Lecture 14
Thur. 10.9.2014

Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3, Unit 7 Ch. 1-2)

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulations of pendulum and cycloidulum)
1D-HO phase-space control (Simulation of "Catcher in the Eye")

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations
Poincare identity and Action, Jacobi-Hamilton *equations*
How Classicists might have "derived" quantum equations
Huygen's contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
("Color-Quantization" simulations: Davis-Heller "Color-Quantization" or "Classical Chromodynamics"*)*

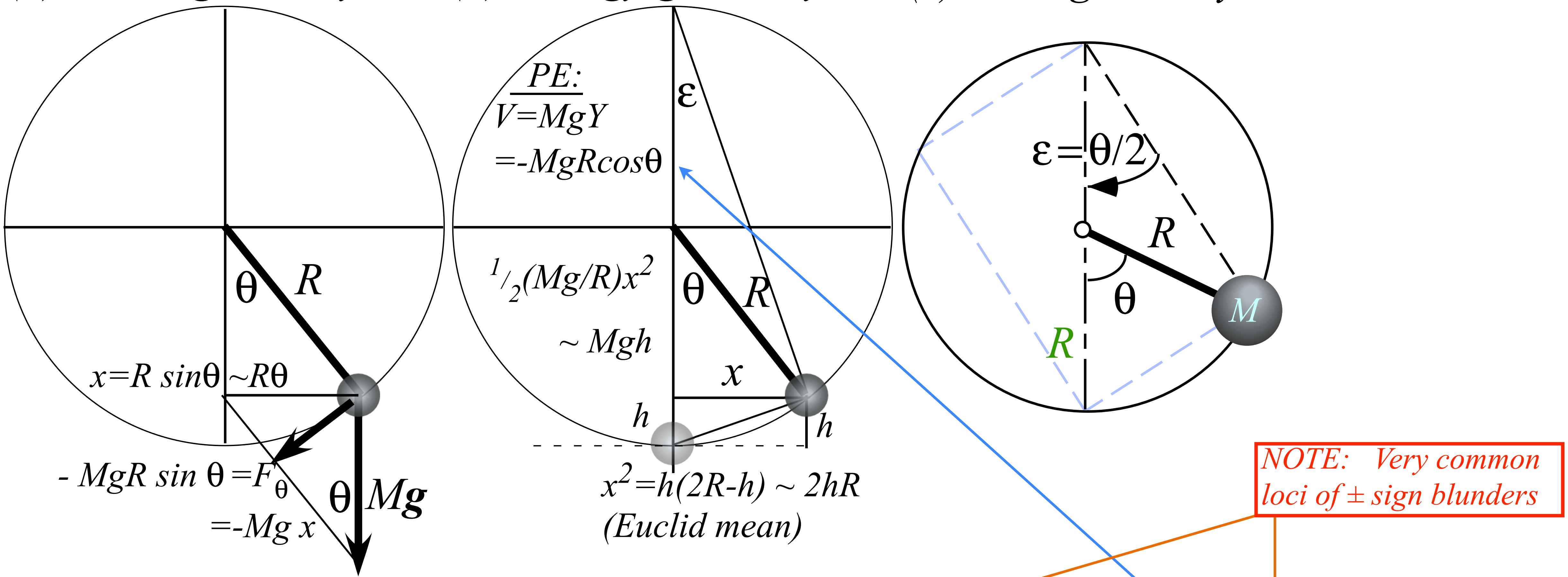
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)

1D-HO phase-space control (Simulation of "Catcher in the Eye")

1D Pendulum and phase plot

(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot

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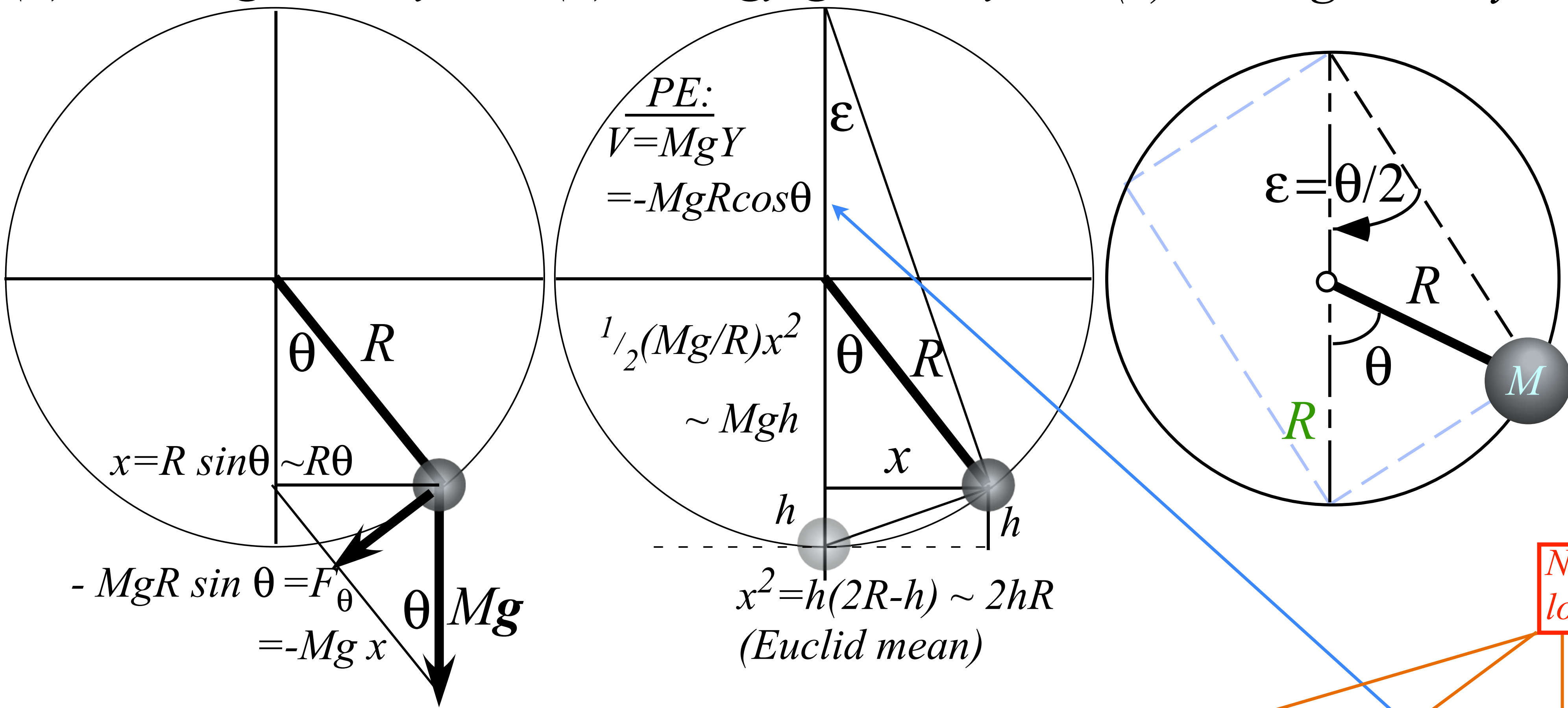


NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

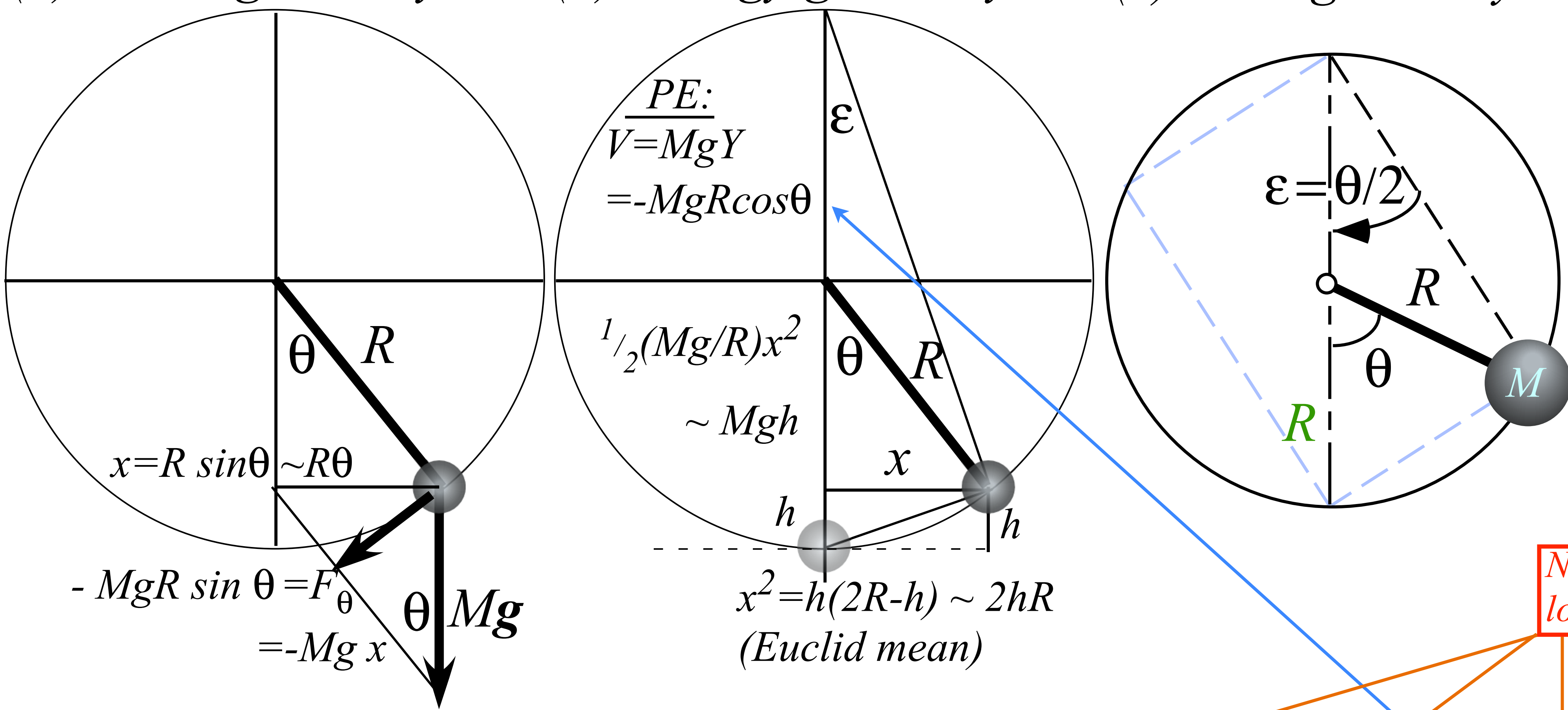
$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta^{\,2} + U(\theta) = \frac{1}{2I}p_\theta^{\,2} - MgR\cos\theta \;=E \quad =const.$$

