AMOP Lecture 3.5
Tue 1.28-Thur 1.30.2014

Relativity of lightwaves and Lorentz-Minkowski coordinates IV.

(Ch. 0-3 of Unit 8)

5. That "old-time" relativity (Circa 600BCE- 1905CE)
("Bouncing-photons" in smoke & mirrors and Thales, again)
The Ship and Lighthouse saga
Light-conic-sections make invariants
A politically incorrect analogy of rotational transformation and Lorentz transformation
The straight scoop on "angle" and "rapidity" (They're area!)
Galilean velocity addition becomes rapidity addition
Introducing the "Sin-Tan Rosetta Stone" (Thanks, Thales!)
Introducing the stellar aberration angle σ vs. rapidity ρ
How Minkowski's space-time graphs help visualize relativity
Group vs. phase velocity and tangent contacts

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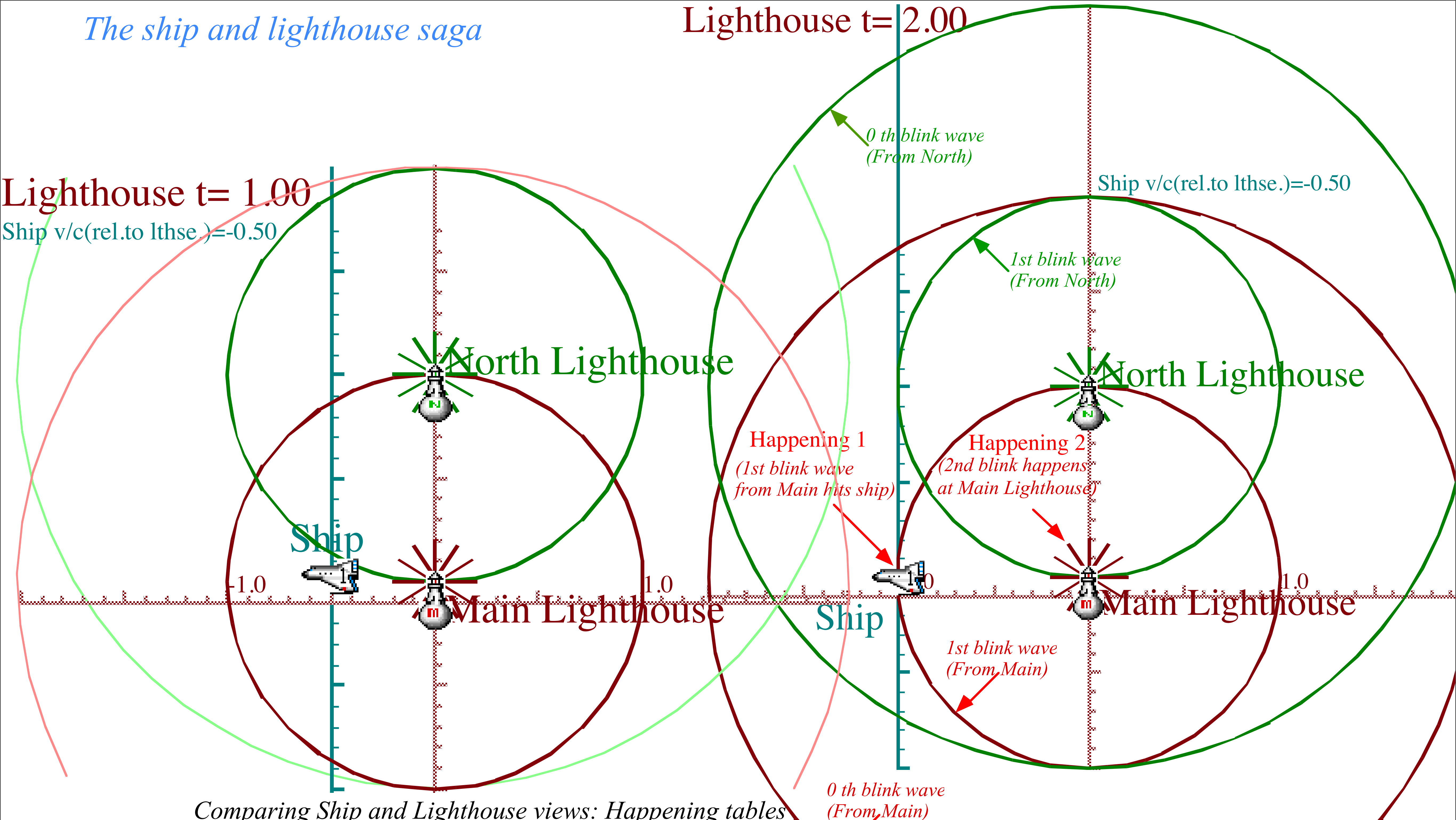


Comparing Ship and Lighthouse views: Happening tables

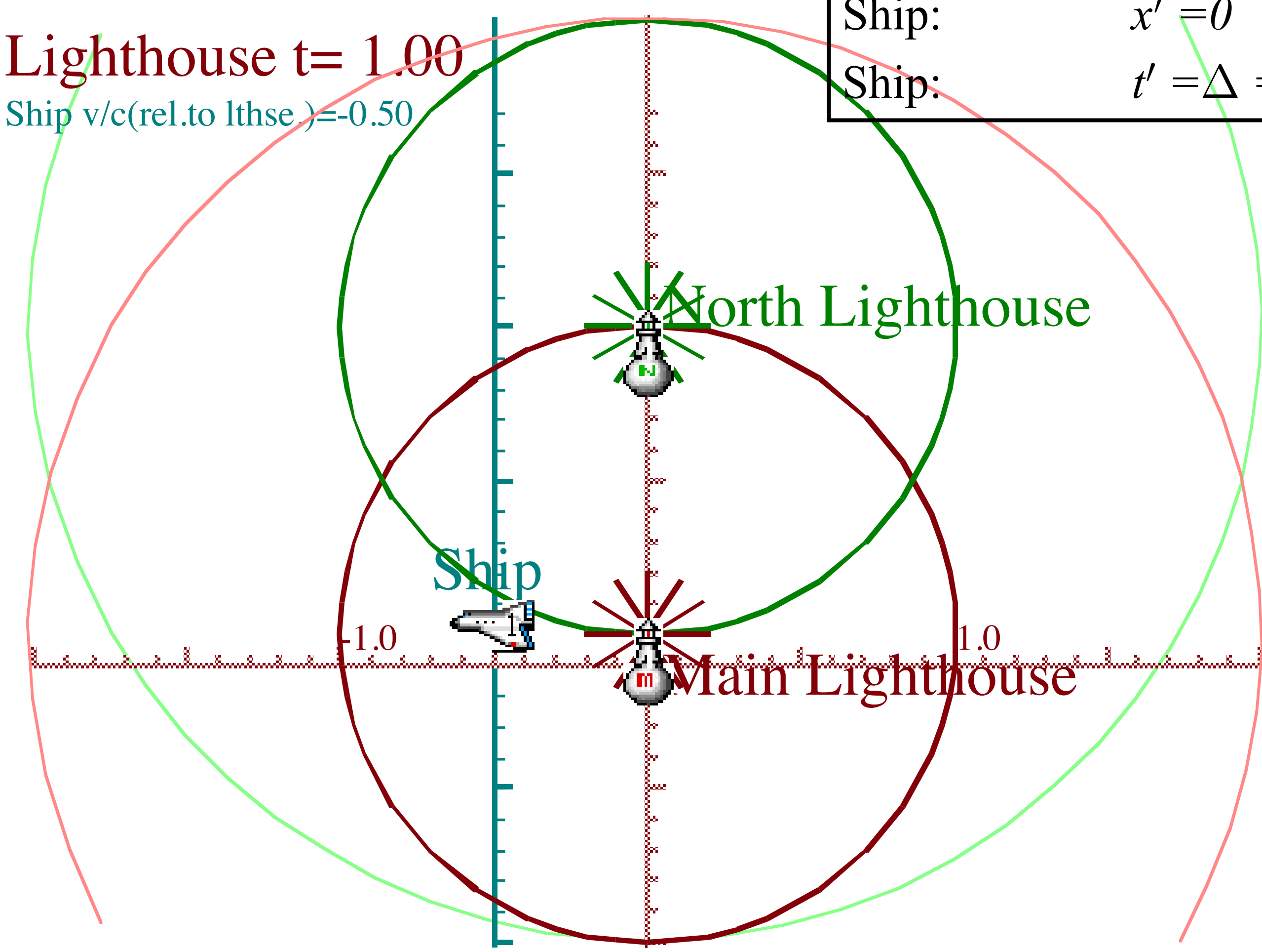
Happening 0: Ship passes Main Lighthouse.	Happening 1: Ship gets hit by first blink from Main Lighthouse.	Happening 2: Main Lighthouse blinks second time.
(Lighthouse space) $x = 0$ (Lighthouse time) $t = 0$	$x = -1.00\ c$ $t = 2.00$	$x = 0$ $t = 2.00$
(Ship space) $x' = 0$ (Ship time) $t' = 0$	$x' = 0$ $t' = 1.75$	$x' = c\,\Delta$ $t' = 2\Delta = 2.30$

Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.

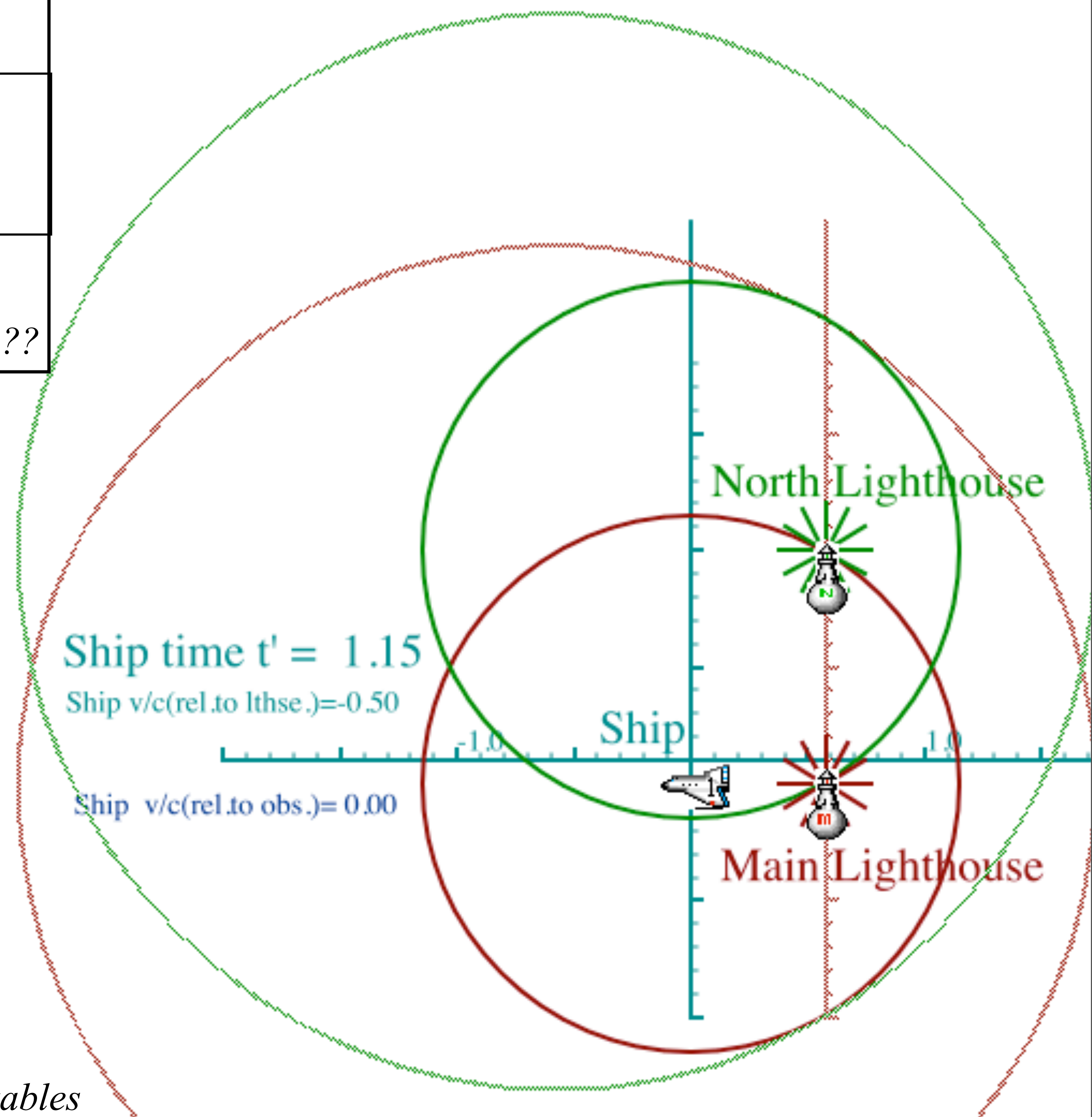
The ship and lighthouse saga

Happening 0.5:
Main Lite
blinks first time.

Lighthouse:	$x = 0$
Lighthouse:	$t = 1.00$
Ship:	$x' = 0$
Ship:	$t' = \Delta = ???$



Ship Time $t' = \Delta = ??$



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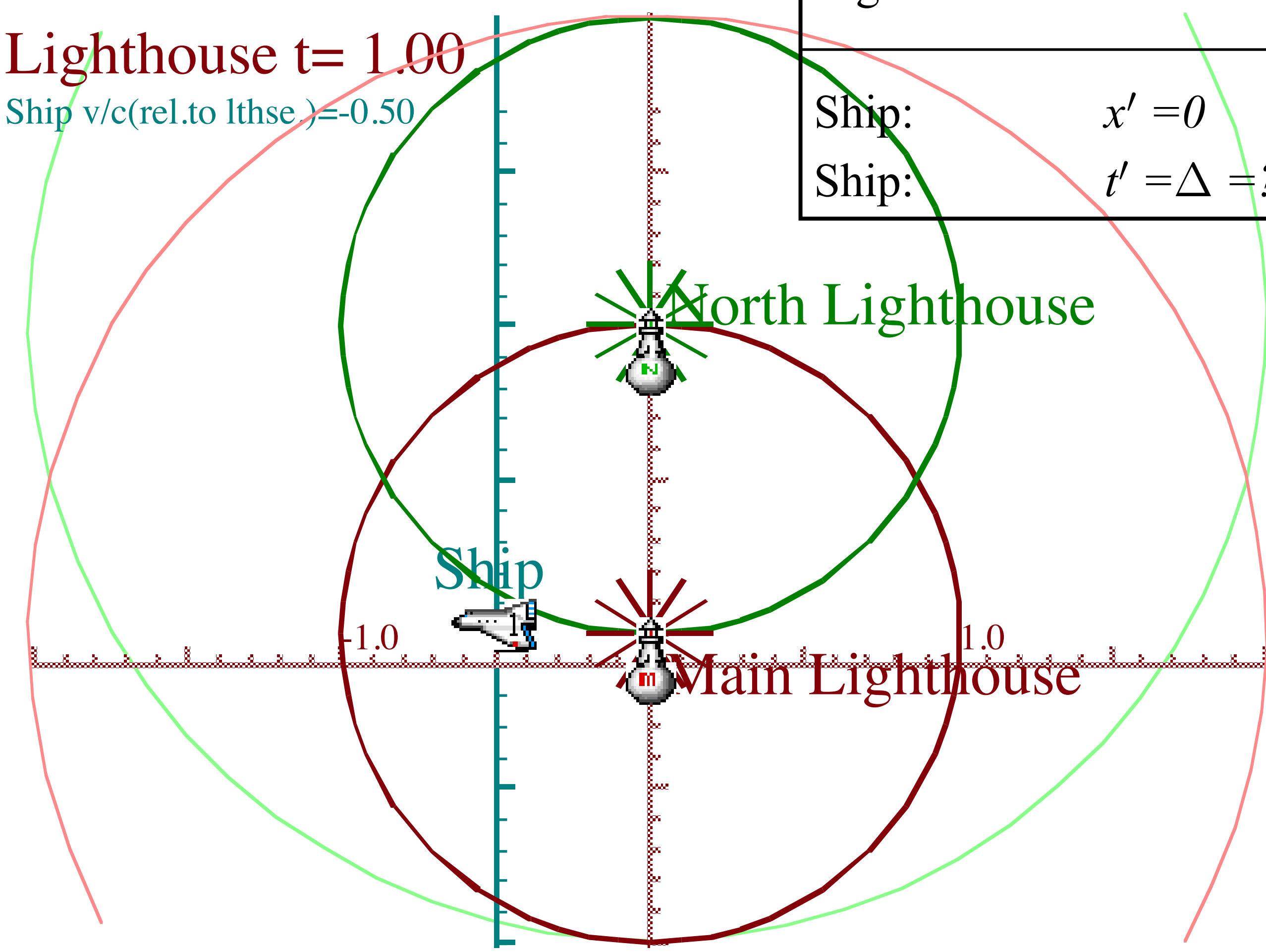

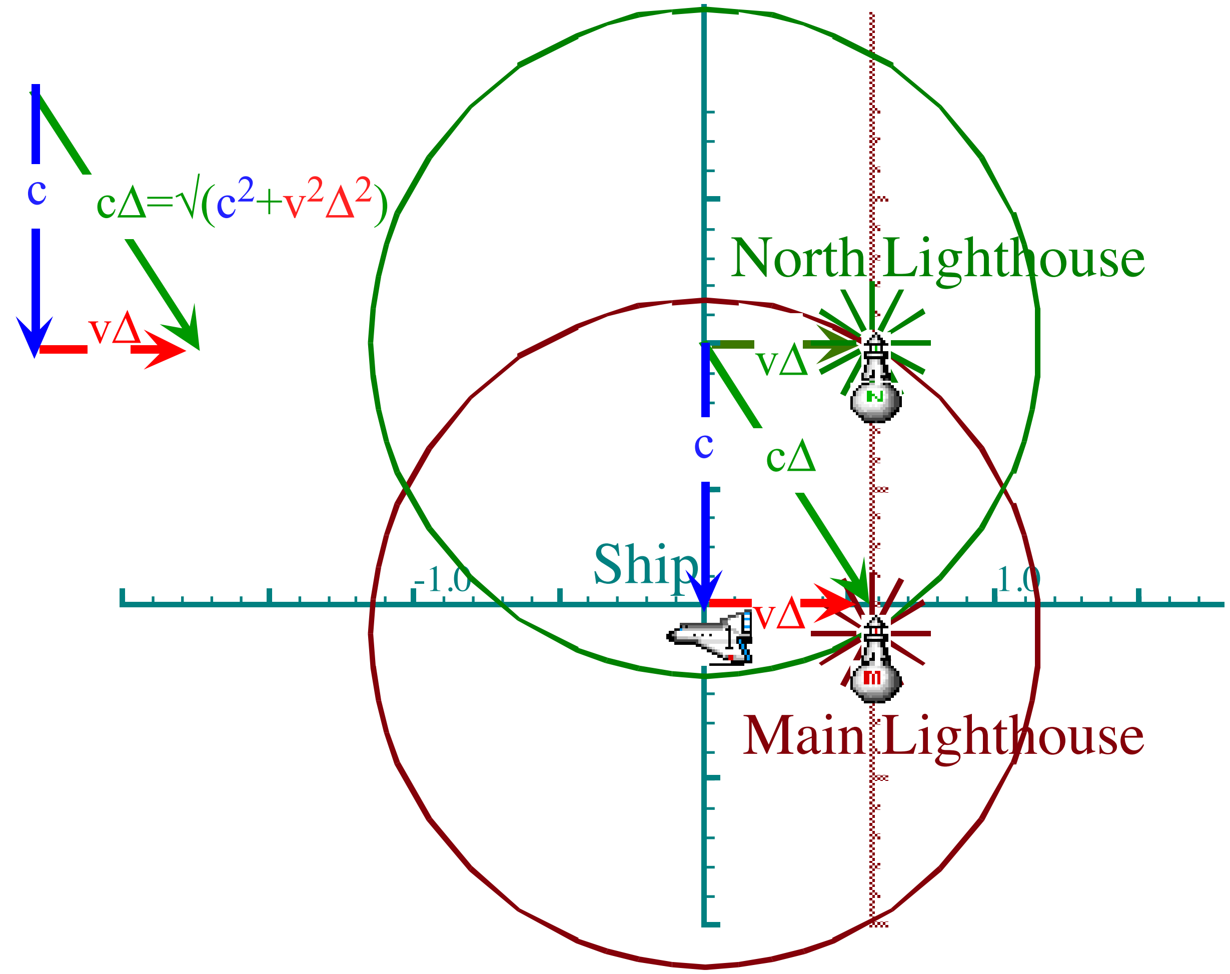

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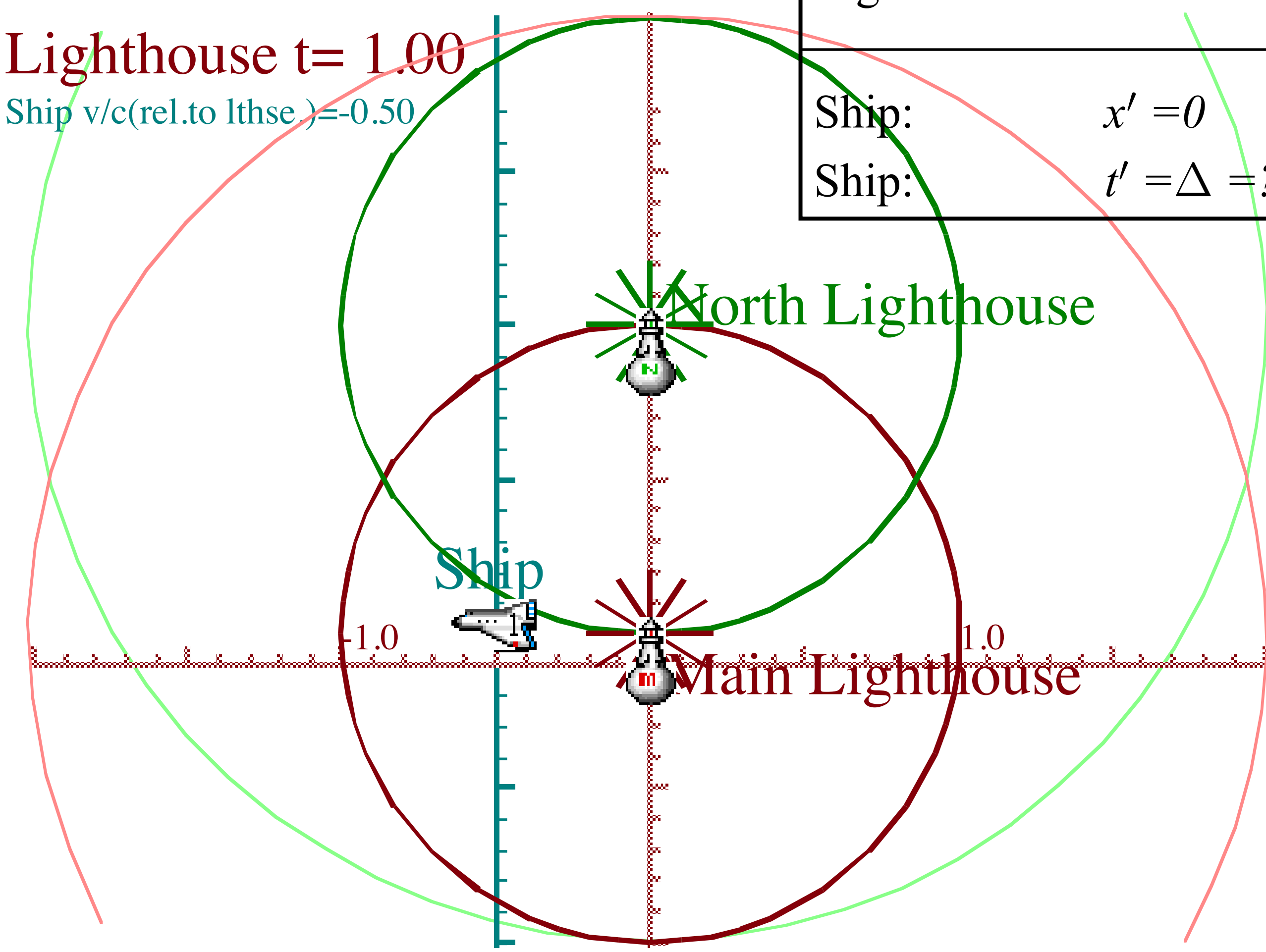
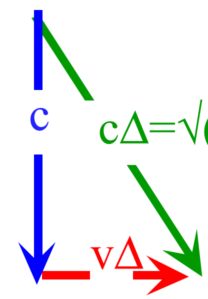
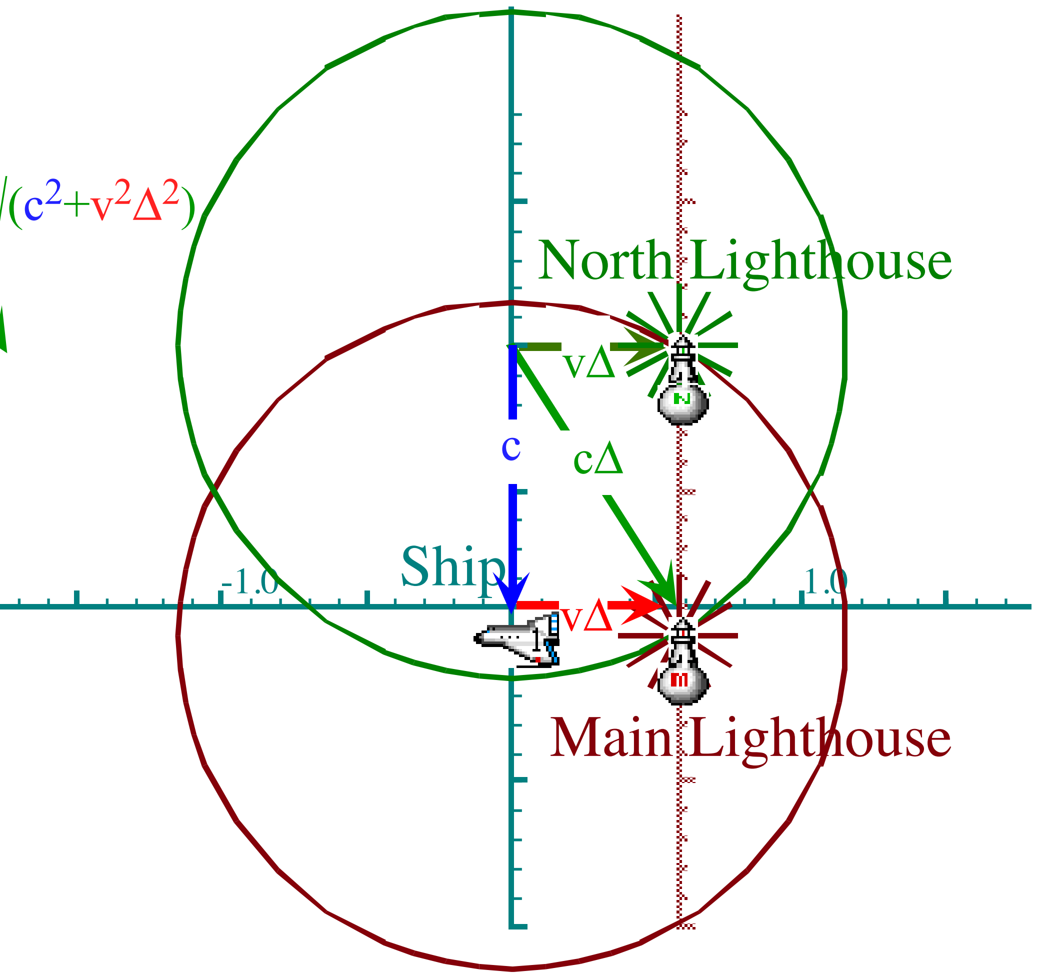
Ship Time $t' = \Delta = ???$

