a}^{\dagger r}\mathbf{a}^r$ *case*

$$\mathbf{a}^n\mathbf{a}^{\dagger n} = \sum_{r=0}^{n}\binom{n}{r}\frac{n!}{r!}\mathbf{a}^{\dagger r}\mathbf{a}^r = n!\left(\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \frac{n(n-1)(n-3)}{3!\cdot 3!}\mathbf{a}^{\dagger 3}\mathbf{a}^3 + \ldots\right)$$

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$$\left|n\right\rangle = \frac{\mathbf{a}^{\dagger n}\left|0\right\rangle}{const.}, \quad \text{where:} \quad 1 = \left\langle n \middle| n \right\rangle = \frac{\left\langle 0\right|\mathbf{a}^{n}\mathbf{a}^{\dagger n}\left|0\right\rangle}{(const.)^{2}}$$

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Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2! \cdot 2!} \mathbf{a}^{\dagger 2} \mathbf{a}^2 + \ldots \right)$

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Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots \right)$

$$\left| n \right\rangle = \frac{\mathbf{a}^{\dagger n}\left| 0 \right\rangle}{const.}, \quad \text{where:} \quad 1 = \left\langle n \middle| n \right\rangle = \frac{\left\langle 0 \right| \mathbf{a}^n \mathbf{a}^{\dagger n} \left| 0 \right\rangle}{(const.)^2} = n!\frac{\left\langle 0 \right| \mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + .. \left| 0 \right\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

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Root-factorial normalization

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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Root-factorial normalization

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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Apply destruction $\mathbf{a}$:

$$\mathbf{a}\left| n \right\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}\left|0\right\rangle}{\sqrt{n!}}$$

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Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

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$$\mathbf{a}^\dagger |n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

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Feynman's mnemonic rule: Larger of two quanta goes in radical factor

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$$\langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

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(Here is a case where $\mathbf{a}^\dagger \mathbf{a}$ does not quite equal $\mathbf{a}\,\mathbf{a}^\dagger$)

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(Welcome to ∞-dimensional... quantum space!)

1-D **a**†**a** *algebra of U(1) representations*

Creation-Destruction **a**†**a** *algebra*

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculations*

→ *Number operator and Hamiltonian operator* ←

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D **a**†**a** *algebra of U(2) representations and R(3) angular momentum operators*

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots \right)$

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Root-factorial normalization

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

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Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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$$\langle \mathbf{a}^\dagger \rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \quad \text{where:} \quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n\mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

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Root-factorial normalization

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

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Hamiltonian operator

$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\,\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

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$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\,\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H}\rangle = \hbar\omega\langle \mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega\begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

➡ *Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$* ⬅

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Operator for position **x**: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^{\dagger}}{2}$

Operator for momentum **p**: $\sqrt{\frac{1}{2\hbar M\omega}}\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^{\dagger}}{2i}$

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$$\overline{\mathbf{x}^2}\Big|_n = \langle n|\mathbf{x}^2|n\rangle = \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2|n\rangle$$

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Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

$$(\Delta q)^2 = \overline{(q-\bar{q})^2} \qquad \text{or:}\quad \Delta q = \sqrt{\overline{(q-\bar{q})^2}}$$

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$$= \frac{\hbar}{2M\omega}\quad(2n+1)$$

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Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

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Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

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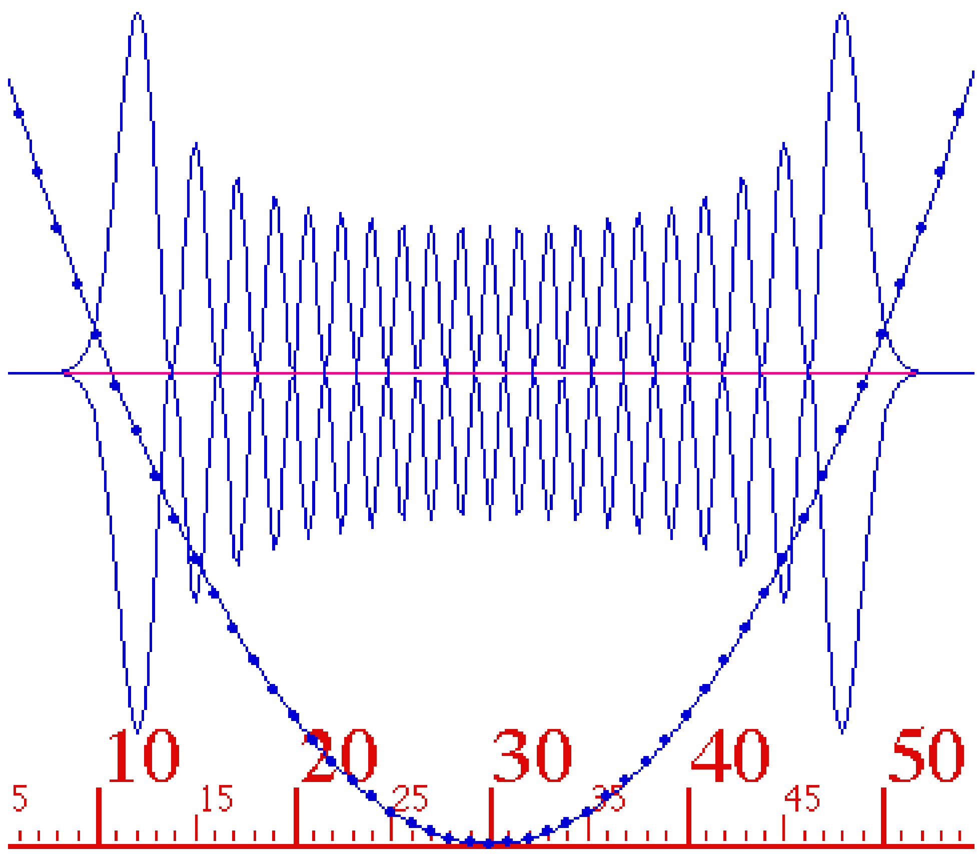
Heisenberg minimum uncertainty product occurs for the *0*-quantum (ground) eigenstate.

$$\boxed{(\Delta x\cdot\Delta p)|_0 = \frac{\hbar}{2}}$$

We pause for sobering considerations of the quantum world *vs.* the classical one.
Consider a “high”-quantum ($n=20$) eigenstate wavefunction:



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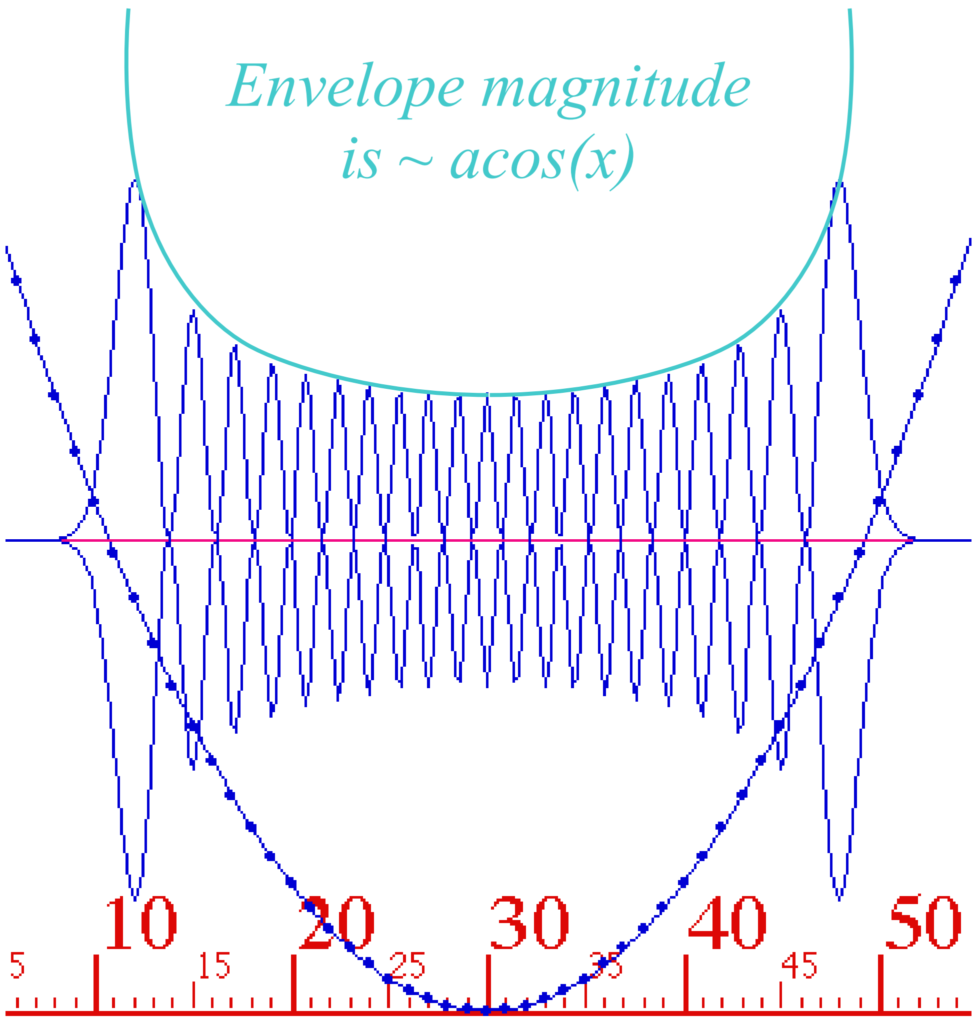

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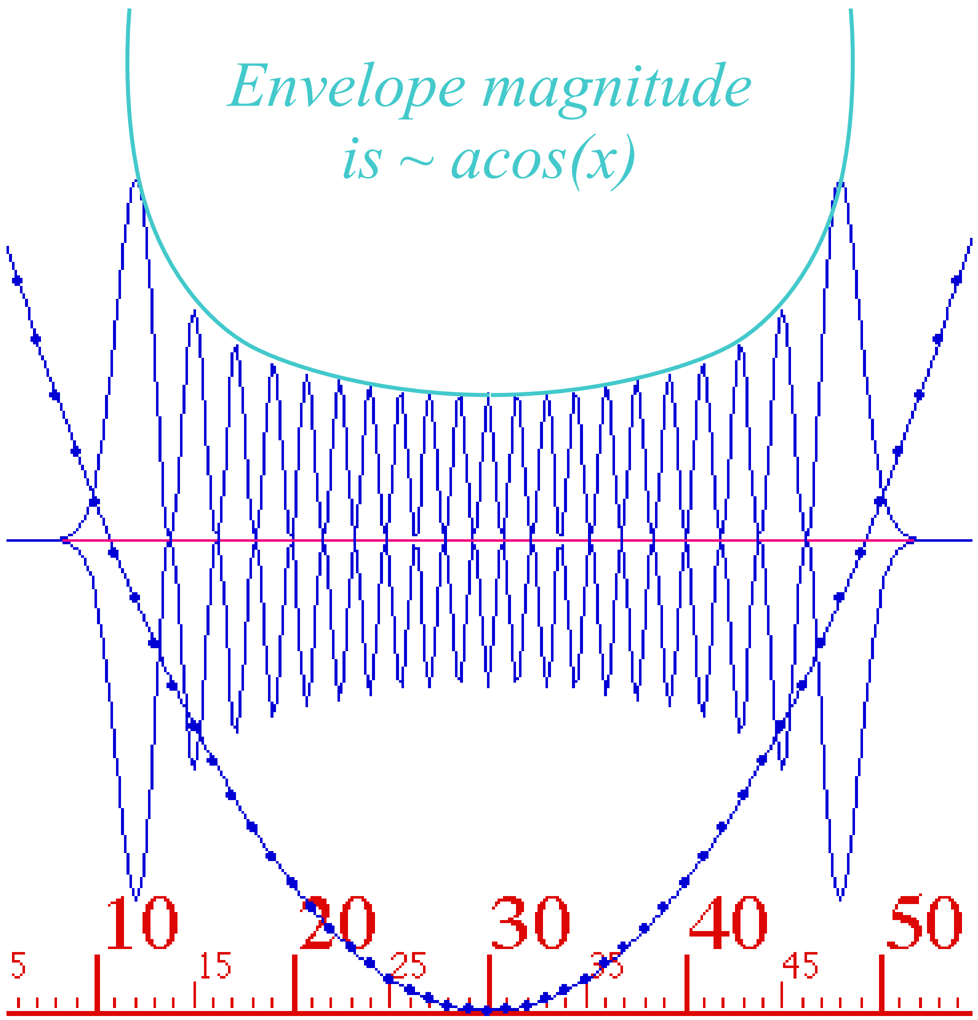

$n=20$ wave is still a long way from a classical energy value of *1 Joule*.
For a *1 Hz* oscillator, *1 Joule* would take a quantum number of roughly
$n = 100,000,000,000,000,000,000,000,000,000,000,000 = 10^{35}$

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

→ *Harmonic oscillator beat dynamics of mixed states* ←

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Harmonic oscillator beat dynamics of mixed states

$$|\Psi\rangle = |0\rangle\langle 0|\Psi\rangle + |1\rangle\langle 1|\Psi\rangle = |0\rangle\Psi_0 + |1\rangle\Psi_1$$

$$\Psi(x) = \langle x|\Psi\rangle = \langle x|0\rangle\langle 0|\Psi\rangle + \langle x|1\rangle\langle 1|\Psi\rangle = \psi_0(x)\,\Psi 0 + \psi_1(x)\,\Psi 1$$

The time dependence $\Psi(x,t)$ of the mixed wave is then

$$\Psi(x,t) = \psi_0(x)\,e^{-i\omega_0 t}\,\Psi_0 + \psi_1(x)\,e^{-i\omega_1 t}\,\Psi_1 = (\psi_0(x)\,e^{-i\omega_0 t} + \psi_1(x)\,e^{-i\omega_1 t})/\sqrt{2}$$

$$\left|\Psi(x,t)\right| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^*\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

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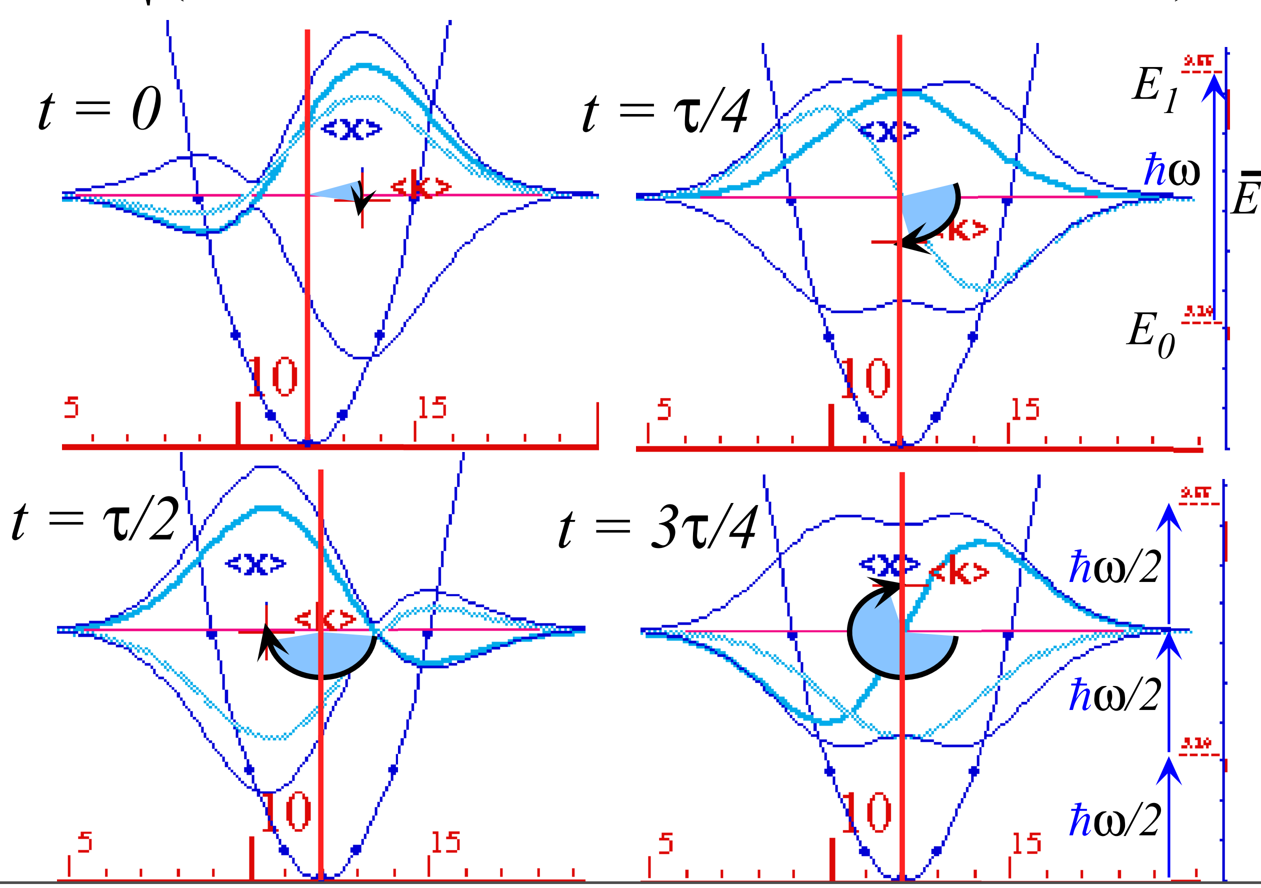

