AMOP Lecture 14
Tue 4.01 - Thur 4.03 2014

Based on QTCA Lectures 24-25
Group Theory in Quantum Mechanics

Rotational symmetry U(2)⊂U(3) and O(3)

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)
(PSDS - Ch. 5, 7)

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
  Key Lie theorems

Angular momentum magnitude and uncertainty
  Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
  Applications of R(3) rotation and U(2) representations
    Molecular and nuclear wavefunctions
    Molecular and nuclear eigenlevels
    Generalized Stern-Gerlach and transformation matrices
  Angular momentum cones and high J properties

As of March 31, 2014

Links to the current Harter-Soft LearnIt web apps for Physics

Bold links have default redirect pages. *Italics* are not yet meant for production.Red: the final stages of testing.

List of *production* Harter-Soft Web Apps & Textbooks (For public)

Classical Mechanics with a Bang! - URL is "http://www.uark.edu/ua/modphys/markup/CMwBangWeb.html"
Quantum Theory for the Computer Age - URL is "http://www.uark.edu/ua/modphys/markup/QTCAWeb.html"
LearnIt Web Applications - URL is "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"

Individual web-apps for current classes:

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
MolVibes - Production; URL is "http://www.uark.edu/ua/modphys/markup/MolVibesWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

The old relativity website (2005):
Relativity - Pirelli Entrant - Production; URL is "**http://www.uark.edu/ua/pirelli**" or "http://www.uark.edu/ua/pirelli/html/default.html"

Newer relativity web-apps currently being developed (2013-)
RelativIt Production; URL is "http://www.uark.edu/ua/modphys/markup/RelativItWeb.html"
RelaWavity Production; URL is "http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html"

Additional classical wep-apps:
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"
WaveIt Production; URL is "http://www.uark.edu/ua/modphys/markup/WaveItWeb.html"

Link to master list of all Harter-Soft Web Apps & Textbooks (Prod, Testing, & Developement)

http://www.uark.edu/ua/modphys/testing/markup/Harter-SoftWebApps.html

Review : *2-D* $a^\dagger a$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators s_+ and s_-
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
  Key Lie theorems

Angular momentum magnitude and uncertainty
  Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
  Applications of R(3) rotation and U(2) representations
    Molecular and nuclear wavefunctions
    Molecular and nuclear eigenstates
    Generalized Stern-Gerlach and transformation matrices
  Angular momentum cones and high J properties

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

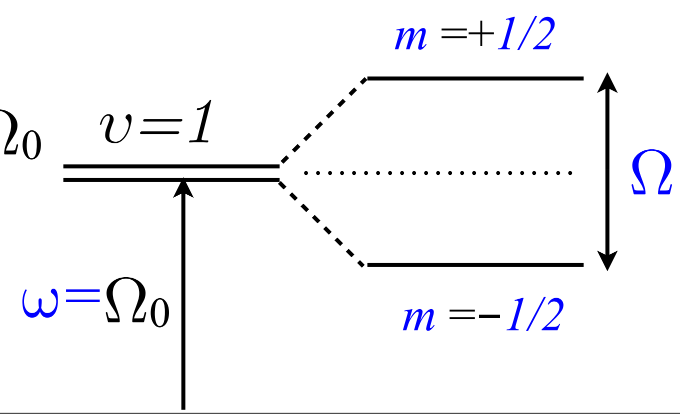
$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0(n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$

$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

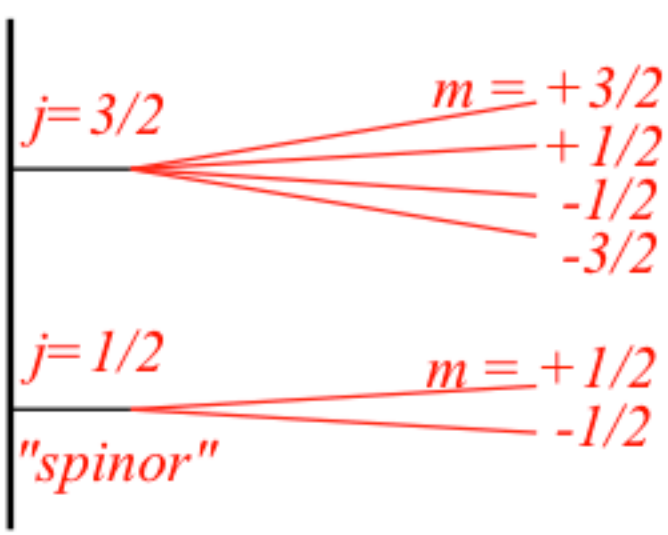
Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$

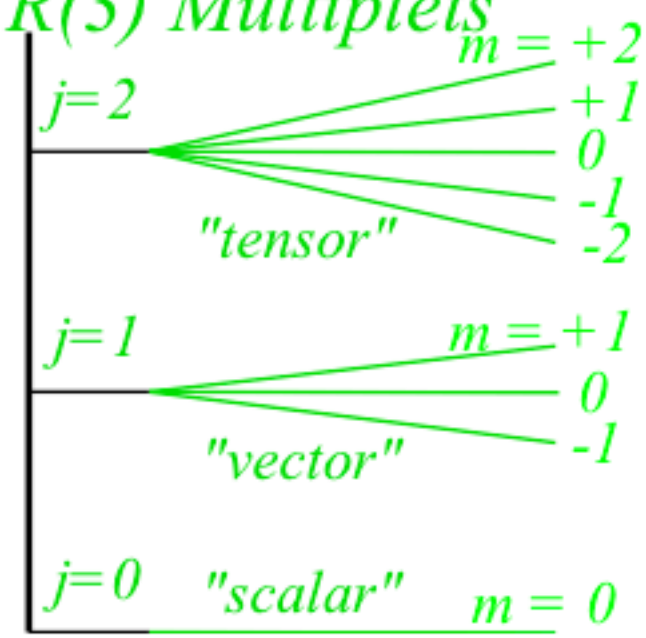
	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

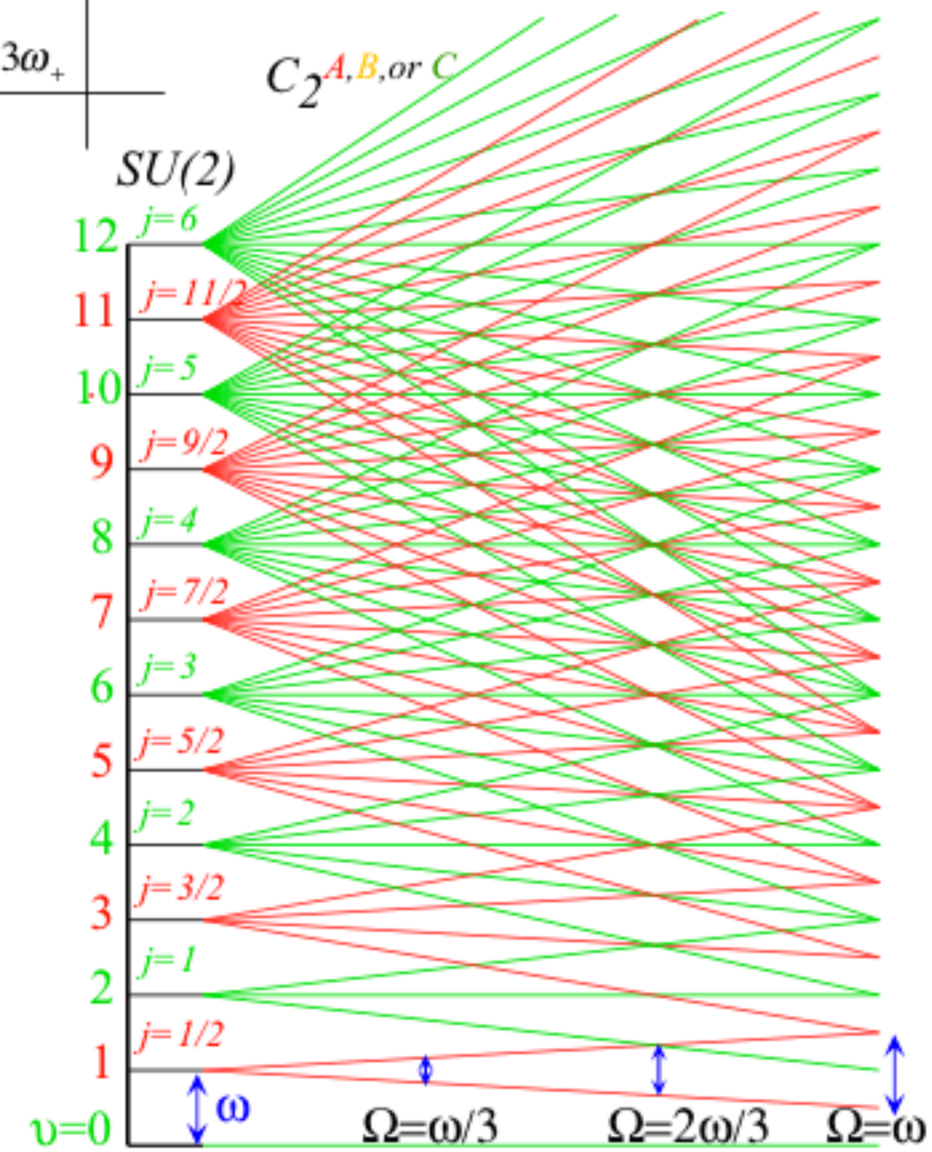
$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

SU(2) Multiplets



R(3) Multiplets





Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

Angular momentum generators by U(2) analysis

(υ=*1*) or (j=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (j=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

(υ=*3*) or (j=*3/2*) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

(υ=*3*) or (*j*=*3/2*) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

(υ=*2j*) or (2*j*+*1*)-by-(2*j*+*1*) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{s}_0 = \mathbf{1}$ and spin operators $\{\mathbf{s}_X, \mathbf{s}_Y, \mathbf{s}_Z\}$.

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

($\upsilon=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

($\upsilon=2j$) or ($2j+1$)-by-($2j+1$) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[(\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right]/2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \;, \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \ , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right) = \tfrac{1}{2}\left(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow\right)$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{(\frac{1}{2})} = D^{(\frac{1}{2})}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{(\frac{1}{2})} = D^{(\frac{1}{2})}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \ , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{(\frac{1}{2})} = D^{(\frac{1}{2})}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin ↑

$$|1\rangle = |\uparrow\rangle = \begin{vmatrix} 1/2 \\ +1/2 \end{matrix}\Big\rangle = \mathbf{a}_1^\dagger|0\rangle = \mathbf{a}_\uparrow^\dagger|0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin ↓

creates up-spin ↑

to <u>raise</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin ↓

$$|2\rangle = |\downarrow\rangle = \begin{vmatrix} 1/2 \\ -1/2 \end{matrix}\Big\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin ↑

creates dn-spin ↓

to <u>lower</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1, n_2\rangle$

Oscillator total quanta: $\upsilon = (n_1 + n_2)$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\;0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\;0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$SU(2) \subset U(2)$ oscillators vs. $R(3) \subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1, n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon = (n_1 + n_2)$ Rotor total momenta: $j = \upsilon/2$ and z-momenta: $m = (n_1 - n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j = \upsilon/2 = (n_1 - n_2)/2$

$m = (n_1 - n_2)/2$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$...

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\;n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\;n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle = \frac{(\mathbf{a}_1^\dagger)^{j+m}(\mathbf{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

U(2) boson oscillator states = *U(2)* spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{n_1}(\mathbf{a}_\downarrow^\dagger)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{j+m}(\mathbf{a}_\downarrow^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\; n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle\end{array}\right\} \mathbf{s}_Z\left|{j \atop m}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|{j \atop m}\right\rangle=\frac{n_1-n_2}{2}\left|{j \atop m}\right\rangle=m\left|{j \atop m}\right\rangle$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{(\mathbf{a}_1^\dagger)^{j+m}(\mathbf{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$|n_\uparrow n_\downarrow\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{n_1}(\mathbf{a}_\downarrow^\dagger)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}|0\,0\rangle = \frac{(\mathbf{a}_\uparrow^\dagger)^{j+m}(\mathbf{a}_\downarrow^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\;n_2-1\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\;n_2+1\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{array}\right\} \mathbf{s}_Z\left|{j \atop m}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|{j \atop m}\right\rangle=\frac{n_1-n_2}{2}\left|{j \atop m}\right\rangle=m\left|{j \atop m}\right\rangle$$

j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix}\cdot & \frac{\sqrt2}{2} & \cdot\\ \frac{\sqrt2}{2} & \cdot & \frac{\sqrt2}{2}\\ \cdot & \frac{\sqrt2}{2} & \cdot\end{pmatrix}+i\begin{pmatrix}\cdot & -i\frac{\sqrt2}{2} & \cdot\\ i\frac{\sqrt2}{2} & \cdot & -i\frac{\sqrt2}{2}\\ \cdot & i\frac{\sqrt2}{2} & \cdot\end{pmatrix}=\begin{pmatrix}\cdot & \sqrt2 & \cdot\\ 0 & \cdot & \sqrt2\\ \cdot & 0 & \cdot\end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix}1 & \cdot & \cdot\\ \cdot & 0 & \cdot\\ \cdot & \cdot & -1\end{pmatrix}$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$\left|n_1 n_2\right\rangle=\frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}\left|0\,0\right\rangle=\frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle=\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$\left|n_\uparrow n_\downarrow\right\rangle=\frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}\left|0\,0\right\rangle=\frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\,0\right\rangle=\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\;n_2-1\right\rangle\Rightarrow\mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\;n_2+1\right\rangle\Rightarrow\mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle\end{array}\right\}\;\mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt2}{2} & \cdot \\ \frac{\sqrt2}{2} & \cdot & \frac{\sqrt2}{2} \\ \cdot & \frac{\sqrt2}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt2}{2} & \cdot \\ i\frac{\sqrt2}{2} & \cdot & -i\frac{\sqrt2}{2} \\ \cdot & i\frac{\sqrt2}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt2 & \cdot \\ 0 & \cdot & \sqrt2 \\ \cdot & 0 & \cdot \end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^{\frac32}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt3 & \cdot & \cdot \\ 0 & \cdot & \sqrt4 & \cdot \\ \cdot & 0 & \cdot & \sqrt3 \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac32}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac32}(\mathbf{s}_Z)=\begin{pmatrix} \frac32 & \cdot & \cdot & \cdot \\ \cdot & \frac12 & \cdot & \cdot \\ \cdot & \cdot & -\frac12 & \cdot \\ \cdot & \cdot & \cdot & -\frac32 \end{pmatrix}$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = $R(3)$ spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{j+m}\left(\mathbf{a}_2^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{j \atop m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

$n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

$$|n_\uparrow n_\downarrow\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_2!}}|0\,0\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{j \atop m}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|n_1+1\;n_2-1\rangle \Rightarrow \mathbf{s}_+\left|{j \atop m}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|{j \atop m+1}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\;n_2+1\rangle \Rightarrow \mathbf{s}_-\left|{j \atop m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{j \atop m-1}\right\rangle$$

$$\left.\begin{aligned}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle&=n_1|n_1 n_2\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle&=n_2|n_1 n_2\rangle\end{aligned}\right\} \mathbf{s}_Z\left|{j \atop m}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|{j \atop m}\right\rangle=\frac{n_1-n_2}{2}\left|{j \atop m}\right\rangle=m\left|{j \atop m}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix}\cdot & \frac{\sqrt{2}}{2} & \cdot\\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2}\\ \cdot & \frac{\sqrt{2}}{2} & \cdot\end{pmatrix}+i\begin{pmatrix}\cdot & -i\frac{\sqrt{2}}{2} & \cdot\\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2}\\ \cdot & i\frac{\sqrt{2}}{2} & \cdot\end{pmatrix}=\begin{pmatrix}\cdot & \sqrt{2} & \cdot\\ 0 & \cdot & \sqrt{2}\\ \cdot & 0 & \cdot\end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix}1 & \cdot & \cdot\\ \cdot & 0 & \cdot\\ \cdot & \cdot & -1\end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix}\cdot & \sqrt{3} & \cdot & \cdot\\ 0 & \cdot & \sqrt{4} & \cdot\\ \cdot & 0 & \cdot & \sqrt{3}\\ \cdot & \cdot & 0 & \cdot\end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix}\frac{3}{2} & \cdot & \cdot & \cdot\\ \cdot & \frac{1}{2} & \cdot & \cdot\\ \cdot & \cdot & -\frac{1}{2} & \cdot\\ \cdot & \cdot & \cdot & -\frac{3}{2}\end{pmatrix}$$

j=2 tensor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix}\cdot & \sqrt{4} & \cdot & \cdot & \cdot\\ 0 & \cdot & \sqrt{3} & \cdot & \cdot\\ \cdot & 0 & \cdot & \sqrt{3} & \cdot\\ \cdot & \cdot & 0 & \cdot & \sqrt{4}\\ \cdot & \cdot & \cdot & 0 & \cdot\end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger,\qquad D^2(\mathbf{s}_Z)=\begin{pmatrix}2 & \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & 0 & \cdot & \cdot\\ \cdot & \cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & \cdot & \cdot & -2\end{pmatrix}$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\, \varepsilon_{\alpha\beta\gamma} \mathbf{s}_\gamma$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\, \varepsilon_{\alpha\beta\gamma} \mathbf{s}_\gamma$

$\boldsymbol{\sigma}_X \boldsymbol{\sigma}_Y = i \boldsymbol{\sigma}_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\boldsymbol{\sigma}_Z \boldsymbol{\sigma}_X = i \boldsymbol{\sigma}_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\boldsymbol{\sigma}_Y \boldsymbol{\sigma}_Z = i \boldsymbol{\sigma}_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\,\mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*.

$[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha/2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha\mathbf{s}_\beta - \mathbf{s}_\beta\mathbf{s}_\alpha = [\mathbf{s}_\alpha,\mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X\sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X,\mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z\sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z,\mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y\sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y,\mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z,\mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z,\mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12} \approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and: $\mathbf{s}_- = \mathbf{e}_{21} \approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha/2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\, \varepsilon_{\alpha\beta\gamma} \mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\boldsymbol{\sigma}_X \boldsymbol{\sigma}_Y = i\boldsymbol{\sigma}_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\boldsymbol{\sigma}_Z \boldsymbol{\sigma}_X = i\boldsymbol{\sigma}_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\boldsymbol{\sigma}_Y \boldsymbol{\sigma}_Z = i\boldsymbol{\sigma}_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*.

$[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha\mathbf{s}_\beta - \mathbf{s}_\beta\mathbf{s}_\alpha = [\mathbf{s}_\alpha,\mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X\sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X,\mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z\sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z,\mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y\sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y,\mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z,\mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z,\mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha/2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

General eigen-commutation theorem:

If Hamiltonian $\mathbf{H}$ (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:

$[\mathbf{H}, \mathbf{a}^\dagger_n] = \omega_n\,\mathbf{a}^\dagger_n$ and $[\mathbf{H}, \mathbf{a}_m] = \omega_m \mathbf{a}_m$, then $\mathbf{H}$ is a combination $\omega_n \mathbf{a}^\dagger_n \mathbf{a}_n$ of number operators.