$$c^2\Delta^2 = c^2 + v^2\Delta^2$$

$$\left(c^2 - v^2\right)\Delta^2 = c^2$$







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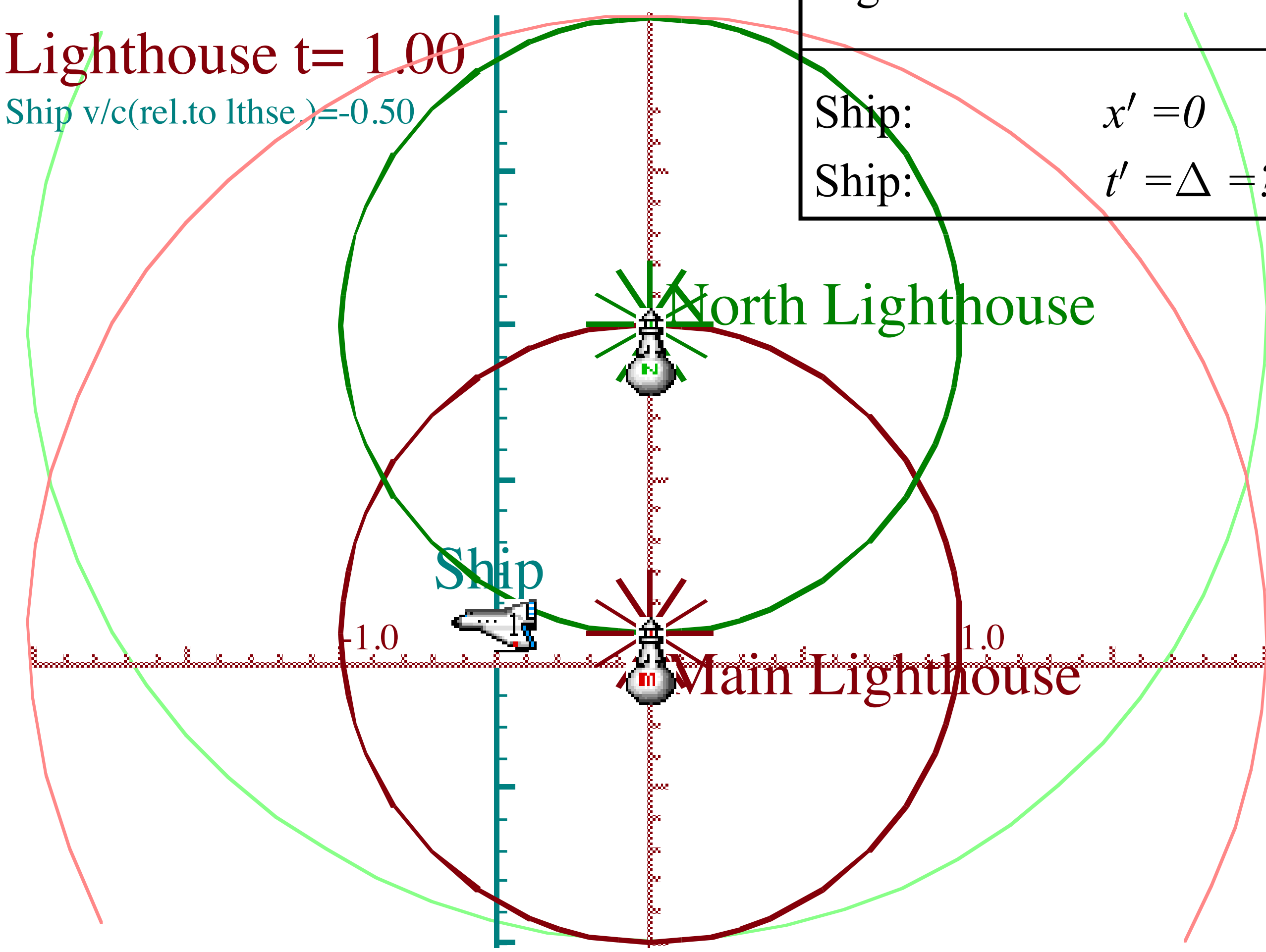
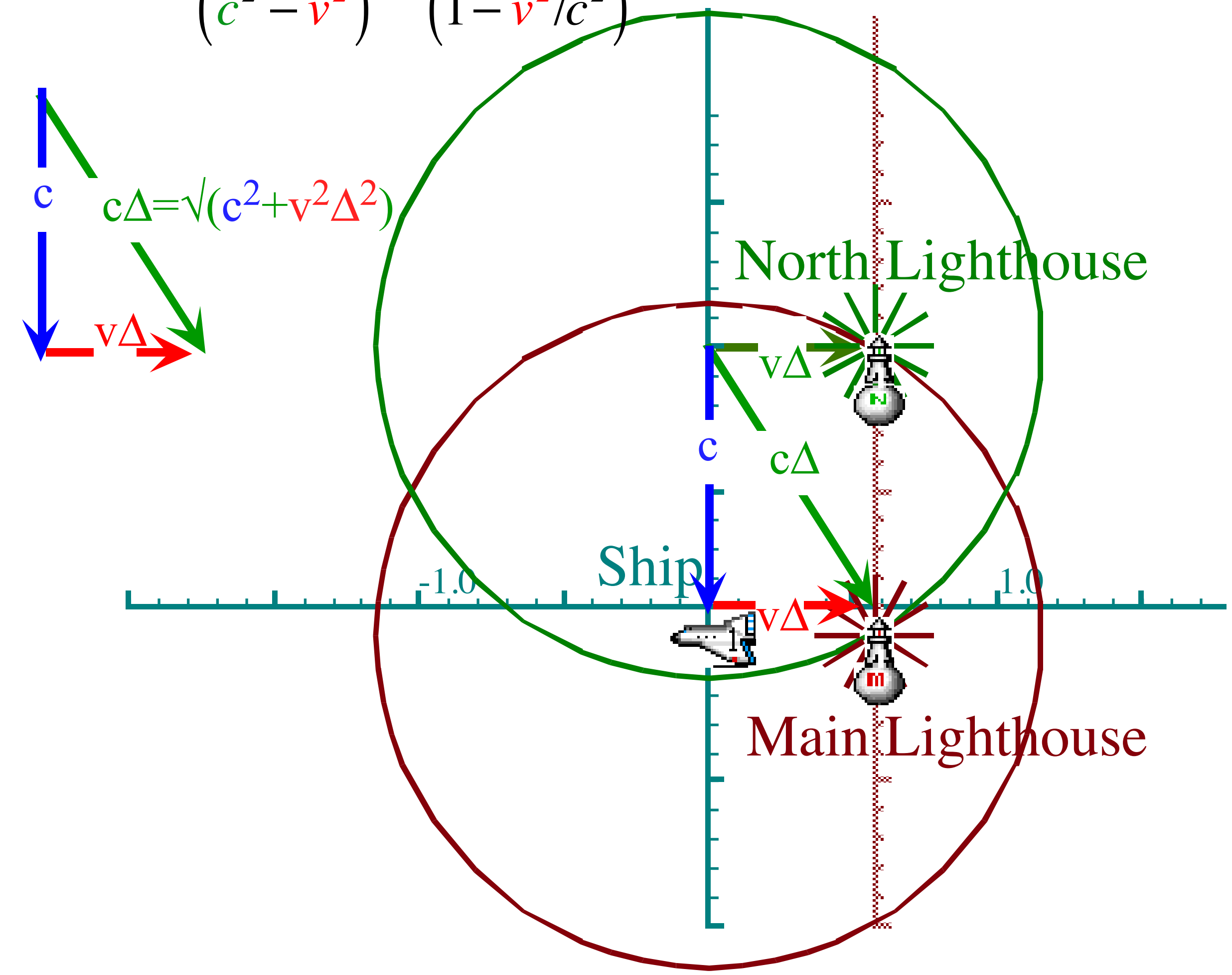
Ship Time $t' = \Delta = 1/\sqrt{(1-v^2/c^2)} = \cosh \rho$

$$c^2\Delta^2 = c^2 + v^2\Delta^2$$

$$\left(c^2 - v^2\right)\Delta^2 = c^2$$

$$\Delta^2 = \frac{c^2}{\left(c^2 - v^2\right)} = \frac{1}{\left(1 - v^2/c^2\right)}$$





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The ship and lighthouse saga

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Lighthouse:	$x = 0$
Lighthouse:	$t = 1.00$
Ship:	$x' = 0$
Ship:	$t' = \Delta = 1.15$

Ship Time $t' = \Delta = 1/\sqrt{(1-v^2/c^2)} = \cosh\rho = 1.15$

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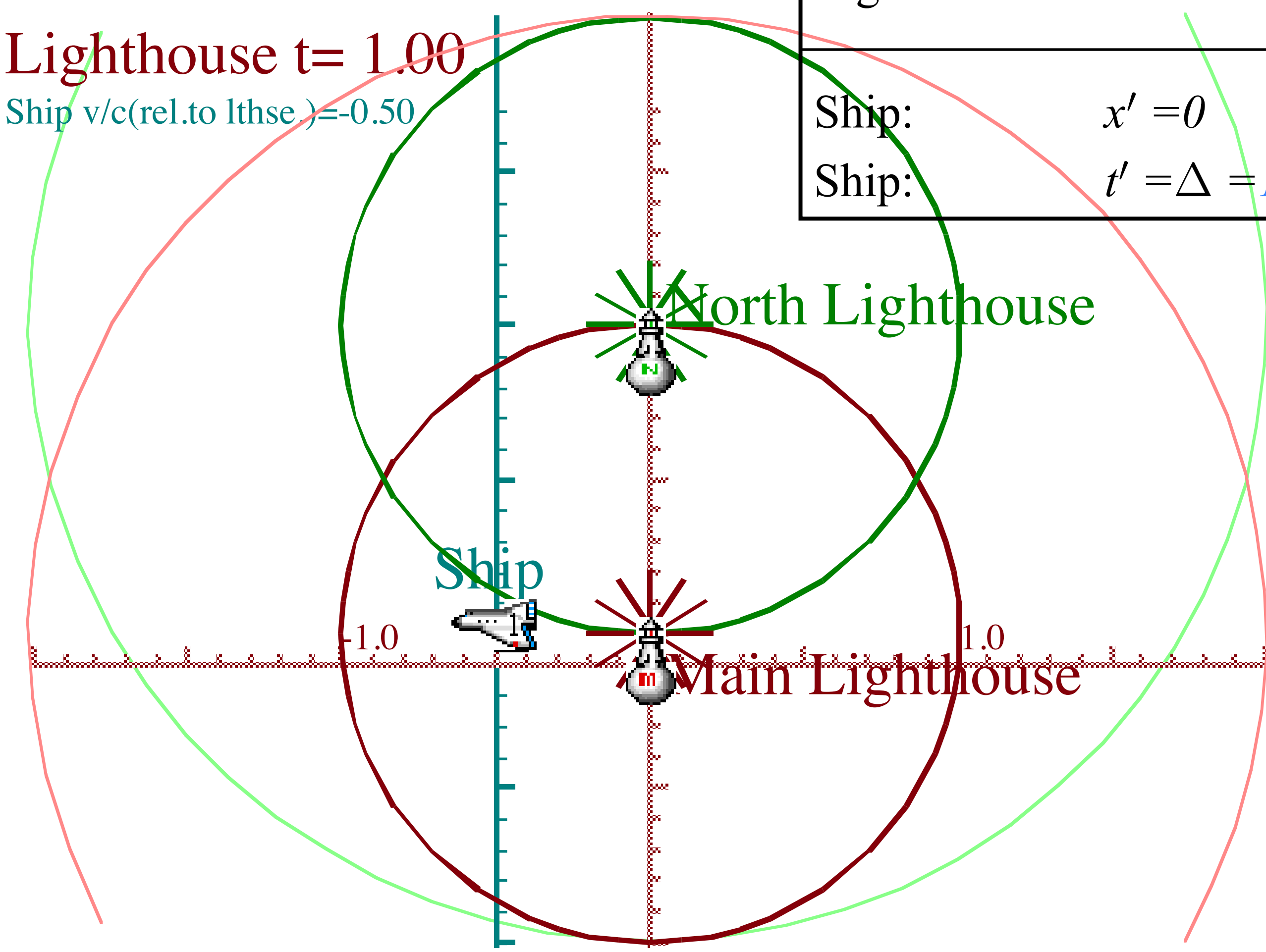

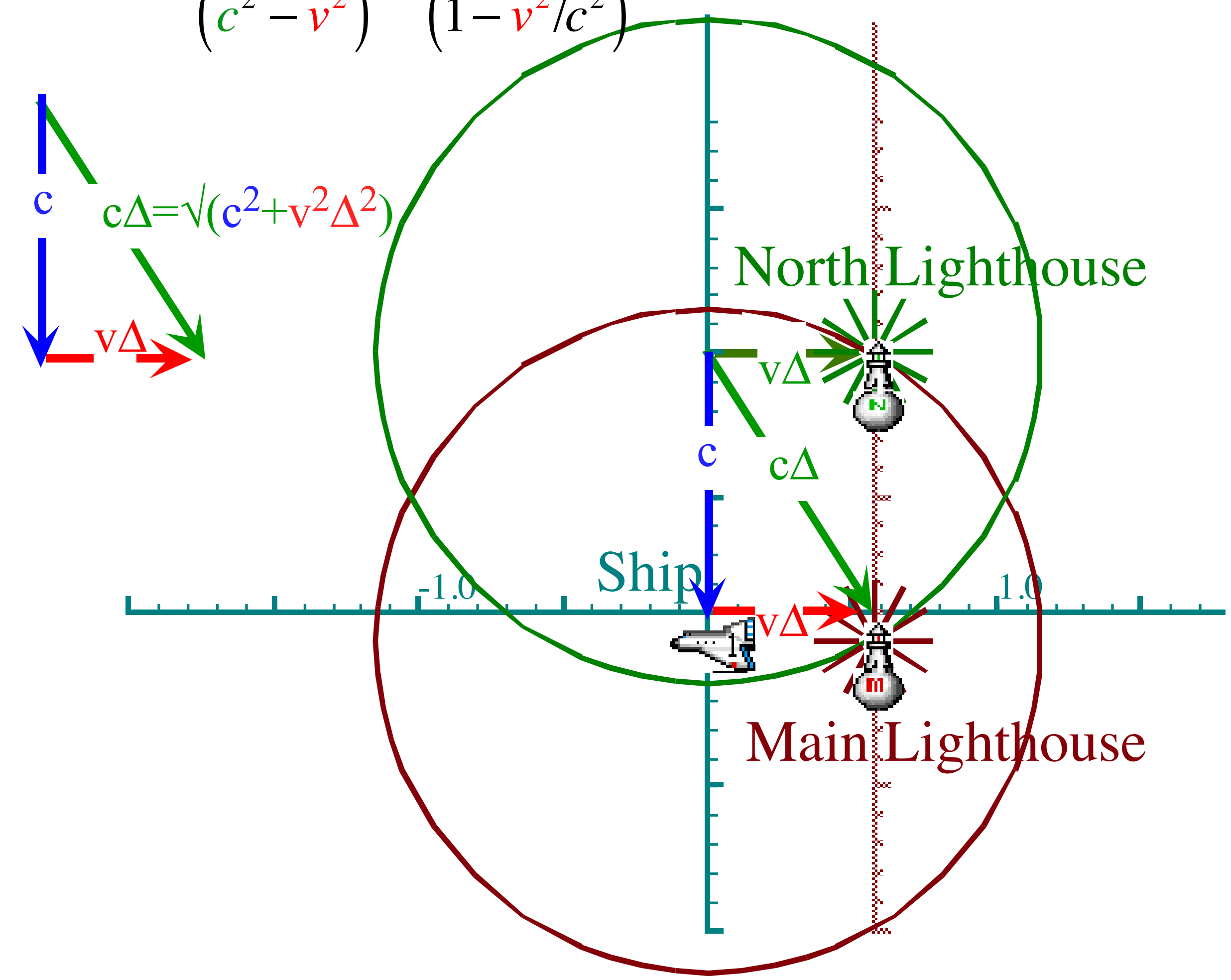

For u/c=1/2

$\Delta = 1/\sqrt{(1-1/4)} = 2/\sqrt{3} = 1.15..$

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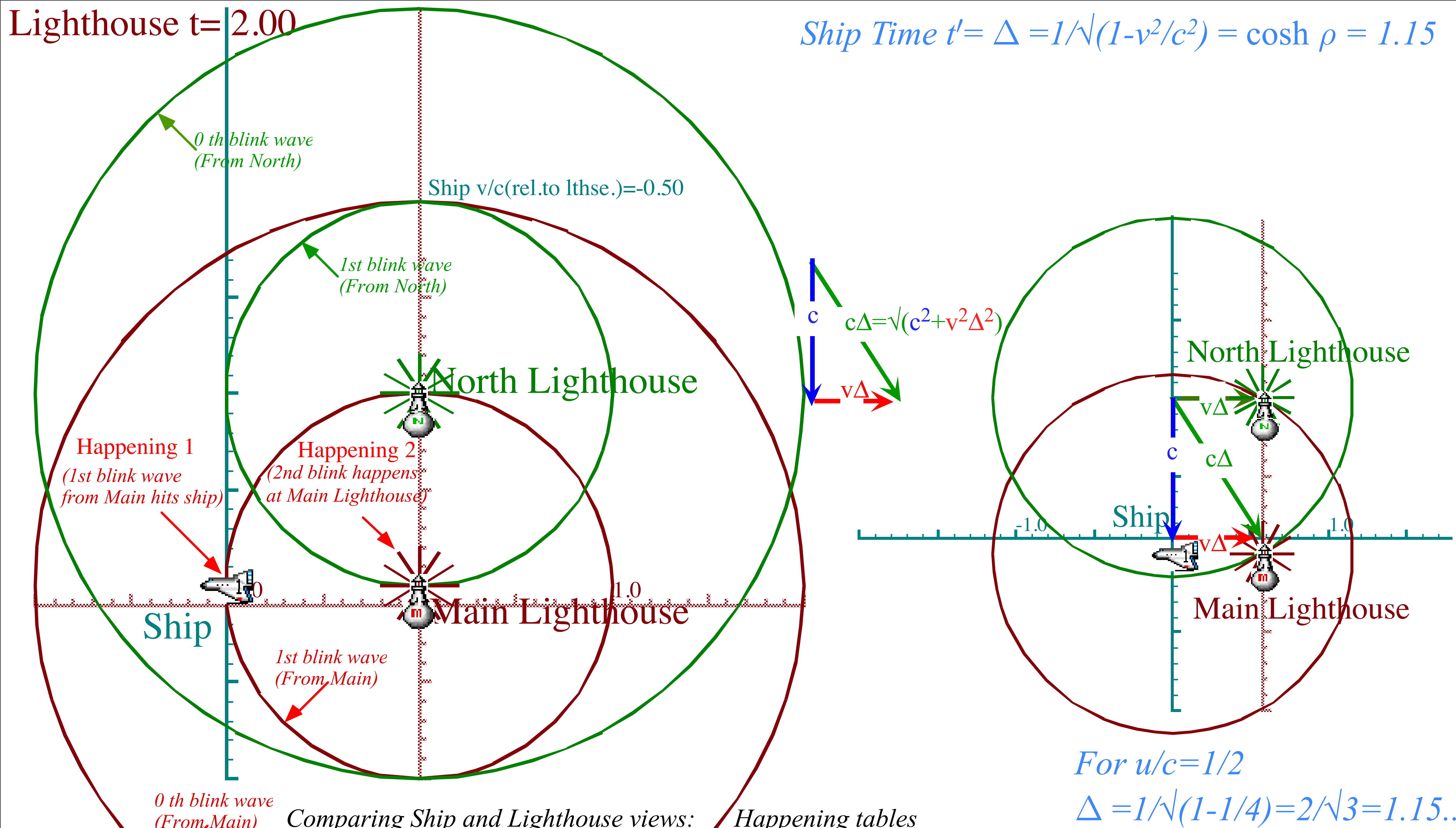

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Lighthouse t= 2.00

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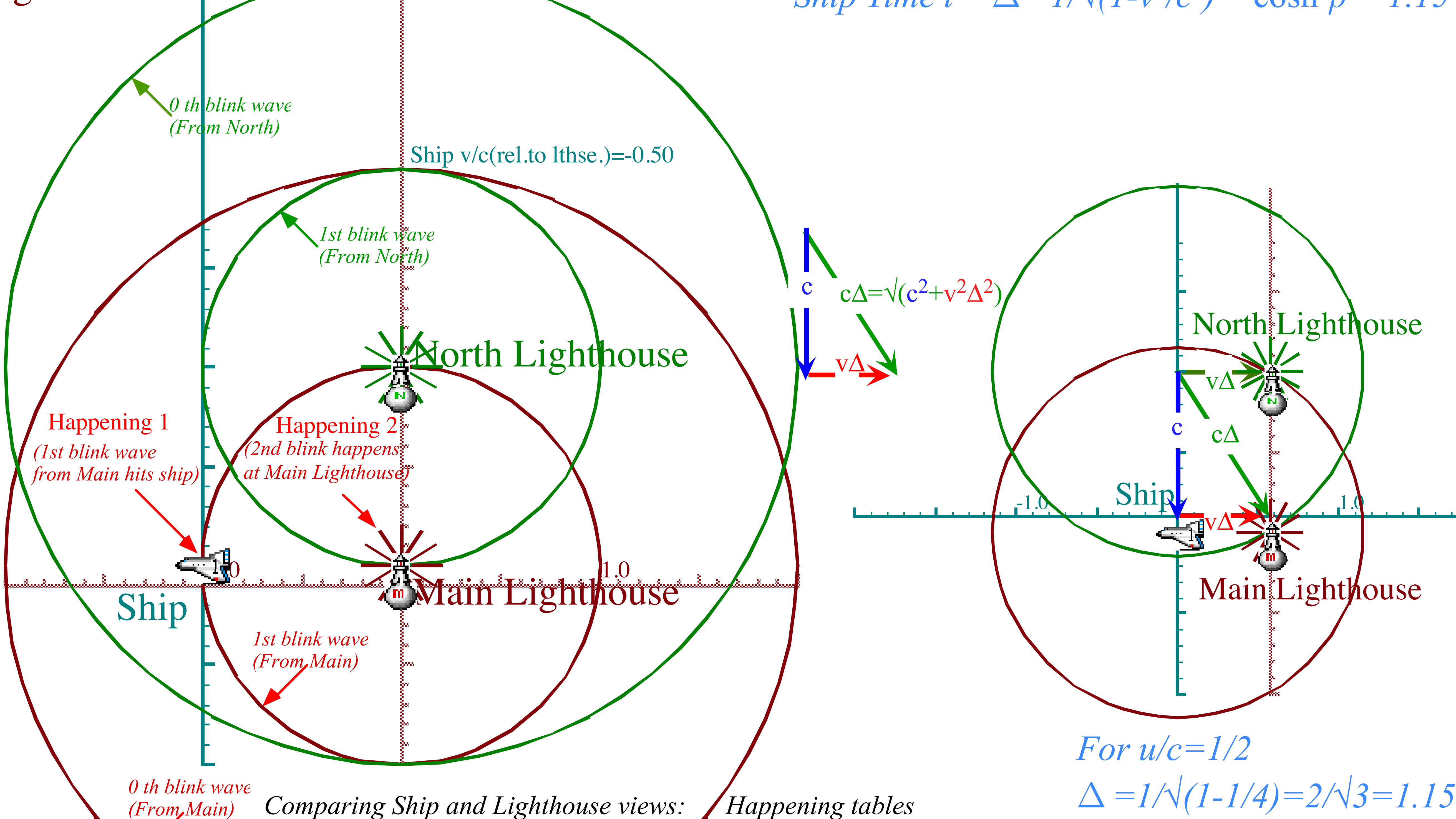