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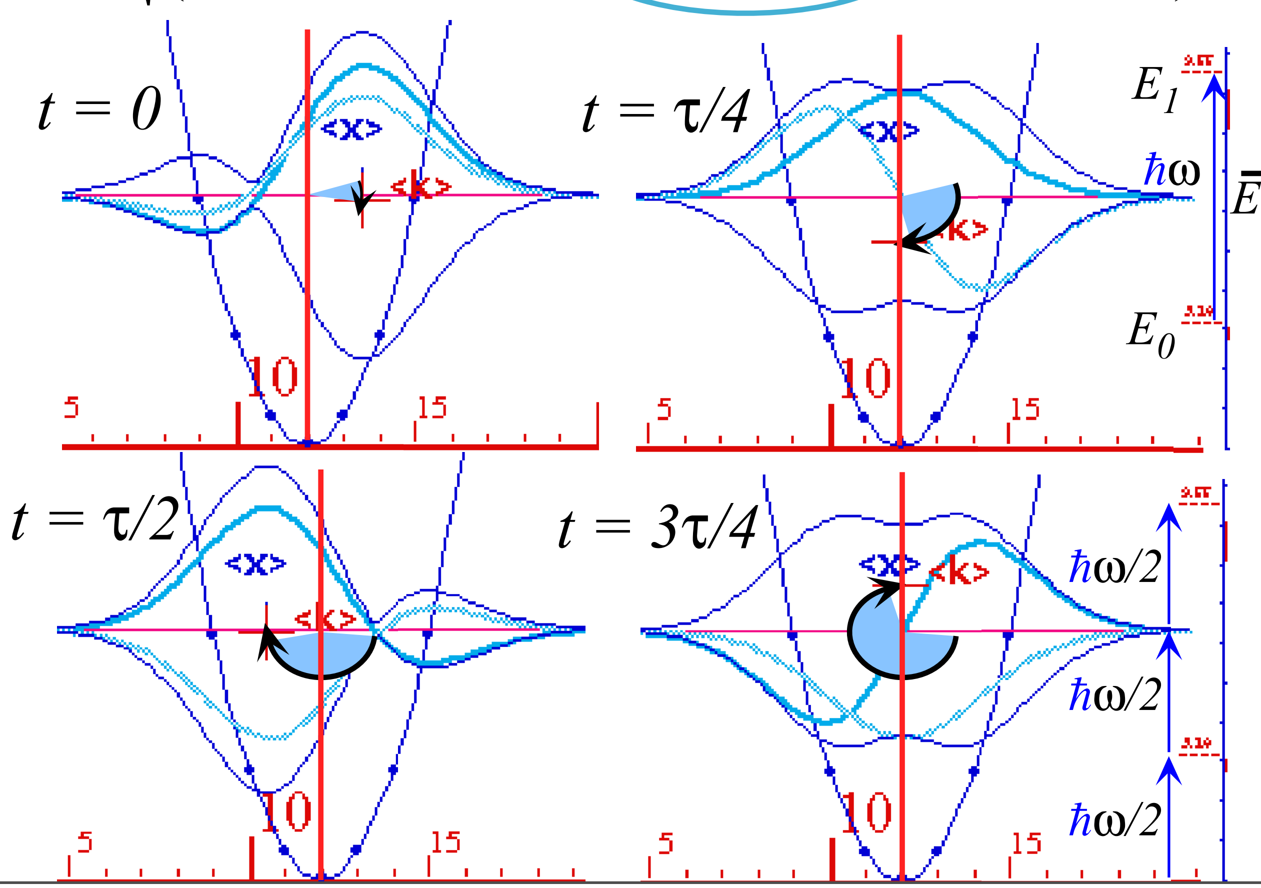
Need some *overlap*
somewhere
to get some *wiggle*



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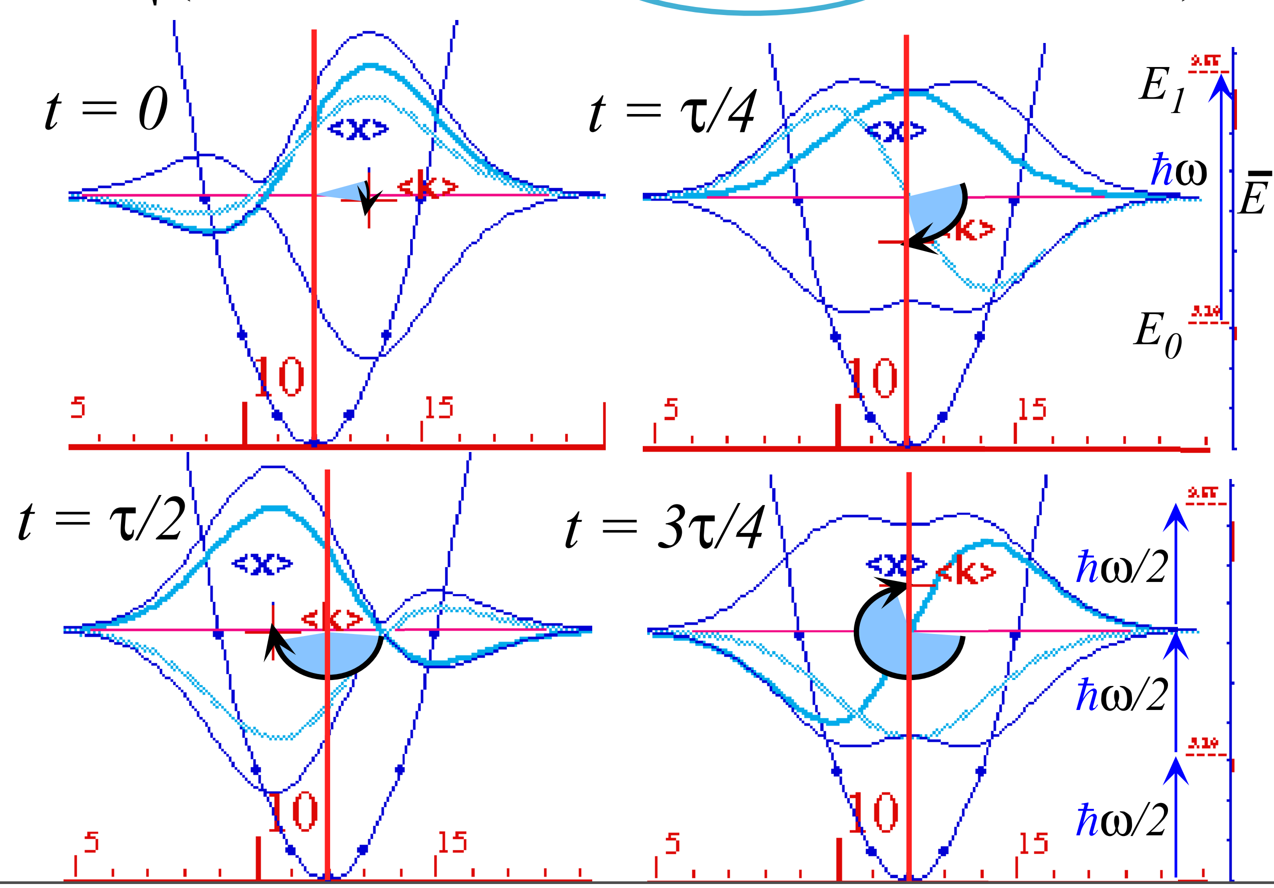

Beat frequency is eigenfrequency difference

$$\omega_{beat} = \omega_1 - \omega_0 = \omega$$

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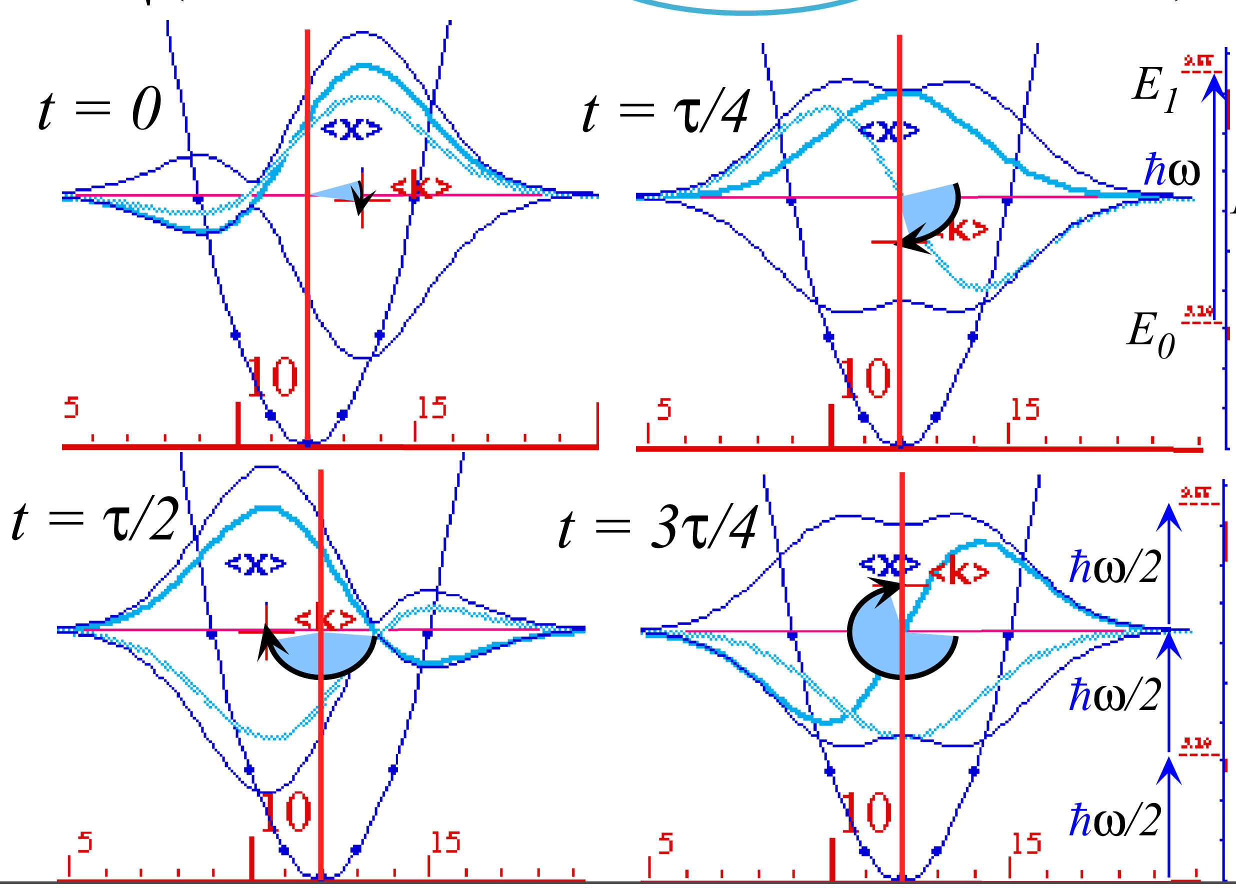

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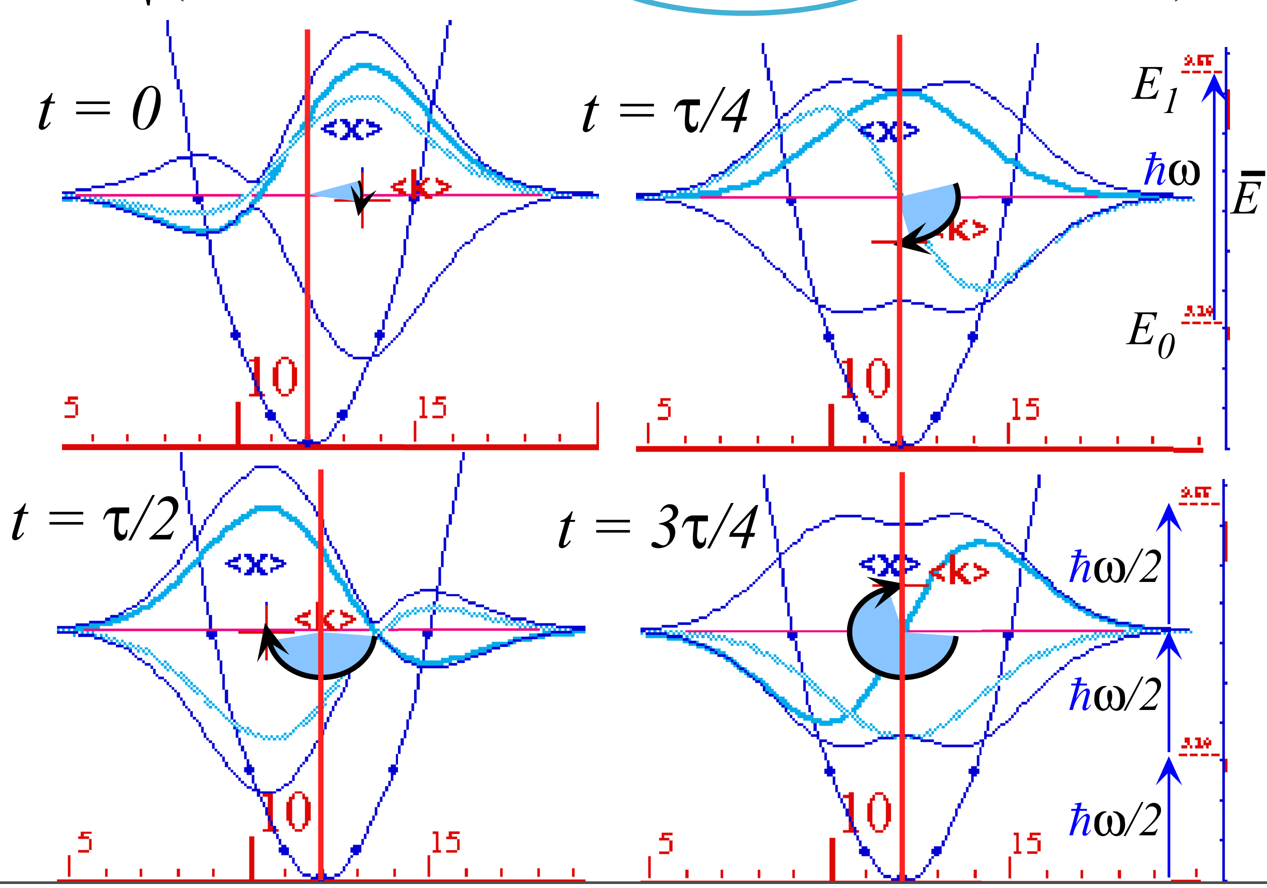

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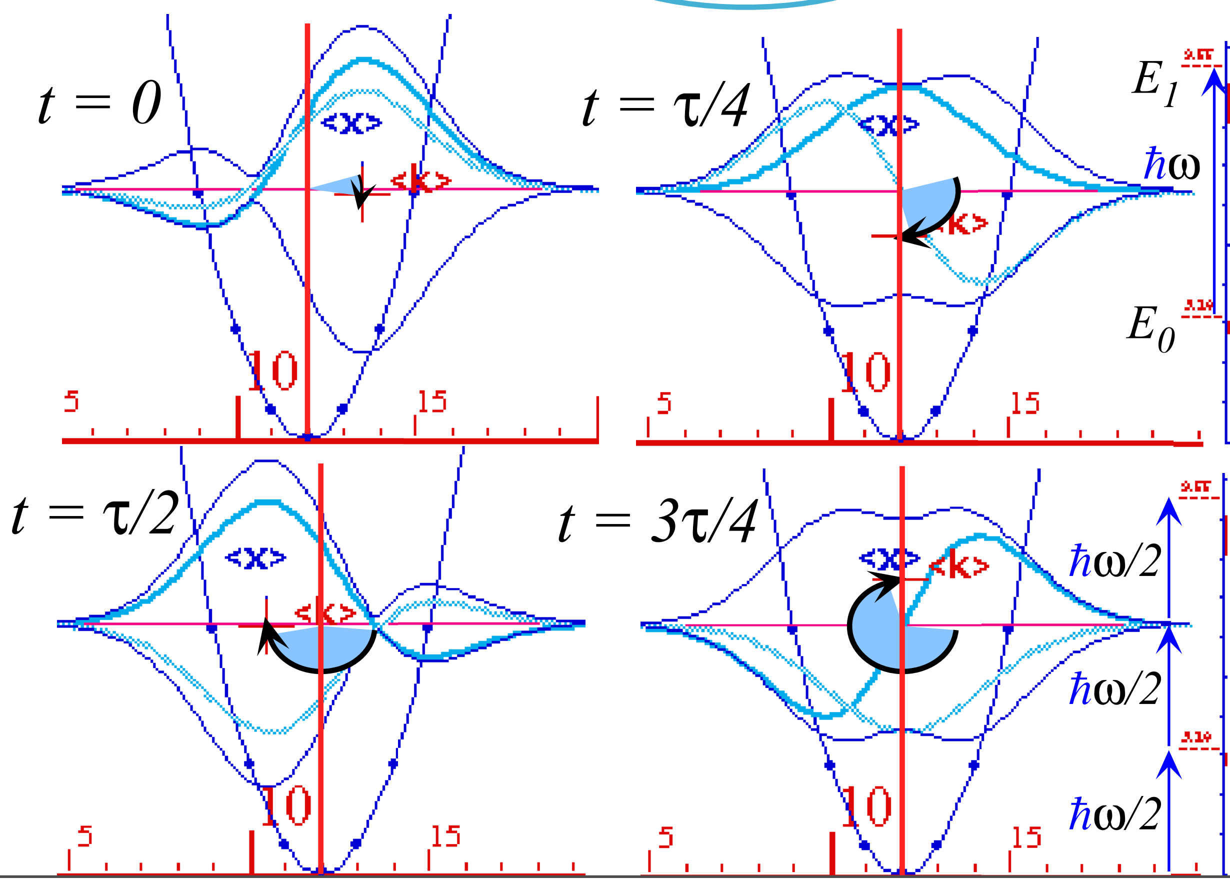

