Lecture 17
Thur. 10.21.2014

Introducing GCC Lagrangian `a la Trebuchet Dynamics

(Ch. 1-3 of Unit 2 and Unit 3)

The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. ***273****, 66 (July 1995))*
- *The medieval ingenium (9th to 14th century) and modern re-enactments*
- *Human kinesthetics and sports kinesiology*

Cartesian to GCC transformations (Mostly Unit 2.)
- *Jacobian relations*
- *Kinetic energy calculation*
- *Dynamic metric tensor* γ_{mn}

Geometric and topological properties of GCC transformations (Mostly Unit 3.)
- *Multivalued functionality and connections*
- *Covariant and contravariant relations*
- *Metric tensors*

Chapter 1. The Trebuchet: A dream problem for Galileo?

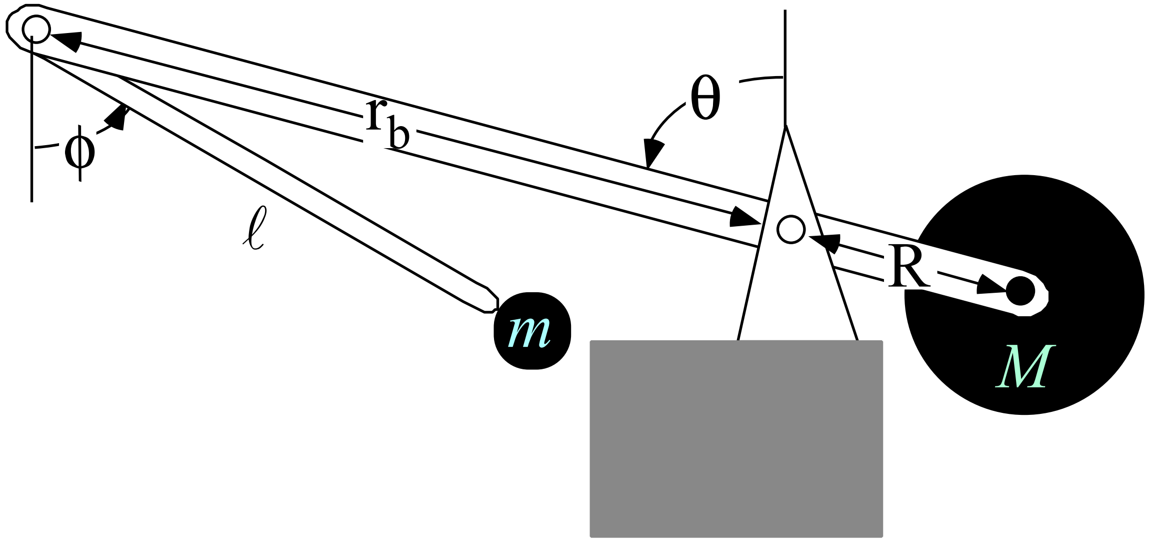
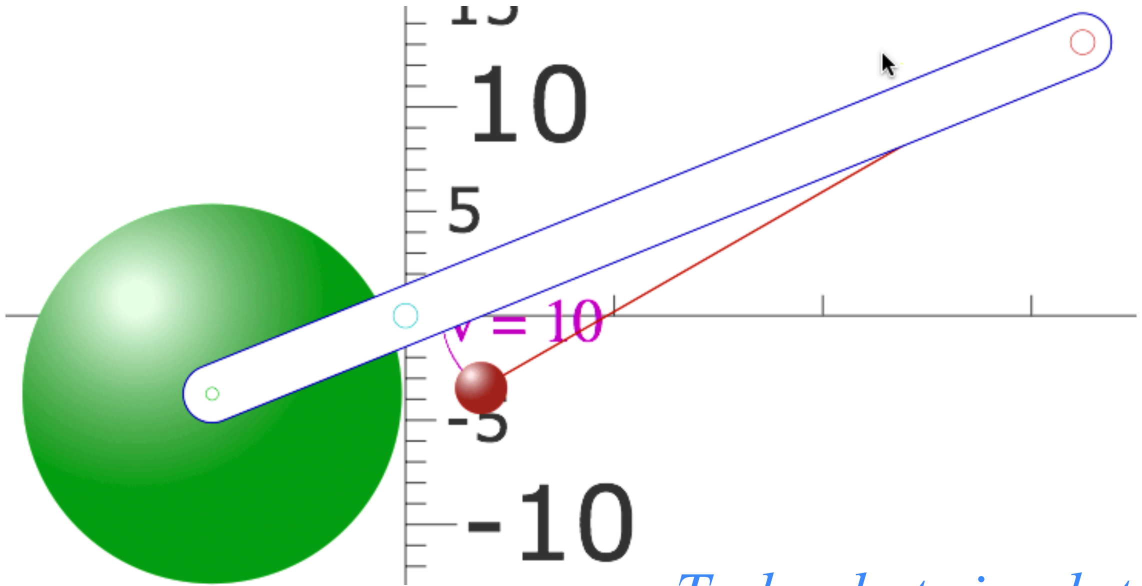


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html

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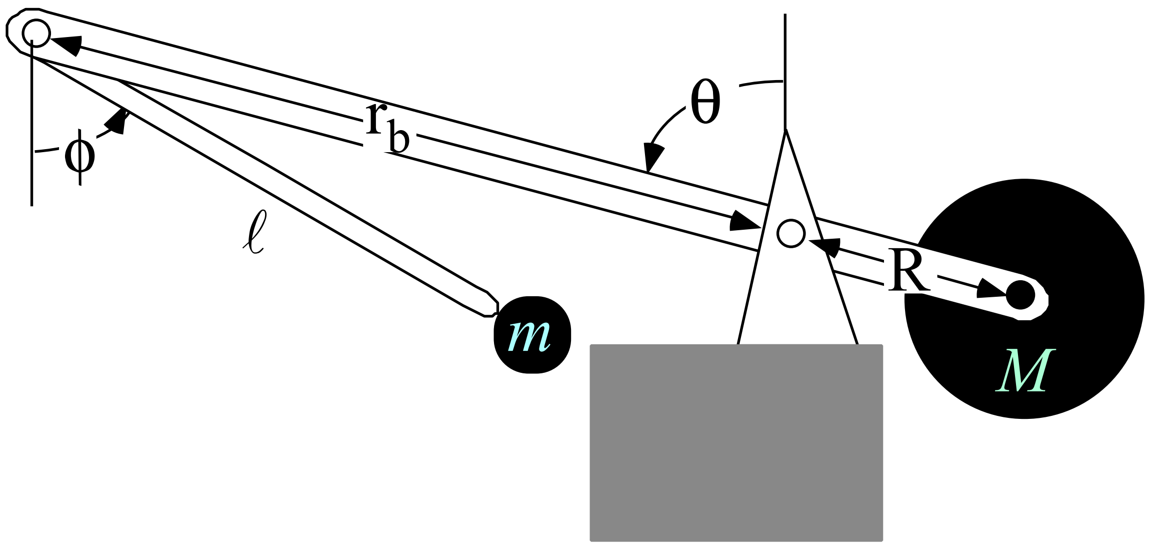