1D Pendulum and phase plot

(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



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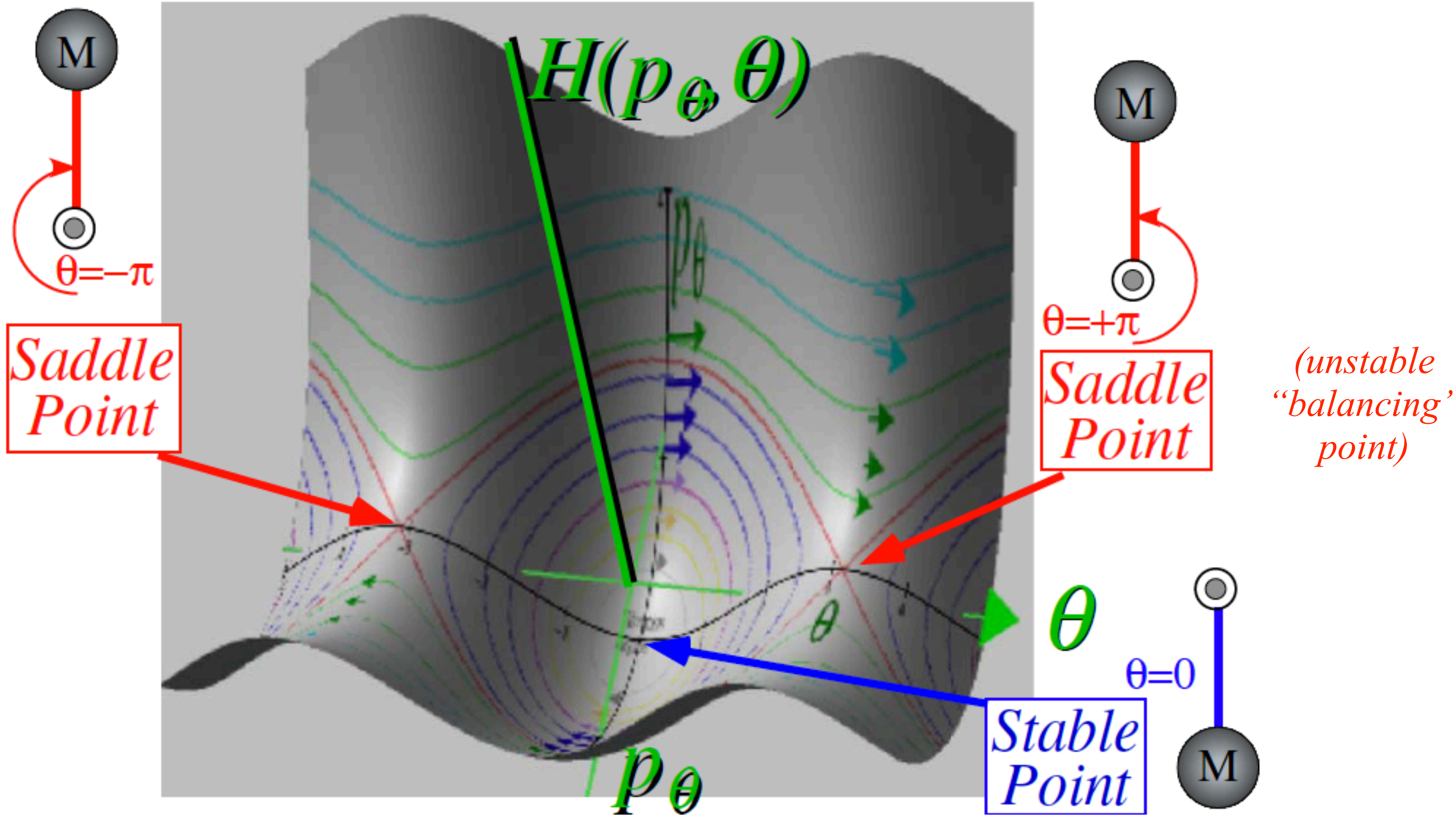
Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

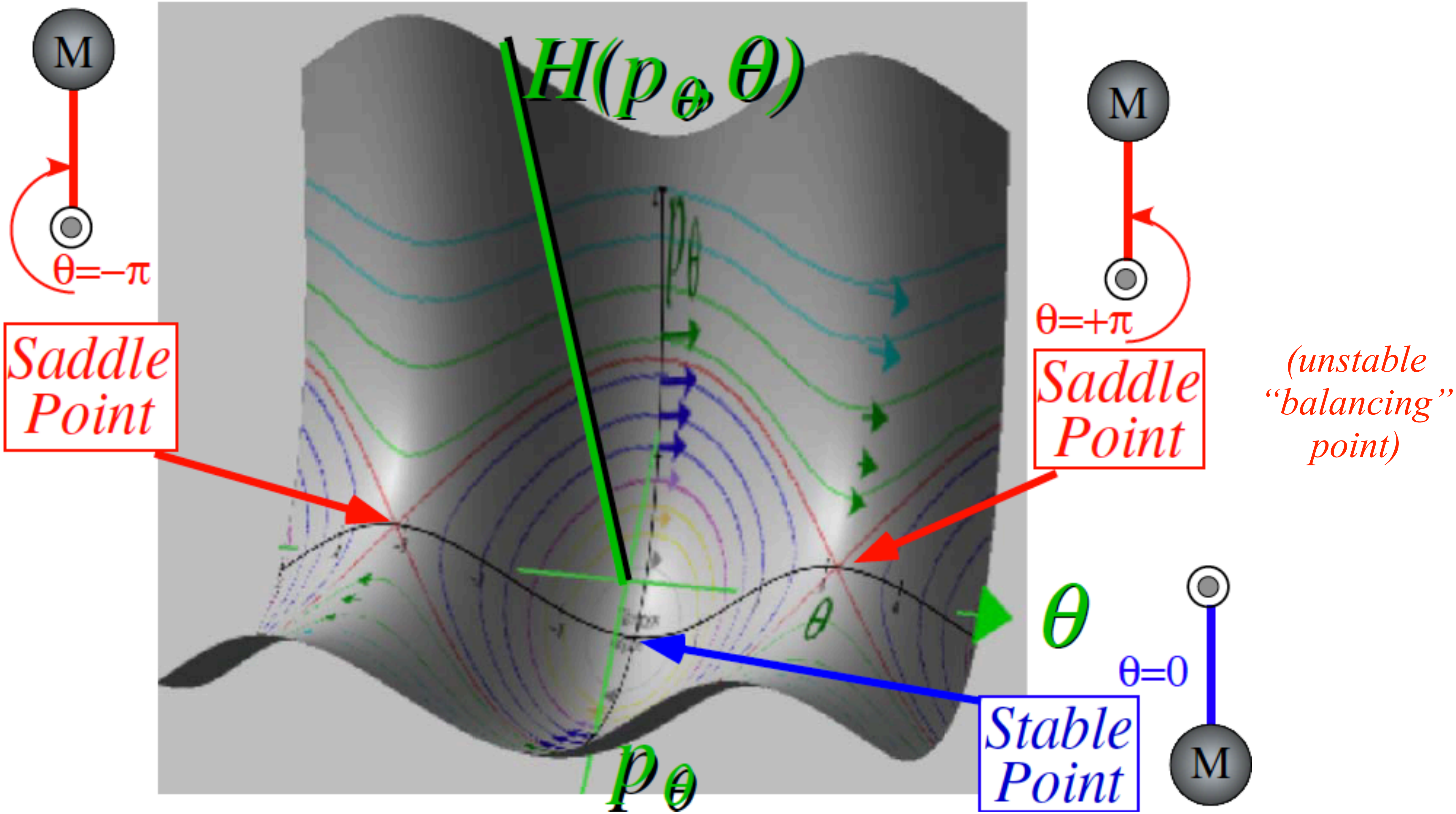
$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;=\; E \quad =const.$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$



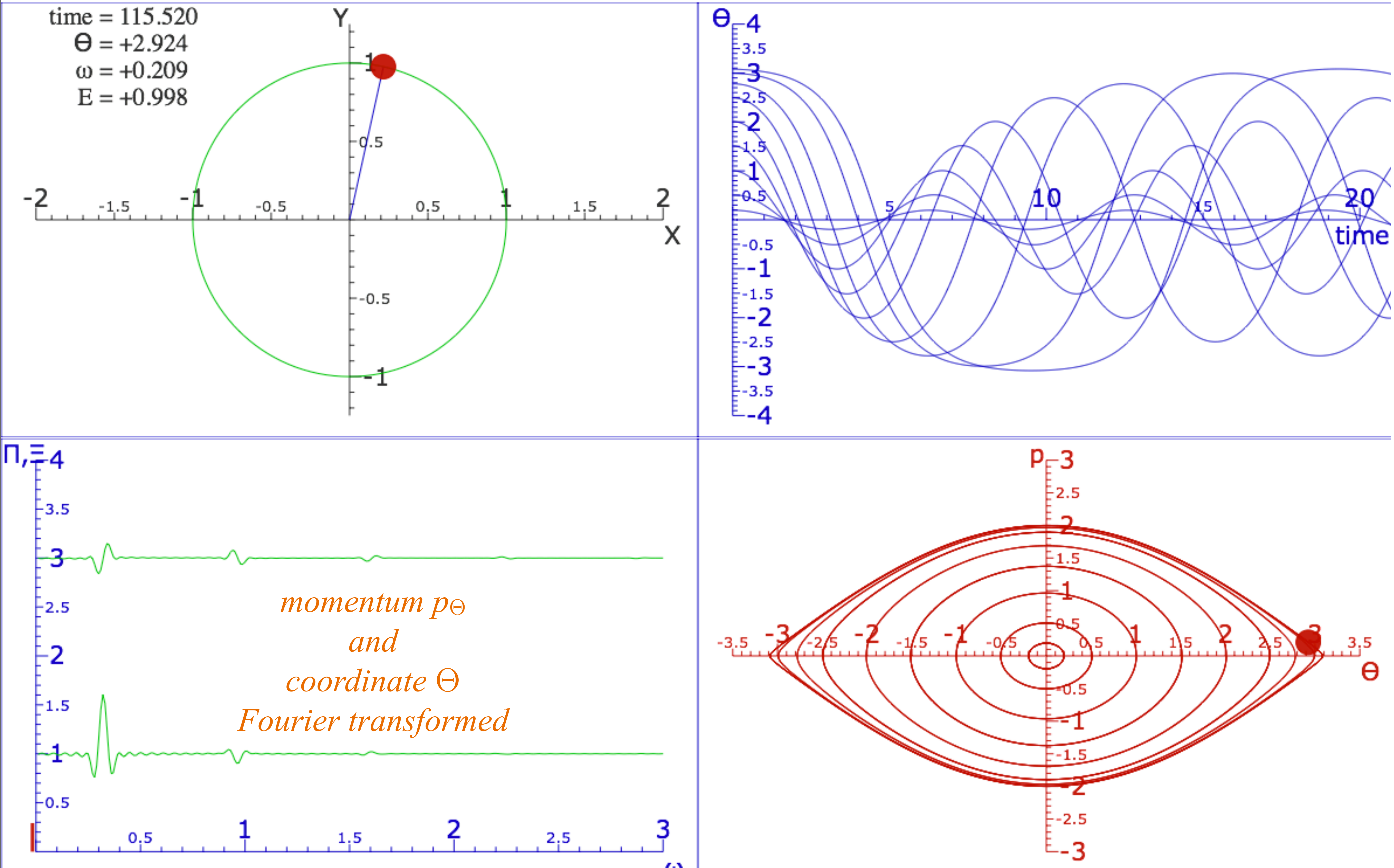
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I} {p_\theta}^2 - MgR\cos\theta \;, \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e_H} \times (-\nabla H) = (\overrightarrow{\text{H-axis}}) \times (\overrightarrow{\text{fall line}}), \text{ where:} \begin{cases} (\overrightarrow{\text{H-axis}}) = \mathbf{e_H} = \mathbf{e_q} \times \mathbf{e_p} \\ (\overrightarrow{\text{fall line}}) = -\nabla H \end{cases}$$

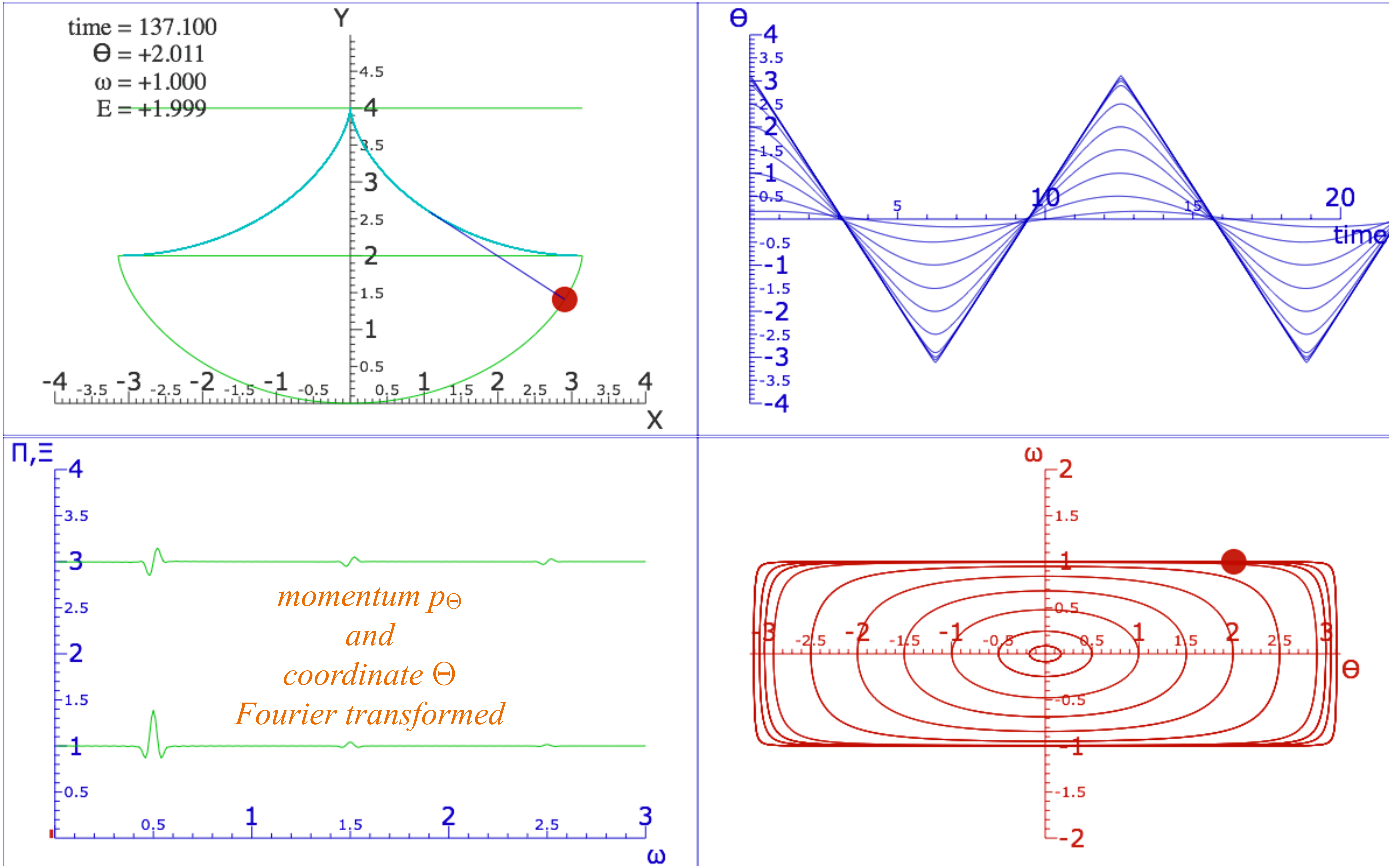
(Simulations of pendulum)