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(Ship space) $x' = 0$ (Ship time) $t' = 0$	$x' = 0$ $t' = (v+c)\Delta/c$	$x' = 2v\Delta$ $t' = 2\Delta$

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Lecture 24 ended here

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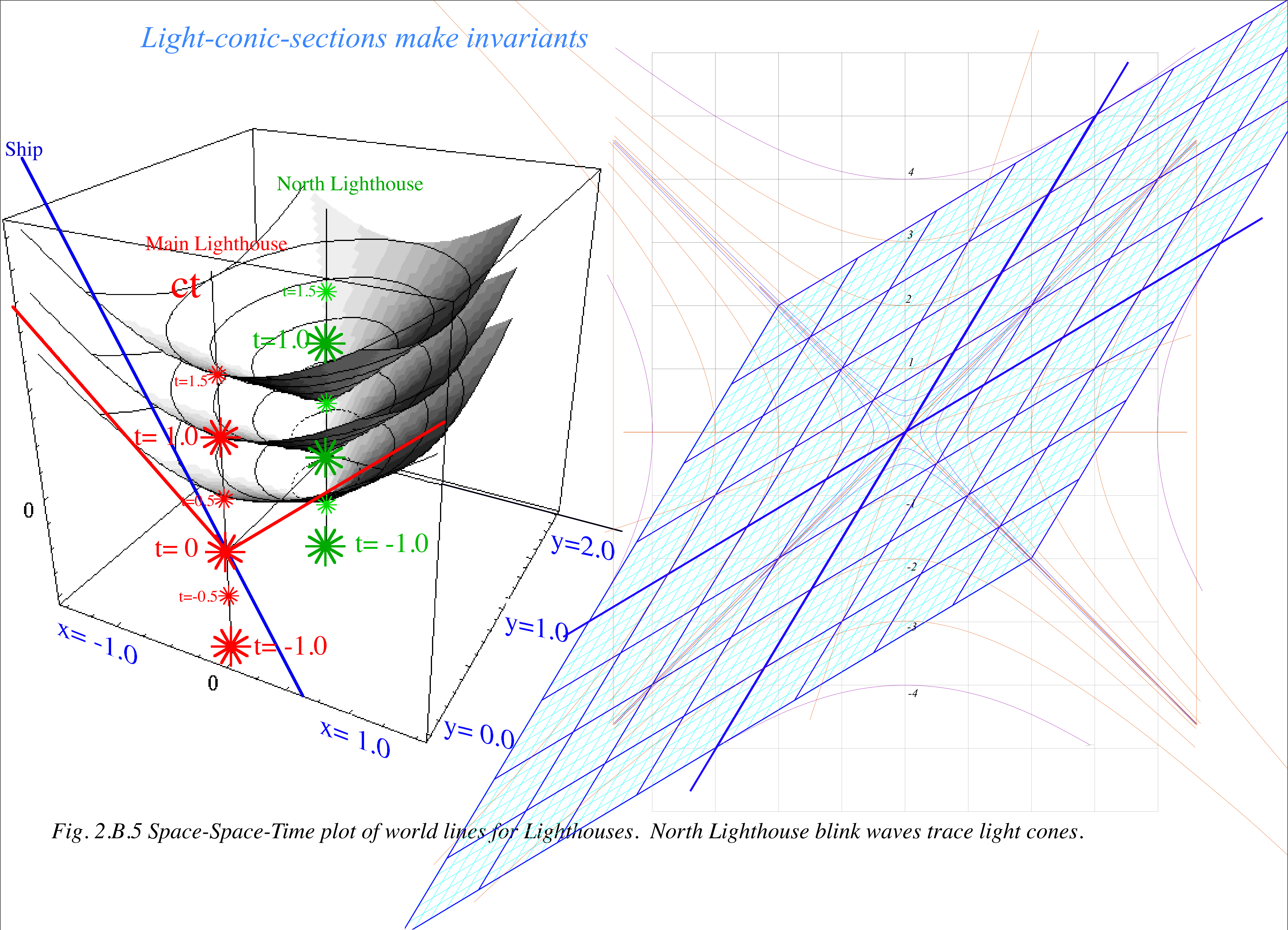

Fig. 2.B.5 Space-Space-Time plot of world lines for Lighthouses. North Lighthouse blink waves trace light cones.

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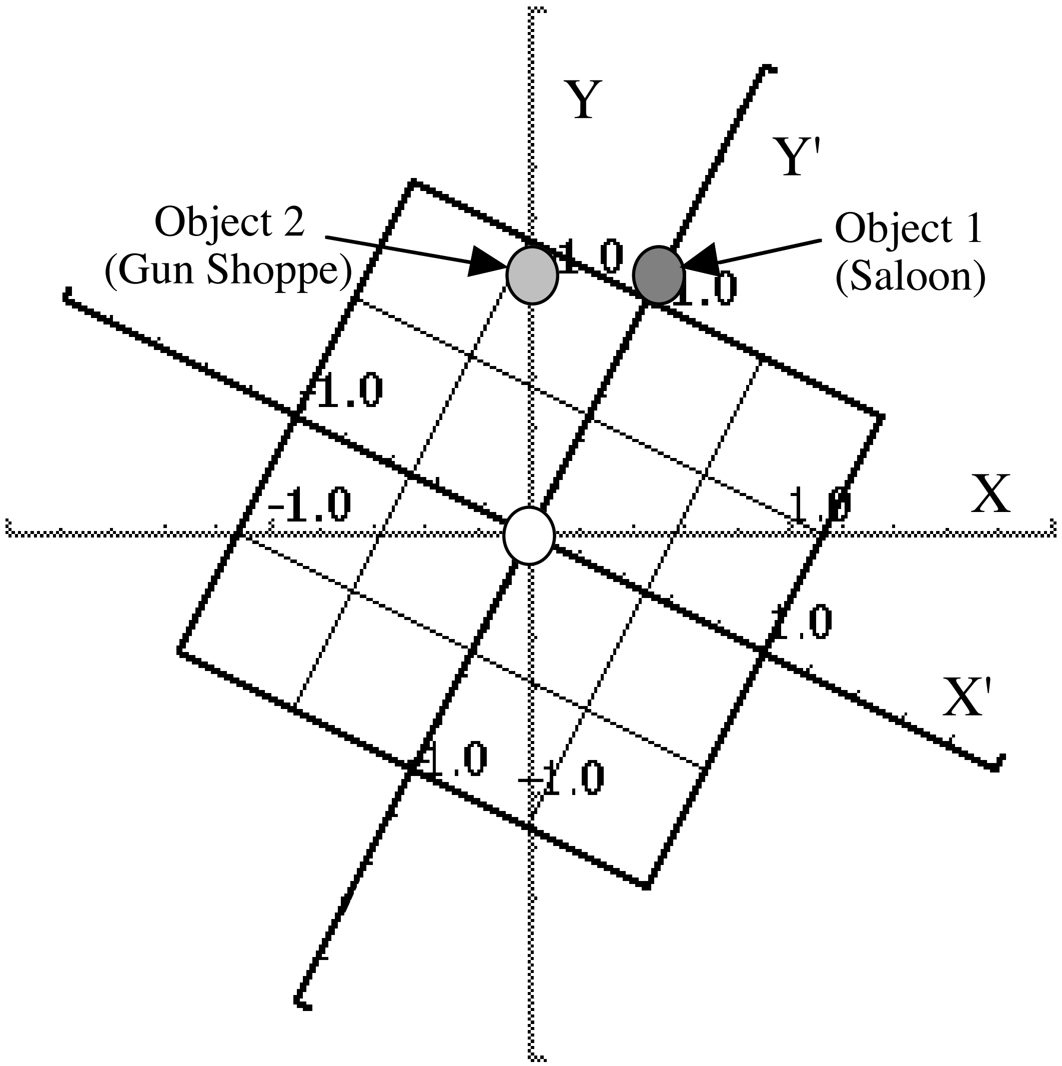

Object 0: Town Square.	Object 1: Saloon.	Object 2: Gun Shoppe.
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(French surveyor) $x' = 0$ $y' = 0$	$x' = 0$ $y' = 1.1$	$x' = -0.45$ $y' = 0.89$

A politically incorrect analogy of rotational transformation and Lorentz transformation

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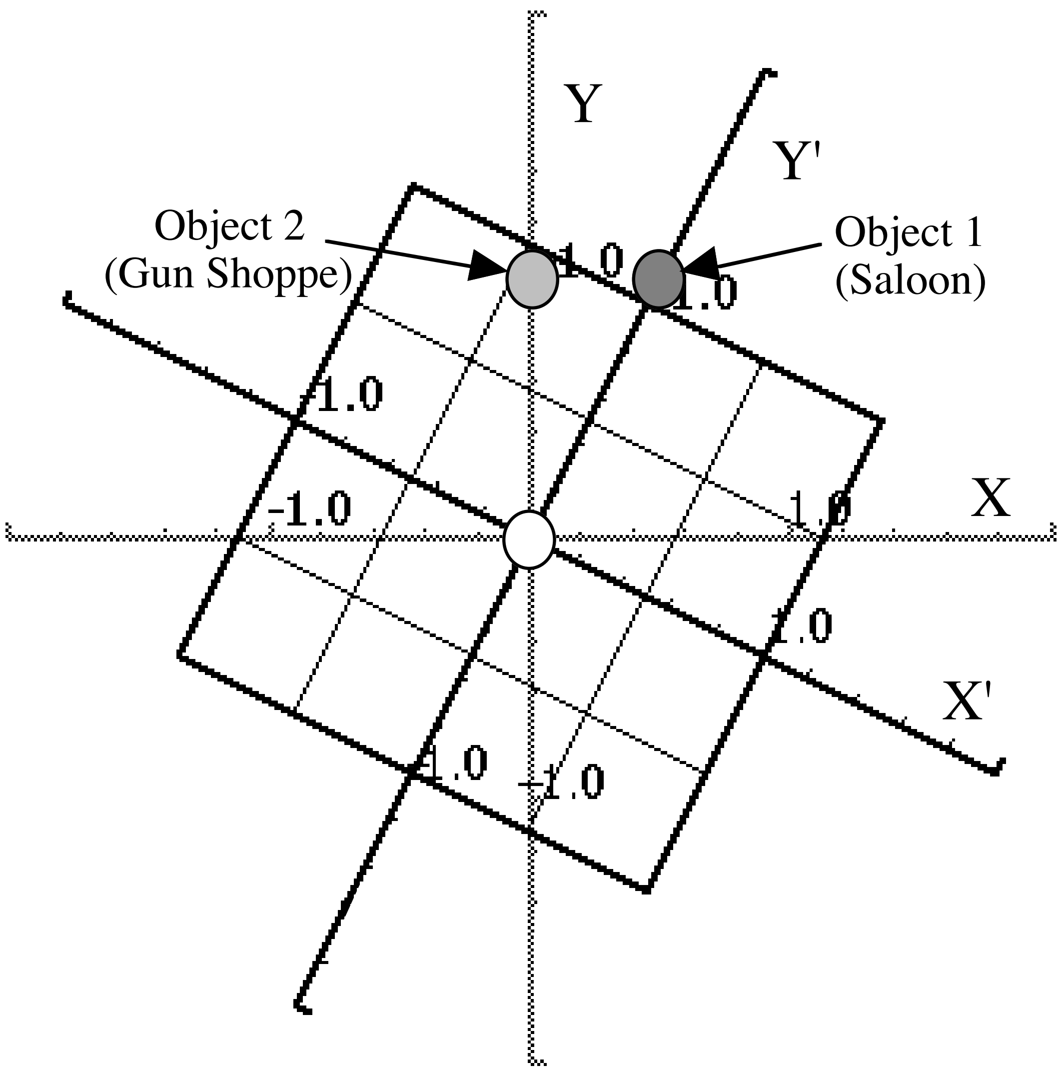

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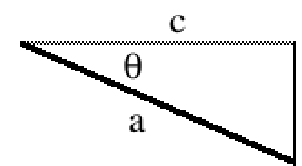
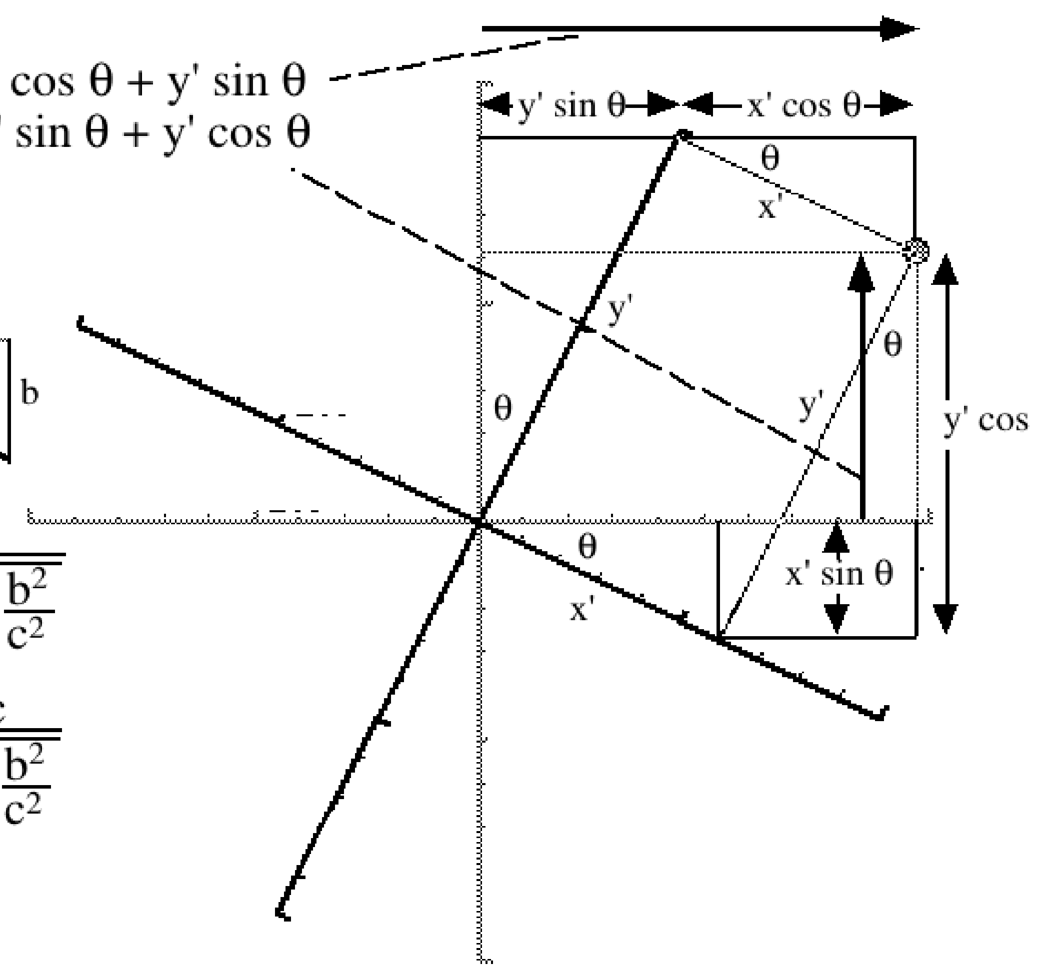
$$x = x' \cos\theta + y' \sin\theta$$
$$y = -x' \sin\theta + y' \cos\theta$$





$$\cos\theta = \frac{1}{\sqrt{1+\frac{b^2}{c^2}}}$$

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$$x' = x\cos\theta - y\sin\theta = \frac{x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{-(b/c)y}{\sqrt{1+\frac{b^2}{c^2}}}$$

$$y' = x\sin\theta + y\cos\theta = \frac{(b/c)x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{y}{\sqrt{1+\frac{b^2}{c^2}}}$$

A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.

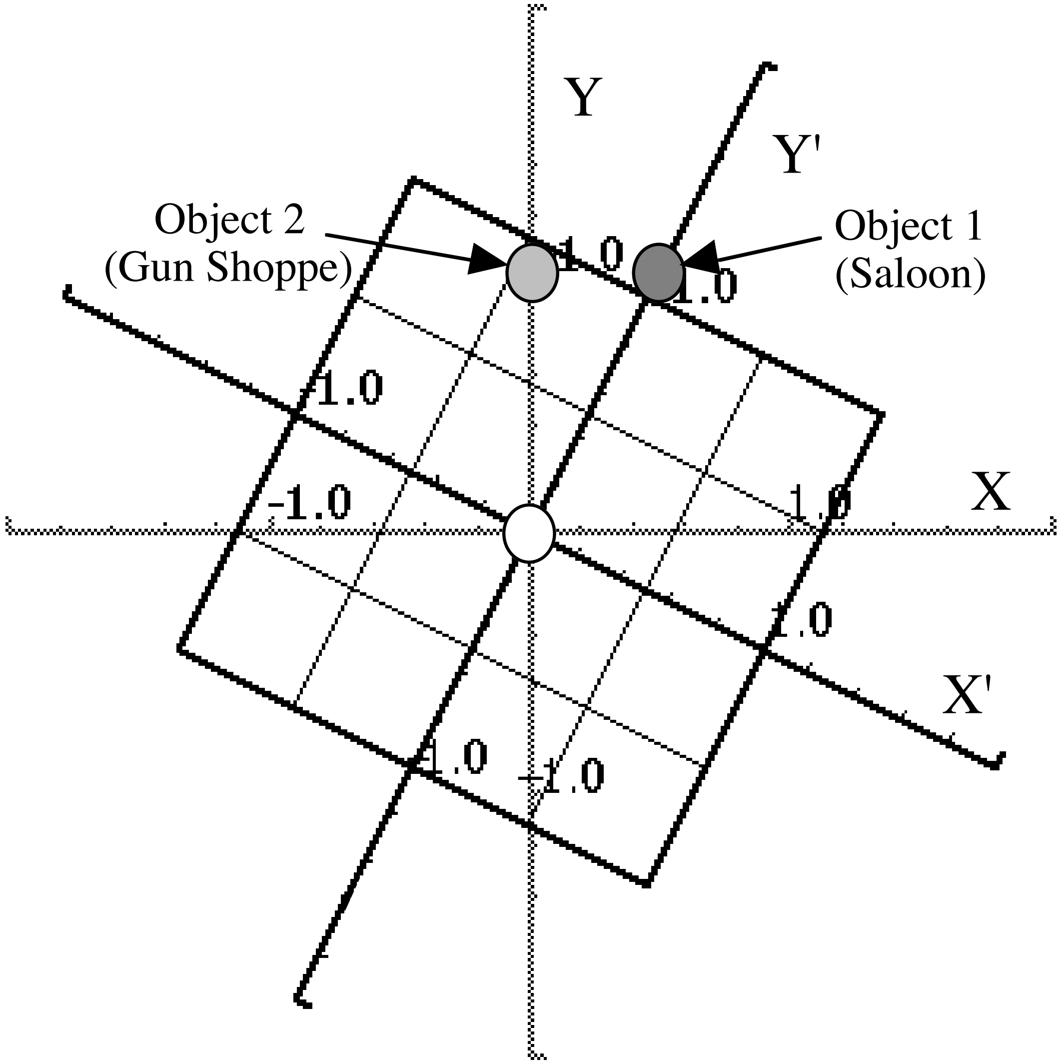

Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.

Reminder: Component-based derivation is *clumsy!*

$$x = x' \cos\theta + y' \sin\theta$$
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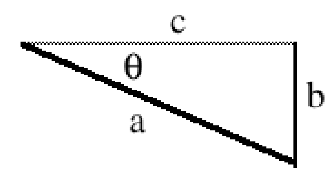

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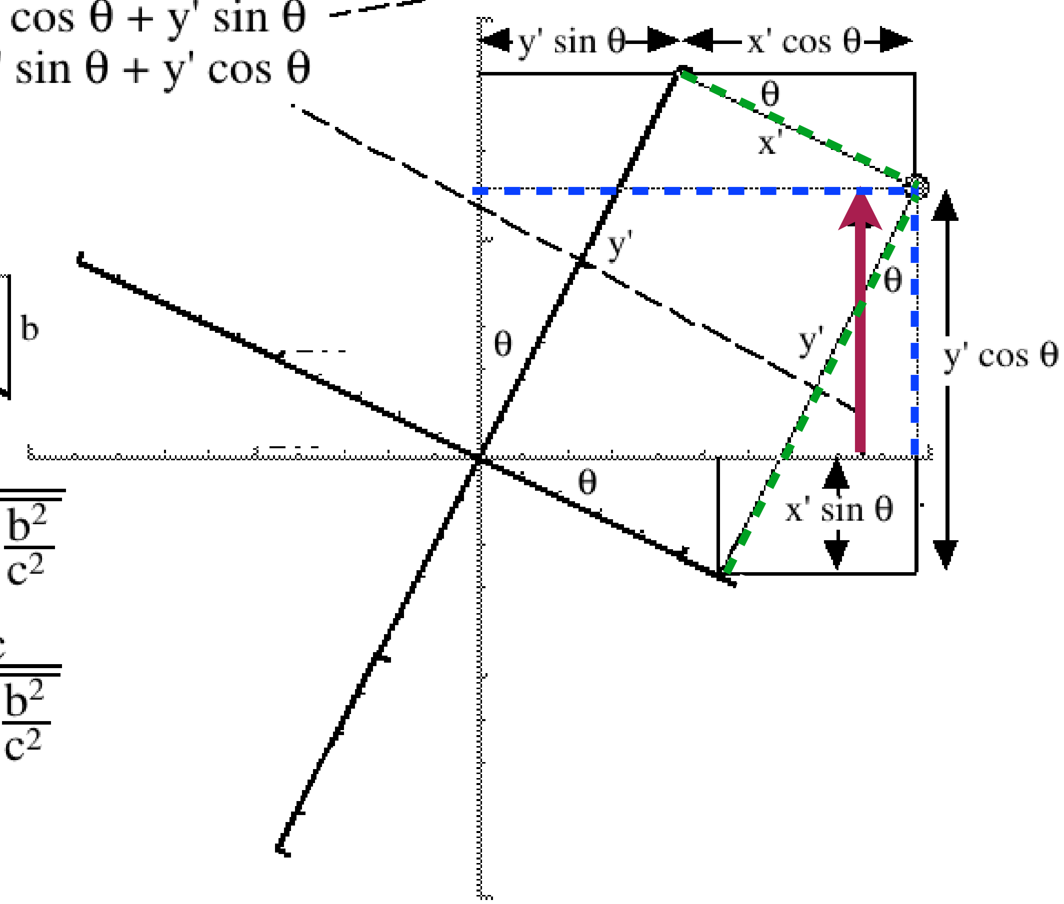