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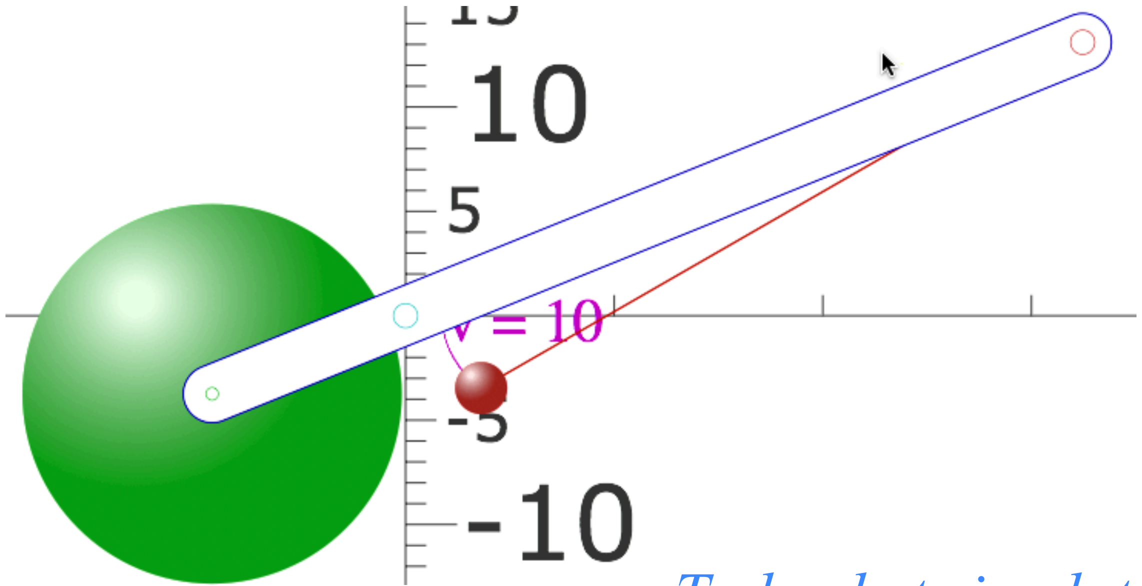

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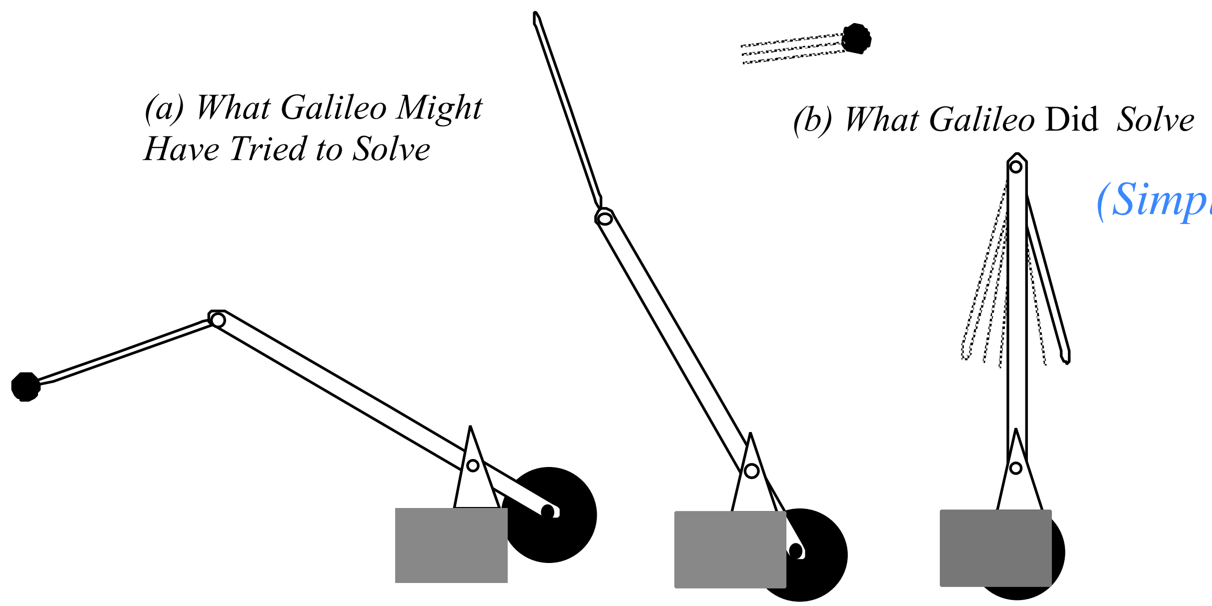