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha/2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha\mathbf{s}_\beta - \mathbf{s}_\beta\mathbf{s}_\alpha = [\mathbf{s}_\alpha,\mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X\sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X,\mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z\sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z,\mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y\sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y,\mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z,\mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z,\mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\quad \approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2,\, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2,\, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+,\mathbf{s}_-] = [\mathbf{e}_{12},\mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

General eigen-commutation theorem:

If Hamiltonian **H** (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:

$[\mathbf{H},\mathbf{a}^\dagger_n] = \omega_n\,\mathbf{a}^\dagger_n$ and $[\mathbf{H},\mathbf{a}_m] = \omega_m\mathbf{a}_m$, then **H** is a combination $\omega_n\mathbf{a}^\dagger_n\mathbf{a}_n$ of number operators.

$$\mathbf{H} = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n = \omega_1 \mathbf{a}^\dagger_1 \mathbf{a}_1 + \omega_2 \mathbf{a}^\dagger_2 \mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

General eigen-commutation theorem:

If Hamiltonian $\mathbf{H}$ (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:
$[\mathbf{H}, \mathbf{a}^\dagger_n] = \omega_n\,\mathbf{a}^\dagger_n$ and $[\mathbf{H}, \mathbf{a}_m] = \omega_m \mathbf{a}_m$, then $\mathbf{H}$ is a combination $\omega_n \mathbf{a}^\dagger_n \mathbf{a}_n$ of number operators.

$$\mathbf{H} = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n = \omega_1 \mathbf{a}^\dagger_1 \mathbf{a}_1 + \omega_2 \mathbf{a}^\dagger_2 \mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

$n_1 = j+m$

$n_2 = j-m$

U(2) Oscillator eigensolutions:

$$\mathbf{H}|n_1 n_2\rangle = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n |n_1 n_2\rangle = (\omega_1 n_1 + \omega_2 n_2)|n_1 n_2\rangle = (\omega_1(j+m) + \omega_2(j-m))\left|{j \atop m}\right\rangle$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

➡ *Angular momentum magnitude and uncertainty* ⬅
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X{}^2 + \mathbf{s}_Y{}^2 + \mathbf{s}_Z{}^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z{}^2$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_{+}=\mathbf{a}_1^{\dagger}\mathbf{a}_2$

$\mathbf{s}_{-}=\mathbf{a}_2^{\dagger}\mathbf{a}_1$

$\mathbf{s}_Z=\frac{1}{2}\left(\mathbf{a}_1^{\dagger}\mathbf{a}_1-\mathbf{a}_2^{\dagger}\mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

j=1/2 fundamental matrices square up not to $(1/2)^2 = 1/4$ but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}+\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of $\mathbf{a}$-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_{+}=\mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_{-}=\mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z=\frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared **s•s** and *Z*-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

j=1/2 fundamental matrices square up not to $(1/2)^2 = 1/4$ but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives **s•s** as number operators.
(Normal order: *left←creation*, *destruct→right*.)

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^{\ 2} + \mathbf{s}_Y^{\ 2} + \mathbf{s}_Z^{\ 2} = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^{\ 2}$$

j=1/2 fundamental matrices square up not to *(1/2)*^{2} *=1/4* but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.

(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^{\,2} + \mathbf{s}_Y^{\,2} + \mathbf{s}_Z^{\,2} = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^{\,2}$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Eigenvalue formula is then found.

$$\mathbf{S}\bullet\mathbf{S}\left|n_1 n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1 - n_2\right)\left(n_1 - n_2\right)\right]\left|n_1 n_2\right\rangle$$

$$= \frac{1}{4}\left[2\left(n_1+n_2\right) + 4n_1n_2 \qquad + n_1^{\,2} - 2n_1n_2 + n_2^{\,2}\right]\left|n_1 n_2\right\rangle$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared **s•s** and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X{}^2 + \mathbf{s}_Y{}^2 + \mathbf{s}_Z{}^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z{}^2$$

j=1/2 fundamental matrices square up not to $(1/2)^2 = 1/4$ but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives **s•s** as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Eigenvalue formula is then found.

$$\mathbf{S}\bullet\mathbf{S}\left|n_1n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$

$$= \frac{1}{4}\left[2\left(n_1+n_2\right) + 4n_1n_2 \qquad + n_1{}^2 - 2n_1n_2 + n_2{}^2\right]\left|n_1n_2\right\rangle = \left[\frac{\left(n_1+n_2\right)}{2} + \frac{\left(n_1+n_2\right)^2}{4}\right]\left|n_1n_2\right\rangle$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_{+}=\mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_{-}=\mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z=\frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^{\,2} + \mathbf{s}_Y^{\,2} + \mathbf{s}_Z^{\,2} = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^{\,2}$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1+2\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

$n_1 = j+m$

$n_2 = j-m$

Eigenvalue formula is then found.

Recall from p.24: $j = \upsilon/2 = (n_1+n_2)/2$, $m = (n_1-n_2)/2$

$$\mathbf{S}\bullet\mathbf{S}|n_1n_2\rangle=\frac{1}{4}\left[2(n_2+1)n_1+2(n_1+1)n_2+(n_1-n_2)(n_1-n_2)\right]|n_1n_2\rangle$$

$$=\frac{1}{4}\left[2(n_1+n_2)+4n_1n_2\qquad+n_1^{\,2}-2n_1n_2+n_2^{\,2}\right]|n_1n_2\rangle=\left[\frac{(n_1+n_2)}{2}+\frac{(n_1+n_2)^2}{4}\right]|n_1n_2\rangle$$

R(3) angular quantum number $j=(n_1+n_2)/2$ determines $\mathbf{s}\bullet\mathbf{s}$ eigenvalues.

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^{\,2} + \mathbf{s}_Y^{\,2} + \mathbf{s}_Z^{\,2} = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^{\,2}$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

$n_1 = j+m$

$n_2 = j-m$

Eigenvalue formula is then found.

Recall from p.24: $j = \upsilon/2 = (n_1+n_2)/2$, $m = (n_1-n_2)/2$

$$\mathbf{S}\bullet\mathbf{S}|n_1n_2\rangle = \frac{1}{4}\left[2(n_2+1)n_1 + 2(n_1+1)n_2 + (n_1-n_2)(n_1-n_2)\right]|n_1n_2\rangle$$

$$= \frac{1}{4}\left[2(n_1+n_2) + 4n_1n_2 \qquad + n_1^{\,2} - 2n_1n_2 + n_2^{\,2}\right]|n_1n_2\rangle = \left[\frac{(n_1+n_2)}{2} + \frac{(n_1+n_2)^2}{4}\right]|n_1n_2\rangle$$

R(3) angular quantum number $j=(n_1+n_2)/2$ determines $\mathbf{s}\bullet\mathbf{s}$ eigenvalues.

$$\mathbf{S}\bullet\mathbf{S}|n_1n_2\rangle = \left[\frac{(n_1+n_2)^2}{4} + \frac{(n_1+n_2)}{2}\right]|n_1n_2\rangle = \left[j^{\,2} + j\right]|n_1n_2\rangle = j(j+1)|n_1n_2\rangle$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_{+}=\mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_{-}=\mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z=\frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X{}^2 + \mathbf{s}_Y{}^2 + \mathbf{s}_Z{}^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z{}^2$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In terms of $\mathbf{a}$-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger{}_n = \mathbf{a}^\dagger{}_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.

(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1+2\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

$n_1 = j+m$

$n_2 = j-m$

Eigenvalue formula is then found.

Recall from p.24: $j = \upsilon/2 = (n_1+n_2)/2$, $m = (n_1-n_2)/2$

$$\mathbf{S}\bullet\mathbf{S}\left|n_1n_2\right\rangle=\frac{1}{4}\left[2(n_2+1)n_1+2(n_1+1)n_2+(n_1-n_2)(n_1-n_2)\right]\left|n_1n_2\right\rangle$$

$$=\frac{1}{4}\left[2(n_1+n_2)+4n_1n_2 \qquad +n_1{}^2-2n_1n_2+n_2{}^2\right]\left|n_1n_2\right\rangle=\left[\frac{(n_1+n_2)}{2}+\frac{(n_1+n_2)^2}{4}\right]\left|n_1n_2\right\rangle$$

R(3) angular quantum number $j=(n_1+n_2)/2$ determines $\mathbf{s}\bullet\mathbf{s}$ eigenvalues.

$$\mathbf{S}\bullet\mathbf{S}\left|n_1n_2\right\rangle=\left[\frac{(n_1+n_2)^2}{4}+\frac{(n_1+n_2)}{2}\right]\left|n_1n_2\right\rangle=\left[j^2+j\right]\left|n_1n_2\right\rangle=j(j+1)\left|n_1n_2\right\rangle$$

For large j:

Magnitude of angular momentum $|\mathbf{s}|$ approaches $j+1/2$: $\quad |\mathbf{s}|\left|{}^{j}_{m}\right\rangle=\sqrt{\mathbf{s}\bullet\mathbf{s}}\left|{}^{j}_{m}\right\rangle=\sqrt{j(j+1)}\left|{}^{j}_{m}\right\rangle\cong\left(j+\frac{1}{2}\right)\left|{}^{j}_{m}\right\rangle$

Review : *2-D* **a**†**a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum uncertainty angle

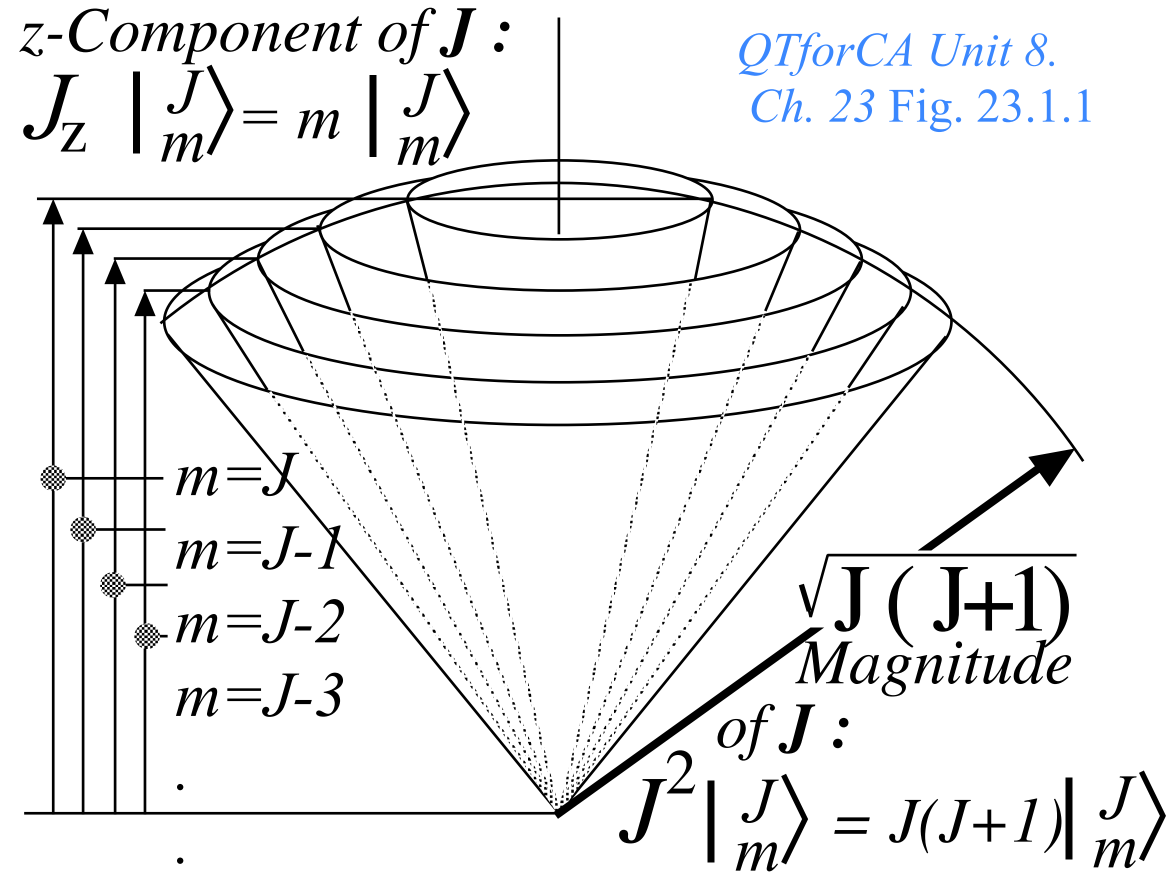
The *angular momentum uncertainty angle* Θ^j_m is given by:

$$\Theta^j_m = \arccos\left(\frac{m}{\sqrt{j\left(j+1\right)}}\right)$$

Angular momentum uncertainty angle

The *angular momentum uncertainty angle* Θ^j_m is given by:

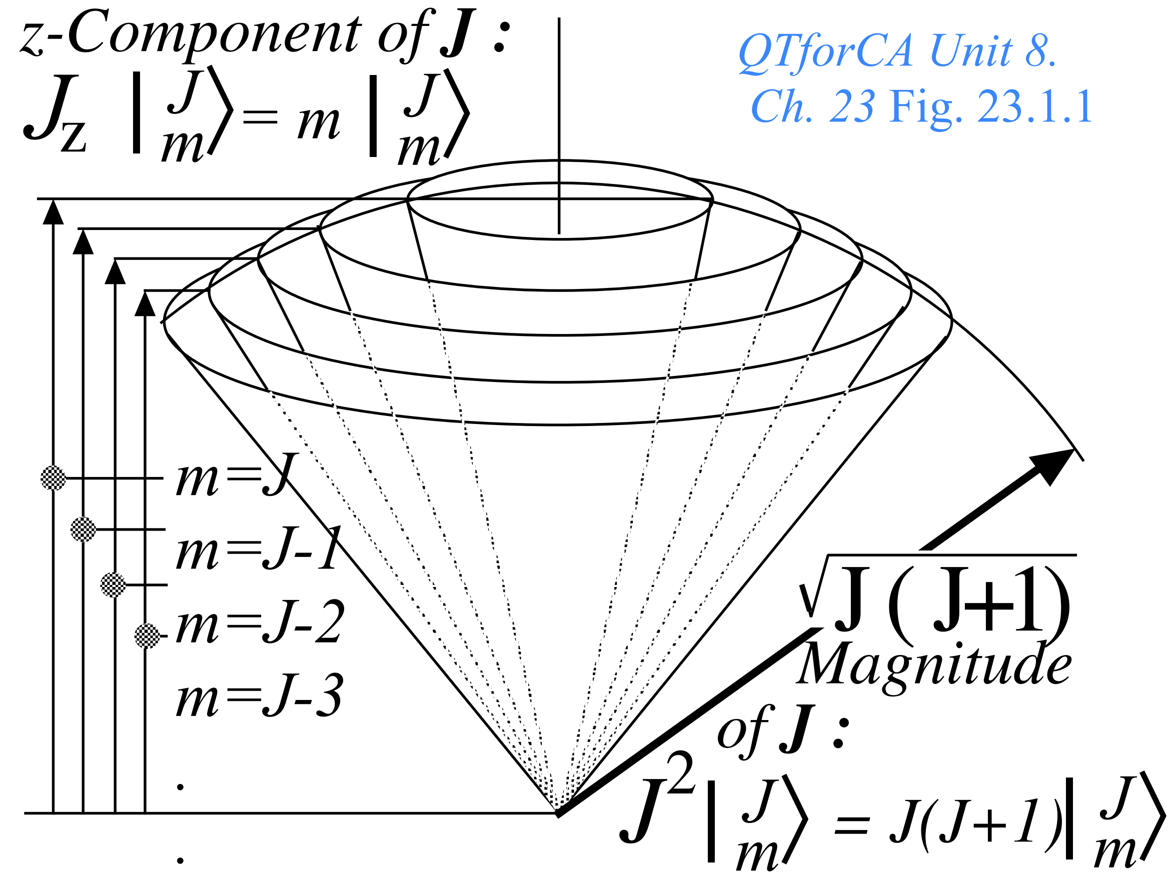
$$\Theta^j_m = \arccos\left(\frac{m}{\sqrt{j(j+1)}}\right)$$



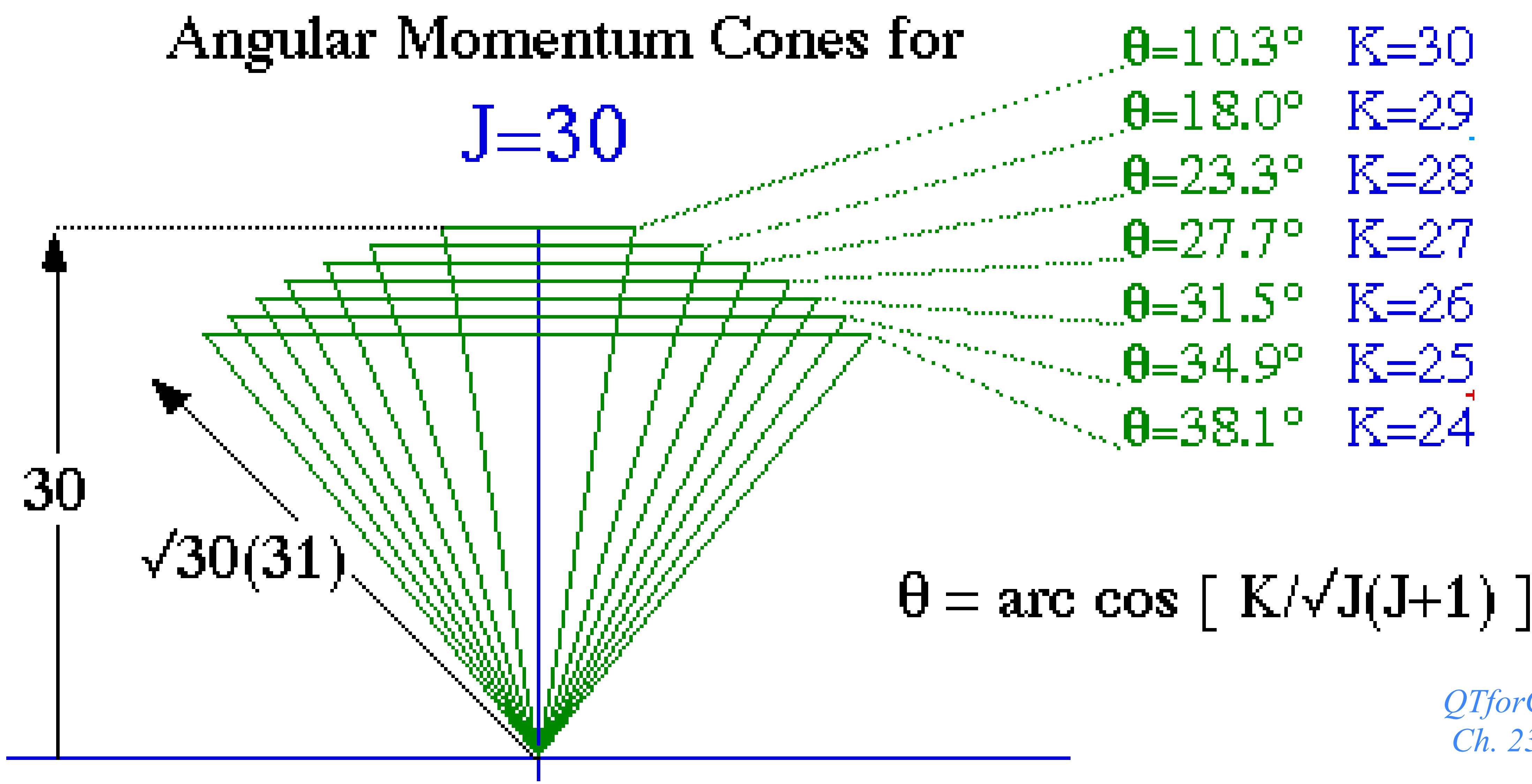
Angular momentum uncertainty angle

The *angular momentum uncertainty angle* Θ^j_m is given by:

$$\Theta^j_m = \arccos\left(\frac{m}{\sqrt{j(j+1)}}\right)$$



QTforCA Unit 8. Ch. 23 Fig. 23.1.1



QTforCA Unit 8. Ch. 23 Fig. 23.1.2

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

→ *Generating R(3) rotation and U(2) representations* ←
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

