Lecture 25

Parametric Resonance and Multi-particle Wave Modes

(Ch. 7-8 of Unit 4 11.27.12)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)
- Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)
- Schrodinger wave equation related to Parametric resonance dynamics
  - Electronic band theory and analogous mechanics

Wave resonance in cyclic symmetry
- Harmonic oscillator with cyclic C_2 symmetry
  - C_2 symmetric (B-type) modes
- Harmonic oscillator with cyclic C_3 symmetry
  - C_3 symmetric spectral decomposition by 3rd roots of unity
  - Resolving C_3 projectors and moving wave modes
  - Dispersion functions and standing waves
- C_6 symmetric mode model:Distant neighbor coupling
  - C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ...)
- C_N symmetric mode models: Made-to order dispersion functions
  - Quadratic dispersion models: Super-beats and fractional revivals
  - Phase arithmetic

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

Relating C_N symmetric H and K matrices to differential wave operators

Two Kinds of Resonance

Linear or *additive resonance*.

Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos\left(\omega_s t\right)$$

Chapter 4.2 study of FDHO

(Here damping $\Gamma \cong 0$*)*

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Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Nonlinear or *multiplicative resonance.*
Example: oscillating magnetic **B**-field is applied to a cyclotron orbit.

$$\ddot{x}+\left(\omega_0^2 + B\cos\left(\omega_s t\right)\right)x = 0$$

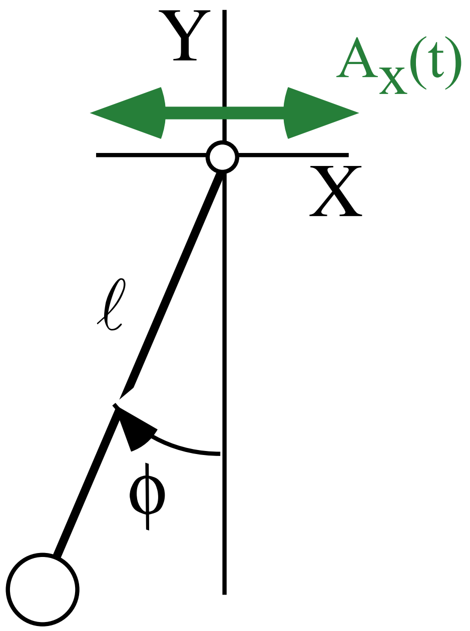
Chapter 4.7

Also called *parametric resonance.*
(Frequency parameter or spring constant $k=m\omega^2$ is being stimulated.)

Coupled Rotation and Translation (Throwing)

Early non-human (or in-human) machines: trebuchets, whips.. (3000 BCE-1542 CE)

X-stimulated pendulum:
(Quasi-Linear Resonance)



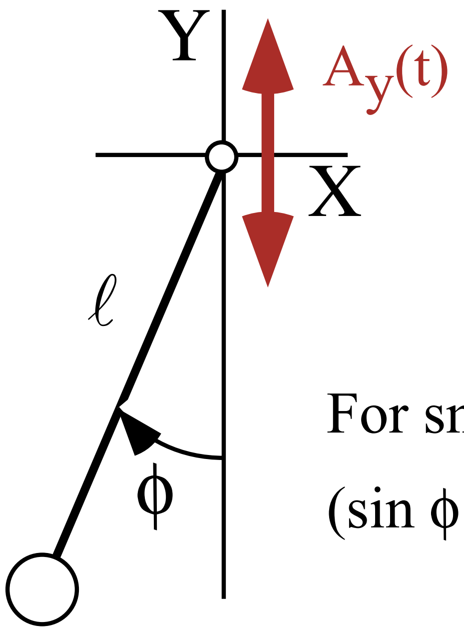
For small ϕ
($\cos\phi \sim 1$) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2}+\frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with Γ=0)

Y-stimulated pendulum:
(Non-Linear Resonance)

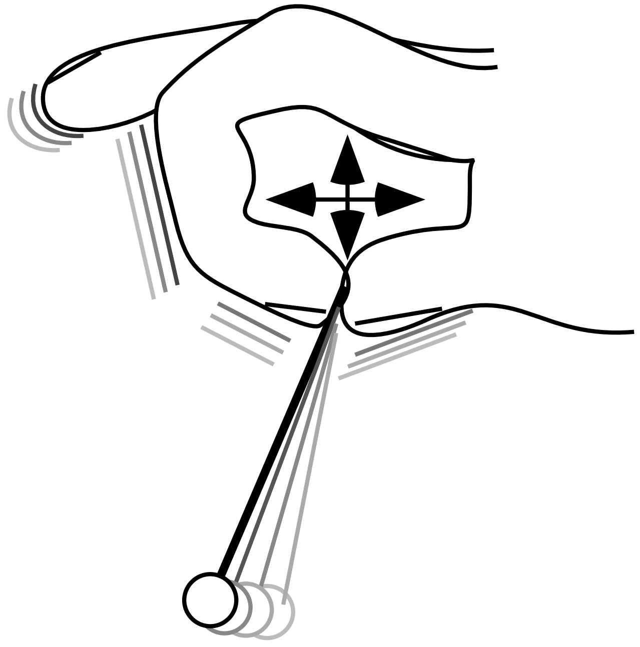


For small ϕ
($\sin\phi \sim\phi$) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{A_y(t)}{\ell}\right)\phi = 0$$

A Schrodinger-like equation
(Time *t* replaces coord. *x*)

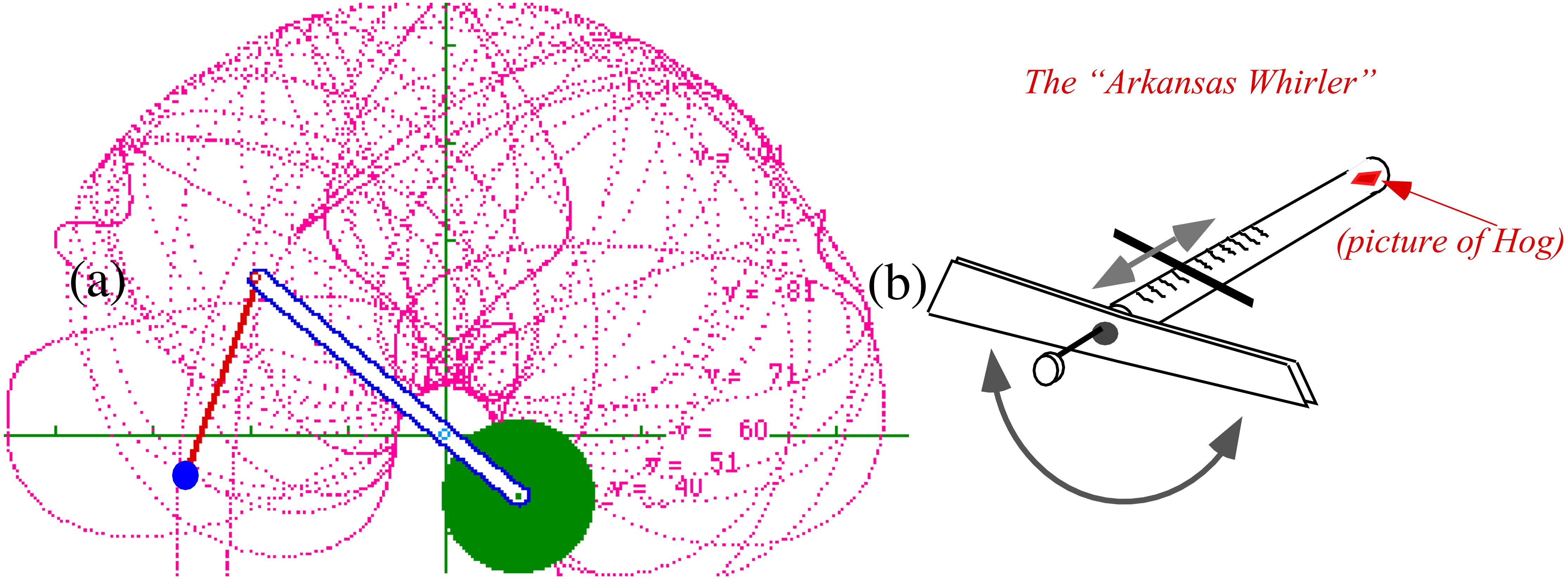


General ϕ :

(1542-2012 CE)

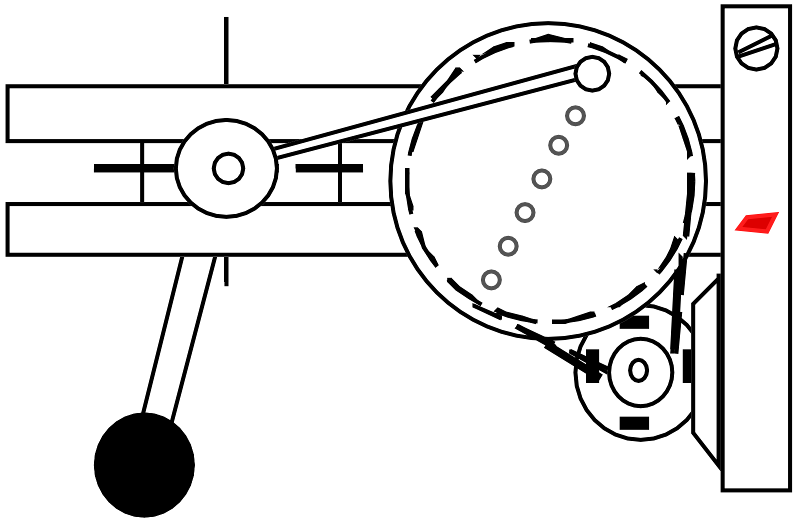
General case: A Nasty equation! $$\frac{d^2\phi}{dt^2}+\frac{g+A_y(t)}{\ell}\sin\phi+\frac{A_x(t)}{\ell}\cos\phi = 0$$

Coupled Rotation and Translation (Throwing)

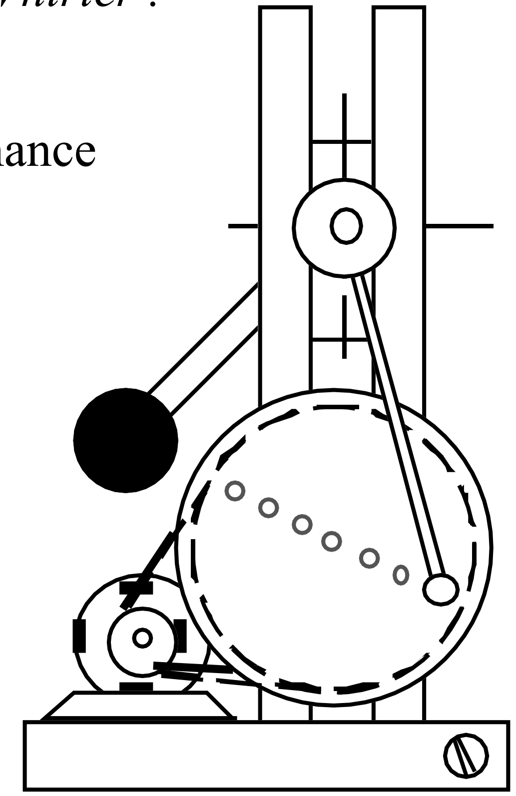


Chaotic motion from both linear and non-linear resonance (a) Trebuchet, (b) Whirler .

Positioned for linear resonance



Positioned for nonlinear resonance



Schrodinger Equation Parametric Resonance Related to Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation

$$\frac{d^2\phi}{dx^2}+\left(E-V(x)\right)\phi=0$$

With periodic potential

$$V(x)=-V_0\cos(Nx)$$

Mathieu Equation

$$\frac{d^2\phi}{dx^2}+\left(E+V_0\cos(Nx)\right)\phi=0$$

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{A_y\left(t\right)}{\ell}\right)\phi=0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y\left(t\right)=\omega_y{}^2 A_y\cos(\omega_y t)$$

$$Nx=\omega_y t$$

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{\omega_y{}^2 A_y}{\ell}\cos(\omega_y t)\right)\phi=0$$

Connection Relations

$$\frac{N}{\omega_y}dx=dt \rightarrow \frac{N^2}{\omega_y{}^2}dx^2=dt^2$$

(Let N=2 to get edge modes)

$$\frac{d^2\phi}{dx^2}+\frac{N^2}{\omega_y{}^2}\left(\frac{g}{\ell}+\frac{\omega_y{}^2 A_y}{\ell}\cos(Nx)\right)\phi=0$$

$$\boxed{E=\frac{N^2}{\omega_y{}^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

$$\boxed{V_0=\frac{N^2 A_y}{\ell}}$$

QM Potential V_0-A_y Amplitude Connection

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

- *Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)*
- *Schrodinger wave equation related to Parametric resonance dynamics*
  - ➔ *Electronic band theory and analogous mechanics*

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus B is zero

$$-\frac{d^2\phi}{dx^2} = E\phi \qquad\qquad -\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x | M \rangle = \phi_M(x) = \frac{e^{\pm iMx}}{\sqrt{2\pi}},\text{ where: } E=M^2 \qquad\qquad \langle t | \omega \rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}},\text{ where: } \omega_0=\sqrt{\frac{g}{\ell}}$$

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Bohr has *periodic boundary conditions* *x* between *0* and *L*

$$\phi(0) = \phi(L) \Rightarrow e^{iML} = 1, \text{ or: } M = \frac{2\pi m}{L}$$

Pendulum repeats perfectly after a time *T*.

$$\phi(0) = \phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0 = \frac{2\pi m}{T}$$

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Limit *L*=2π=*T* for both analogies. Then the allowed energies and frequencies follow

$$E = m^2 = 0,1,4,9,16... \qquad\qquad \omega_0 = m = 0,\pm1,\pm2,\pm3,\pm4,...$$

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Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2}+V\cos(nx)\phi=E\phi\ , \qquad \left(\mathbf{D}+\mathbf{V}\right)|\phi\rangle=E|\phi\rangle$$

Fourier representation is with $\langle j|\mathbf{D}|k\rangle=j^2\delta_j^k$

$$\Sigma\langle j|\left(\mathbf{D}+\mathbf{V}\right)|k\rangle\langle k|\phi\rangle=E\langle j|\phi\rangle$$

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus B is zero

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Fourier representation is with $\left\langle j\middle|\mathbf{D}\middle|k\right\rangle = j^2\delta_j^k$ and $\left\langle j\middle|\mathbf{V}\middle|k\right\rangle = \int_0^{2\pi} dx \frac{e^{-ijx}}{\sqrt{2\pi}} V\cos(nx)\frac{e^{-ikx}}{\sqrt{2\pi}} = \int_0^{2\pi} dx \frac{e^{-i(j-k)x}}{2\pi} V \frac{e^{-inx}+e^{inx}}{2}$

$$\Sigma\left\langle j\middle|(\mathbf{D}+\mathbf{V})\middle|k\right\rangle\left\langle k\middle|\phi\right\rangle = E\left\langle j\middle|\phi\right\rangle \qquad\qquad = \frac{V}{2}\left(\delta_j^{k+n} + \delta_j^{k-n}\right)$$

Electronic band theory and analogous mechanics

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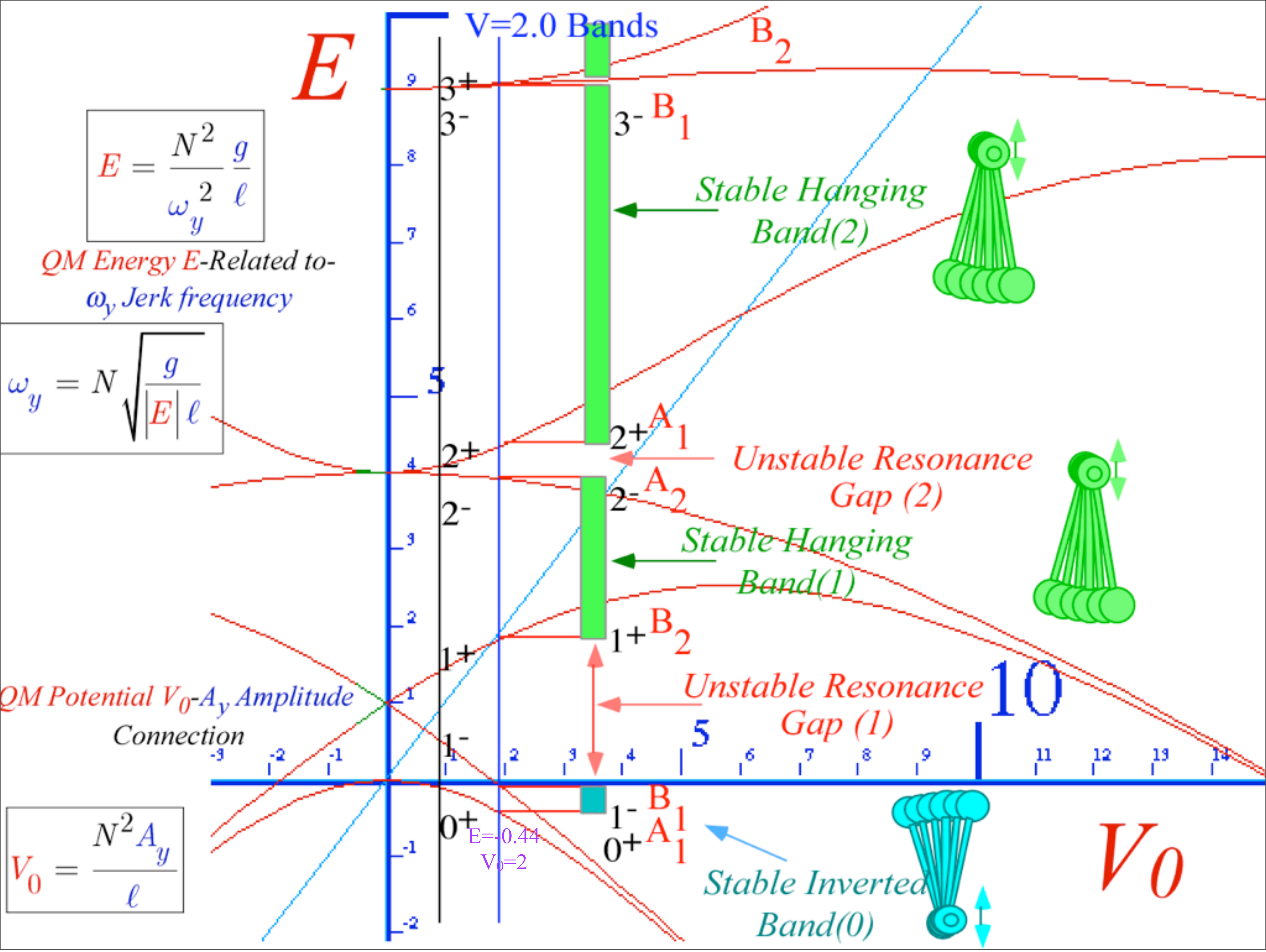
$$\Sigma\langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle \qquad\qquad = \frac{V}{2}\left(\delta_j^{k+n}+\delta_j^{k-n}\right)$$

$\langle j|(\mathbf{D}+\mathbf{V})|k\rangle =$ (for j and k even) $\qquad \langle j|(\mathbf{D}+\mathbf{V})|k\rangle =$ (for j and k odd)

$\cdots\ |-6\rangle,|-4\rangle,|-2\rangle,\ |0\rangle,\ |2\rangle,\ |4\rangle,\ |6\rangle,\cdots \qquad \cdots\ |-7\rangle,|-5\rangle,|-3\rangle,|-1\rangle,\ |1\rangle,\ |3\rangle,\ |5\rangle,\cdots$

$$\begin{pmatrix}
\ddots &&&&&&&& \\
& 6^2 & v &&&&&& \\
& v & 4^2 & v &&&&& \\
&& v & 2^2 & v &&&& \\
&&& v & 0 & v &&& \\
&&&& v & 2^2 & v && \\
&&&&& v & 4^2 & v & \\
&&&&&& v & 6^2 & \\
&&&&&&&& \ddots
\end{pmatrix},
\begin{pmatrix}
\ddots &&&&&&&& \\
& 7^2 & v &&&&&& \\
& v & 5^2 & v &&&&& \\
&& v & 3^2 & v &&&& \\
&&& v & 1^2 & v &&& \\
&&&& v & 1^2 & v && \\
&&&&& v & 3^2 & v & \\
&&&&&& v & 5^2 & \\
&&&&&&&& \ddots
\end{pmatrix}$$

V=2.0 Bands



$$E = \frac{N^2}{\omega_y^2}\frac{g}{\ell}$$

QM Energy E-Related to-
ω_y Jerk frequency

$$\omega_y = N\sqrt{\frac{g}{|E|\ell}}$$

QM Potential V_0-A_y Amplitude
Connection

$$V_0 = \frac{N^2 A_y}{\ell}$$

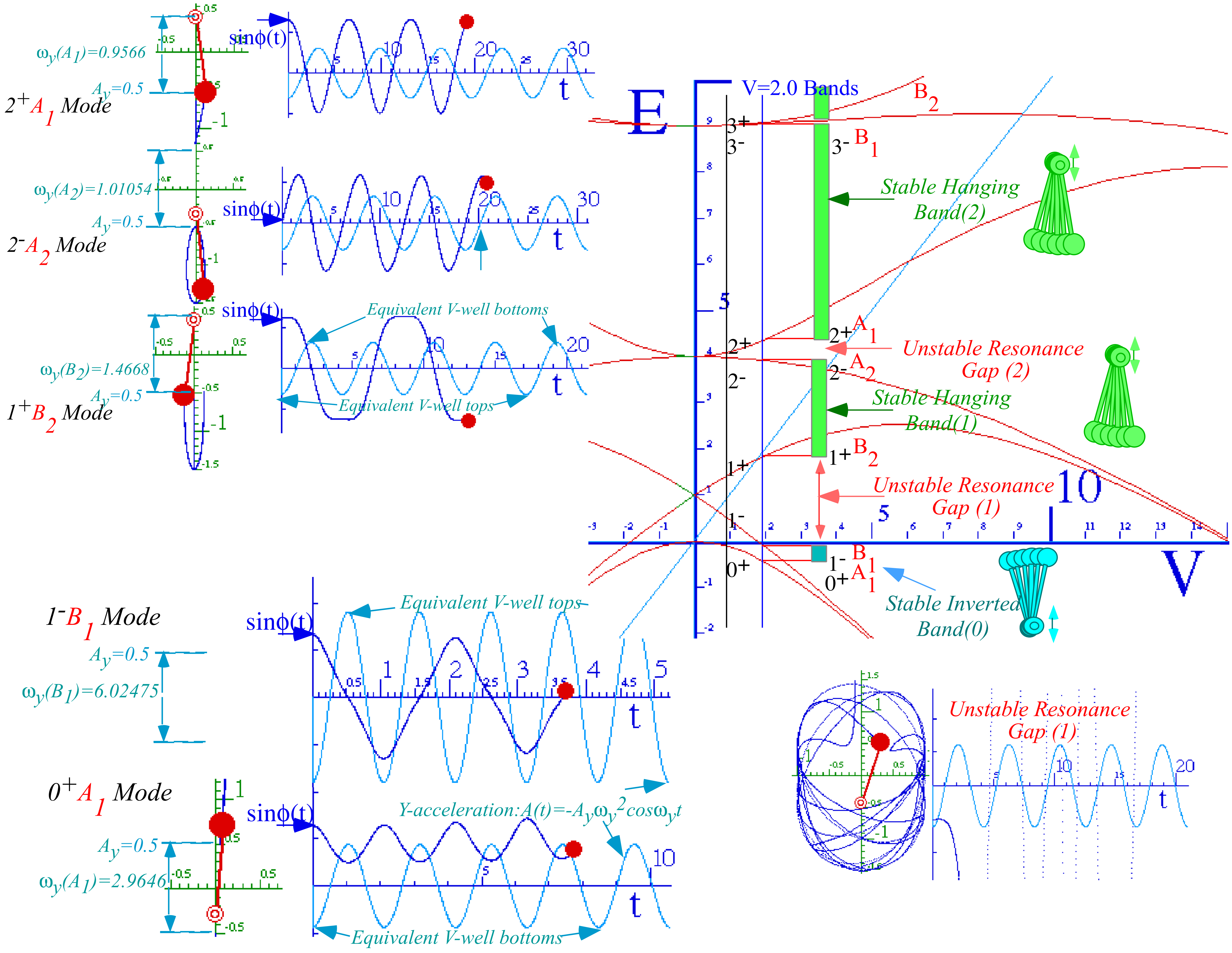
Stable Hanging Band(2)

Unstable Resonance Gap (2)

Stable Hanging Band(1)

Unstable Resonance Gap (1)

Stable Inverted Band(0)


2+A1 Mode
ωy(A1)=0.9566
Ay=0.5
sinφ(t)
2-A2 Mode
ωy(A2)=1.01054
Ay=0.5
sinφ(t)
1+B2 Mode
ωy(B2)=1.4668
Ay=0.5
sinφ(t)
Equivalent V-well bottoms
Equivalent V-well tops
t
E
V=2.0 Bands
B2
3+
3-
3- B1
Stable Hanging Band(2)
2+ A1
2+
Unstable Resonance Gap (2)
2- A2
2-
Stable Hanging Band(1)
1+ B2
1+
Unstable Resonance Gap (1)
1-
1- B1
0+
0+ A1
Stable Inverted Band(0)
V
1-B1 Mode
Ay=0.5
ωy(B1)=6.02475
sinφ(t)
Equivalent V-well tops
0+A1 Mode
Ay=0.5
ωy(A1)=2.9646
sinφ(t)
Y-acceleration:A(t)=-Ayωy²cosωyt
Equivalent V-well bottoms
Unstable Resonance Gap (1)
t

Wave resonance in cyclic symmetry

- ➔ *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
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Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

$$= (A^2+B^2)\cdot\mathbf{1} \qquad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$=A\cdot\mathbf{1}\qquad +B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

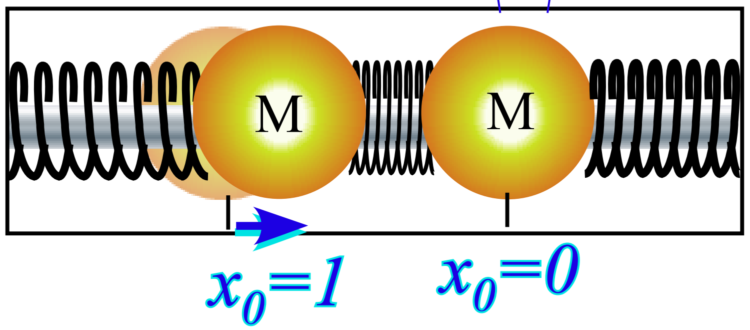
$$=(A^2+B^2)\cdot\mathbf{1}\qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry $\boldsymbol{\sigma}_B$ defined by $(\boldsymbol{\sigma}_B)^2=\mathbf{1}$ in C_2 group product table.