(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed fictitious) problem

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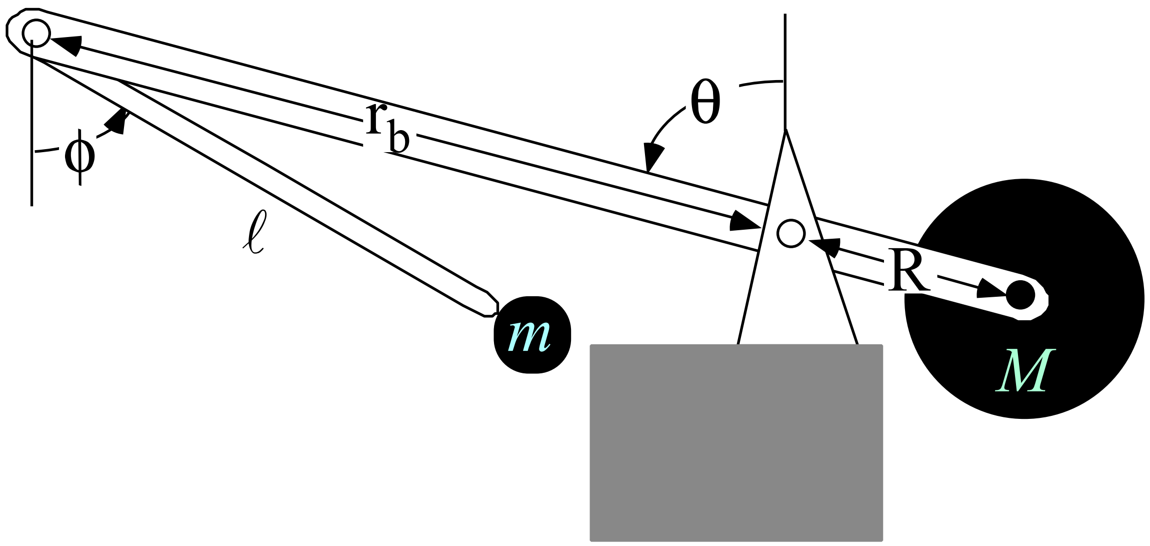

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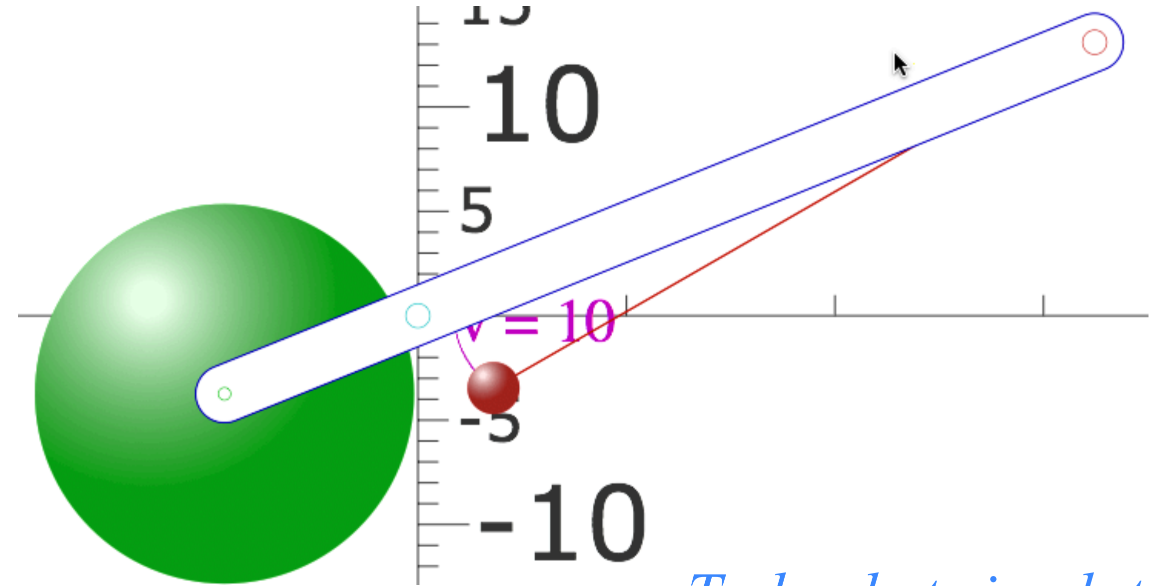

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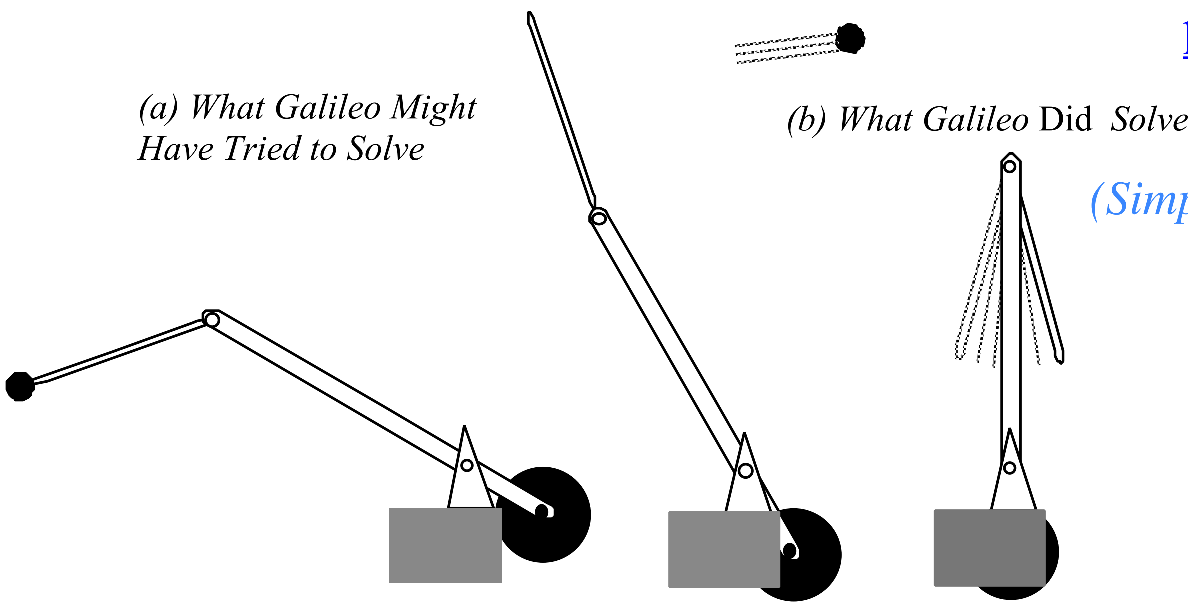