http://www.uark.edu/ua/modphys/markup/PendulumWeb.html

See also: http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html

(Simulations of cycloidulum)



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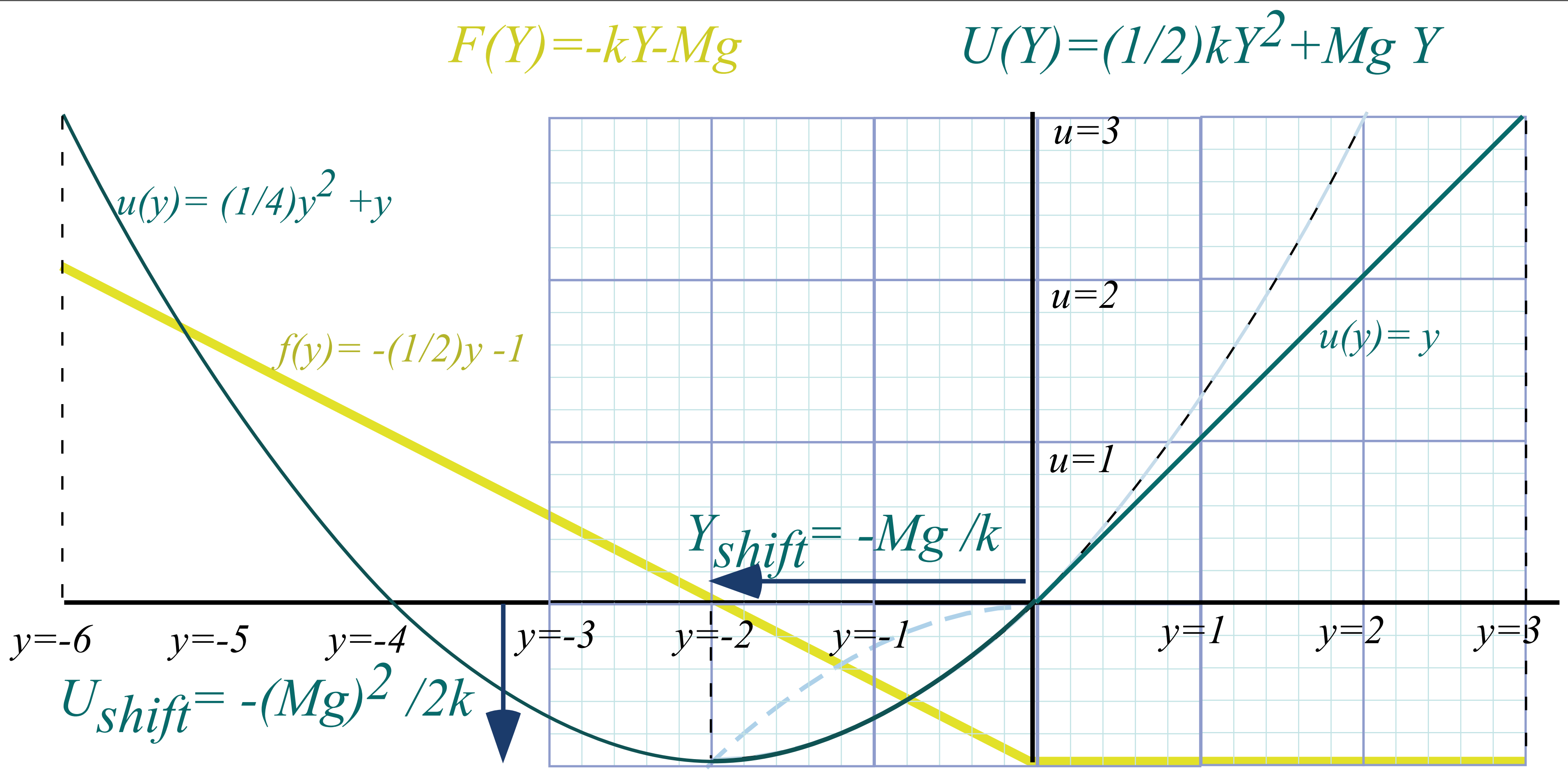
Examples of Hamiltonian dynamics and phase plots

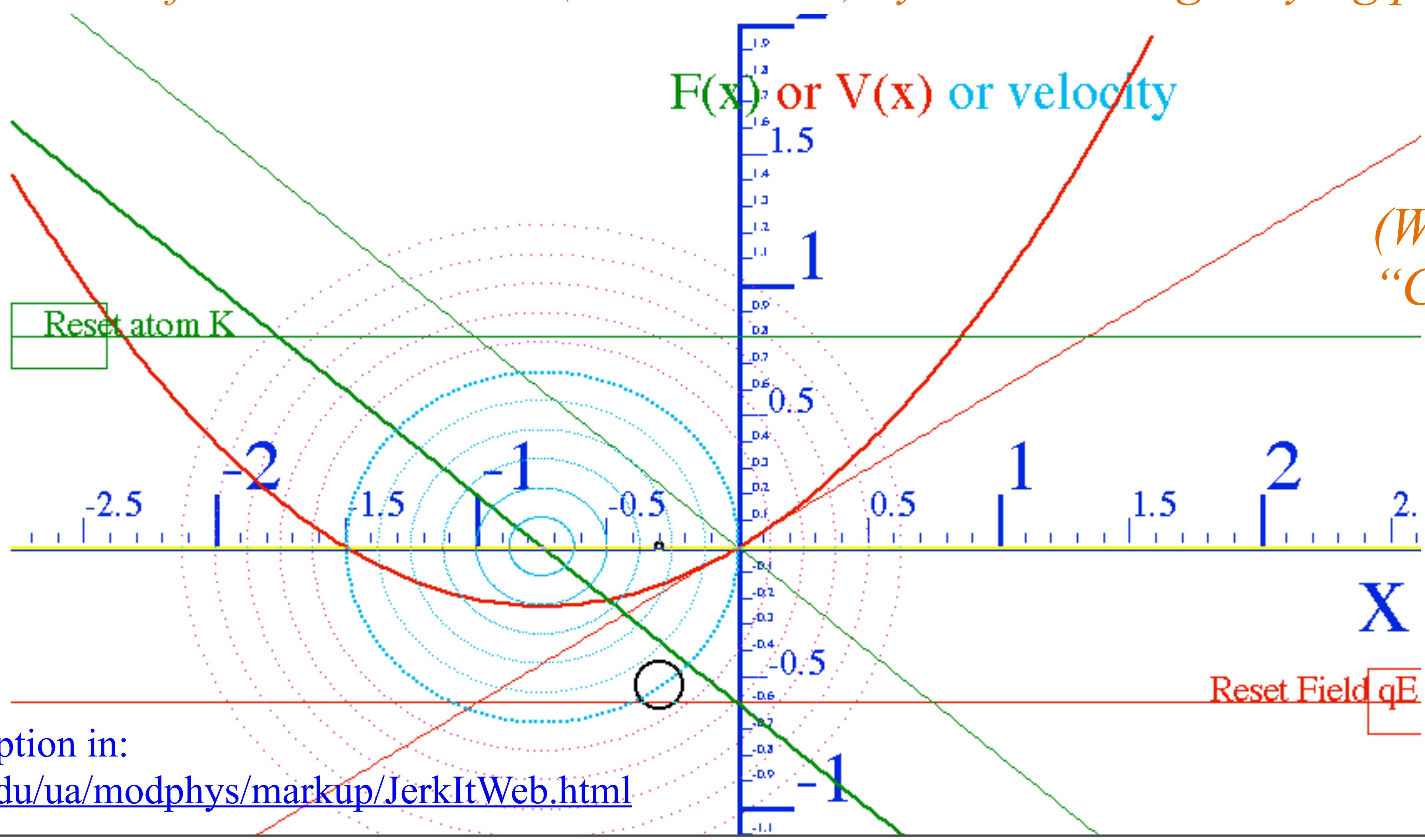
1D Pendulum and phase plot (Simulation)

→ ***Phase control*** *(Simulation of "Catcher in the Eye"))*



Unit 1
Fig. 7.4

Simulation of atomic classical (or semi-classical) dynamics using varying phase control



(We call this process "Catcher in the Eye")

Simulation is an option in:
http://www.uark.edu/ua/modphys/markup/JerkItWeb.html

Exploring phase space and Lagrangian mechanics more deeply

→ ***A weird "derivation" of Lagrange's equations***

Poincare identity and Action, Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

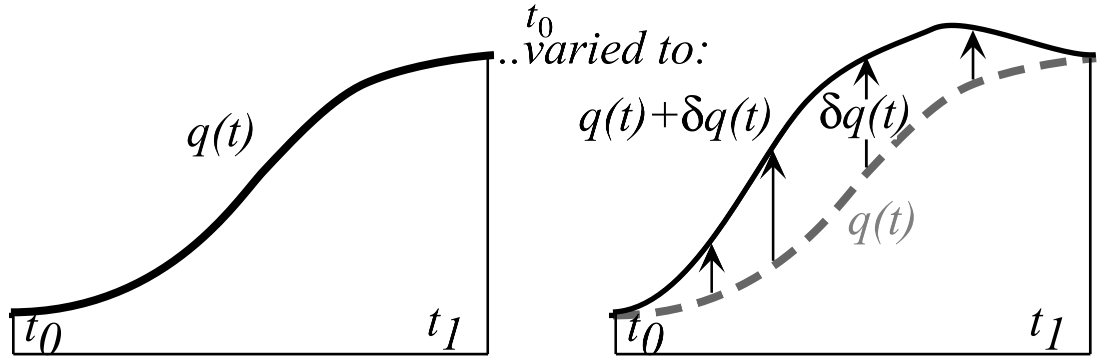
Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange "derivation" of Lagrange's equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q)=\int_{t_0}^{t_1} dt\ L(q(t),\dot{q}(t),t).$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1. There we demand it not vary at all.(1)

$$\delta q(t_0)=0=\delta q(t_1) \qquad (1)$$

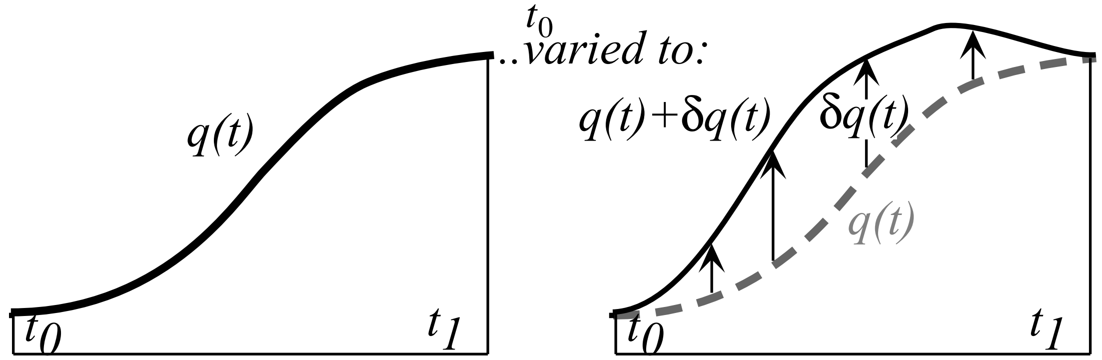
1st order L(q+δq) approximate:

$$S(q+\delta q)=\int_{t_0}^{t_1} dt\left[L(q,\dot{q},t)+\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right] \text{ where: } \delta\dot{q}=\frac{d}{dt}\delta q$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q)=\int_{t_0}^{t_1} dt\; L\left(q(t),\dot{q}(t),t\right).$$