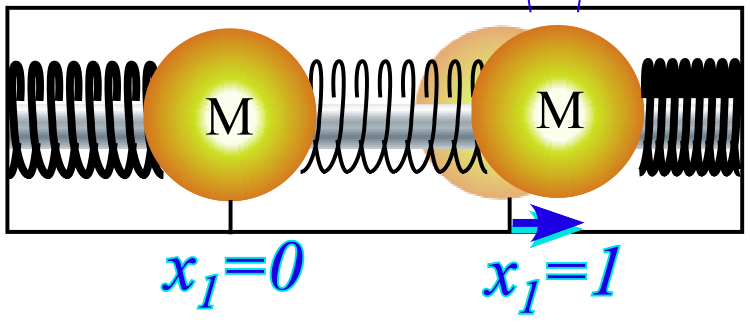
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

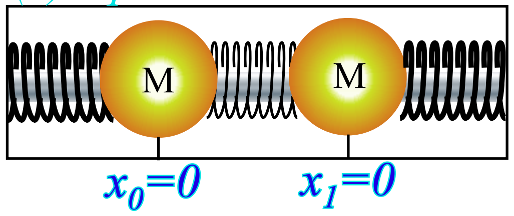


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|\text{-}1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

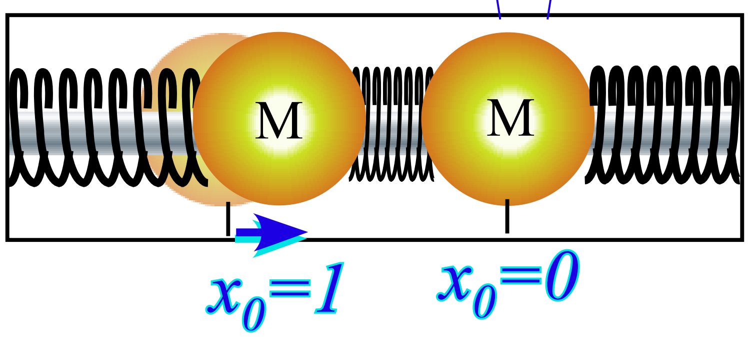
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1} \qquad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

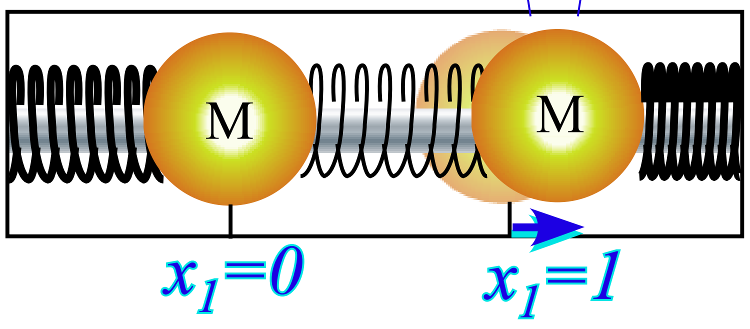
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

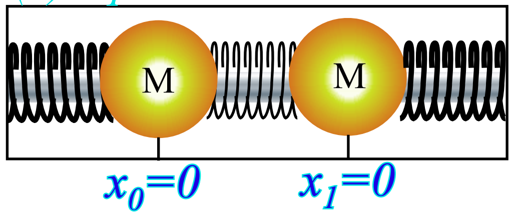


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2\mathbf{-1=0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

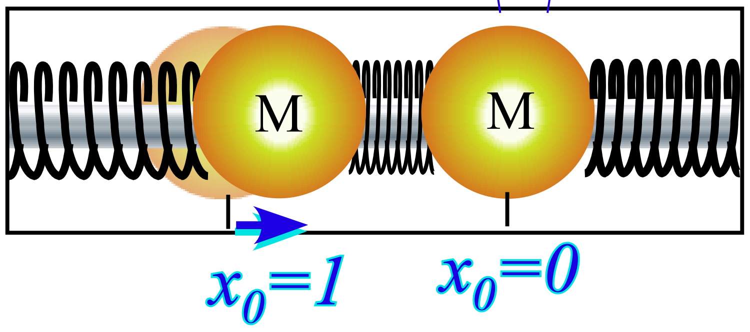
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

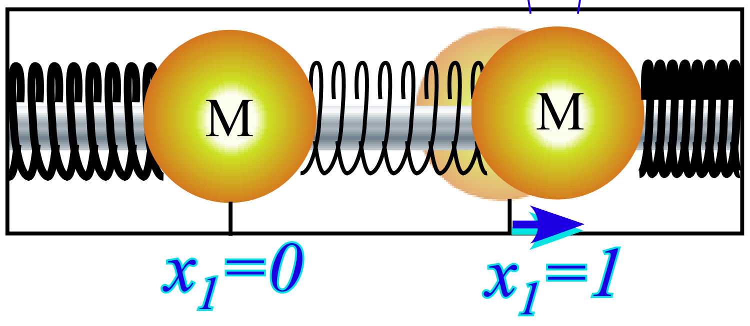
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

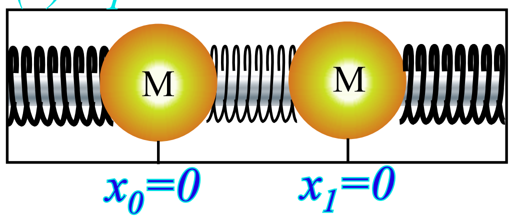


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

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$$=(A^2+B^2)\cdot\mathbf{1}\qquad+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

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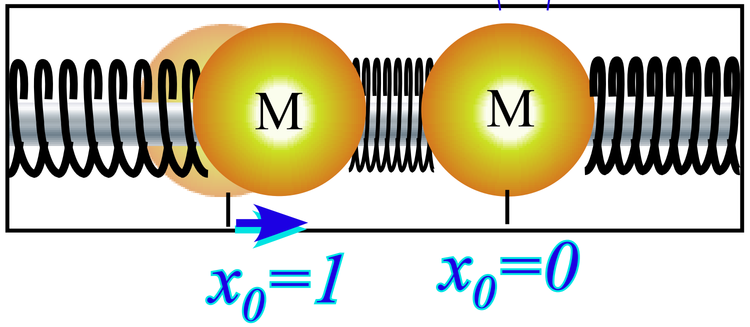

(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

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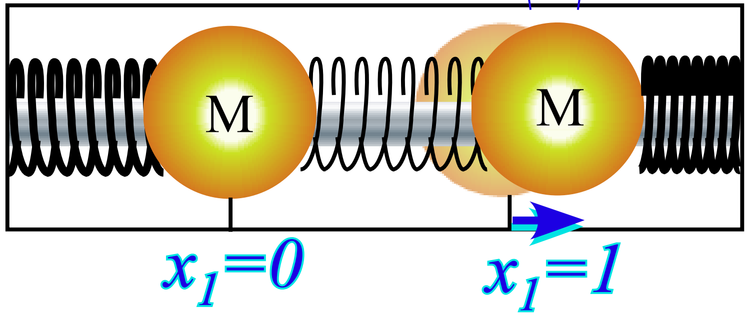
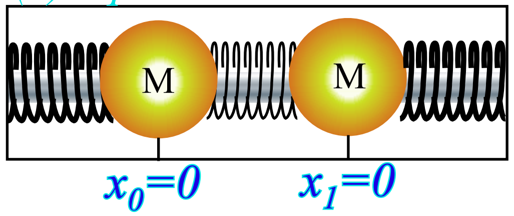


(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

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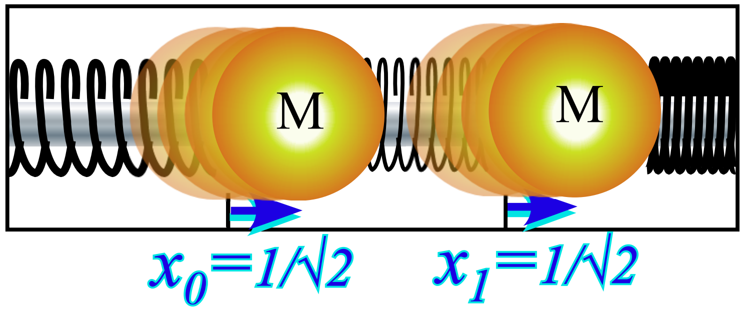
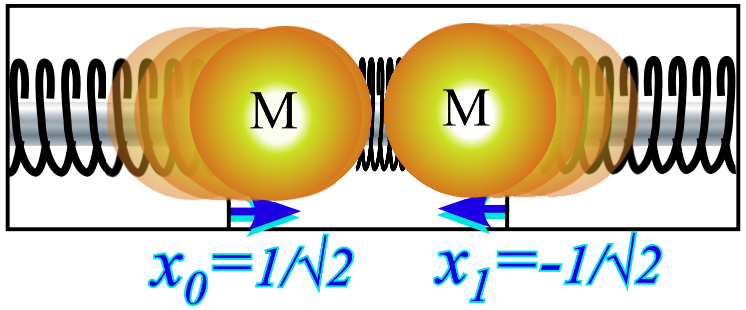
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



Wave resonance in cyclic symmetry

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$$=(A^2+B^2)\cdot\mathbf{1}\qquad+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

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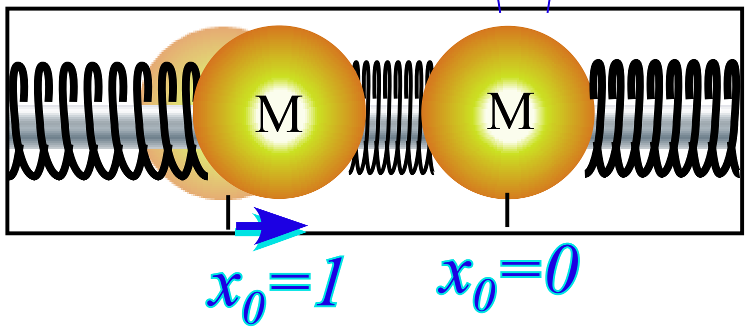
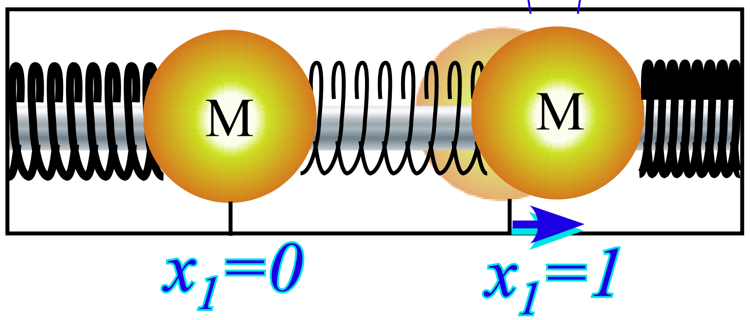
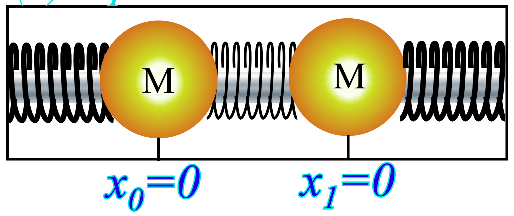


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



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C_2 symmetry (B-type) modes

Mode state projection:

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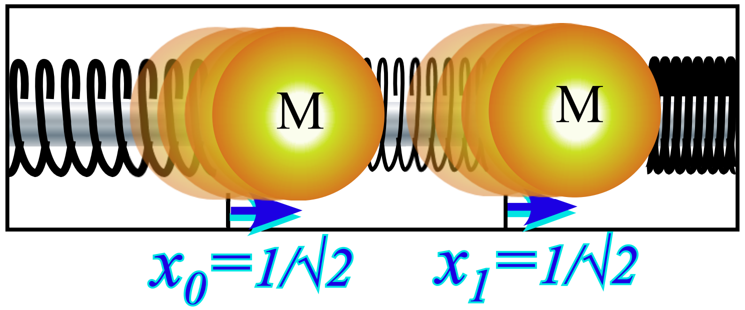
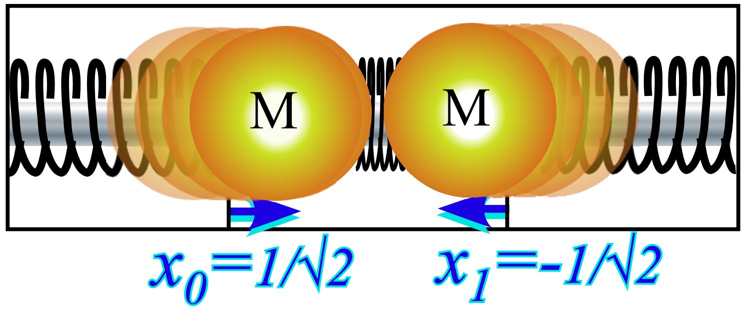


$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

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C_2	1	σ_B
1	1	σ_B
σ_B	σ_B	1

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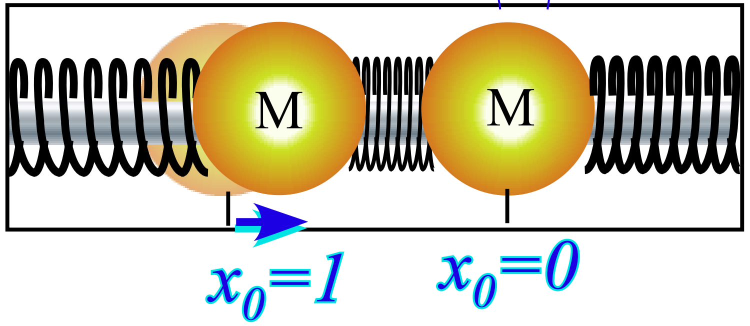
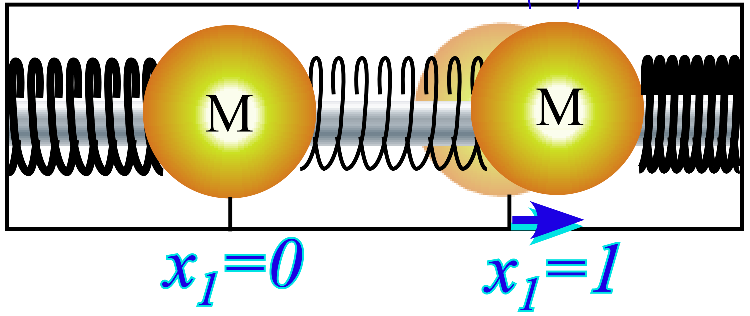
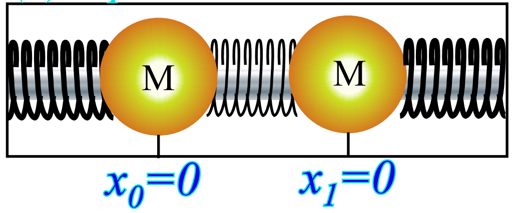


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



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C_2 symmetry (B-type) modes

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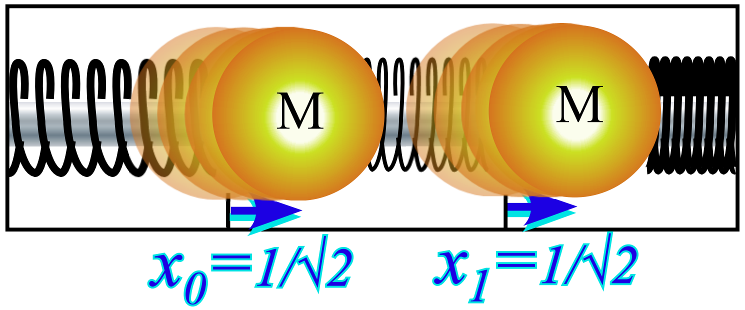
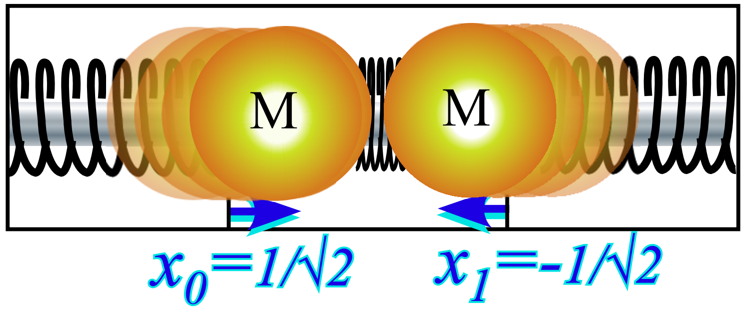


$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

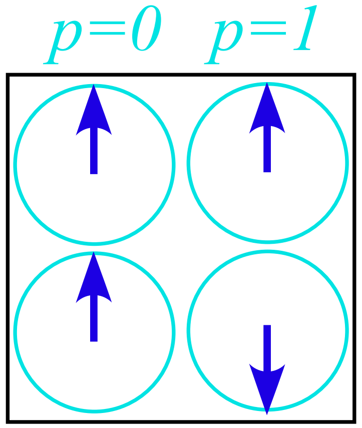
(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

C_2 mode phase & character tables

p= position point (modulo-2)



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

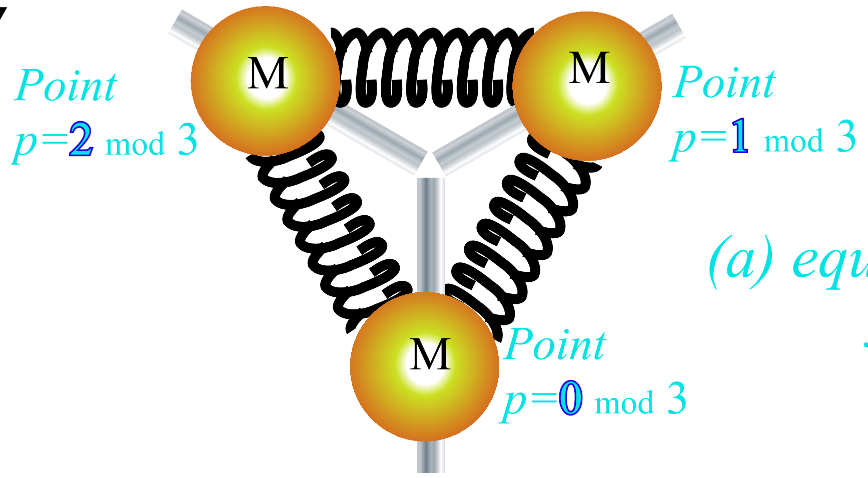
- *Wave resonance in cyclic symmetry*
  - *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
  - ➤ *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
  - *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .*
  - *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2\\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0\end{pmatrix} = r_0 \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + r_1 \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} + r_2 \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\mathbf{H} = r_0 \cdot \mathbf{1} + r_1 \cdot \mathbf{r}^1 + r_2 \cdot \mathbf{r}^2$$

$$\mathbf{r}^0=\mathbf{1}$$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

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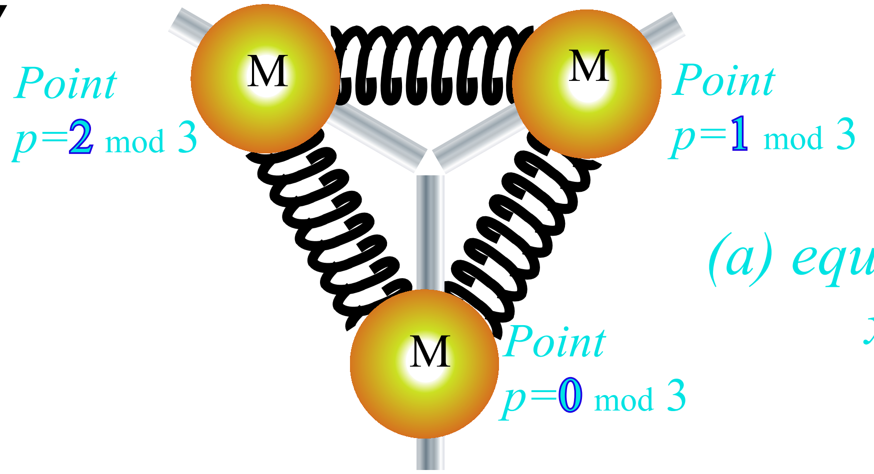
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



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H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

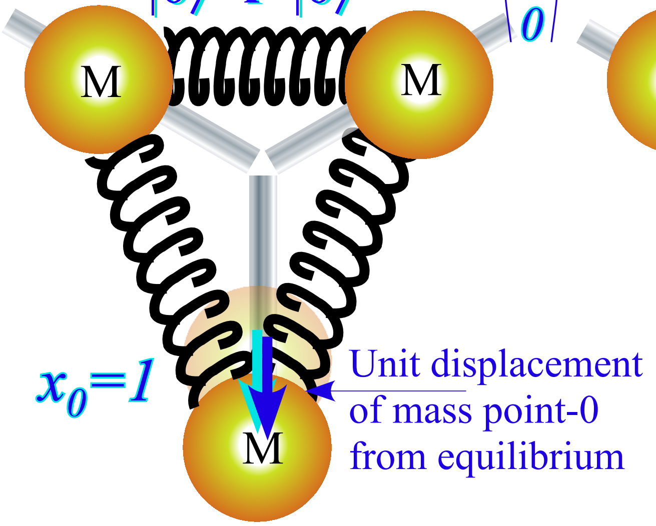
$$\mathbf{H} \quad = \quad r_0 \cdot \mathbf{1} \quad + r_1 \cdot \mathbf{r}^1 \quad + r_2 \cdot \mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

C_3 unit base states

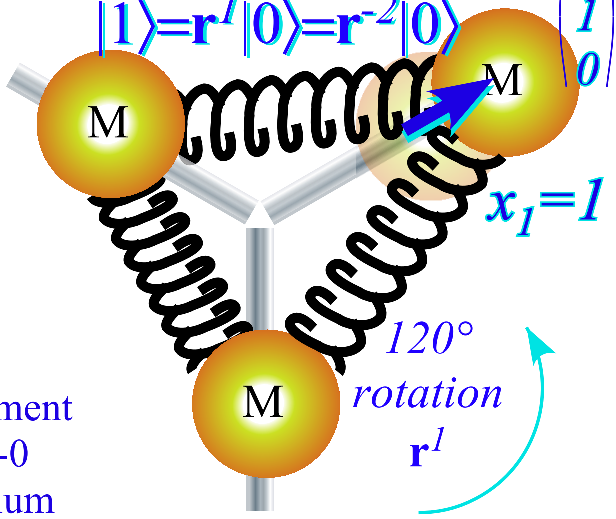
(p=0) unit base state $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$|0\rangle=\mathbf{r}^0|0\rangle$



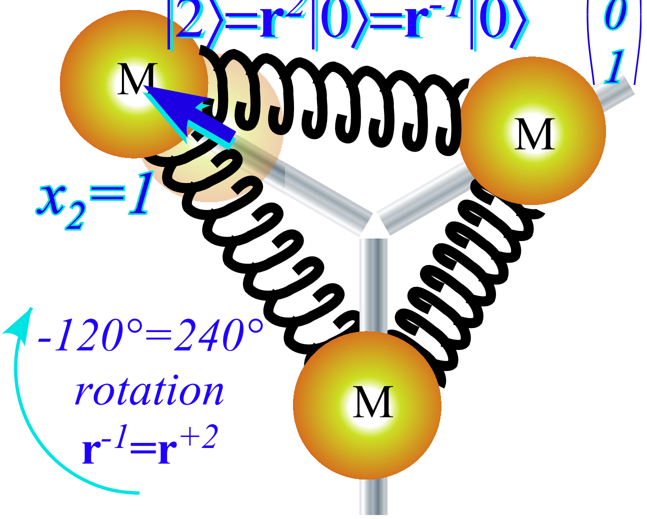
(p=1) unit base state $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$|1\rangle=\mathbf{r}^1|0\rangle=\mathbf{r}^{-2}|0\rangle$



(p=2) unit base state $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$|2\rangle=\mathbf{r}^2|0\rangle=\mathbf{r}^{-1}|0\rangle$



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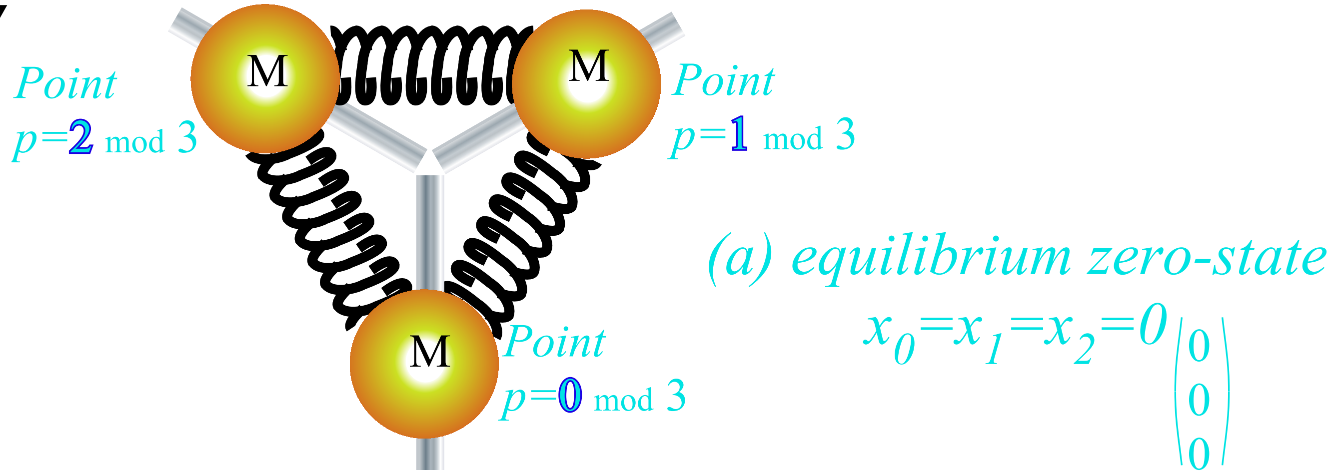
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



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$x_0=x_1=x_2=0$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$

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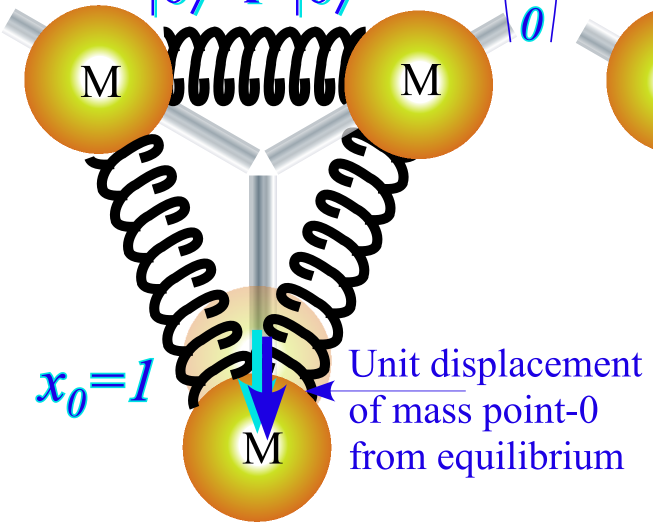
$$\begin{pmatrix} r_0 & r_1 & r_2\\ r_2 & r_0 & r_1\\ r_1 & r_2 & r_0\end{pmatrix} = r_0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + r_1\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} + r_2\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

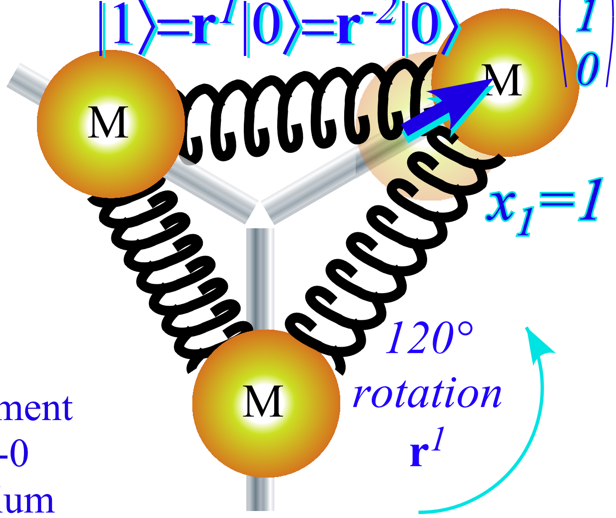
C_3 unit base states

(p=0) unit base state $\begin{pmatrix}1\\0\\0\end{pmatrix}$
$|0\rangle=\mathbf{r}^0|0\rangle$

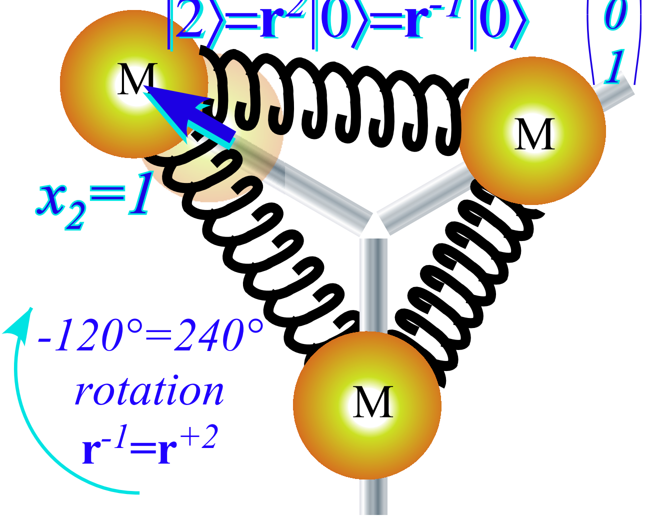


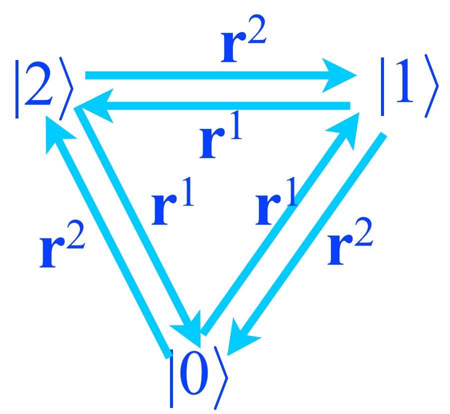
Unit displacement
of mass point-0
from equilibrium

(p=1) unit base state $\begin{pmatrix}0\\1\\0\end{pmatrix}$
$|1\rangle=\mathbf{r}^1|0\rangle=\mathbf{r}^{-2}|0\rangle$



(p=2) unit base state $\begin{pmatrix}0\\0\\1\end{pmatrix}$
$|2\rangle=\mathbf{r}^2|0\rangle=\mathbf{r}^{-1}|0\rangle$





$\mathbf{r}^0=\mathbf{1}$
$=\mathbf{r}^3$

Each **H**-matrix coupling constant $r_p=\{r_0, r_1, r_2\}$ is amplitude of its operator power $\mathbf{r}^p=\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$

Wave resonance in cyclic symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - ➔ *C_3 symmetric spectral decomposition by 3rd roots of unity*
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- *C_6 symmetric mode model:Distant neighbor coupling*
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- *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

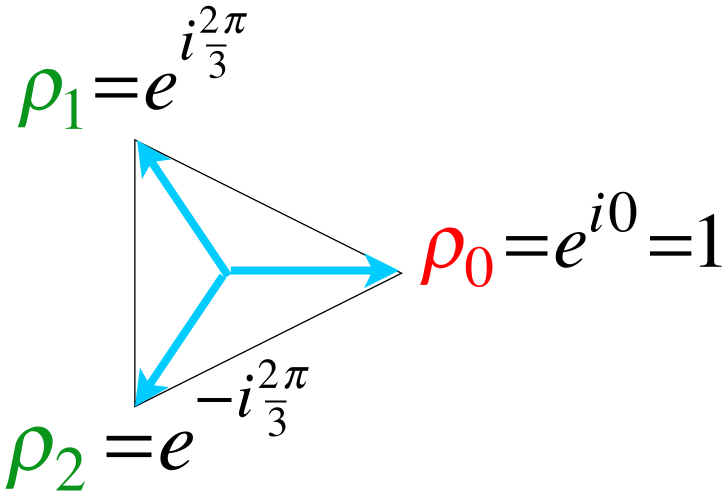
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3\mathbf{-1=0}$ and resolves to factors of *3^{rd} roots of unity* $\rho_m=e^{im2\pi/3}$.