A fundamental (spin-1/2) Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

$$\mathbf{a}^\dagger_{2'} = D^{1/2}_{12}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{22}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_2$$

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

$$\mathbf{a}^\dagger_{2'} = D^{1/2}_{12}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{22}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_2$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\upsilon=2j$)-oscillator state

($\upsilon=2j$)-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{(\mathbf{a}^\dagger_{1'})^{j+n}(\mathbf{a}^\dagger_{2'})^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \frac{(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2)^{j+n}(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle$$

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

$$\mathbf{a}^\dagger_{2'} = D^{1/2}_{12}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{22}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_2$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\upsilon=2j$)-oscillator state

($\upsilon=2j$)-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\left(\mathbf{a}^\dagger_{1'}\right)^{j+n}\left(\mathbf{a}^\dagger_{2'}\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \frac{\left(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2\right)^{j+n}\left(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle$$

Binomial expansion is a double sum over binomial coefficients: $\binom{n}{k} = n!/k!(n-k)!$

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\sum_{\ell}\sum_{k}\binom{j+n}{\ell}\left(D_{11}\mathbf{a}^\dagger_1\right)^{\ell}\left(D_{21}\mathbf{a}^\dagger_2\right)^{j+n-\ell}\binom{j-n}{k}\left(D_{12}\mathbf{a}^\dagger_1\right)^{k}\left(D_{22}\mathbf{a}^\dagger_2\right)^{j-n-k}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \sqrt{(j+n)!(j-n)!}\frac{\sum_{\ell}\sum_{k}\left(D_{11}\mathbf{a}^\dagger_1\right)^{\ell}\left(D_{21}\mathbf{a}^\dagger_2\right)^{j+n-\ell}\left(D_{12}\mathbf{a}^\dagger_1\right)^{k}\left(D_{22}\mathbf{a}^\dagger_2\right)^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}|0\,0\rangle$$

Generating R(3) rotation and U(2) representations

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

$$\mathbf{a}^\dagger_{2'} = D^{1/2}_{12}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{22}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_2$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a $(\upsilon=2j)$-oscillator state

$(\upsilon=2j)$-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{(\mathbf{a}^\dagger_{1'})^{j+n}(\mathbf{a}^\dagger_{2'})^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \frac{(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2)^{j+n}(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle$$

Binomial expansion is a double sum over binomial coefficients: $\binom{n}{k} = n!/k!(n-k)!$

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\sum_\ell\sum_k \binom{j+n}{\ell}(D_{11}\mathbf{a}^\dagger_1)^\ell(D_{21}\mathbf{a}^\dagger_2)^{j+n-\ell}\binom{j-n}{k}(D_{12}\mathbf{a}^\dagger_1)^k(D_{22}\mathbf{a}^\dagger_2)^{j-n-k}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell\sum_k (D_{11}\mathbf{a}^\dagger_1)^\ell(D_{21}\mathbf{a}^\dagger_2)^{j+n-\ell}(D_{12}\mathbf{a}^\dagger_1)^k(D_{22}\mathbf{a}^\dagger_2)^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}|0\,0\rangle$$

Let $\mathbf{a}^\dagger$-operator powers be $j\pm m$ forms : $j+m=\ell+k$, $j-m=2j-\ell-k$ so $\ell=j+m-k$ and $j+n-\ell=n-m+k$

$$= \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell\sum_k (D_{11})^\ell(D_{21})^{j+n-\ell}(D_{12})^k(D_{22})^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}(\mathbf{a}^\dagger_1)^{\ell+k}(\mathbf{a}^\dagger_2)^{2j-\ell-k}|0\,0\rangle = \sqrt{(j+n)!(j-n)!}\frac{\sum_m\sum_k (D_{11})^{j+m-k}(D_{21})^{n-m+k}(D_{12})^k(D_{22})^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}(\mathbf{a}^\dagger_1)^{j+m}(\mathbf{a}^\dagger_2)^{j-m}|0\,0\rangle$$

A fundamental (spin-1/2) irep $D^{(1/2)}(\alpha\beta\gamma)$ of Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

$$\mathbf{a}^\dagger_{2'} = D^{1/2}_{12}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{22}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_2$$

Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\upsilon=2j$)-oscillator state

($\upsilon=2j$)-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\left(\mathbf{a}^\dagger_{1'}\right)^{j+n}\left(\mathbf{a}^\dagger_{2'}\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \frac{\left(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2\right)^{j+n}\left(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle$$

This gives general *irreducible representation of U(2)* :

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\sum_k \frac{\left(D_{11}\right)^{j+m-k}\left(D_{21}\right)^{n-m+k}\left(D_{12}\right)^{k}\left(D_{22}\right)^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

And general *SU(2) irreducible representation for Euler angles (αβγ)*.

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\sum_k \frac{(-1)^k\left(\cos\frac{\beta}{2}\right)^{2j+m-n-2k}\left(\sin\frac{\beta}{2}\right)^{n-m+2k} e^{-i(m\alpha+n\gamma)}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
➡ *Applications of R(3) rotation and U(2) representations* ⬅
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Applications of R(3) rotation and U(2) representations

Vector (j=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Applications of R(3) rotation and U(2) representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$

$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^{ℓ}_{m}.

$$Y_m^{\ell}(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta \quad=\quad \cos\theta \quad=\quad z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

Applications of R(3) rotation and U(2) representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y_m^\ell(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix.

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \end{pmatrix}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$
$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta=\cos\theta=z/r$$
$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle\end{pmatrix}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$

$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole ($j=\ell=1$) waves

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1\end{pmatrix}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle\end{pmatrix}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2}$, $\sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2}$, $\sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2}$,

$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$

$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta$$

$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) waves

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \end{pmatrix}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applying T-matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} D^1_{x,x}(\alpha\beta\gamma) & D^1_{x,y} & D^1_{x,z} \\ D^1_{y,x} & D^1_{y,y} & D^1_{y,z} \\ D^1_{z,x} & D^1_{z,y} & D^1_{z,z} \end{pmatrix}=$$

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

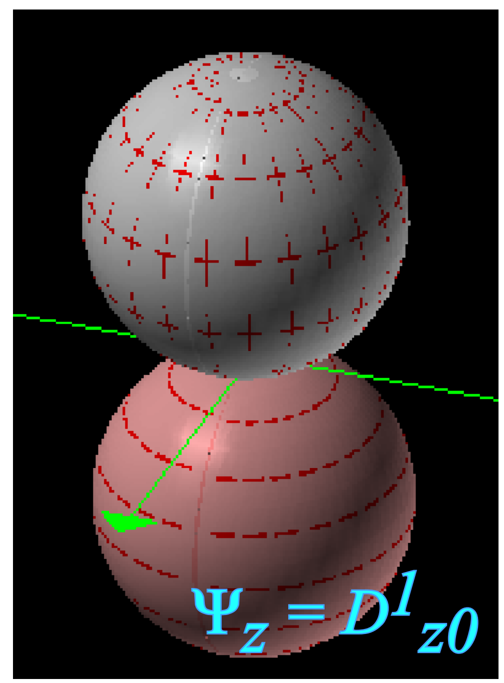
$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^{ℓ}_{m}.

$$Y^{\ell}_{m}(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

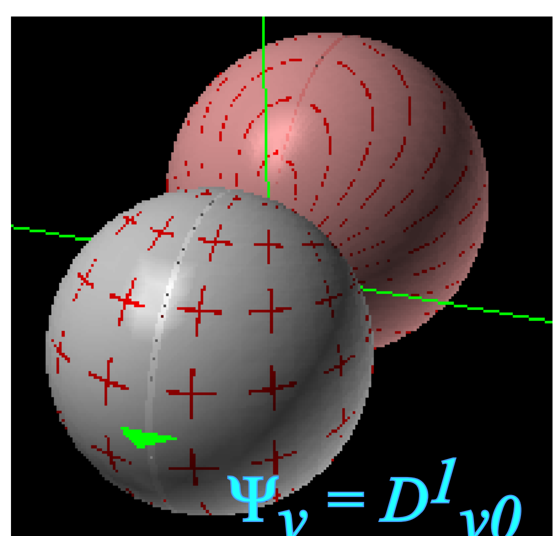
Dipole (j=ℓ=1) waves

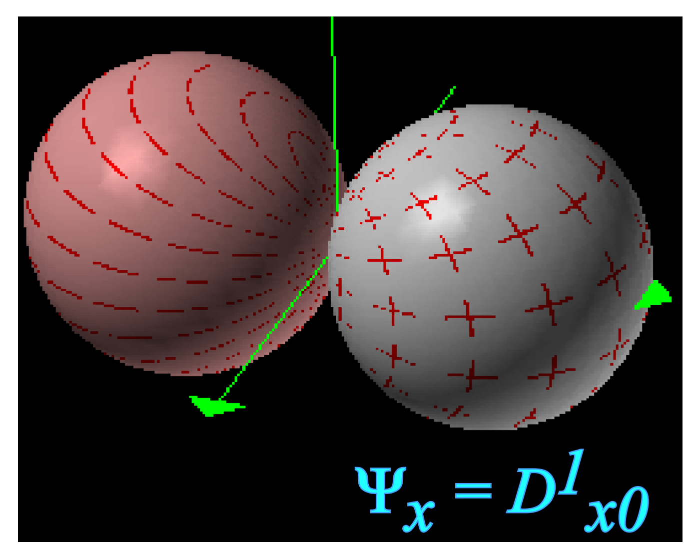
$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$
$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$
$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

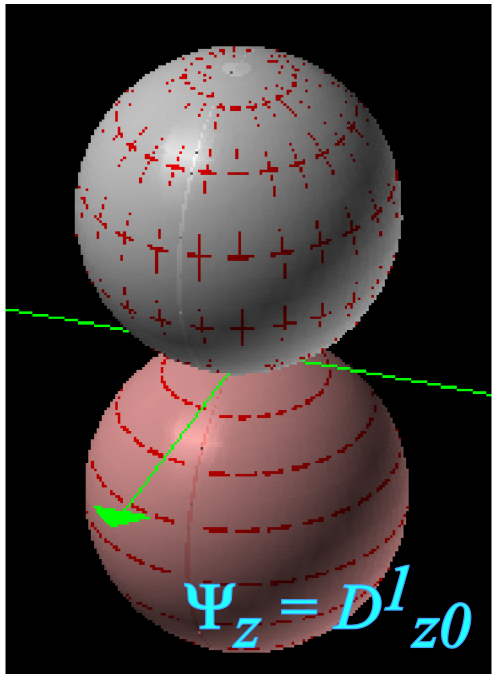


j = 1
Standing
p-Waves

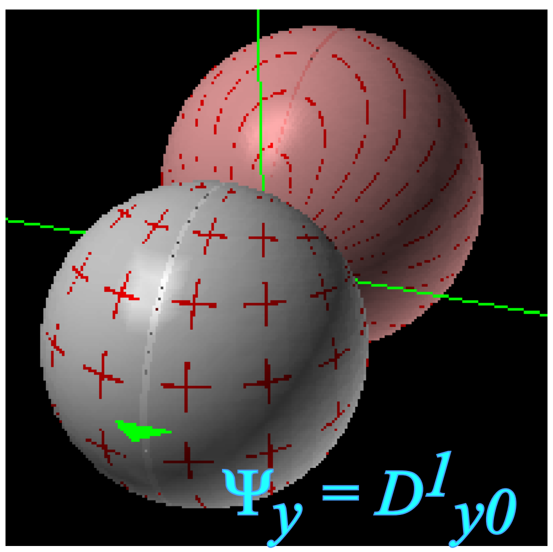
$$\Psi^1_x(\phi,\theta)=D^1_{x,z}(\phi,\theta,0)=\cos\phi\sin\theta$$
$$\Psi^1_y(\phi,\theta)=D^1_{y,z}(\phi,\theta,0)=\sin\phi\sin\theta$$
$$\Psi^1_z(\phi,\theta)=D^1_{z,z}(\phi,\theta,0)=\cos\theta$$

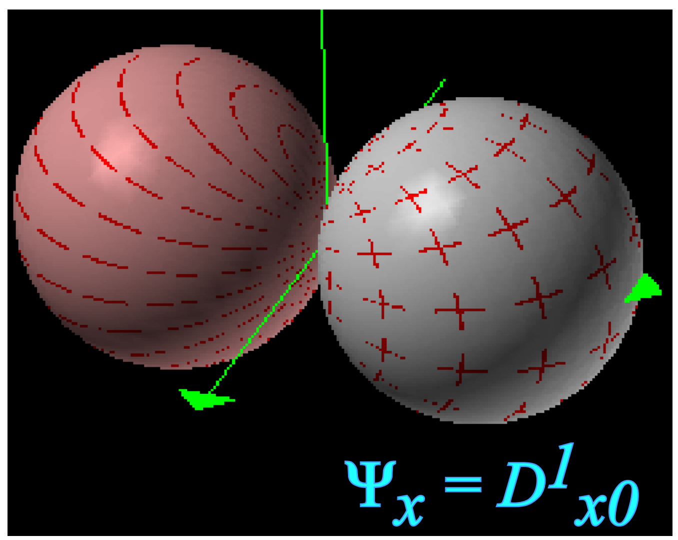






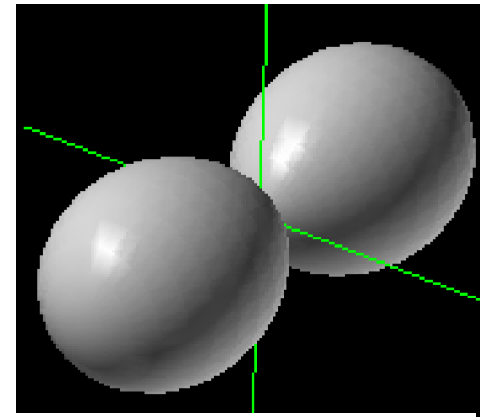
j = 1
Standing
p-Waves



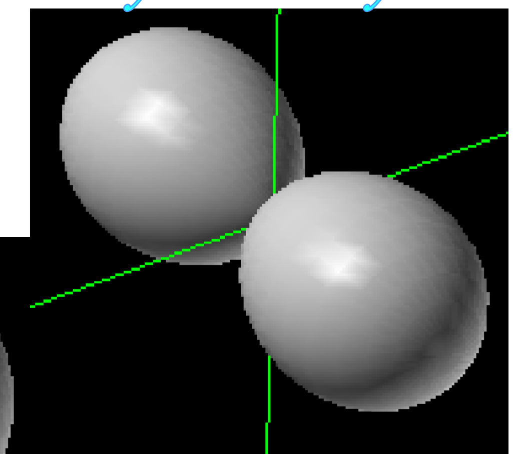


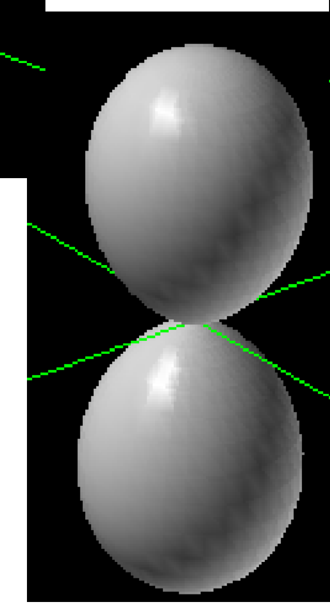
$$\Psi^1_x(\phi,\theta) = D^1_{x,z}(\phi,\theta,0)$$
$$= \cos\phi\sin\theta$$
$$\Psi^1_y(\phi,\theta) = D^1_{y,z}(\phi,\theta,0)$$
$$= \sin\phi\sin\theta$$
$$\Psi^1_z(\phi,\theta) = D^1_{z,z}(\phi,\theta,0)$$
$$= \cos\theta$$

$|\Psi_x|^2 = |D^1{}_{x0}|^2$



$|\Psi_y|^2 = |D^1{}_{y0}|^2$



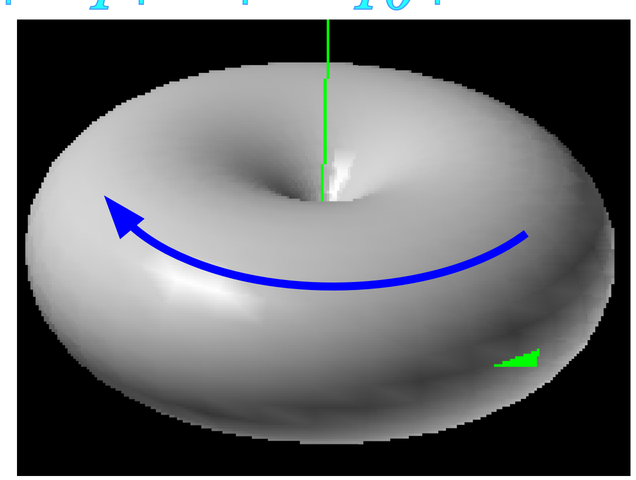


$|\Psi_z|^2 = |D^1{}_{z0}|^2$

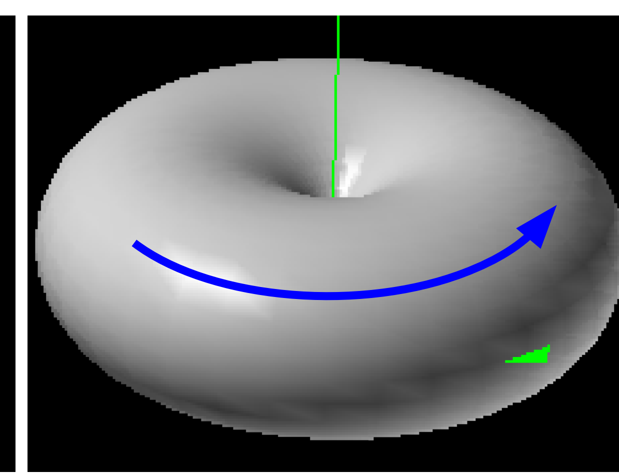
Standing p-Wave Distributions

Moving p-Wave Distributions

$|\Psi_{-1}|^2 = |D^1{}_{-10}|^2$



$|\Psi_1|^2 = |D^1{}_{10}|^2$



Tensor (j=ℓ=2) representation

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Applications of R(3) rotation and U(2) representations

Tensor (j=ℓ=2) representation

$$D^2(\alpha\beta0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Spherical 2^k-*multipole functions* $X^k{}_q$ or *X-functions* are D^*-functions times the k^{th} power of radius (r^k).