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Replace $\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

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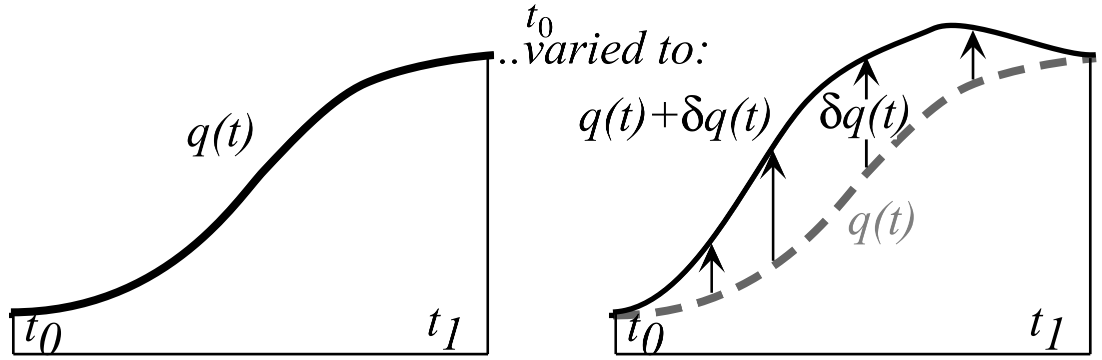

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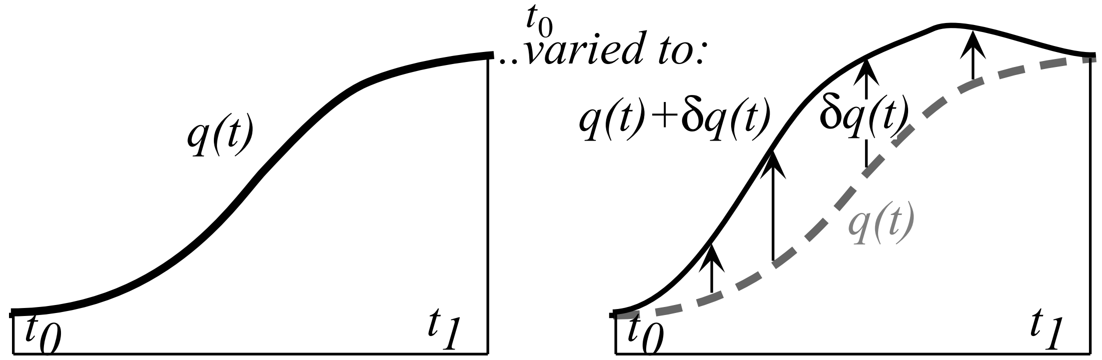
A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt\ L(q(t), \dot{q}(t), t).$$



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1st order $L(q+\delta q)$ approximate:

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] \quad \text{where: } \delta\dot{q} = \frac{d}{dt}\delta q$$

Replace $\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q \right] + \int_{t_0}^{t_1} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)$$

$$= \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Bigg|_{t_0}^{t_1}$$

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Variational calculus finds extreme (minimum or maximum) values to entire integrals

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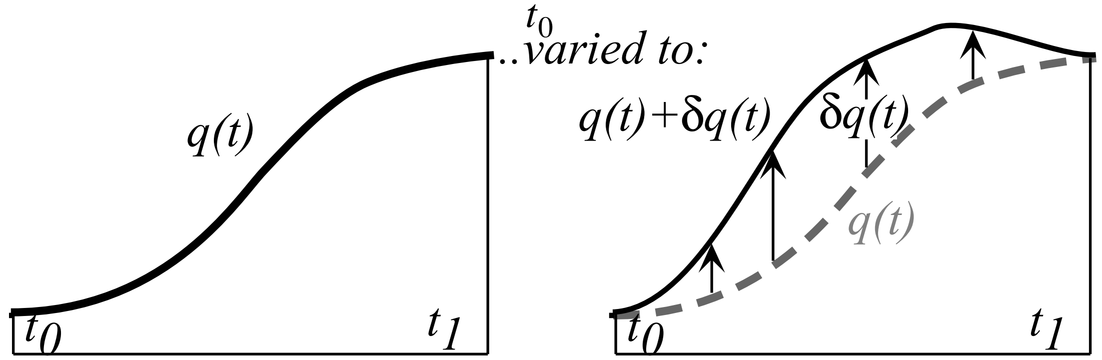

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$$= \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Big|_{t_0}^{t_1}$$

due to requiring (1)

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Euler-Lagrange equation(s)

A weird "derivation" of Lagrange's equations

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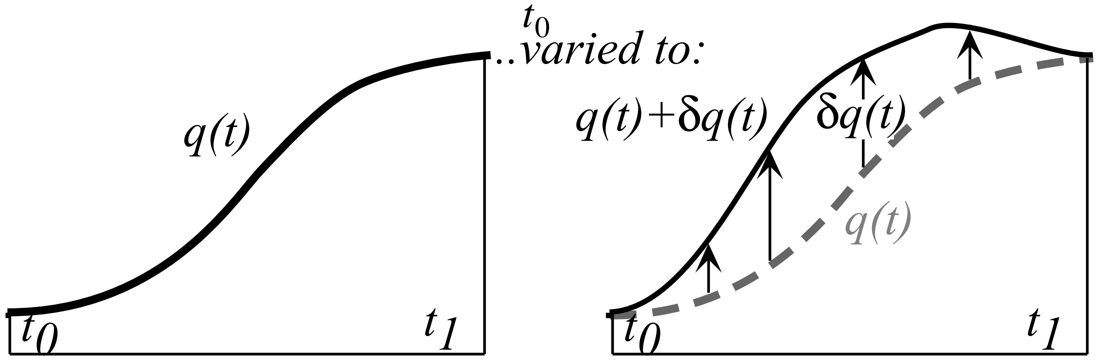

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$$=\int_{t_0}^{t_1} dt\ L(q,\dot q,t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right]\delta q+\left(\frac{\partial L}{\partial q}\delta q\right)\Big|_{t_0}^{t_1}$$

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$$\delta S=0\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)-\frac{\partial L}{\partial q}=0$$

Euler-Lagrange equation(s)

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian L = T - U???

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

➡ ***Poincare identity and Action,*** Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \left(\mathbf{v}=\frac{d\mathbf{r}}{dt} \textit{ implies: } \mathbf{v}\cdot dt = d\mathbf{r}\right)$$

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This is the time differential dS of *action* $S = \int L\cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

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$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 8 shows *DeBroglie law* $\mathbf{p}=\hbar\mathbf{k}$ and *Planck law* $H=\hbar\omega$ make *quantum plane wave phase* Φ:

$$\Phi = S/\hbar = \int L\cdot dt/\hbar$$

Legendre-Poincare identity and Action

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$$\Phi = S/\hbar = \int L\cdot dt/\hbar$$

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

Legendre-Poincare identity and Action

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$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

Q:When is the *Action*-differential dS integrable?
A: A differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

This is the time differential dS of *action* $S = \int L\cdot dt$ whose time derivative is rate L of *quantum phase*.

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Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* *∂-symmetry of W:*

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y\partial x} = \frac{\partial^2 W}{\partial x\partial y} = \frac{\partial f_y}{\partial x}$$

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

This is the time differential dS of *action* $S = \int L\cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 8 shows *DeBroglie law* $\mathbf{p}=\hbar\mathbf{k}$ and *Planck law* $H=\hbar\omega$ make *quantum plane wave phase* Φ:

$$\Phi = S/\hbar = \int L\cdot dt/\hbar$$

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* ∂-symmetry *of* W:

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y\partial x} = \frac{\partial^2 W}{\partial x\partial y} = \frac{\partial f_y}{\partial x}$$

Q: When is the *Action*-differential dS integrable?

A: Differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

dS is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -H$

These conditions are known as Jacobi-Hamilton *equations*

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action, Jacobi-Hamilton *equations*

→ ***How Classicists might have "derived" quantum equations***

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

dS is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -H$

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These conditions are known as Jacobi-Hamilton *equations*

Try 1st **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t)=\frac{\partial}{\partial \mathbf{r}}e^{iS/\hbar}=\frac{\partial(iS/\hbar)}{\partial \mathbf{r}}e^{iS/\hbar}=(i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t)=(i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t)=\mathbf{p}\psi(\mathbf{r},t)}$$

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

dS is integrable if: $\dfrac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\dfrac{\partial S}{\partial t} = -H$

These conditions are known as Jacobi-Hamilton *equations*

Try 1st **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = (i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\psi(\mathbf{r},t)}$$

Try 1st t-derivative of wave ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)}$$

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action, Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

→ ***Huygen's contact transformations enforce minimum action***

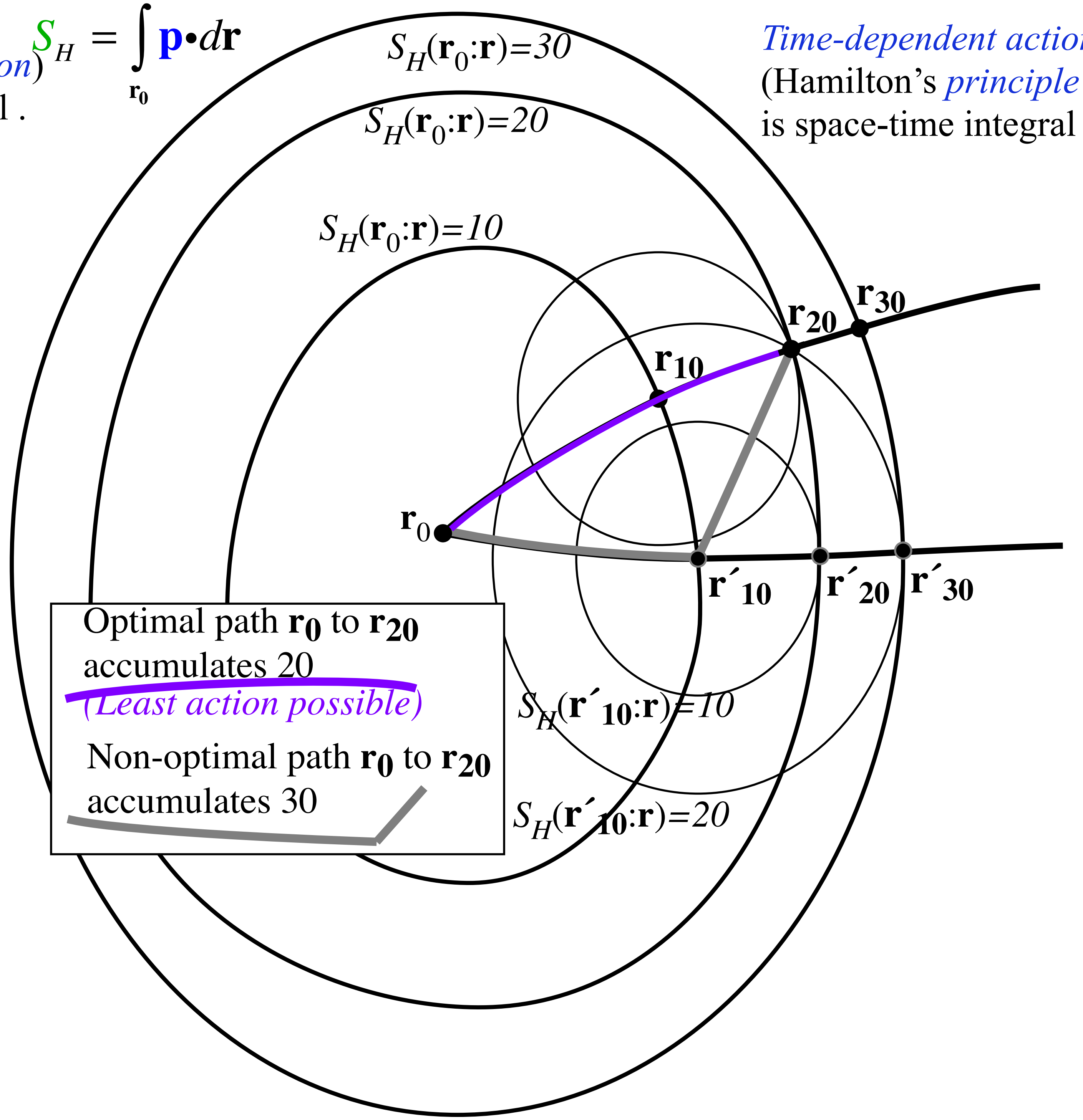
How to do quantum mechanics if you only know classical mechanics

Huygen's contact transformations enforce minimum action

Each point $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
Only ***minimum action*** *path interferes constructively*

Time-independent action $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p}\bullet d\mathbf{r}$
(Hamilton's *reduced action*)
is a purely spatial integral .