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ω is frequency of radiating antenna
of a transmitter or of a receiver, i.e.,
of an emitter or an absorber
(Usually of a dipole symmetry)

Examples of 2-Well system overlap

$$|\Psi(x,t)| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^*\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

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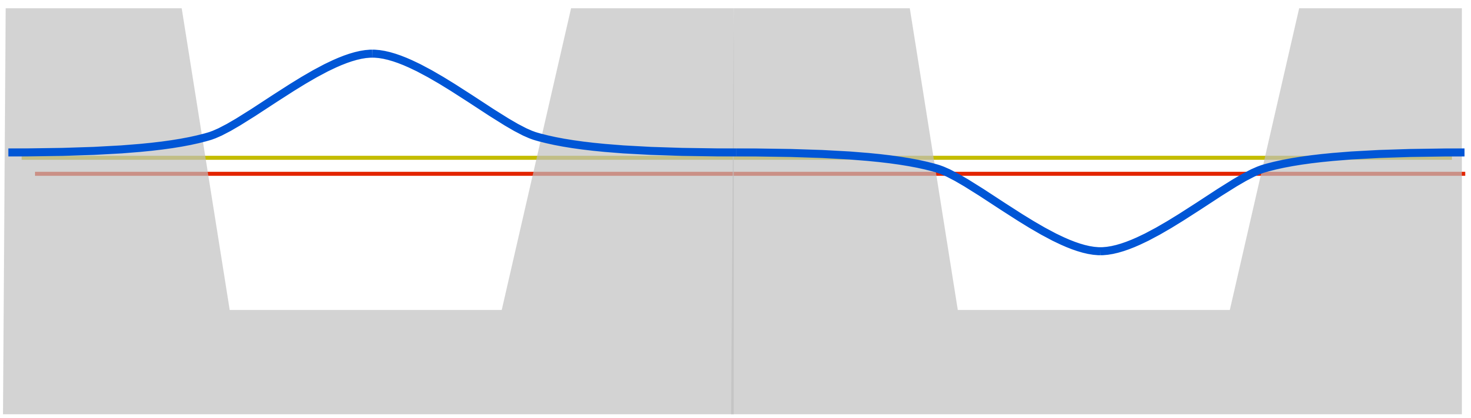
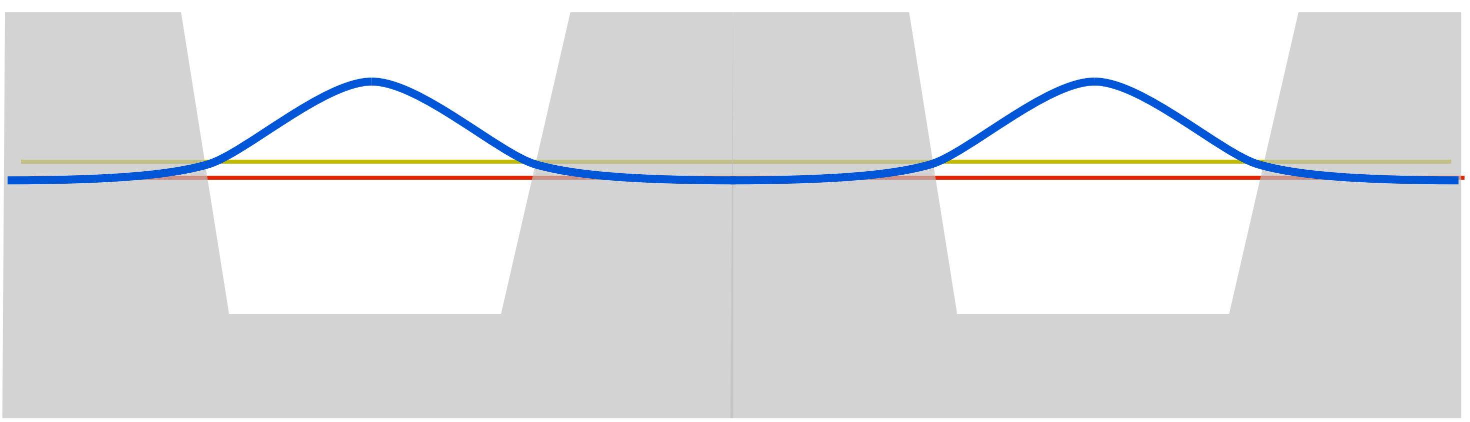
Example of 2-Well system with healthy overlap due to symmetry

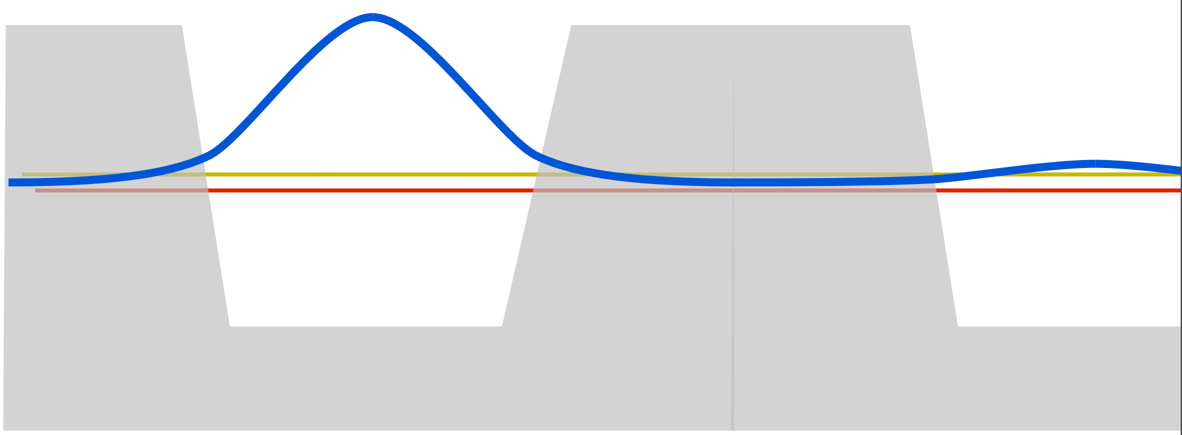
Odd eigenstate $\psi^{(-)}$



Even eigenstate $\psi^{(+)}$



Combination state $\psi^{(+)} + \psi^{(-)}$
has lots of *wiggle*...



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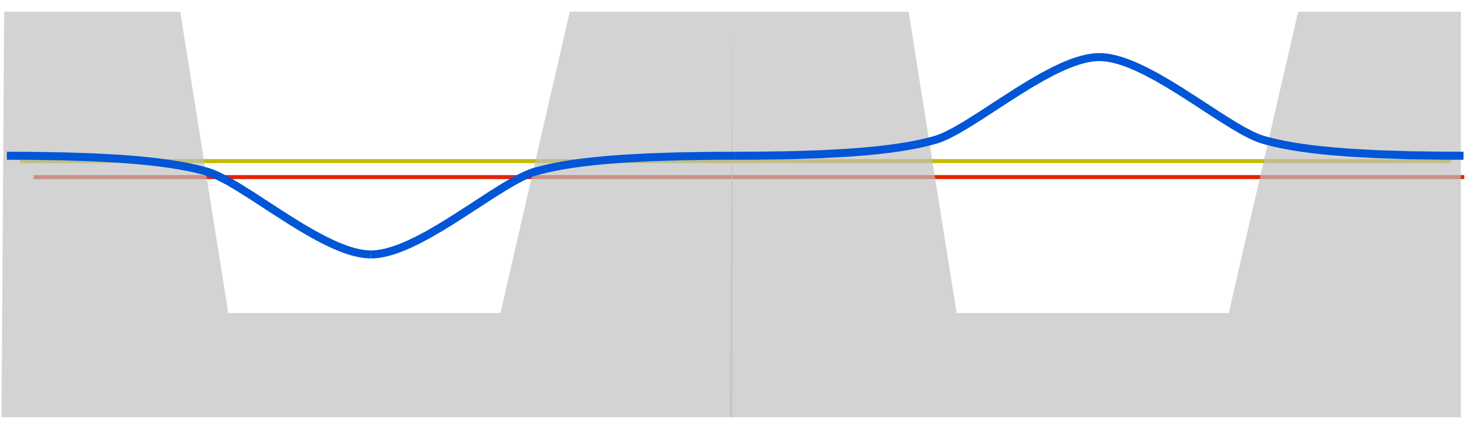
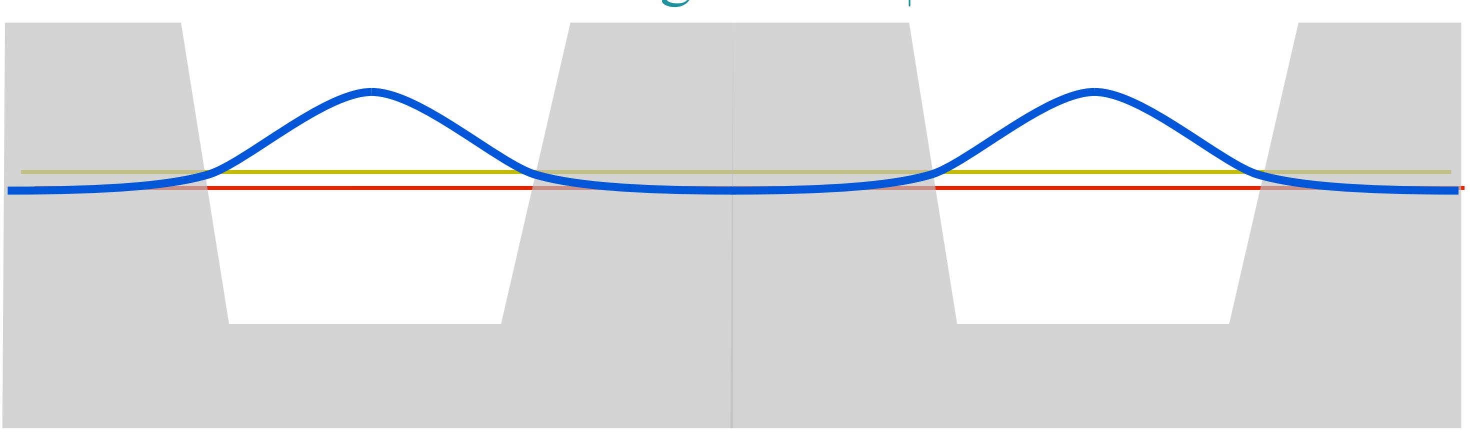
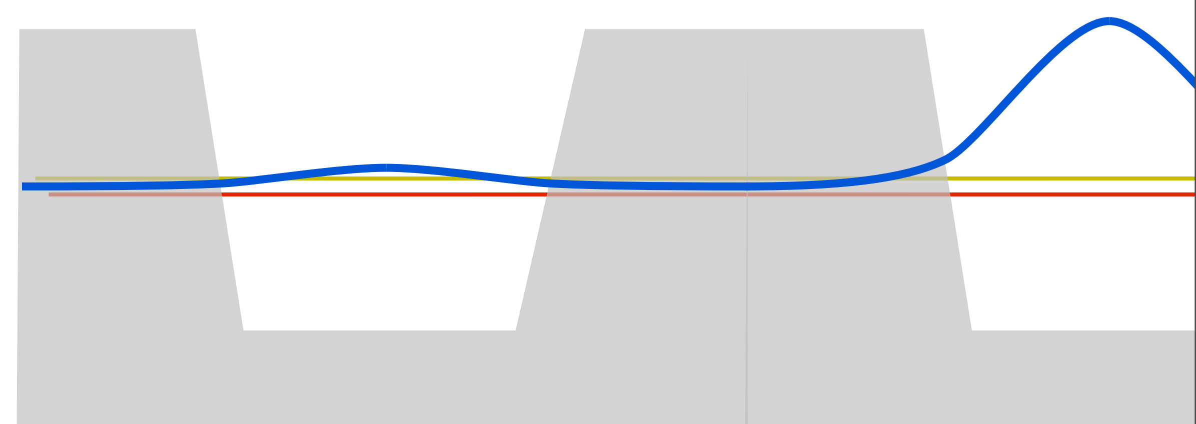


Even eigenstate $\psi^{(+)}$



Combination state $\psi^{(+)} + \psi^{(-)}$ has lots of *wiggle...*



Examples of 2-Well system overlap

$$|\Psi(x,t)| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^* \left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

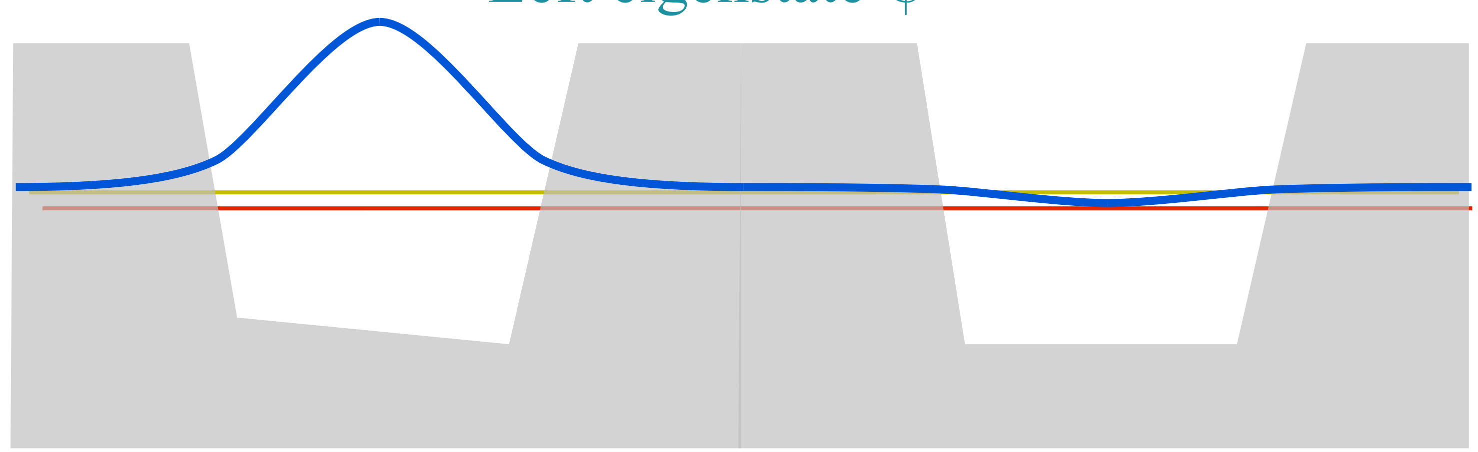
$$= \sqrt{\left(|\psi_0(x)|^2 + |\psi_1(x)|^2 + \psi_0(x)\psi_1(x)\left(e^{i(\omega_1-\omega_0)t} + e^{-i(\omega_1-\omega_0)t}\right)\right)/2}$$

$$= \sqrt{\left(|\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\psi_0(x)\psi_1(x)\cos(\omega_1-\omega_0)t\right)/2}$$

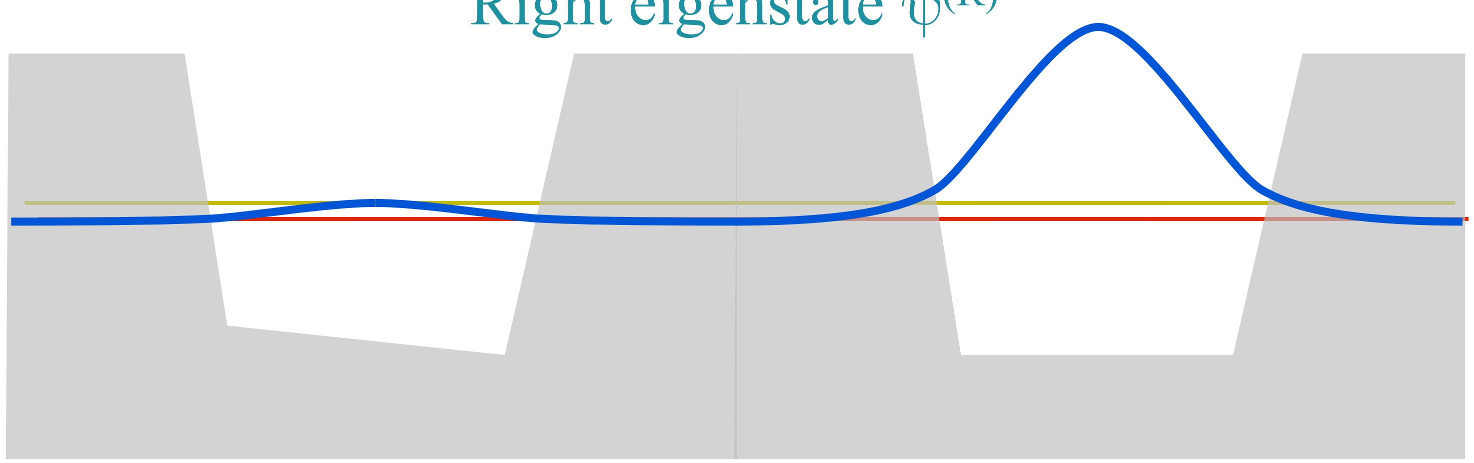
Need some *overlap* somewhere
to get some *wiggle*