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta)=D^{2*}_{2,0}(\phi\theta0)=\sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta=\sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

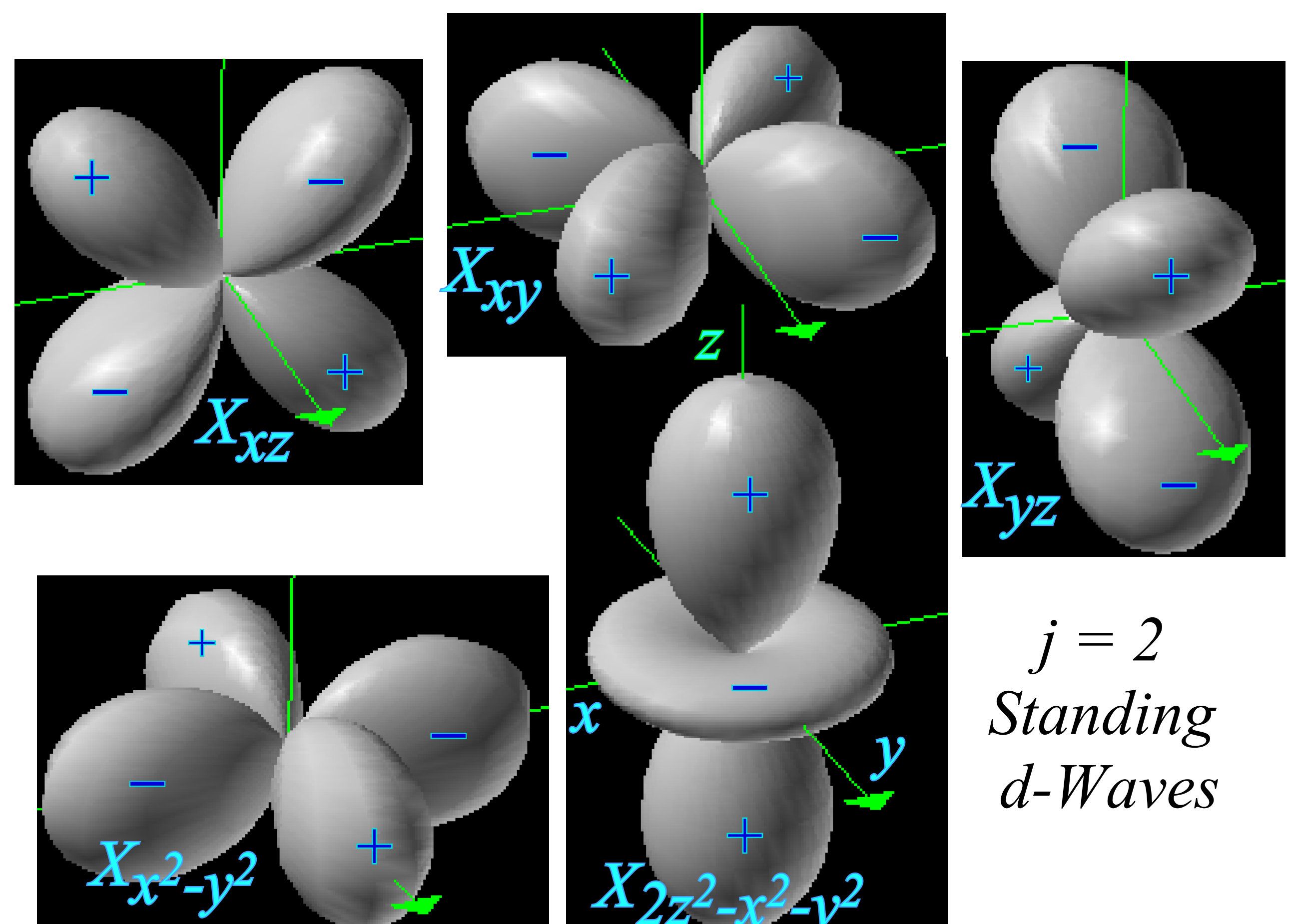
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta)=D^{2*}_{1,0}(\phi\theta0)=-\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta=-\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta)=D^{2*}_{0,0}(\phi\theta0)=\frac{3\cos^2\theta-1}{2}=\frac{3z^2-r^2}{2r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-1}(\phi\theta)=D^{2*}_{-1,0}(\phi\theta0)=\sqrt{\frac{3}{2}}e^{-i\phi}\sin\theta\cos\theta=\sqrt{\frac{3}{2}}\frac{(x-iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta)=D^{2*}_{-2,0}(\phi\theta0)=\sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta=\sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$

$$X^k_q=r^kD^{k*}_{q,0}=\sqrt{\frac{4\pi}{2k+1}}r^kY^k_q$$



j = 2
Standing
d-Waves

Tensor (j=ℓ=2) representation

Spherical 2^k*-multipole functions* $X^k{}_q$ or *X-functions* are *D**-functions times the k^{th} power of radius (r^k).

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta) = D^{2*}_{2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

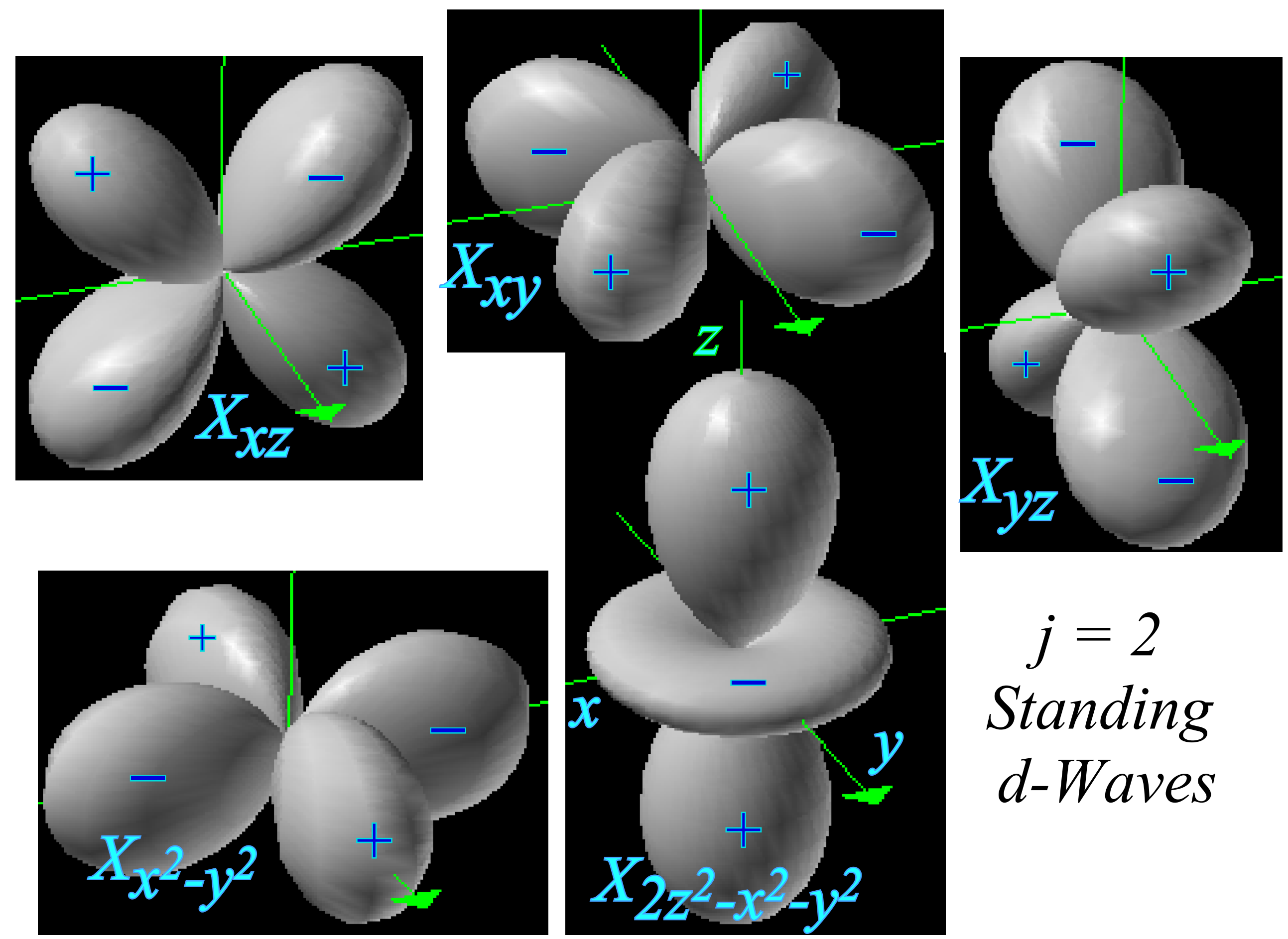
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta) = D^{2*}_{1,0}(\phi\theta 0) = -\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta) = D^{2*}_{0,0}(\phi\theta 0) = \frac{3\cos^2\theta-1}{2} = \frac{3z^2-r^2}{2r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-1}(\phi\theta) = D^{2*}_{-1,0}(\phi\theta 0) = \sqrt{\frac{3}{2}}e^{-i\phi}\sin\theta\cos\theta = \sqrt{\frac{3}{2}}\frac{(x-iy)z}{r^2}$$

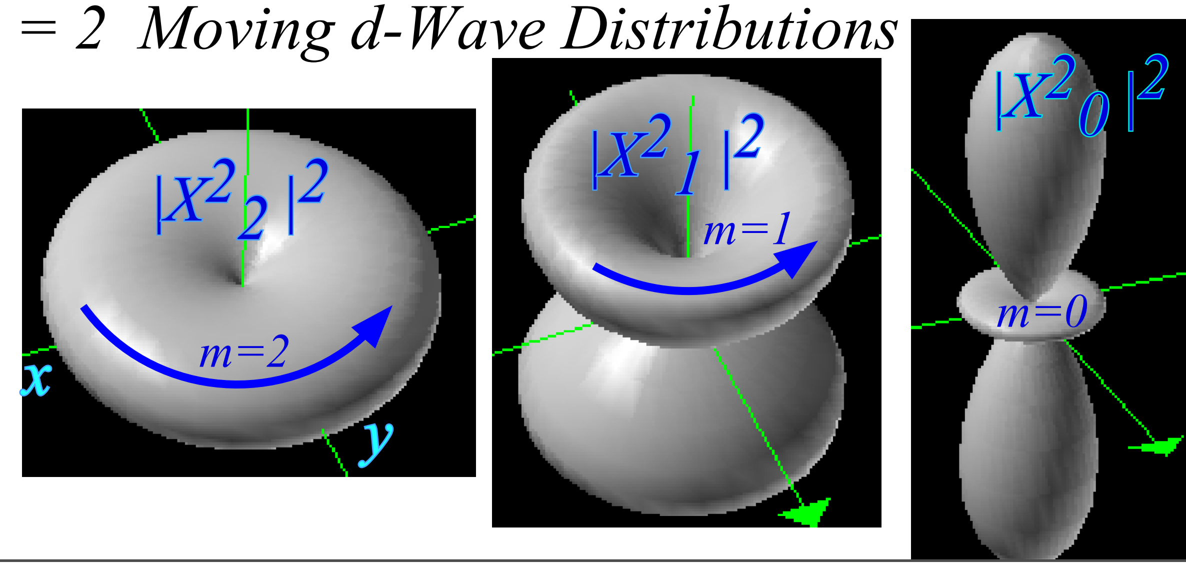
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta) = D^{2*}_{-2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$

$$X^k_q = r^k D^{k*}_{q,0} = \sqrt{\frac{4\pi}{2k+1}}\,r^k Y^k_q$$



j = 2
Standing
d-Waves

j = 2 Moving d-Wave Distributions



Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
➡ *Molecular and nuclear wavefunctions* ⬅
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$for\ R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1$$

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-$j=0, 1, 2,..$ the *R(3)* integral over polar angle β ranges from 0 to π.

$$for\ R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

For integral-$j=1/2, 3/2,..$ the *U(2)* integral over polar angle β ranges from -π to π.

$$for\ SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

For integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$for\ R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

For integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$for\ SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relations* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^\dagger(\alpha\beta\gamma)|000\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

For integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relation* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^{\dagger}(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles $(\alpha\beta\gamma)$.

For integral-$j=0, 1, 2,..$ the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

For integral-$j=1/2, 3/2,..$ the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states $|000\rangle$.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relation* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^{\dagger}(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Left hand (lab-m) and right hand (body-n) quantum numbers apply.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{m'=-j}^{j} D^{j}_{m',m}(\alpha\beta\gamma)\left|{}^{j}_{m',n}\right\rangle \qquad \bar{\mathbf{R}}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{n'=-j}^{j} D^{j*}_{n',n}(\alpha\beta\gamma)\left|{}^{j}_{m,n'}\right\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\textit{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

For integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\textit{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relation* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^{\dagger}(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Left hand (lab-*m*) and right hand (body-*n*) quantum numbers apply.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{m'=-j}^{j} D^{j}_{m',m}(\alpha\beta\gamma)\left|{}^{j}_{m',n}\right\rangle \qquad \bar{\mathbf{R}}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{n'=-j}^{j} D^{j*}_{n',n}(\alpha\beta\gamma)\left|{}^{j}_{m,n'}\right\rangle$$

Same applies to the generators $\mathbf{s}_Z$ or $\mathbf{J}_Z$ of *SU(2)* or *R(3)*.

$$\mathbf{s}_Z\left|{}^{j}_{m,n}\right\rangle = m\left|{}^{j}_{m,n}\right\rangle \qquad \bar{\mathbf{s}}_Z\left|{}^{j}_{m,n}\right\rangle = -n\left|{}^{j}_{m,n}\right\rangle$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
➡ *Molecular and nuclear eigenlevels* ⬅
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

$$\mathbf{s}_{\bar{Z}} \left| \begin{smallmatrix} j \\ m,n \end{smallmatrix} \right\rangle = +n \left| \begin{smallmatrix} j \\ m,n \end{smallmatrix} \right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

$$\mathbf{s}_{\bar{Z}}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle = +n\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + A\mathbf{J}_{\bar{Z}}^2$$ (Molecular spin is labeled **J** instead of **s**)

The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

$$\mathbf{s}_{\bar{Z}}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle = +n\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + A\mathbf{J}_{\bar{Z}}^2$$

(Molecular spin is labeled $\mathbf{J}$ instead of $\mathbf{s}$)

Constants are inverse moments of inertia. $\frac{1}{2I_{\bar{X}}} = B = \frac{1}{2I_{\bar{Y}}}$, $A = \frac{1}{2I_{\bar{Z}}}$

The reversed sign is a nuisance, so let us define *reversed momentum operators* that give a positive sign.

$$\mathbf{s}_{\bar{Z}}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle = +n\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + A\mathbf{J}_{\bar{Z}}^2$$ (Molecular spin is labeled $\mathbf{J}$ instead of $\mathbf{s}$)

Constants are inverse moments of inertia. $\frac{1}{2I_{\bar{X}}} = B = \frac{1}{2I_{\bar{Y}}}$, $A = \frac{1}{2I_{\bar{Z}}}$

Hamiltonian can be rewritten in terms of two commuting observables, the $\mathbf{J}_{Z}$ and $\mathbf{J}^2$ operators.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + B\mathbf{J}_{\bar{Z}}^2 + (A-B)\mathbf{J}_{\bar{Z}}^2 = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2$$

The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

$$\mathbf{s}_{\bar{Z}}\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle = +n\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}^2_{\bar{X}} + B\mathbf{J}^2_{\bar{Y}} + A\mathbf{J}^2_{\bar{Z}}$$

(Molecular spin is labeled $\mathbf{J}$ instead of $\mathbf{s}$)

Constants are inverse moments of inertia.

$$\frac{1}{2I_{\bar{X}}} = B = \frac{1}{2I_{\bar{Y}}}\ , \qquad A = \frac{1}{2I_{\bar{Z}}}$$

Hamiltonian can be rewritten in terms of two commuting observables, the $\mathbf{J}_Z$ and $\mathbf{J}^2$ operators.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}^2_{\bar{X}} + B\mathbf{J}^2_{\bar{Y}} + B\mathbf{J}^2_{\bar{Z}} + (A-B)\mathbf{J}^2_{\bar{Z}} = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}^2_{\bar{Z}}$$

Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}^2_{\bar{Z}}\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

Eigenvalue and energy level spectrum is shown next.

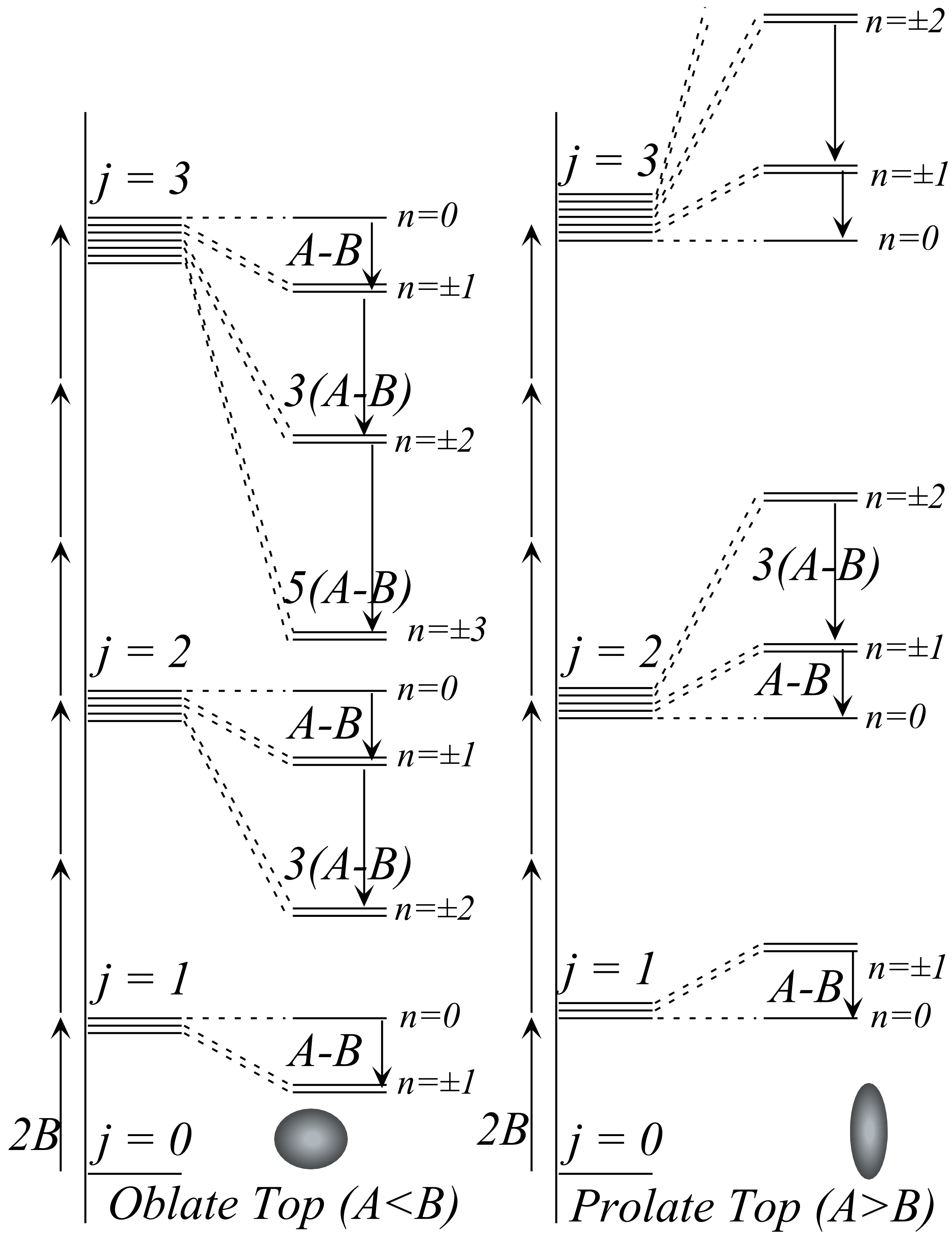
$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + B\mathbf{J}_{\bar{Z}}^2 + (A-B)\mathbf{J}_{\bar{Z}}^2 = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2$$

Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$



QTforCA Unit 8. Ch. 23 Fig. 23.1.3

Applications of R(3) rotation and U(2) representations *Molecular and nuclear eigenlevels*

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + B\mathbf{J}_{\bar{Z}}^2 + (A-B)\mathbf{J}_{\bar{Z}}^2 = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2$$

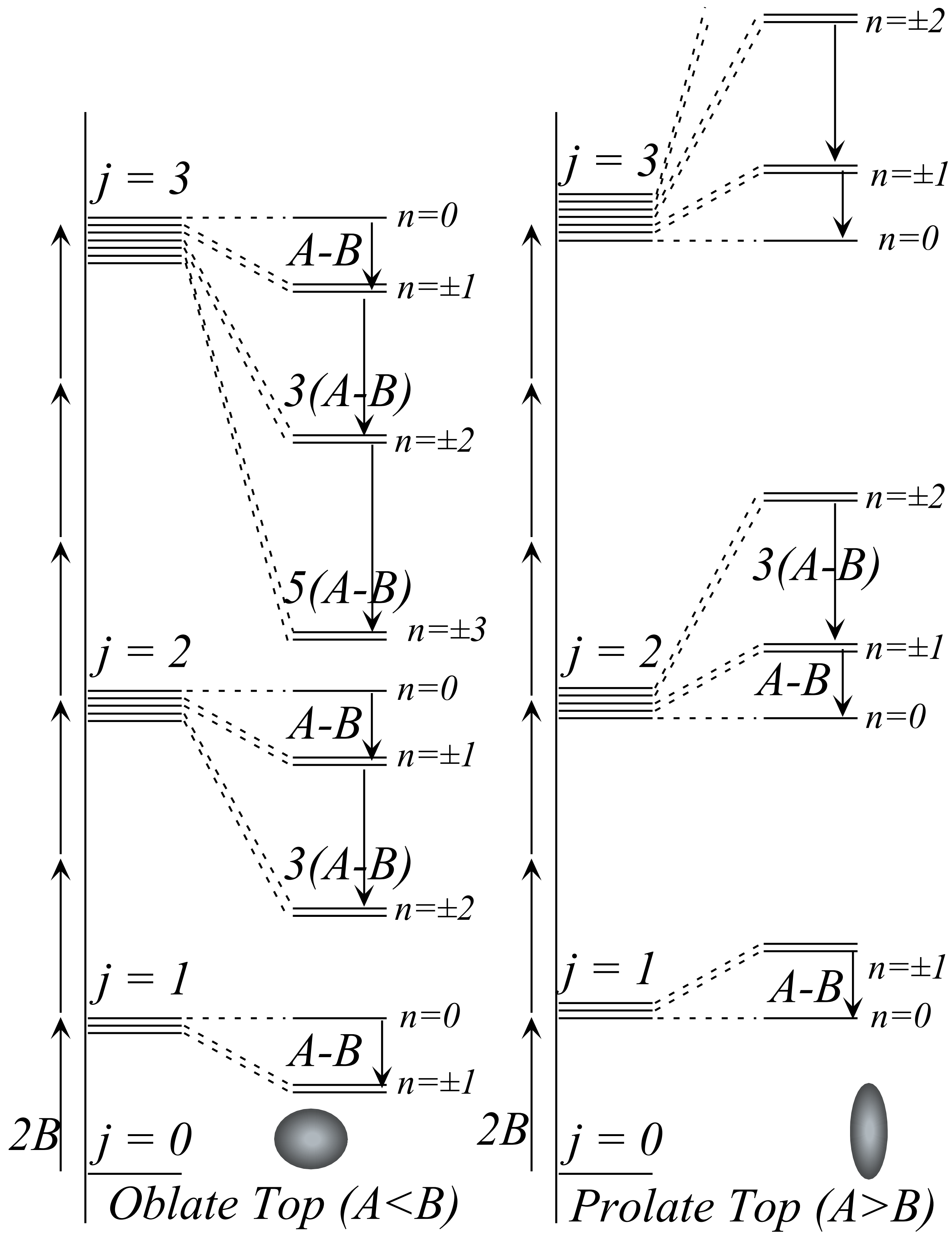
Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

Mock-Mach-Multiplicity
is $(2j+1)^2$ for each j



QTforCA Unit 8. Ch. 23 Fig. 23.1.3

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + B\mathbf{J}_{\bar{Z}}^2 + (A-B)\mathbf{J}_{\bar{Z}}^2 = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2$$

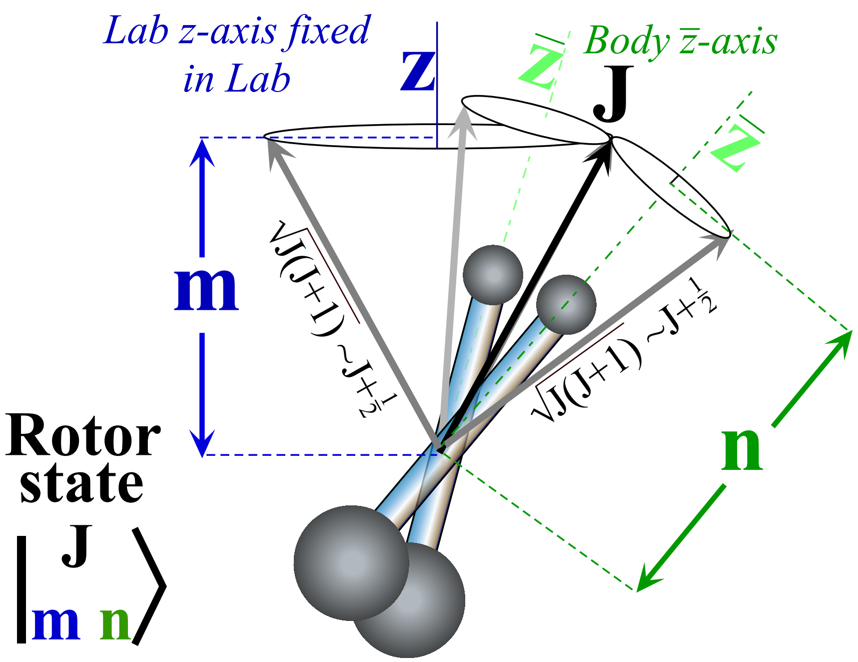
Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

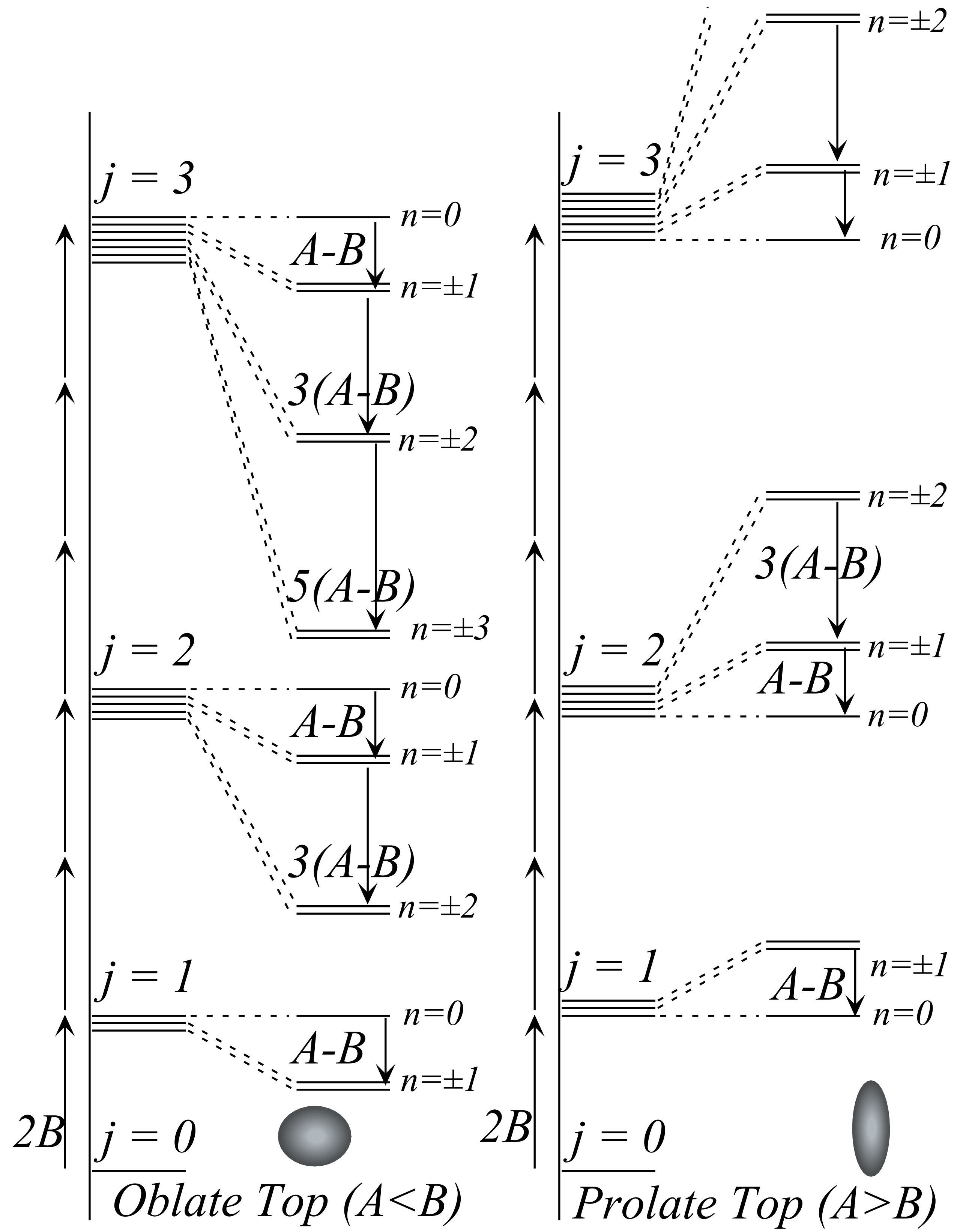
$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

Mock-Mach-Multiplicity is $(2j+1)^2$ for each j



QTforCA Unit 8. Ch. 23 Fig. 23.2.4



QTforCA Unit 8. Ch. 23 Fig. 23.1.3

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + B\mathbf{J}_{\bar{Z}}^2 + (A-B)\mathbf{J}_{\bar{Z}}^2 = B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2$$

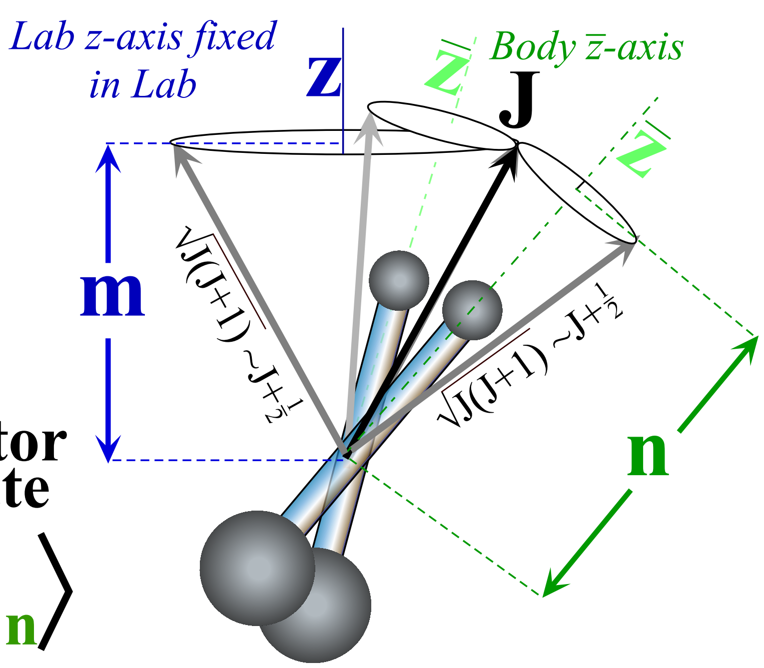
Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

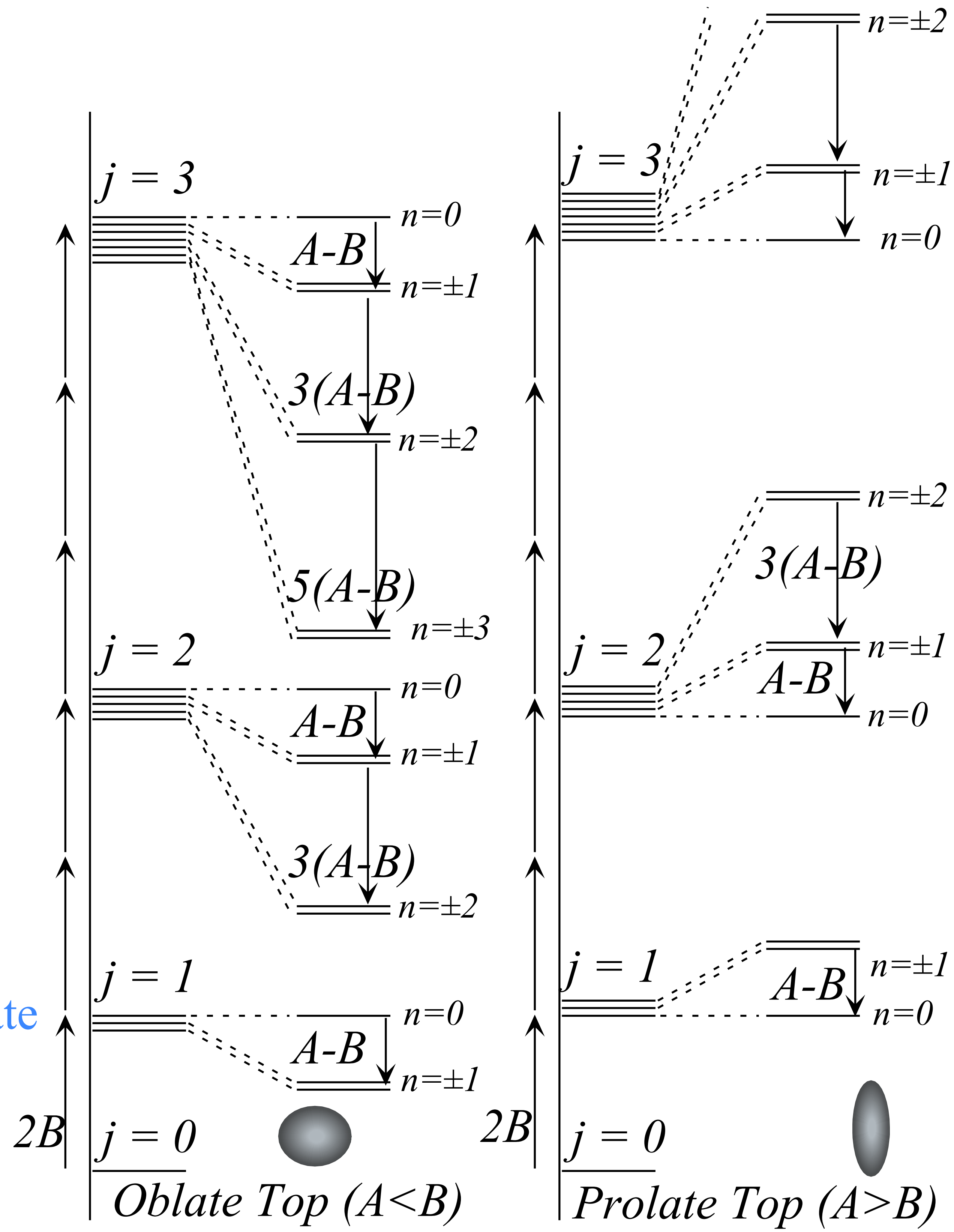
$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{smallmatrix} j \\ m,n \end{smallmatrix}\right\rangle$$

Mock-Mach-Multiplicity
is $(2j+1)^2$ for each j



Even $n=0$ levels are $2j+1$-fold degenerate
If n is non-zero the degeneracy is $4j+2$.



QTforCA Unit 8. Ch. 23 Fig. 23.2.4

QTforCA Unit 8. Ch. 23 Fig. 23.1.3

Applications of R(3) rotation and U(2) representations

Molecular and nuclear eigenlevels

Introducing Racah tensor notation

Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}^2_{\bar{Z}}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

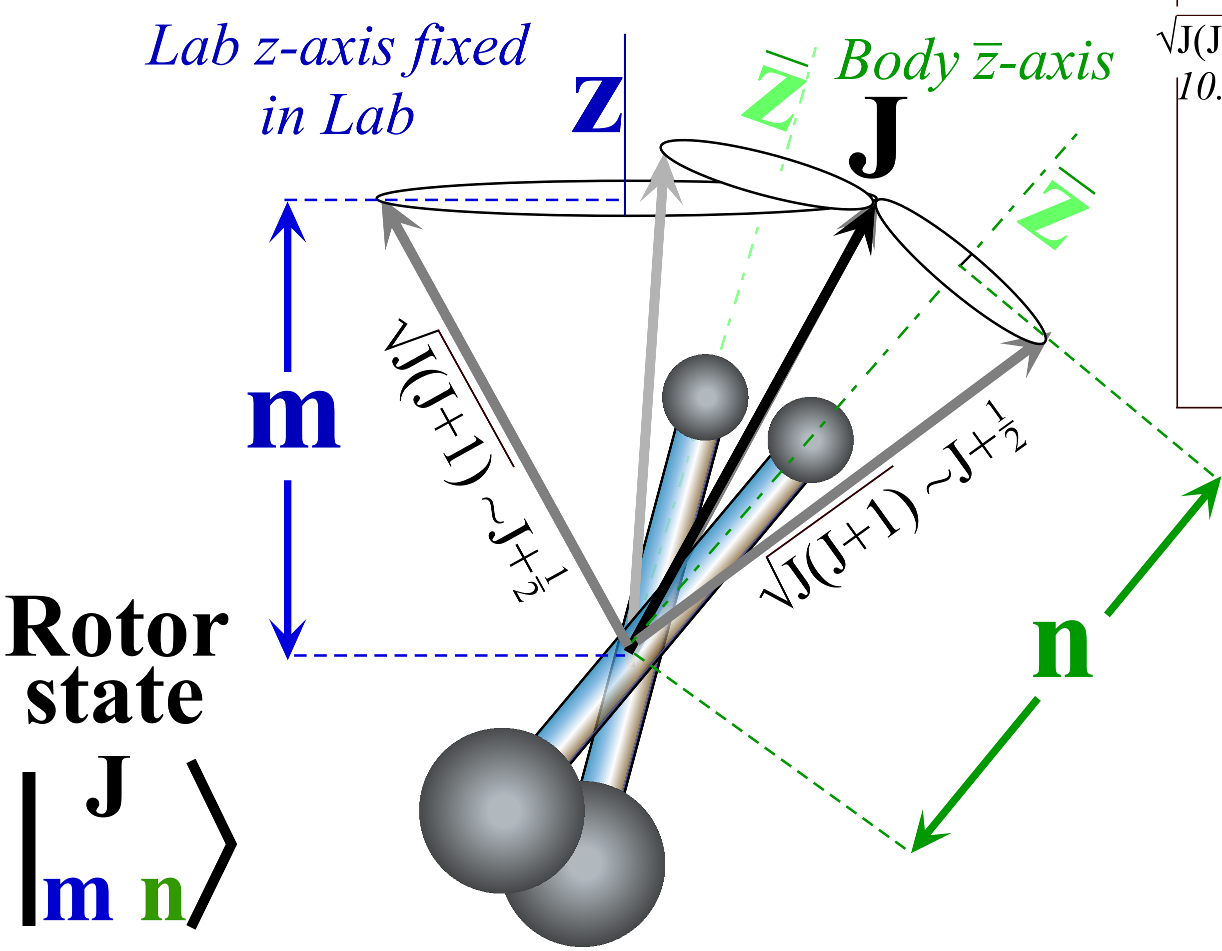
$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