Time-dependent action $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} \left(\mathbf{p}\bullet d\mathbf{r} - H\cdot dt\right)$
(Hamilton's *principle action*)
is space-time integral .



Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

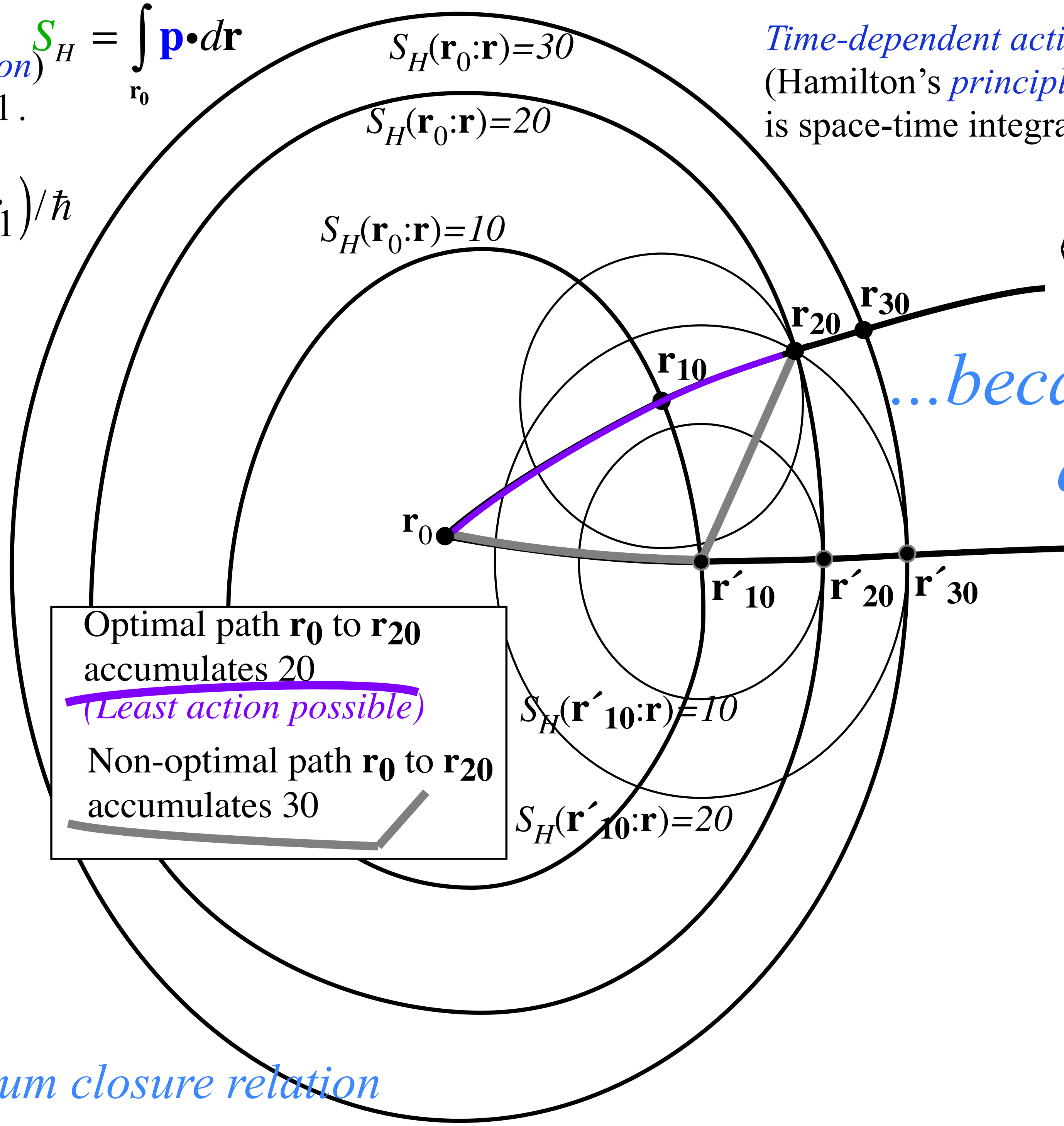
Each point $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
Only ***minimum action*** *path interferes constructively*

Time-independent action $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \bullet d\mathbf{r}$
(Hamilton's *reduced action*)
is a purely spatial integral .

$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1)/\hbar}$$

Time-dependent action $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \bullet d\mathbf{r} - H \cdot dt)$
(Hamilton's *principle action*)
is space-time integral .

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1)/\hbar}$$



...because action *<u>is</u>*
quantum wave phase

Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i\left(S_H(\mathbf{r}_0:\mathbf{r}') + S_H(\mathbf{r}':\mathbf{r}_1)\right)/\hbar} = e^{i S_H(\mathbf{r}_0:\mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

Exploring phase space and Lagrangian mechanics more deeply

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How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

➡ ***How to do quantum mechanics if you only know classical mechanics***

Davis-Heller "Color-Quantization" or "Classical Chromodynamics"

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0 : \mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \bullet d\mathbf{r}$. The integer n ($n = 0, 1, 2, \ldots$) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi\hbar n = hn$$

Numerically integrate Hamilton's equations and Lagrangian L. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L\, dt + Ht \,.$$

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$. The integer n (n = 0, 1, 2,...) is a *quantum number*.

$$1=\langle \mathbf{r}_0|\mathbf{r}_0\rangle = e^{iS_H(\mathbf{r}_0:\mathbf{r}_0)/\hbar} = e^{i\Sigma_H/\hbar}=1 \text{ for: } \Sigma_H = 2\pi\hbar n = hn$$

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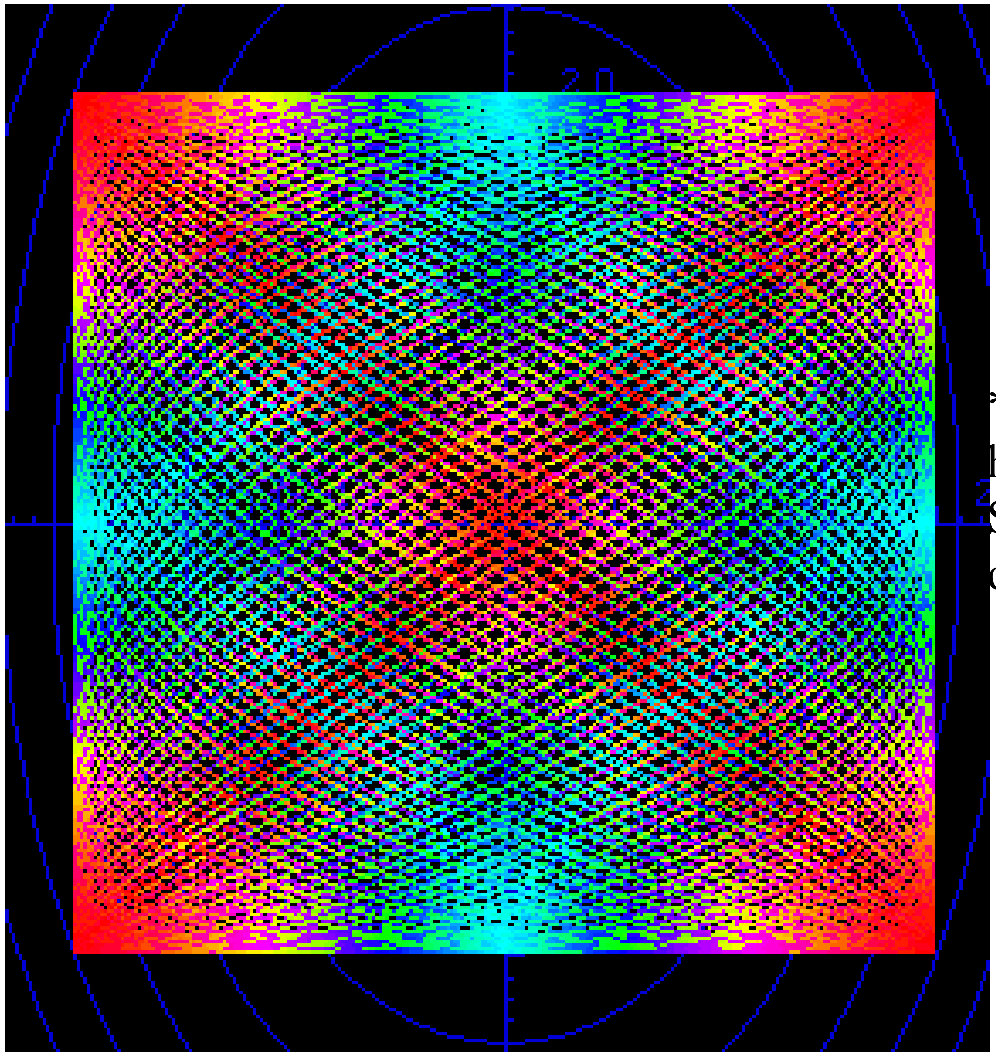
$$S_H(\mathbf{0}:\mathbf{r}) = S_p(\mathbf{0}, 0:\mathbf{r}, t) + Ht = \int_0^t L\,dt + Ht\,.$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* 2π as, for example,
0=red, *π/4=orange*, *π/2=yellow*, *3π/4=green*, *π=cyan* (opposite of red), *5π/4=indigo*, *3π/2=blue*, *7π/4=purple*, *and 2π=red* (full color circle).
Interpolating action on a palette of *32* colors is enough precision for low quanta.



simulation by "Color U(2)"

Unit 1 Fig. 12.13

*closed system has quantized E.
Standing wave has only two phases(±) *cyan* and *red*

Wavepacket and Color-quantization:
M. J. Davis and E. J. Heller, J. Chem. Phys. 75, 246 (1981)

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$. The integer n ($n = 0, 1, 2,...$) is a *quantum number*.