Example of 2-Well system with *un*healthy overlap due to broken symmetry

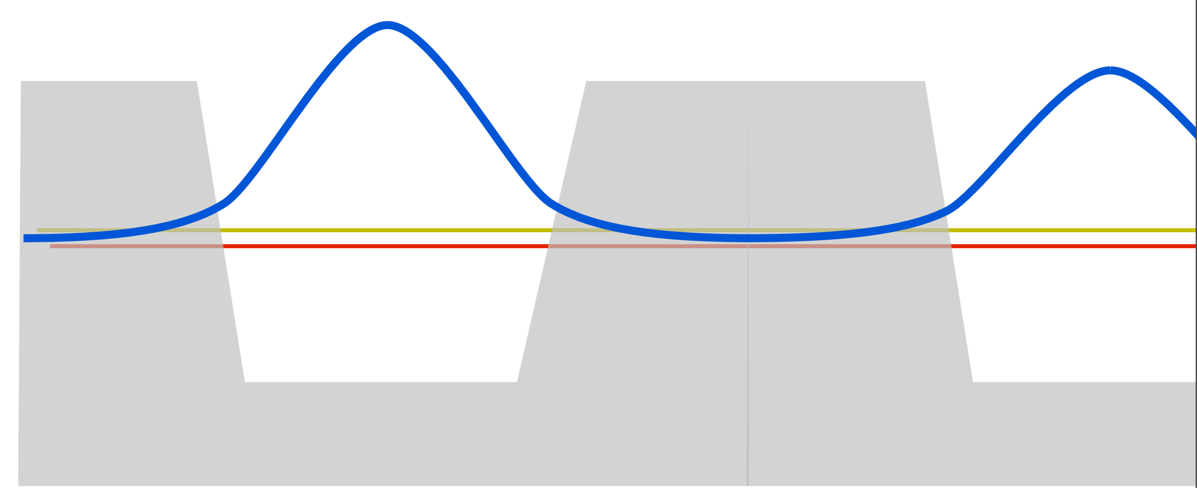
Left eigenstate $\psi^{(L)}$



Right eigenstate $\psi^{(R)}$



Combination state $\psi^{(L)}+\psi^{(R)}$
has very little *wiggle*...



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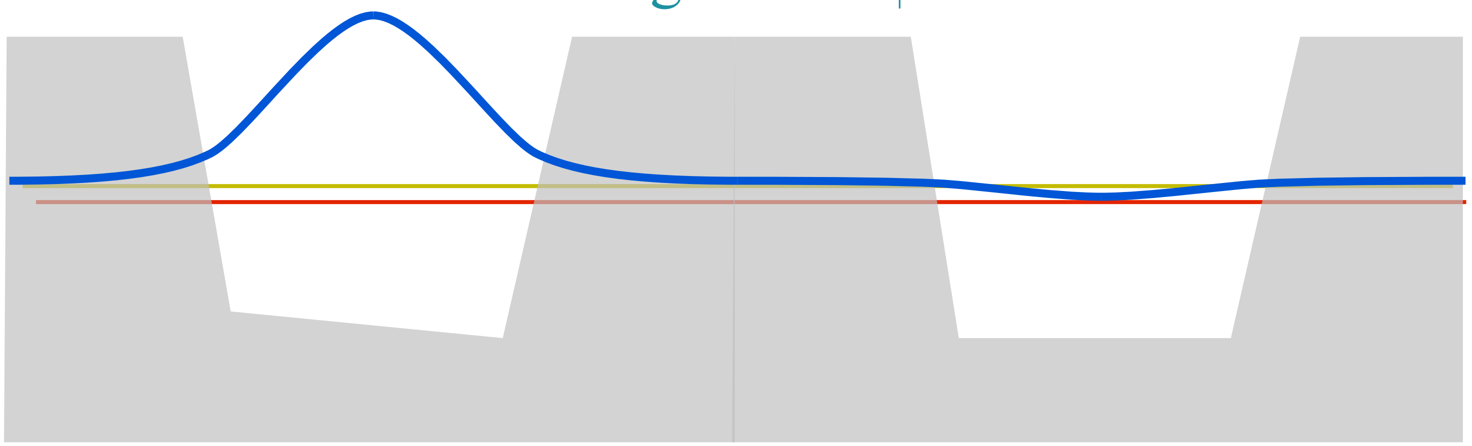

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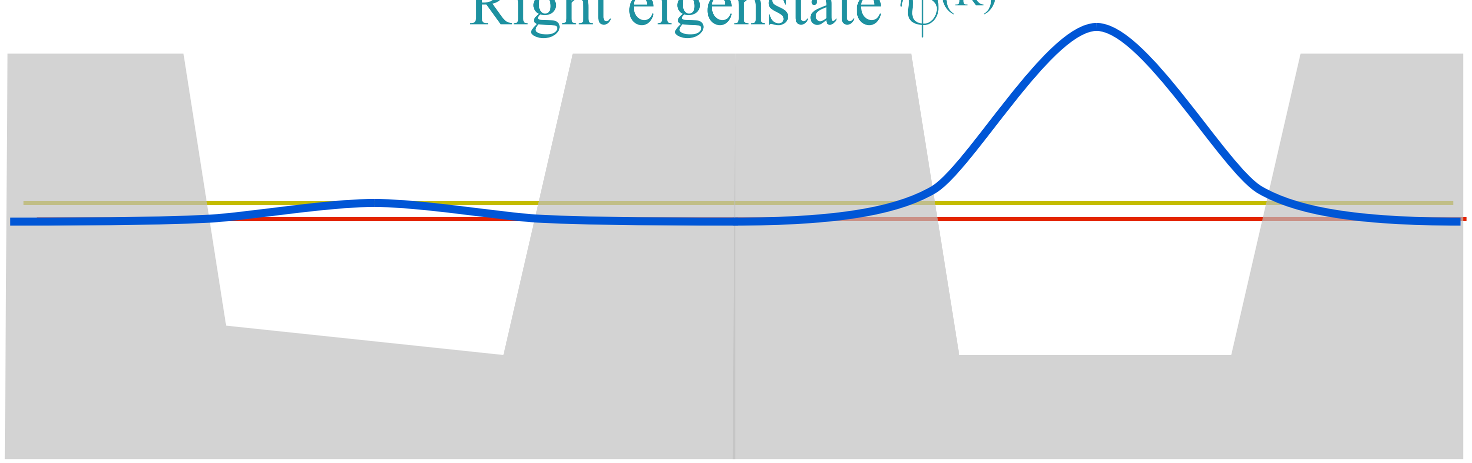

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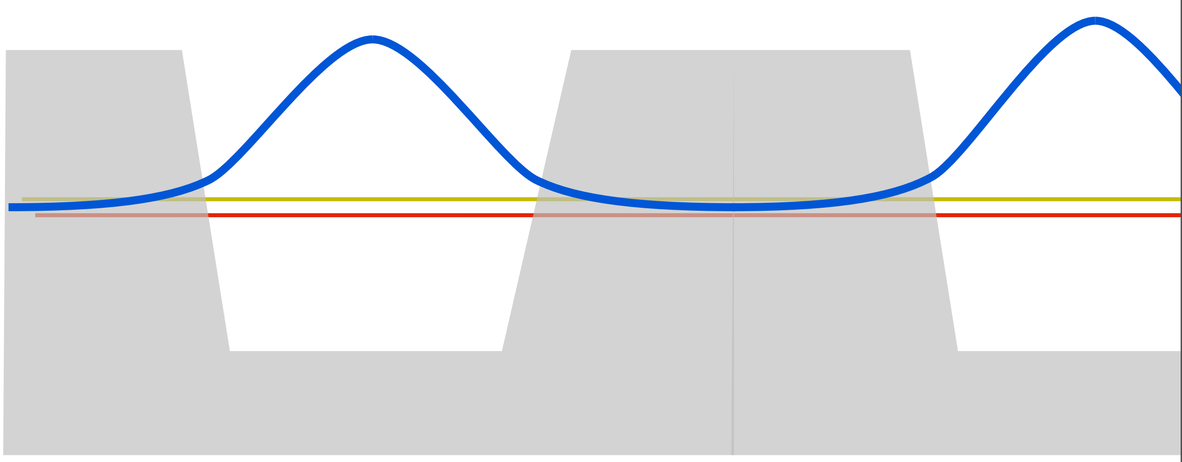

*1-D **a†a** algebra of U(1) representations*

*Creation-Destruction **a†a** algebra*

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*2-D **a†a** algebra of U(2) representations and R(3) angular momentum operators*

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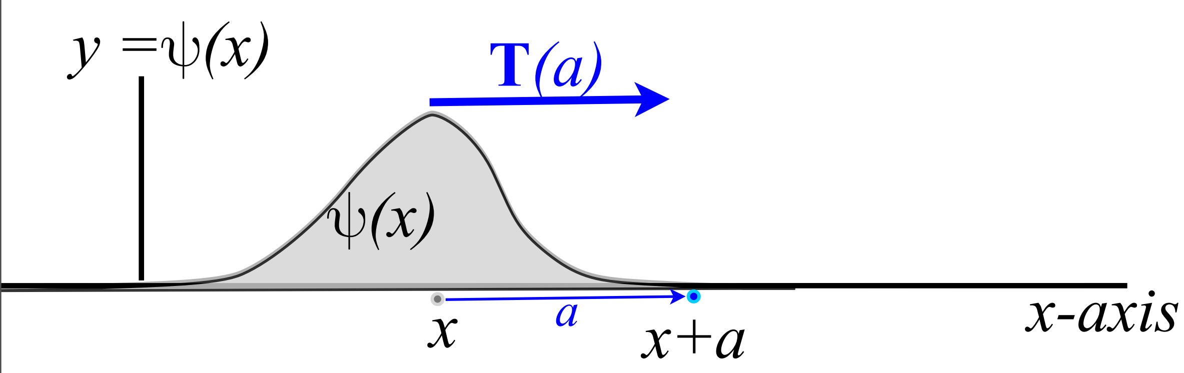
Translation operators and generators: (A "shove")

Translation operator $\mathbf{T}(a)$ shoves x-wavefunctions

$\mathbf{T}(a)\cdot\psi(x) = ?$



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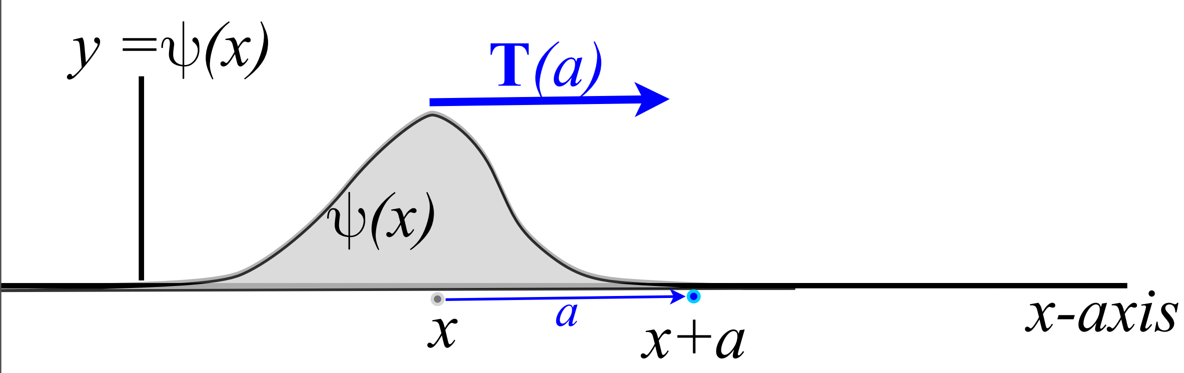
Shoves ψ a-units to right



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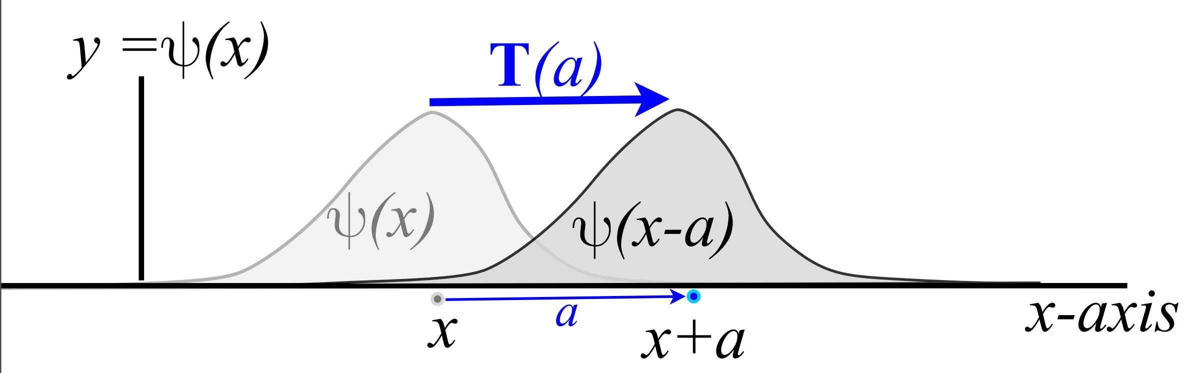

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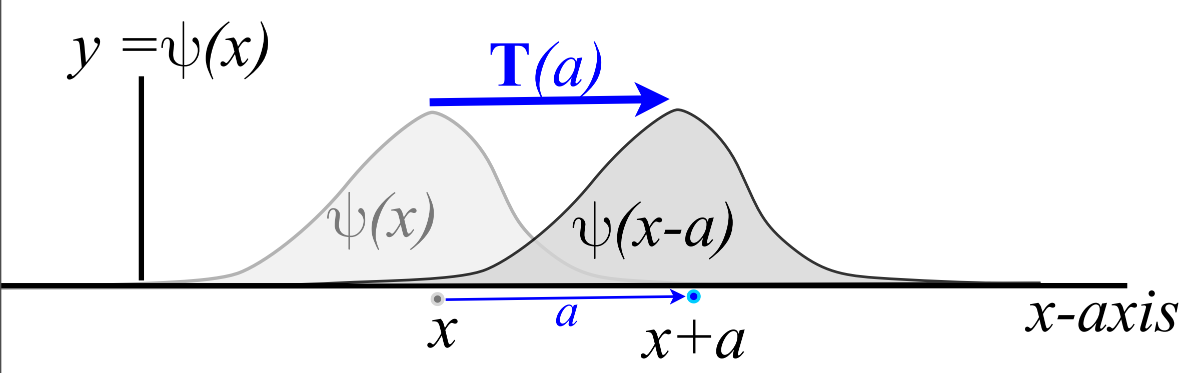

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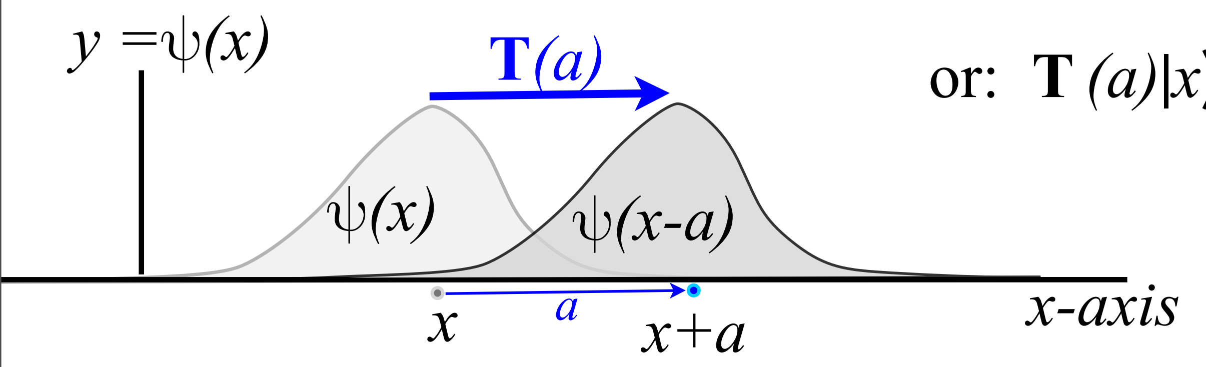

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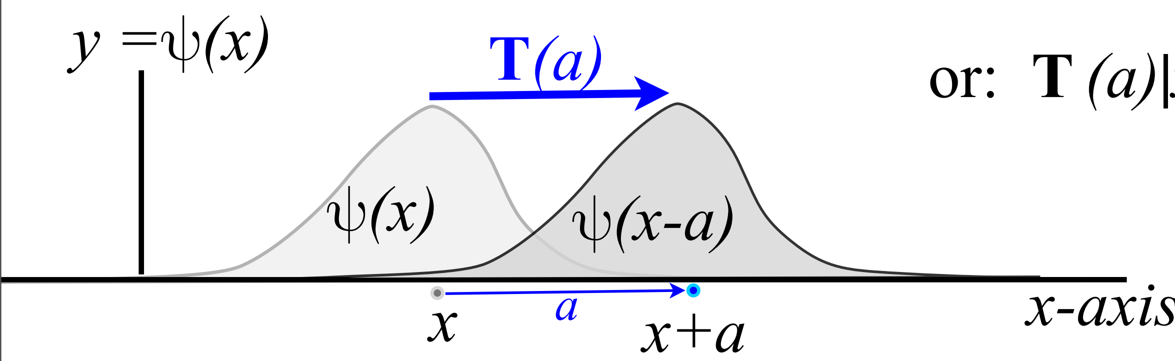