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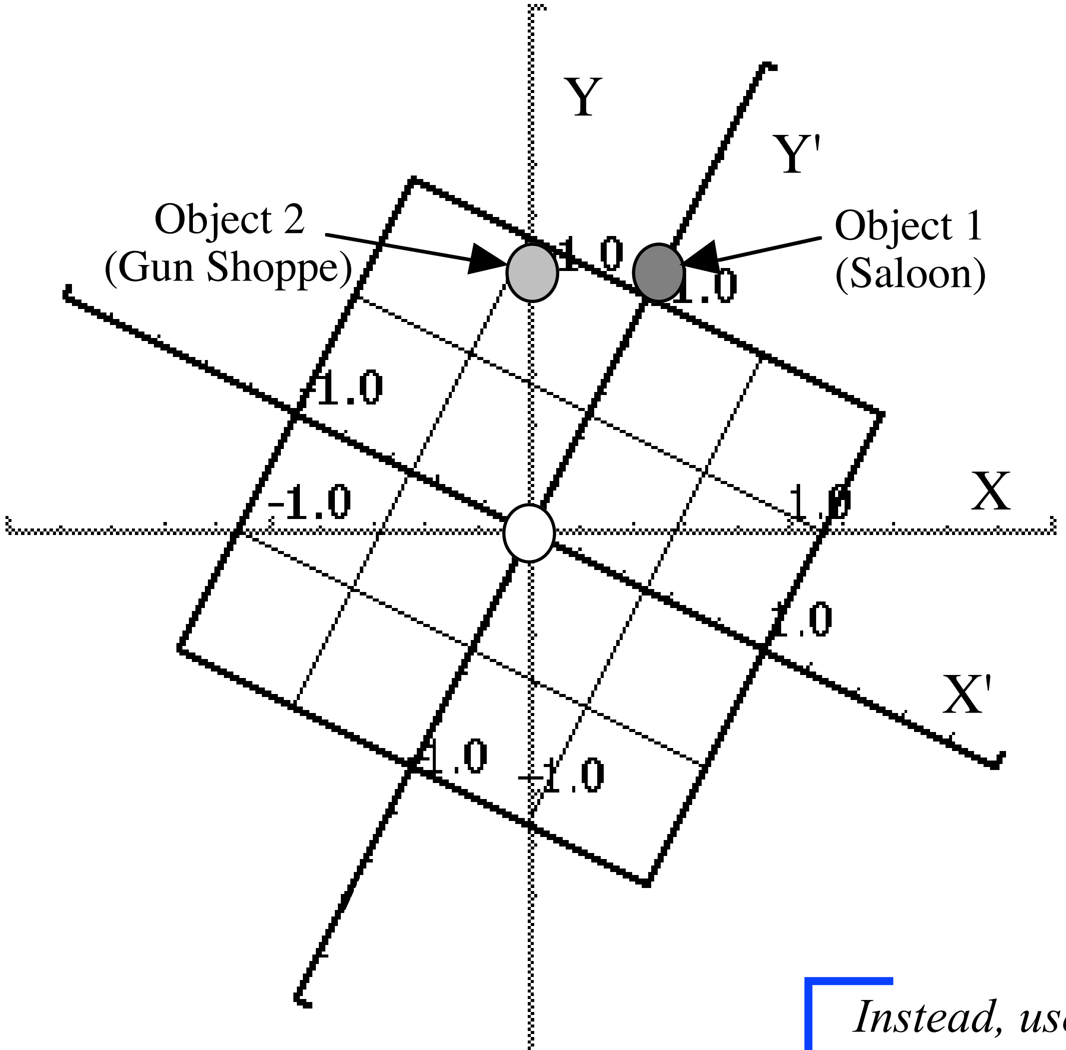

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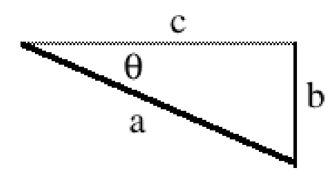

$$\cos\theta = \frac{1}{\sqrt{1+\frac{b^2}{c^2}}}$$

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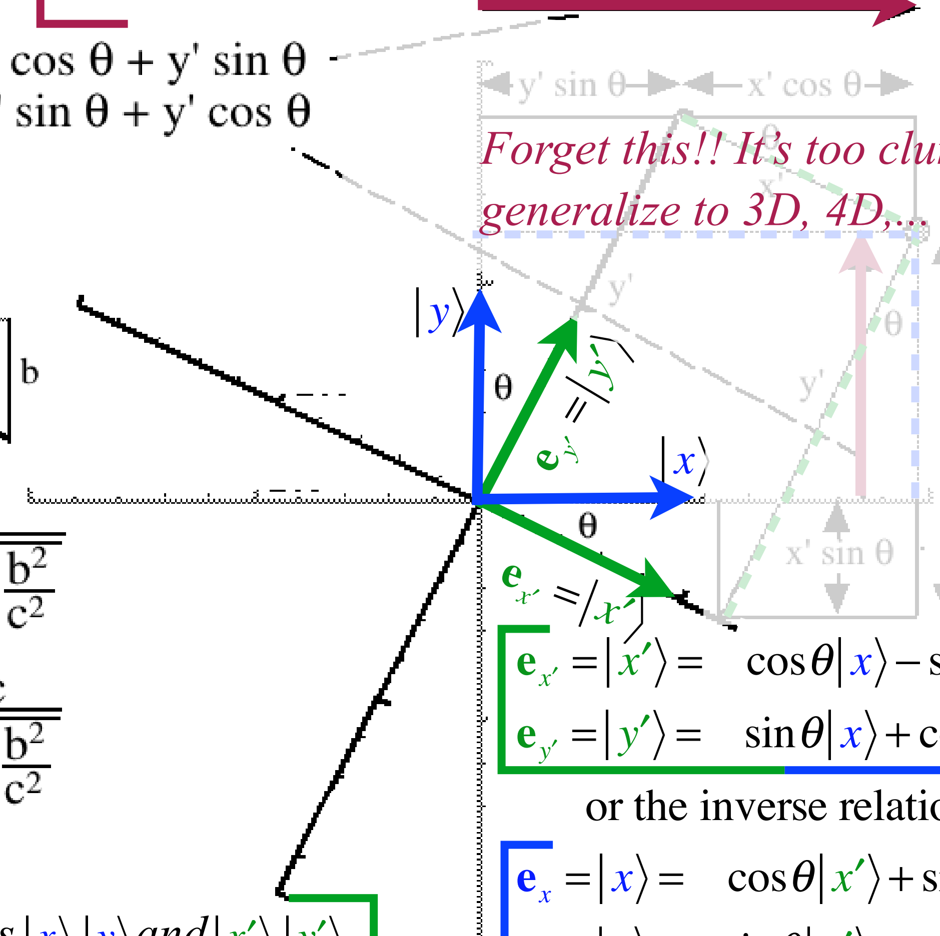

$\mathbf{e}_{x'} = |x'\rangle = \cos\theta|x\rangle - \sin\theta|y\rangle$

$\mathbf{e}_{y'} = |y'\rangle = \sin\theta|x\rangle + \cos\theta|y\rangle$

or the inverse relation:

$\mathbf{e}_{x} = |x\rangle = \cos\theta|x'\rangle + \sin\theta|y'\rangle$

$\mathbf{e}_{y} = |y\rangle = -\sin\theta|x'\rangle + \cos\theta|y'\rangle$

Instead, use Dirac unit vectors $|x\rangle, |y\rangle$ *and* $|x'\rangle, |y'\rangle$

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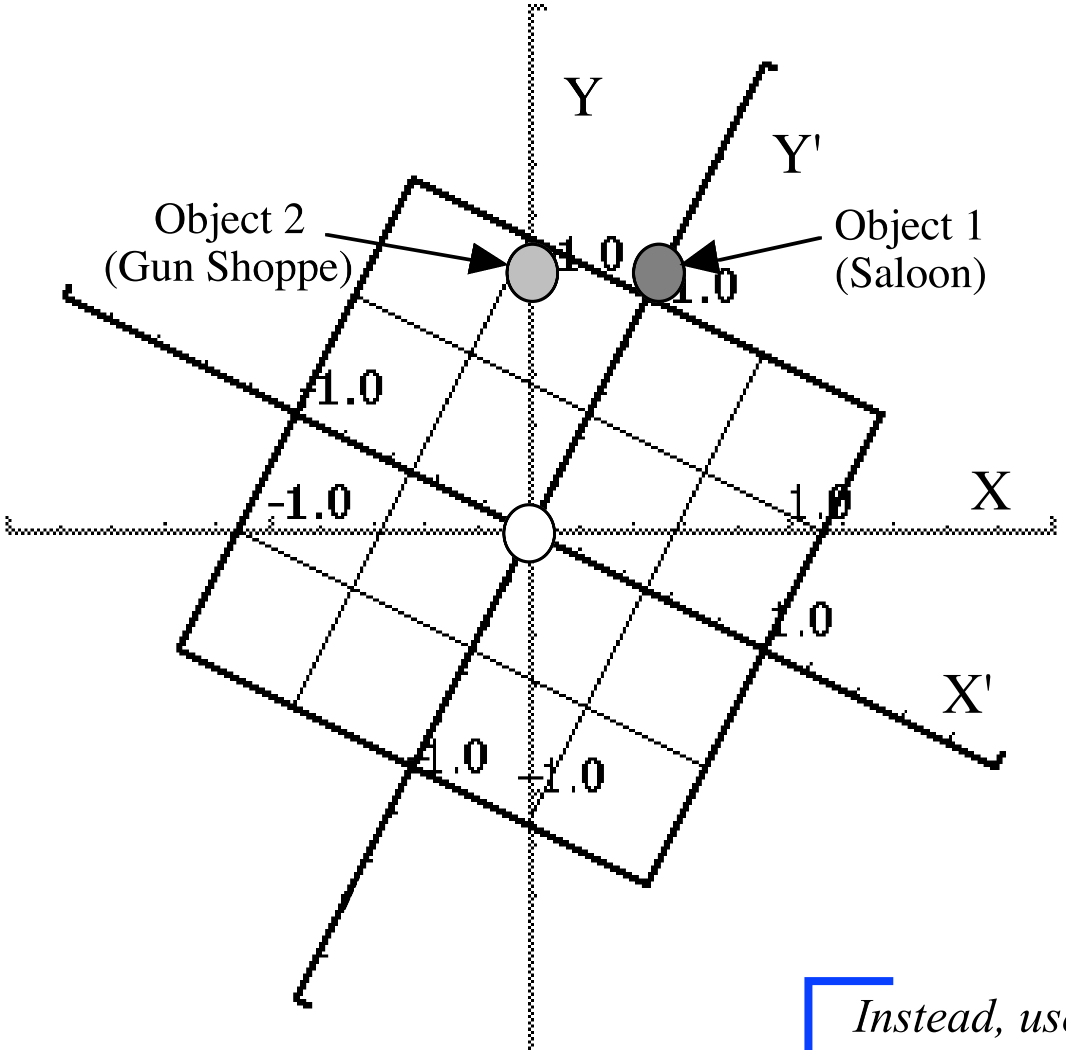

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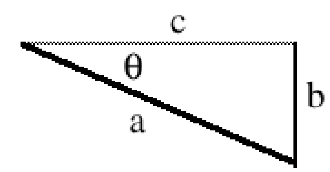

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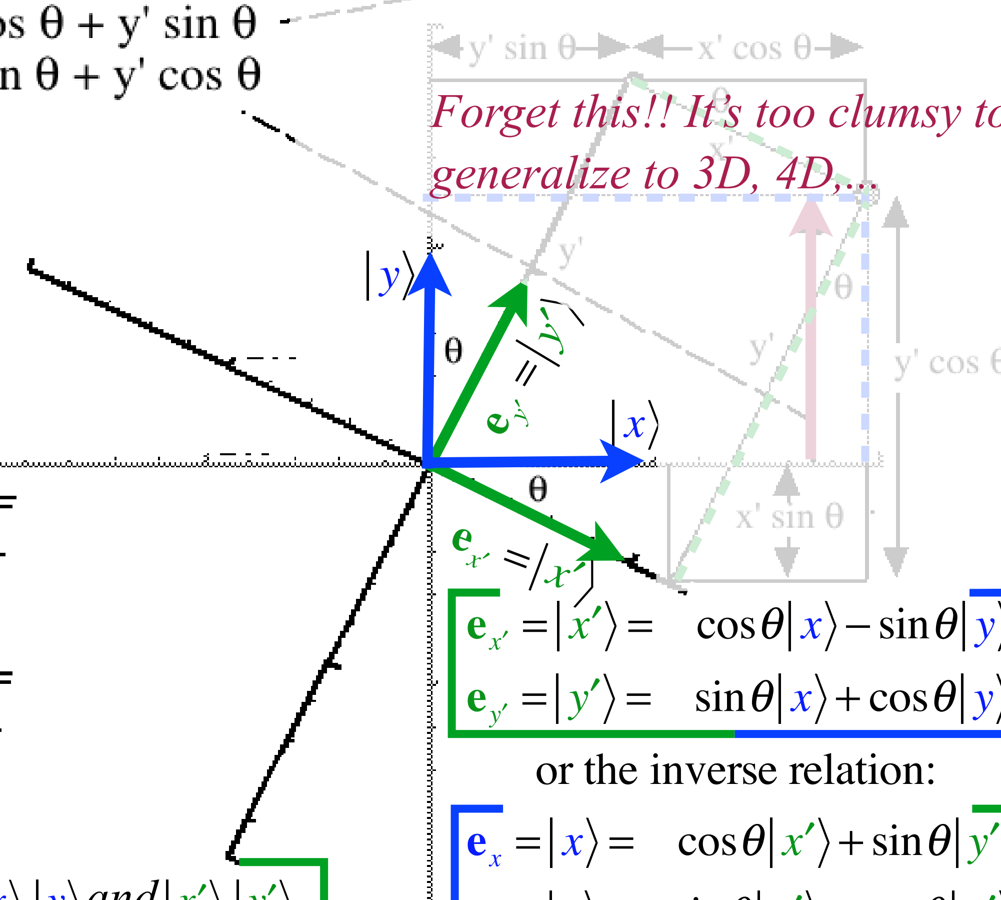

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You may apply (Jacobian) transform matrix:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

or the inverse (Kajobian) transformation:

$$\begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

to *any* vector $\mathbf{V} = |V\rangle = |x\rangle\langle x|V\rangle + |y\rangle\langle y|V\rangle$

$$= |x'\rangle\langle x'|V\rangle + |y'\rangle\langle y'|V\rangle$$

A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.

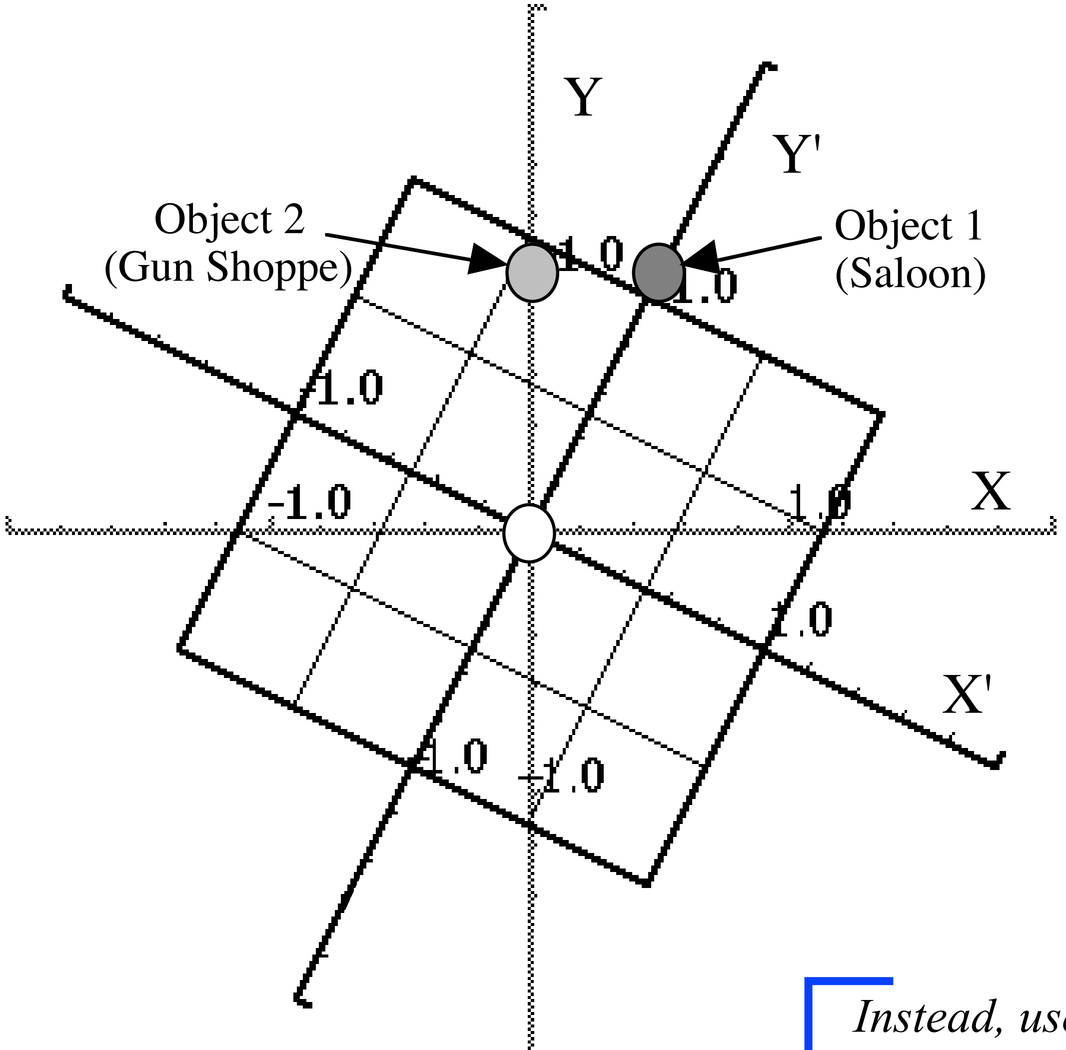
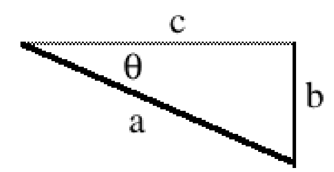


Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.

Reminder: Component-based derivation is clumsy!

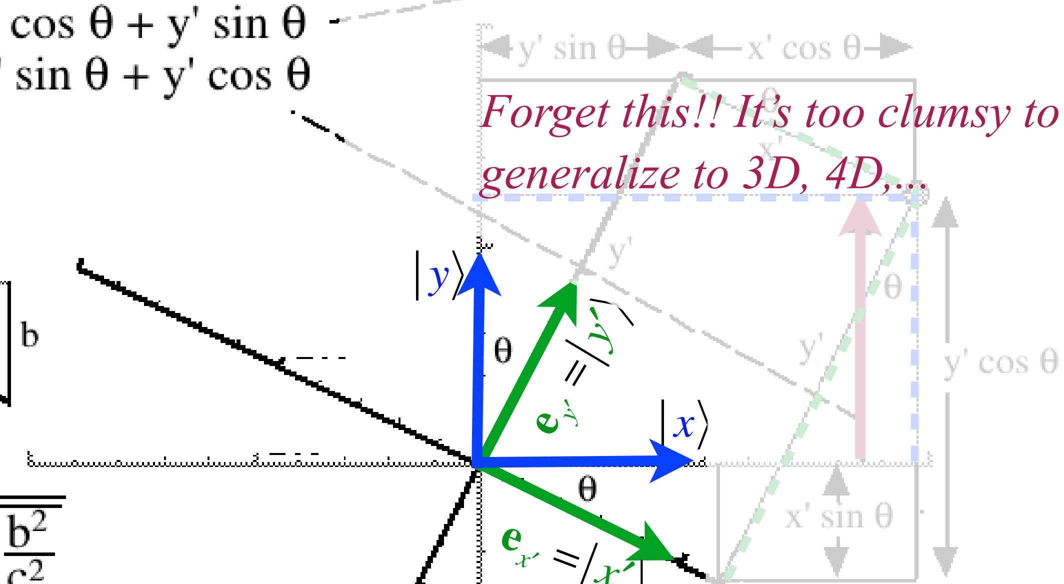
x = x' cos θ + y' sin θ

y = -x' sin θ + y' cos θ



$$\cos\theta = \frac{1}{\sqrt{1+\frac{b^2}{c^2}}}$$

$$\sin\theta = \frac{b/c}{\sqrt{1+\frac{b^2}{c^2}}}$$



$$\mathbf{e}_{x'} = |x'\rangle = \cos\theta|x\rangle - \sin\theta|y\rangle$$

$$\mathbf{e}_{y'} = |y'\rangle = \sin\theta|x\rangle + \cos\theta|y\rangle$$

or the inverse relation:

$$\mathbf{e}_{x} = |x\rangle = \cos\theta|x'\rangle + \sin\theta|y'\rangle$$

$$\mathbf{e}_{y} = |y\rangle = -\sin\theta|x'\rangle + \cos\theta|y'\rangle$$

Instead, use Dirac unit vectors $|x\rangle, |y\rangle$ *and* $|x'\rangle, |y'\rangle$

Object 0: Town Square.	Object 1: Saloon.	Object 2: Gun Shoppe.
(US surveyor) $x = 0$, $y = 0$	$x = 0.5$, $y = 1.0$	$x = 0$, $y = 1.0$
(2nd surveyor) $x' = 0$, $y' = 0$	$x' = 0$, $y' = 1.1$	$x' = -0.45$, $y' = 0.89$

(Jacobian) transformation $\{V_x V_y\}$ from $\{V_{x'} V_{y'}\}$:

$$V_x = \langle x|V\rangle = \langle x|1|V\rangle = \langle x|x'\rangle\langle x'|V\rangle + \langle x|y'\rangle\langle y'|V\rangle$$

$$V_y = \langle y|V\rangle = \langle y|1|V\rangle = \langle y|x'\rangle\langle x'|V\rangle + \langle y|y'\rangle\langle y'|V\rangle$$

You may apply (Jacobian) transform matrix:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

or the inverse (Kajobian) transformation:

$$\begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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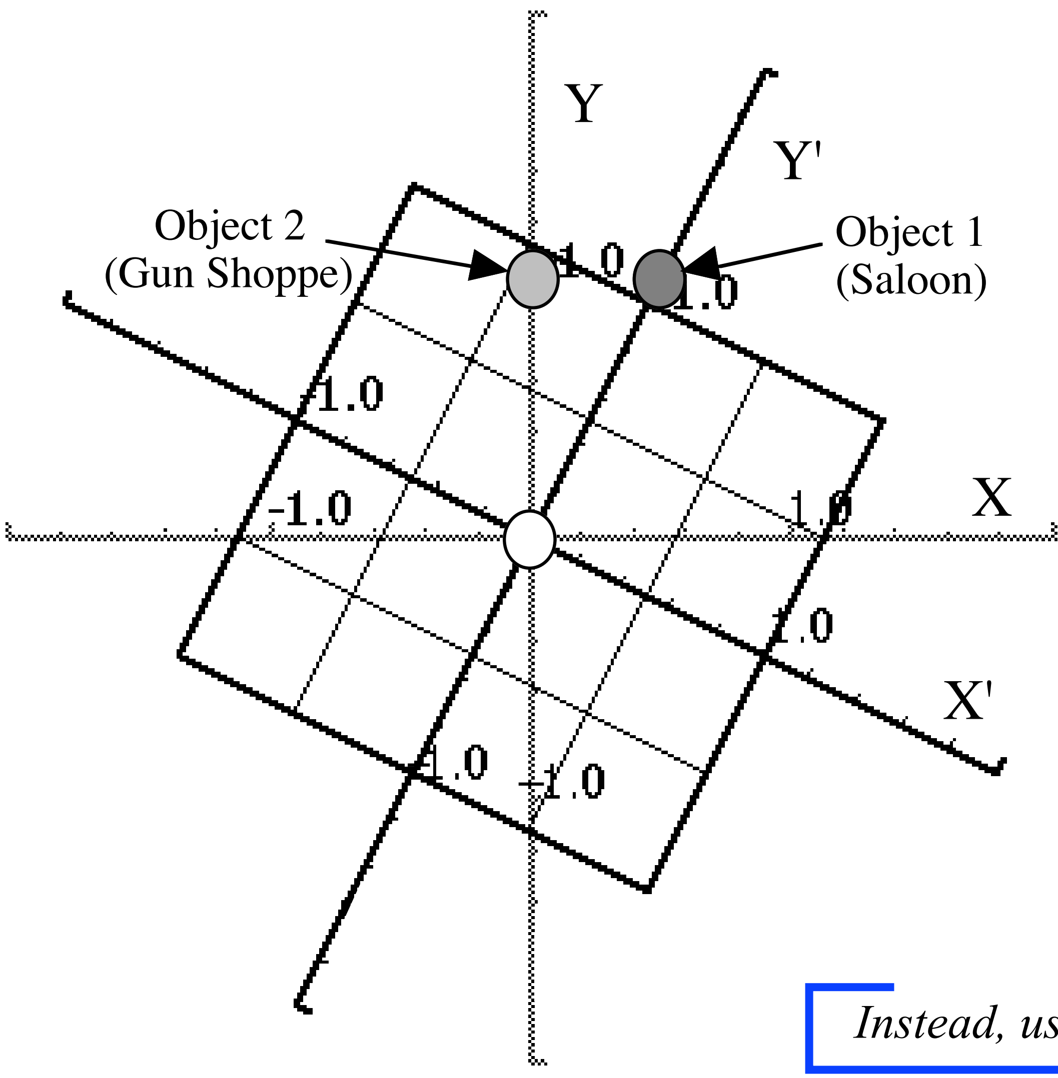

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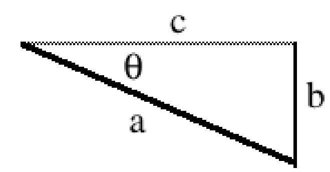
Reminder: Component-based derivation is clumsy!

x = x' cos θ + y' sin θ

y = -x' sin θ + y' cos θ

Forget this!! It's too clumsy to generalize to 3D, 4D,...



$$\cos\theta = \frac{1}{\sqrt{1+\frac{b^2}{c^2}}}$$

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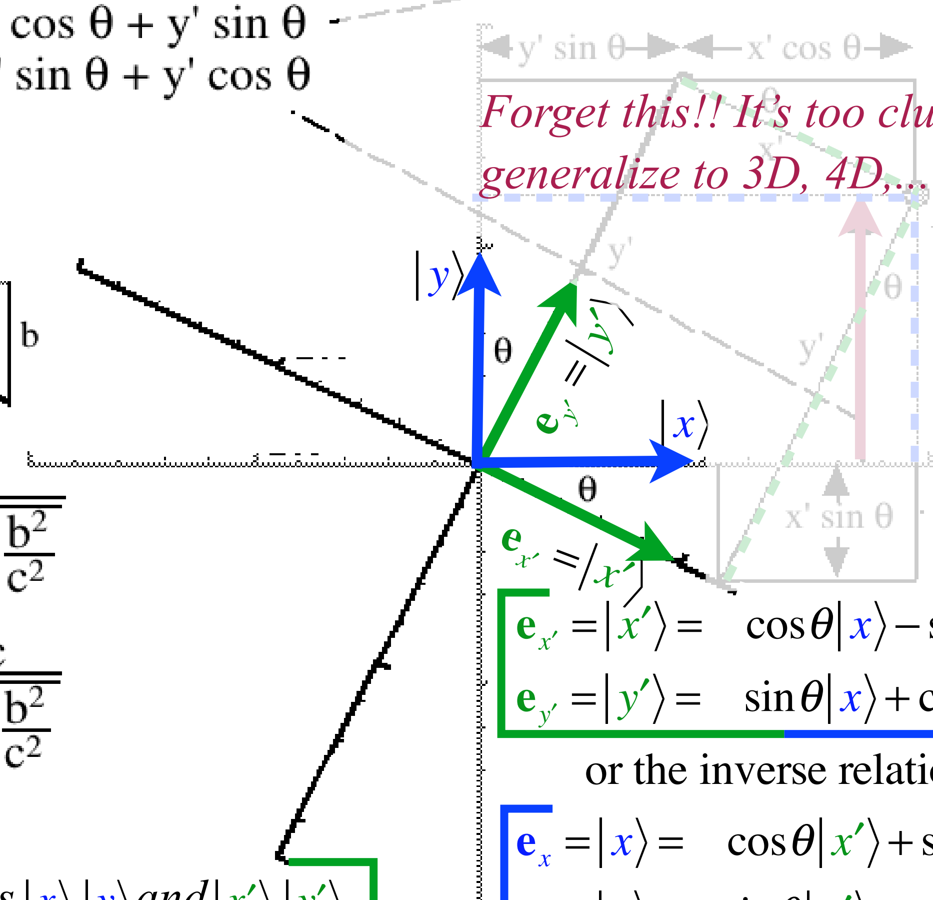

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in matrix form:

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} V_{x'} \\ V_{y'} \end{pmatrix}$$

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to *any* vector $\mathbf{V} = |V\rangle = |x\rangle\langle x|V\rangle + |y\rangle\langle y|V\rangle$

$= |x'\rangle\langle x'|V\rangle + |y'\rangle\langle y'|V\rangle$

PLEASE!

