DAMOP - 2015

Special Relativity and Quantum Mechanics by Ruler and Compass II.

The simplest "molecule": Relativistic mechanics by optical coherence geometry

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(A) Using wave parameters to quickly derive Planck (1900), Einstein (1905), and DeBroglie (1921) formulation

$\upsilon_{phase} = B\cosh\rho = B + \frac{1}{2}B\rho^2$ (for $u \ll c$) $\quad \cosh\rho = 1 + \frac{1}{2}\rho^2 = 1 + \frac{u^2}{2c^2}$

$\kappa_{phase} = B\sinh\rho = B\rho$ (for $u \ll c$) $\quad \sinh\rho = \rho = \frac{u}{c}$

$\frac{u}{c} = \tanh\rho = \rho$ (for $u \ll c$)

$\upsilon_{phase} = B + \frac{1}{2}\frac{B}{c^2}u^2 \Leftarrow$ for $(u \ll c) \Rightarrow \quad \kappa_{phase} = \frac{B}{c^2}u$

(Famous Mc^2 shows up here!)

$h\upsilon_{phase} = hB + \frac{1}{2}\frac{hB}{c^2}u^2 \Leftarrow$ for $(u \ll c) \Rightarrow \quad h\kappa_{phase} = \frac{hB}{c^2}u$

$h\upsilon_{phase} = Mc^2 + \frac{1}{2}Mu^2 \Leftarrow$ for $(u \ll c) \Rightarrow \quad h\kappa_{phase} = Mu$

Base scale : $B = \upsilon_A$ *for* υ_{phase}

(1) Low speed υ_{phase} and κ_{phase} approximations vary with u like Newton's kinetic energy $\frac{1}{2}Mu^2$ and momentum Mu.

(2) So attach scale factor h (or $\hbar N$) to match units

Rescale υ_{phase} by h so: $M = \frac{hB}{c^2}$ or: $hB = Mc^2$

(3) Use *exact* υ_{phase} *and* κ_{phase}

$h\upsilon_{phase} = hB\cosh\rho = Mc^2\cosh\rho$

Planck (1900)

$= \text{Total Energy}: E = \frac{Mc^2}{\sqrt{1-u^2/c^2}}$

Einstein (1905)

Need to replace $\hbar$ with $\hbar N$ to match e.m. energy density $\epsilon_0 E^2 E = \hbar N \nu_{phase}$

$h\omega_{phase} = hB\cosh\rho = \frac{Mc^2}{2\pi}\cosh\rho$

This motivates the "particle" normalization $\int \Psi^*\Psi\, dV = N \quad \Psi = \sqrt{\frac{N}{V}}e^{\frac{i}{\hbar}Et}$

$h\kappa_{phase} = hB\sinh\rho = Mc^2\sinh\rho$

Momentum: $h\kappa_{phase} = p = \frac{Mu}{\sqrt{1-u^2/c^2}}$

DeBroglie (1921)

$cp = \frac{Muc}{\sqrt{1-u^2/c^2}}$

$h\kappa_{phase} = hB\sinh\rho = \frac{Mc^2}{2\pi}\sinh\rho$

$\frac{1}{\sqrt{\beta^{-1}-1}} = \frac{u}{\sqrt{1-\frac{u^2}{c^2}}}$ (old-fashioned notation) $\quad \frac{u}{c} = \frac{c}{\sqrt{1+\frac{c^2}{u^2}}}$

group	$\beta_{GRP} = \frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$\beta_{PHASE}^{Inverse}$
phase	$\beta_{PHASE}^{Inverse} = \frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\upsilon_{phase}}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\kappa_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\beta_{GRP}^{Inverse}$
rapidity ρ	$e^{-\rho}$	$\sinh\rho$	$\cosh\rho$	$\sinh\rho$	$\cosh\rho$	$\cosh\rho$	$e^{+\rho}$
stellar "σ angle"	$1/e^{\sigma}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$1/e^{+\sigma}$
$\beta = \frac{u}{c}$	$\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}$	β	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\sqrt{1-\beta^2}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\sqrt{\beta^{-2}-1}$	$\sqrt{\frac{1-\beta}{1+\beta}}$
value for $\beta = 3/5$	$\frac{1}{2} = 0.5$	$\frac{3}{5} = 0.6$	$\frac{3}{4} = 0.75$	$\frac{4}{5} = 0.80$	$\frac{5}{4} = 1.25$	$\frac{5}{3} = 1.33$	$\frac{2}{3} = 1.67$	$\frac{2}{1} = 2.0$
functions		$\frac{V_{group}}{c}\tanh\rho$	momentum $cp = Mc^2\sinh\rho$	Lagrangian $L = -Mc^2\operatorname{sech}\rho$	Hamiltonian $H = Mc^2\cosh\rho$	DeBroglie $\lambda = c\operatorname{csch}\rho$	$\frac{V_{phase}}{c} = \coth\rho$