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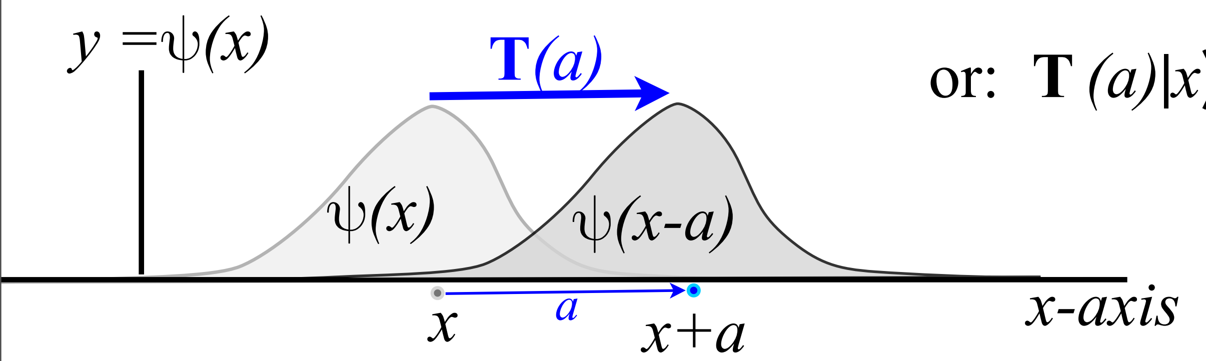

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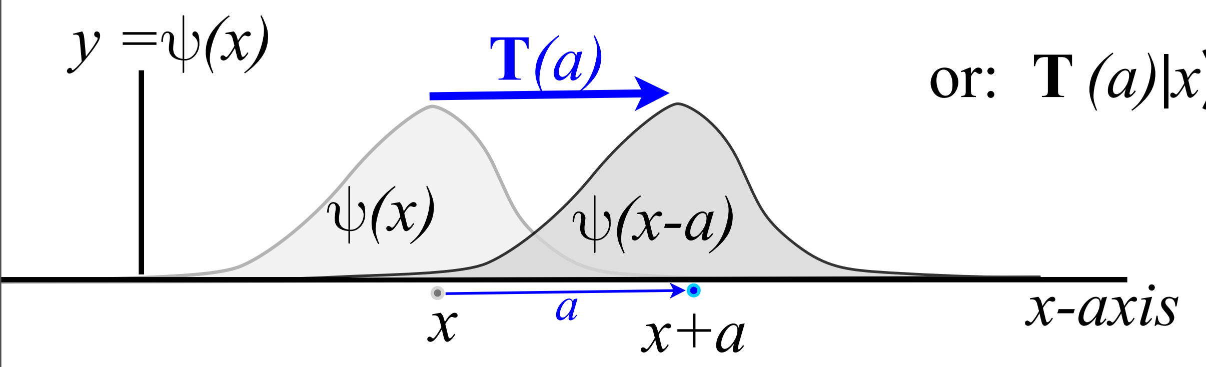

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$$\mathbf{B}(db) = \mathbf{1} + \mathbf{K}\cdot db \quad \text{where: } \mathbf{K} = \left.\frac{\partial \mathbf{B}}{\partial b}\right|_{b=0}$$

is *generator* $\mathbf{K}$ *of boosts*

$$\mathbf{B}(b) = \left(\mathbf{B}(\frac{b}{N})\right)^N = \lim_{N\to\infty}\left(1 + \frac{b}{N}\mathbf{K}\right)^N = e^{b\mathbf{K}}$$

$$\mathbf{B}(b)\cdot\psi(p) = e^{b\mathbf{K}}\cdot\psi(p) = e^{-b\frac{\partial}{\partial p}}\cdot\psi(p)$$

$$=\psi(p) - b\frac{\partial\psi(p)}{\partial p} + \frac{b^2}{2!}\frac{\partial^2\psi(p)}{\partial p^2} - \frac{b^3}{3!}\frac{\partial^3\psi(p)}{\partial p^3} + \ldots$$

$\mathbf{K}$ relates to position $\mathbf{x} \to \hbar i\frac{\partial}{\partial p} = i\frac{\partial}{\partial k}$

$$\mathbf{K} = \frac{i}{\hbar}\mathbf{x} \to -\frac{\partial}{\partial p} = \frac{-1}{\hbar}\frac{\partial}{\partial k}$$

$$\mathbf{B}(b) = e^{b\frac{i}{\hbar}\mathbf{x}} = e^{ib\left(\mathbf{a}^\dagger + \mathbf{a}\right)/\sqrt{2\hbar M\omega}}$$

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators and generators: (A "shove")

Translation operator $\mathbf{T}(a)$ shoves x-wavefunctions

$$\mathbf{T}(a)\cdot\psi(x) = \psi(x-a) = \langle x|\mathbf{T}(a)|\psi\rangle = \langle x-a|\psi\rangle$$

Shoves ψ a-units to right or x-space a-units left

$$\langle x|\mathbf{T}(a) = \langle x-a| \quad \text{or:} \quad \mathbf{T}^\dagger(a)|x\rangle = |x-a\rangle$$

Tiny translation $a \to da$ is identity $\mathbf{1}$ plus $\mathbf{G}\cdot da$

$$\mathbf{T}(da) = \mathbf{1} + \mathbf{G}\cdot da \quad \text{where: } \mathbf{G} = \left.\frac{\partial \mathbf{T}}{\partial a}\right|_{a=0}$$

is *generator* $\mathbf{G}$ *of translations*

$$\mathbf{T}(a) = \left(\mathbf{T}\left(\frac{a}{N}\right)\right)^N = \lim_{N\to\infty}\left(1 + \frac{a}{N}\mathbf{G}\right)^N = e^{a\mathbf{G}}$$

$$\mathbf{T}(a)\cdot\psi(x) = e^{a\mathbf{G}}\cdot\psi(x) = e^{-a\frac{\partial}{\partial x}}\cdot\psi(x)$$

$$= \psi(x) - a\frac{\partial\psi(x)}{\partial x} + \frac{a^2}{2!}\frac{\partial^2\psi(x)}{\partial x^2} - \frac{a^3}{3!}\frac{\partial^3\psi(x)}{\partial x^3} + \dots$$

$\mathbf{G}$ relates to momentum $\mathbf{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}$

$$\mathbf{G} = -\frac{i}{\hbar}\mathbf{p} \to -\frac{\partial}{\partial x}$$

$$\mathbf{T}(a) = e^{-a\frac{i}{\hbar}\mathbf{p}} = e^{a\left(\mathbf{a}^\dagger - \mathbf{a}\right)\sqrt{M\omega/2\hbar}}$$

Check $\mathbf{T}(a)$ on plane-wave with $p=\hbar k$ — *Bottom Line*

$$\mathbf{T}(a)e^{ikx} = e^{-ia\mathbf{p}/\hbar}e^{ikx} = e^{-iak}e^{ikx} = e^{ik(x-a)}$$

Boost operators and generators: (A "kick")

Boost operator $\mathbf{B}(b)$ boosts p-wavefunctions

$$\mathbf{B}(b)\cdot\psi(p) = \psi(p-b) = \langle x|\mathbf{B}(b)|\psi\rangle = \langle p-b|\psi\rangle$$

Increases momentum of ket-state by b units

$$\langle p|\mathbf{B}(b) = \langle p-b| \text{ , or: } \mathbf{B}^\dagger(b)|p\rangle = |p-b\rangle$$

Tiny boost $b \to db$ is identity $\mathbf{1}$ plus $\mathbf{K}\cdot db$

$$\mathbf{B}(db) = \mathbf{1} + \mathbf{K}\cdot db \quad \text{where: } \mathbf{K} = \left.\frac{\partial \mathbf{B}}{\partial b}\right|_{b=0}$$

is *generator* $\mathbf{K}$ *of boosts*

$$\mathbf{B}(b) = \left(\mathbf{B}\left(\frac{b}{N}\right)\right)^N = \lim_{N\to\infty}\left(1 + \frac{b}{N}\mathbf{K}\right)^N = e^{b\mathbf{K}}$$

$$\mathbf{B}(b)\cdot\psi(p) = e^{b\mathbf{K}}\cdot\psi(p) = e^{-b\frac{\partial}{\partial p}}\cdot\psi(p)$$

$$= \psi(p) - b\frac{\partial\psi(p)}{\partial p} + \frac{b^2}{2!}\frac{\partial^2\psi(p)}{\partial p^2} - \frac{b^3}{3!}\frac{\partial^3\psi(p)}{\partial p^3} + \dots$$

$\mathbf{K}$ relates to position $\mathbf{x} \to \hbar i\frac{\partial}{\partial p} = i\frac{\partial}{\partial k}$

$$\mathbf{K} = \frac{i}{\hbar}\mathbf{x} \to -\frac{\partial}{\partial p} = \frac{-1}{\hbar}\frac{\partial}{\partial k}$$

$$\mathbf{B}(b) = e^{b\frac{i}{\hbar}\mathbf{x}} = e^{ib\left(\mathbf{a}^\dagger + \mathbf{a}\right)/\sqrt{2\hbar M\omega}}$$

Check $\mathbf{B}(b)$ on plane-wave with $p=\hbar k$

$$\mathbf{B}(b)e^{ikx} = e^{ib\mathbf{x}/\hbar}e^{ikx} = e^{ibx/\hbar}e^{ikx} = e^{i(k+b/\hbar)x}$$

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

- *Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra*
- *Eigenstate creationism (and destruction)*
  - *Vacuum state*
  - *1st excited state*
- *Normal ordering for matrix calculation*
  - *Commutator derivative identities*
  - *Binomial expansion identities*
- *Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations*
  - *Number operator and Hamiltonian operator*
  - *Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$*
  - *Harmonic oscillator beat dynamics of mixed states*
- *Oscillator coherent states ("Shoved" and "kicked" states)*
  - *Translation operators vs. boost operators*
  - ➡ *Applying boost-translation combinations* ⬅
  - *Time evolution of coherent state*
  - *Properties of coherent state and "squeezed" states*

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Applying boost-translation combinations

T(a) and **B**(b) operations do not commute. Q. Which should come first? ??

Applying boost-translation combinations

T(*a*) and **B**(*b*) operations do not commute. Q. Which should come first? $\mathbf{T}(a) = e^{-ia\mathbf{p}/\hbar}$ *or* $\mathbf{B}(b) = e^{ib\mathbf{x}/\hbar}$??

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A. Neither and Both.

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A. Neither and Both. Define a *combined boost-translation operation:* $\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}$

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$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2}=e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2}\text{ , where: }\left[\mathbf{A},\left[\mathbf{A},\mathbf{B}\right]\right]=\mathbf{0}=\left[\mathbf{B},\left[\mathbf{A},\mathbf{B}\right]\right]$$

(left as an exercise)

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$$=\mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar}=\mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

Reordering only affects the overall phase.

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$$=\mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar}=\mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

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$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib(\mathbf{a}^\dagger+\mathbf{a})/\sqrt{2\hbar M\omega}+a(\mathbf{a}^\dagger-\mathbf{a})\sqrt{M\omega/2\hbar}}$$

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Complex *phasor coordinate* $\alpha(a,b)$ is defined by:

$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib(\mathbf{a}^\dagger+\mathbf{a})/\sqrt{2\hbar M\omega}+a(\mathbf{a}^\dagger-\mathbf{a})\sqrt{M\omega/2\hbar}}$$

$$=e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}}=e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}}=e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

$$\alpha(a,b)=a\sqrt{M\omega/2\hbar}+ib/\sqrt{2\hbar M\omega}=\left[a+i\frac{b}{M\omega}\right]\sqrt{M\omega/2\hbar}$$

Applying boost-translation combinations

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Complex ***phasor coordinate α(a,b)*** is defined by:

$$\alpha(a,b)=a\sqrt{M\omega/2\hbar}+ib/\sqrt{2\hbar M\omega}=\left[a+i\frac{b}{M\omega}\right]\sqrt{M\omega/2\hbar}$$

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$$=e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}}=e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}}=e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

Coherent wavepacket state $|\alpha(x_0,p_0)\rangle$: $\left|\alpha_0(x_0,p_0)\right\rangle=\mathbf{C}(x_0,p_0)|0\rangle=e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$

Applying boost-translation combinations

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Coherent wavepacket state $|\alpha(x_0,p_0)\rangle$: $\left|\alpha_0(x_0,p_0)\right\rangle=\mathbf{C}(x_0,p_0)|0\rangle=e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$

$$=e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}e^{-\alpha_0^*\mathbf{a}}|0\rangle$$

Applying boost-translation combinations

T(*a*) and **B**(*b*) operations do not commute. Q. Which should come first? $\mathbf{T}(a)=e^{-ia\mathbf{p}/\hbar}$ *or* $\mathbf{B}(b)=e^{ib\mathbf{x}/\hbar}$??

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Coherent wavepacket state $|\alpha(x_0,p_0)\rangle$:

$$\left|\alpha_0(x_0,p_0)\right\rangle=\mathbf{C}(x_0,p_0)|0\rangle=e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$$

$$=e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}e^{-\alpha_0^*\mathbf{a}}|0\rangle$$

$$=e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}|0\rangle$$

(since: $\mathbf{a}|0\rangle=\mathbf{0}$)

Applying boost-translation combinations

$\mathbf{T}(a)$ and $\mathbf{B}(b)$ operations do not commute. Q. Which should come first? $\mathbf{T}(a)=e^{-ia\mathbf{p}/\hbar}$ *or* $\mathbf{B}(b)=e^{ib\mathbf{x}/\hbar}$??

A. Neither and Both. Define a ***combined boost-translation operation:*** $\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}$

(More like Darboux rotation $e^{-i\boldsymbol{\Theta}\cdot\mathbf{J}/\hbar}$ than Euler rotation with three factors $e^{-i\mathbf{J}_z\alpha/\hbar}e^{-i\mathbf{J}_y\beta/\hbar}e^{-i\mathbf{J}_z\gamma/\hbar}$)

May evaluate with *Baker-Campbell-Hausdorf identity* since $[\mathbf{x},\mathbf{p}]=i\hbar\mathbf{1}$ and $[[\mathbf{x},\mathbf{p}],\mathbf{x}]=[[\mathbf{x},\mathbf{p}],\mathbf{p}]=\mathbf{0}$.

$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2}=e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2}\text{ , where: }\left[\mathbf{A},[\mathbf{A},\mathbf{B}]\right]=\mathbf{0}=\left[\mathbf{B},[\mathbf{A},\mathbf{B}]\right]$$ (left as an exercise)

$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-ab[\mathbf{x},\mathbf{p}]/2\hbar^2}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-iab/2\hbar}$$

$$=\mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar}=\mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

Reordering only affects the overall phase.