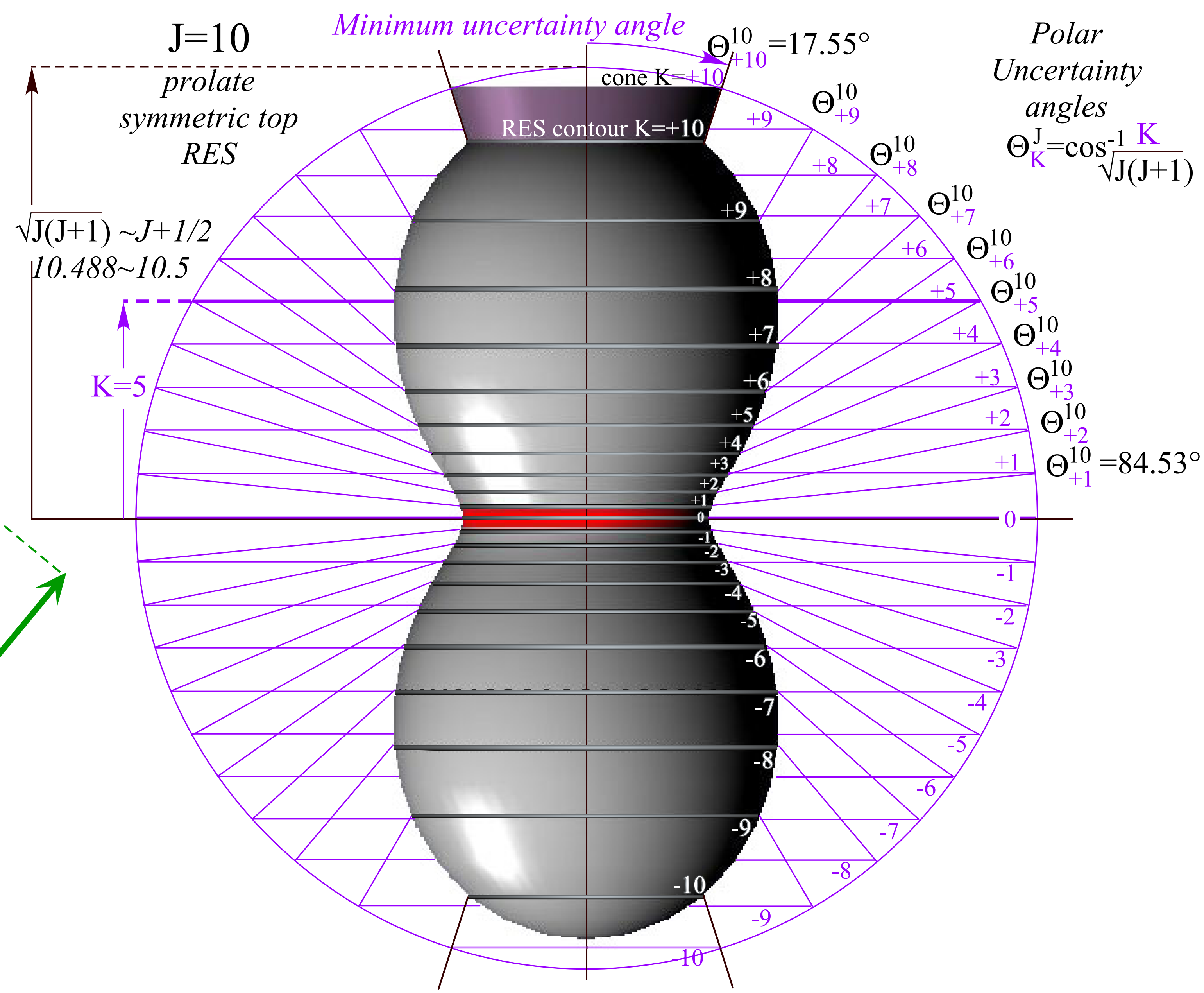
$$T^0_0 = \mathrm{J}\cdot\mathrm{J} = \langle J\rangle^2 \quad = \quad \left(J_x^2 + J_y^2 + J_y^2\right),$$

$$T^2_0 = \tfrac{1}{2}\langle J\rangle^2\left(3\cos^2\beta - 1\right) = \tfrac{1}{2}\left(2J_z^2 - J_x^2 - J_y^2\right),$$

$$H = B\ T^0_0 + \frac{2}{3}(A-B)\ T^2_0$$

Mock-Mach-Multiplicity
is *(2j+1)*^{2} for each *j*





QTforCA Unit 8. Ch. 23 Fig. 23.2.4

Int.J.Molecular Science 1.4.(2013) Fig.1 p. 730

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

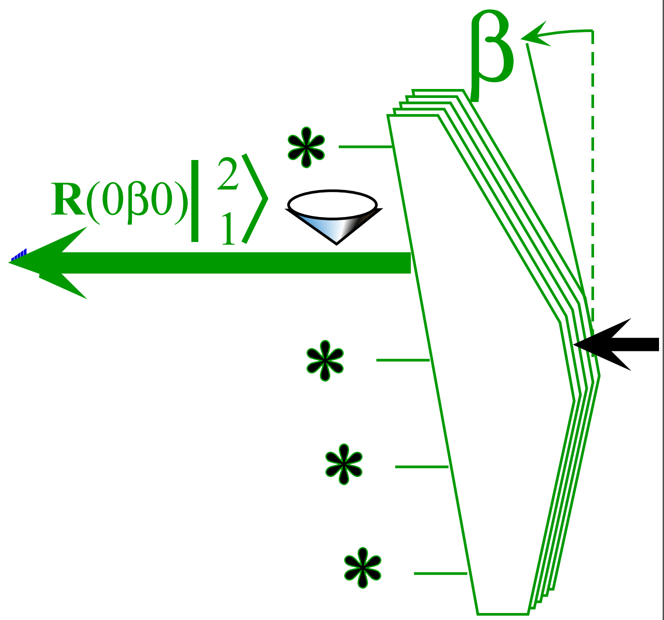
Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
➡ *Generalized Stern-Gerlach and transformation matrices* ⬅
Angular momentum cones and high J properties

Generalized Stern-Gerlach and transformation matrices

Polarization analysis Suppose a spin-j state $\mathbf{R}(0\beta 0)\,|{}^{j=2}_{m=1}\rangle$ exits an analyzer rotated by β

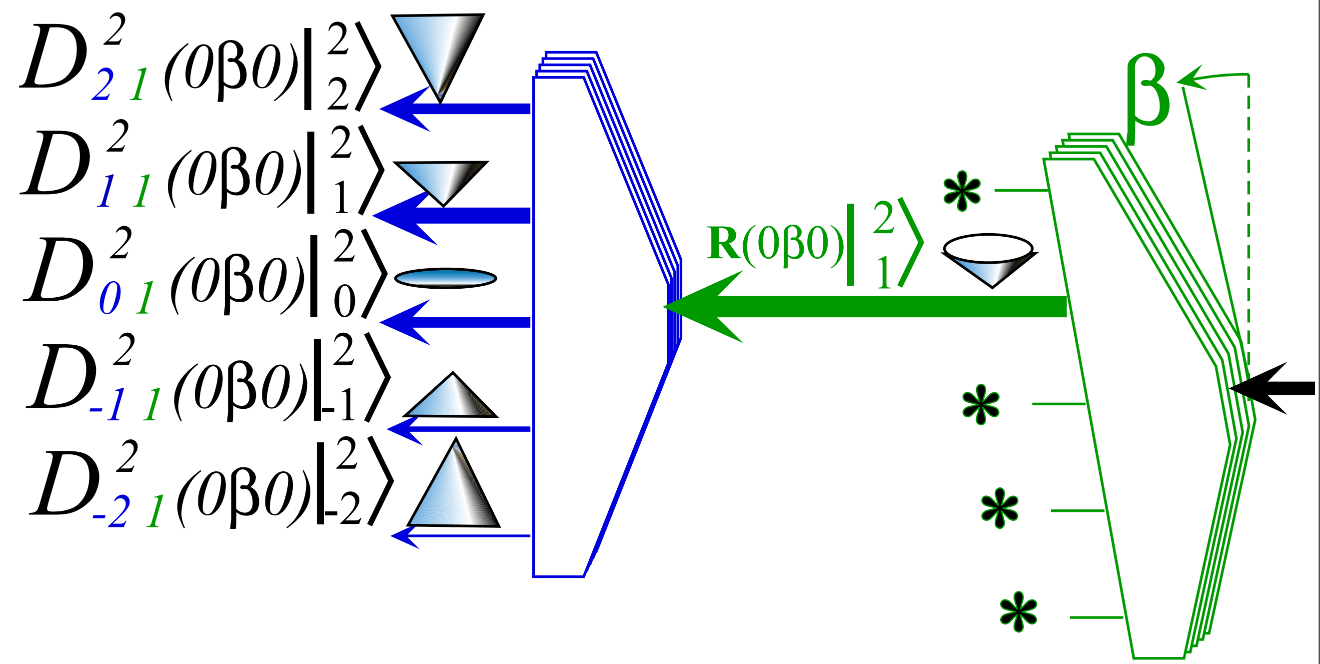


Generalized Stern-Gerlach and transformation matrices

Polarization analysis Suppose a spin-j state $\mathbf{R}(0\beta 0)\,|^{j=2}_{m=1}\rangle$ exits an analyzer rotated by β and then enters a vertical ($\beta=0$) analyzer and forced to choose from unrotated states $|^{j=2}_{m'}\rangle$

$$\mathbf{R}(0\beta 0)\left|{j \atop m}\right\rangle = \sum_{m'=-j}^{j}\left|{j \atop m'}\right\rangle\left\langle{j \atop m'}\right|\mathbf{R}(0\beta 0)\left|{j \atop m}\right\rangle$$

$$= \sum_{m'=-j}^{j}\left|{j \atop m'}\right\rangle D^{j}_{m'm}(0\beta 0)$$



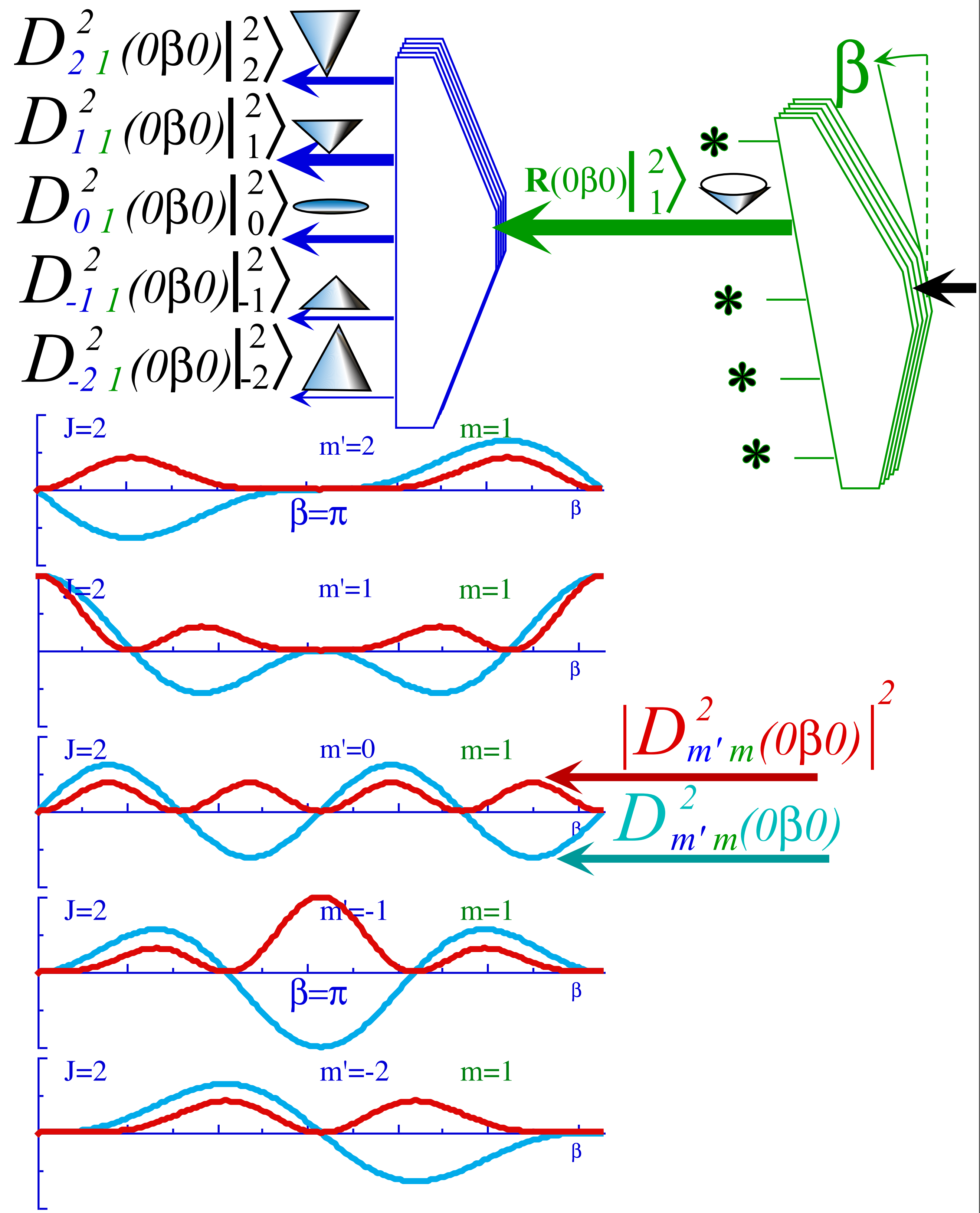
QTforCA Unit 8. Ch. 23 Fig. 23.2.1

Generalized Stern-Gerlach and transformation matrices

Polarization analysis Suppose a spin-j state $\mathbf{R}(0\beta 0)\left|{}^{j=2}_{m=1}\right\rangle$ exits an analyzer rotated by β and then enters a vertical ($\beta=0$) analyzer and forced to choose from unrotated states $\left|{}^{j=2}_{m'}\right\rangle$

$$\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle = \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle\left\langle{}^{j}_{m'}\right|\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle$$

$$= \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle D^{j}_{m'm}(0\beta 0)$$



Overlap of state $\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}_{1}\right\rangle$ with unrotated $\left|{}^{j=2}_{m'}\right\rangle$ is the corresponding D-matrix element.

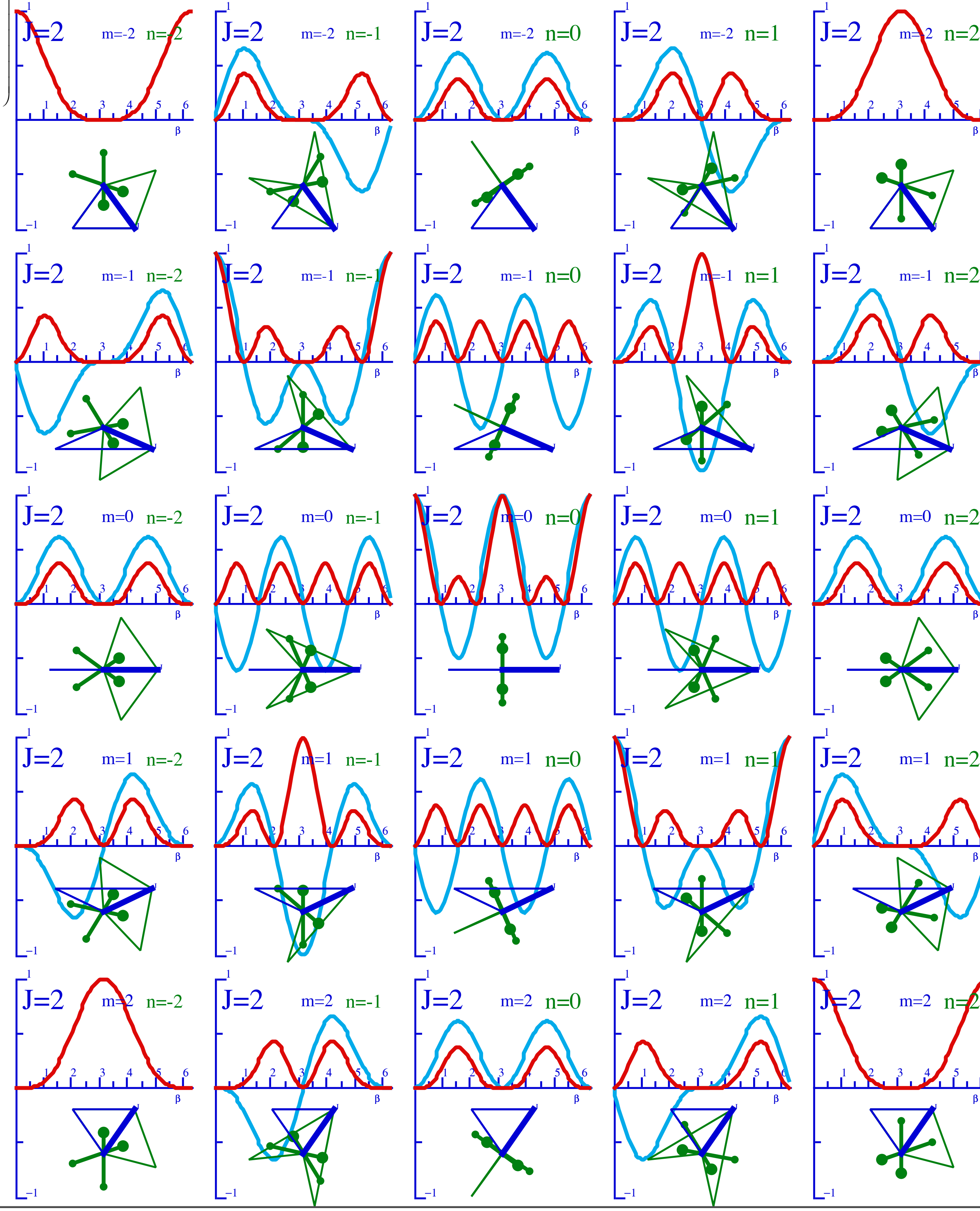
$$\left\langle{}^{j'}_{m'}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}_{1}\right\rangle = \delta^{j'2}D^{2}_{m'1}(\alpha\beta\gamma) = \left\langle{}^{j'}_{m'}\middle|{}^{2}_{1}\right\rangle_{R}$$

QTforCA Unit 8. Ch. 23 Fig. 23.2.1

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

$$\mathbf{R}(0\beta 0)\left|\begin{matrix} j \\ m\end{matrix}\right\rangle = \sum_{m'=-j}^{j}\left|\begin{matrix} j \\ m'\end{matrix}\right\rangle\left\langle\begin{matrix} j \\ m'\end{matrix}\right|\mathbf{R}(0\beta 0)\left|\begin{matrix} j \\ m\end{matrix}\right\rangle$$