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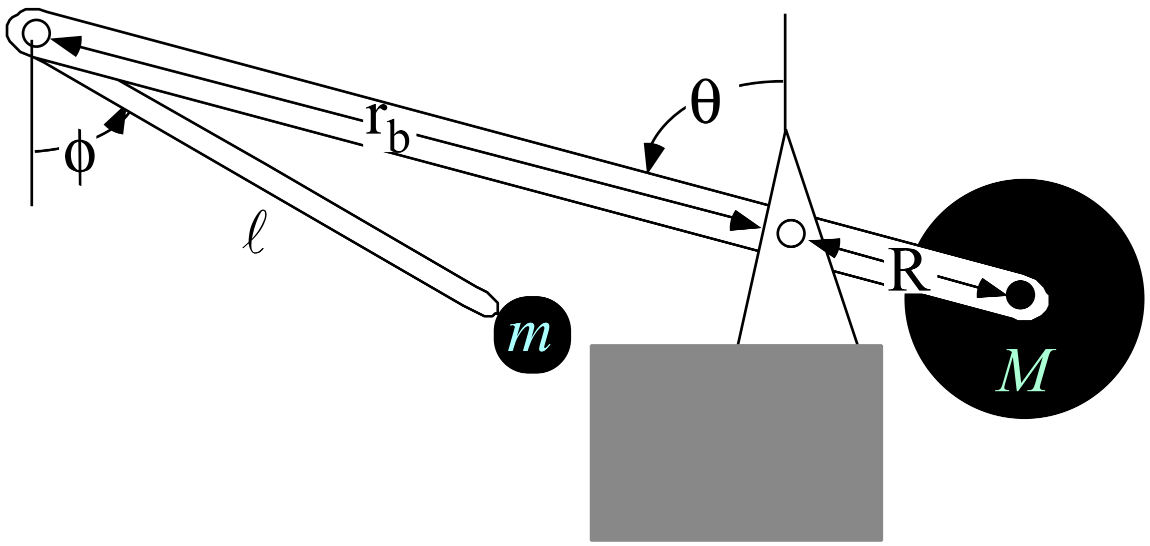

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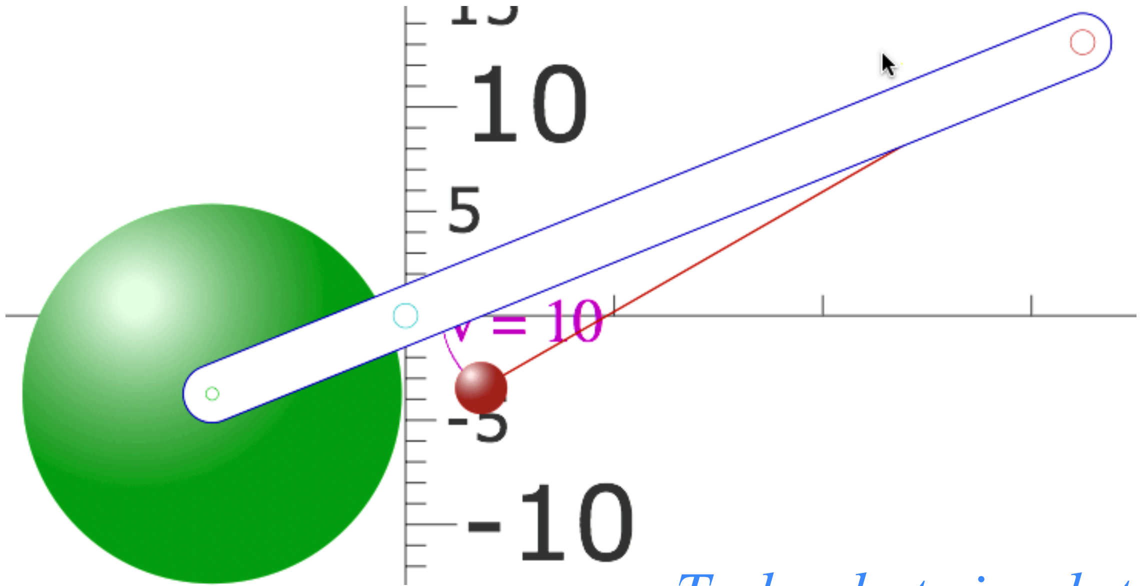

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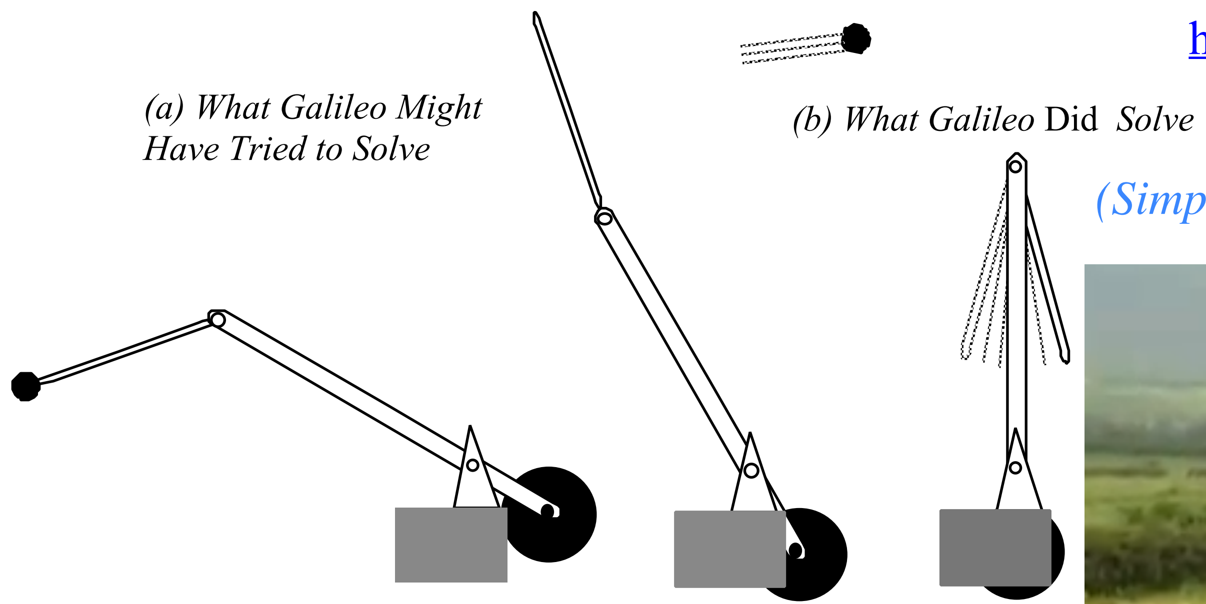


Fig. 2.1.2 Galileo's (supposed) problem

(Simple pendulum dynamics)

The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. ***273****, 66 (July 1995))*