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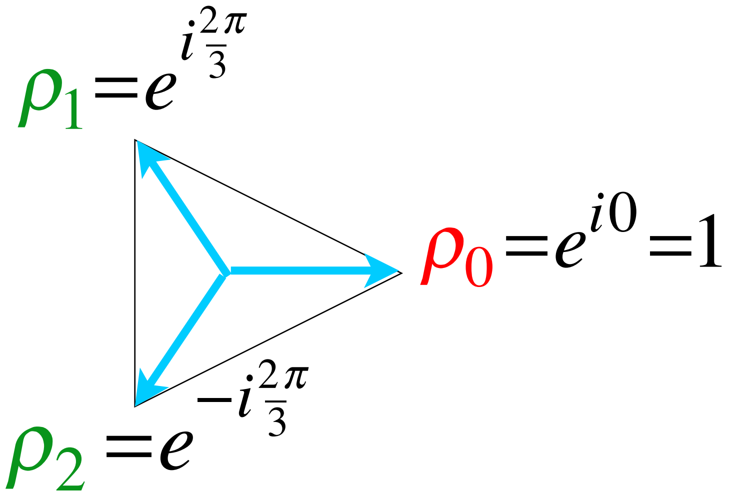
$$\mathbf{1}=\mathbf{r}^3 \; implies: \;\; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \; where: \;\; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.



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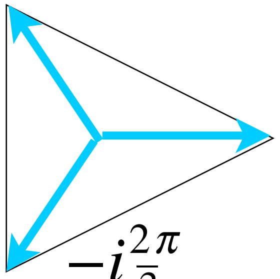
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\rho_1=e^{i\frac{2\pi}{3}}$$



$$\rho_0=e^{i0}=1$$

$$\rho_2=e^{-i\frac{2\pi}{3}}$$

C_3 Spectral resolution: 3^{rd} roots of unity

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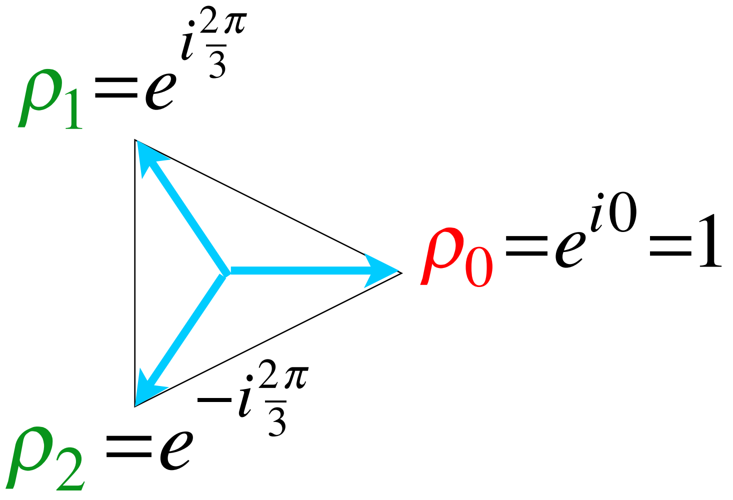

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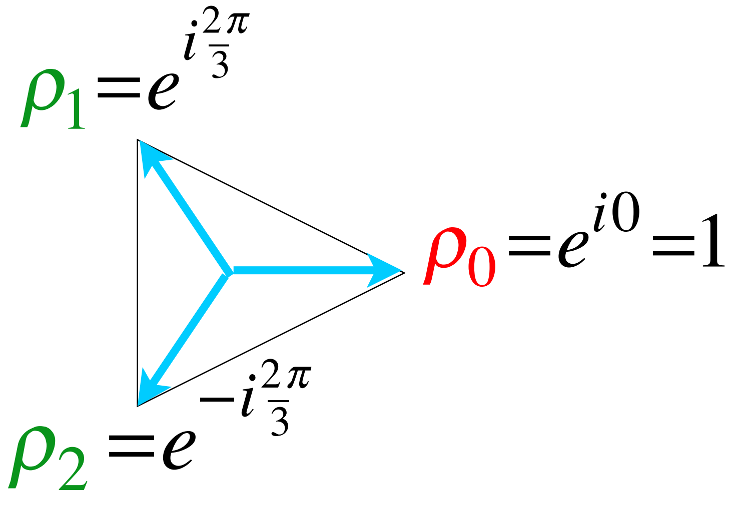

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$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

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We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

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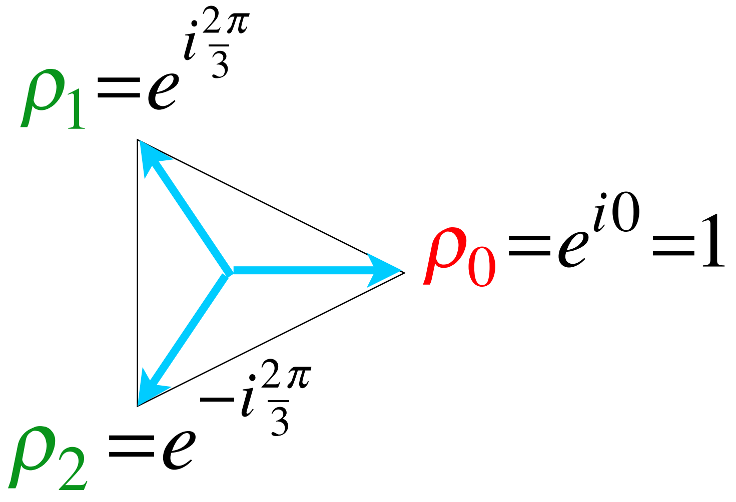

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Easy to resolve spectral projectors $\mathbf{P}^{(m)}$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

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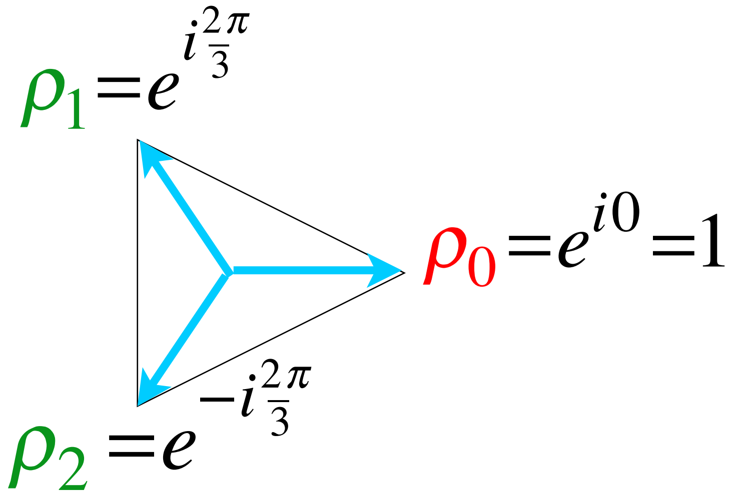

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$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

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$$\langle(0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1 \quad 1 \quad 1\;)$$

$$\langle(1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1 \quad e^{-i2\pi/3} \quad e^{+i2\pi/3})$$

$$\langle(2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(1 \quad e^{+i2\pi/3} \quad e^{-i2\pi/3})$$

(m_3) means: *m-modulo-3* (Details follow)

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
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$$\mathbf{P}^{(1)} = \tfrac{1}{3}(\mathbf{r}^0 + \rho_1^* \mathbf{r}^1 + \rho_2^* \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{-i2\pi/3}\mathbf{r}^1 + e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)} = \tfrac{1}{3}(\mathbf{r}^0 + \rho_2^* \mathbf{r}^1 + \rho_1^* \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{+i2\pi/3}\mathbf{r}^1 + e^{-i2\pi/3}\mathbf{r}^2)$$

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$$\langle (2_3)| = \langle 0|\mathbf{P}^{(2)}\sqrt{3} = \sqrt{\tfrac{1}{3}}\,(1 \quad e^{+i2\pi/3} \quad e^{-i2\pi/3})$$

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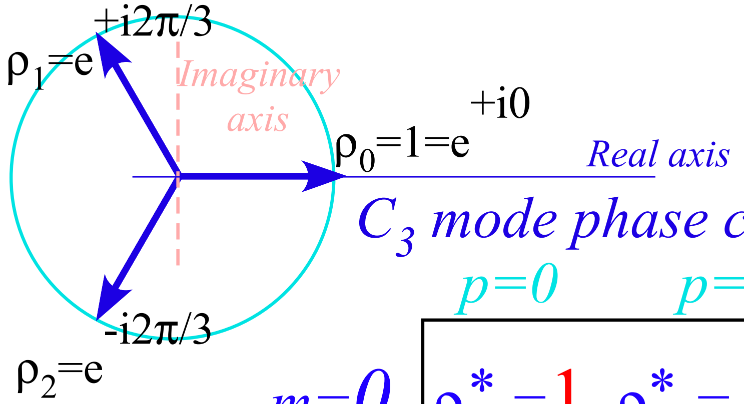
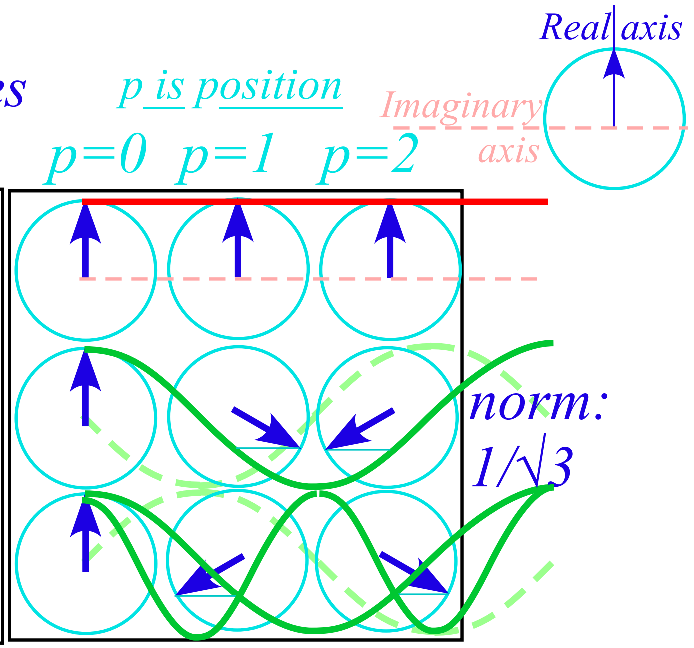


C_3 mode phase character tables

wave-number $m=$ "momentum"

	p=0	p=1	p=2
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

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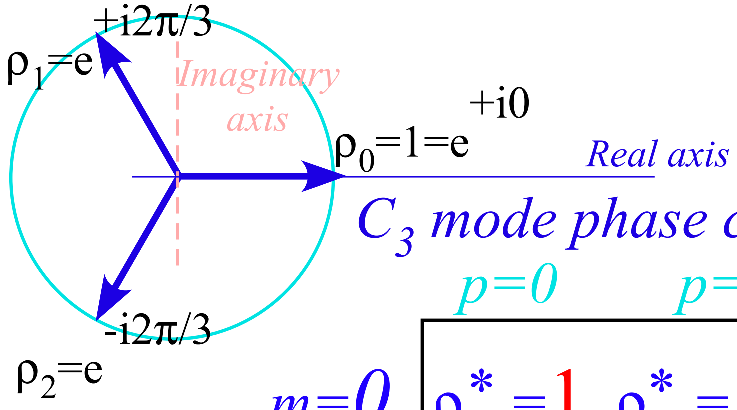

(m_3) means: *m-modulo-3* (Details follow)

C_3 mode phase character tables

wave-number m= "momentum"

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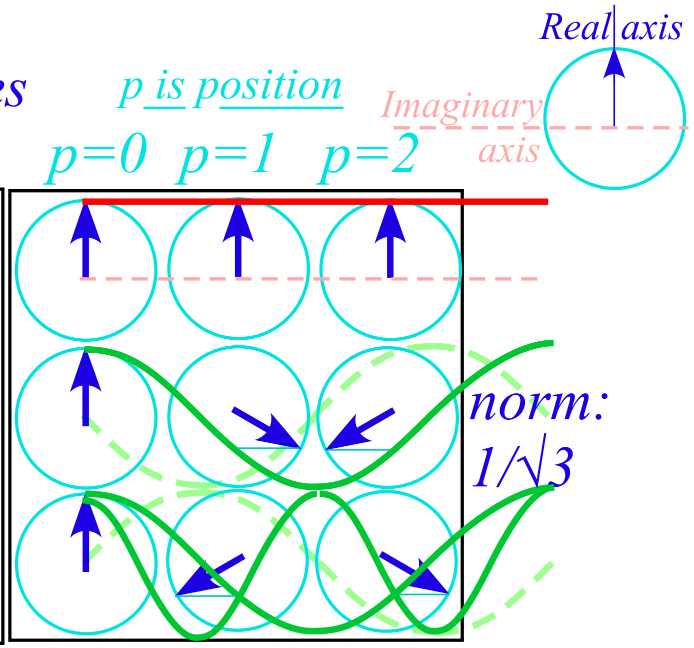

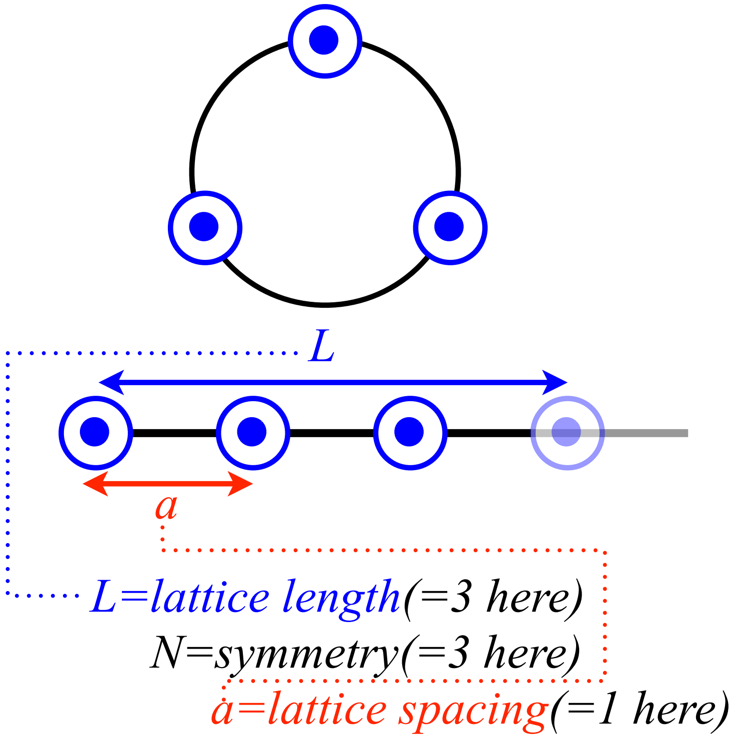

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

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$$\langle(0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

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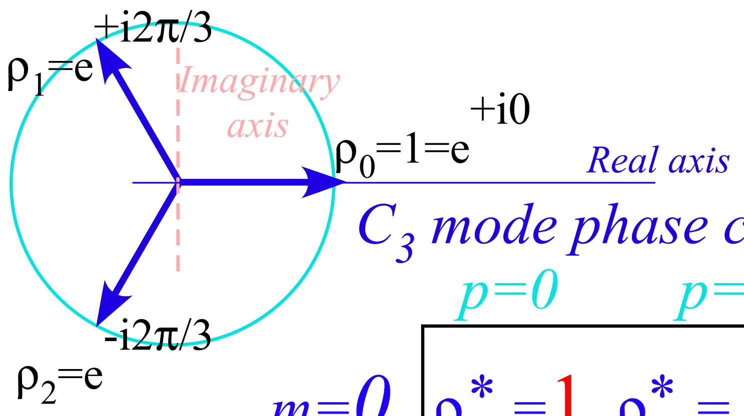

C_3 mode phase character tables

wave-number m= “momentum”

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
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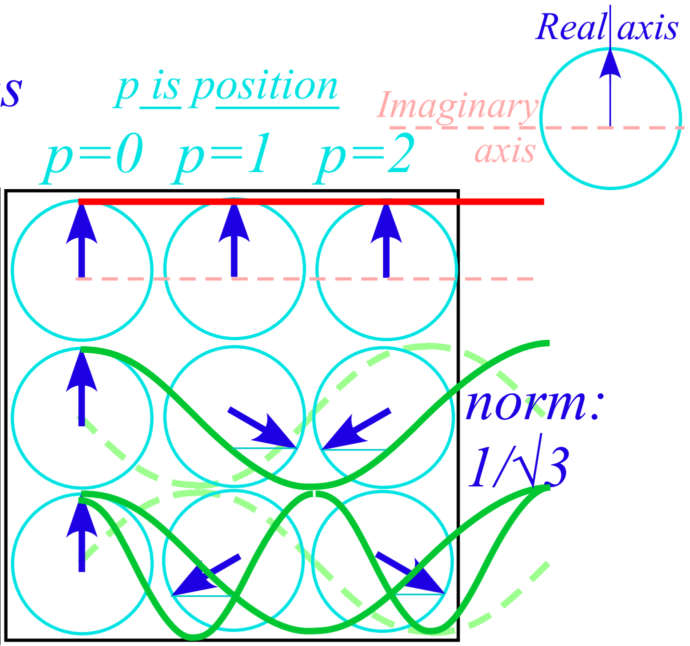

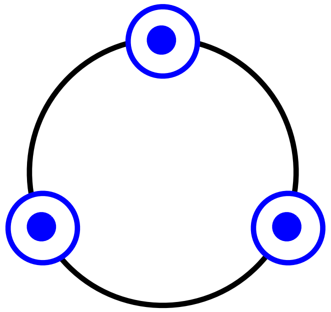

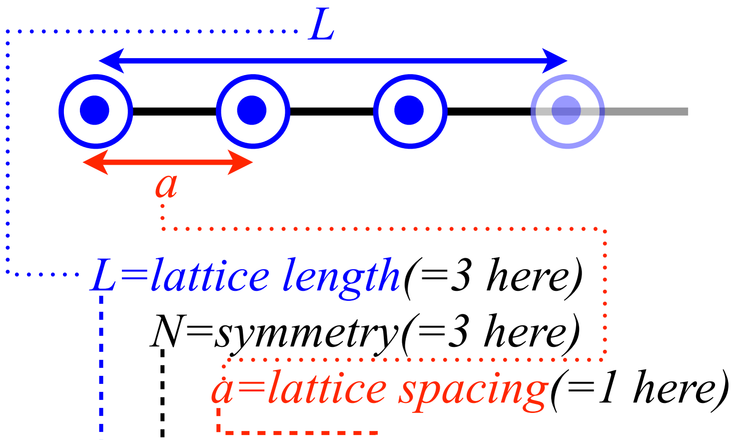

Two distinct types of “quantum” numbers.

p=0,1,or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator’s *position point p*.

m=0,1,or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. ($L=Na=3$)

wavelength $\lambda_m=2\pi/k_m= L/m$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

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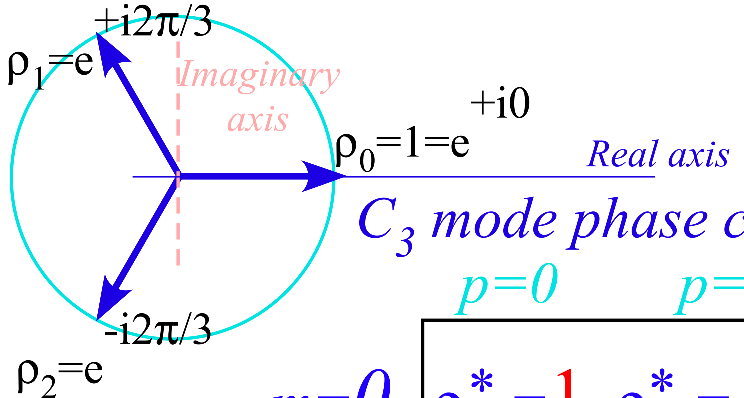

C_3 *mode phase character tables*

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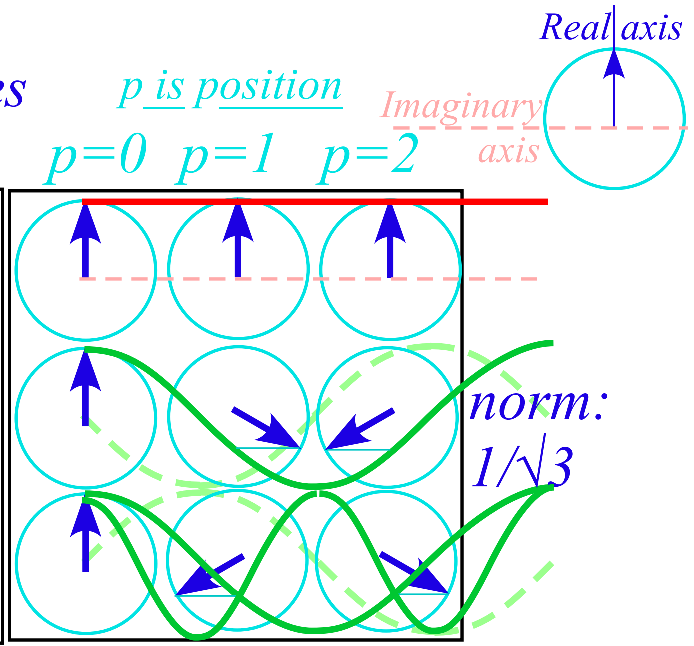
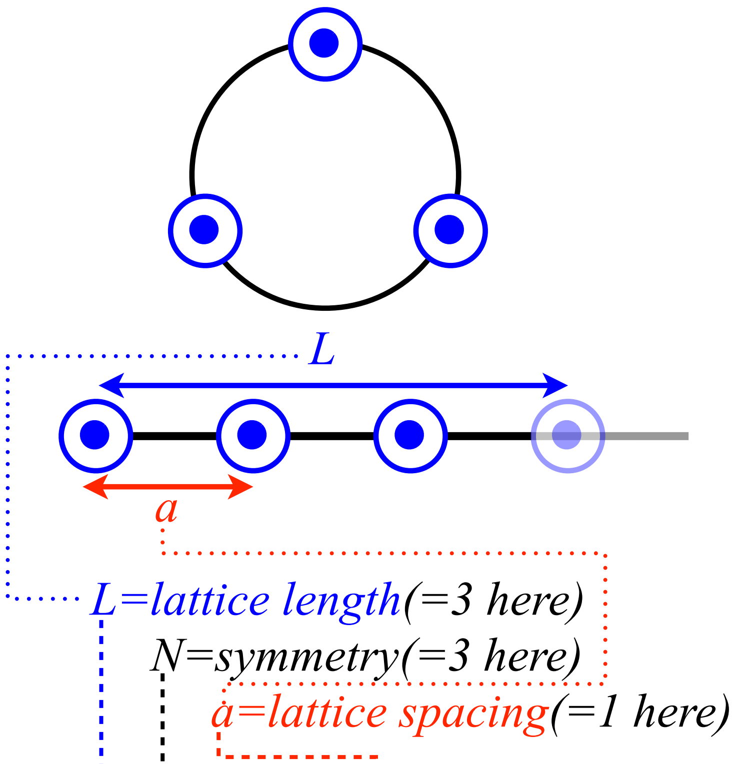
	p=0	p=1	p=2
m=0_3	ρ^*_{00}=1	ρ^*_{01}= 1	ρ^*_{02}= 1
m=1_3	ρ^*_{10}=1	ρ^*_{11}=e$^{-i2\pi/3}$	ρ^*_{12}=e$^{i2\pi/3}$
m=2_3	ρ^*_{20}=1	ρ^*_{21}=e$^{i2\pi/3}$	ρ^*_{22}=e$^{-i2\pi/3}$



(m_3) means: *m-modulo-3* (Details follow)



Two distinct types of “quantum” numbers.

p=0,1,or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator’s *position point p*.

m=0,1,or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. ($L=Na=3$)

wavelength $\lambda_m=2\pi/k_m=L/m$

Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always 0,1,or 2, or else -1,0,or 1, or else -2,-1,or 0, *etc.,* depending on choice of origin.

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

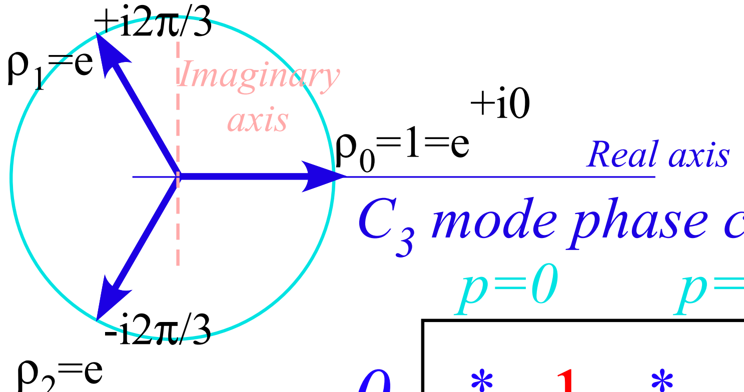
$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

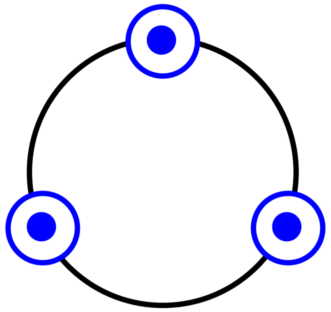
$$\langle (0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\ 1 \quad 1 \quad 1\ )$$

$$\langle (1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\ 1 \quad e^{-i2\pi/3} \quad e^{+i2\pi/3})$$

$$\langle (2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(1 \quad e^{+i2\pi/3} \quad e^{-i2\pi/3})$$



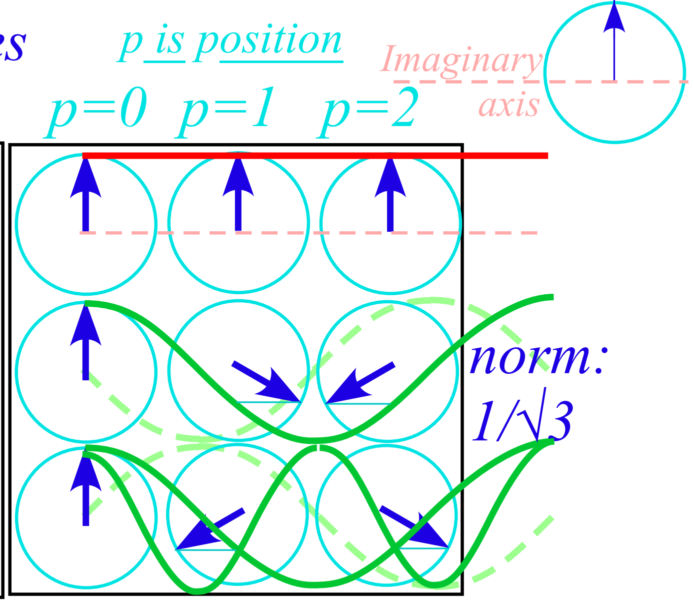
(m_3) means: *m-modulo-3* (Details follow)

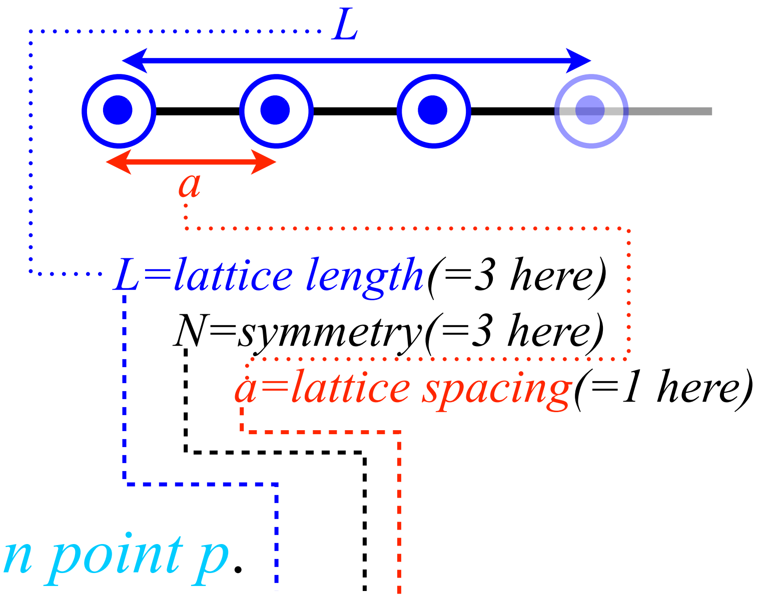


C_3 mode phase character tables

wave-number m= "momentum"

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





L=lattice length(=3 here)

N=symmetry(=3 here)

a=lattice spacing(=1 here)

Two distinct types of "quantum" numbers.

p=0,1,or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator's *position point p*.

m=0,1,or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. $(L=Na=3)$

wavelength $\lambda_m=2\pi/k_m= L/m$

Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always *0,1*,or *2,* or else *-1,0*,or *1,* or else *-2,-1*,or *0, etc.,* depending on choice of origin.

For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}= e^{i4\cdot 2\pi/3}= e^{i1\cdot 2\pi/3}\, e^{i2\pi}= e^{i2\pi/3}=\rho_1$.

That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3=1*, the remainder of *4* divided by *3*.)