Do NOT ***ever*** *write*

this:

$$\mathbf{e}_{x'} = |x'\rangle = \cos\theta|x\rangle - \sin\theta|y\rangle$$
$$\mathbf{e}_{y'} = |y'\rangle = \sin\theta|x\rangle + \cos\theta|y\rangle$$

like this:

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(This is an abstract definition.)

like this:

This is GARBAGE!

$$\begin{pmatrix} \mathbf{e}_{x'} \\ \mathbf{e}_{y'} \end{pmatrix} = \begin{pmatrix} |x'\rangle \\ |y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix}$$

PLEASE!

Do NOT ever write this:

$$\mathbf{e}_{x'} = |x'\rangle = \cos\theta|x\rangle - \sin\theta|y\rangle \equiv \mathbf{R}|x\rangle$$
$$\mathbf{e}_{y'} = |y'\rangle = \sin\theta|x\rangle + \cos\theta|y\rangle \equiv \mathbf{R}|y\rangle$$

(This is an abstract definition.)

like this:

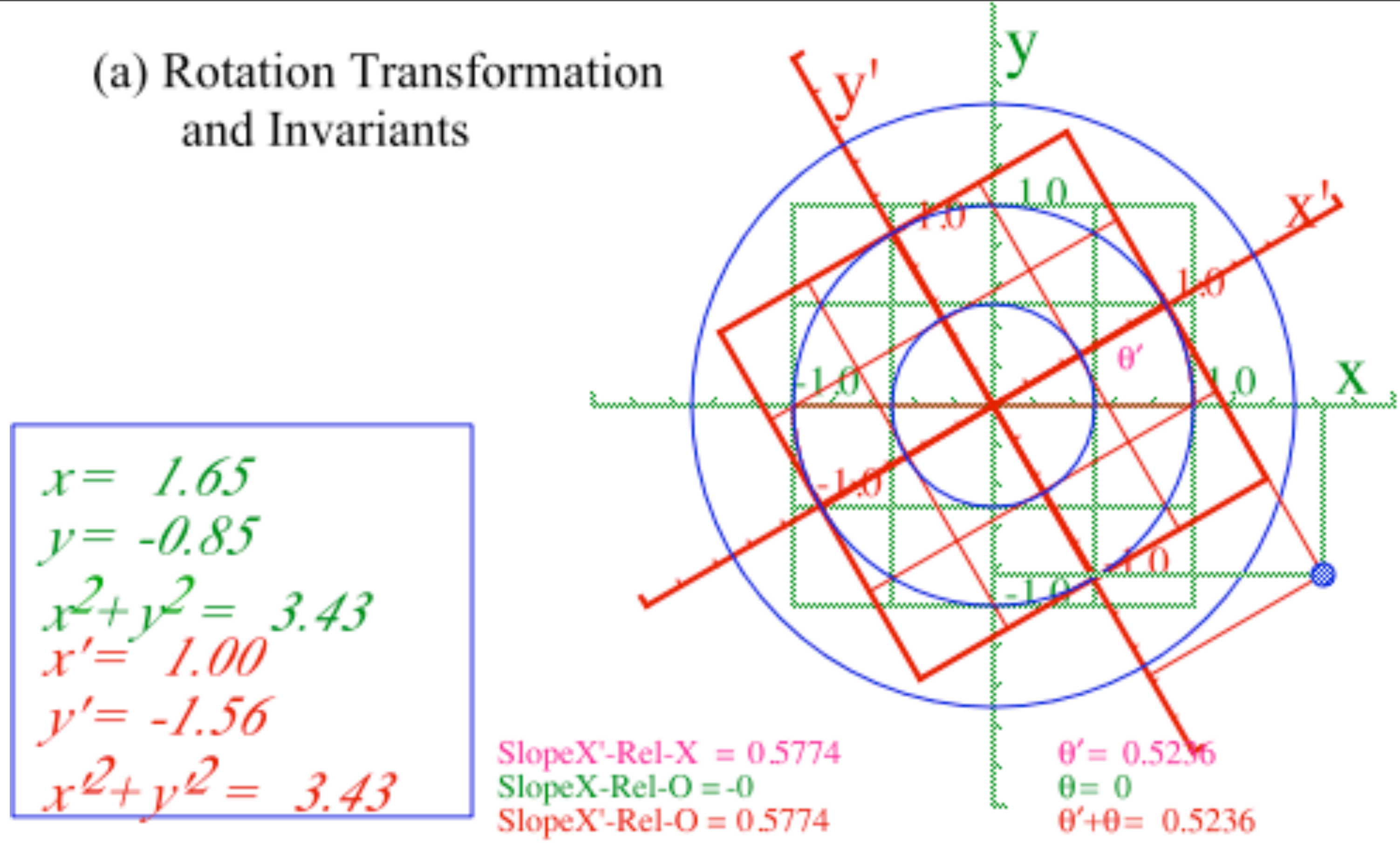
$$\begin{pmatrix} \mathbf{e}_{x'} \\ \mathbf{e}_{y'} \end{pmatrix} = \begin{pmatrix} |x'\rangle \\ |y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix}$$

(This is GARBAGE!)

Here is a matrix representation of abstract definitions: $|x'\rangle \equiv \mathbf{R}|x\rangle$, $|y'\rangle \equiv \mathbf{R}|y\rangle$

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} V_{x'} \\ V_{y'} \end{pmatrix} = \begin{pmatrix} \langle x|\mathbf{R}|x\rangle & \langle x|\mathbf{R}|y\rangle \\ \langle y|\mathbf{R}|x\rangle & \langle y|\mathbf{R}|y\rangle \end{pmatrix} \begin{pmatrix} V_{x'} \\ V_{y'} \end{pmatrix} = \begin{pmatrix} \langle x'|\mathbf{R}|x'\rangle & \langle x'|\mathbf{R}|y'\rangle \\ \langle y'|\mathbf{R}|x'\rangle & \langle y'|\mathbf{R}|y'\rangle \end{pmatrix} \begin{pmatrix} V_{x'} \\ V_{y'} \end{pmatrix}$$

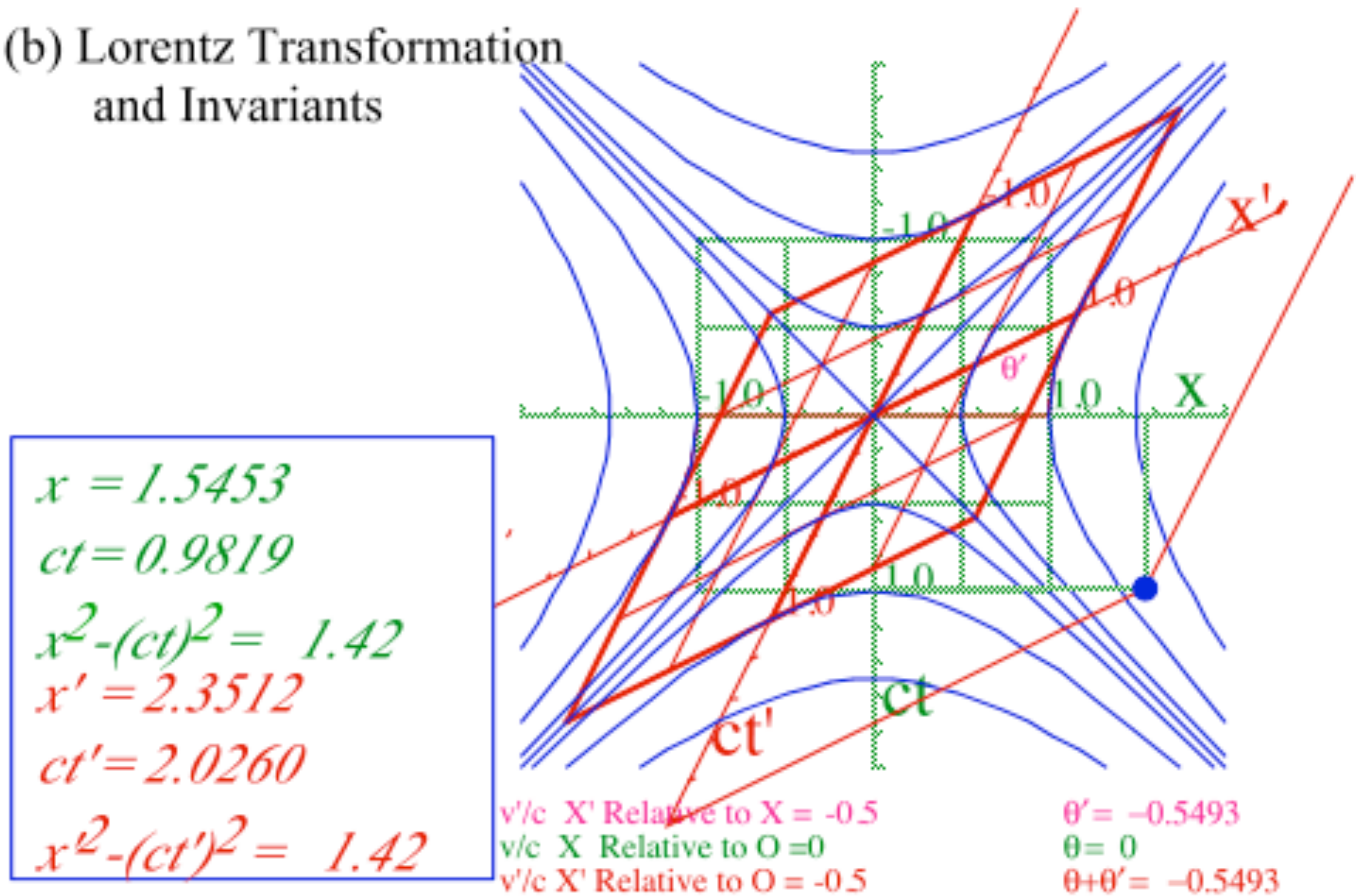
(a) Rotation Transformation and Invariants



$$x' = x\cos\theta - y\sin\theta = \frac{x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{-(b/c)y}{\sqrt{1+\frac{b^2}{c^2}}}$$

$$y' = x\sin\theta + y\cos\theta = \frac{(b/c)x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{y}{\sqrt{1+\frac{b^2}{c^2}}}$$

(b) Lorentz Transformation and Invariants



$$x' = \frac{x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v}{c}ct}{\sqrt{1-\frac{v^2}{c^2}}} = x\cosh\rho + y\sinh\rho$$

$$ct' = \frac{\frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{ct}{\sqrt{1-\frac{v^2}{c^2}}} = x\sinh\rho + y\cosh\rho$$

5. That "old-time" relativity (Circa 600BCE- 1905CE)

("Bouncing-photons" in smoke & mirrors and Thales, again)

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➡ The straight scoop on "angle" and "rapidity" (They're area!)

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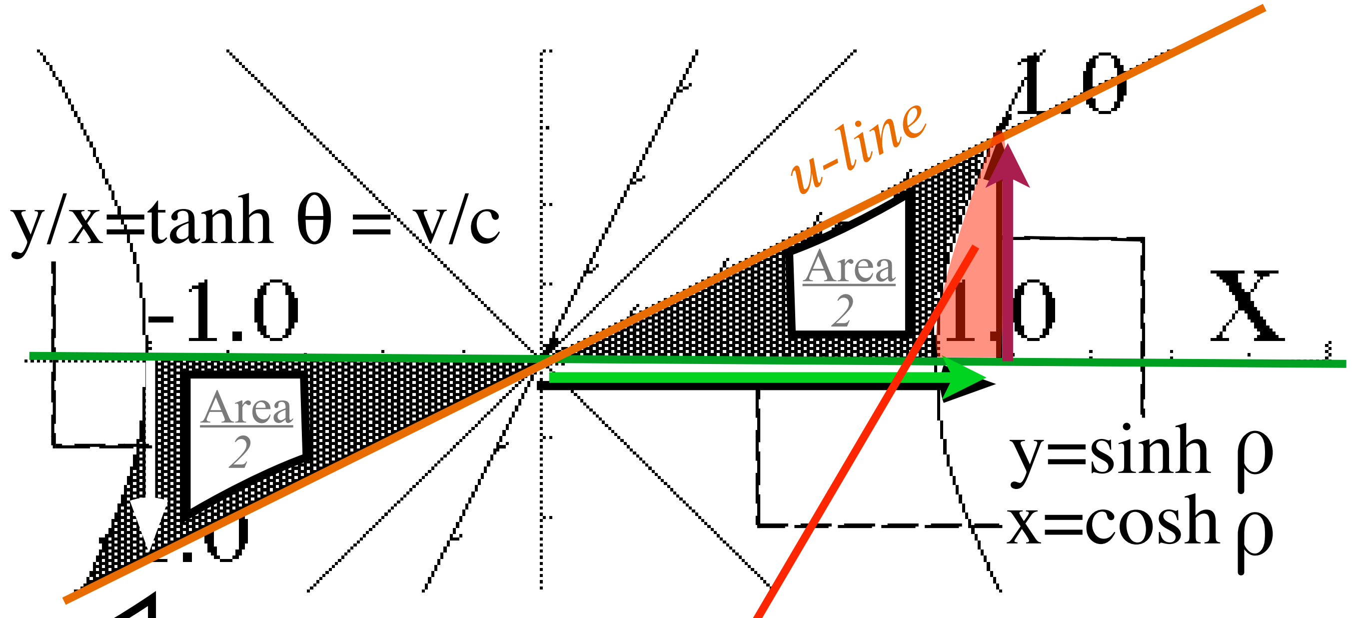
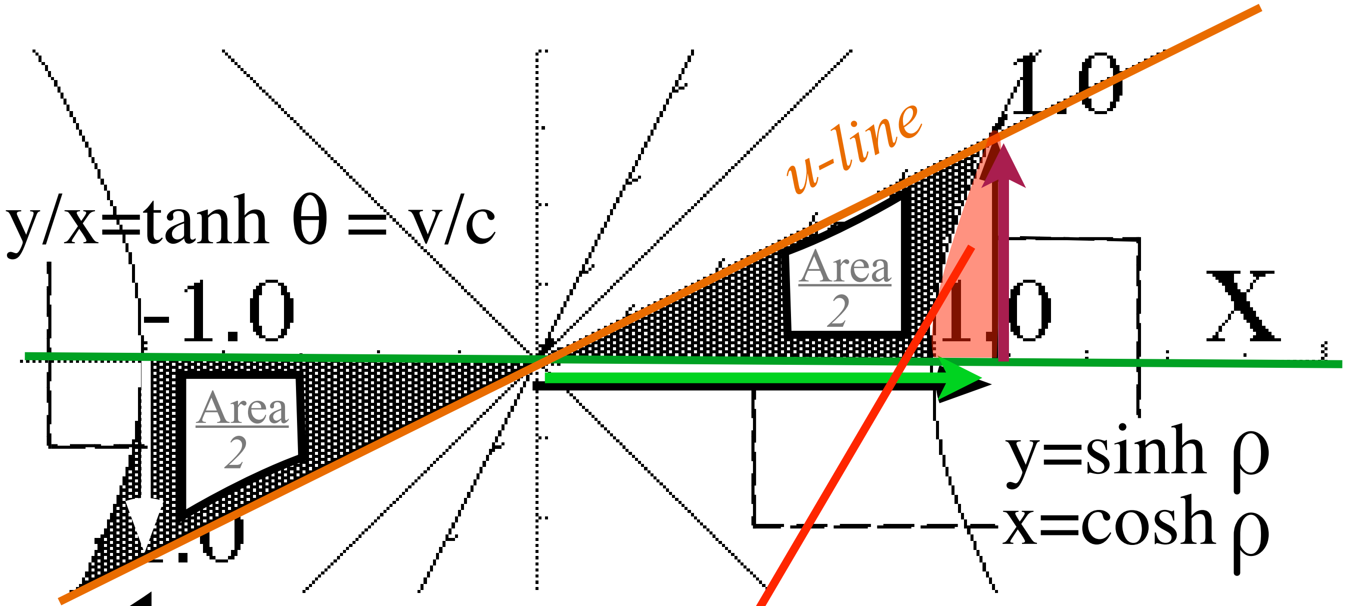


$\frac{\text{Area}}{2} = \frac{1}{2}\,\textit{base}\cdot\textit{altitude} - \text{area under curve} = \frac{1}{2}xy - \int y\,dx$

The "Area" being calculated is the *total* **Gray Area** *between hyperbola pairs, X axis, and sloping u-line*

The straight scoop on "angle" and "rapidity" (They're area!)



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$$\frac{Area}{2} = \frac{1}{2} base \cdot altitude - \text{area under curve} = \frac{1}{2}xy - \int y\,dx$$

$$\frac{Area}{2} = \frac{1}{2}\sinh\rho\cosh\rho - \int \sinh\rho\, d(\cosh\rho)$$

Useful hyperbolic identities

$$\sinh^2\rho = \left(\frac{e^{\rho}-e^{-\rho}}{2}\right)^2 = \frac{1}{4}\left(e^{2\rho}+e^{-2\rho}-2\right) = \frac{\cosh 2\rho - 1}{2}$$

$$\sinh\rho\cosh\rho = \left(\frac{e^{\rho}-e^{-\rho}}{2}\right)\left(\frac{e^{\rho}+e^{-\rho}}{2}\right) = \frac{1}{4}\left(e^{2\rho}-e^{-2\rho}\right) = \frac{1}{2}\sinh 2\rho$$

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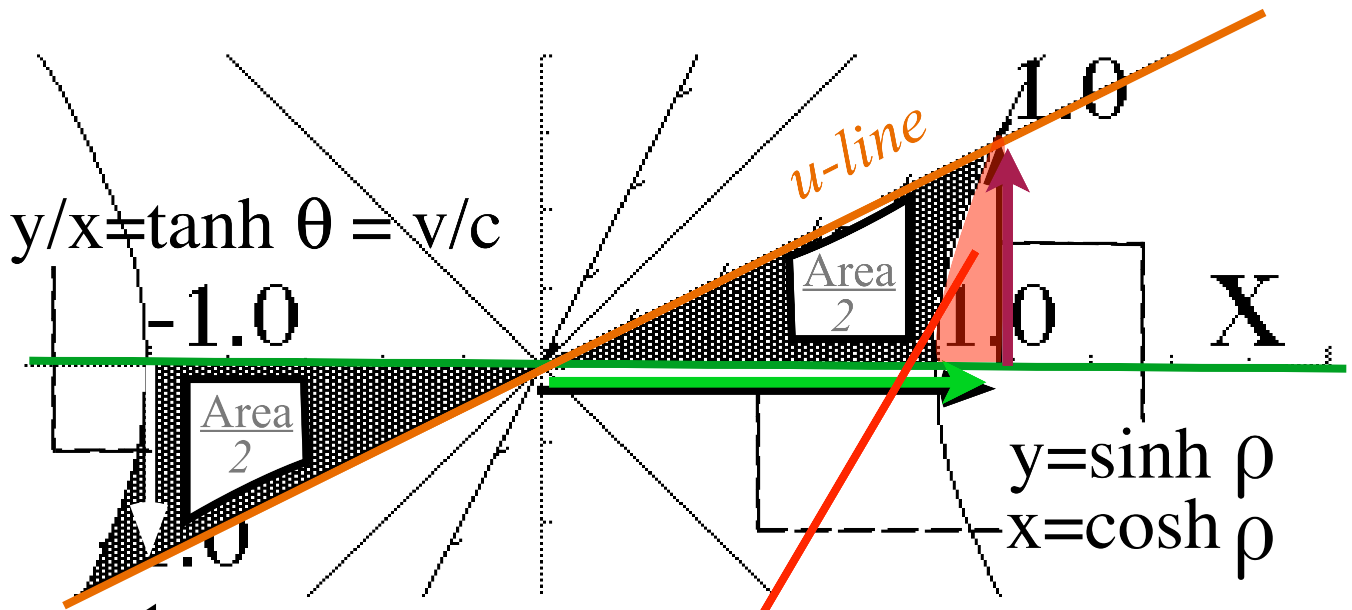

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$$\int \cosh a\rho\, d\rho = \frac{1}{a}\sinh a\rho$$

$$\frac{Area}{2} = \frac{1}{2}\sinh\rho\cosh\rho - \int \sinh^2\rho\, d\rho = \frac{1}{4}\sinh 2\rho - \int \frac{\cosh 2\rho - 1}{2}\, d\rho$$