(B) Definition(s) of mass for relativity and quantum theory

(1) *Rest Mass* M_{rest} *(Einstein's mass)* Defines invariant hyperbola(s)

$h\upsilon_A = Mc^2 = h\kappa_A$

$\frac{h\upsilon_{phase}}{c^2} = M_{rest} = \frac{h\kappa_{phase}}{c}$ Rest Mass

$E = \pm\sqrt{(Mc^2)^2 + (cp)^2}$

Given: *Energy*: $E = Mc^2\cosh\rho = h\upsilon_{phase}$

momentum: $cp = Mc^2\sinh\rho = h\kappa_{phase}$

Group velocity: $u = c\tanh\rho = \frac{d\upsilon}{d\kappa}$

(2) *Momentum Mass* M_{mom} *(Galileo's mass)* Defined by ratio p/u of relativistic momentum to group velocity.

$M_{mom} \equiv \frac{p}{u} = \frac{M_{rest}c\sinh\rho}{c\tanh\rho}$

$= M_{rest}\cosh\rho = \frac{M_{rest}}{\sqrt{1-u^2/c^2}}$ Momentum Mass

Limiting cases: $M_{mom} \xrightarrow[u \to c]{} M_{rest}e^{\rho}/2$ $\quad M_{mom} \xrightarrow[u \ll c]{} M_{rest}$

(3) *Effective Mass* M_{eff} *(Newton's mass)* Defined by ratio $F/a = dp/du$ of relativistic force to acceleration.

That is ratio of change $dp = Mc\cosh\rho\, d\rho$ in momentum to change $du = c\operatorname{sech}^2\rho\, d\rho$ in velocity

$M_{eff} \equiv \frac{dp}{du} = M_{rest}\frac{c\cosh\rho}{c\operatorname{sech}^2\rho} = M_{rest}\cosh^3\rho$ Effective Mass

Limiting cases: $M_{eff} \xrightarrow[u \to c]{} M_{rest}e^{3\rho}/2$ $\quad M_{eff} \xrightarrow[u \ll c]{} M_{rest}$

More common derivation using group velocity: $u \equiv V_{group} = \frac{d\omega}{dk} = \frac{d\upsilon}{d\kappa}$

$M_{eff} \equiv \frac{dp}{du} = \frac{\hbar dk}{dV_{group}} = \frac{\hbar}{\frac{d}{dk}\frac{d\omega}{dk}} = \frac{\hbar}{\frac{d^2\omega}{dk^2}} = \frac{M_{rest}}{\left(1-u^2/c^2\right)^{3/2}} = M_{rest}\cosh^3\rho$ Effective Mass

general wave formula to accompany $V_{group} = \frac{d\omega}{dk}$

(C) Defining phase Φ, action $S = \hbar\Phi$, Hamiltonian, and Lagrangian

(1) Define *Lagrangian* L in terms of phase $\Phi = kx - \omega t = k'x' - \omega' t'$ for $k = k_{phase}$ and $\omega = \omega_{phase}$.

$L = \frac{dS}{dt} \equiv \hbar\frac{d\Phi}{dt} = \hbar k\frac{dx}{dt} - \hbar\omega$ $\quad \hbar \equiv \frac{h}{2\pi}$

(2) Use DeBroglie-momentum $p = \hbar k$ relation and Planck-energy $E = \hbar\omega$ relation

$L = \frac{dS}{dt} \equiv \hbar\frac{d\Phi}{dt} = \hbar k\frac{dx}{dt} - \hbar\omega = p\frac{dx}{dt} - E \equiv p\dot{x} - E \equiv pu - H = L$ Legendre transformation

(3) Use relawavity relations: *Group velocity*: $u = \frac{dx}{dt} = c\tanh\rho$, *Rest energy*: $\hbar\omega_A = Mc^2 = \hbar\kappa_A$

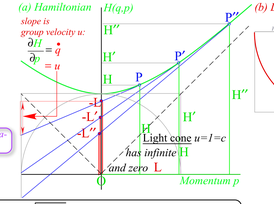
Momentum: $p = \hbar\kappa_{phase} = cp = \hbar\omega_A\sinh\rho$

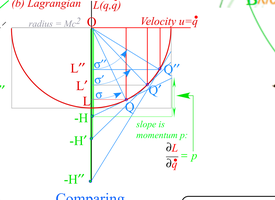
Hamiltonian: $H = \hbar\omega_{phase} = E = \hbar\omega_A\cosh\rho$