- *Wave resonance in cyclic symmetry*
  - *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
  - *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - ➔ *Dispersion functions and standing waves*
  - *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
  - *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\left\langle m\middle|\mathbf{H}\middle|m\right\rangle = \left\langle m\middle|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2\middle|m\right\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\ 2\pi/3}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\left\langle m\middle|\mathbf{H}\middle|m\right\rangle = \left\langle m\middle| r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 \middle|m\right\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

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$$\left\langle m\right|\mathbf{H}\left|m\right\rangle = \left\langle m\right| r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 \left|m\right\rangle = r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\, \frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\, \frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\, \mathbf{r}^p \,|m\rangle = e^{\,i\, m\cdot p\, 2\pi/3}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

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$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$

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H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\;2\pi/3}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0)\\ r_0 - r & (\text{for } m = \pm 1)\end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r\\ r & r_0 & r\\ r & r & r_0\end{pmatrix}\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}$$

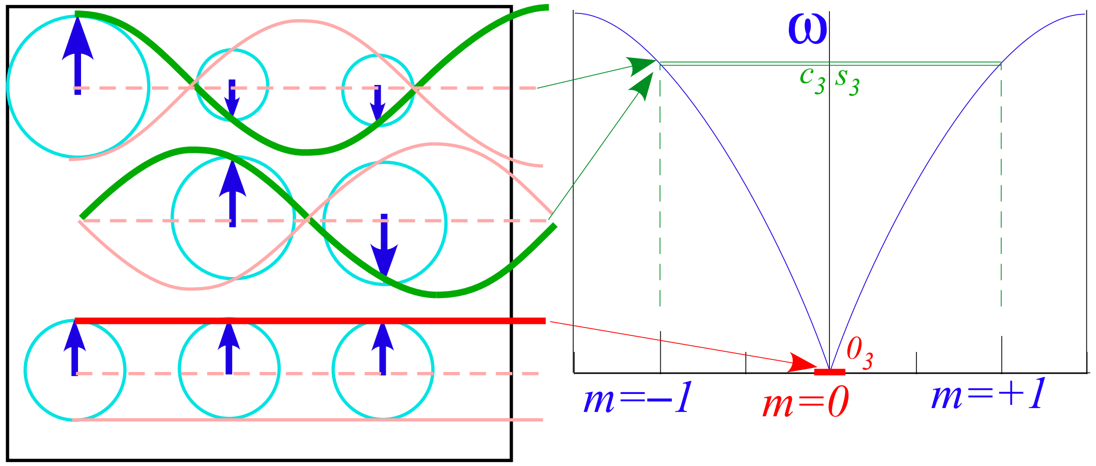
K-eigenvalues:

$$\begin{pmatrix} K & -k & -k\\ -k & K & -k\\ -k & -k & K\end{pmatrix}\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\|(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ e^{+i2\pi/3}\\ e^{-i2\pi/3}\end{pmatrix}$	$\|c_3\rangle = \frac{\|(+1)_3\rangle + \|(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\ -1\\ -1\end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\|(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ e^{-i2\pi/3}\\ e^{+i2\pi/3}\end{pmatrix}$	$\|s_3\rangle = \frac{\|(+1)_3\rangle - \|(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ +1\\ -1\end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\|(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

C_3 *standing wave modes and eigenfrequencies of* **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

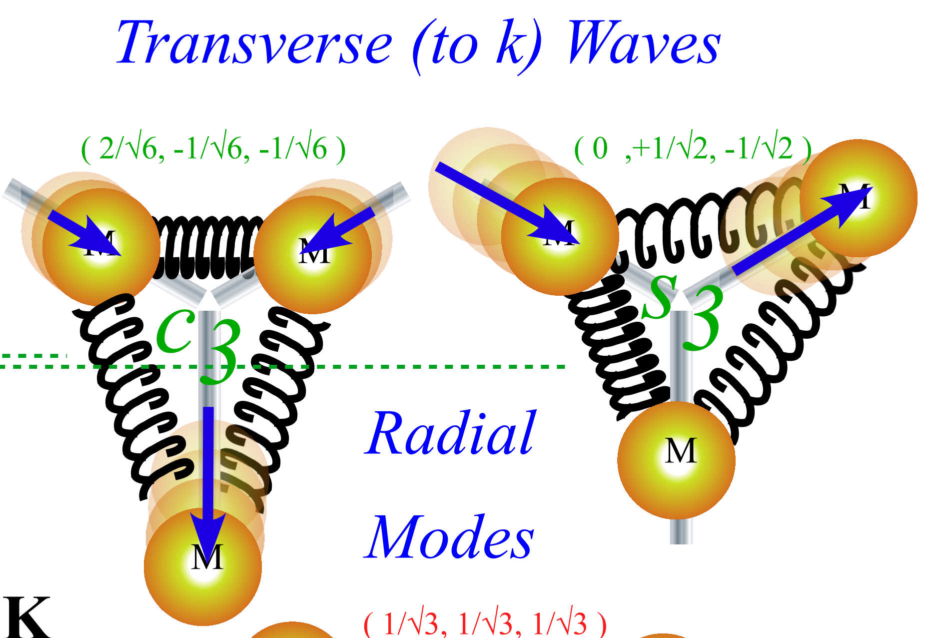
H-eigenvalues:

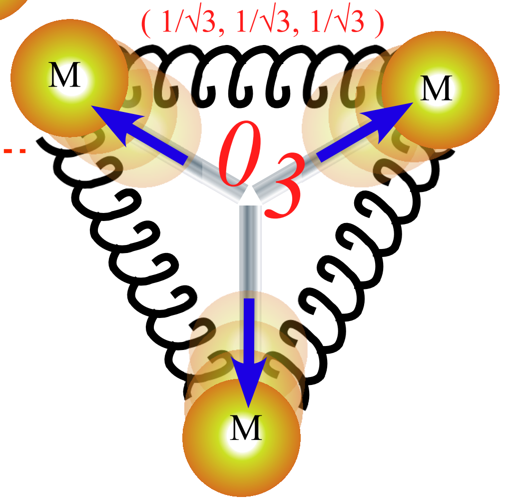
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

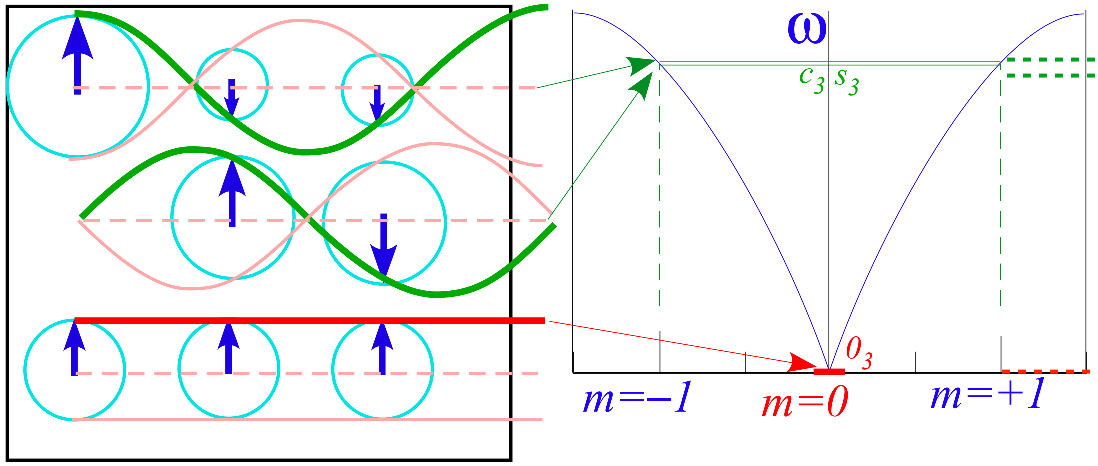
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}$ – *eigenfrequencies*	$\mathbf{K}$ – *eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$





C_3 *standing wave modes and eigenfrequencies of* **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle=\langle m|r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2|m\rangle=r_0e^{i\,m\cdot 0\,\frac{2\pi}{3}}+r_1e^{i\,m\cdot 1\,\frac{2\pi}{3}}+r_2e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\ \mathbf{r}^p\ |m\rangle=e^{\,i\,m\cdot p\,2\pi/3}$$

$$=r_0e^{i\,m\cdot 0\,\frac{2\pi}{3}}+r(e^{i\frac{2\pi m}{3}}+e^{-i\frac{2\pi m}{3}})=r_0+2r\cos(\tfrac{2\pi m}{3})=\begin{cases}r_0+2r & (\text{for } m=0)\\ r_0-r & (\text{for } m=\pm 1)\end{cases}$$

H-eigenvalues:

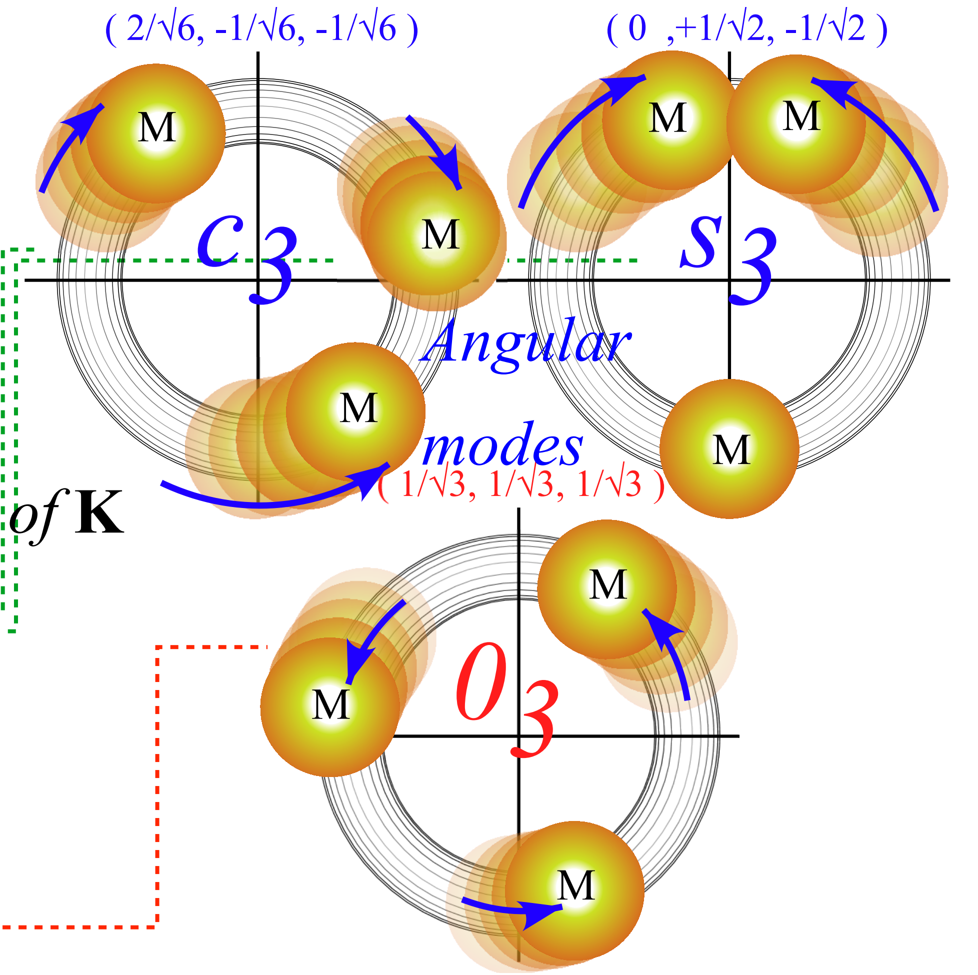
$$\begin{pmatrix}r_0 & r & r\\ r & r_0 & r\\ r & r & r_0\end{pmatrix}\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}=\left(r_0+2r\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix}K & \text{-}k & \text{-}k\\ \text{-}k & K & \text{-}k\\ \text{-}k & \text{-}k & K\end{pmatrix}\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}=\left(K-2k\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix}1\\ e^{i\frac{2m\pi}{3}}\\ e^{-i\frac{2m\pi}{3}}\end{pmatrix}$$

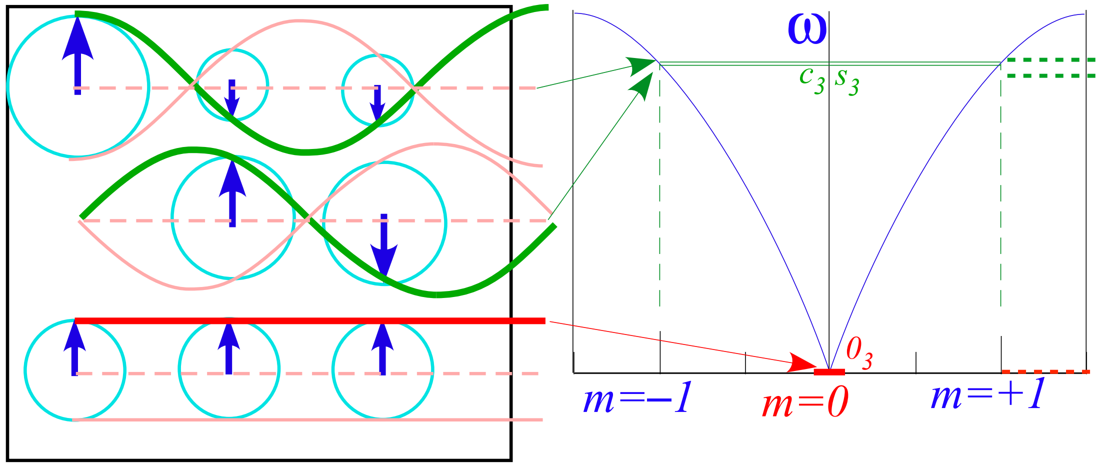
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle=\frac{1}{\sqrt3}\begin{pmatrix}1\\ e^{+i2\pi/3}\\ e^{-i2\pi/3}\end{pmatrix}$	$\lvert c_3\rangle=\frac{\lvert(+1)_3\rangle+\lvert(-1)_3\rangle}{\sqrt2}=\frac{1}{\sqrt6}\begin{pmatrix}2\\ -1\\ -1\end{pmatrix}$	$r_0+2r\cos(\frac{2m\pi}{3})$ $=r_0-r$	$\sqrt{k_0-2k\cos(\frac{2m\pi}{3})}$ $=\sqrt{k_0+k}$
$\lvert(-1)_3\rangle=\frac{1}{\sqrt3}\begin{pmatrix}1\\ e^{-i2\pi/3}\\ e^{+i2\pi/3}\end{pmatrix}$	$\lvert s_3\rangle=\frac{\lvert(+1)_3\rangle-\lvert(-1)_3\rangle}{i\sqrt2}=\frac{1}{\sqrt2}\begin{pmatrix}0\\ +1\\ -1\end{pmatrix}$	$r_0+2r\cos(\frac{-2m\pi}{3})$ $=r_0-r$	$\sqrt{k_0-2k\cos(\frac{2m\pi}{3})}$ $=\sqrt{k_0+k}$
	$\lvert(0)_3\rangle=\frac{1}{\sqrt3}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	r_0+2r	$\sqrt{k_0-2k}$

Longitudinal (to k) Waves



	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3

C_3 standing wave modes and eigenfrequencies of **K**



- *Wave resonance in cyclic symmetry*
    - *Harmonic oscillator with cyclic C_2 symmetry*
        - *C_2 symmetric (B-type) modes*
    - *Harmonic oscillator with cyclic C_3 symmetry*
        - *C_3 symmetric spectral decomposition by 3rd roots of unity*
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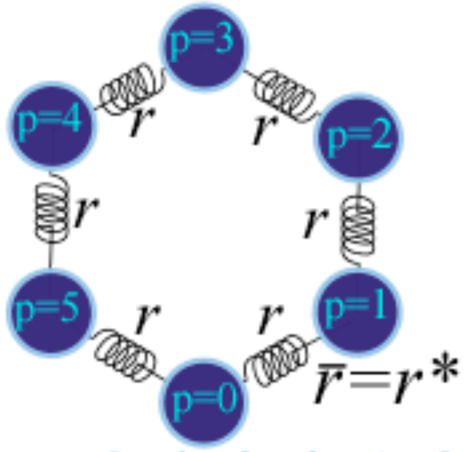
        - *Dispersion functions and standing waves*
    - ➡ *C_6 symmetric mode model:Distant neighbor coupling*
        - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
    - *C_N symmetric mode models: Made-to order dispersion functions*
        - *Quadratic dispersion models: Super-beats and fractional revivals*
        - *Phase arithmetic*

C_6 Symmetric Mode Model: Distant neighbor coupling

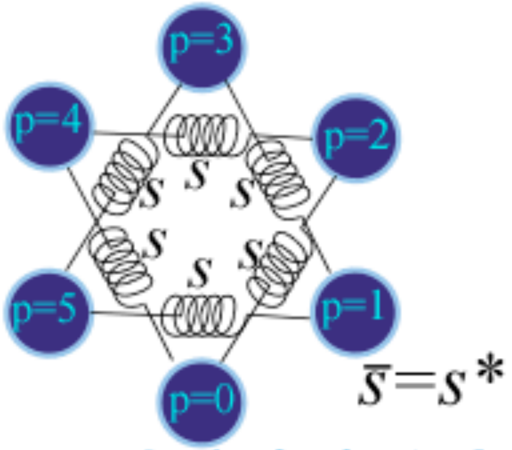
(a) 1st Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{cc} \begin{matrix} p=0 & 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ \begin{pmatrix} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

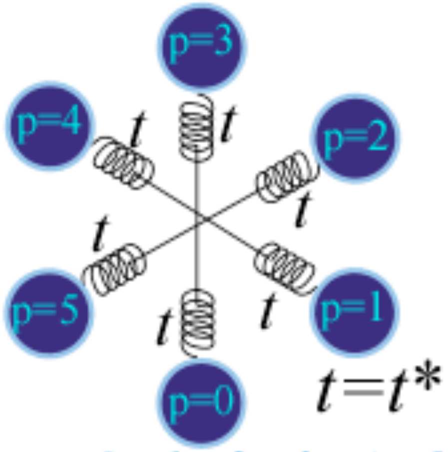
(b) 2nd Neighbor C_6



$$\mathbf{H}^{B2(6)}=\begin{array}{cc} \begin{matrix} p=0 & 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ \begin{pmatrix} H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot \\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s} \\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot \\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s \\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot \\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

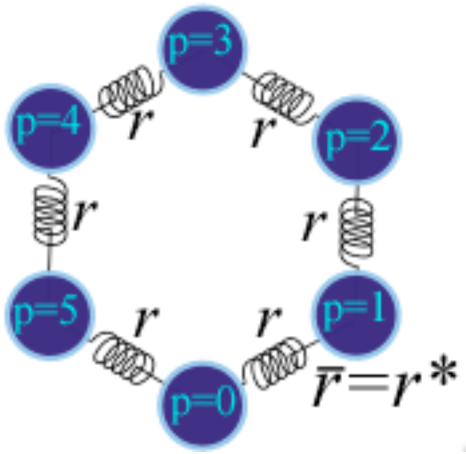
$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$

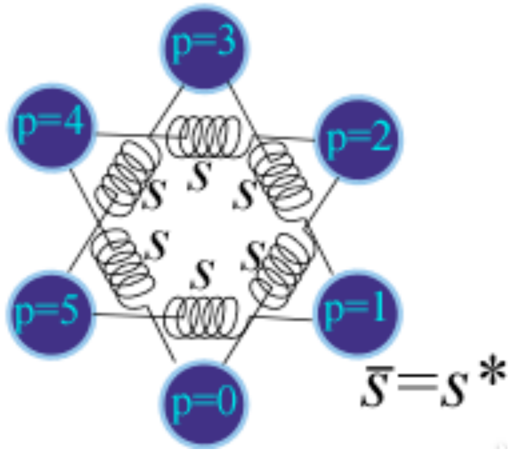
(c) 3rd Neighbor C_6

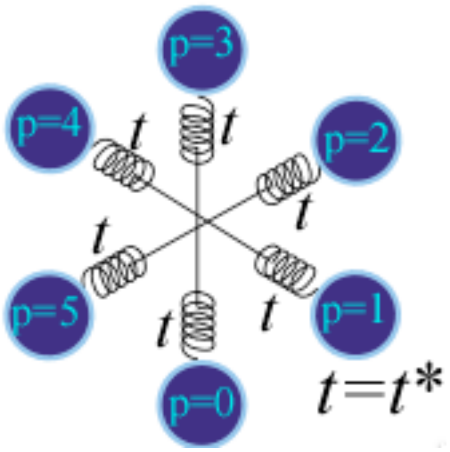


$$\mathbf{H}^{B3(6)}=\begin{array}{cc} \begin{matrix} p=0 & 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ \begin{pmatrix} H_3 & \cdot & \cdot & -t & \cdot & \cdot \\ \cdot & H_3 & \cdot & \cdot & -t & \cdot \\ \cdot & \cdot & H_3 & \cdot & \cdot & -t \\ -t & \cdot & \cdot & H_3 & \cdot & \cdot \\ \cdot & -t & \cdot & \cdot & H_3 & \cdot \\ \cdot & \cdot & -t & \cdot & \cdot & H_3 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$



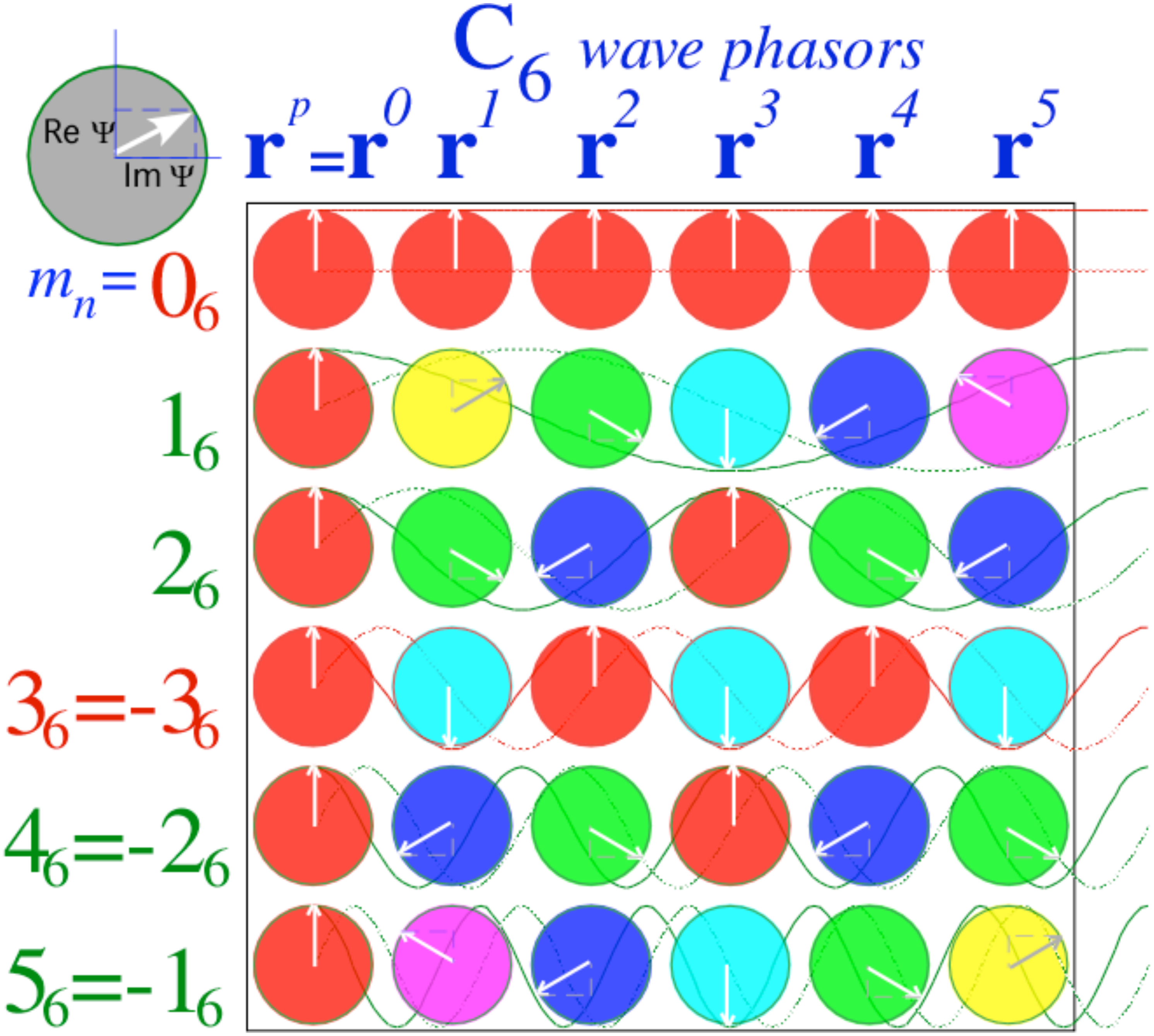




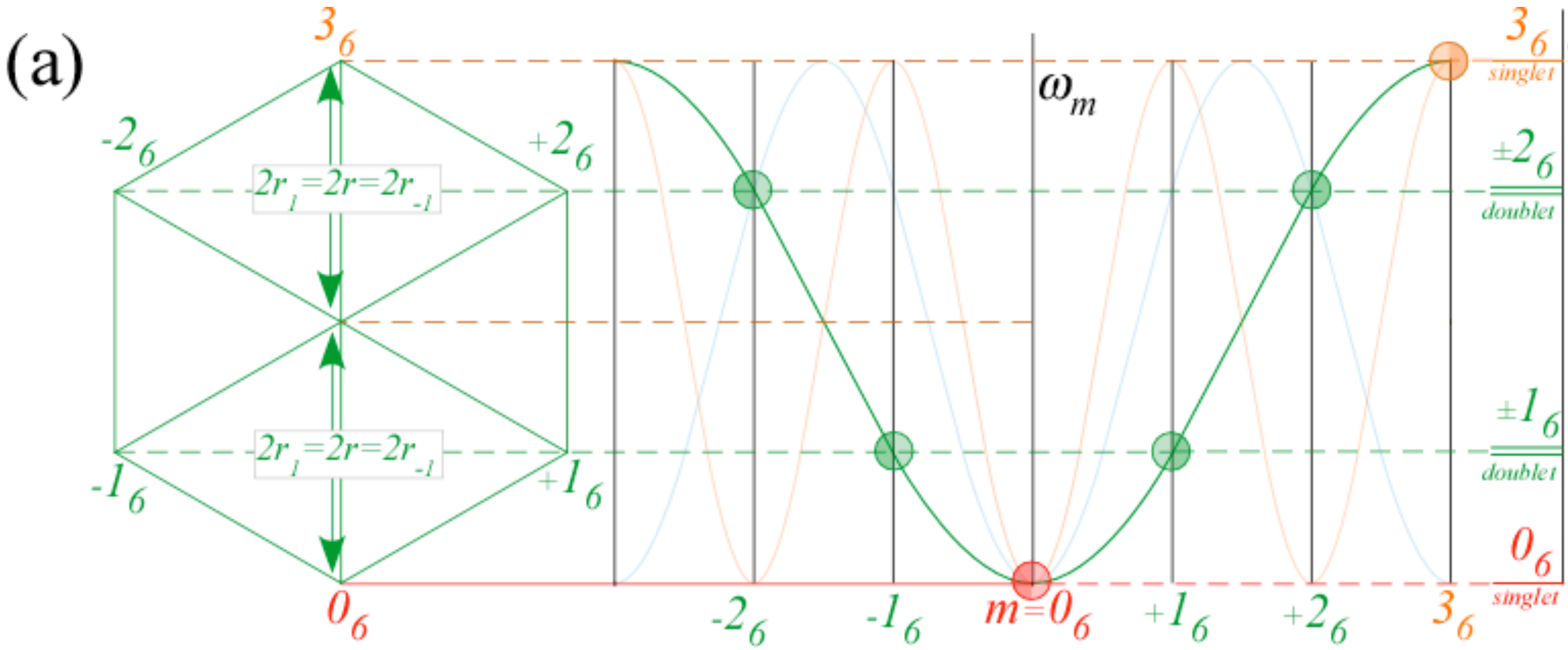
C_6 Spectral resolution: 6th roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

$$Wavefunction:\ \Psi^m(x_p)=\chi_p^{m*}=D^{m*}(\mathbf{r}^p)$$



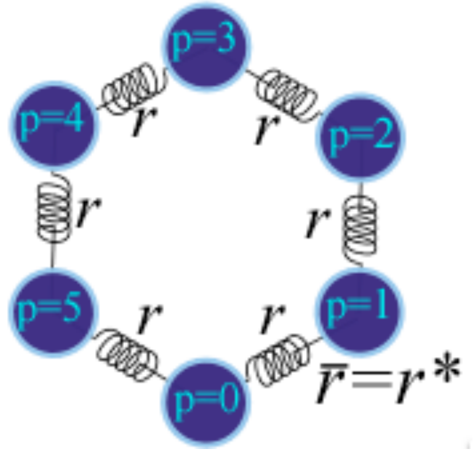
C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion

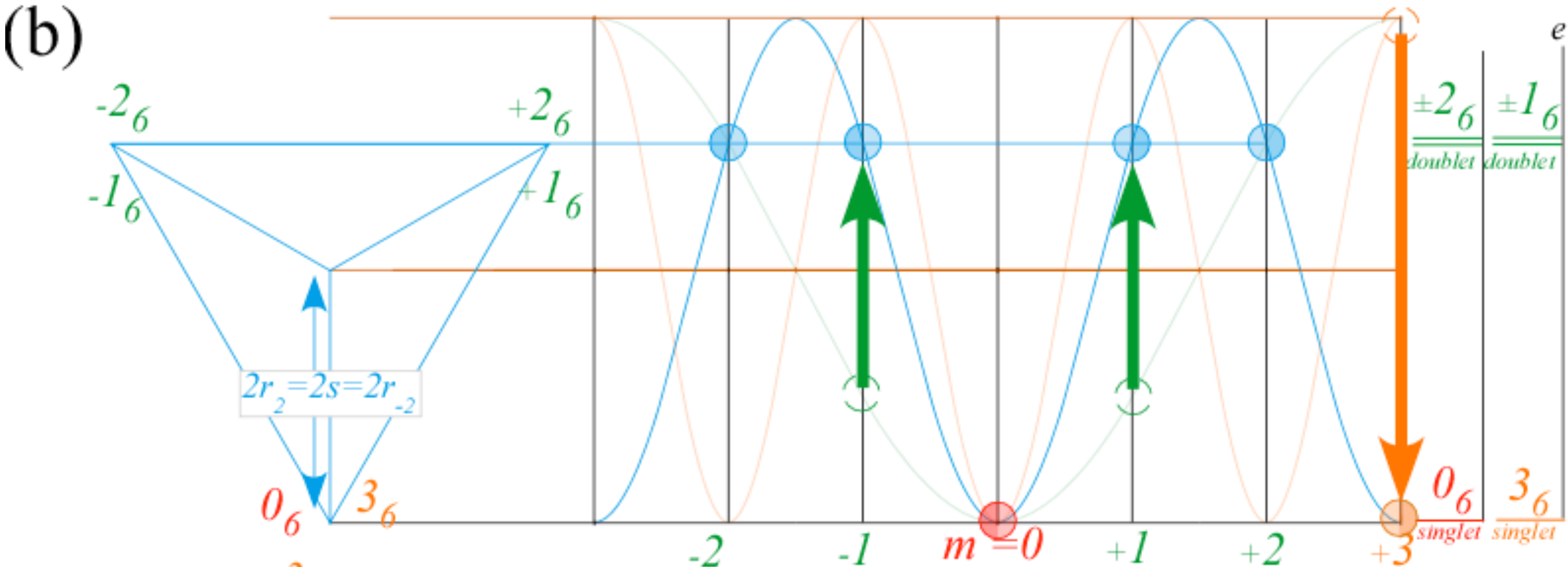


eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc|}
p= & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & H_1 & -r & \cdot & \cdot & \cdot & -r \\
1 & -r & H_1 & -r & \cdot & \cdot & \cdot \\
2 & \cdot & -r & H_1 & -r & \cdot & \cdot \\
3 & \cdot & \cdot & -r & H_1 & -r & \cdot \\
4 & \cdot & \cdot & \cdot & -r & H_1 & -r \\
5 & -r & \cdot & \cdot & \cdot & -r & H_1
\end{array}$$