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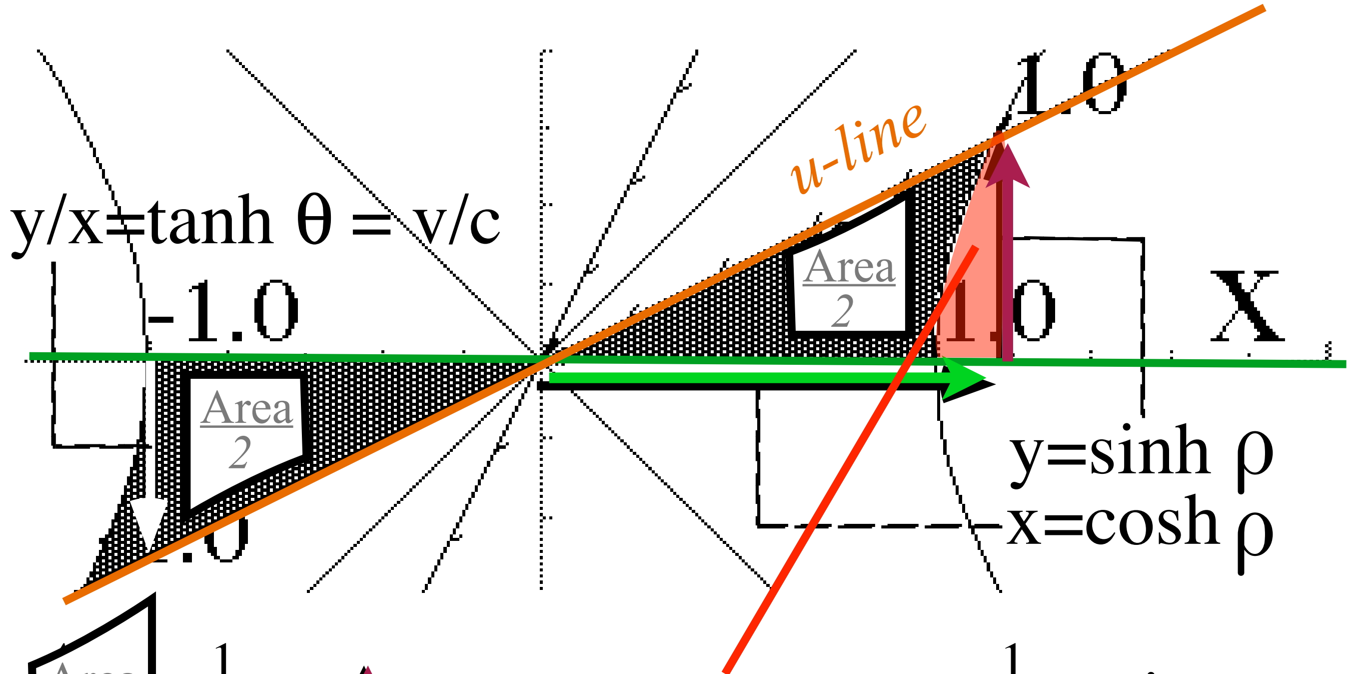

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$$= \frac{1}{4}\sinh 2\rho - \frac{1}{4}\sinh 2\rho + \int \frac{1}{2} d\rho$$

$$= \frac{\rho}{2}$$

Amazing result: Area = ρ is rapidity

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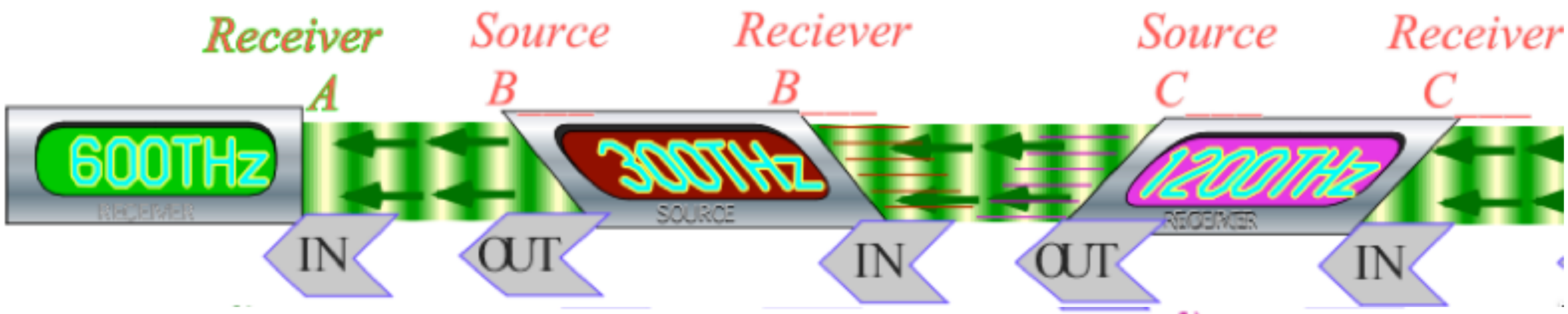
From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires *geometric* Doppler transform: $e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB}+\rho_{BC}}$

Easy to combine frame velocities using *rapidity addition:* $\rho_{u+v} = \rho_u + \rho_v$



$\frac{\upsilon_A}{\upsilon_B} = b_{AB} = e^{\rho_{AB}} = 2$

$\rho_{AB} = ln(2) = 0.69$

$\frac{\upsilon_B}{\upsilon_C} = b_{BC} = e^{\rho_{BC}} = 1/4$

$\rho_{BC} = ln(1/4) = -1.38$

$\frac{\upsilon_C}{\upsilon_A} = b_{CA} = e^{\rho_{CA}} = 2$

$\rho_{CA} = ln(2) = 0.69$

$$\rho_{AB} + \rho_{BC} = \rho_{AC} = -\rho_{CA}$$

$$0.69 - 1.38 = -0.69$$

Galilean velocity addition becomes rapidity addition

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$$\frac{u'}{c} = \tanh(\rho_u + \rho_v) = \frac{\tanh\rho_u + \tanh\rho_v}{1 + \tanh\rho_u \tanh\rho_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c}\frac{v}{c}}$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

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No longer does *(1/2+1/2)c* equal *(1)c…*

Relativistic result is: $$\frac{\frac{1}{2}+\frac{1}{2}}{1+\frac{1}{2}\frac{1}{2}}c = \frac{1}{1+\frac{1}{4}}c = \frac{1}{\frac{5}{4}}c = \frac{4}{5}c$$

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...but, *(1/2+1)c* <u>does</u> equal *(1)c…*

$$\frac{\frac{1}{2}+1}{1+\frac{1}{2}1}c = c$$

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(a) Circlular Functions
(plane geometry)

www.uark.edu/ua/pirelli/php/complex_phasors_1.php

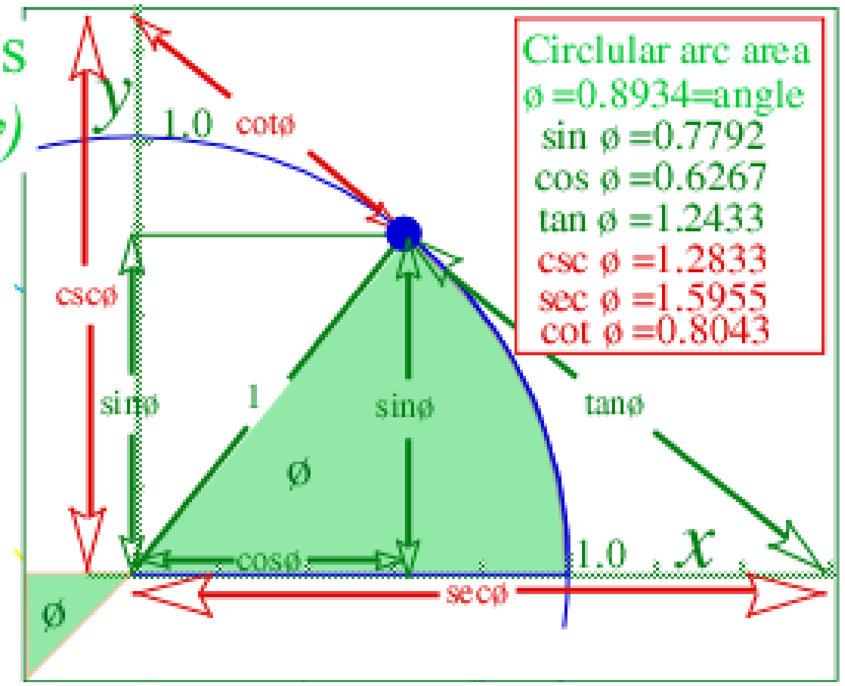

Fig. C.2-3
and
Fig. 5.4
in Unit 2

Introducing the "Sin-Tan Rosetta Stone" *NOTE: Angle ϕ is now called stellar aberration angle σ*

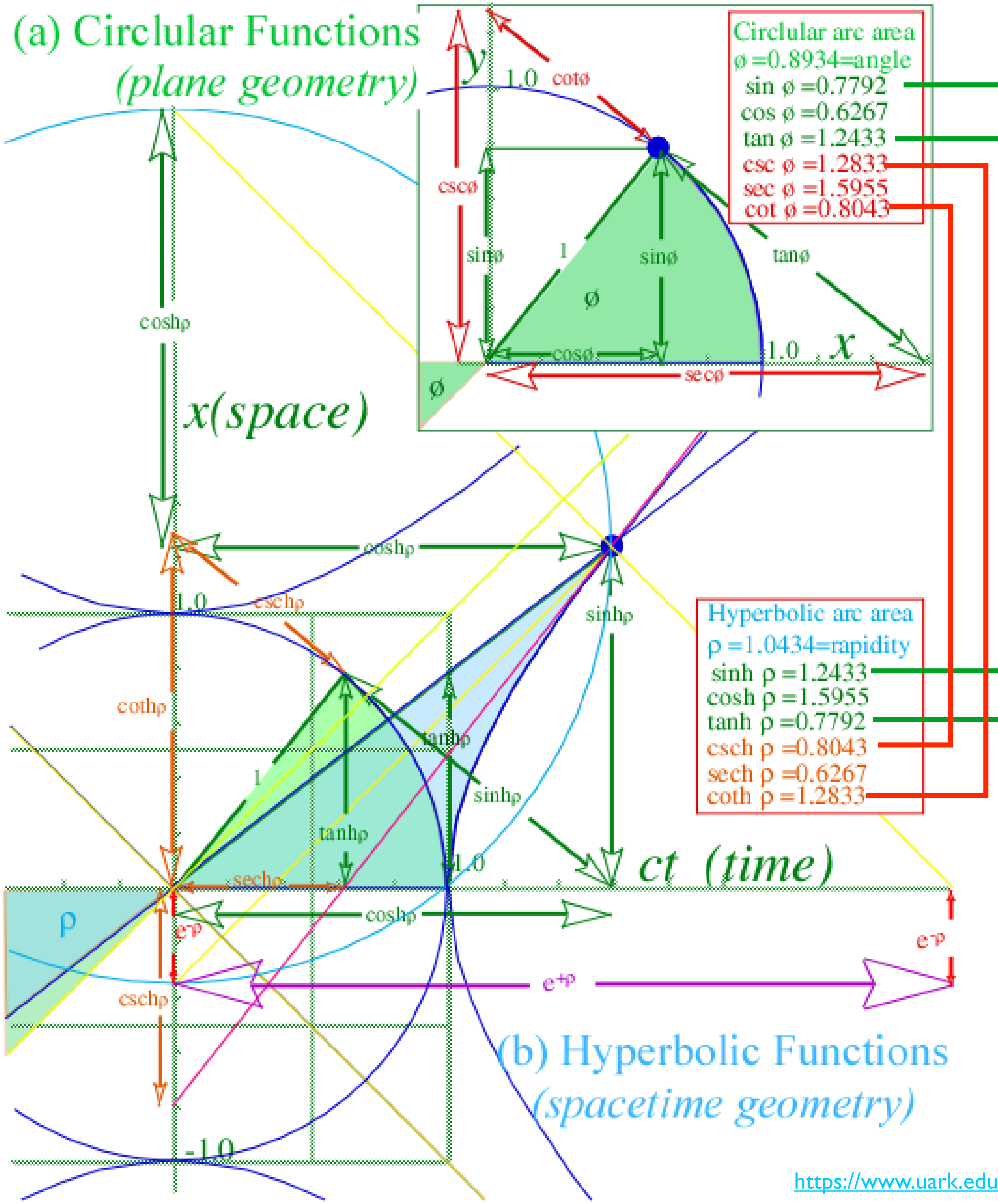

Fig. C.2-3 and Fig. 5.4 in Unit 2

https://www.uark.edu/ua/pirelli/php/hyper_constrct.php

Introducing the "Sin-Tan Rosetta Stone"

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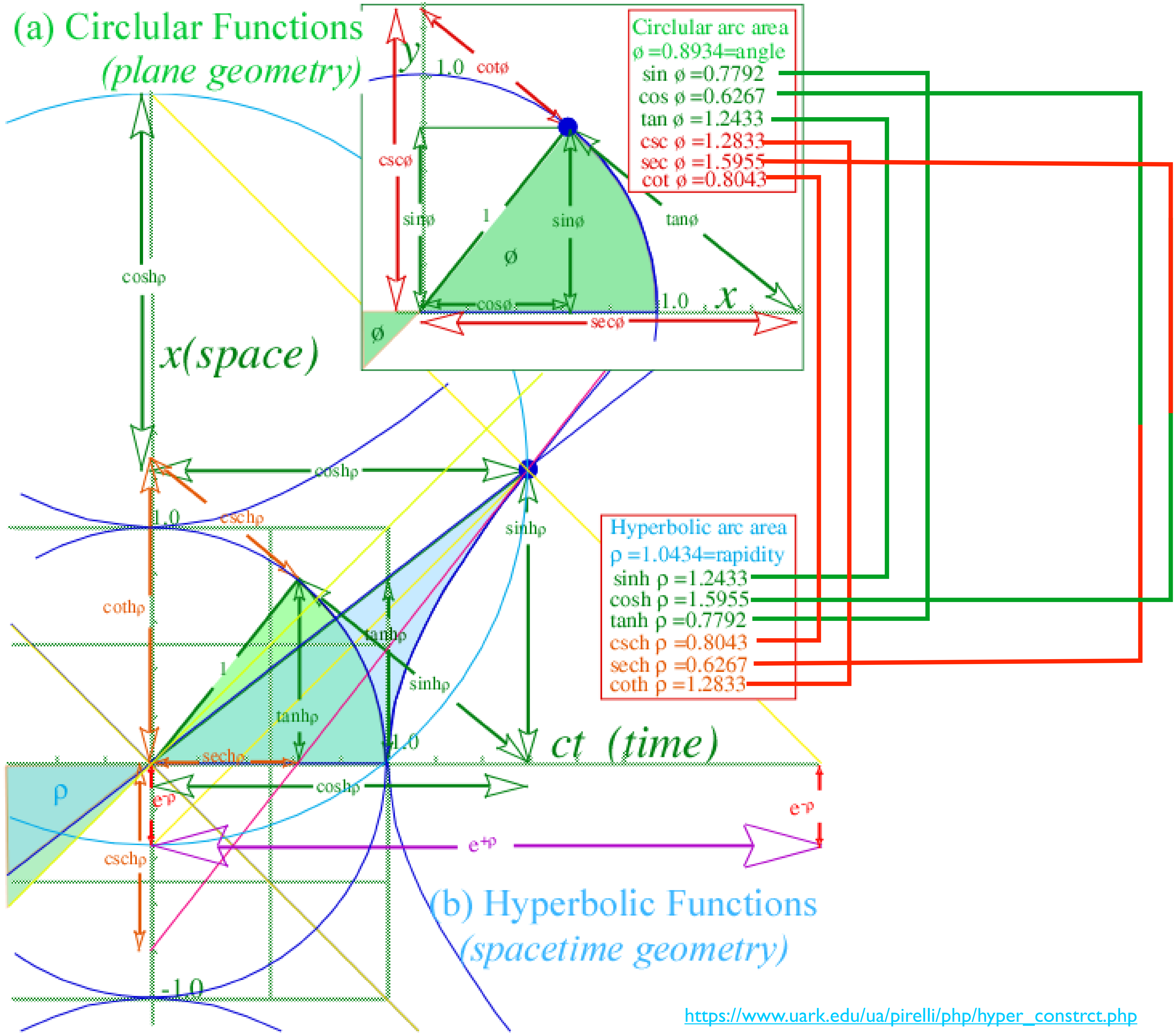

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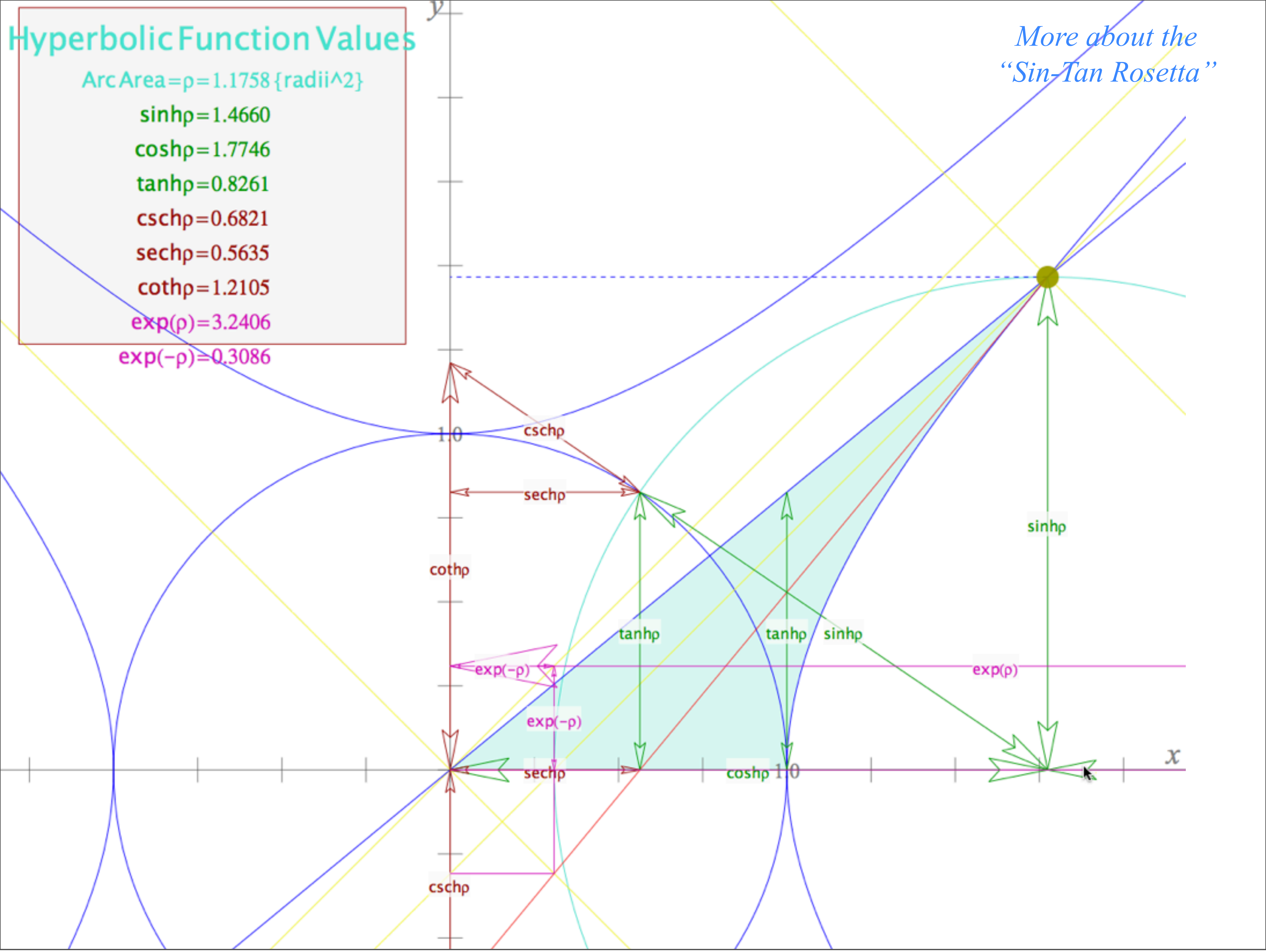
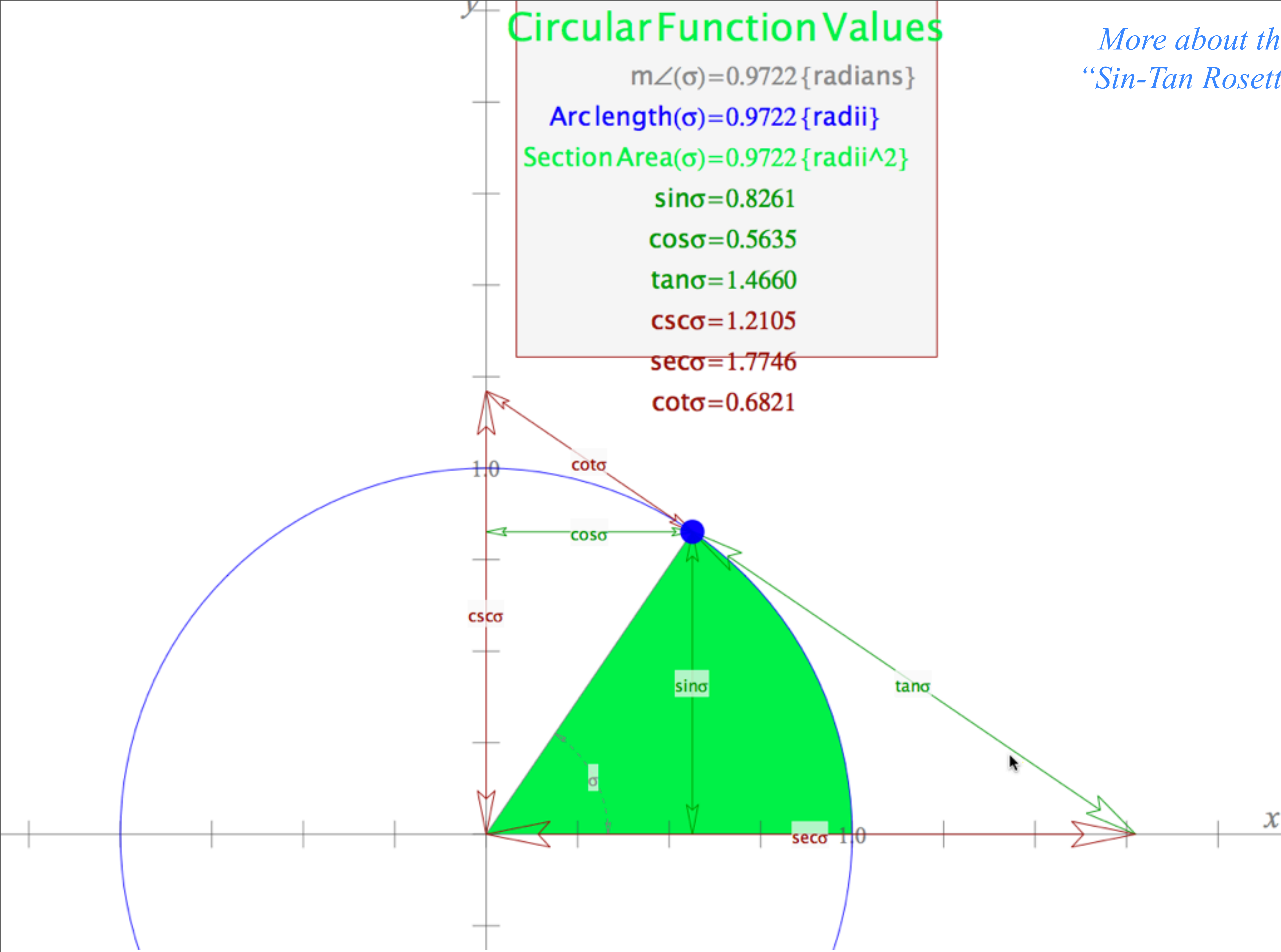
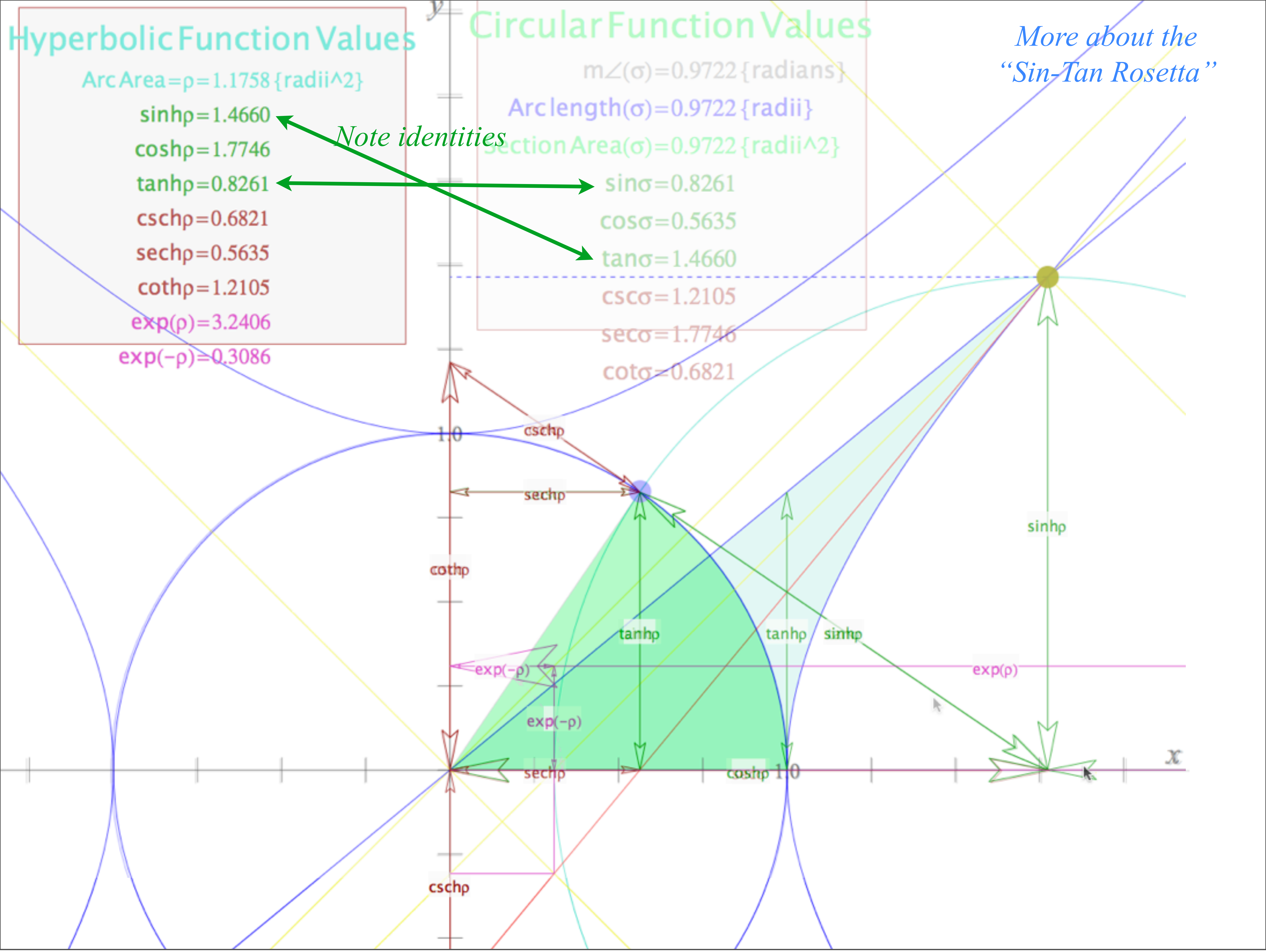