(4) $L = pu - H = (Mc\sinh\rho)(c\tanh\rho) - Mc^2\cosh\rho$

$= Mc^2\frac{\sinh^2\rho - \cosh^2\rho}{\cosh\rho} = -Mc^2\operatorname{sech}\rho$

Hamiltonian Legendre transforms to Lagrangian





Comparing Lagrangian $L(\text{velocity } u)$ with Hamiltonian $H(\text{momentum } p)$

$L = \hbar\frac{d\Phi}{dt} = -Mc^2\sqrt{1-\frac{u^2}{c^2}} = -Mc^2\operatorname{sech}\rho = -Mc^2\cos\sigma$

$H = \hbar\upsilon_{phase} = Mc^2\Big/\sqrt{1-\frac{u^2}{c^2}} = Mc^2\cosh\rho = Mc^2\sec\sigma$

(5) $\Phi = kx - \omega t \quad H = Mc^2\sqrt{1+\sinh^2\rho} = Mc^2\sqrt{1+(cp)^2}$

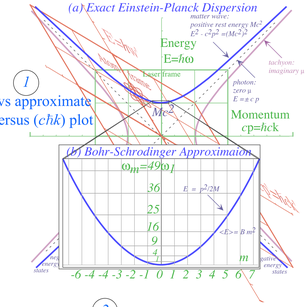
$u = V_{group} = c\tanh\rho = c\sin\sigma$

$cp = c\hbar\kappa_{phase} = Mc^2\sinh\rho = Mc^2\tan\sigma$ (6)

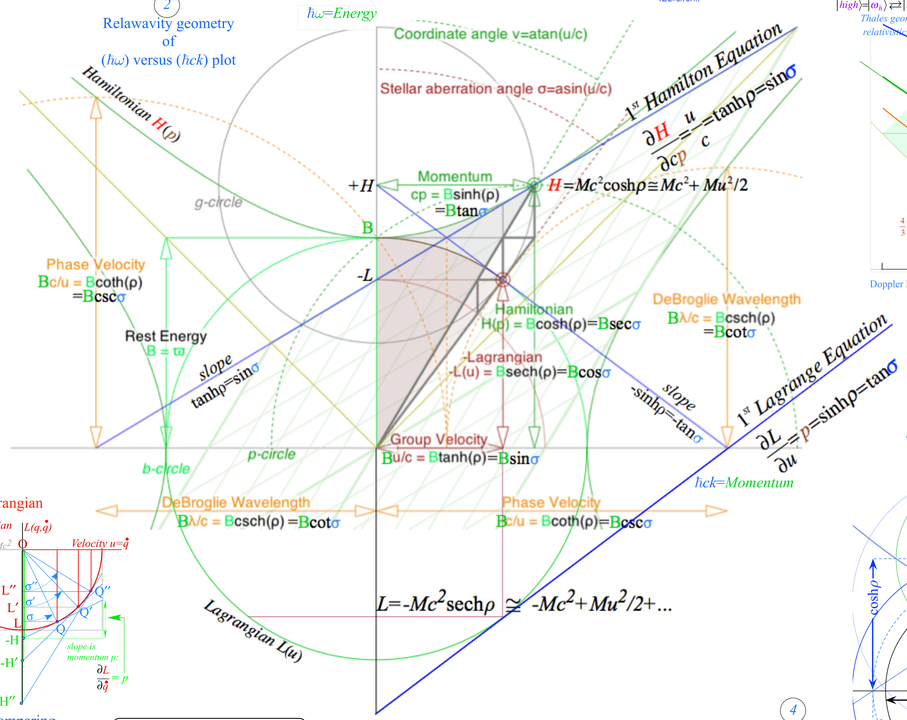
(D) Geometry and plots of "Relawavity" variables

$\sinh\rho = \tan\sigma$, $\cosh\rho = \sec\sigma$, $\coth\rho = \csc\sigma$, $\tanh\rho = \sin\sigma$, $\operatorname{sech}\rho = \cos\sigma$, $\operatorname{csch}\rho = \cot\sigma$.