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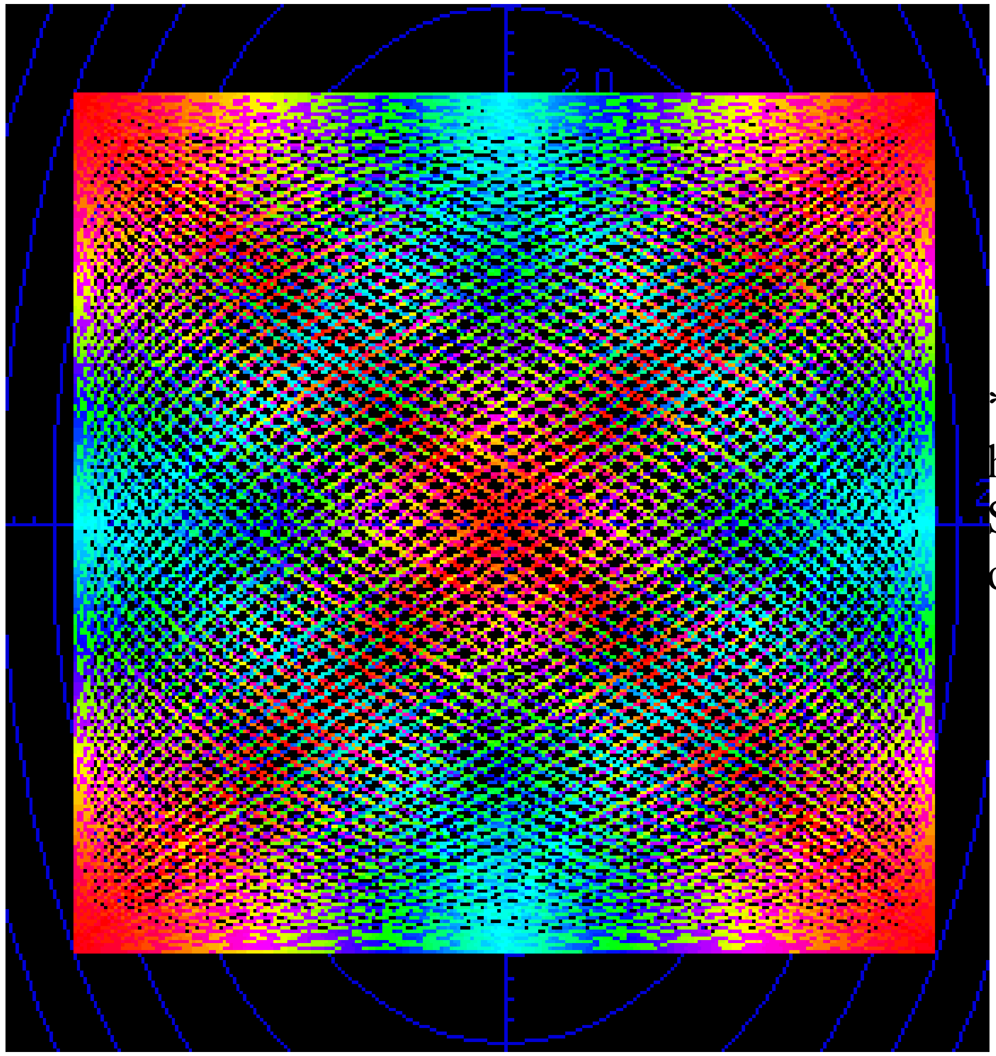
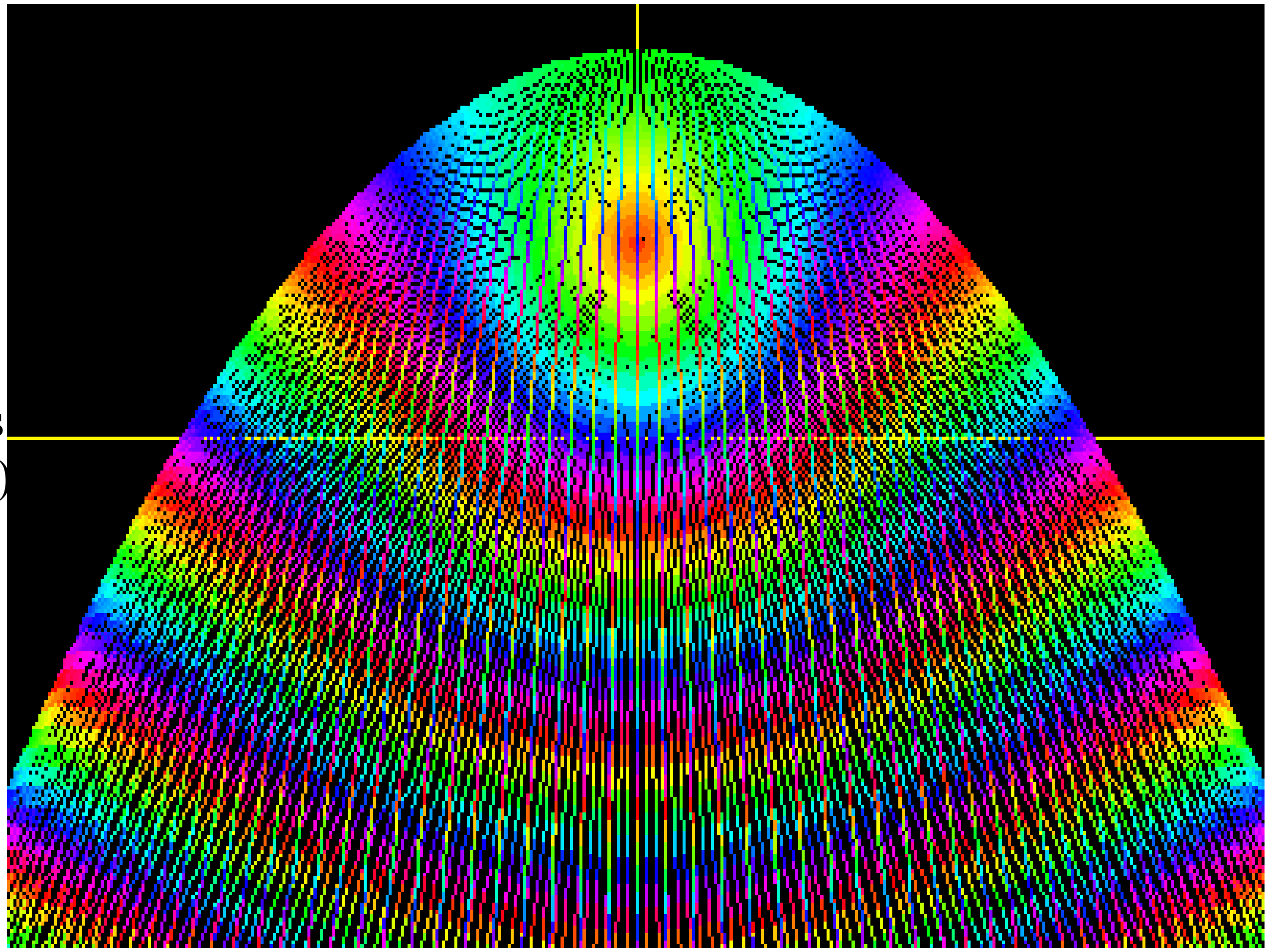


simulation by "Color U(2)"

Unit 1 Fig. 12.13

*closed system has quantized E
Standing wave has only two phases(±) *cyan* and *red*

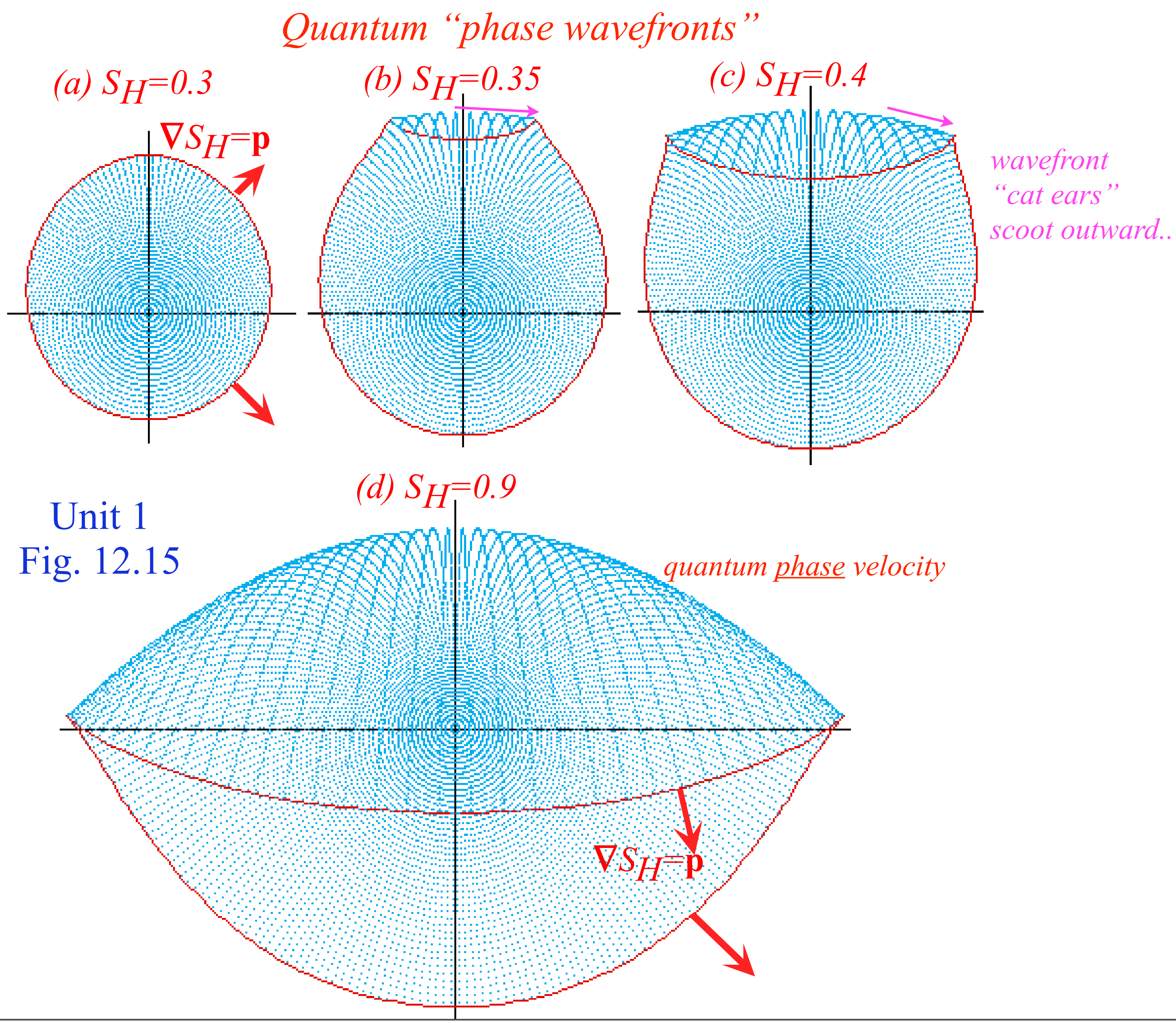
*open system has continuous energy



Unit 1 Fig. 12.14 *simulation by "CoulIt"*

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-H dt$.

$$V_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$



Unit 1
Fig. 12.15

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

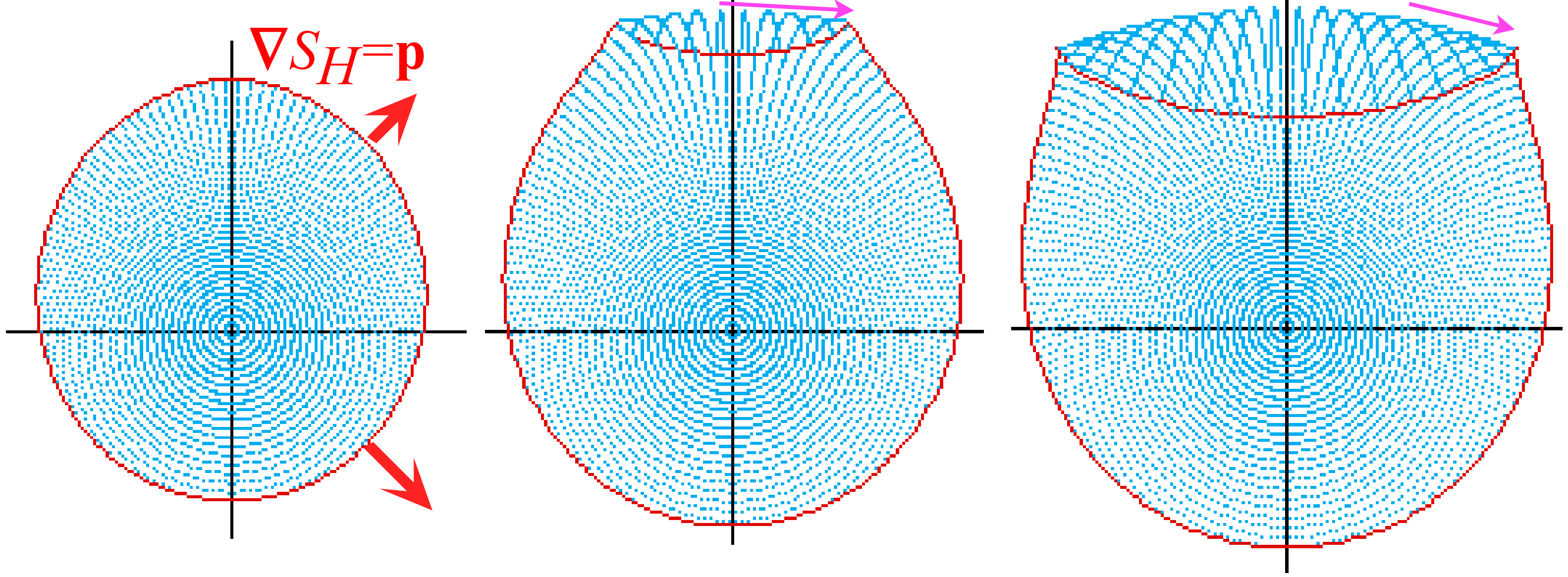
This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"