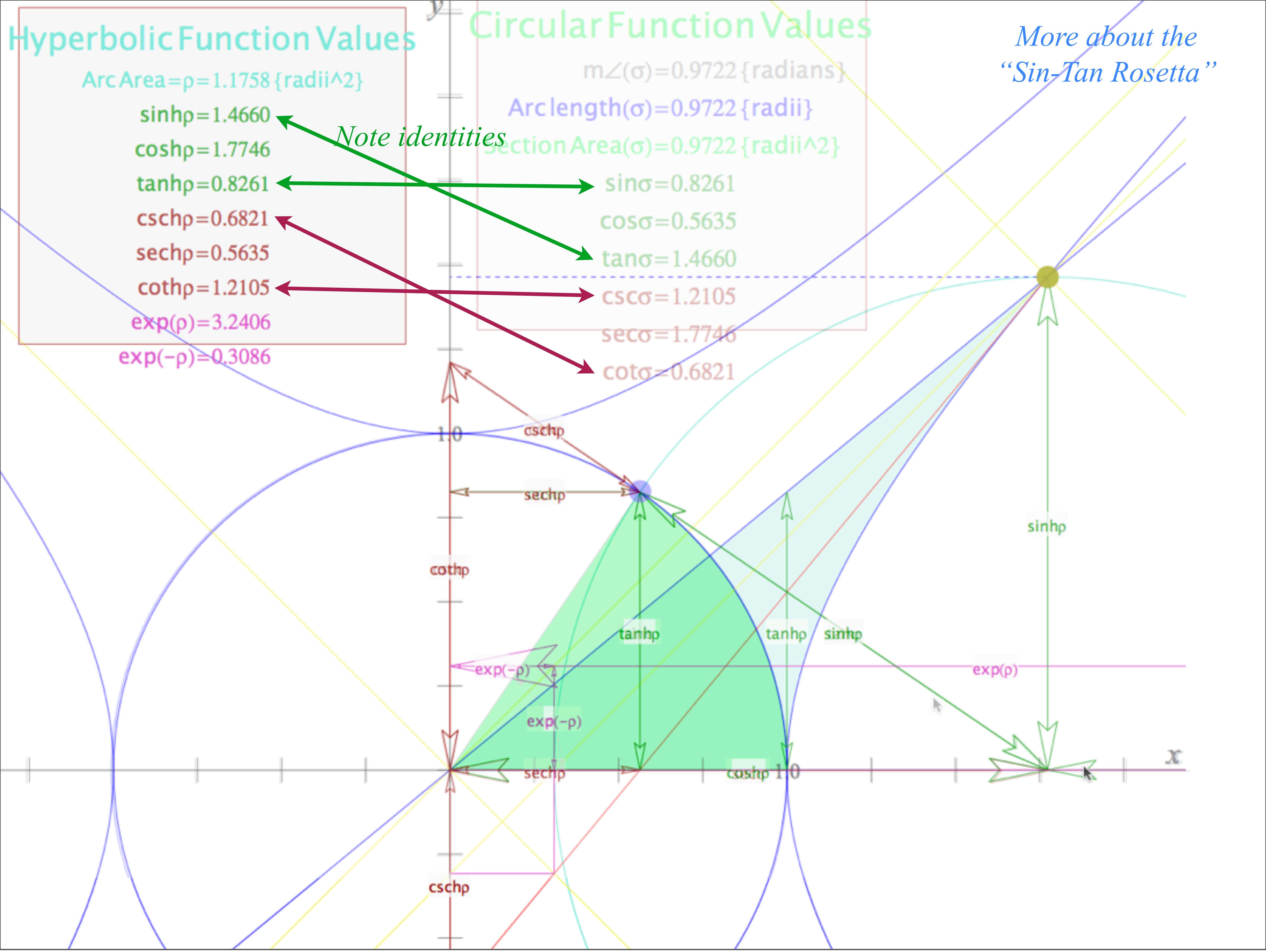
Hyperbolic Function Values
ArcArea=ρ=1.1758 {radii^2}
sinhρ=1.4660
coshρ=1.7746
tanhρ=0.8261
cschρ=0.6821
sechρ=0.5635
cothρ=1.2105
exp(ρ)=3.2406
exp(−ρ)=0.3086
More about the
"Sin-Tan Rosetta"
y
x
1.0
1.0
cschρ
sechρ
cothρ
tanhρ
tanhρ
sinhρ
sinhρ
exp(−ρ)
exp(ρ)
exp(−ρ)
sechρ
coshρ
cschρ

More about the "Sin-Tan Rosetta"

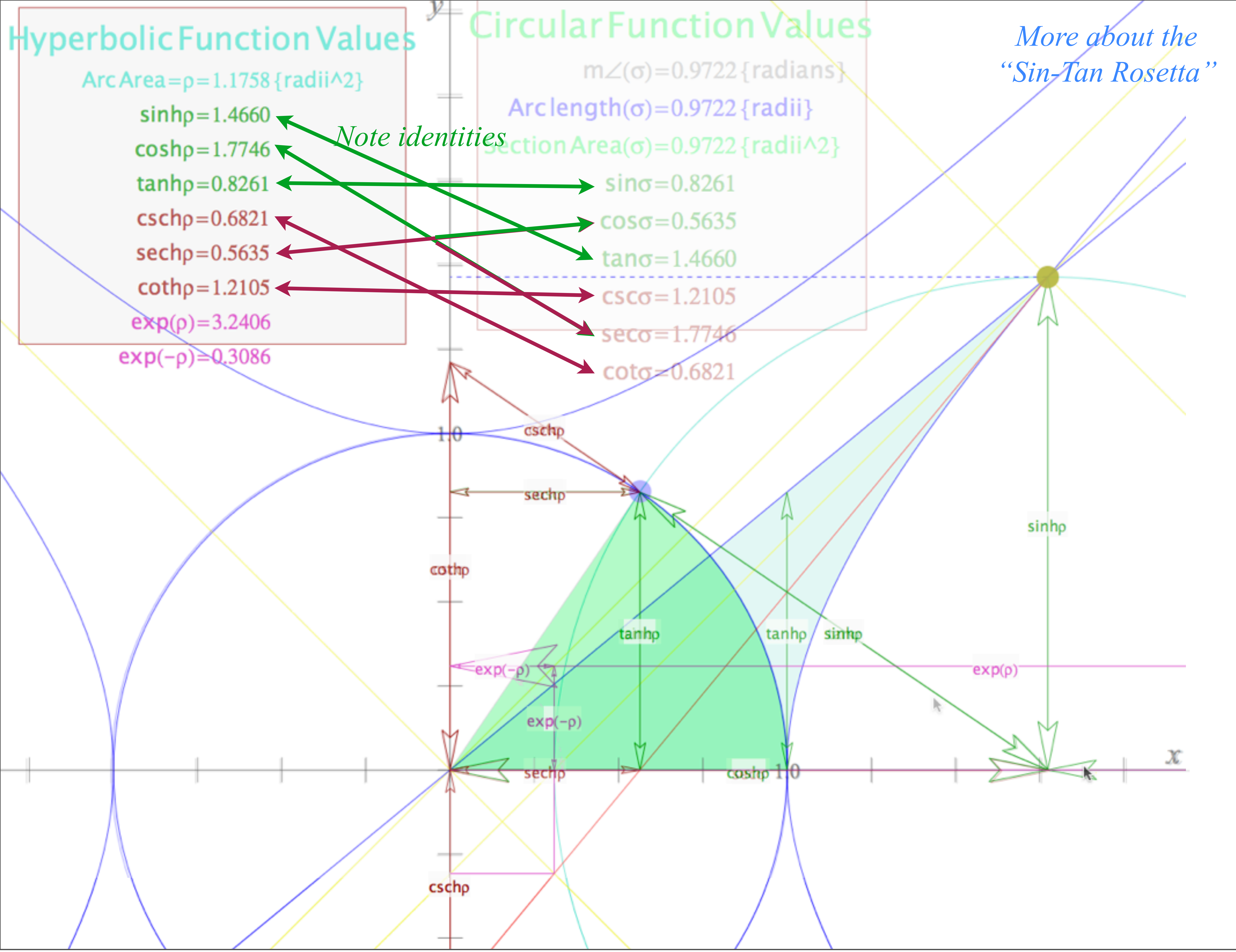


More about the "Sin-Tan Rosetta"




Hyperbolic Function Values
Arc Area=ρ=1.1758{radii^2}
sinhρ=1.4660
coshρ=1.7746
tanhρ=0.8261
cschρ=0.6821
sechρ=0.5635
cothρ=1.2105
exp(ρ)=3.2406
exp(−ρ)=0.3086
Circular Function Values
m∠(σ)=0.9722{radians}
Arc length(σ)=0.9722{radii}
Section Area(σ)=0.9722{radii^2}
sinσ=0.8261
cosσ=0.5635
tanσ=1.4660
cscσ=1.2105
secσ=1.7746
cotσ=0.6821
Note identities
More about the
"Sin-Tan Rosetta"
y
x
1.0
cschρ
sechρ
cothρ
tanhρ
tanhρ
sinhρ
sinhρ
exp(−ρ)
exp(−ρ)
exp(ρ)
sechρ
coshρ
1.0
cschρ

More about the "Sin-Tan Rosetta"



5. That "old-time" relativity *(Circa 600BCE- 1905CE)*

("Bouncing-photons" in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga

Light-conic-sections make invariants

A politically incorrect analogy of rotational transformation and Lorentz transformation

The straight scoop on "angle" and "rapidity" (They're area!)

Galilean velocity addition becomes rapidity addition

Introducing the "Sin-Tan Rosetta Stone" (Thanks, Thales!)

➡ *Introducing the stellar aberration angle σ vs. rapidity ρ*

How Minkowski's space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts

Introducing the stellar aberration angle σ vs. rapidity ρ

Together, rapidity ρ=ln b and stellar aberration angle σ are parameters of relative velocity

The rapidity ρ=ln b is based on longitudinal wave Doppler shift $b=e^{\rho}$ defined by $u/c=\tanh(\rho)$.

At low speed: $u/c\sim\rho$.

The stellar aberration angle σ is based on the transverse wave rotation $R=e^{i\sigma}$ defined by $u/c=\sin(\sigma)$.

At low speed: $u/c\sim\sigma$.

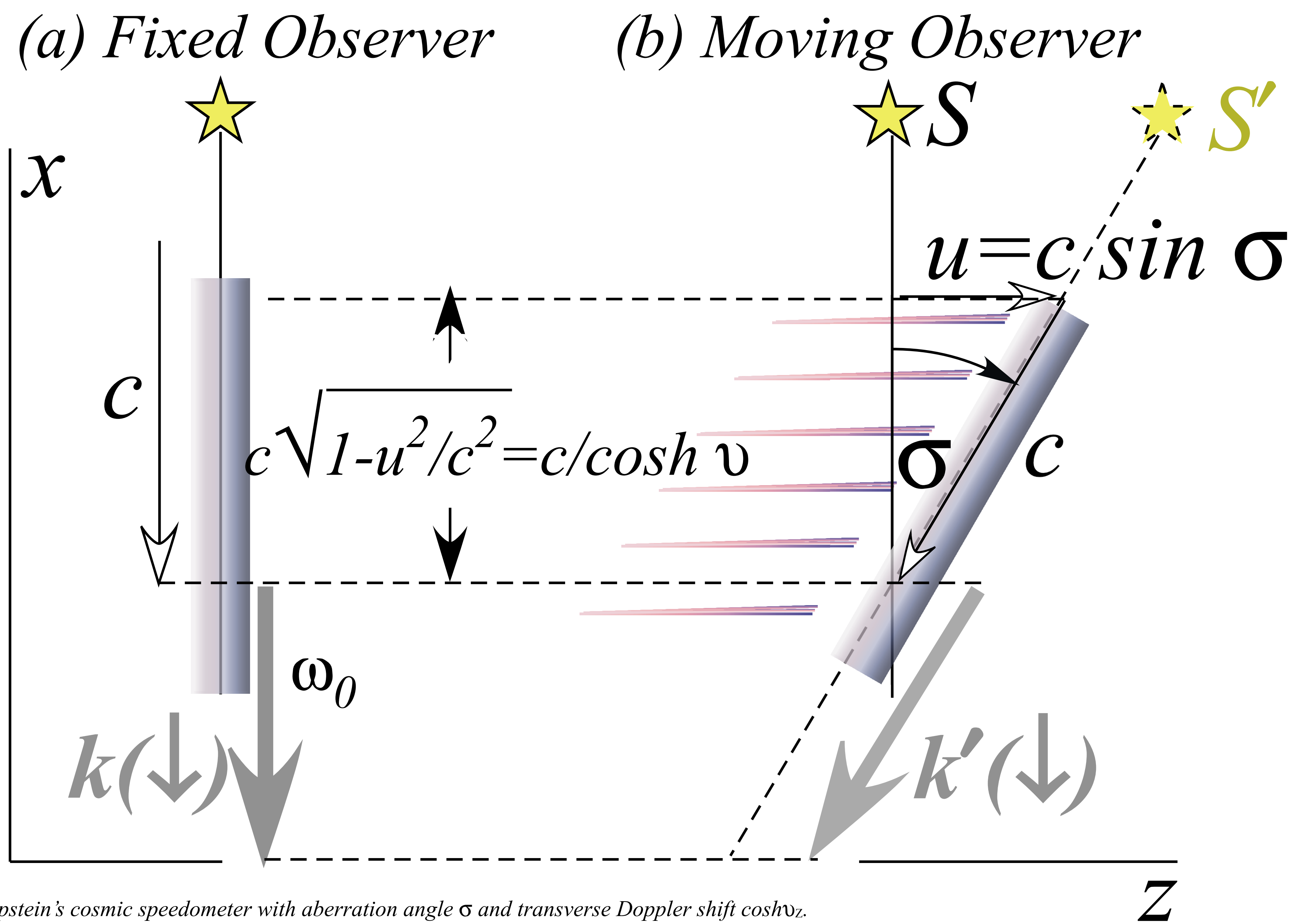


Fig. 5.6 Epstein's cosmic speedometer with aberration angle σ and transverse Doppler shift $\cosh\upsilon_Z$.

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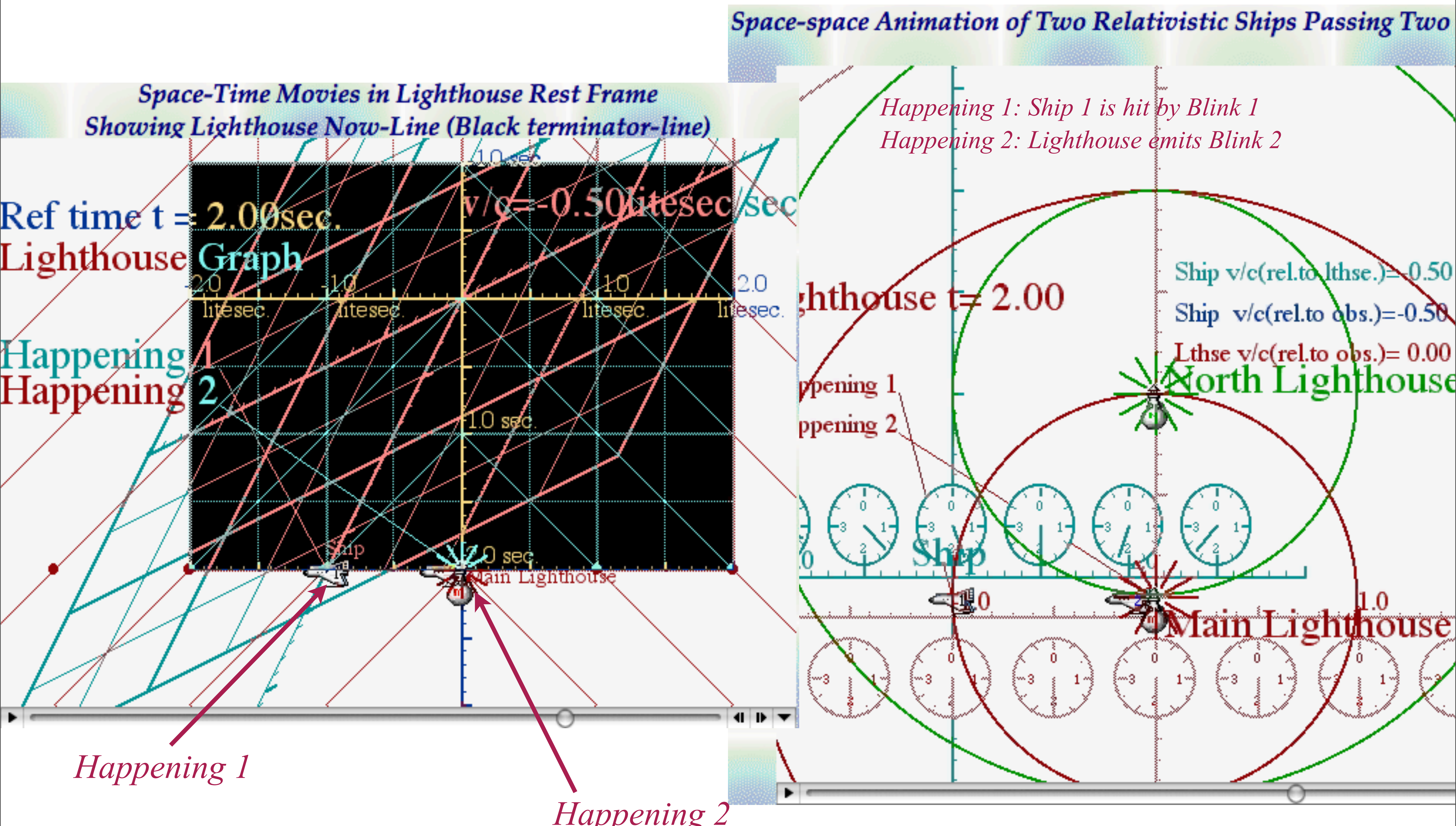
→ *How Minkowski's space-time graphs help visualize relativity*

Group vs. phase velocity and tangent contacts

How Minkowski's space-time graphs help visualize relativity

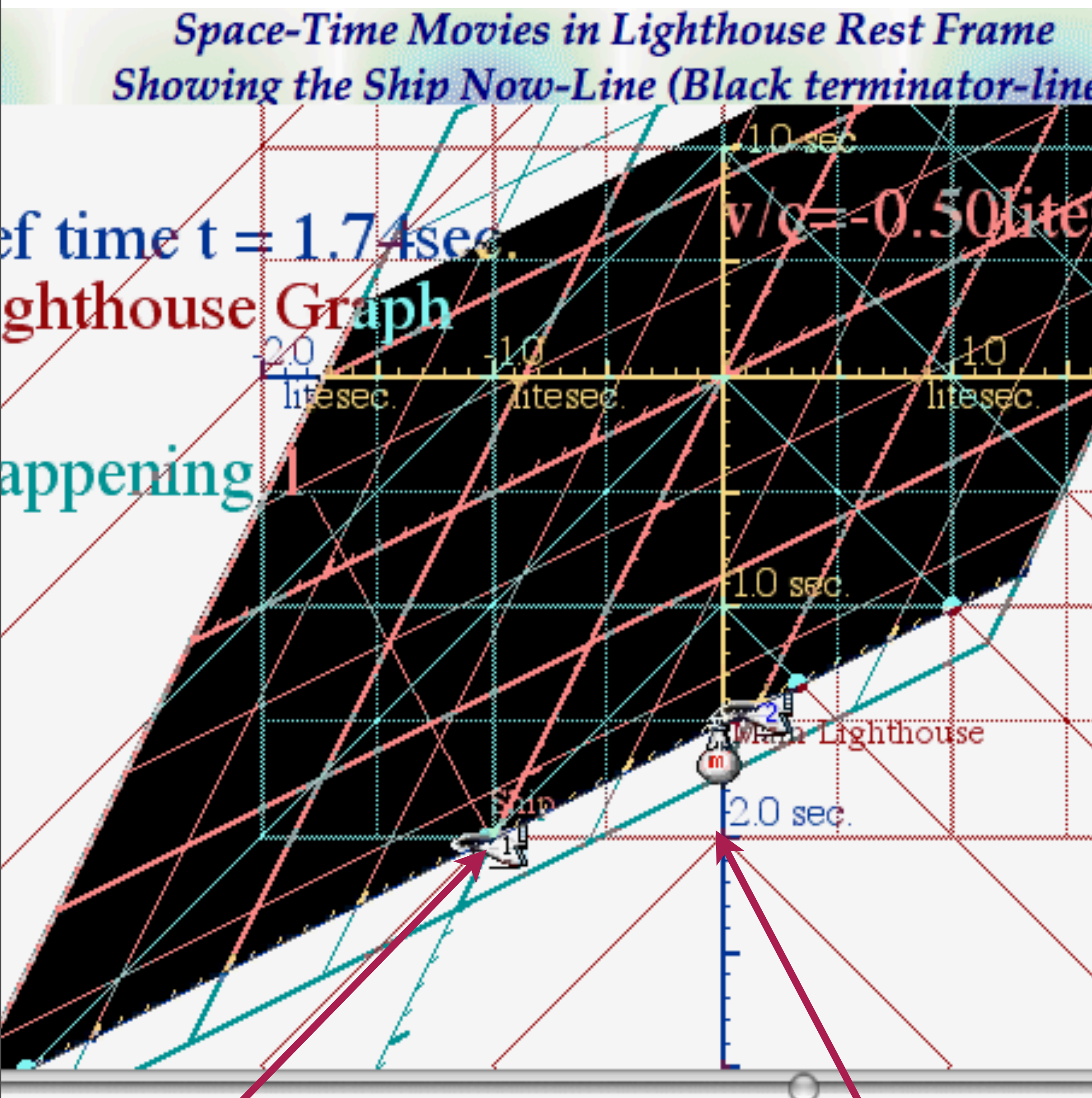
Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at t=2.00sec.



www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php

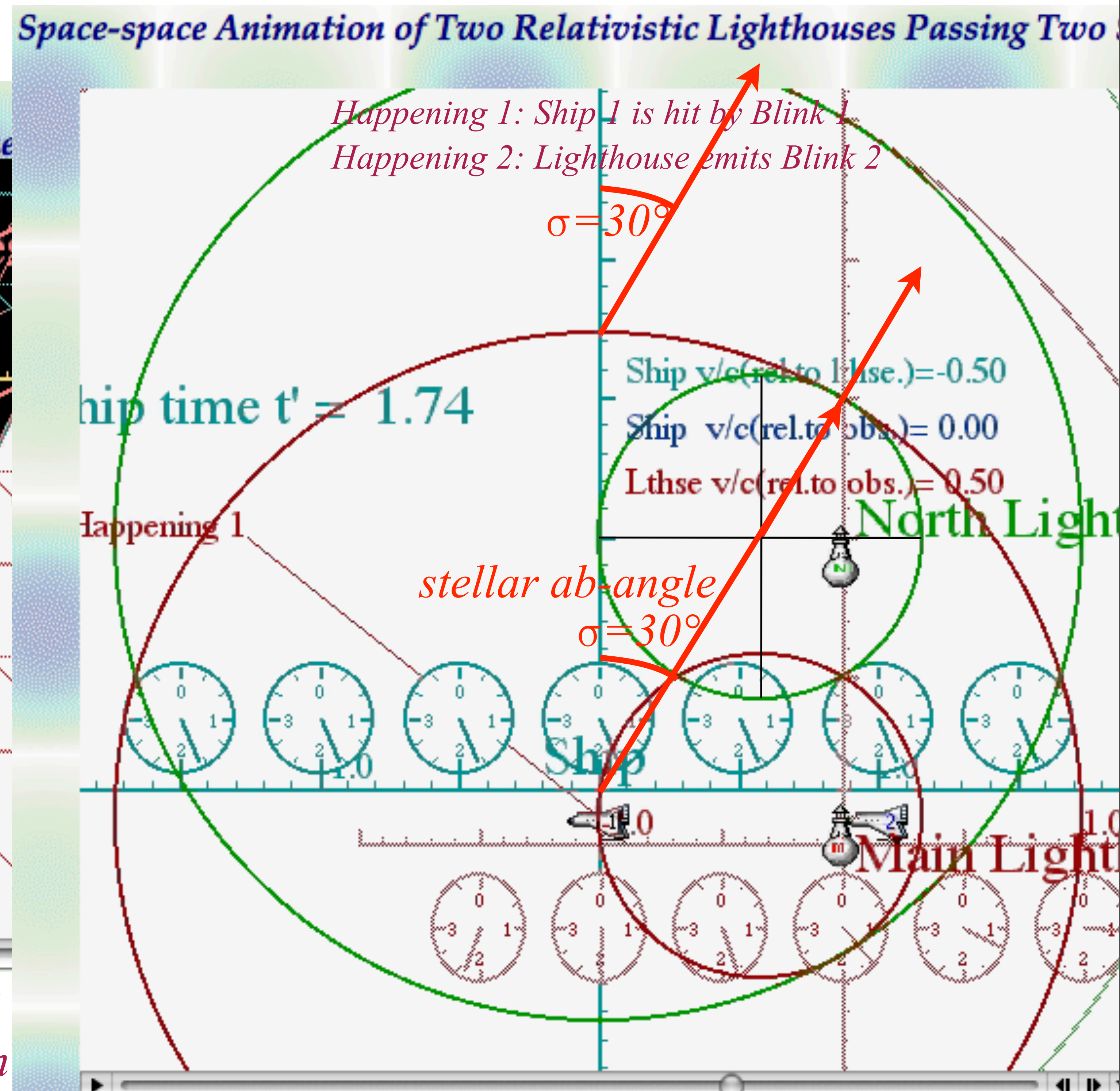
How Minkowski's space-time graphs help visualize relativity (Here:r=atanh(1/2)=0.549,

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at t=2.00sec.
...but, in Ship frame Happening 1 is at t′=1.74 and Happening 2 is at t′=2.30sec.