Complex *phasor coordinate* $\alpha(a,b)$ is defined by:

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Applying boost-translation combinations

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1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

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→ *Time evolution of coherent state* ←

Properties of coherent state and "squeezed" states

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Time evolution of coherent state: $\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\left|n\right\rangle$

Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0)=\mathrm{e}^{-i\mathbf{H}t/\hbar}$

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Evolution simplifies to a variable-α_0 coherent state with a *time dependent phasor coordinate* α_t:

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-i\omega t/2}\left|\alpha_t(x_t,p_t)\right\rangle$$

where:

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$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

(x_t,p_t) mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

Real and imaginary parts (x_t and $p_t/M\omega$) of α_t go clockwise on phasor circle

1-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

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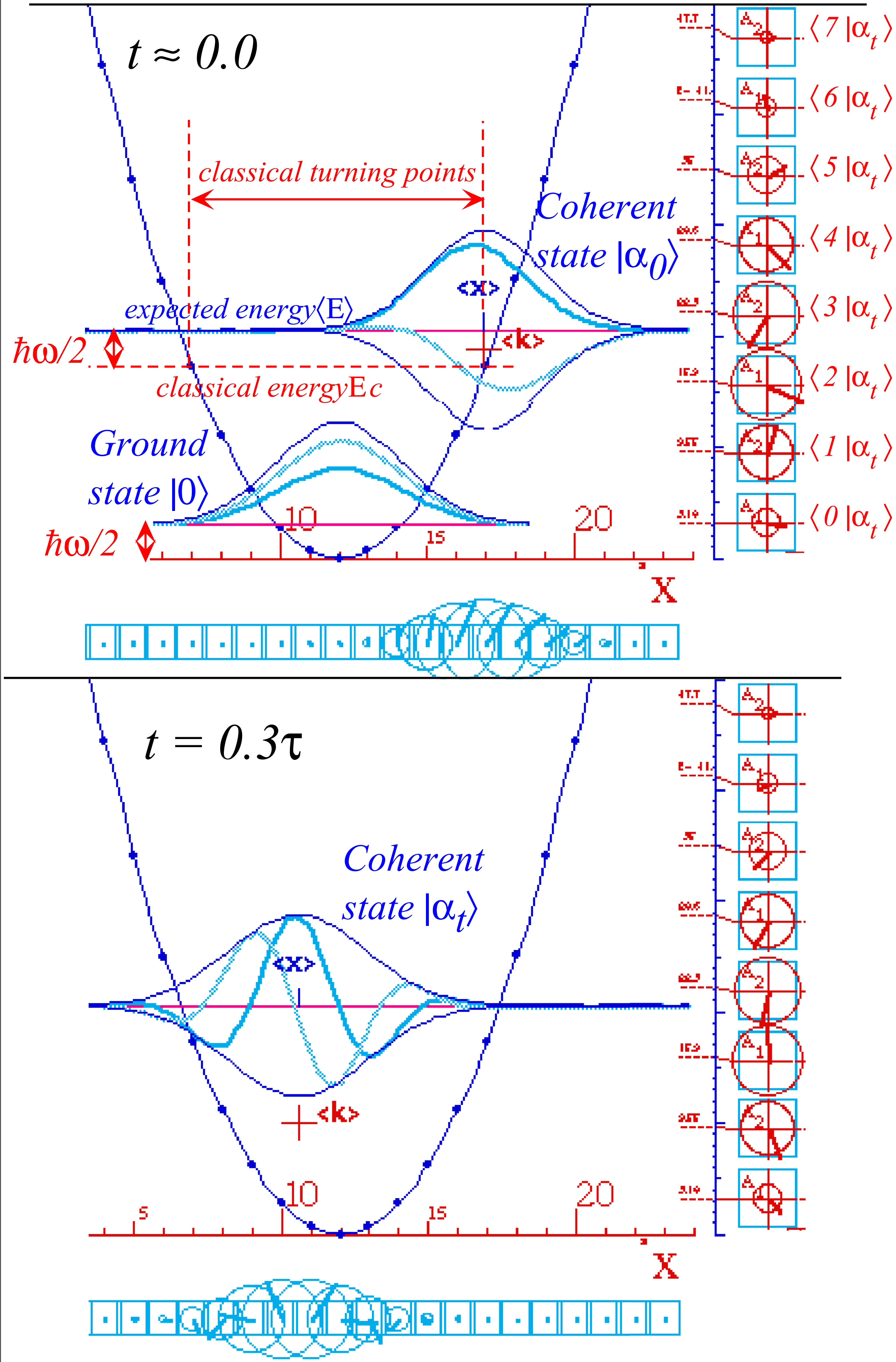
➡ *Properties of coherent state and "squeezed" states* ⬅

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

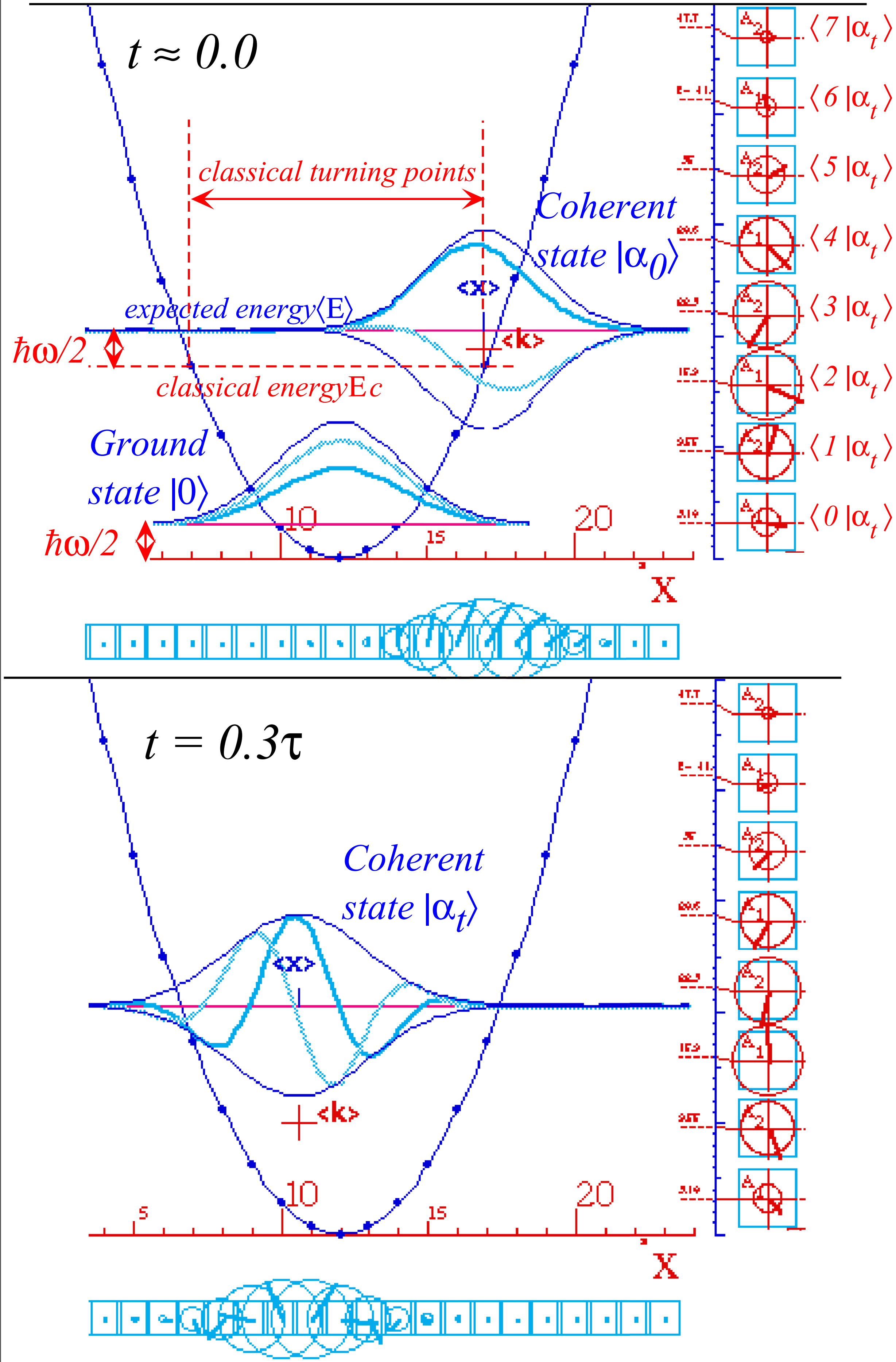
Properties of coherent state



Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

$$\mathbf{a}|\alpha_0(x_0,p_0)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$

Properties of coherent state



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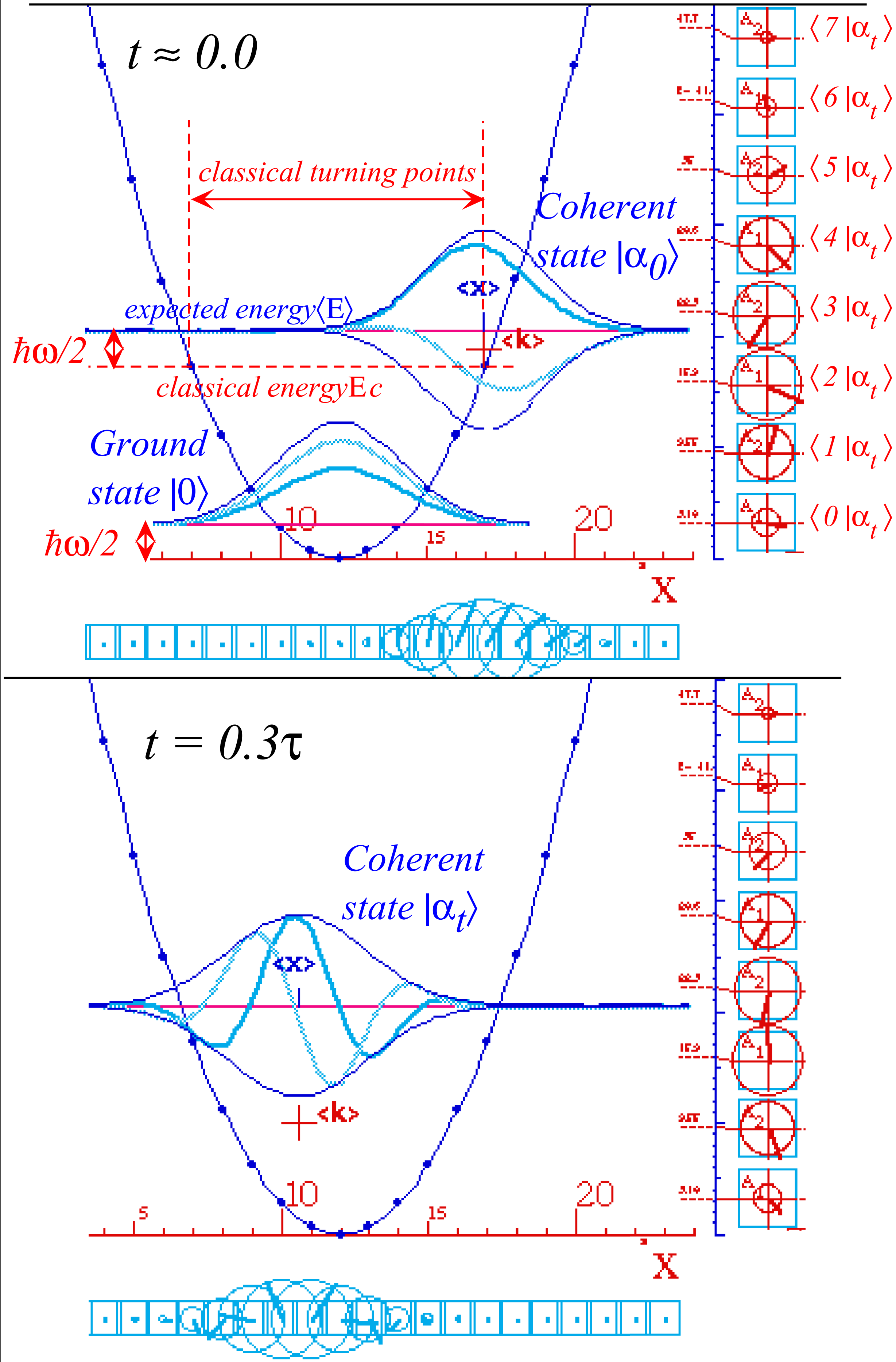

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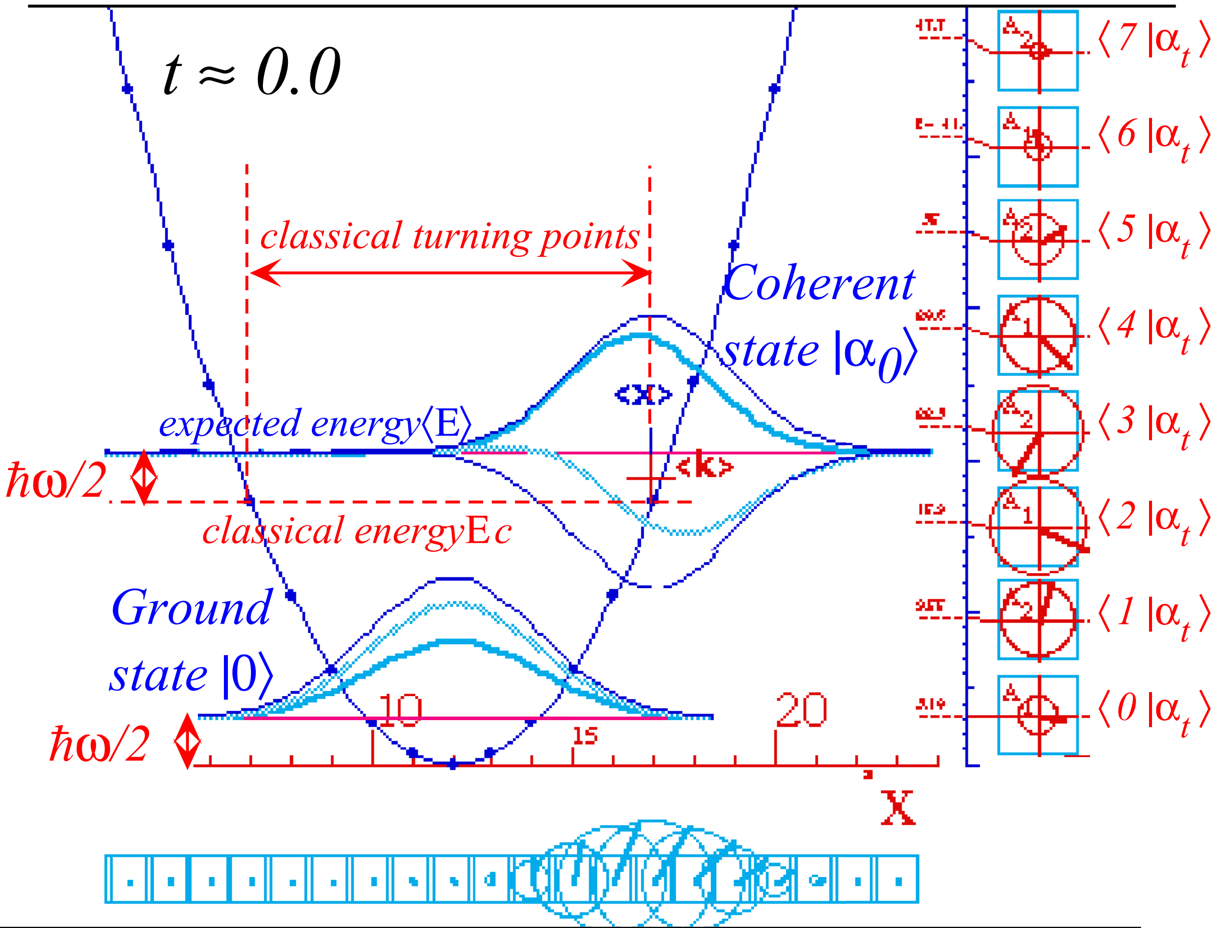