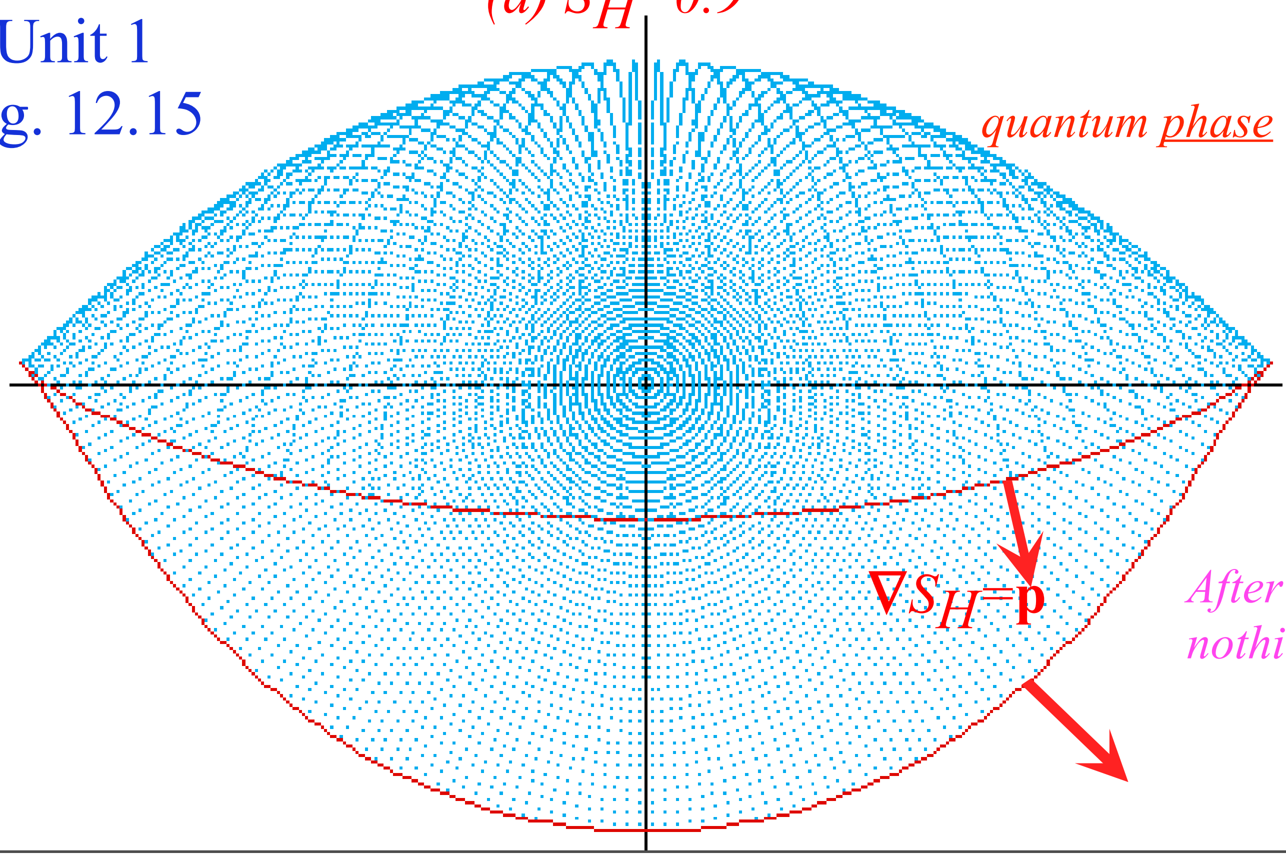
(a) $S_H=0.3$ *(b)* $S_H=0.35$ *(c)* $S_H=0.4$



wavefront "cat ears" scoot outward..

(d) $S_H=0.9$

Unit 1
Fig. 12.15



quantum *<u>phase</u>* *velocity*

After a while …
nothing left but a smile!

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

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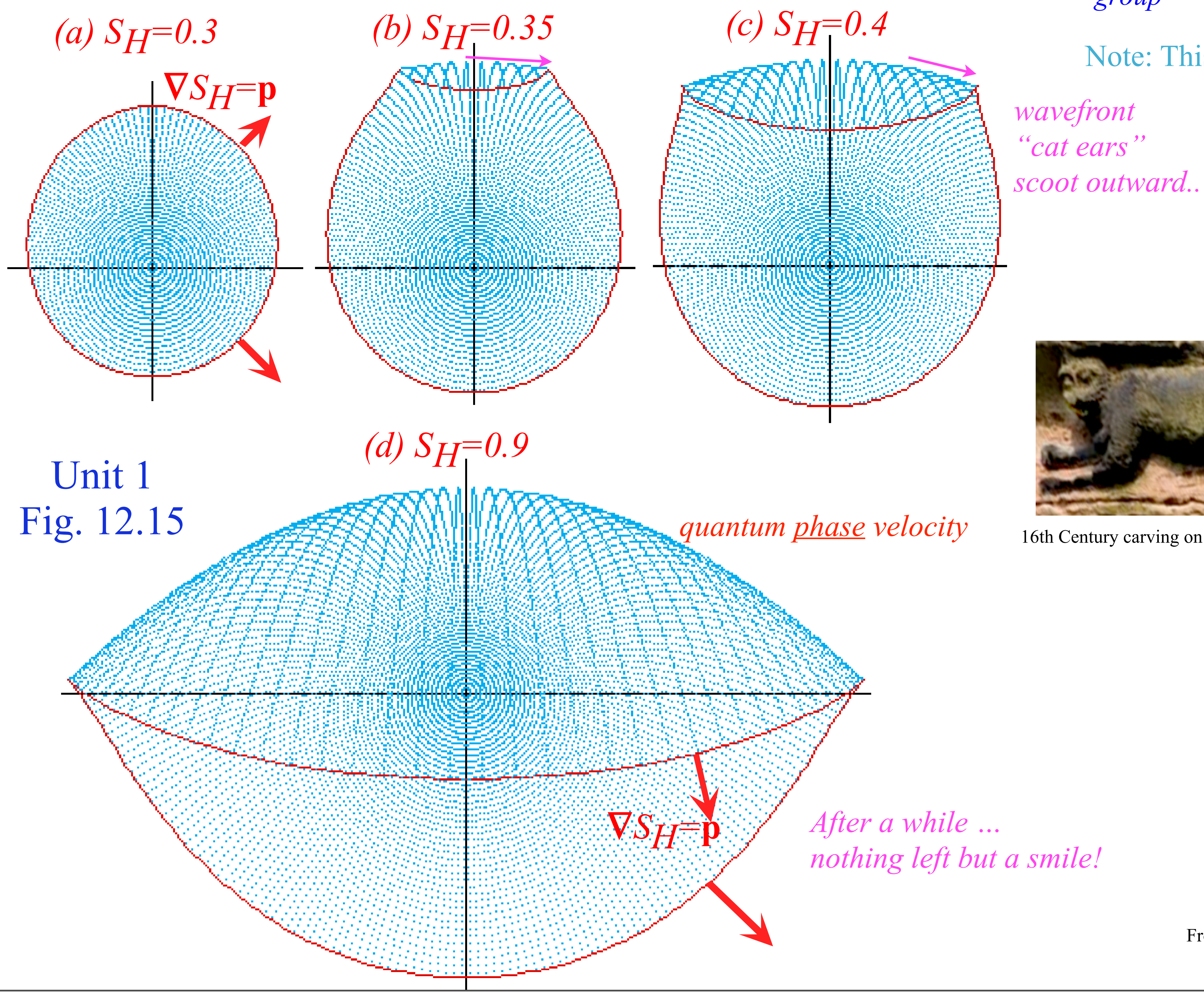
Quantum "phase wavefronts"



Unit 1
Fig. 12.15

16th Century carving on St. Wifred's in Grappenhall

...on St. Nicolas

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)

A moving wave has a *quantum phase velocity* found by setting $S = const.$ or $dS(0,0{:}\mathbf{r},t) = 0 = \mathbf{p}\cdot d\mathbf{r} - H dt$.

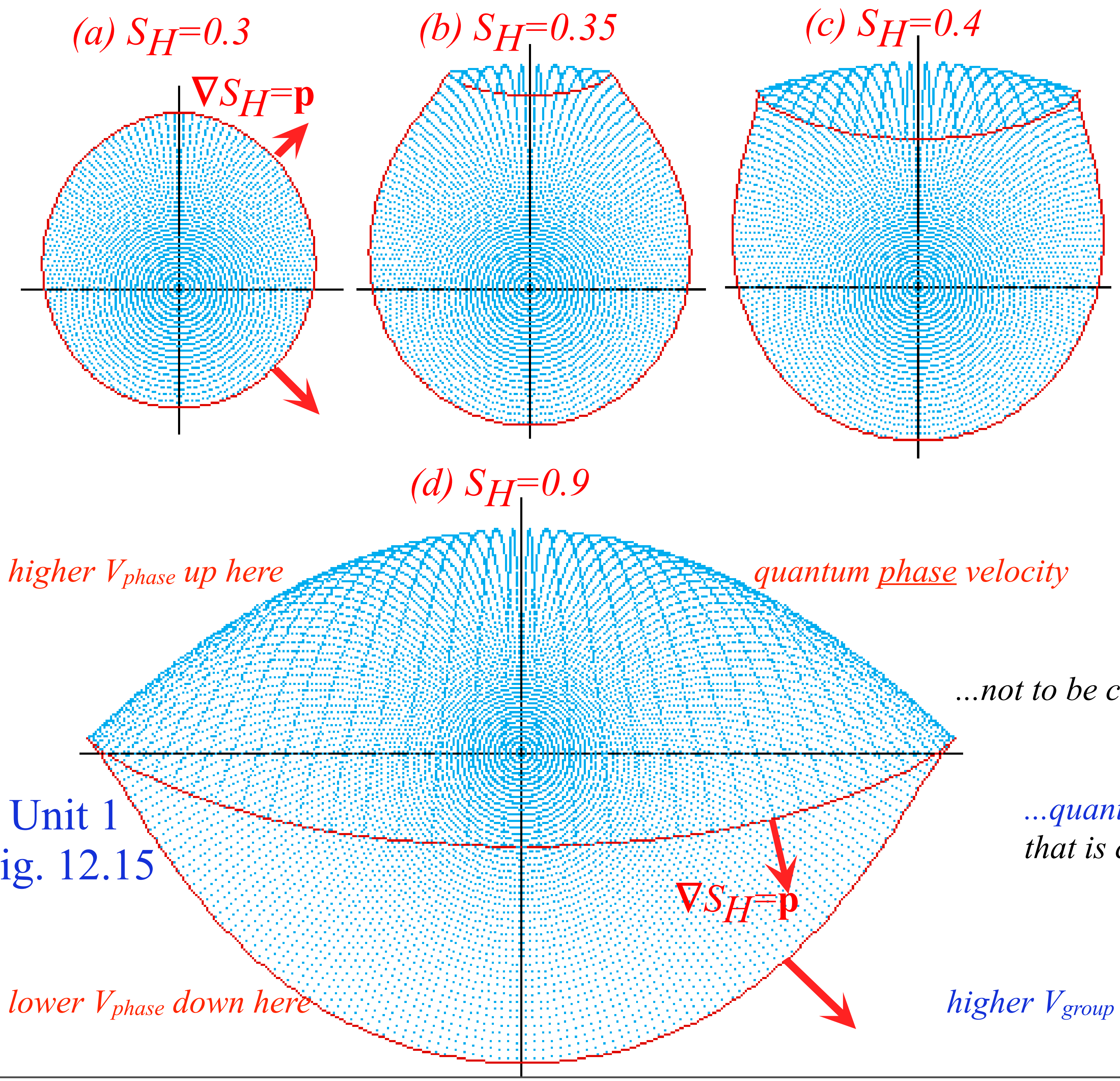
$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