1^{st} Neighbor H

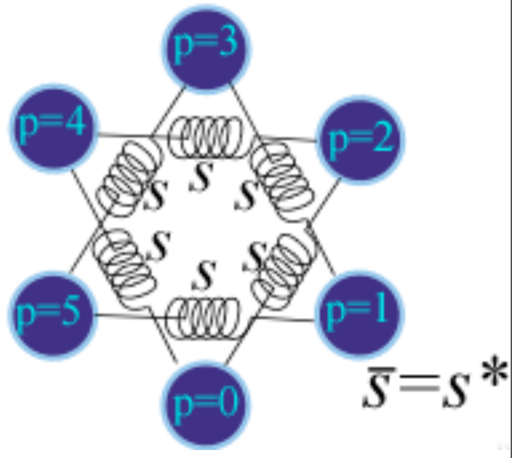


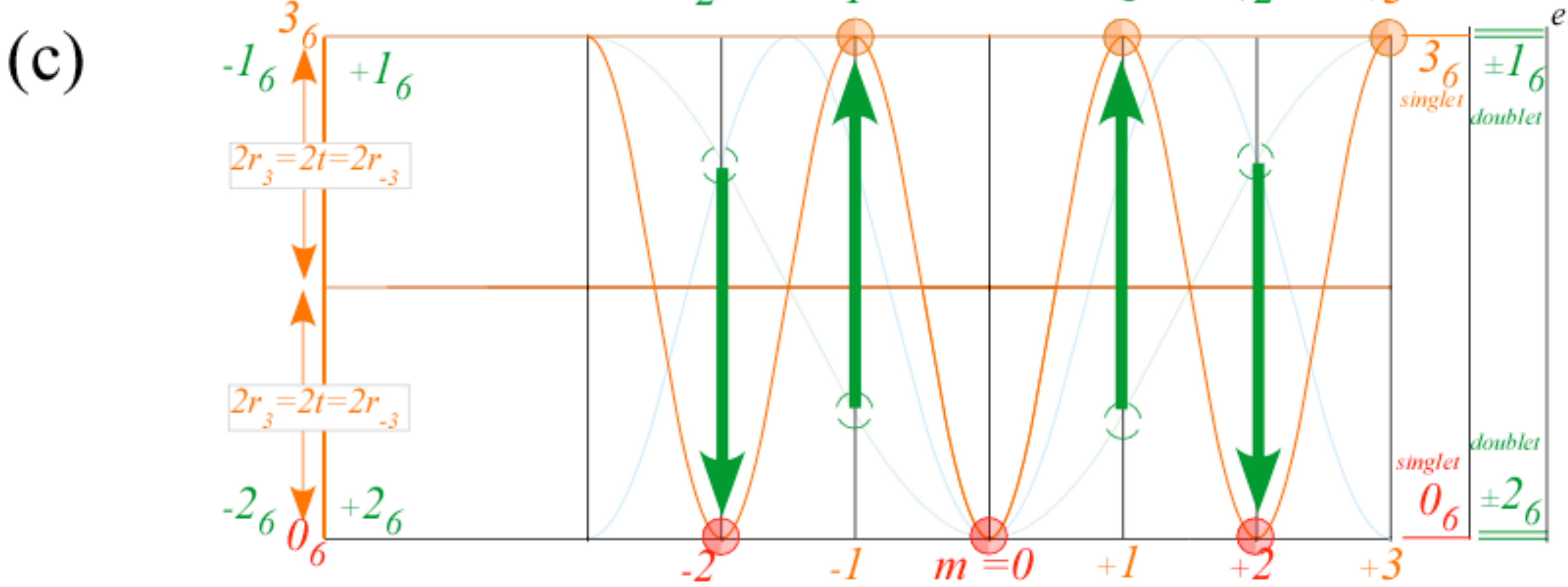


eigenvalues of $\mathbf{H}^{B2(6)}$

$$\begin{array}{c|cccccc|}
p= & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & H_2 & \cdot & -s & \cdot & -s & \cdot \\
1 & \cdot & H_2 & \cdot & -s & \cdot & -s \\
2 & -s & \cdot & H_2 & \cdot & -s & \cdot \\
3 & \cdot & -s & \cdot & H_2 & \cdot & -s \\
4 & -s & \cdot & -s & \cdot & H_2 & \cdot \\
5 & \cdot & -s & \cdot & -s & \cdot & H_2
\end{array}$$

2^{nd} Neighbor H

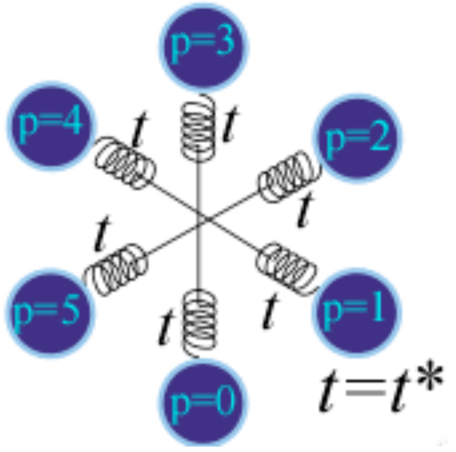




eigenvalues of $\mathbf{H}^{B3(6)}$

$$\begin{array}{c|cccccc|}
p= & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & H_3 & \cdot & \cdot & -t & \cdot & \cdot \\
1 & \cdot & H_3 & \cdot & \cdot & -t & \cdot \\
2 & \cdot & \cdot & H_3 & \cdot & \cdot & -t \\
3 & -t & \cdot & \cdot & H_3 & \cdot & \cdot \\
4 & \cdot & -t & \cdot & \cdot & H_3 & \cdot \\
5 & \cdot & \cdot & -t & \cdot & \cdot & H_3
\end{array}$$

3^{rd} Neighbor H



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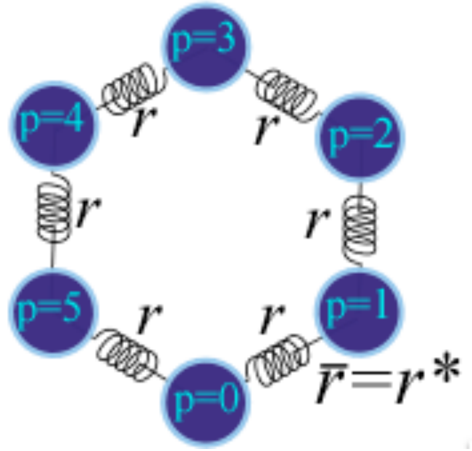
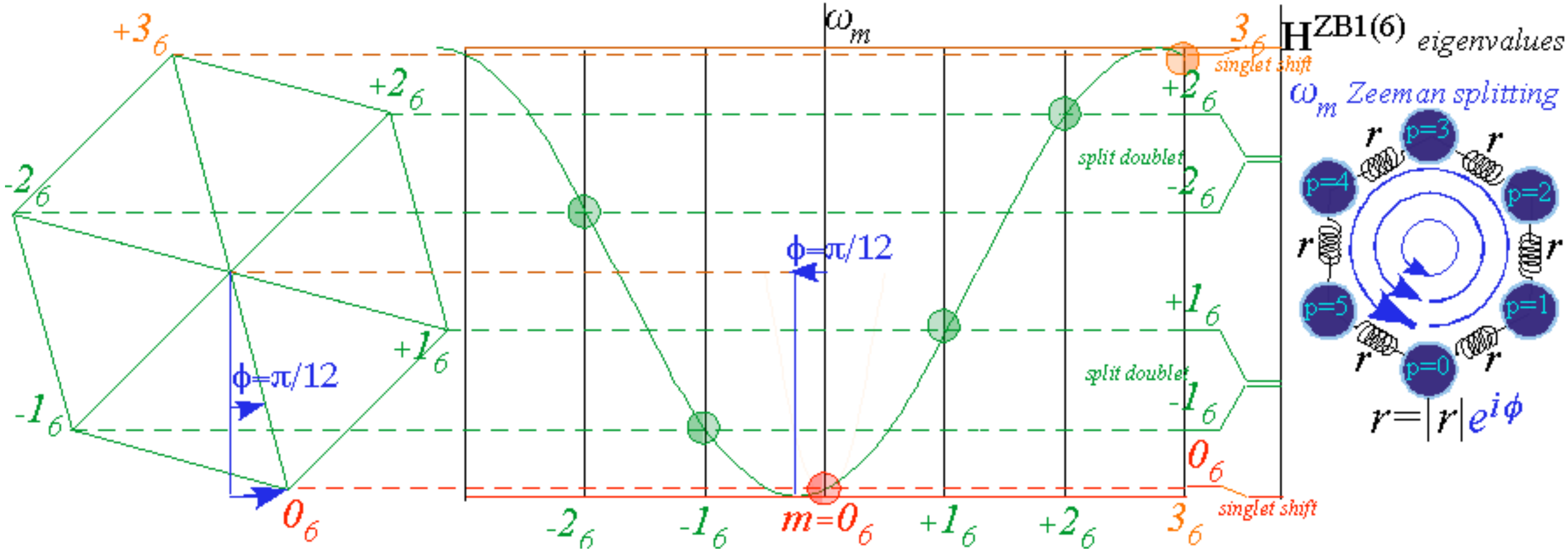
C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,

1st Neighbor H





Wave resonance in cyclic symmetry

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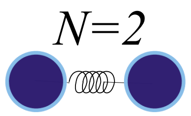
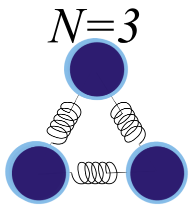
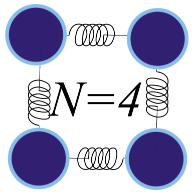
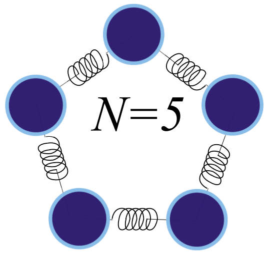
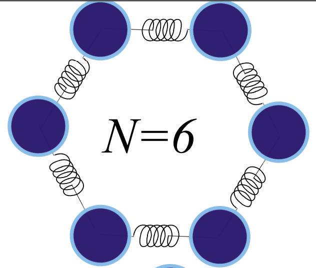
C_N Symmetric Mode Models:

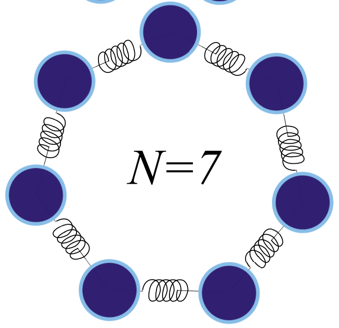


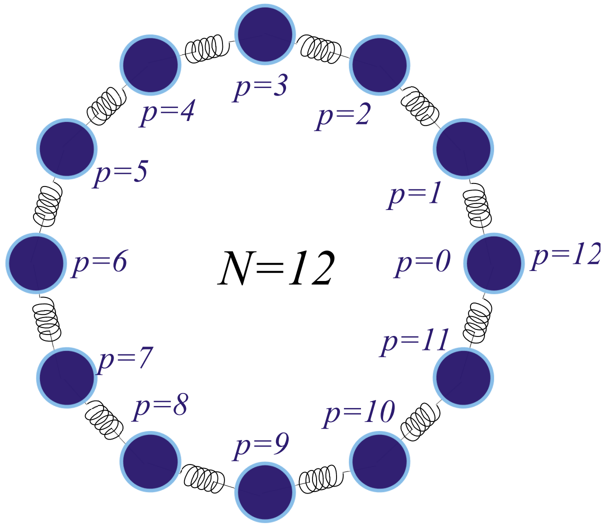




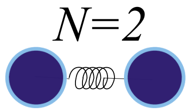


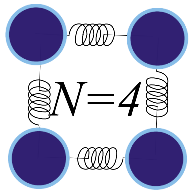


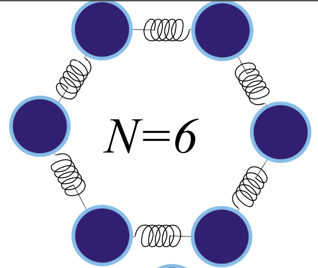


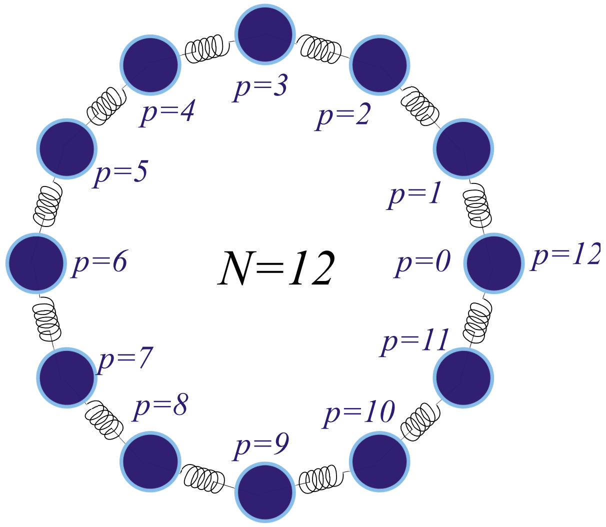


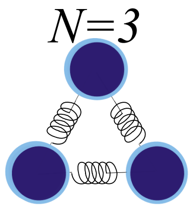
C_N Symmetric Mode Models:

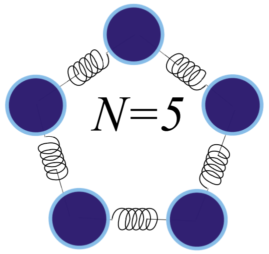


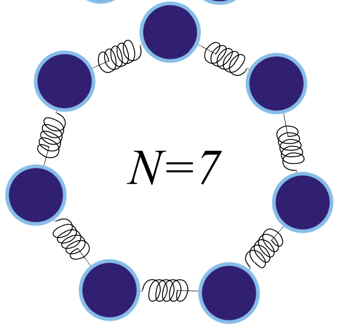












1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

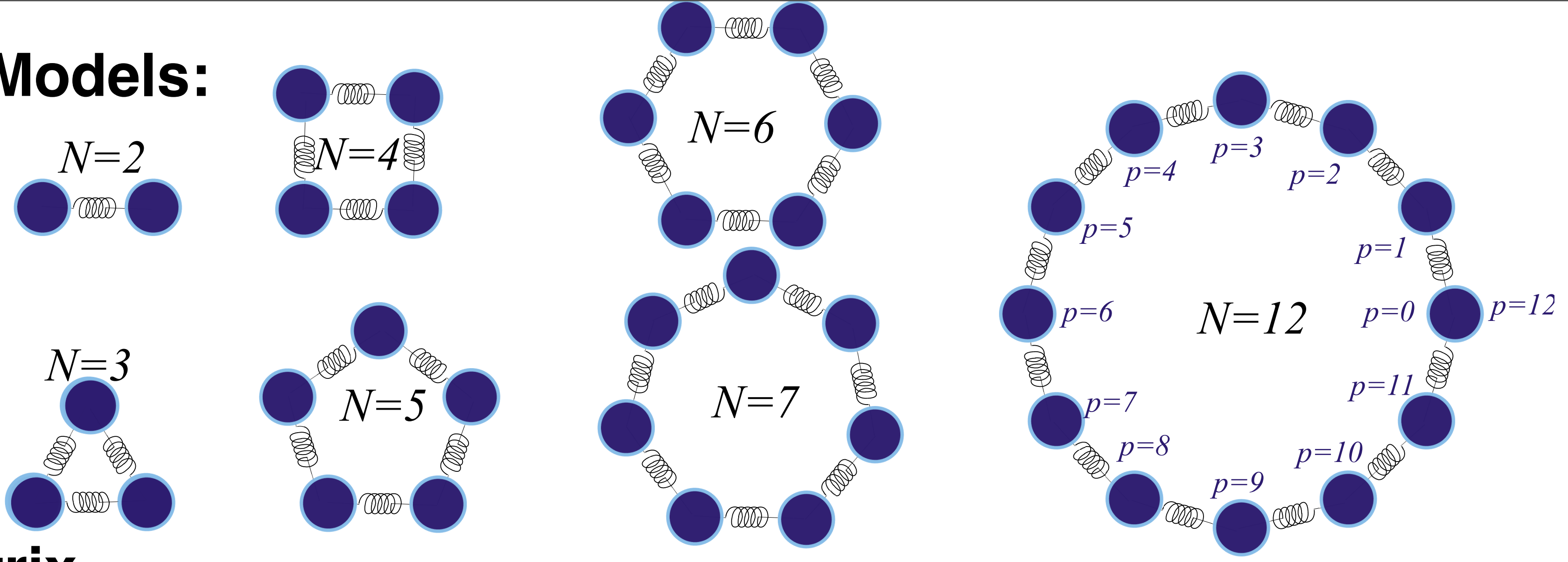
where:

$$K = k + 2k_{12}$$

$$k = \frac{Mg}{\ell}$$

$$(\cdot) = 0$$

C_N Symmetric Mode Models:



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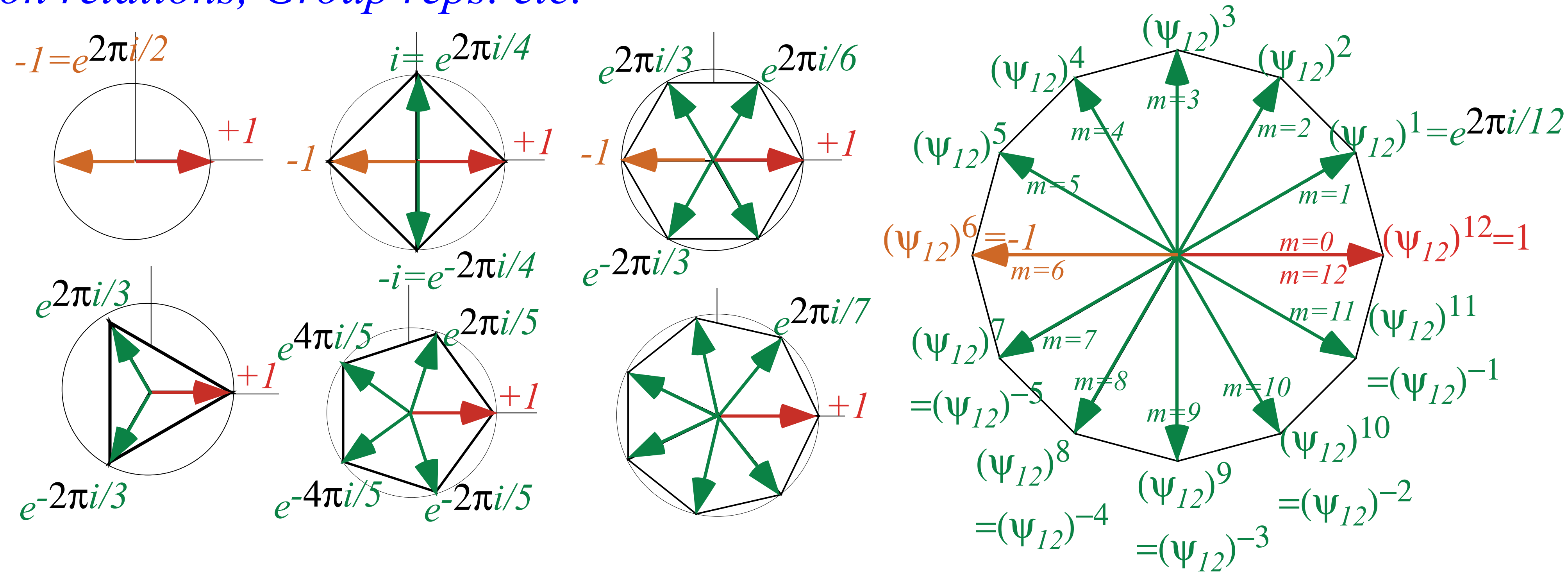
where:

$$K = k + 2k_{12}$$

$$k = \frac{Mg}{\ell}$$

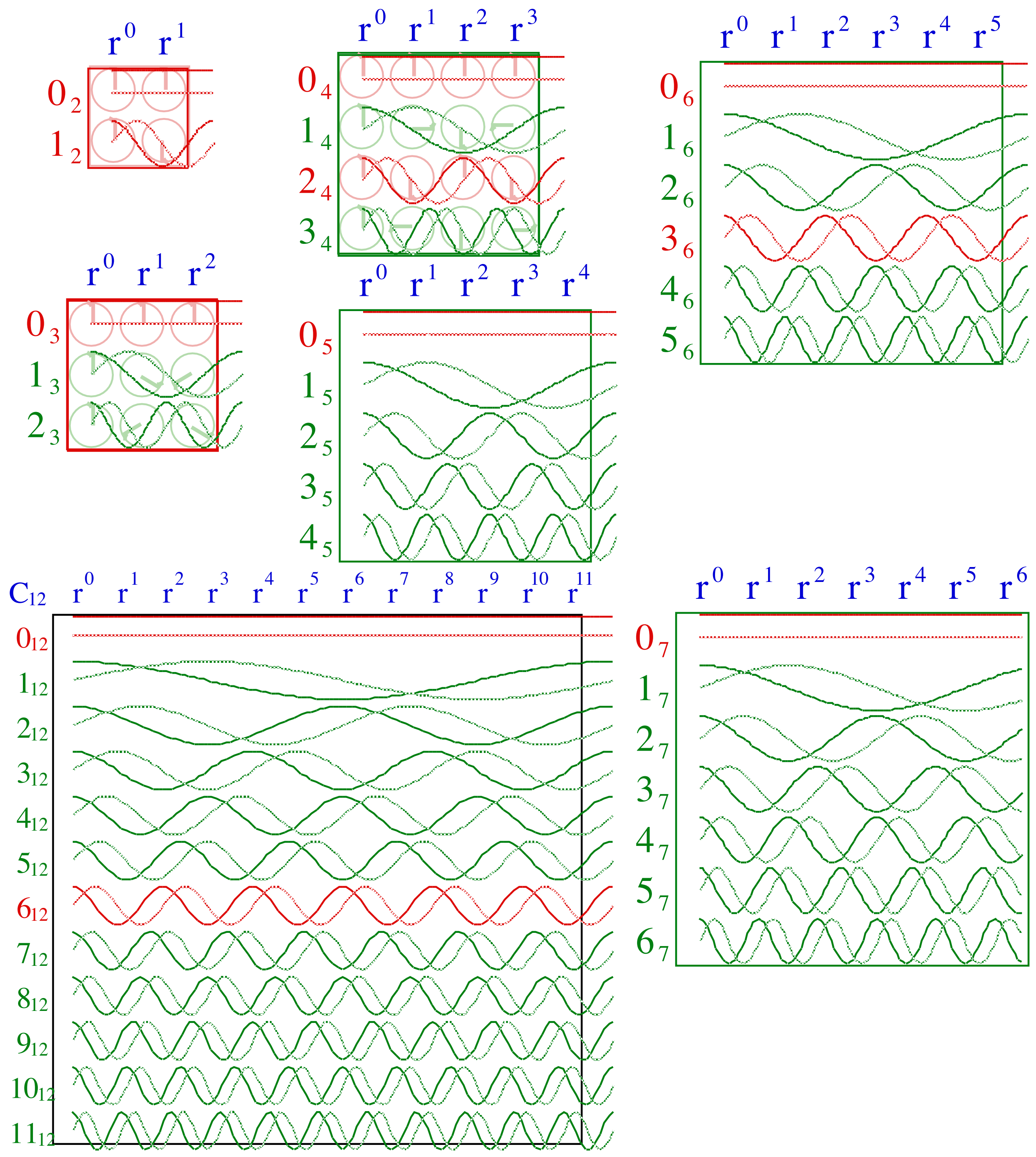
$$(\cdot) = 0$$

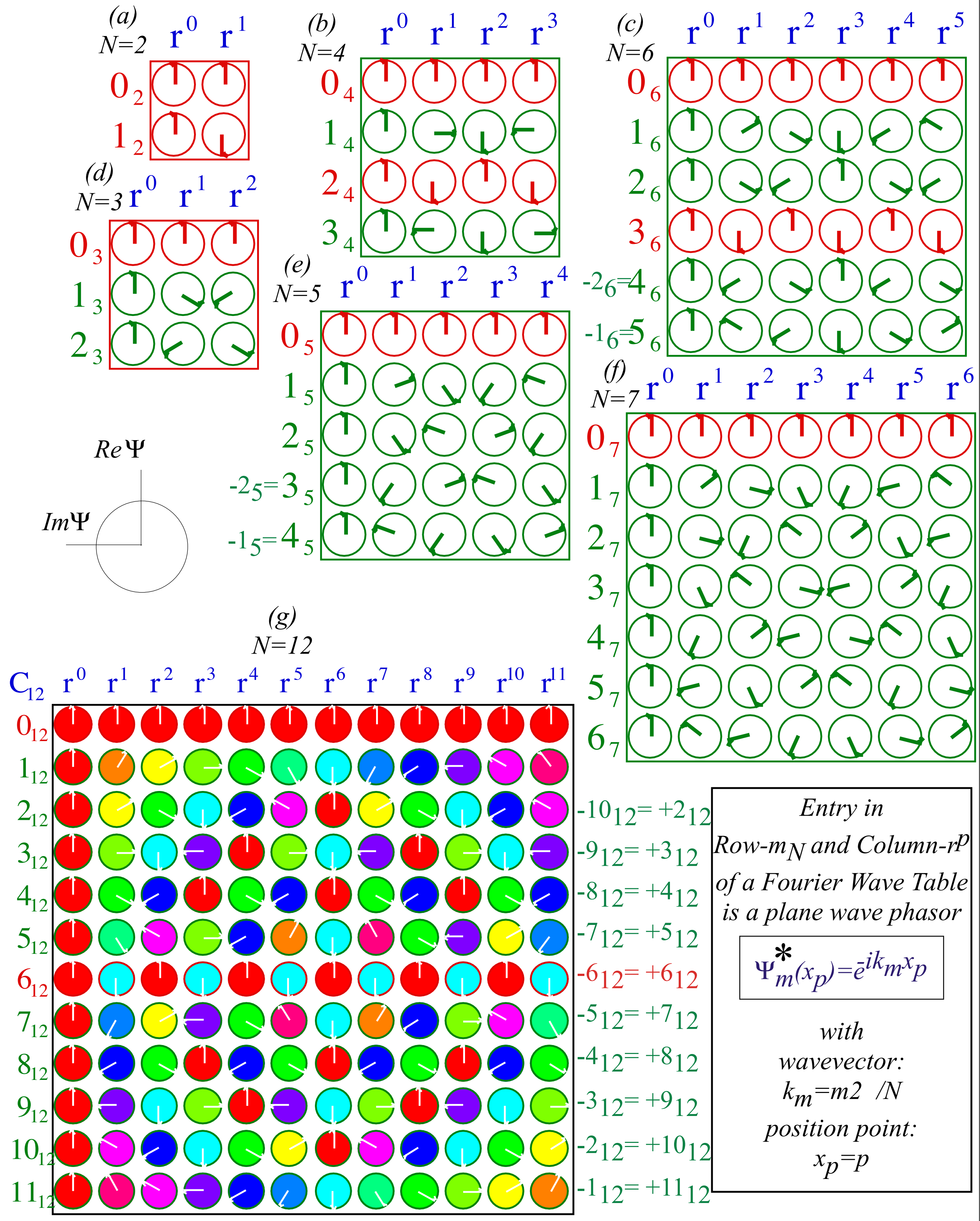
Nth roots of 1 $e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*



C_N Symmetric Mode Models:

N^{th} roots of 1 $e^{i\,m\cdot p\,2\pi/N}=\langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*