The medieval ingenium (9th to 14th century) and modern re-enactments

→ *Human kinesthetics and sports kinesiology*

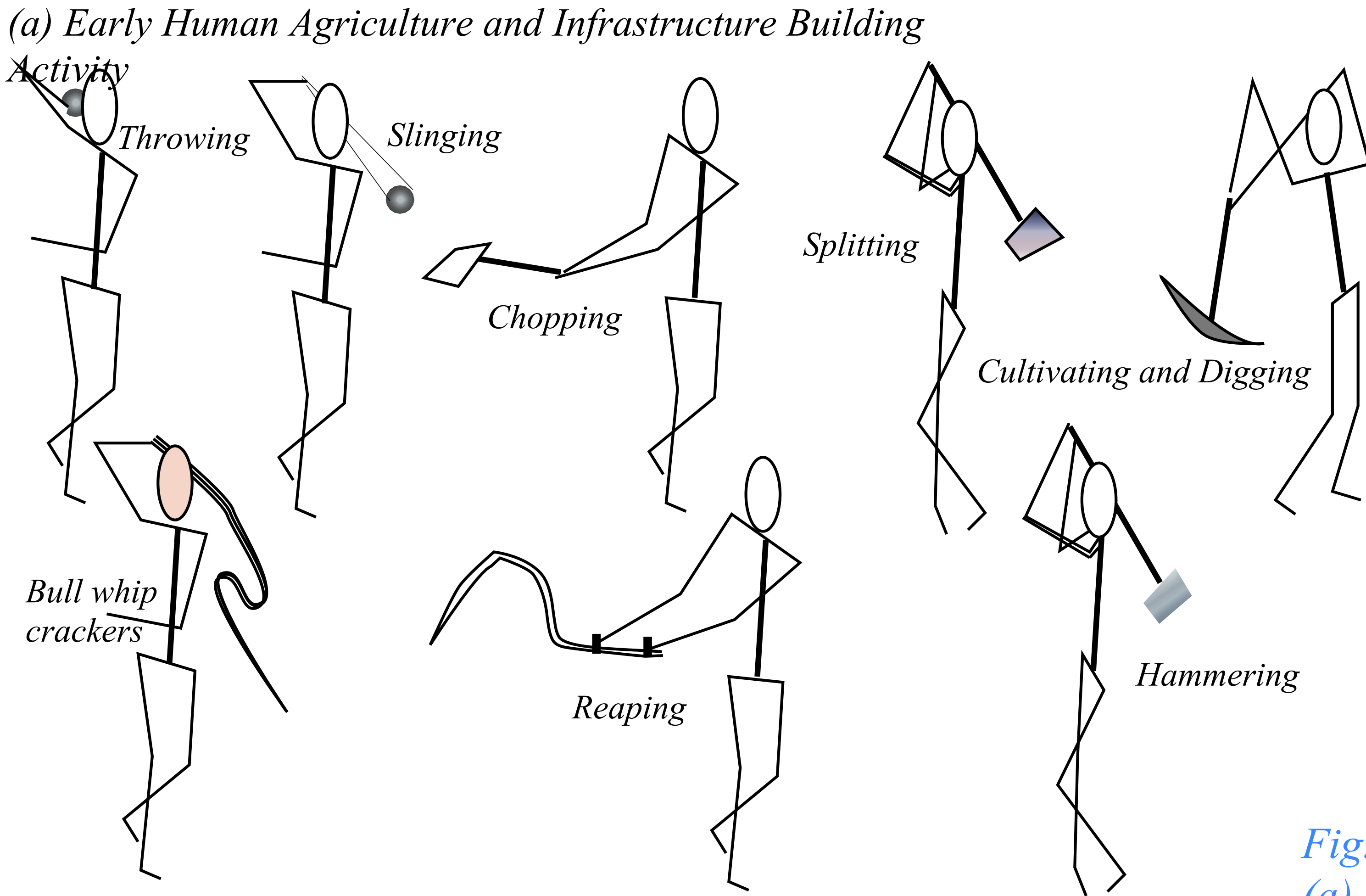


Fig. 2.1.3 Trebuchet-like motion of humans. (a) Early work.