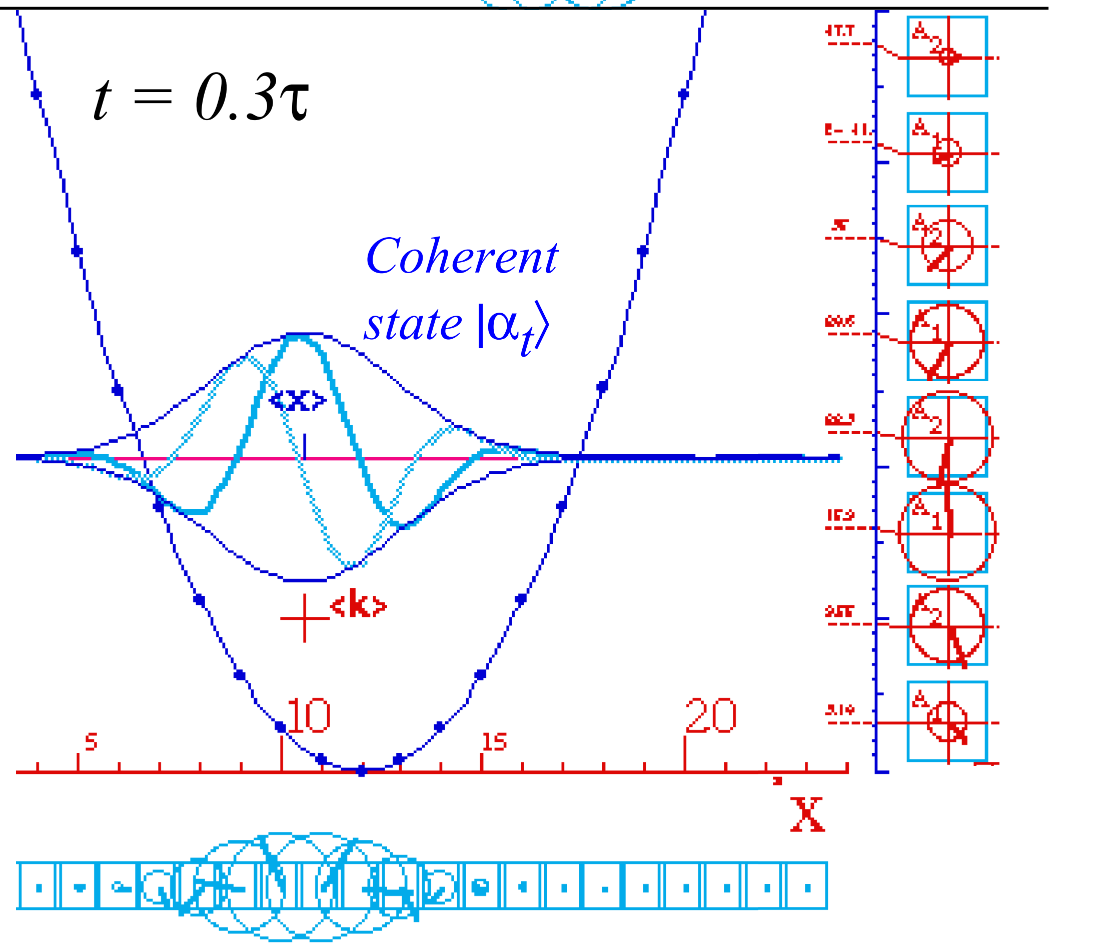

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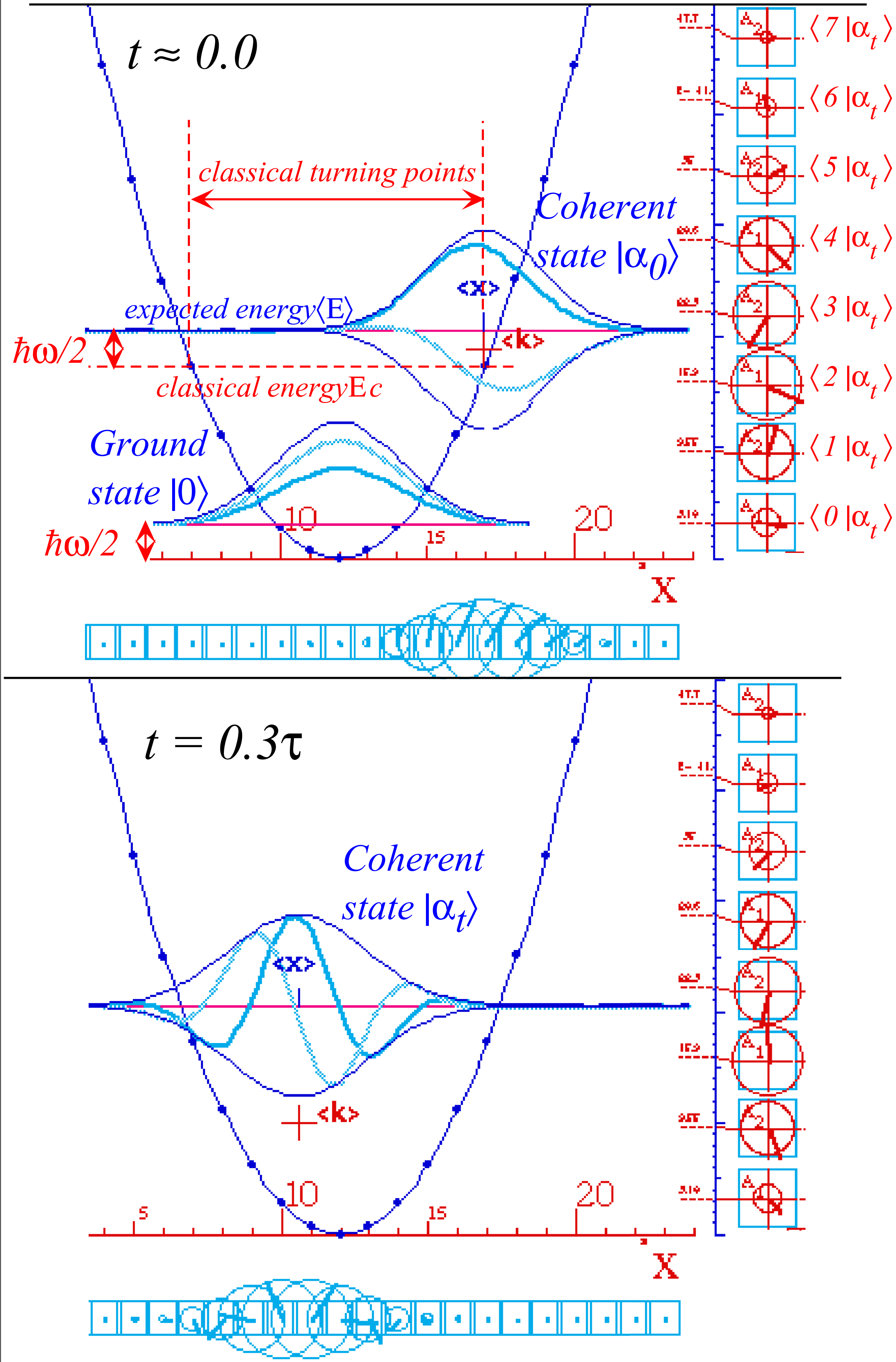

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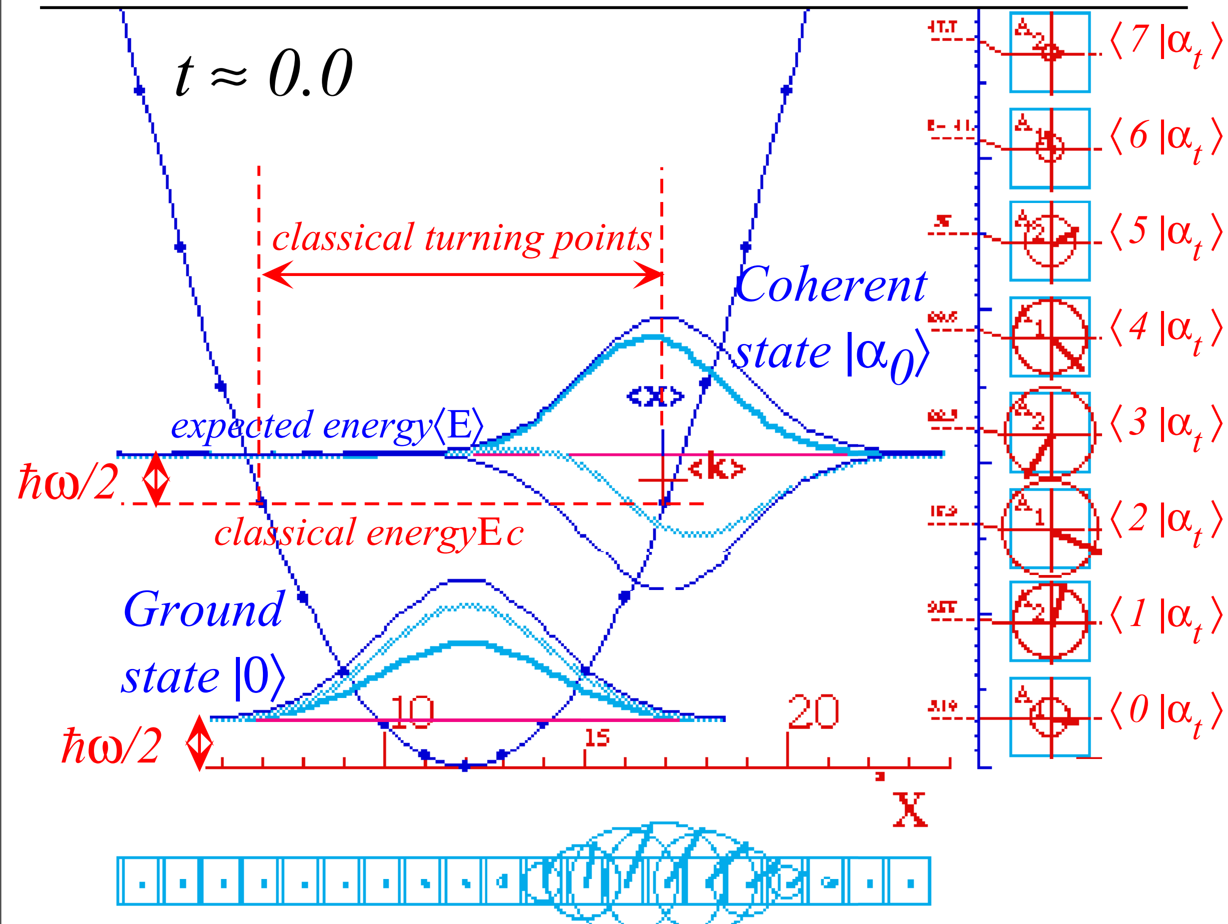
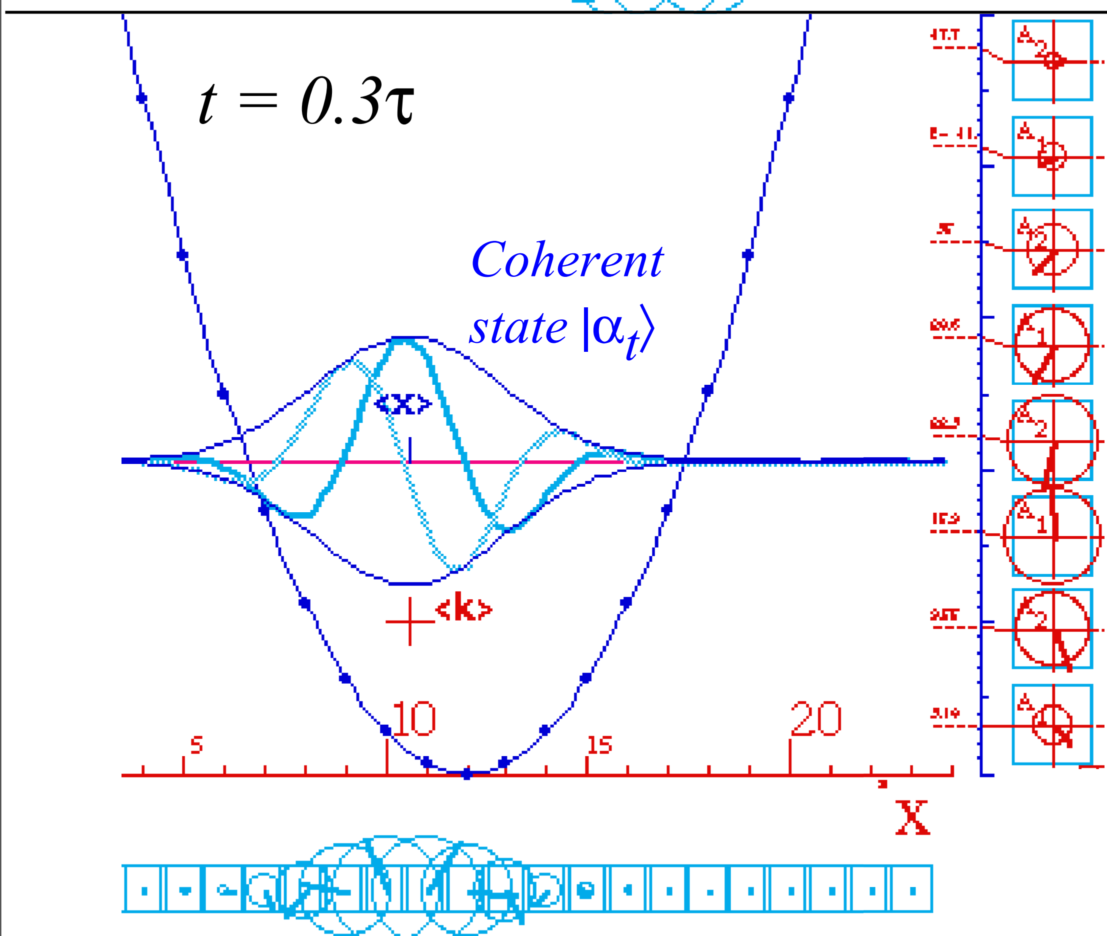
Properties of coherent state





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nemonic 1: Right $|\alpha\rangle$ *is eigenvector of destruction-operator*

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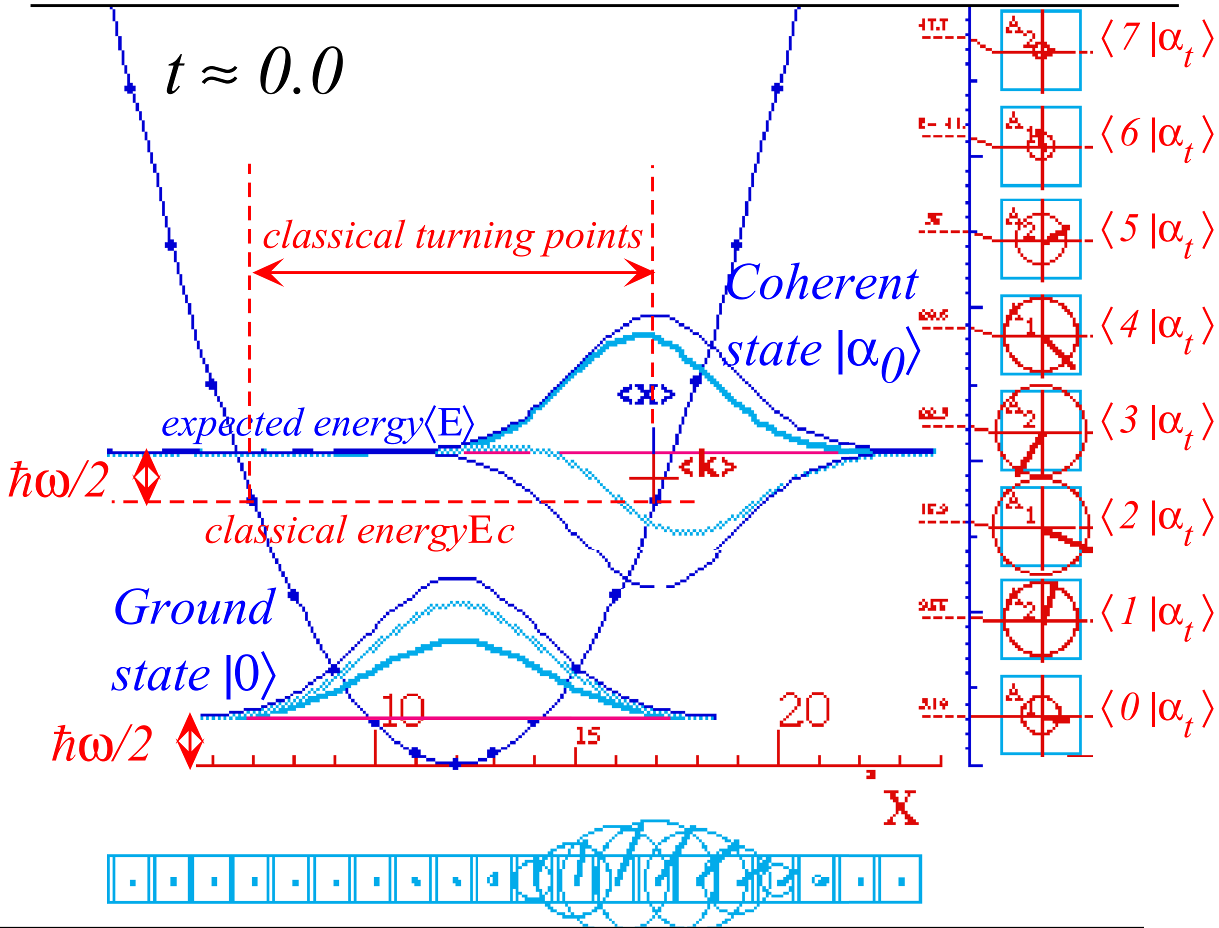
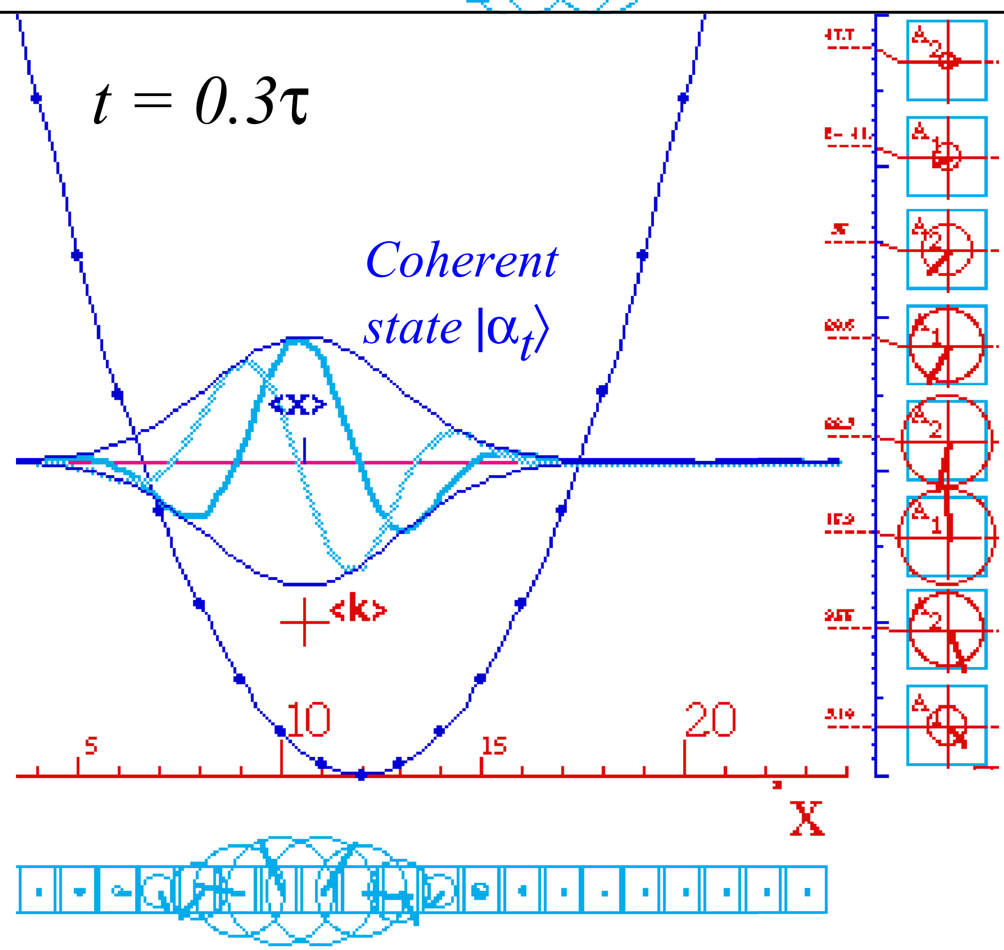
nemonic 2: Left $\langle\alpha|$ *is eigenvector of creation-operator*