Happening 1

Happening 2
won't happen
'til t=2.00

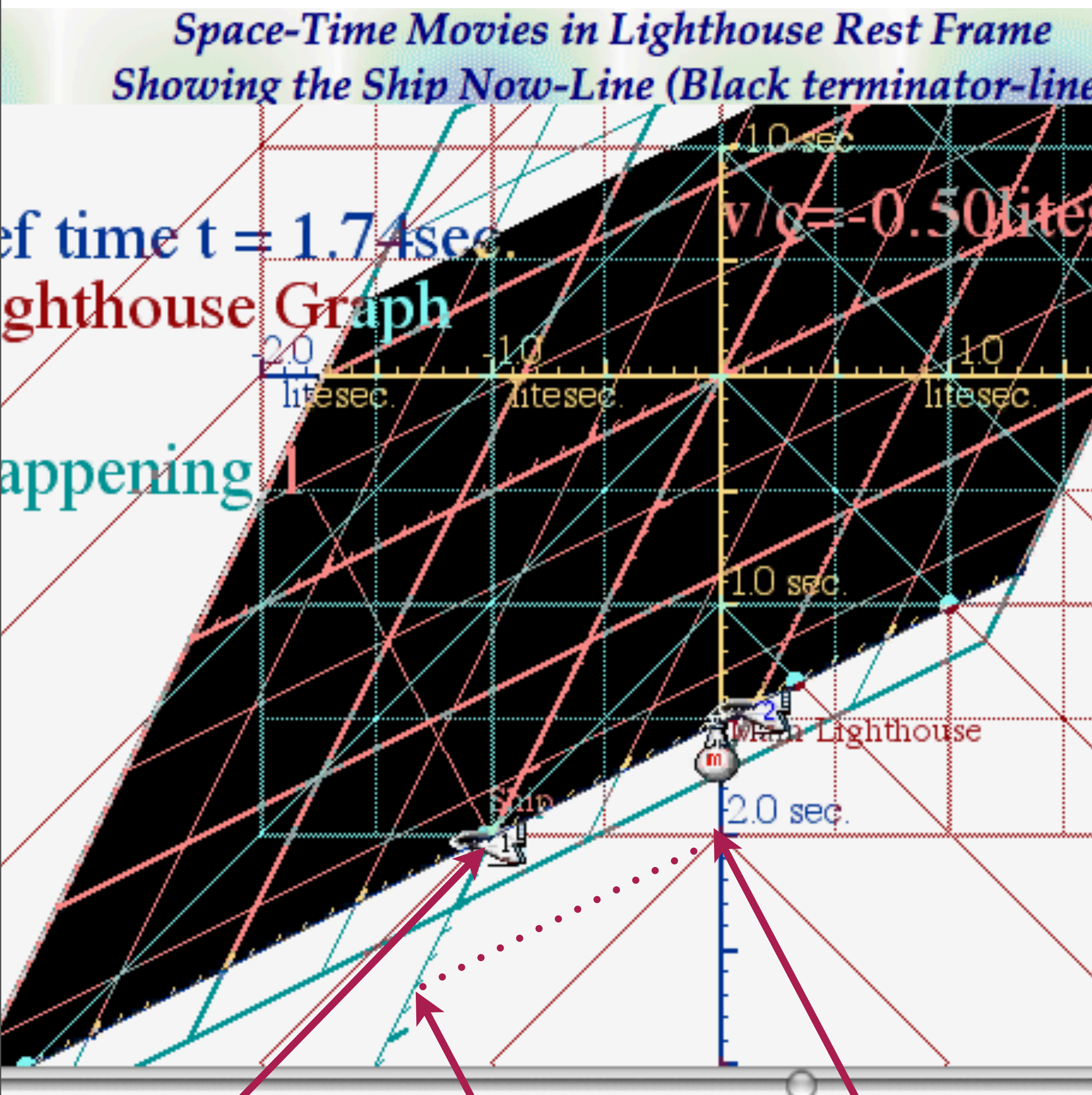


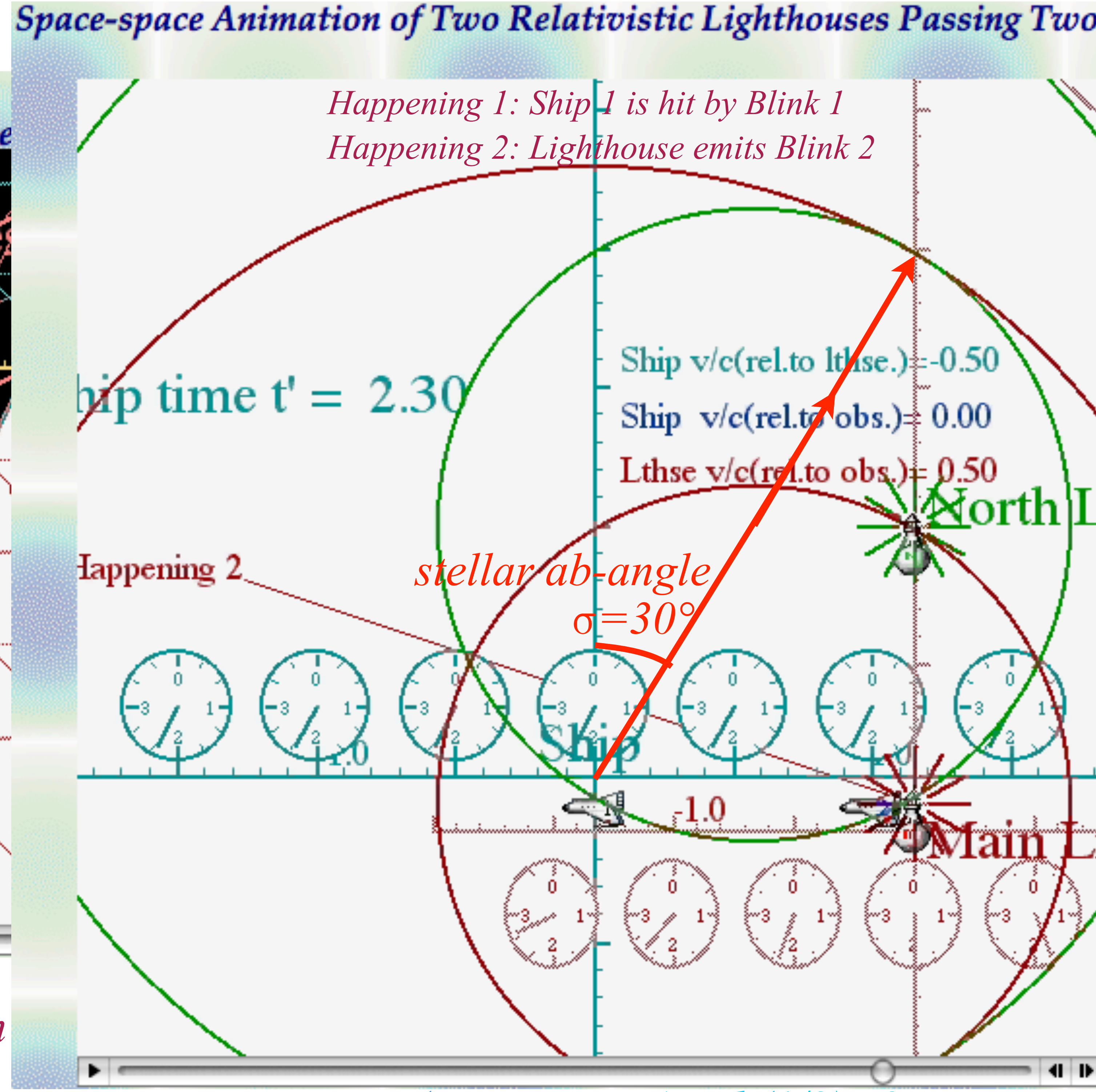
(Here: ρ=Atanh(1/2)=0.55,
and: σ=Asin(1/2)=0.52 or 30°)

www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at t=2.00sec.

...but, in Ship frame Happening 1 is at t′=1.74 and Happening 2 is at t′=2.30sec.





Happening 1

Happening 2 won't happen 'til t=2.00

That is t′=2.30 ship time

(Here: ρ=Atanh(1/2)=0.55,

and: σ=Asin(1/2)=0.52 or 30°)

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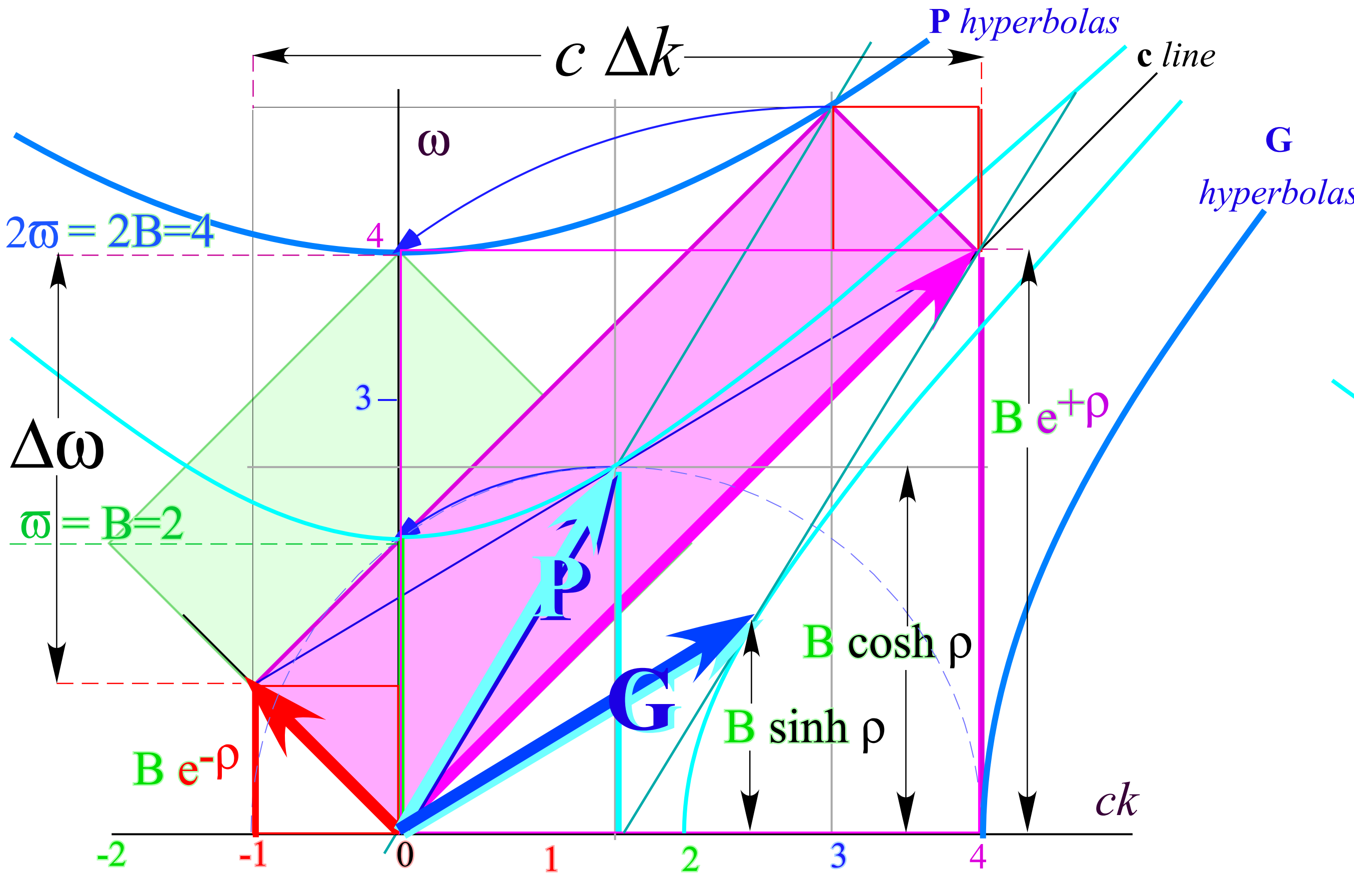
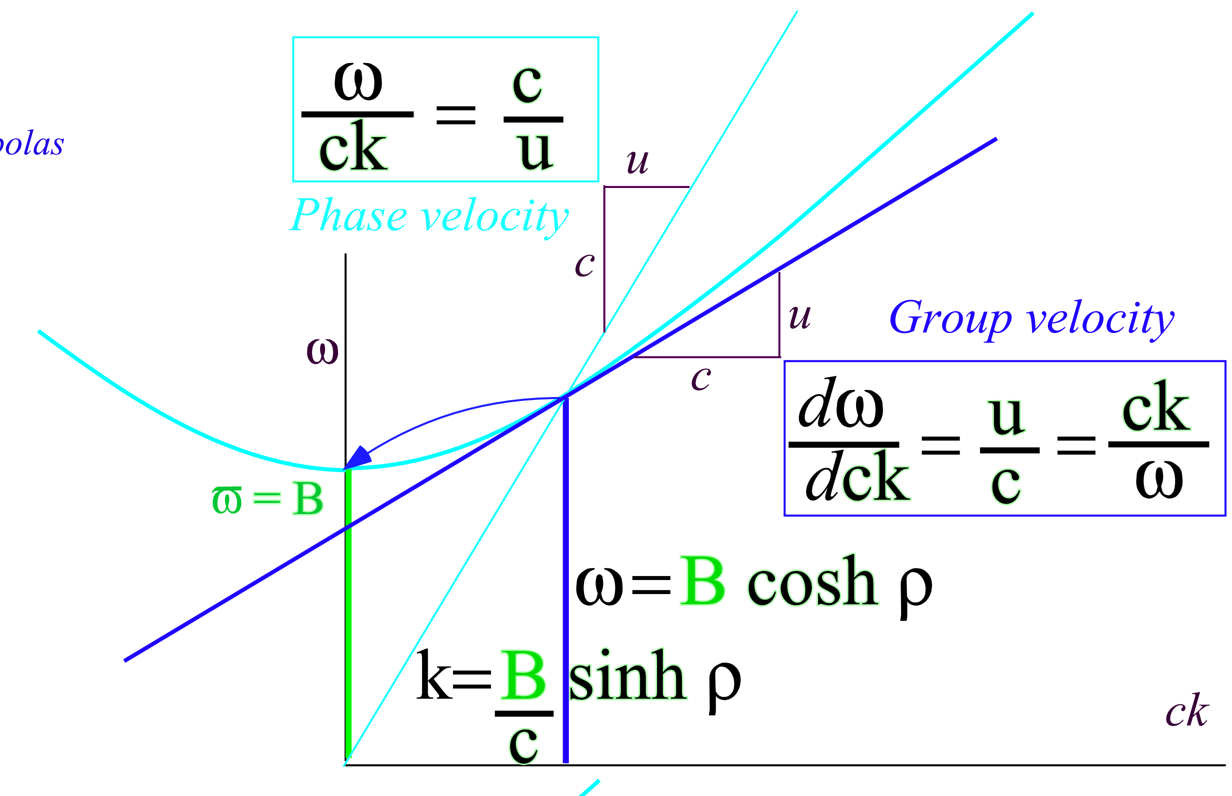
How Minkowski's space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts

Group velocity u and phase velocity c^2/u are hyperbolic tangent slopes

(From Fig. 2.3.4)

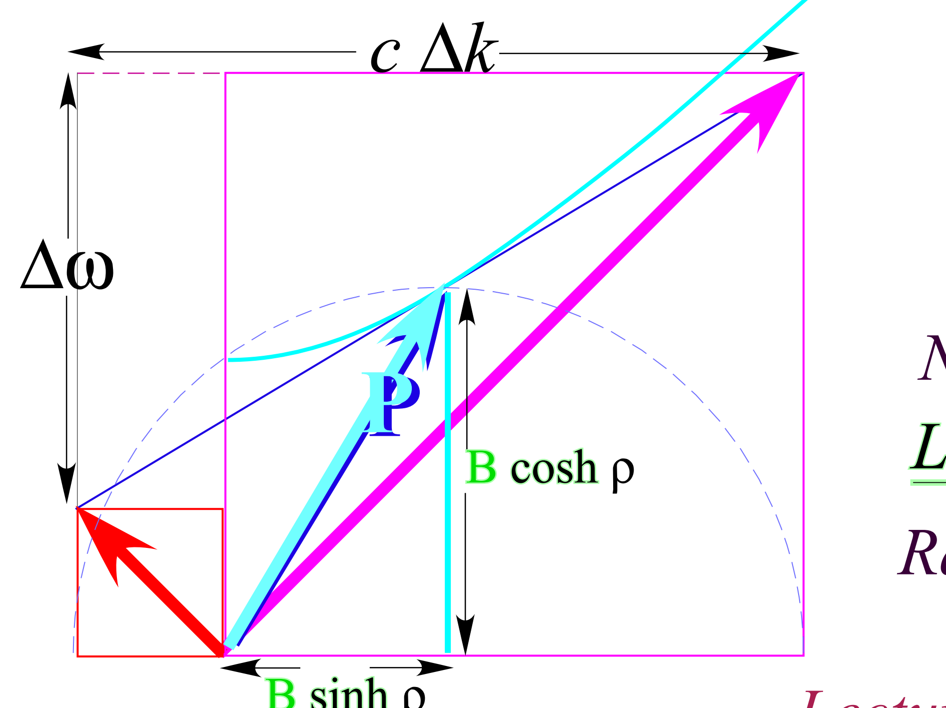




Rare but important case where

$$\frac{d\omega}{dk} = \frac{\Delta\omega}{\Delta k}$$

with LARGE Δk
(not infinitesimal)



Relativistic group wave speed u=c tanh ρ

Newtonian speed u~cρ
Low speed approximation
Rapidity ρ approaches u/c

Lecture 25 ended here

Group vs. phase velocity and tangent contacts

Phase velocity

$$\frac{c^2}{u} = u_{phase} = \frac{\omega}{k} = \frac{c^2 \Delta k}{\Delta\omega} = c \coth\rho$$

Phase-line slope

$$\frac{\omega}{ck} = \frac{c}{u} = \frac{u_{phase}}{c} = \frac{c\Delta k}{\Delta\omega}$$

$$= \coth\rho$$

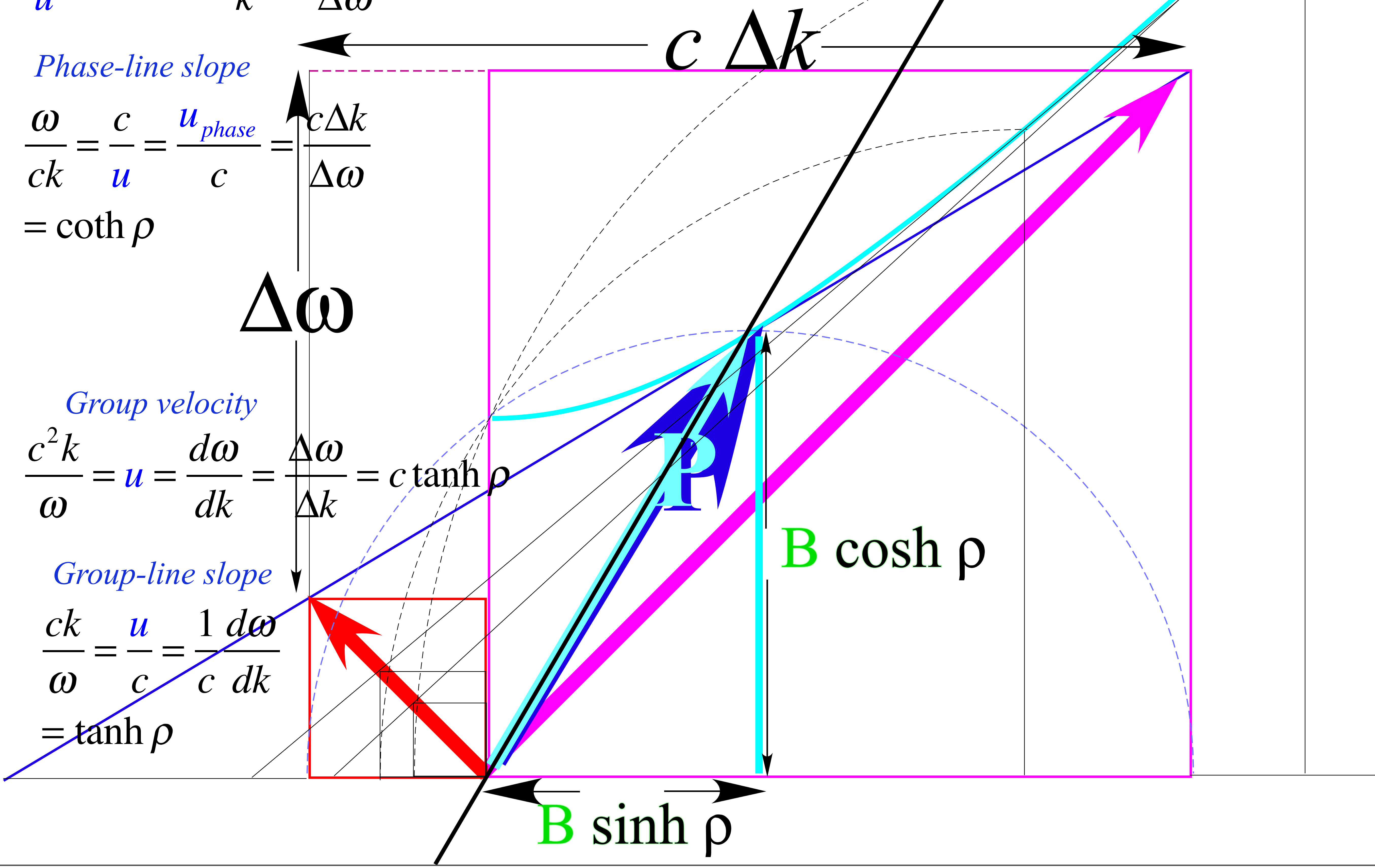
Group velocity

$$\frac{c^2 k}{\omega} = u = \frac{d\omega}{dk} = \frac{\Delta\omega}{\Delta k} = c \tanh\rho$$

Group-line slope

$$\frac{ck}{\omega} = \frac{u}{c} = \frac{1}{c}\frac{d\omega}{dk}$$

$$= \tanh\rho$$



Group vs. phase velocity and tangent contacts