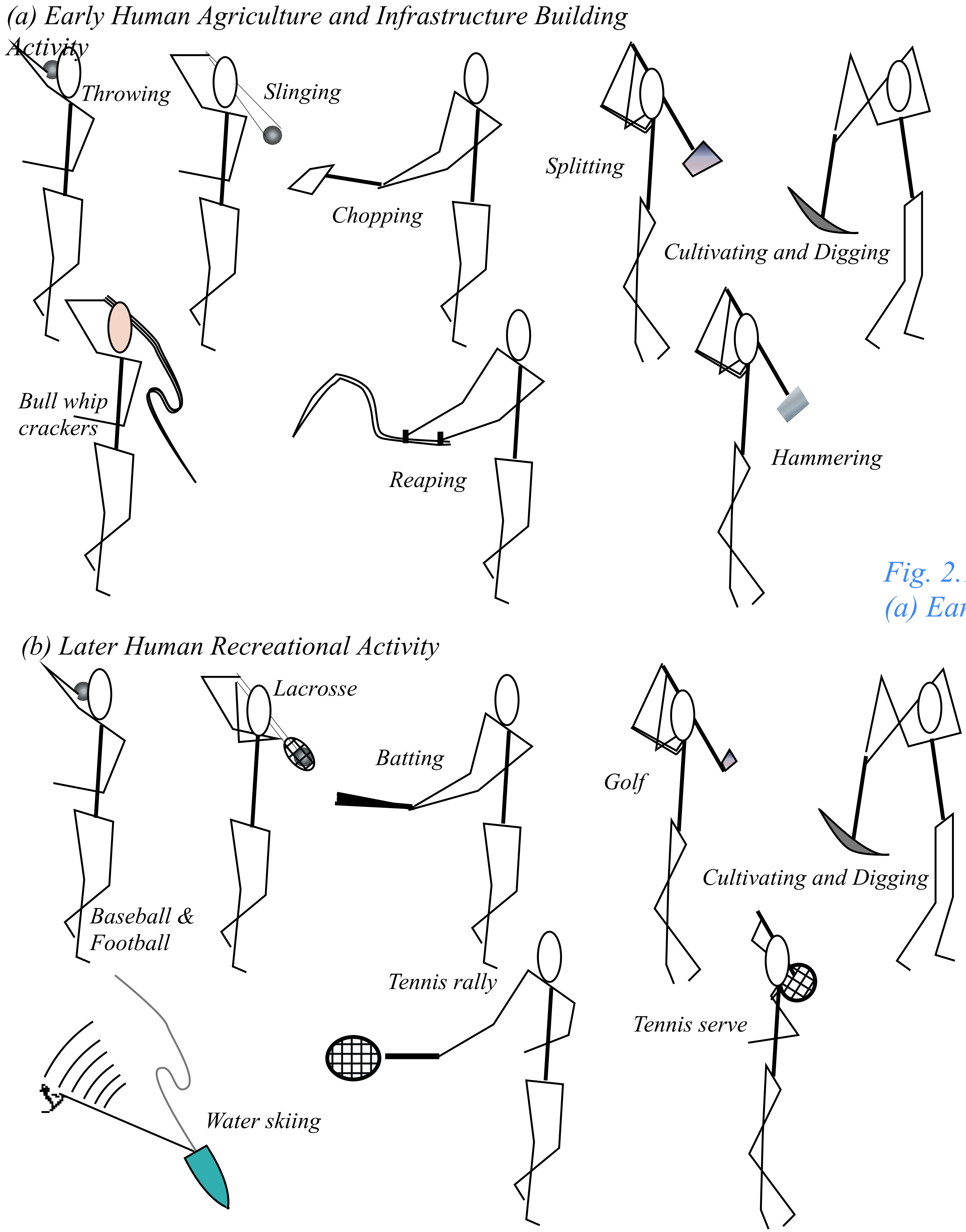


Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work. (b) Later recreational kinesthetics.

Cartesian to GCC transformations

→ *Jacobian relations*

Kinetic energy calculation

Dynamic metric tensor γ_{mn}

Coordinates of mass M (Driving weight):

Coordinates of mass m (Payload or projectile):

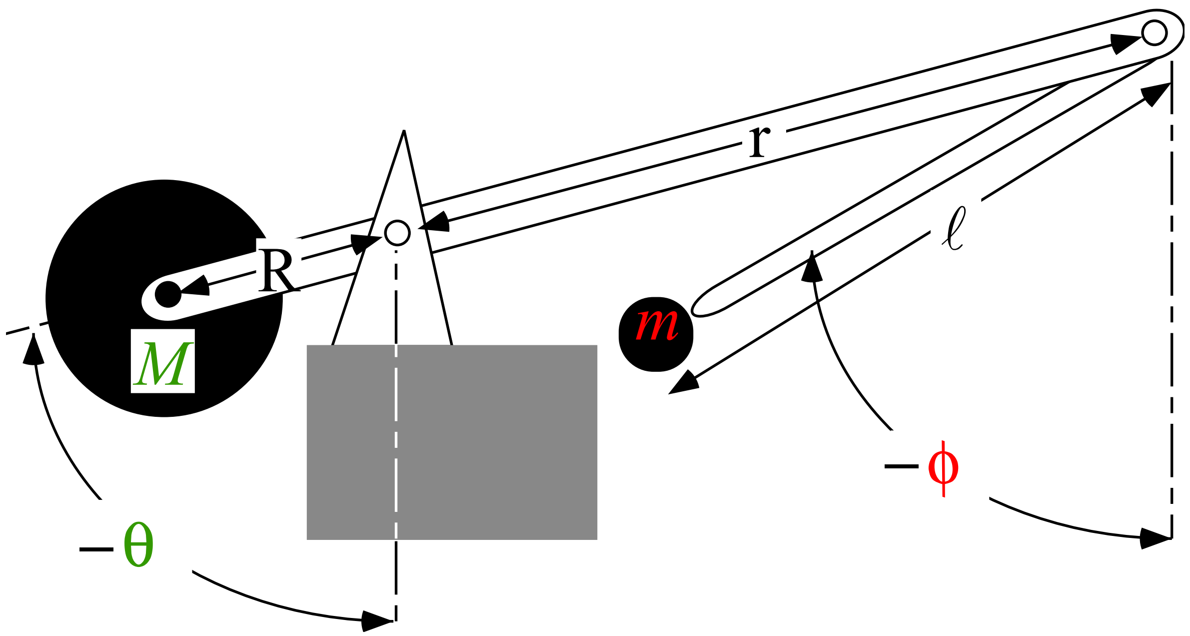
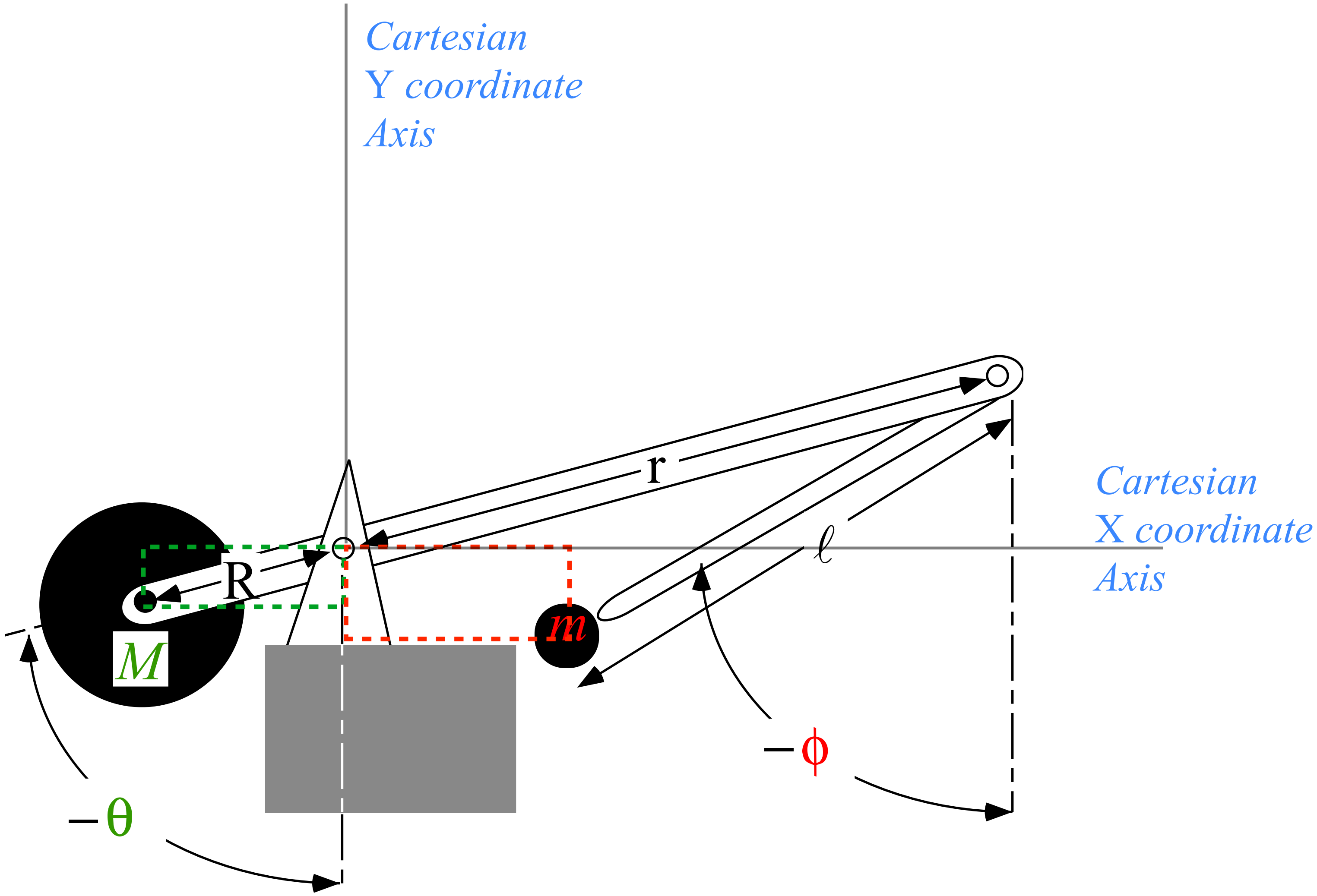