Properties of coherent state





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nemonic 2: Left $\langle\alpha|$ is eigenvector of creation-operator

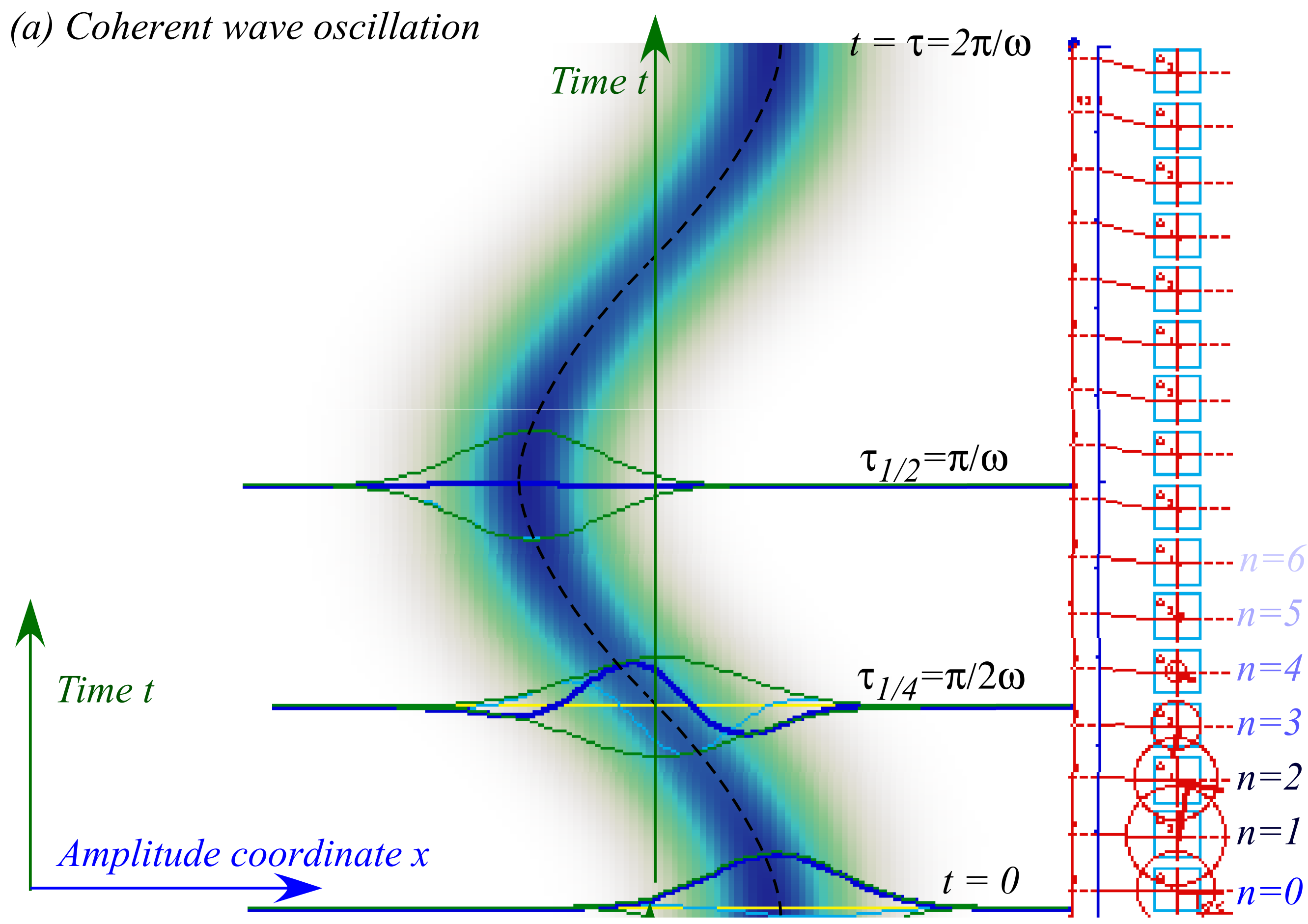
Expected quantum energy has simple time independent form.

$$\langle E\rangle\big|_{\alpha_0} = \left\langle\alpha_0(x_0,p_0)\right|\mathbf{H}\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \left\langle\alpha_0(x_0,p_0)\right|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)\left|\alpha_0(x_0,p_0)\right\rangle$$

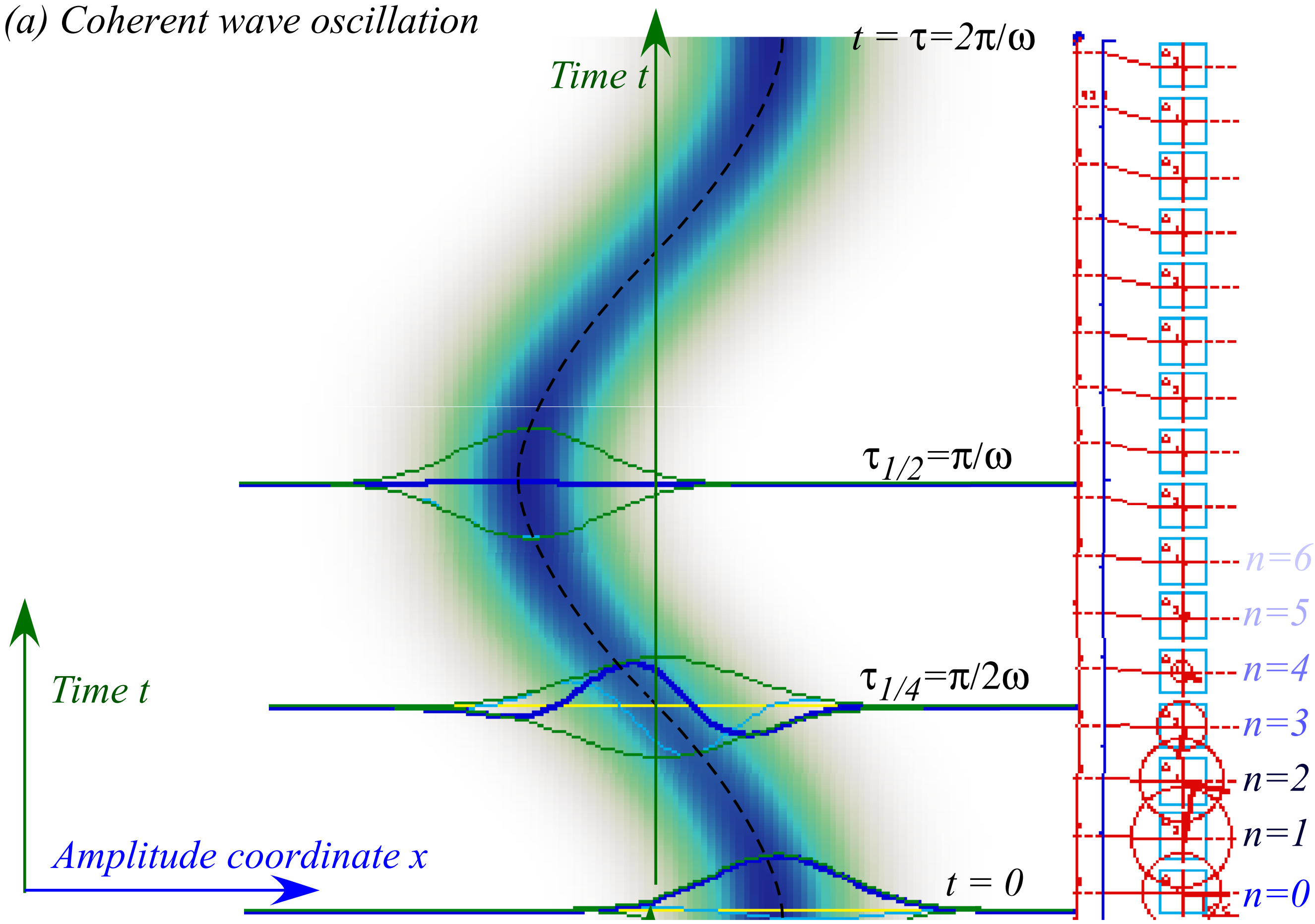
$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

Properties of "squeezed" coherent states



Yeah! Cosine trajectory!

Properties of "squeezed" coherent states

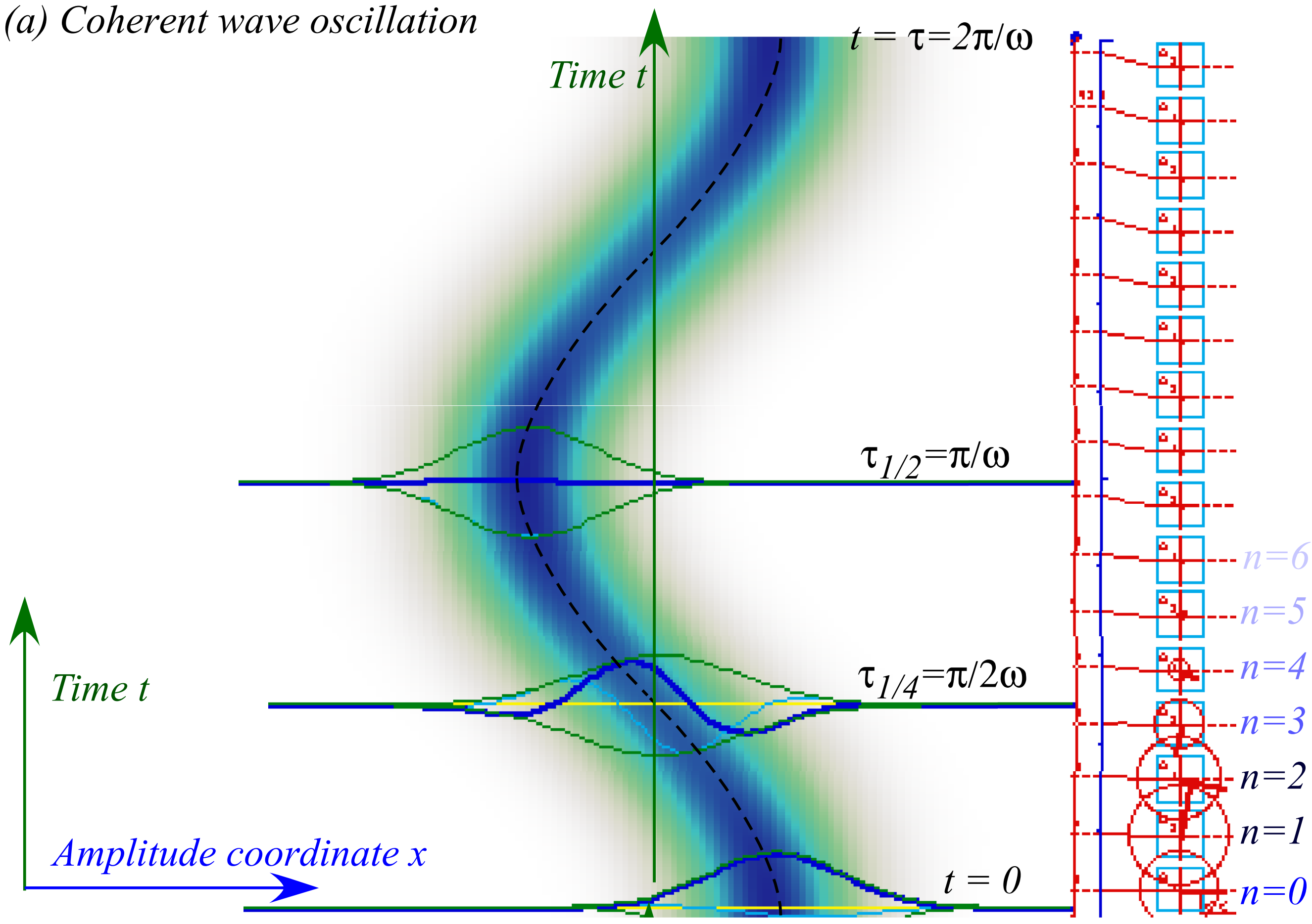


Yeah! Cosine trajectory!

$$\langle \alpha_0(x_0,p_0)|\mathbf{x}|\alpha_0(x_0,p_0)\rangle = \sqrt{\frac{\hbar}{2M\omega}}\langle \alpha_0(x_0,p_0)|\left(\mathbf{a}+\mathbf{a}^\dagger\right)|\alpha_0(x_0,p_0)\rangle$$

$$= \sqrt{\frac{\hbar}{2M\omega}}\left(\alpha_0+\alpha_0^*\right) = x_0$$

Properties of "squeezed" coherent states



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$$= \sqrt{\frac{\hbar}{2M\omega}}\left(\alpha_0+\alpha_0^*\right) = x_0$$

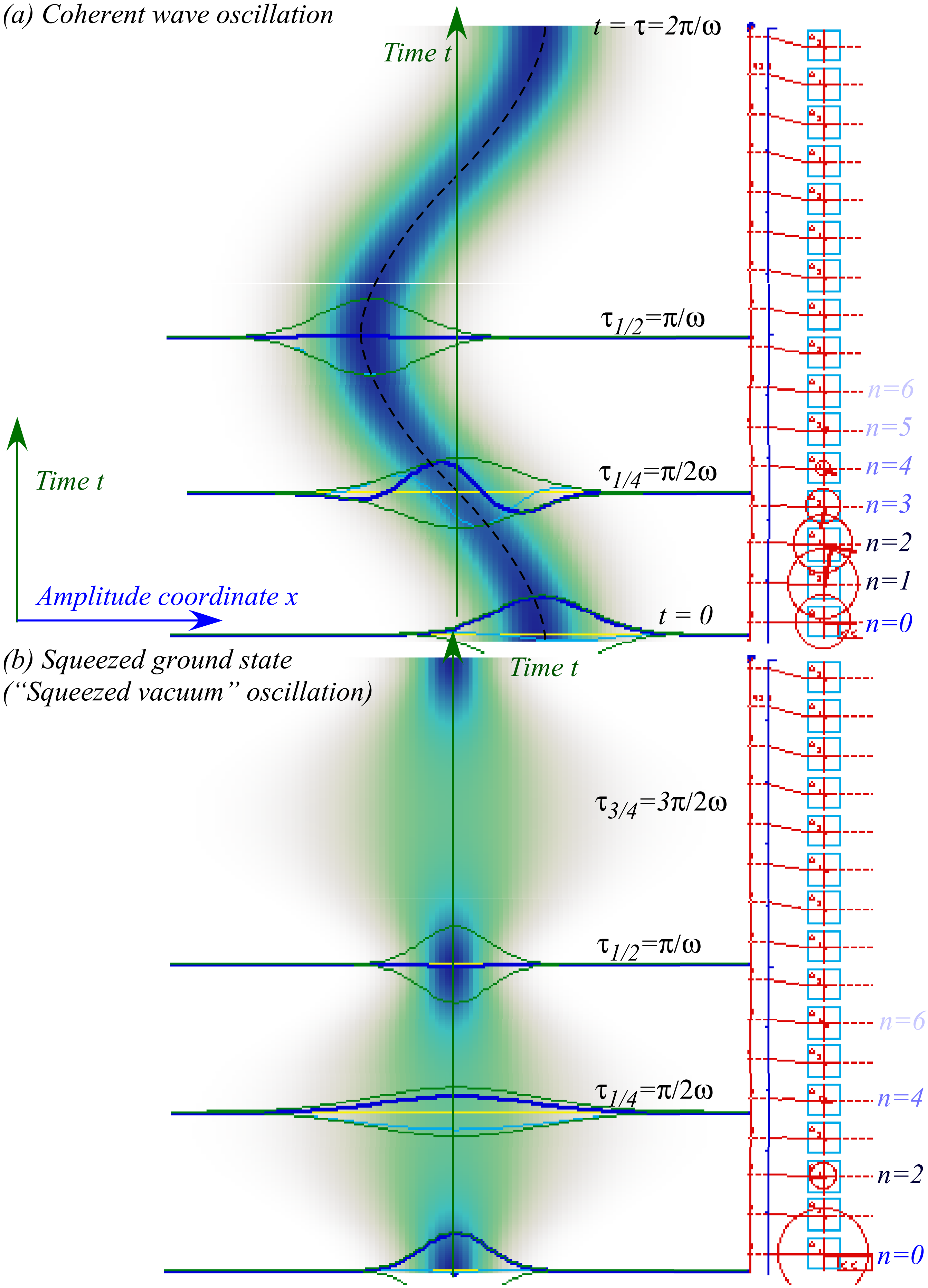
$$\langle \alpha_t(x_t,p_t)|\mathbf{x}|\alpha_t(x_t,p_t)\rangle = \sqrt{\frac{\hbar}{2M\omega}}\left(\alpha_t+\alpha_t^*\right) = x_t$$

$$\alpha_t(x_t,p_t) = e^{-i\omega t}\ \alpha_0(x_0,p_0)$$

$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

...and HO phasor

Properties of "squeezed" coherent states



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...and HO phasor

what happens if you apply operators with non-linear "tensor" exponents exp(s$\mathbf{x}^2$), exp(f $\mathbf{p}^2$), etc.

Properties of “squeezed” coherent states