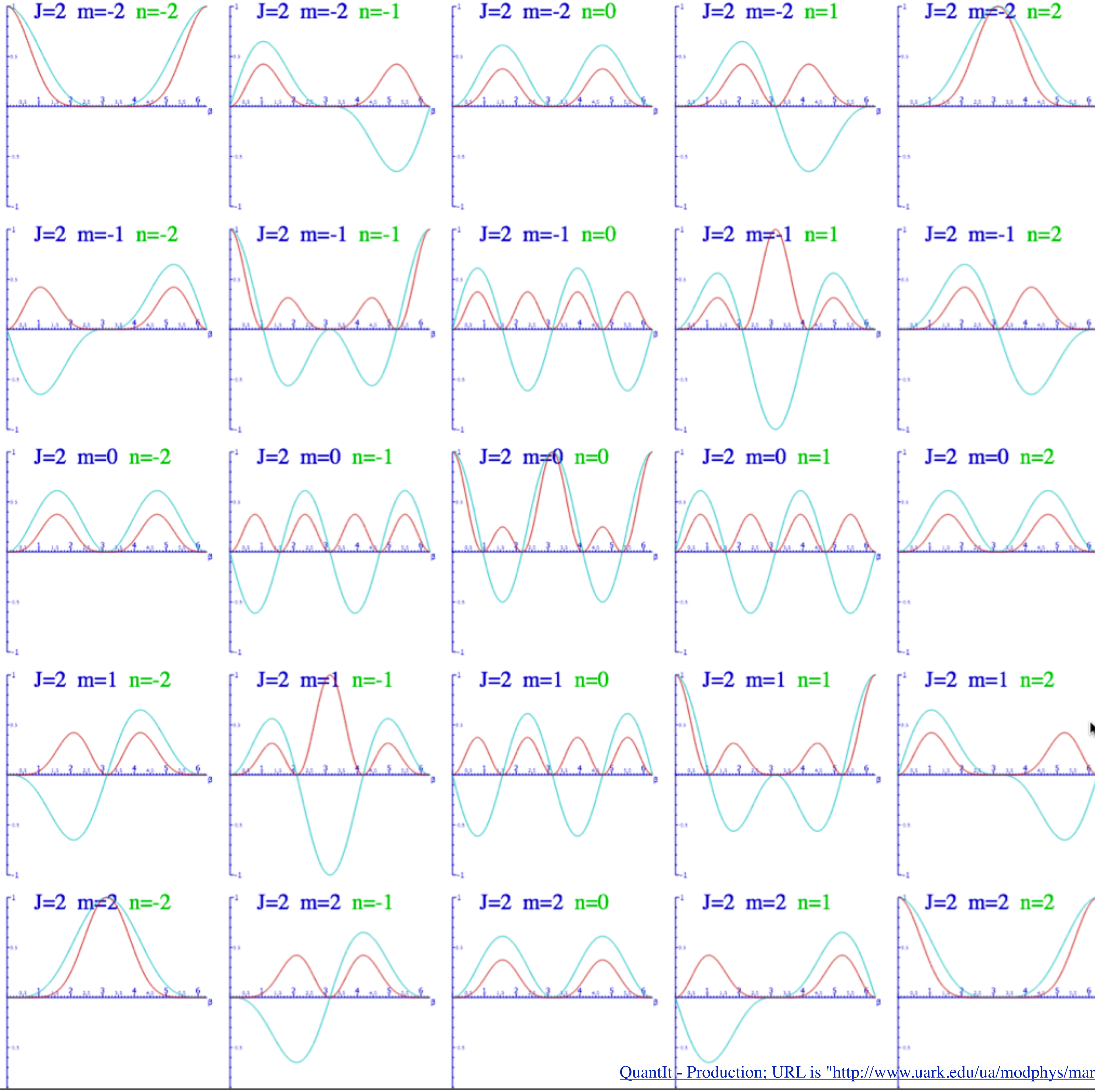
$$= \sum_{m'=-j}^{j}\left|\begin{matrix} j \\ m'\end{matrix}\right\rangle D^j_{m'm}(0\beta 0)$$



Overlap of state $\mathbf{R}(\alpha\beta\gamma)\,|{}^{2}{}_{1}\rangle$ with unrotated $|{}^{j=2}{}_{m'}\rangle$ is the corresponding D-matrix element.

$$\left\langle\begin{matrix} j' \\ m'\end{matrix}\right|\mathbf{R}(\alpha\beta\gamma)\left|\begin{matrix} 2 \\ 1\end{matrix}\right\rangle = \delta^{j'2}D^2_{m'1}(\alpha\beta\gamma) = \left\langle\begin{matrix} j' \\ m'\end{matrix}\middle|\begin{matrix} 2 \\ 1\end{matrix}\right\rangle_R$$

QTforCA Unit 8. Ch. 23 Fig. 23.2.5



QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

Review : *2-D* **a†a** *algebra of U(2) representations*

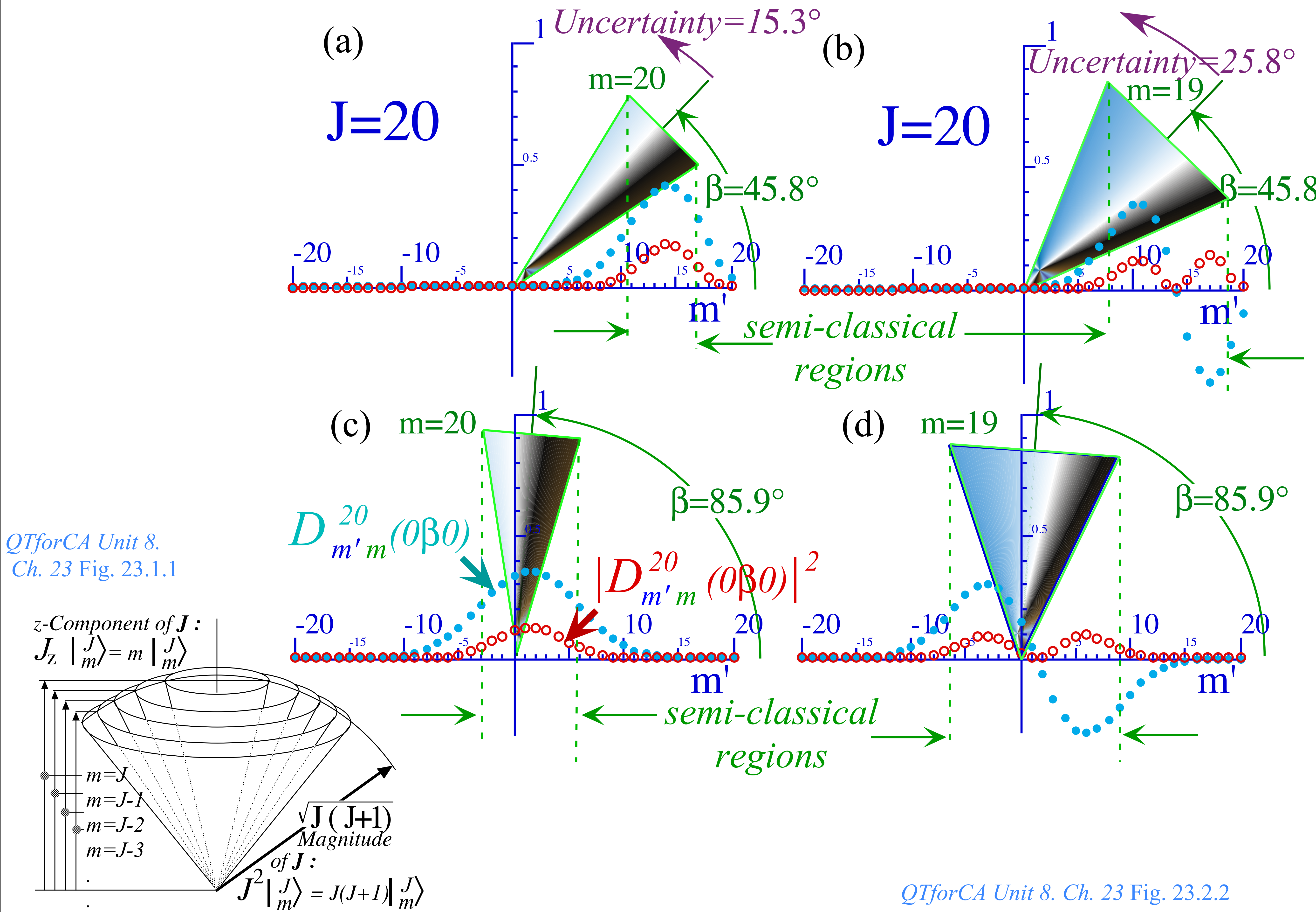
Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

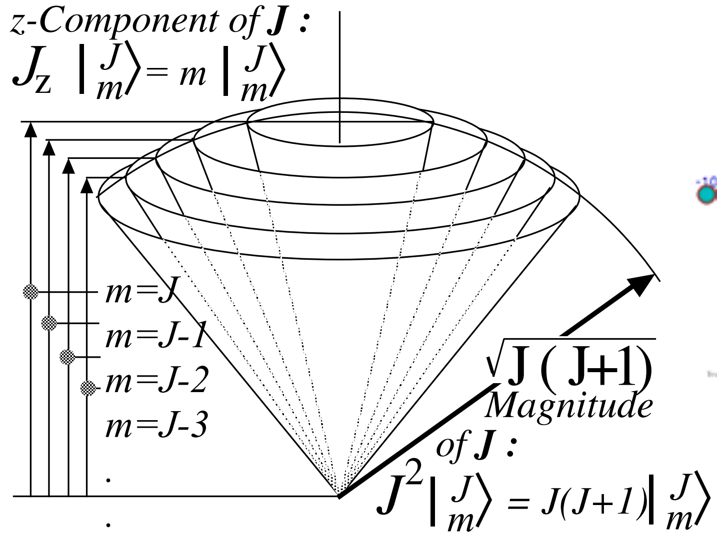
Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

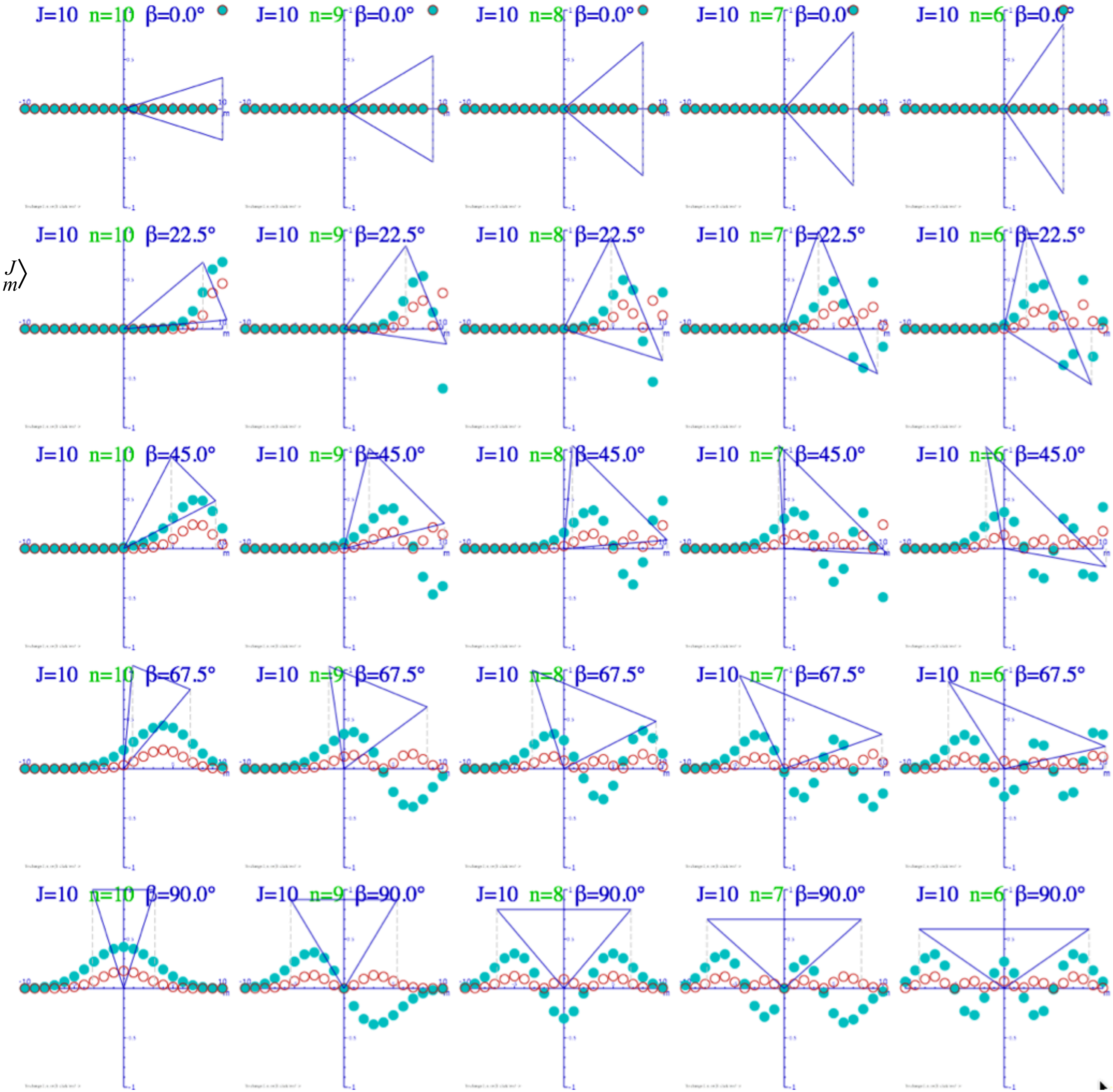
Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
➡ *Angular momentum cones and high J properties* ⬅

Angular momentum cones and high J properties

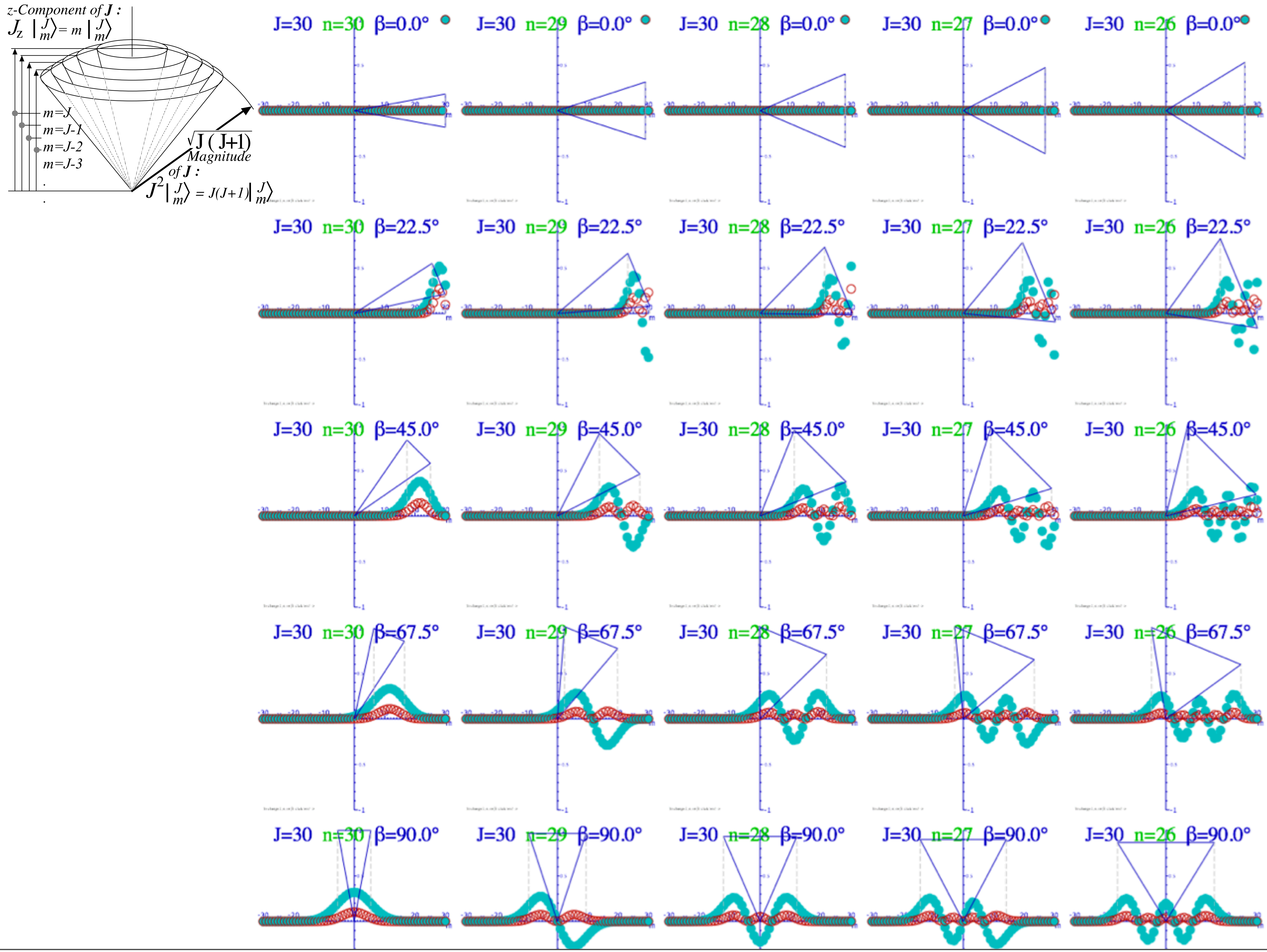




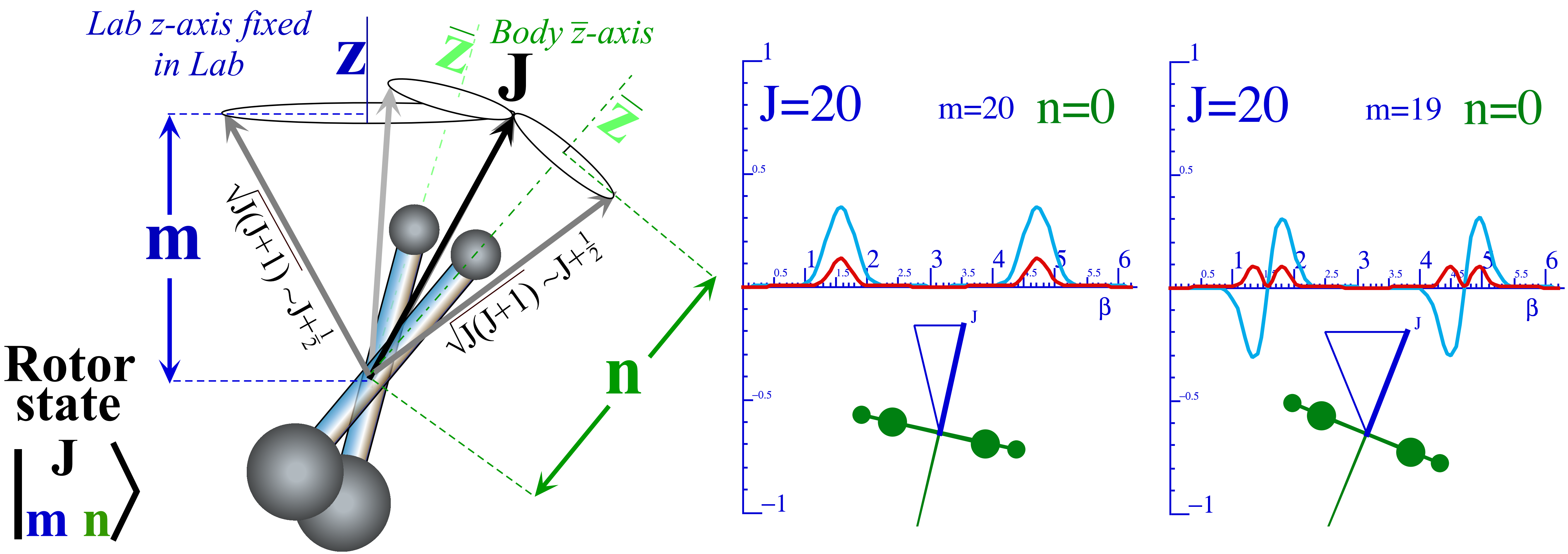
z-Component of J :
$J_z \left|{J \atop m}\right\rangle = m \left|{J \atop m}\right\rangle$
m=J
m=J-1
m=J-2
m=J-3
$\sqrt{J(J+1)}$
Magnitude of J :
$J^2\left|{J \atop m}\right\rangle = J(J+1)\left|{J \atop m}\right\rangle$