Fig. 2.2.1 Cartesian coordinates related to trebuchet angles θ and ϕ.

Coordinates of mass *M (Driving weight):*

Coordinates of mass *m (Payload or projectile):*



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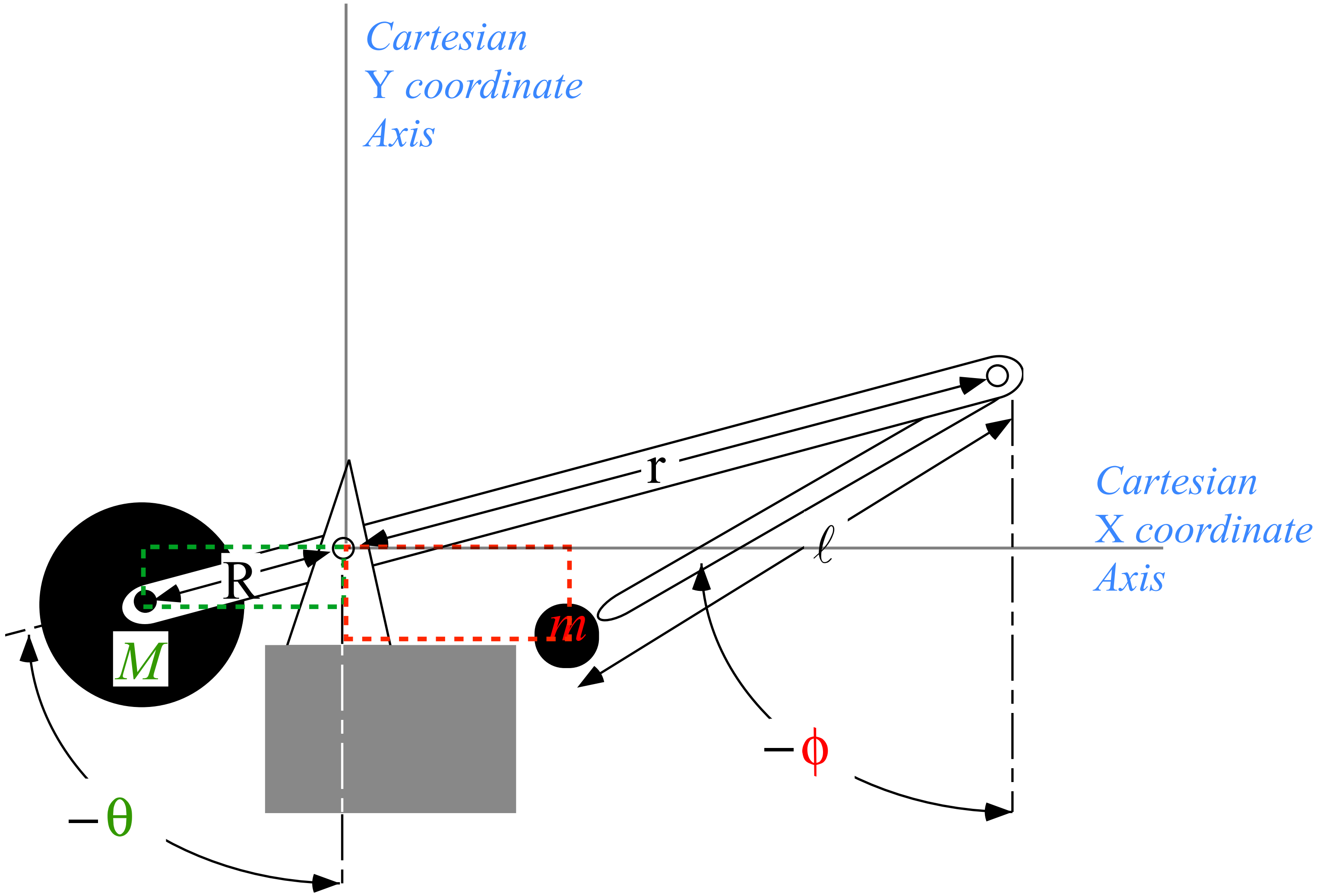