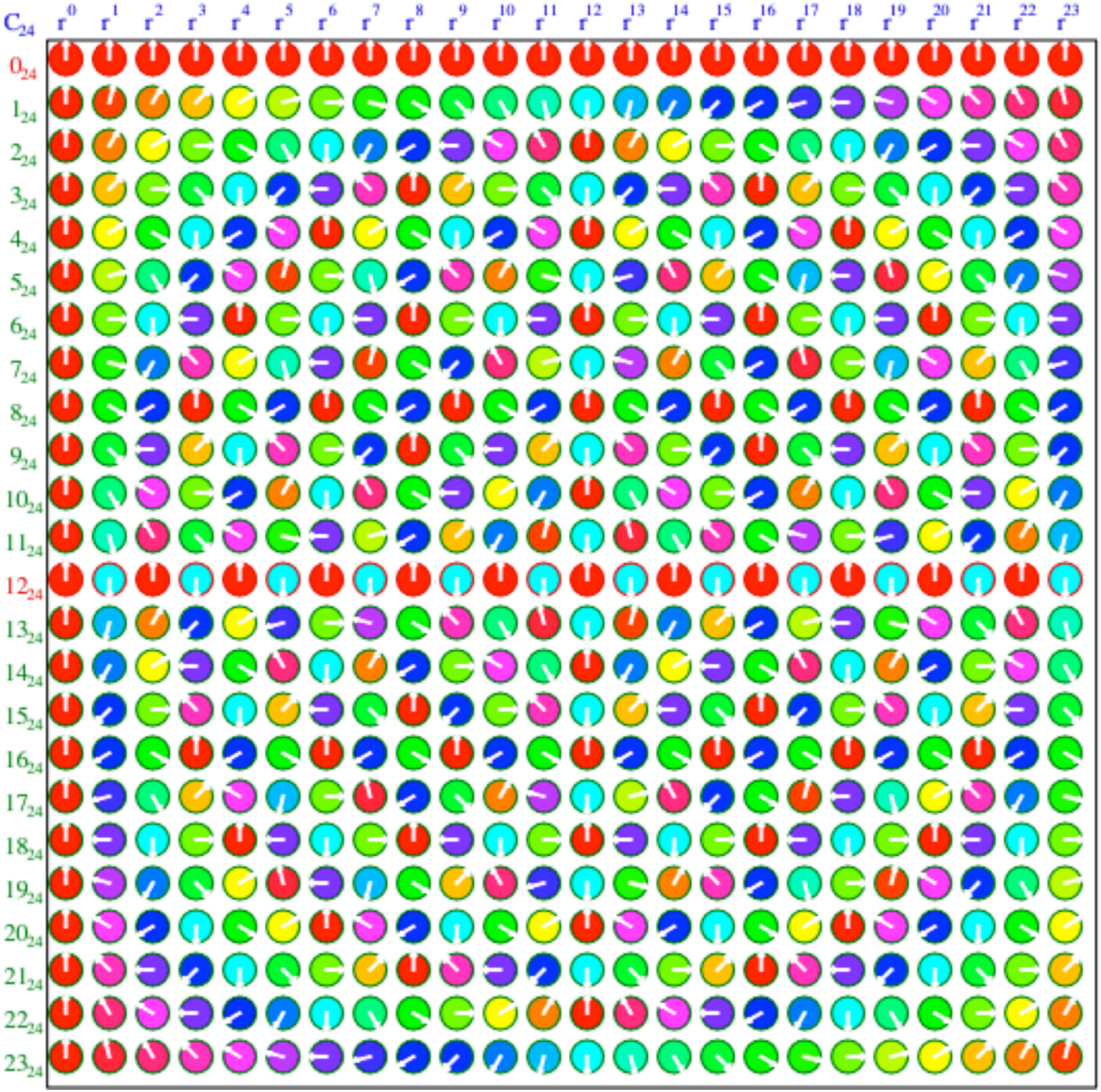
Entry in Row-m_N and Column-r^p of a Fourier Wave Table is a plane wave phasor

$$\Psi^*_m(x_p)=e^{-ik_m x_p}$$

with wavevector:
$k_m=m2\pi/N$
position point:
$x_p=p$

C_{24} Symmetric Modes in

Fourier transformation matrix



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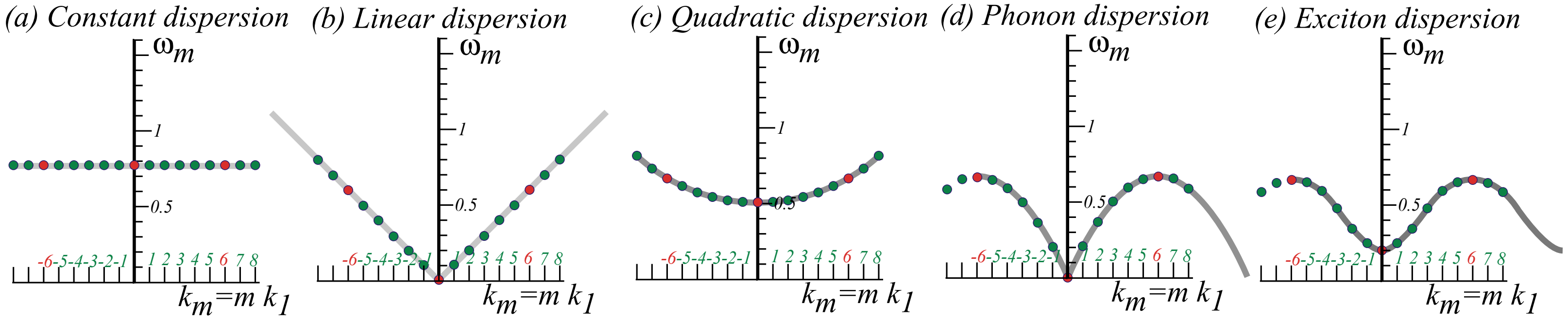
C_N Symmetric Mode Models: Made-to-Order Dispersion *(and wave dynamics)*

(Making pure linear $\omega = ck$, quadratic $\omega = ck^2$, etc. ?)

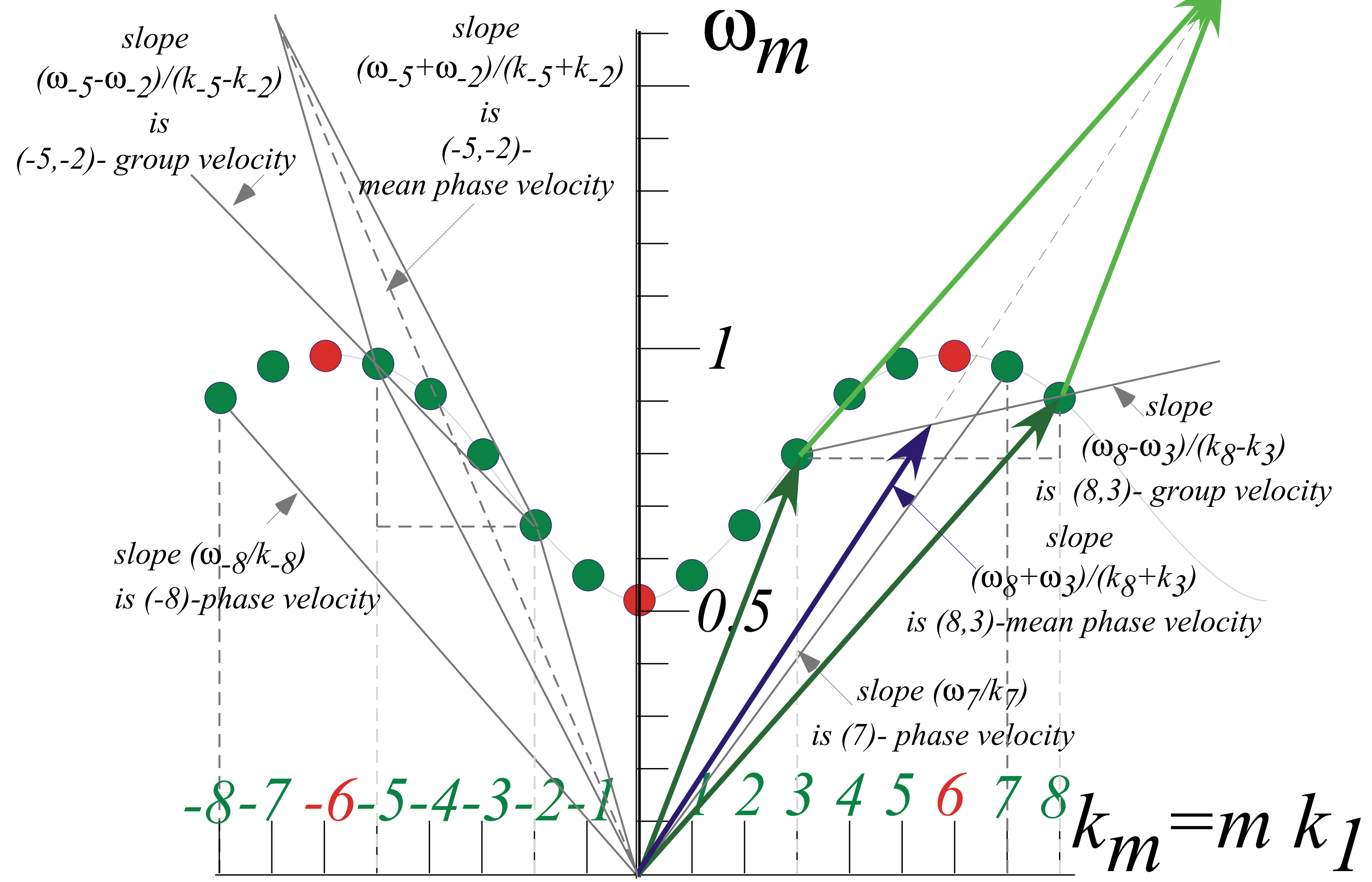
Archetypical Examples of Dispersion Functions



Applications:

Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis *Xmas lights*	*Light in vacuum (Exactly)* *Sound (Approximately)*	*Light in fiber (Approx)* *Non-relativistic* *Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids* *Relativistic matter* *(If exact hyperbola)*

Reading Wave Velocity From Dispersion Function by (k,ω) Vectors



$$a = k_a \cdot x - \omega_a \cdot t$$

$$b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right)$$

$$= e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2} \right)$$

Things determined by

Dispersion $\omega = \omega(k)$

Individual phase velocity:

$$V_{phase-1} = \frac{\omega(k)}{k}$$

Pairwise phase velocity:

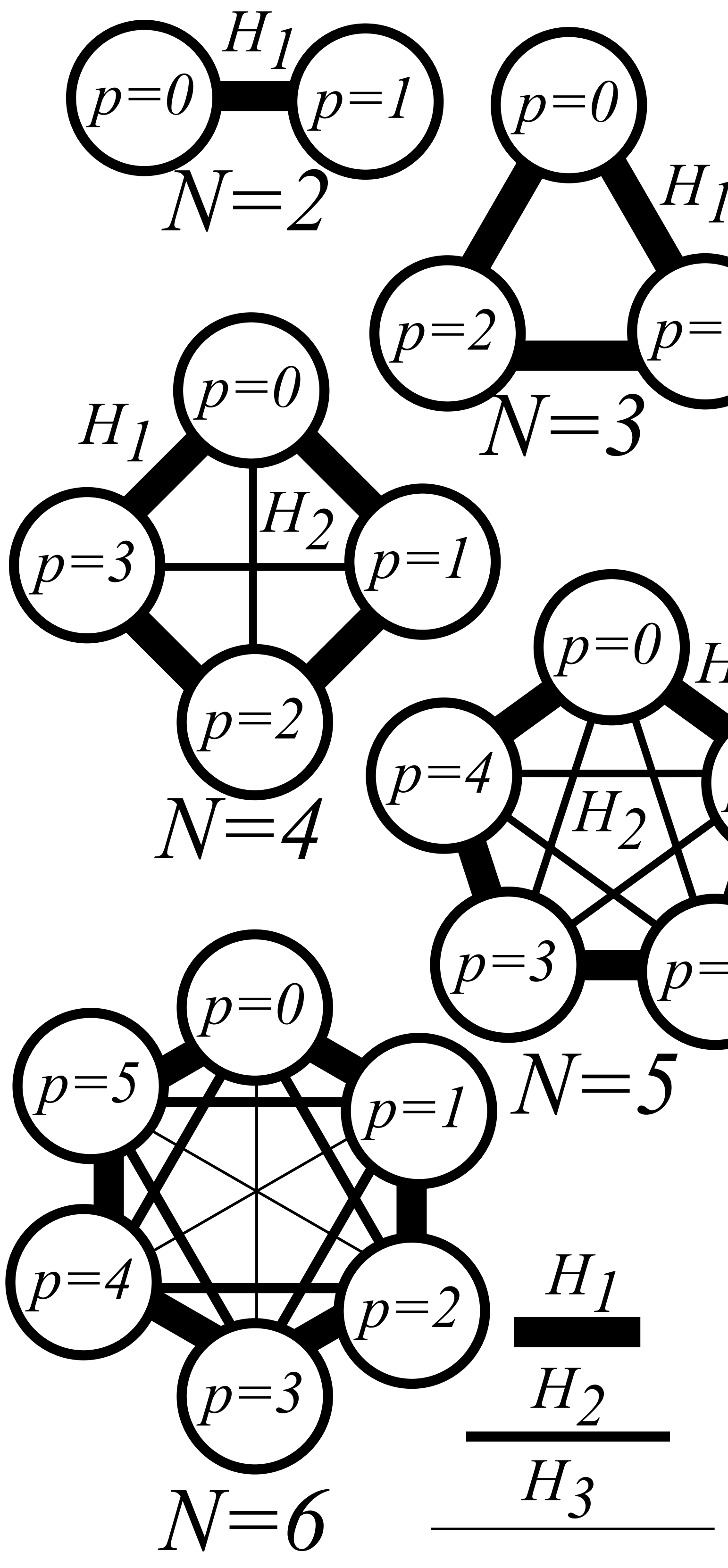
$$V_{phase-2} = \frac{\omega(k_a) + \omega(k_b)}{k_a + k_b}$$

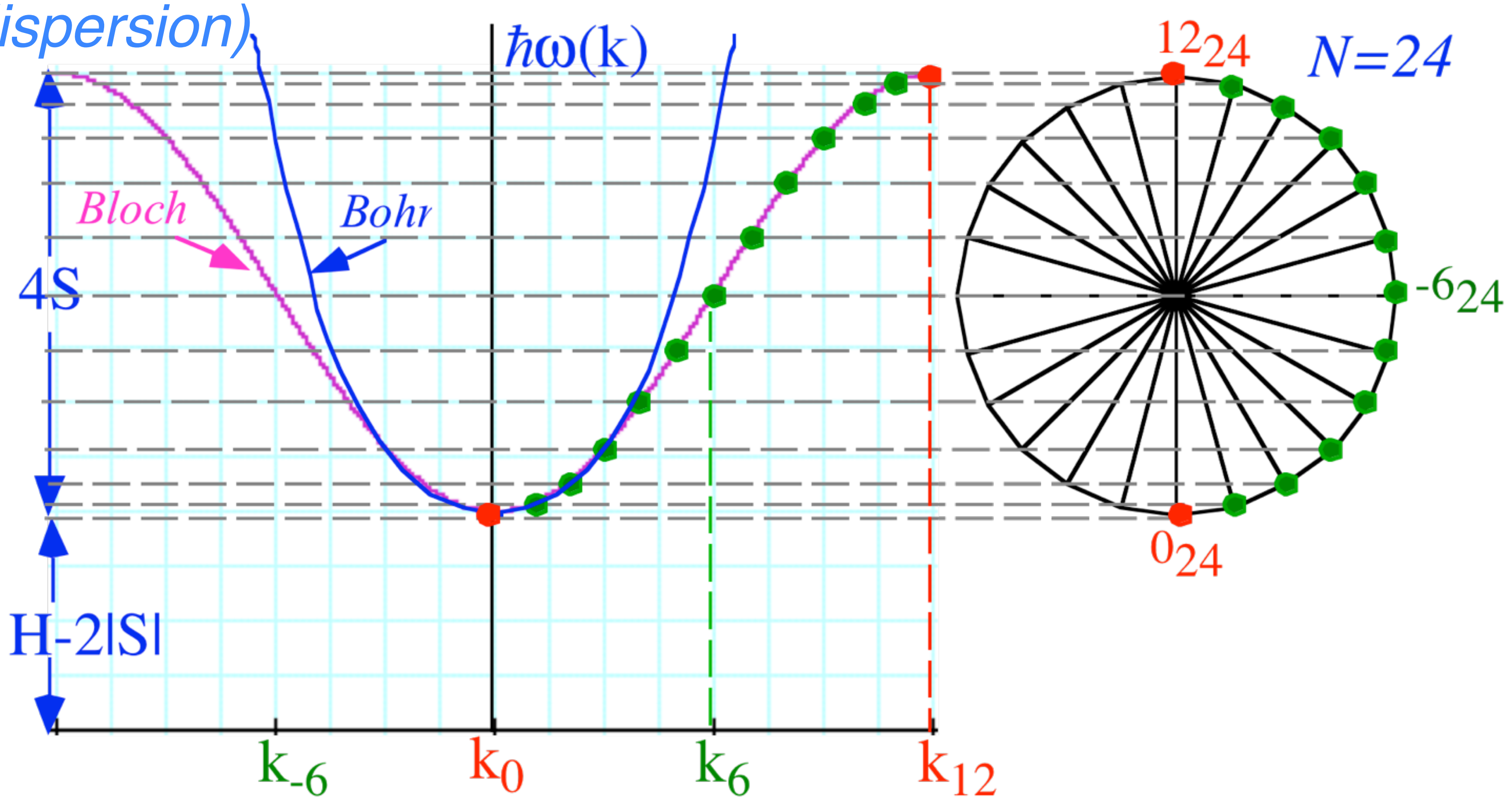
Pairwise group velocity:

$$V_{group-2} = \frac{\omega(k_a) - \omega(k_b)}{k_a - k_b}$$

C_N Symmetric Mode Models: Made-to-Order Dispersion

Making pure quadratic $\omega=ck^2$ (Bohr dispersion)



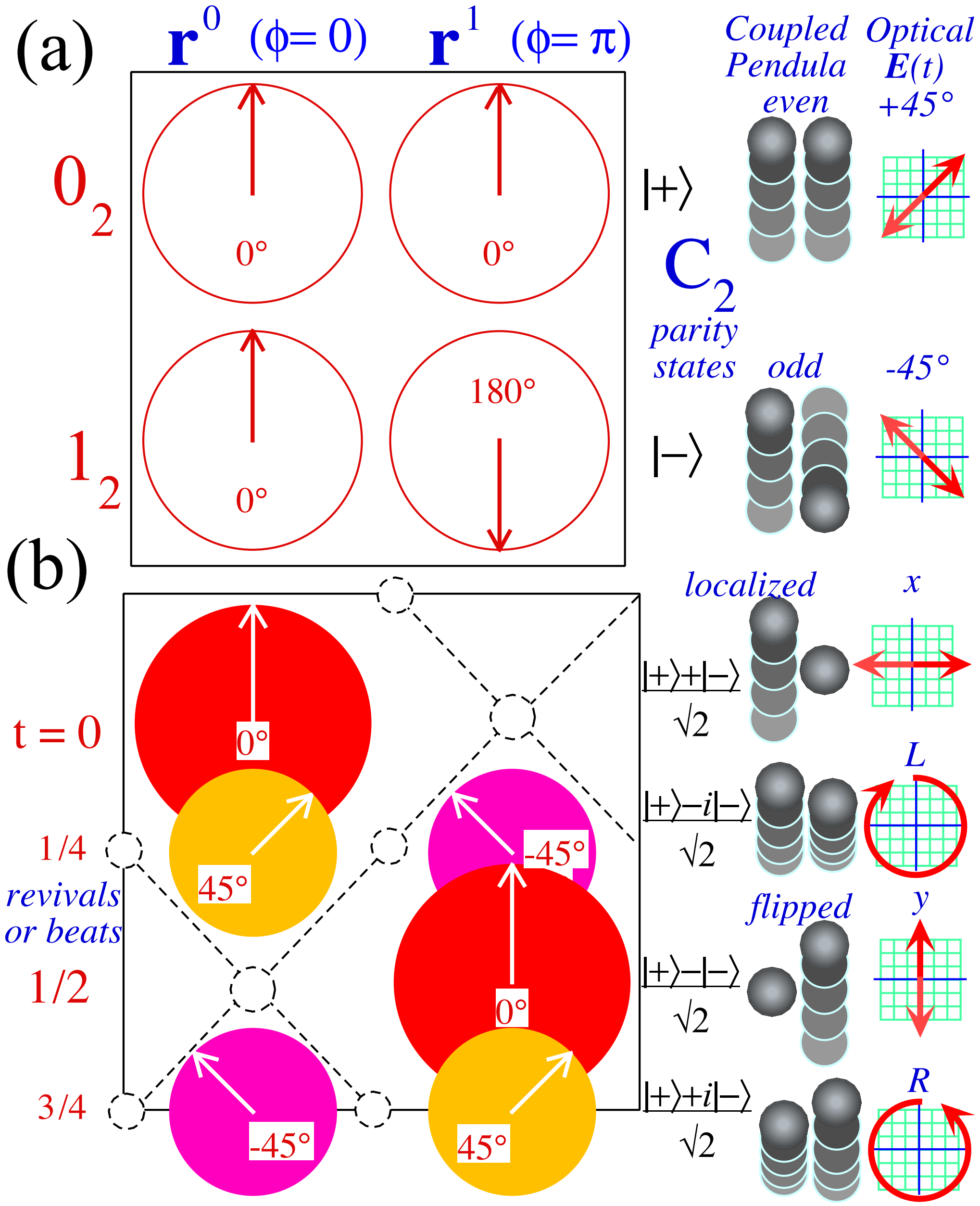


	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
N=2	1/2	-1/2							
N=3	2/3	-1/3							
N=4	3/2	-1	1/2						
N=5	2	-1.1708	0.1708						
N=6	19/6	-2	2/3	-1/2					
N=7	4	-2.393	0.51	-0.1171					
N=8	11/2	-3.4142	1	-0.5858	1/2				
N=9	20/3	-4.0165	0.9270	-1/3	0.0895				
N=10	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
N=11	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
N=12	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
N=13	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
N=14	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
N=15	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
N=16	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
N=17	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

C_N Symmetric Mode Models: Made-to-Order Dispersion

Revivals with quadratic $\omega=ck^2$ (Bohr dispersion)

C_2 beats or revivals happen
with most any dispersion

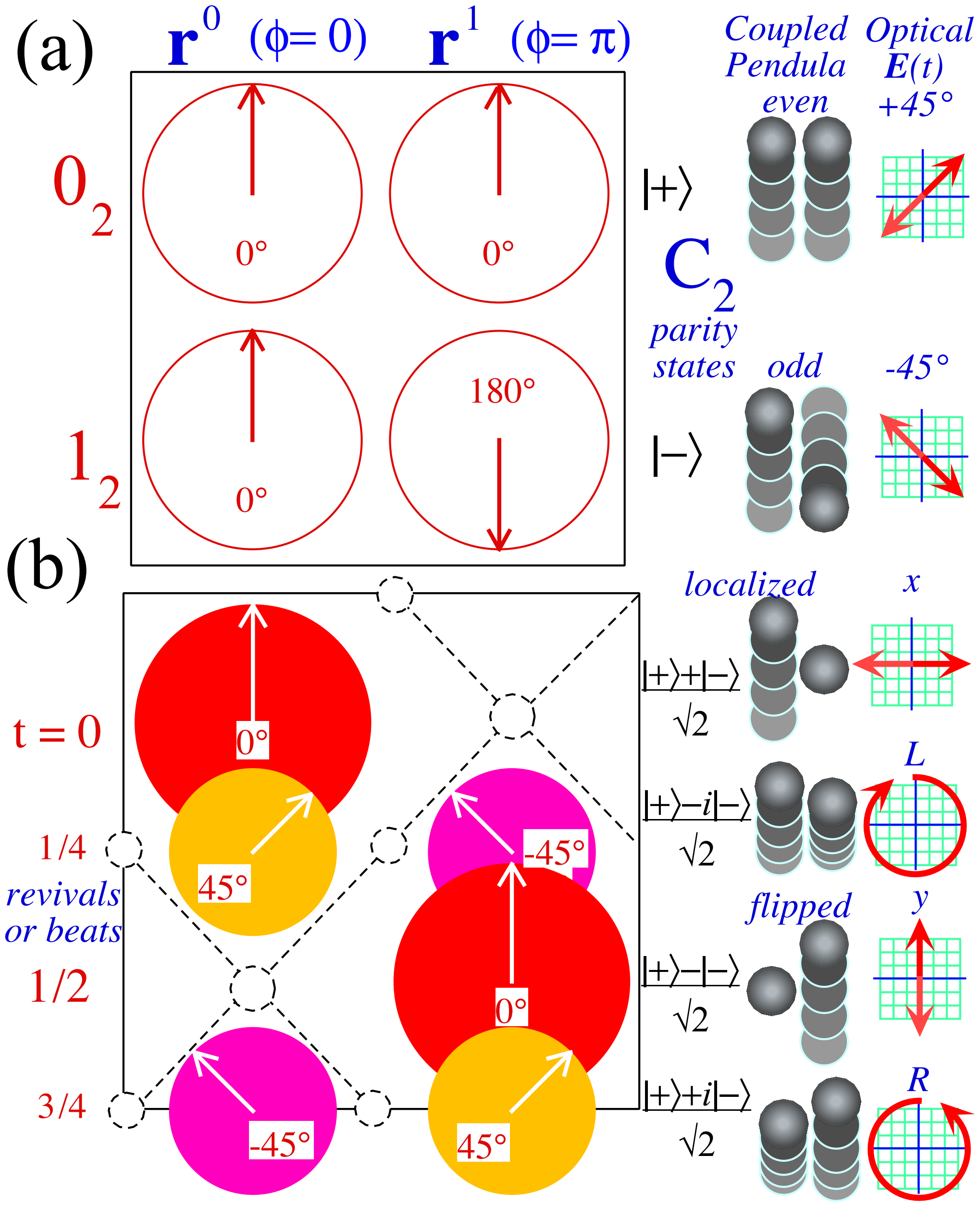


- *Wave resonance in cyclic symmetry*
  - *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
  - *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
  - *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
  - *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - ➔ *Phase arithmetic*

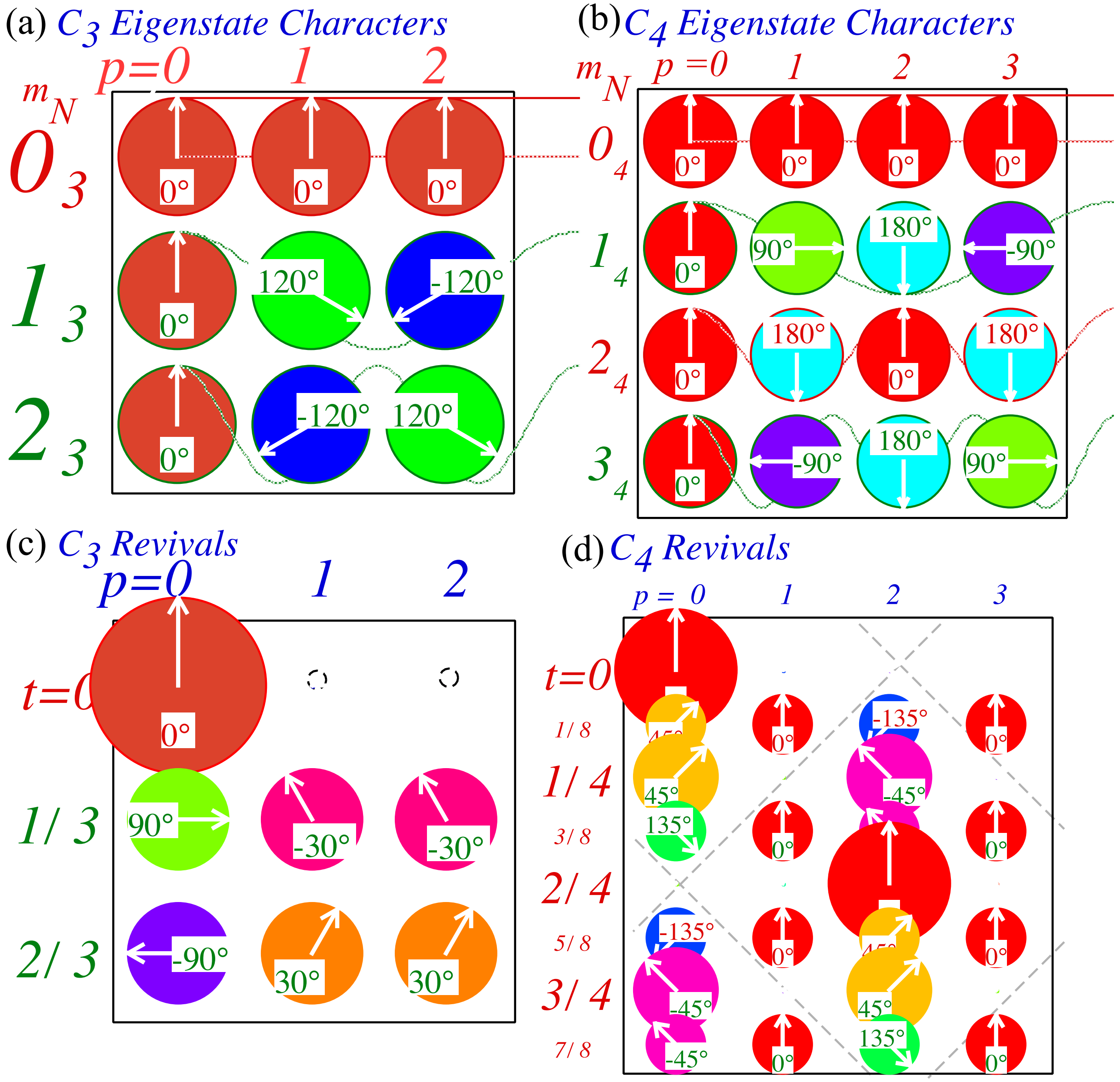
C_N Symmetric Mode Models: Made-to-Order Dispersion

Revivals with quadratic $\omega=ck^2$ (Bohr dispersion)