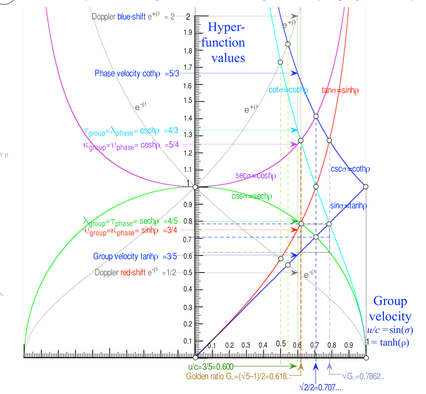
(1) Exact vs approximate ($\hbar\omega$) versus ($c\hbar k$) plot



(2) Relawavity geometry of ($\hbar\omega$) versus ($\hbar ck$) plot



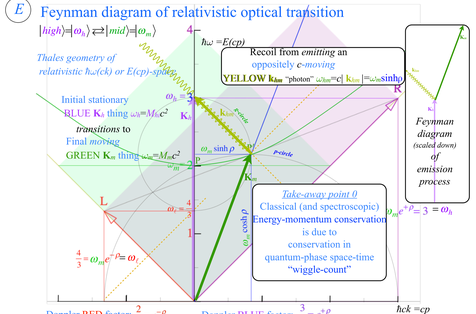
(3) Relawavity variables plotted *versus* Group Velocity $V_{group} = c\tanh\rho$



(E) Feynman diagram of relativistic optical transition

$|high\rangle = |\omega_h\rangle \rightleftarrows |mid\rangle = |\omega_m\rangle \rightleftarrows |low\rangle = |\omega_l\rangle$

Thales geometry of relativistic $\hbar\omega(ck)$ *or* $E(cp)$*-space*

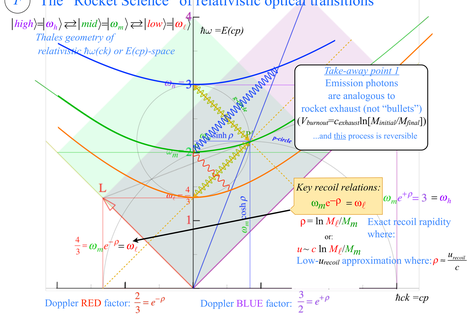


Doppler RED factor: $\frac{2}{3} = e^{-\rho}$ $\quad$ Doppler BLUE factor: $\frac{3}{2} = e^{+\rho}$

(F) The "Rocket Science" of relativistic optical transitions

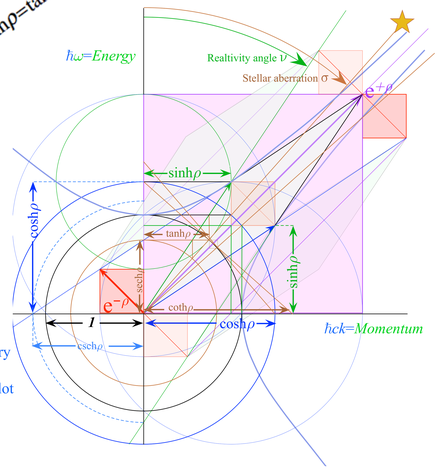
$|high\rangle = |\omega_h\rangle \rightleftarrows |mid\rangle = |\omega_m\rangle \rightleftarrows |low\rangle = |\omega_l\rangle$

Thales geometry of relativistic $\hbar\omega(ck)$ *or* $E(cp)$*-space*

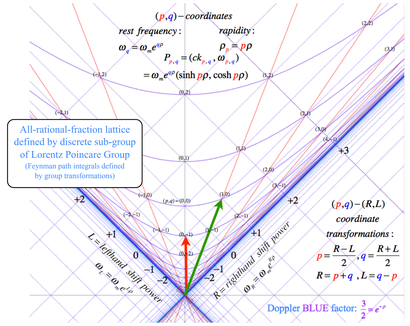
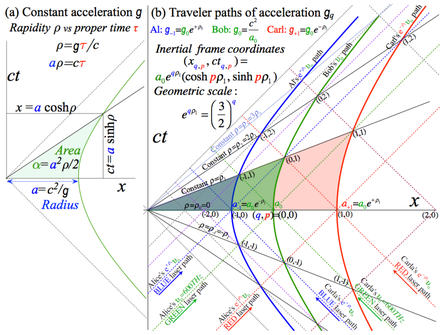
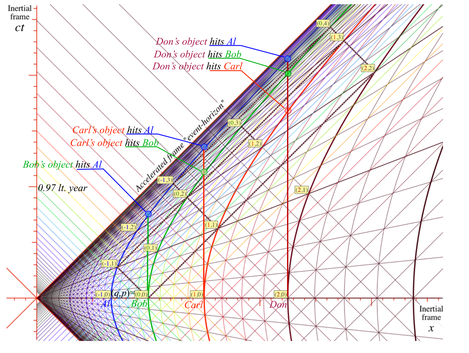


Doppler RED factor: $\frac{2}{3} = e^{-\rho}$ $\quad$ Doppler BLUE factor: $\frac{3}{2} = e^{+\rho}$

(4) Doppler geometry of (ω) versus (ck) plot



(G) Discrete ($\hbar\omega$) versus ($\hbar ck$) plot of Compton scattering



(H) Discrete time (ct) versus space (x) plot of constant acceleration paths