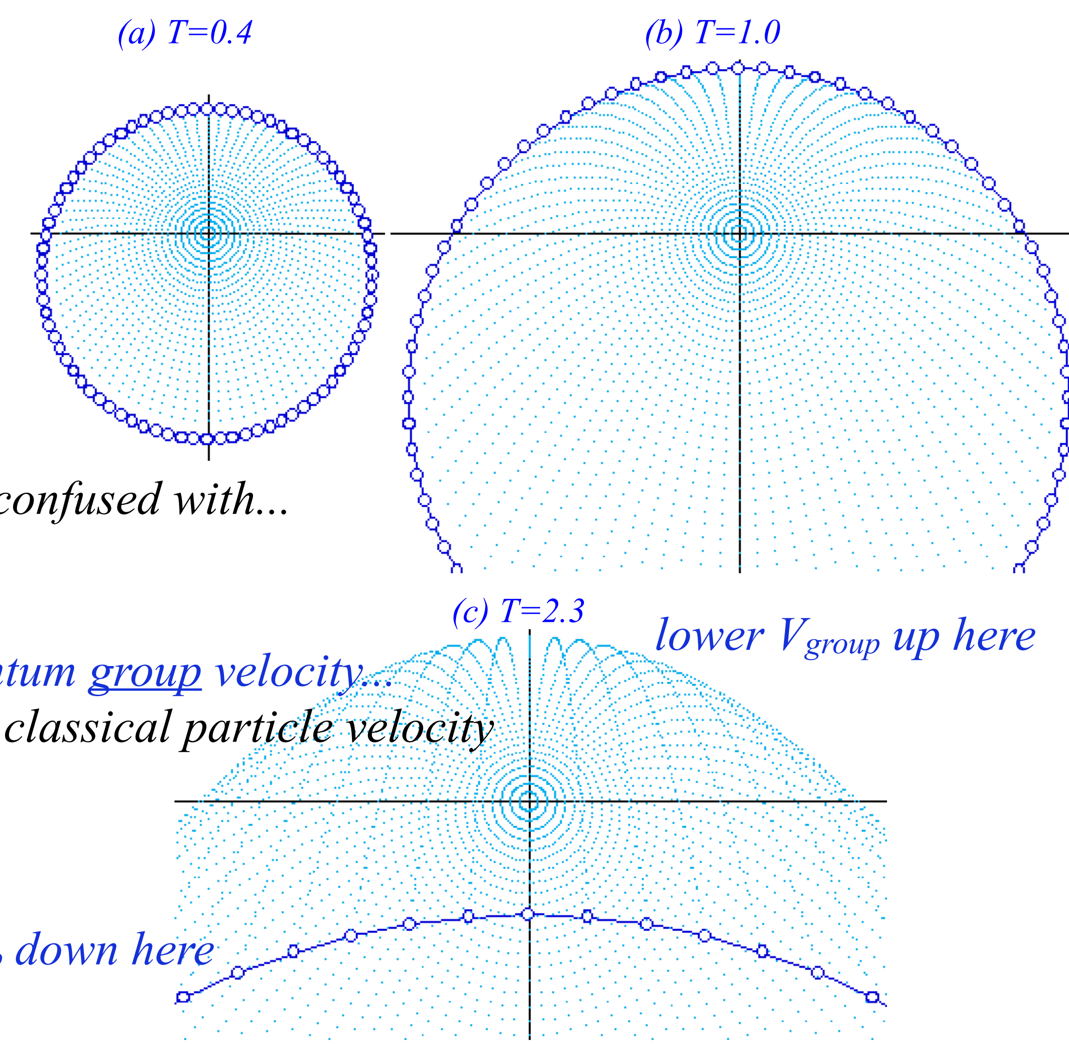
Quantum "phase wavefronts"



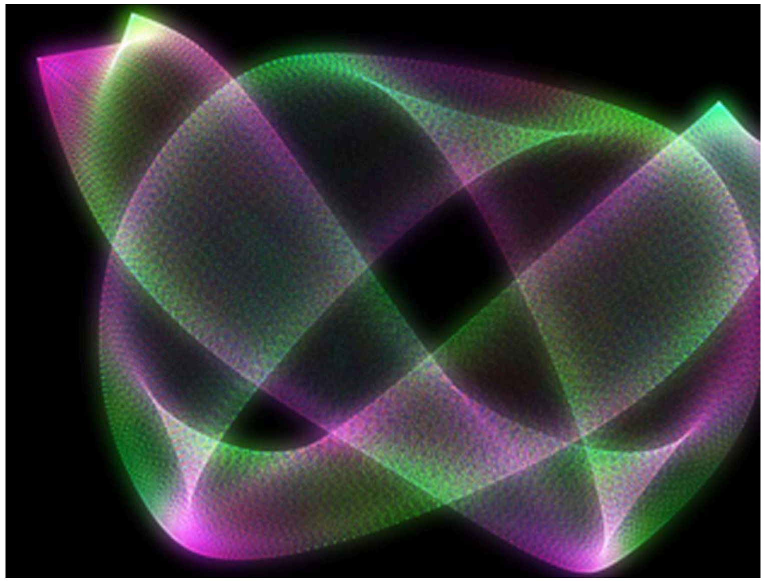
Unit 1
Fig. 12.15

...not to be confused with...

Classical "blast wavefronts"



Check out the Heller Galleries http://www.ericjhellergallery.com/index.pl?page=image;iid=76



Resonance Fine Art

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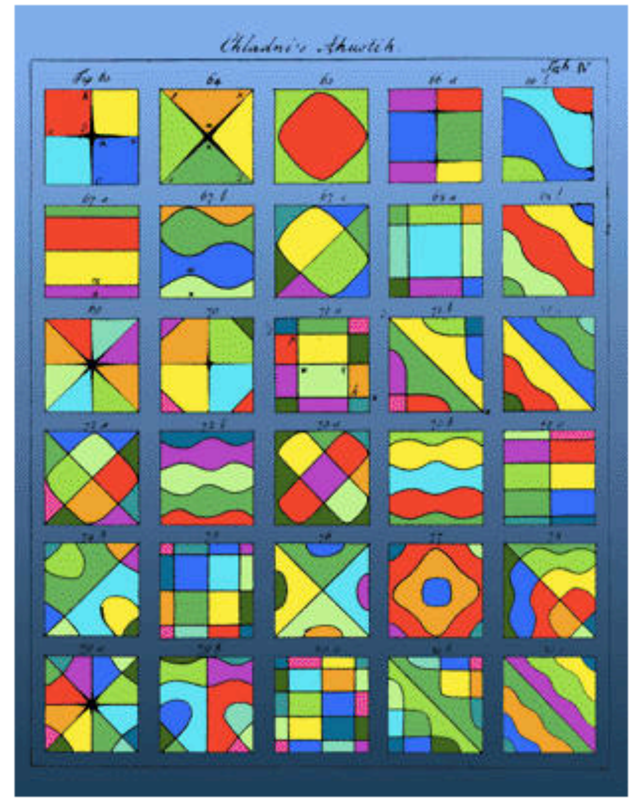
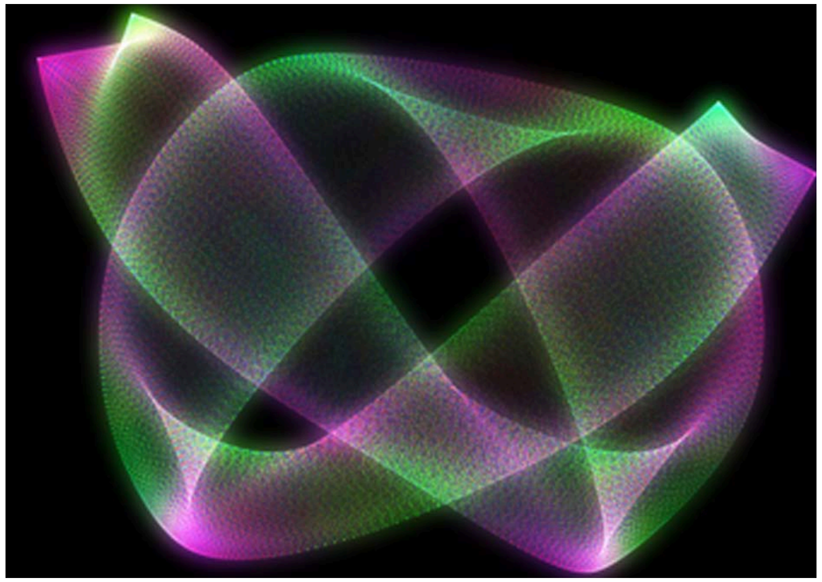
Chladni



The diagrams of Ernst Chladni (1756-1827) are the scientific, artistic, and even the sociological birthplace of the modern field of wave physics and quantum chaos. Educated in Law at the University of Leipzig, and an amateur musician, Chladni soon followed his love of science and wrote one of the first treatises on acoustics, "Discovery of the Theory of Pitch". Chladni had an inspired idea: to make waves in a solid material visible. This he did by getting metal plates to vibrate, stroking them with a violin bow. Sand or a similar substance spread on the surface of the plate naturally settles to the places where the metal vibrates the least, making such places visible. These places are the so-called nodes, which are wavy lines on the surface. The plates vibrate at pure, audible pitches, and each pitch has a unique nodal pattern. Chladni took the trouble to carefully diagram the patterns, which helped to popularize his work. Then he hit the lecture circuit, fascinating audiences in Europe with live demonstrations. This culminated with a command performance for Napoleon, who was so impressed that he offered a prize to anyone who could explain the patterns. More than that, according to Chladni himself, Napoleon remarked that irregularly shaped plate would be much harder to understand! While this was surely also known to Chladni, it is remarkable that Napoleon had this insight. Chladni received a sum of 6000 francs from Napoleon, who also offered 3000 francs to anyone who could explain the patterns. The mathematician Sophie Germain took he prize in 1816, although her solutions were not completed until the work of Kirchoff thirty years later. Even so, the patterns for irregular shapes remained (and to some extent remains) unexplained. Government funding of waves research goes back a long way! (Chladni was also the first to maintain that meteorites were extraterrestrial; before that, the popular theory was that they were of volcanic origin.) One of his diagrams is the basis for image, which is a playfully colored version of Chaldni's original line drawing. Chladni's original work on waves confined to a region was followed by equally remarkable progress a few years later.

Check out the Heller Galleries

http://www.ericjhellergallery.com/index.pl?page=image;iid=76



National Science Foundation (NSF)
Arlington, VA

September-November 2002

Selected images.

http://search.nsf.gov/search?ie=&site=nsf&output=xml_no_dtd&proxyreload=1&client=nsf&lr=&proxystylesheet=http%3A%2F%2Fwww.nsf.gov%2Fsearch%2Fnsf_new.xslt&oe=&btnG.x=0&btnG.y=0&q=eric+heller

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University Museum, University of Arkansas, Fayetteville, AK

October 2002 - December 2002

"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

Check out the Heller Galleries

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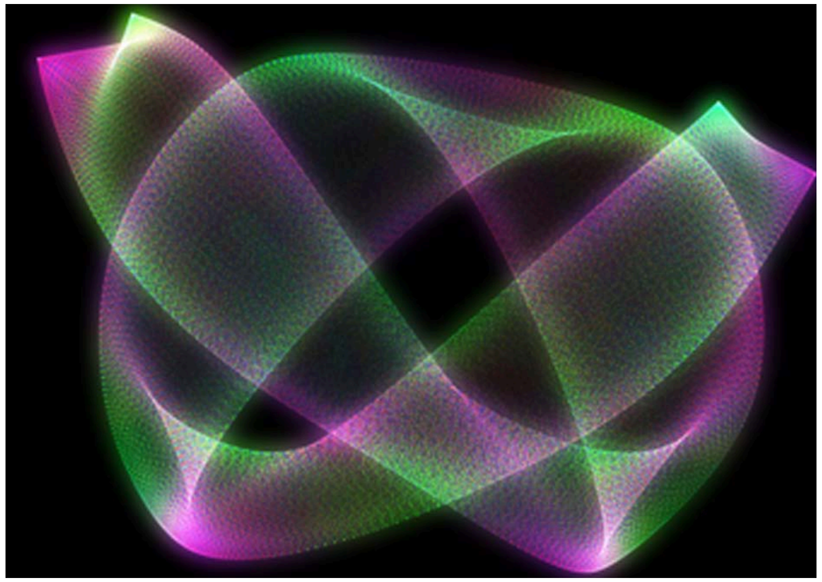

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"Approaching Chaos: Visions from the Quantum Frontier"

Approaching Chaos is supported by a grant from the National Science Foundation and by MIT Museum and the Center for Theoretical Physics at the Massachusetts Institute of Technology.

*UAF *Museum closed after this exhibit*

Lecture 14 ends here
Thur 10.9.2014