C_2 beats or revivals happen
with most any dispersion



C_3 revivals and C_4 revivals
occur with quadratic dispersion

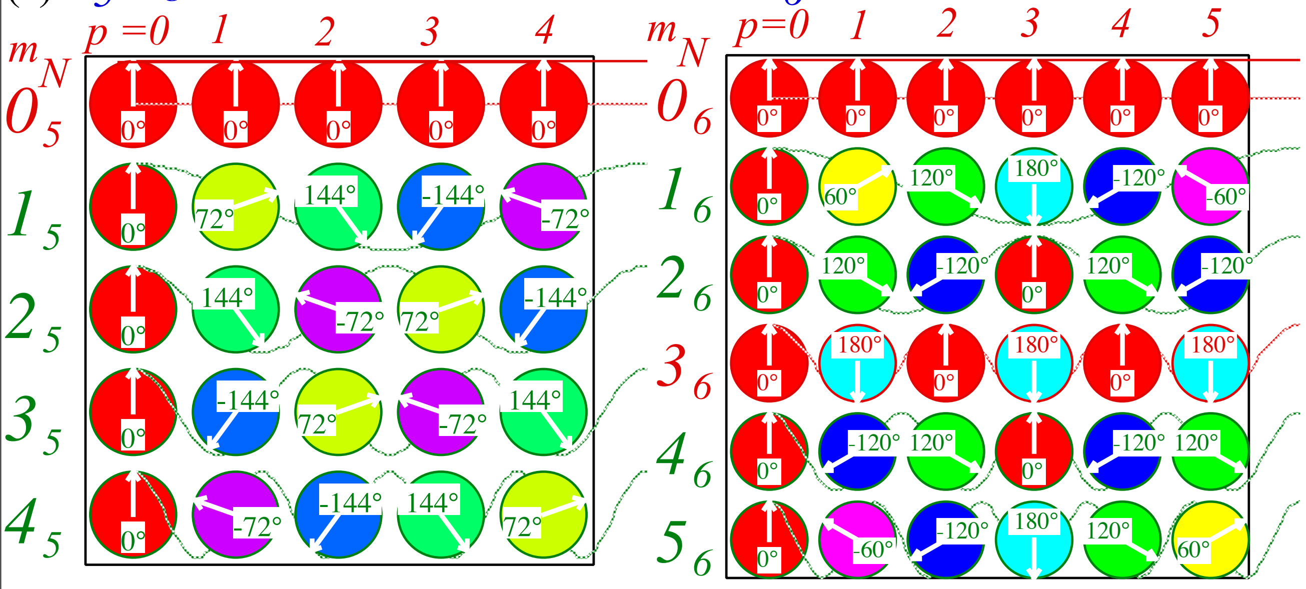


C_N Symmetric Mode Models: Made-to-Order Dispersion

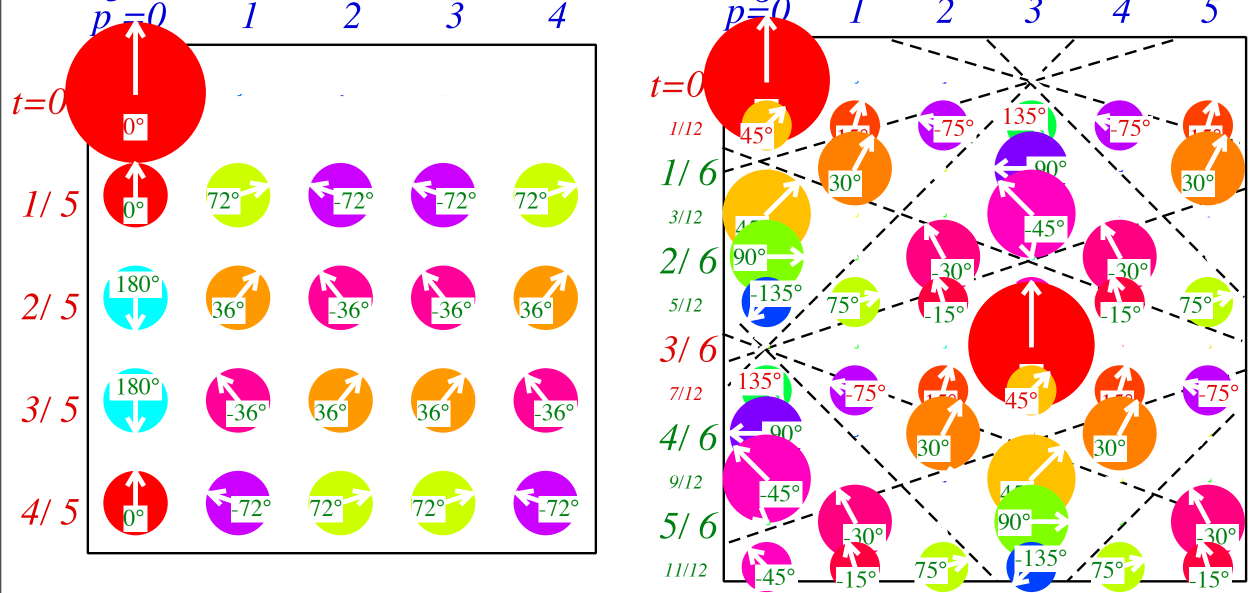
Revivals with quadratic $\omega = ck^2$ (Bohr dispersion)

C_5 revivals and C_6 revivals
occur with quadratic dispersion

(a) C_5 Eigenstate Characters (b) C_6 Eigenstate Characters

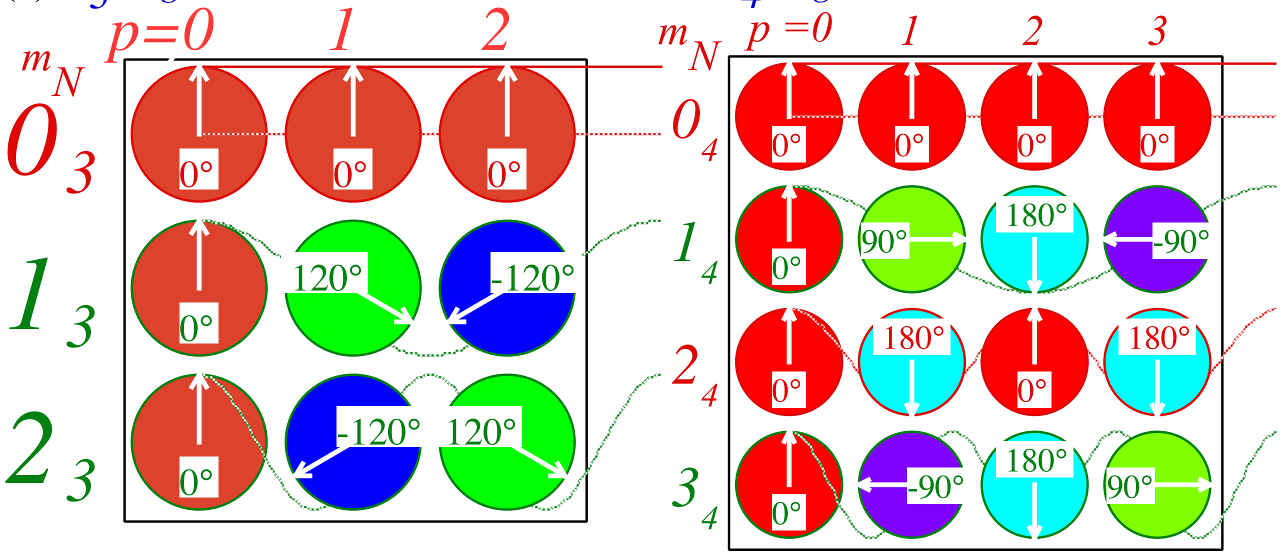


(c) C_5 Revivals (d) C_6 Revivals

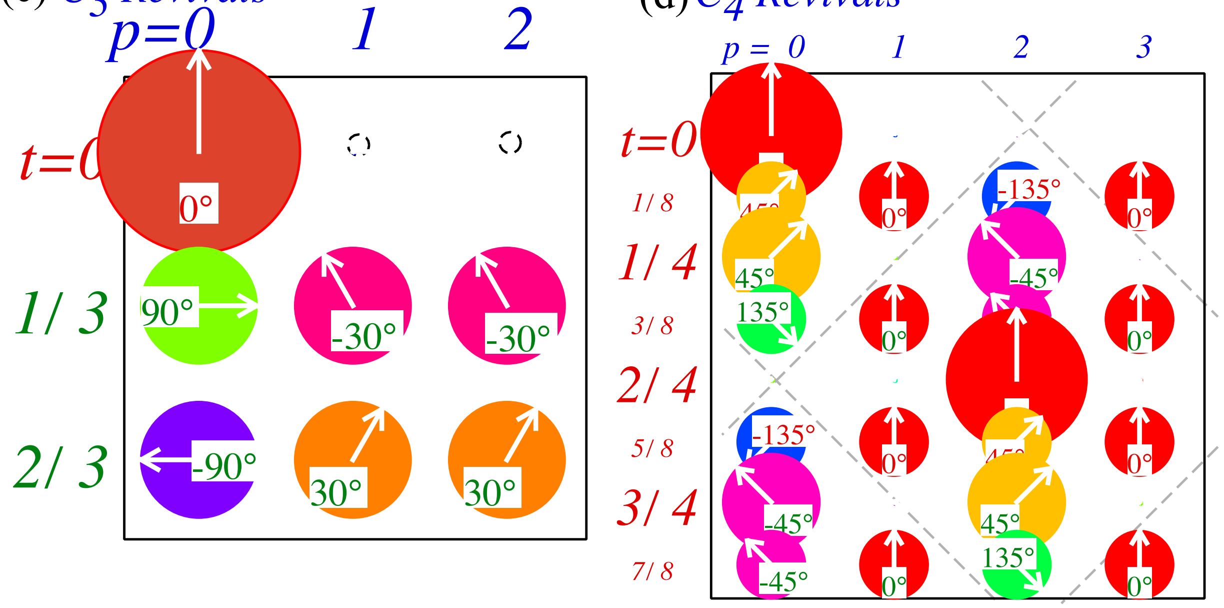


C_3 revivals and C_4 revivals
occur with quadratic dispersion

(a) C_3 Eigenstate Characters (b) C_4 Eigenstate Characters



(c) C_3 Revivals (d) C_4 Revivals



C_N Symmetric Mode Models: Made-to-Order Dispersion

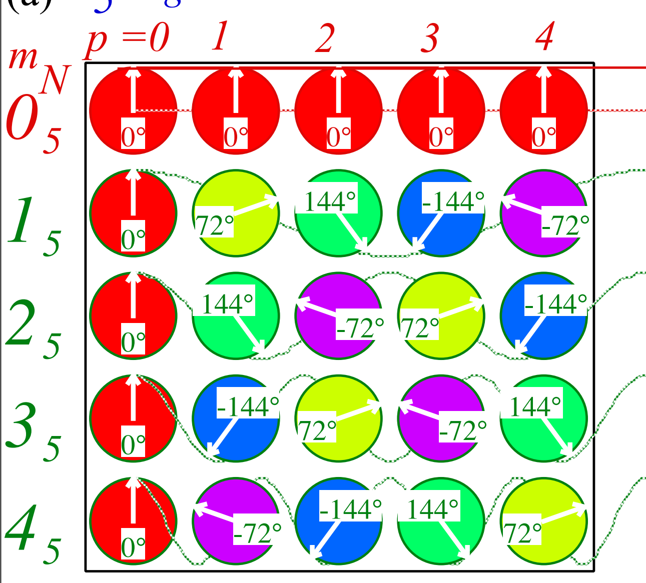
Revivals with quadratic $\omega = ck^2$ (Bohr dispersion)

C_5 revivals and C_6 revivals
occur with quadratic dispersion

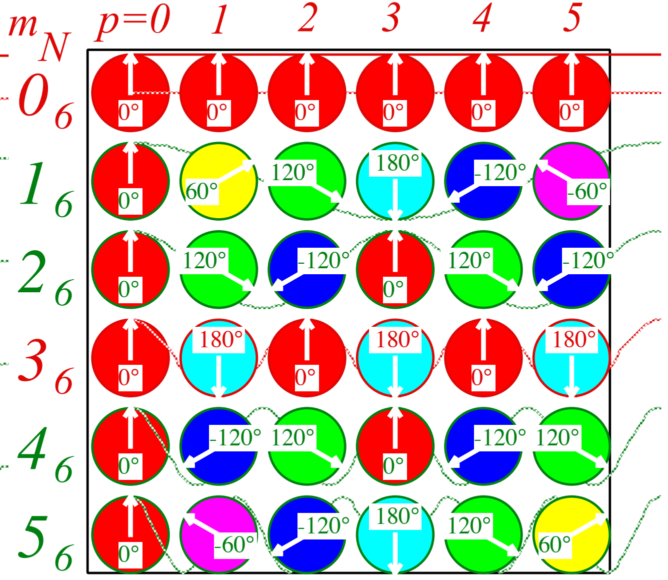
C_{15} revivals occur with quadratic dispersion

first display prime factors then multiples by zeros at site ±1

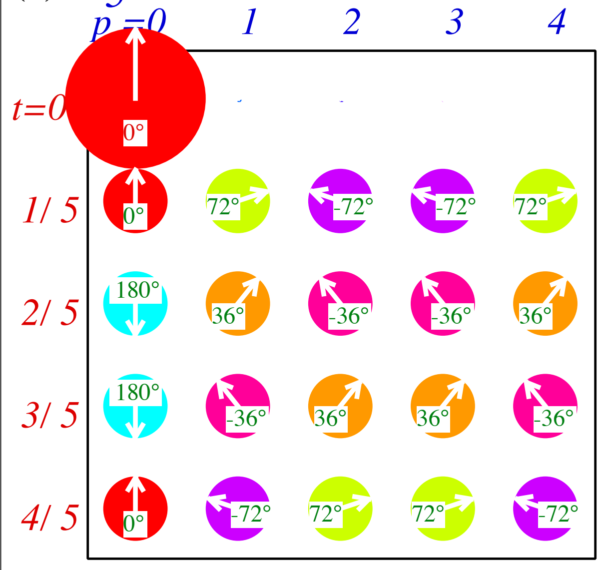
(a) C_5 *Eigenstate Characters*



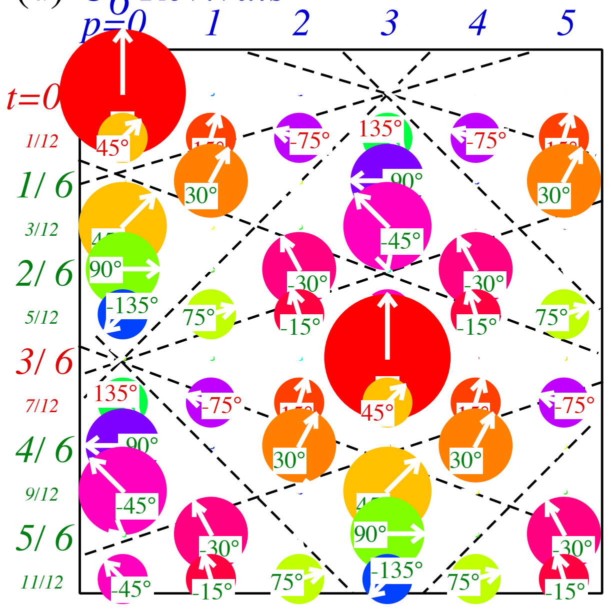
(b) C_6 *Eigenstate Characters*

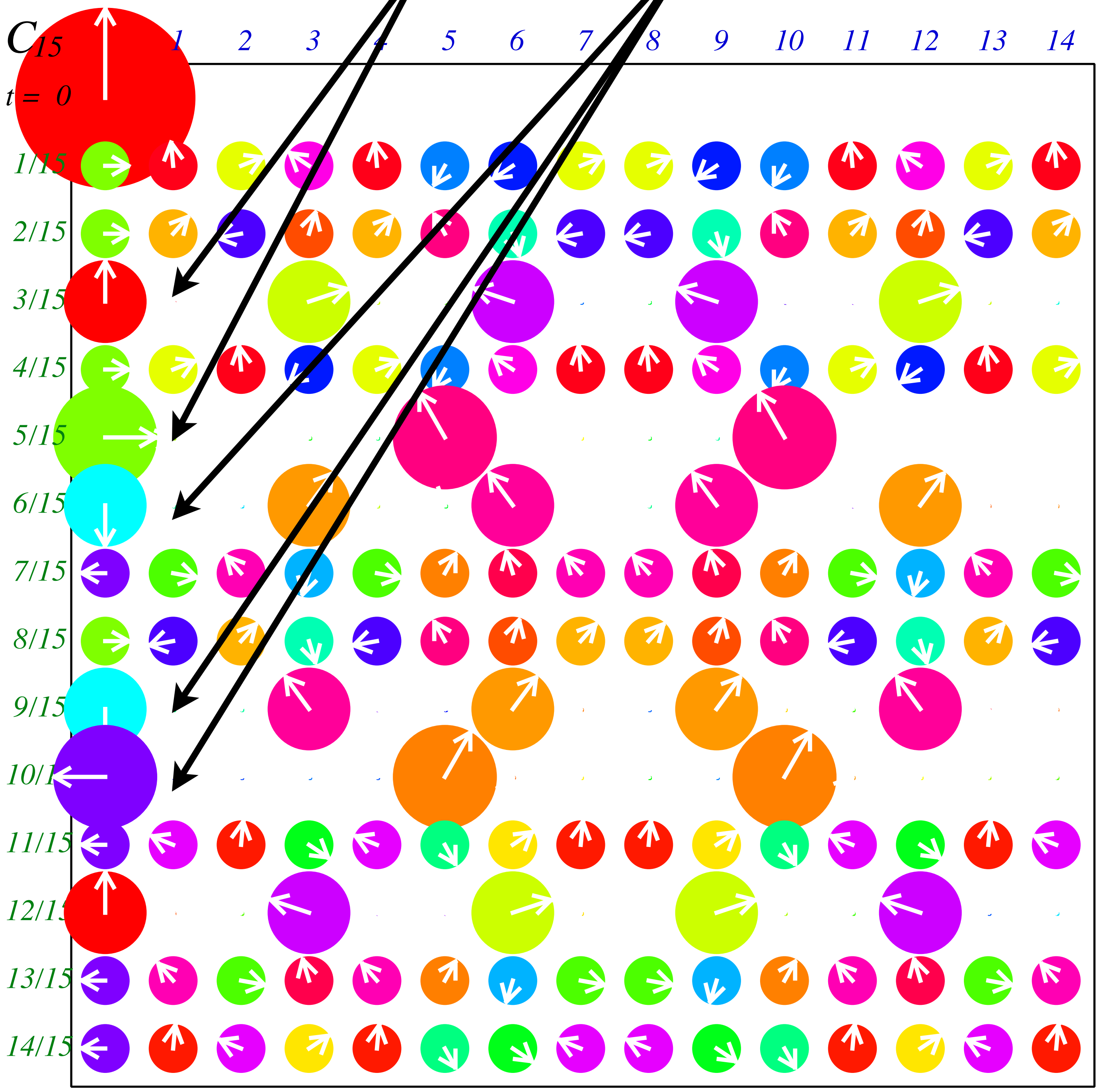


(c) C_5 *Revivals*

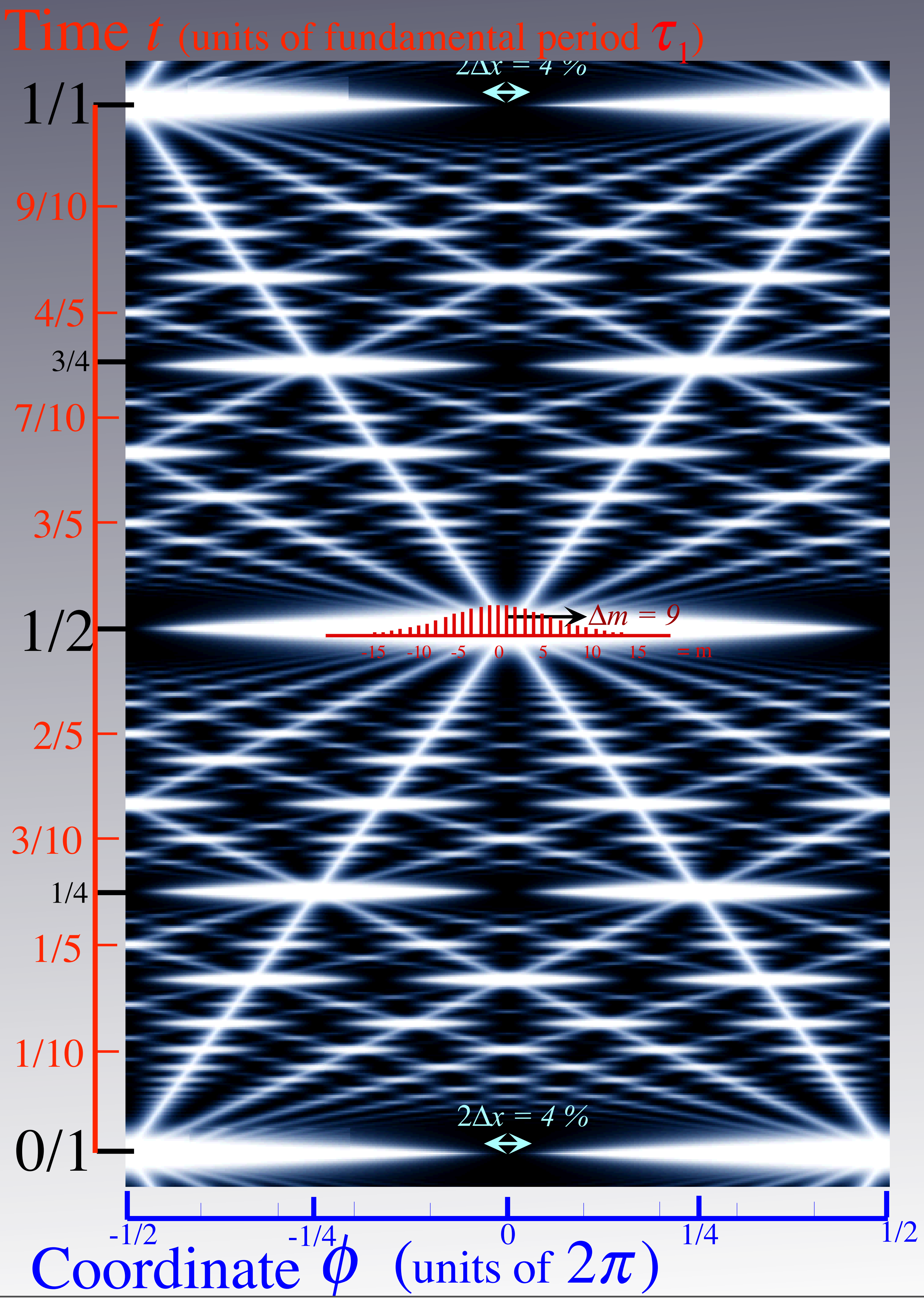


(d) C_6 *Revivals*

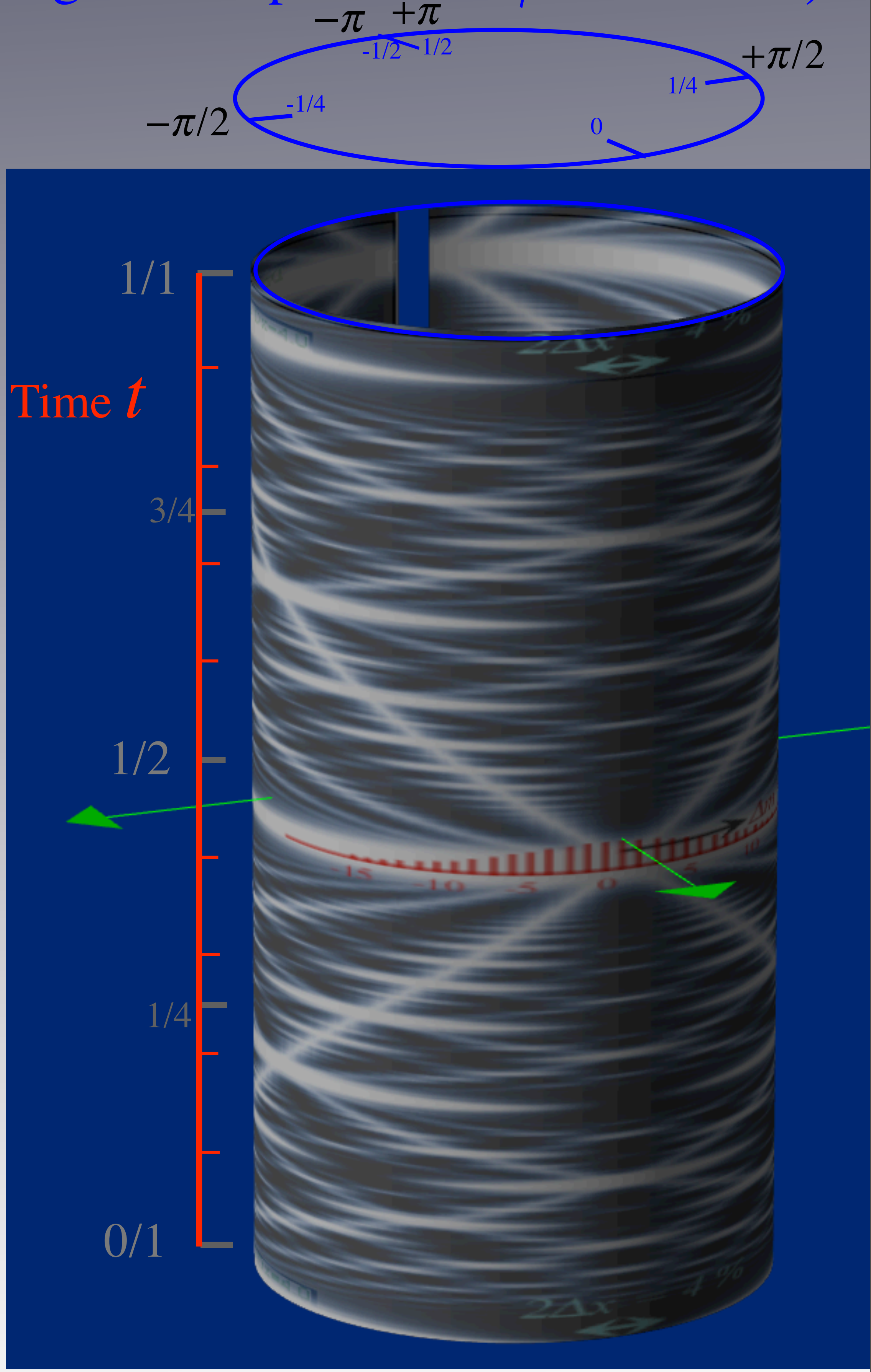




Algebra and geometry of resonant revivals: Farey Sums and Ford Circles



(Imagine "wrap-around" ϕ-coordinate)