J=10 n=10 β=0.0°
J=10 n=9 β=0.0°
J=10 n=8 β=0.0°
J=10 n=7 β=0.0°
J=10 n=6 β=0.0°
J=10 n=10 β=22.5°
J=10 n=9 β=22.5°
J=10 n=8 β=22.5°
J=10 n=7 β=22.5°
J=10 n=6 β=22.5°
J=10 n=10 β=45.0°
J=10 n=9 β=45.0°
J=10 n=8 β=45.0°
J=10 n=7 β=45.0°
J=10 n=6 β=45.0°
J=10 n=10 β=67.5°
J=10 n=9 β=67.5°
J=10 n=8 β=67.5°
J=10 n=7 β=67.5°
J=10 n=6 β=67.5°
J=10 n=10 β=90.0°
J=10 n=9 β=90.0°
J=10 n=8 β=90.0°
J=10 n=7 β=90.0°
J=10 n=6 β=90.0°

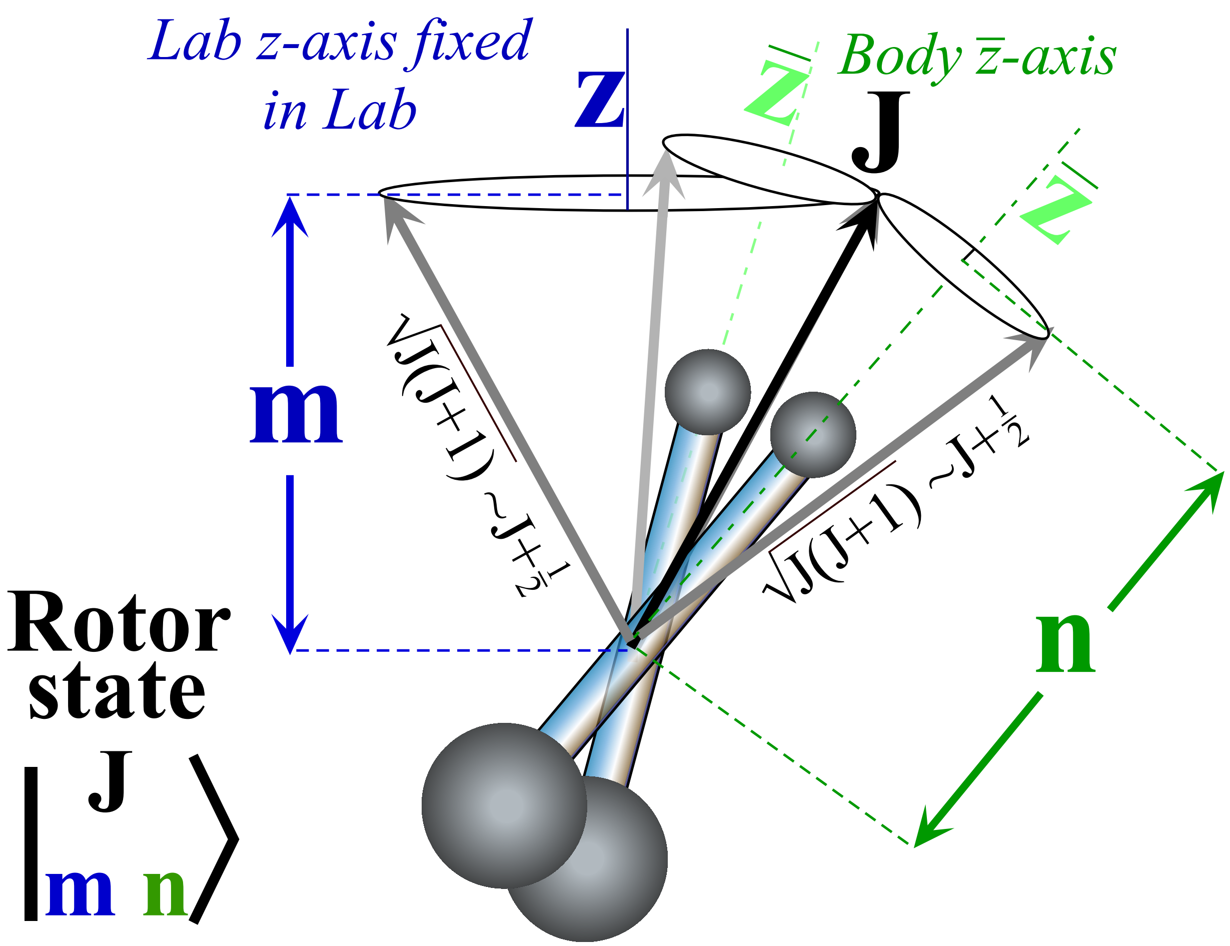


z-Component of J :
$J_z \left|{J \atop m}\right\rangle = m \left|{J \atop m}\right\rangle$
m=J
m=J-1
m=J-2
m=J-3
$\sqrt{J(J+1)}$
Magnitude of J :
$J^2 \left|{J \atop m}\right\rangle = J(J+1) \left|{J \atop m}\right\rangle$
J=30 n=30 β=0.0°
J=30 n=29 β=0.0°
J=30 n=28 β=0.0°
J=30 n=27 β=0.0°
J=30 n=26 β=0.0°
J=30 n=30 β=22.5°
J=30 n=29 β=22.5°
J=30 n=28 β=22.5°
J=30 n=27 β=22.5°
J=30 n=26 β=22.5°
J=30 n=30 β=45.0°
J=30 n=29 β=45.0°
J=30 n=28 β=45.0°
J=30 n=27 β=45.0°
J=30 n=26 β=45.0°
J=30 n=30 β=67.5°
J=30 n=29 β=67.5°
J=30 n=28 β=67.5°
J=30 n=27 β=67.5°
J=30 n=26 β=67.5°
J=30 n=30 β=90.0°
J=30 n=29 β=90.0°
J=30 n=28 β=90.0°
J=30 n=27 β=90.0°
J=30 n=26 β=90.0°

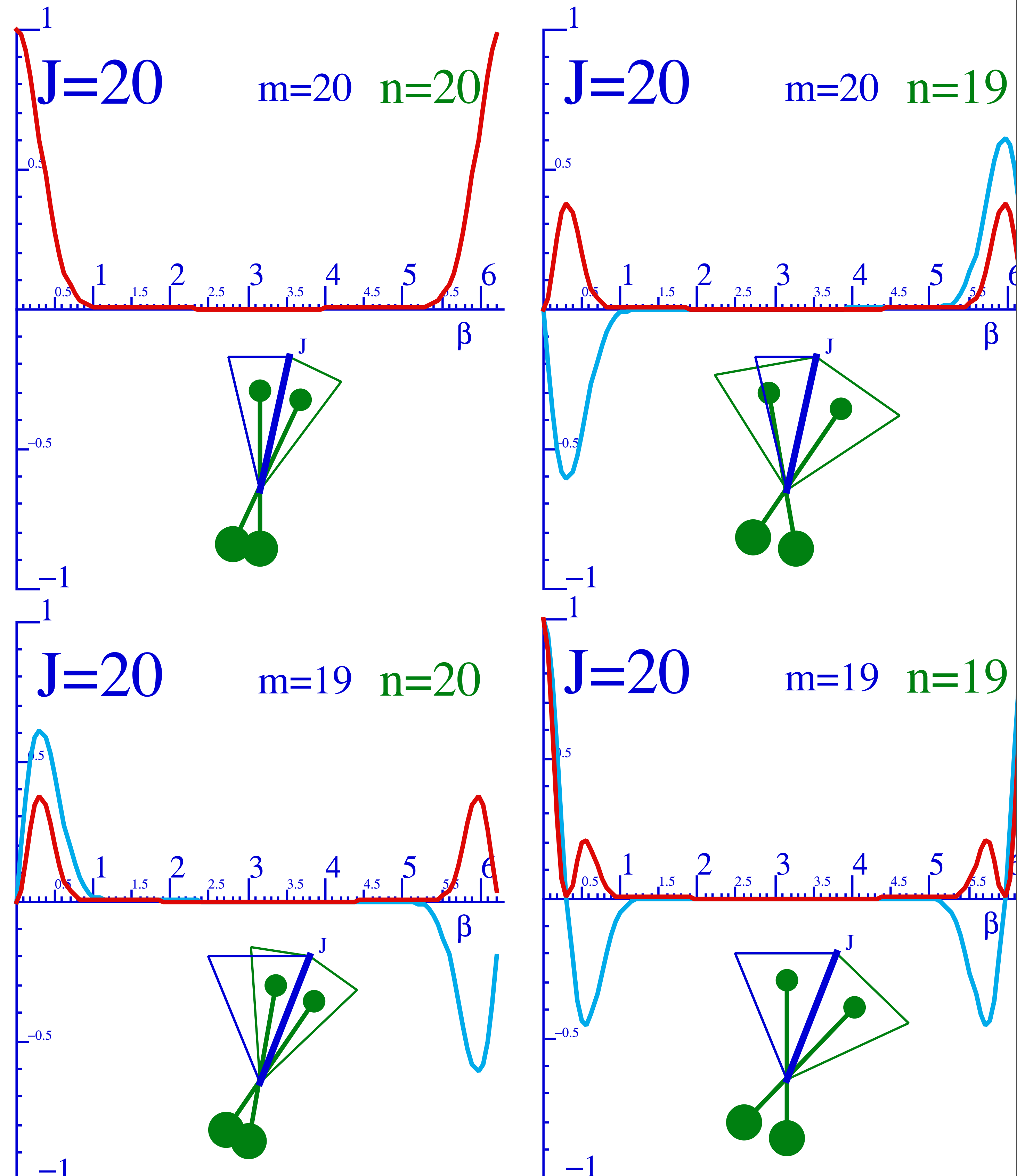


QTforCA Unit 8. Ch. 23 Fig. 23.2.4

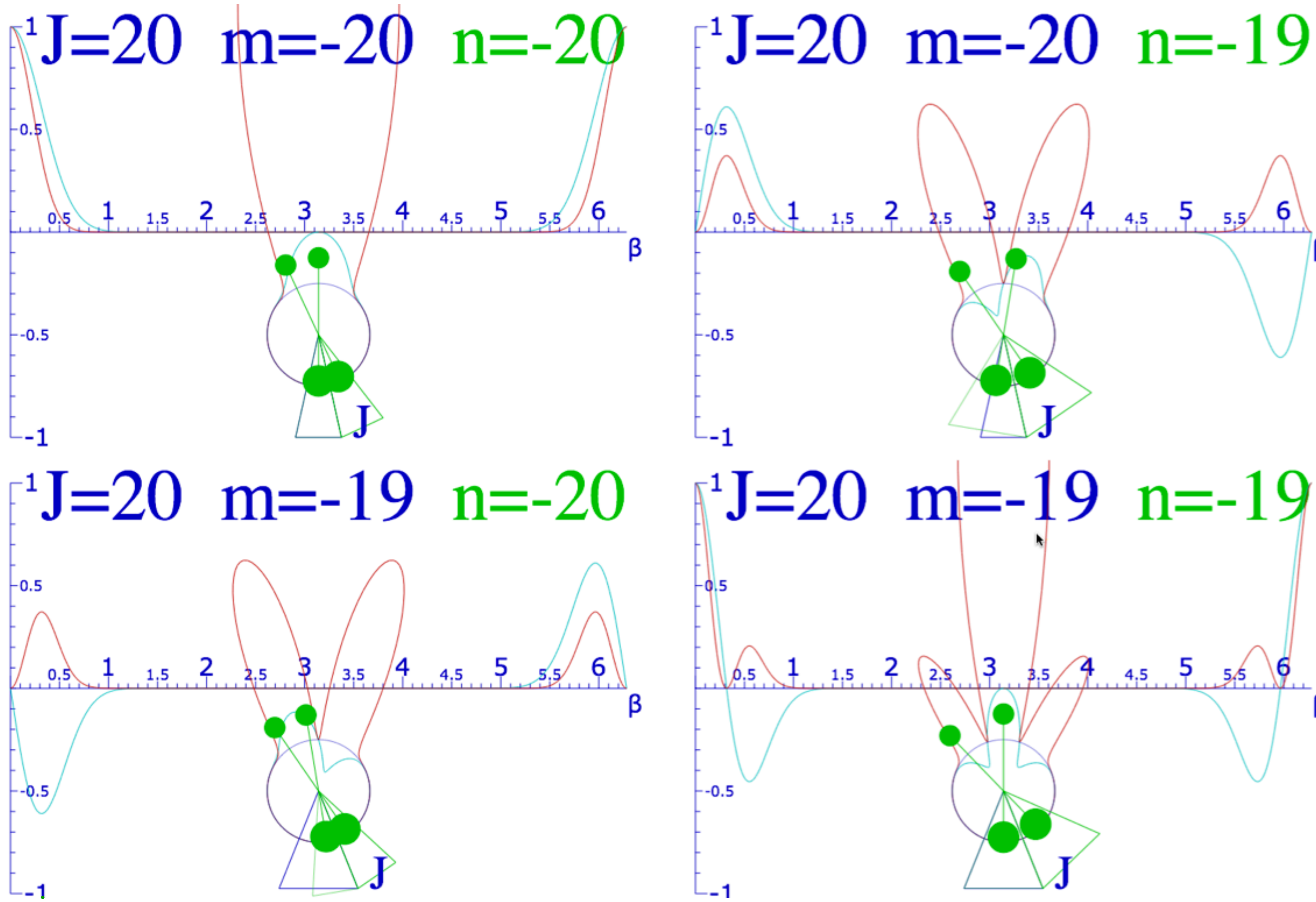
QTforCA Unit 8. Ch. 23 Fig. 23.2.7

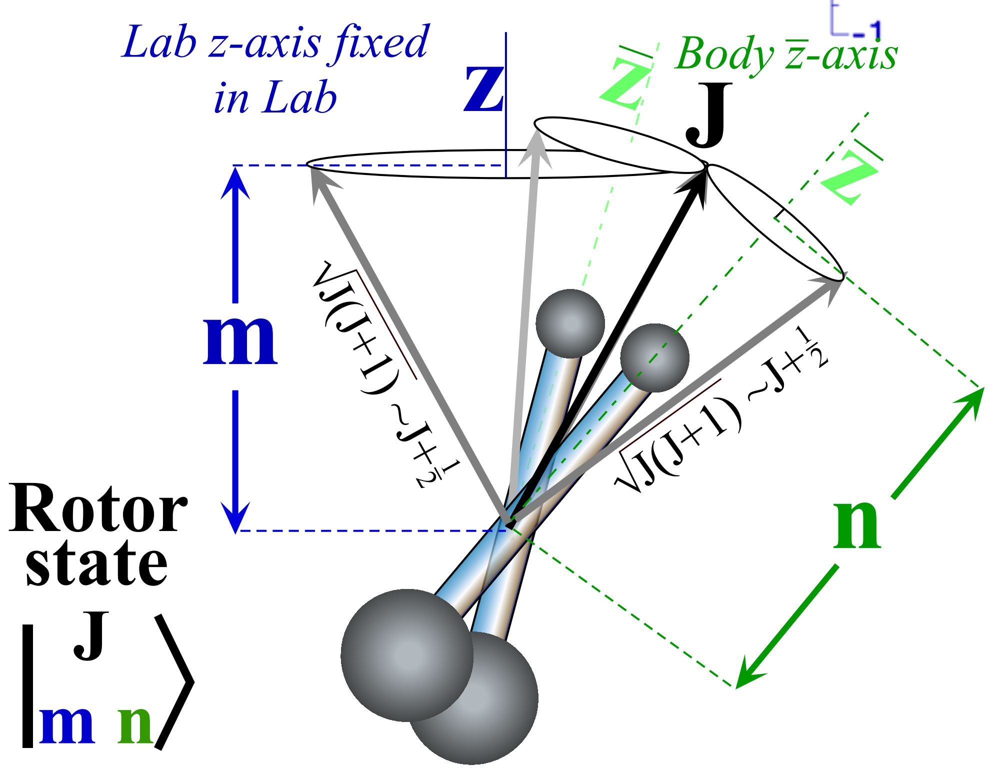


Lab z-axis fixed in Lab
z
Body z̄-axis
z̄
J
m
n
√J(J+1) ~J+½
Rotor state
|J m n⟩



J=20 m=20 n=20
J=20 m=20 n=19
J=20 m=19 n=20
J=20 m=19 n=19
β
J


J=20 m=-20 n=-20
J=20 m=-20 n=-19
J=20 m=-19 n=-20
J=20 m=-19 n=-19
β
J


Lab z-axis fixed in Lab
z
Body $\bar{z}$-axis
J
m
n
$\sqrt{J(J+1)} \sim J+\frac{1}{2}$
$\sqrt{J(J+1)} \sim J+\frac{1}{2}$
Rotor state
$\left|{}^{J}_{m\ n}\right\rangle$

As of March 31, 2014

Links to the current Harter-Soft LearnIt web apps for Physics

Bold links have default redirect pages. ***Italics*** **are not yet meant for production.Red: the final stages of testing.**

List of *production* Harter-Soft Web Apps & Textbooks (For public)

Classical Mechanics with a Bang! - URL is "http://www.uark.edu/ua/modphys/markup/CMwBangWeb.html"
Quantum Theory for the Computer Age - URL is "http://www.uark.edu/ua/modphys/markup/QTCAWeb.html"
LearnIt Web Applications - URL is "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"

Individual web-apps for current classes:

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
MolVibes - Production; URL is "http://www.uark.edu/ua/modphys/markup/MolVibesWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

The old relativity website (2005):
Relativity - Pirelli Entrant - Production; URL is "**http://www.uark.edu/ua/pirelli**" or "http://www.uark.edu/ua/pirelli/html/default.html"

Newer relativity web-apps currently being developed (2013-)
RelativIt Production; URL is "http://www.uark.edu/ua/modphys/markup/RelativItWeb.html"
RelaWavity Production; URL is "http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html"

Additional classical wep-apps:
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"
WaveIt Production; URL is "http://www.uark.edu/ua/modphys/markup/WaveItWeb.html"

Link to master list of all Harter-Soft Web Apps & Textbooks (Prod, Testing, & Developement)

http://www.uark.edu/ua/modphys/testing/markup/Harter-SoftWebApps.html