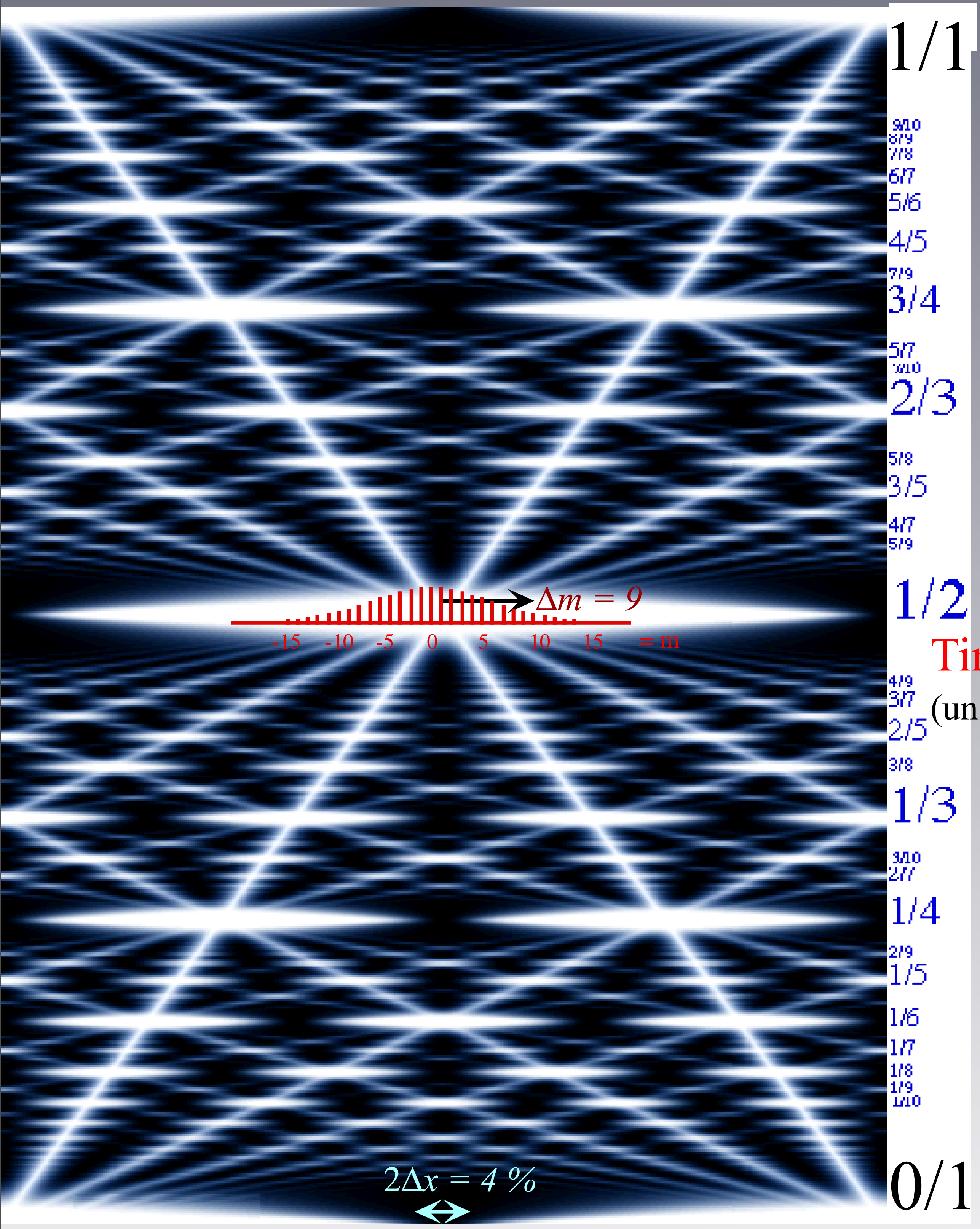


[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

N-level-system and revival-beat wave dynamics

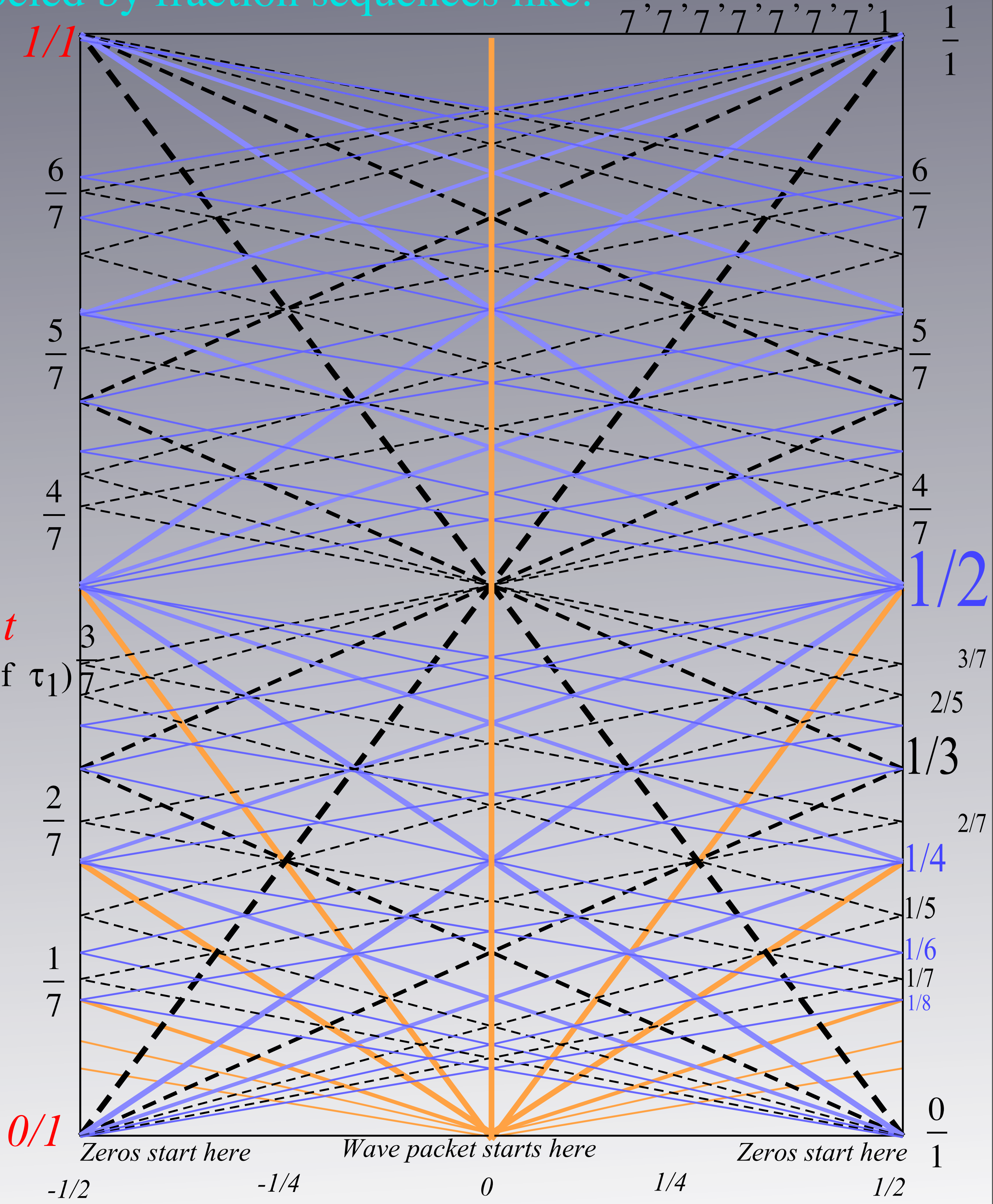
(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$



Time *t*

(units of τ_1)



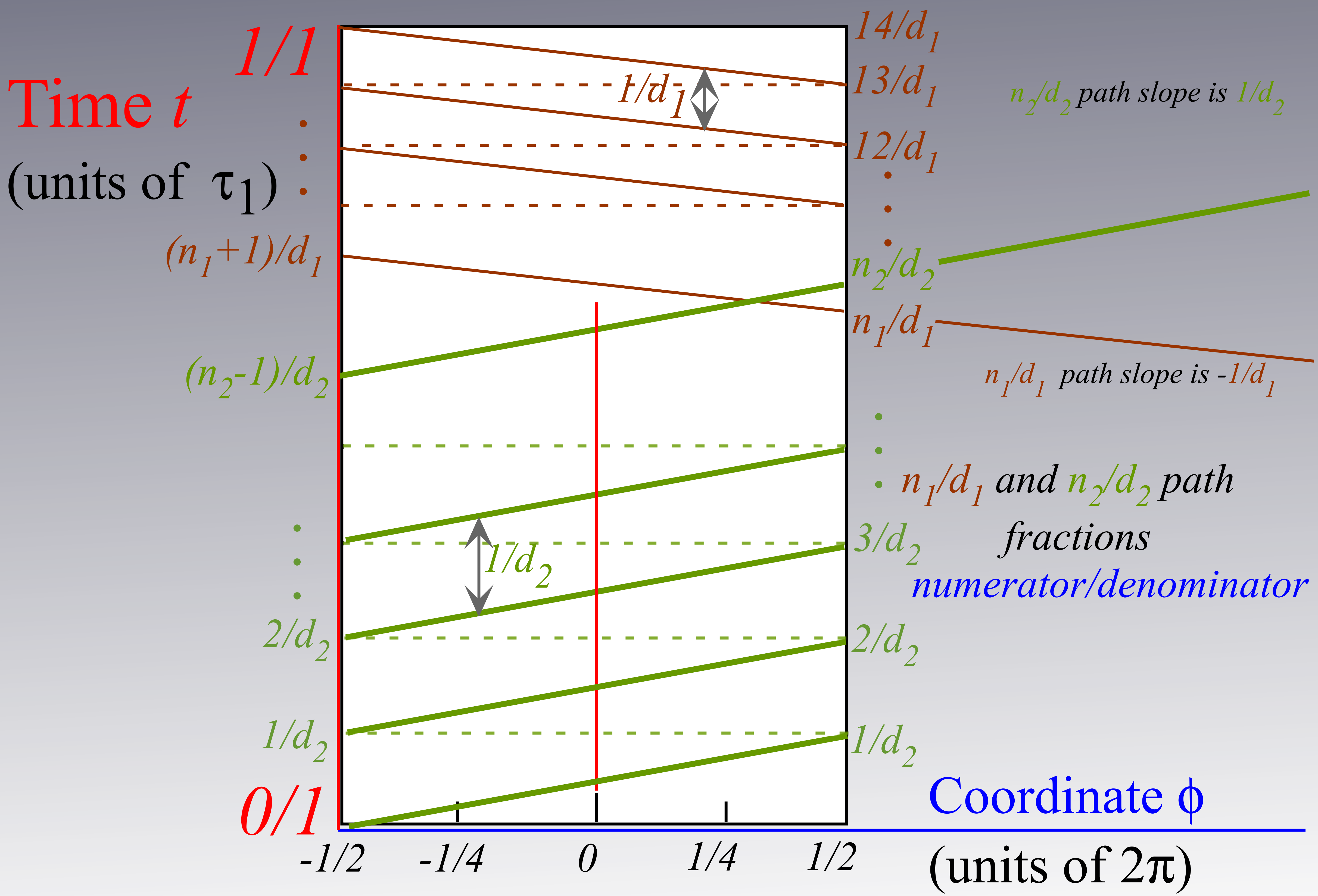
Coordinate ϕ

(units of 2π)

[Harter, J. Mol. Spec. 210, 166-182 (2001)]

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



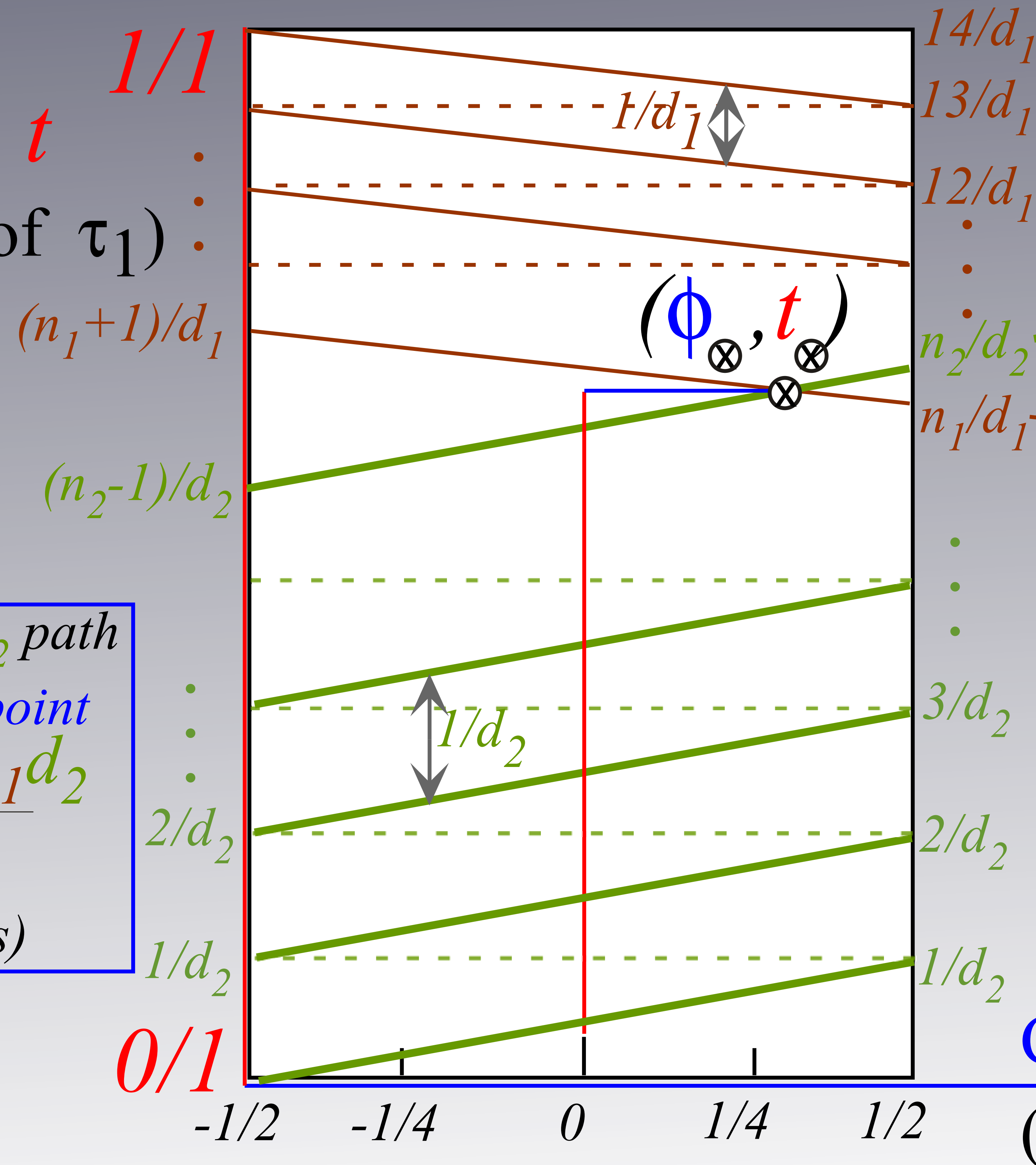
n_1/d_1 and n_2/d_2 path fractions
numerator/denominator

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*

Time t

(units of τ_1)



n_2/d_2 path slope is $1/d_2$

$$\frac{n_2/d_2 - t}{1/2 - \phi} = 1/d_2$$

$$\frac{n_1/d_1 - t}{1/2 - \phi} = -1/d_1$$

n_1/d_1 path slope is $-1/d_1$

*n_1/d_1 and n_2/d_2 path intersection **point***

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

*n_1/d_1 and n_2/d_2 path intersection **time***

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

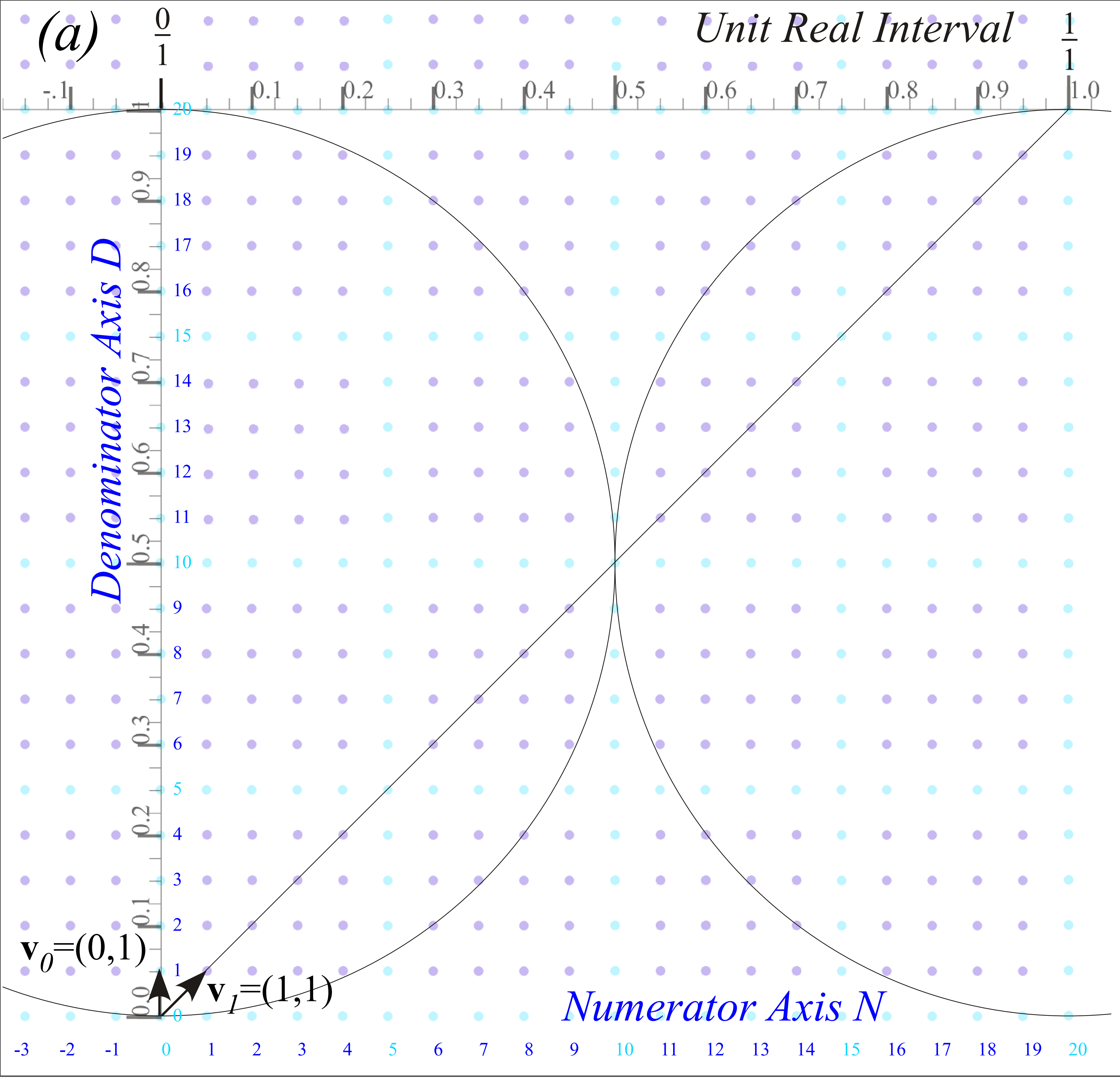
(Farey-Sum)

Coordinate ϕ

(units of 2π)

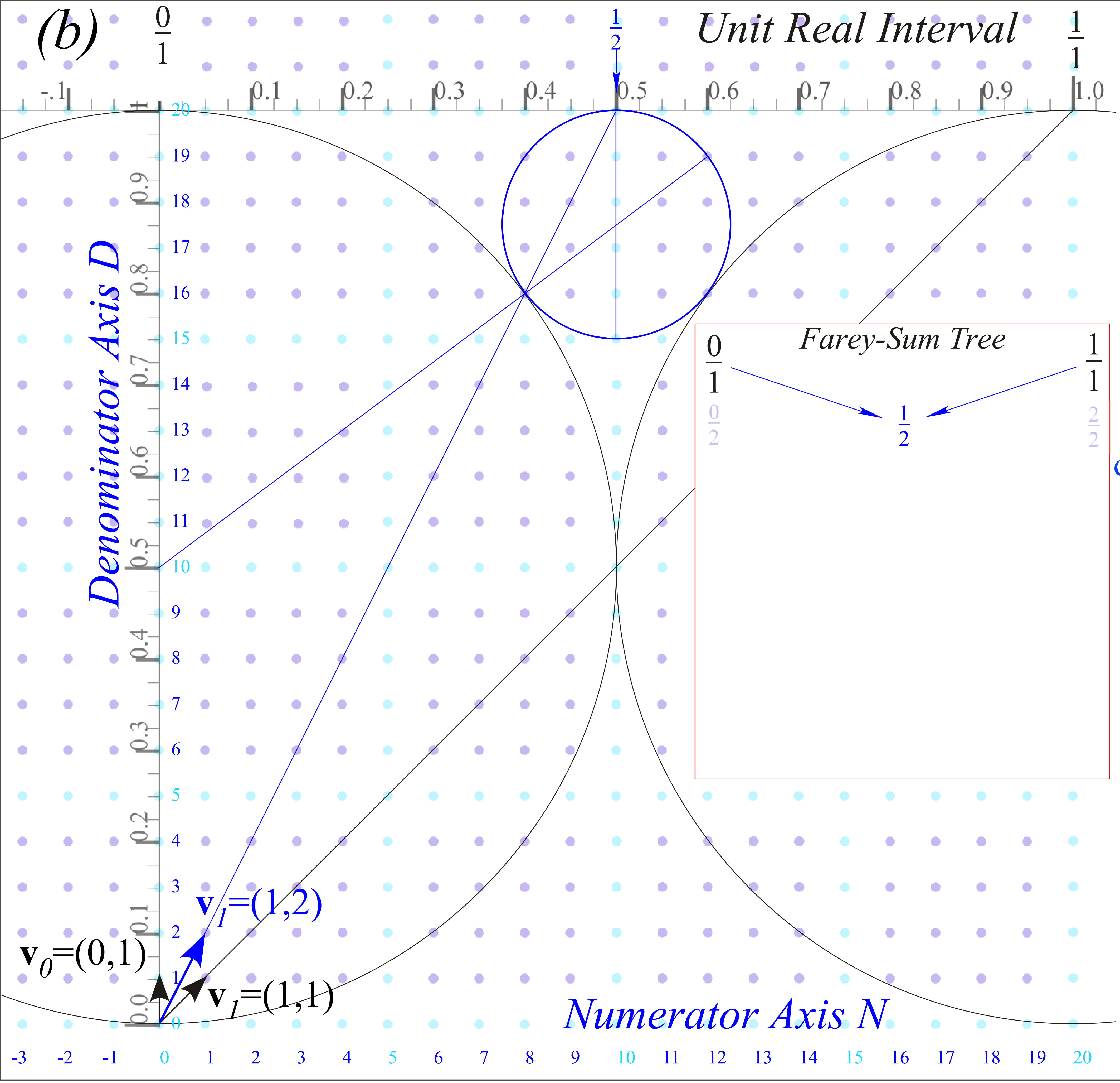
[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]



Farey Sum related to vector sum and *Ford Circles*

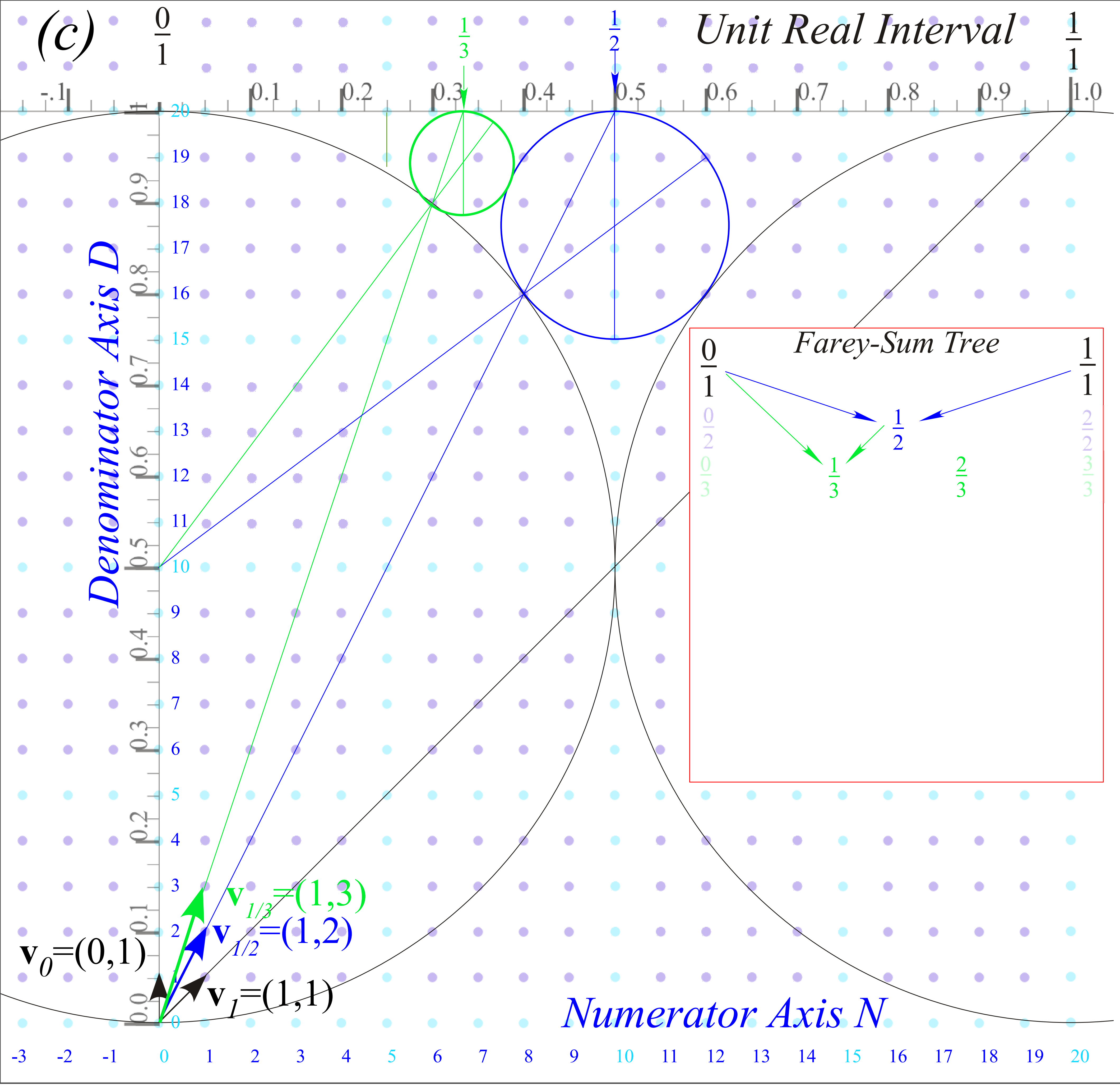
1/1-circle has diameter *1*



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

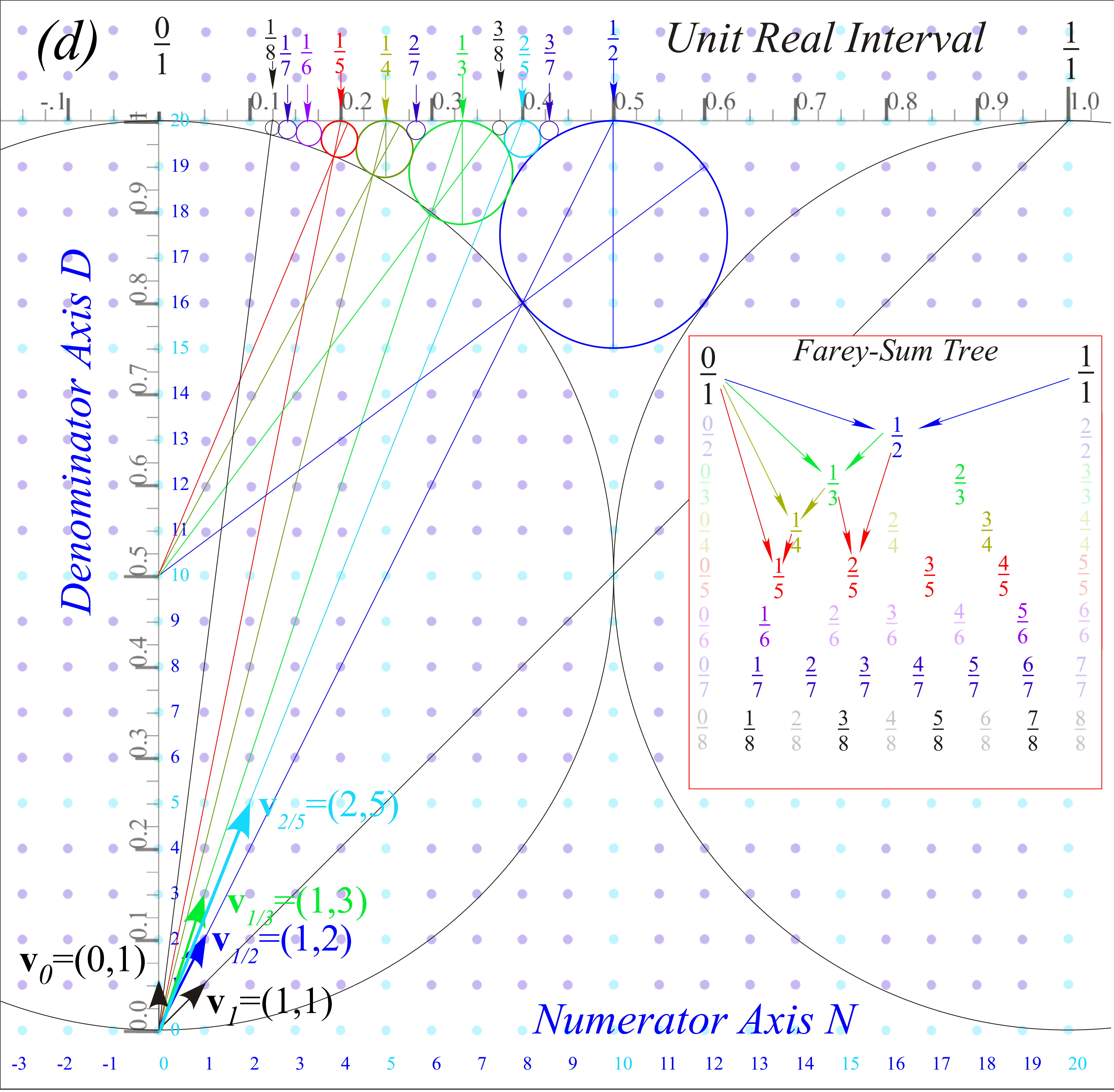
1/2-circle has diameter *1/2²=1/4*



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter *$1/2^2=1/4$*

1/3-circles have diameter *$1/3^2=1/9$*

n/d-circles have diameter *$1/d^2$*

Relating C_N symmetric H and K matrices to differential wave operators

Relating C_N symmetric H and K matrices to wave differential operators

The 1st neighbor **K** matrix relates to a 2nd *finite-difference* matrix of 2nd x-derivative for high C_N.

$$\mathbf{K} = k(2\mathbf{1} - \mathbf{r} - \mathbf{r}^{-1}) \text{ analogous to:} -k\frac{\partial^2}{\partial x^2}$$

1st derivative momentum: $p = \frac{\hbar}{i}\frac{\partial y}{\partial x} \approx \frac{\hbar}{i}\frac{y(x+\Delta x) - y(x)}{(\Delta x)}$

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \frac{\hbar}{i}\begin{pmatrix} \cdot \\ y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ y_4 - y_3 \\ \cdot \end{pmatrix}$$

2nd derivative KE: $2mE = -\hbar^2\frac{\partial^2 y}{\partial x^2} \approx \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2}$

$$-\hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \end{pmatrix}$$

H and **K** matrix equations are finite-difference versions of quantum and classical wave equations.

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathbf{H}|\psi\rangle$ *(**H**-matrix equation)*

$-\frac{\partial^2}{\partial t^2}|y\rangle = \mathbf{K}|y\rangle$ *(**K**-matrix equation)*

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V)|\psi\rangle$ *(Scrodinger equation)*

$-\frac{\partial^2}{\partial t^2}|y\rangle = -k\frac{\partial^2}{\partial x^2}|y\rangle$ *(Classical wave equation)*

Square p^2 gives 1st neighbor **K** matrix.

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Higher order p^3, p^4,.. involve 2nd, 3rd, 4th..neighbor **H**

$$p^4 \cong \begin{pmatrix} \ddots & \vdots & 1 & 0 & & \\ \cdots & 6 & -4 & 1 & 0 & \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ & 0 & 1 & -4 & 6 & -4 \\ & & 0 & 1 & -4 & 6 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\bar{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \ \bar{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\bar{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \bar{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$