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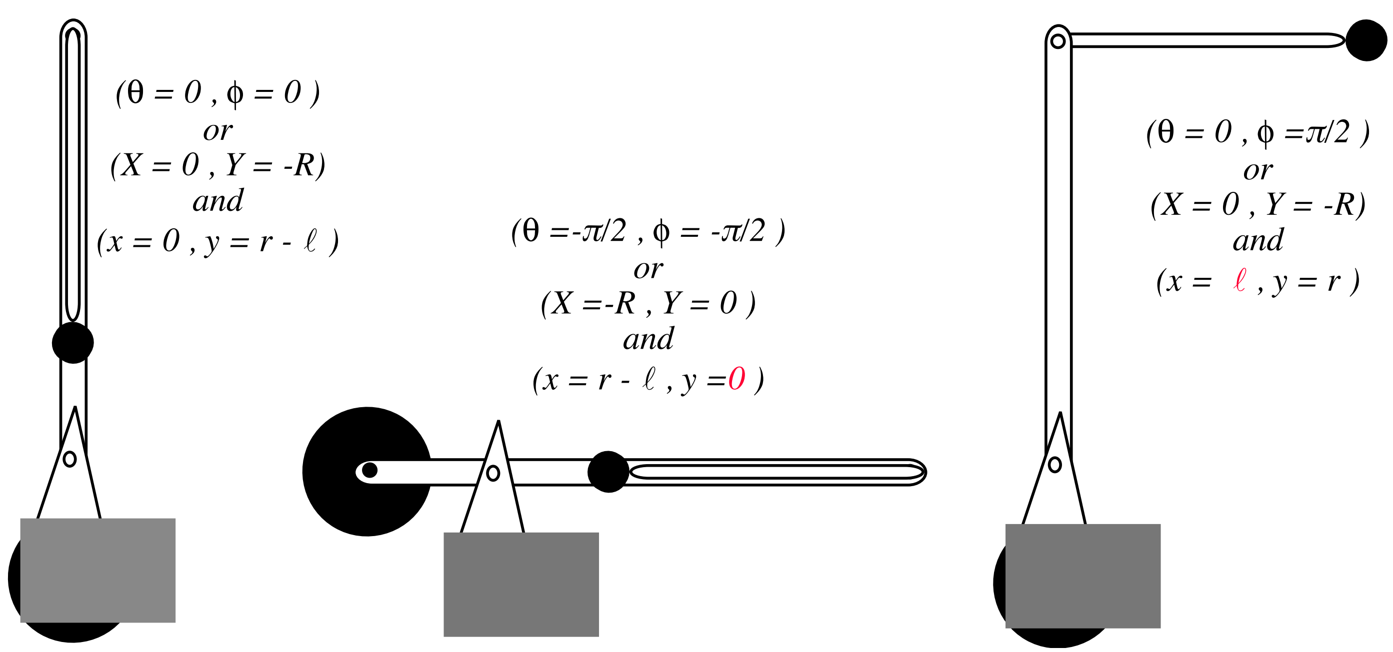
Fig. 2.2.2 Singular positions of the trebuchet



Coordinates of mass M (Driving weight):

$X = R \sin \theta$

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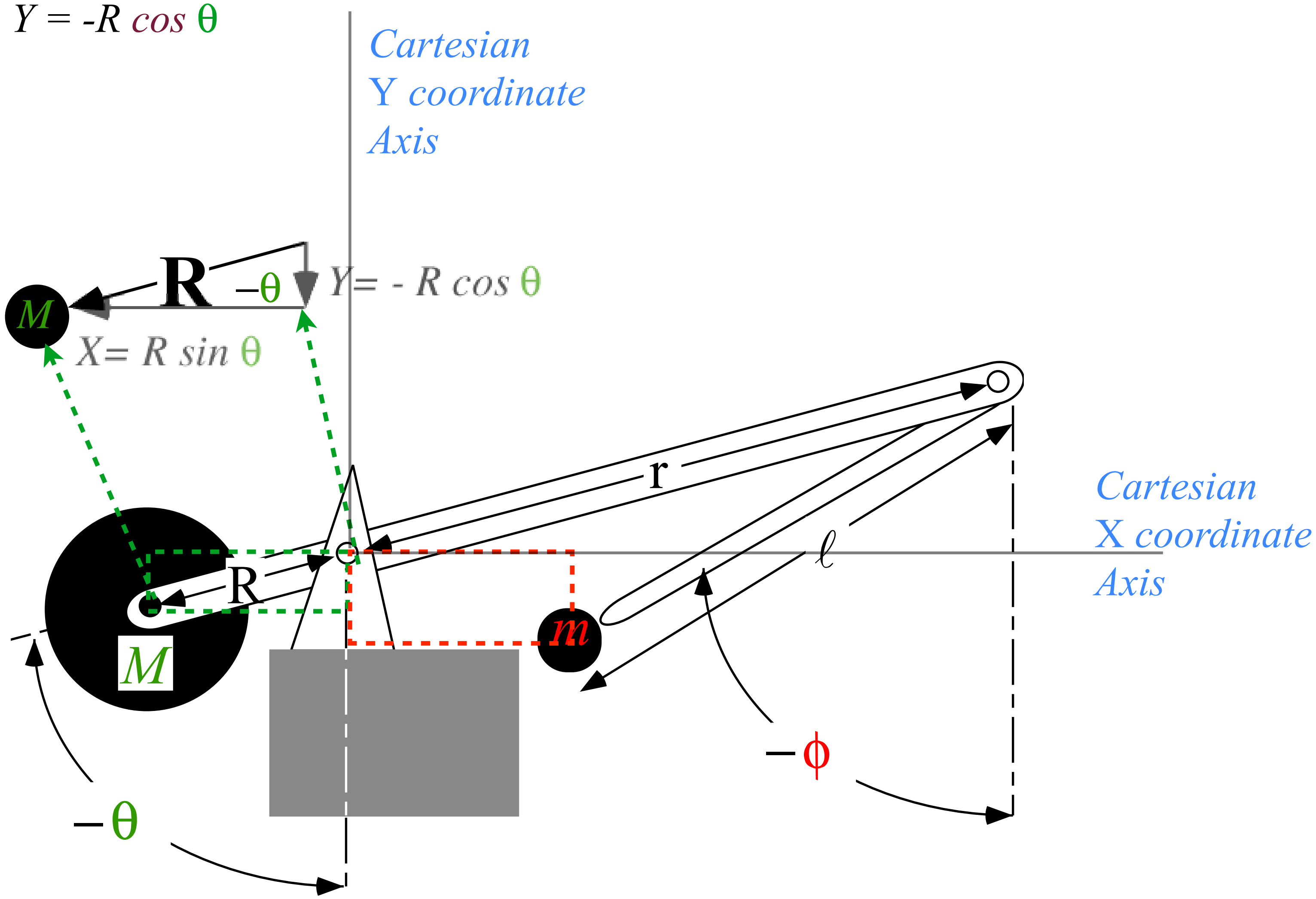

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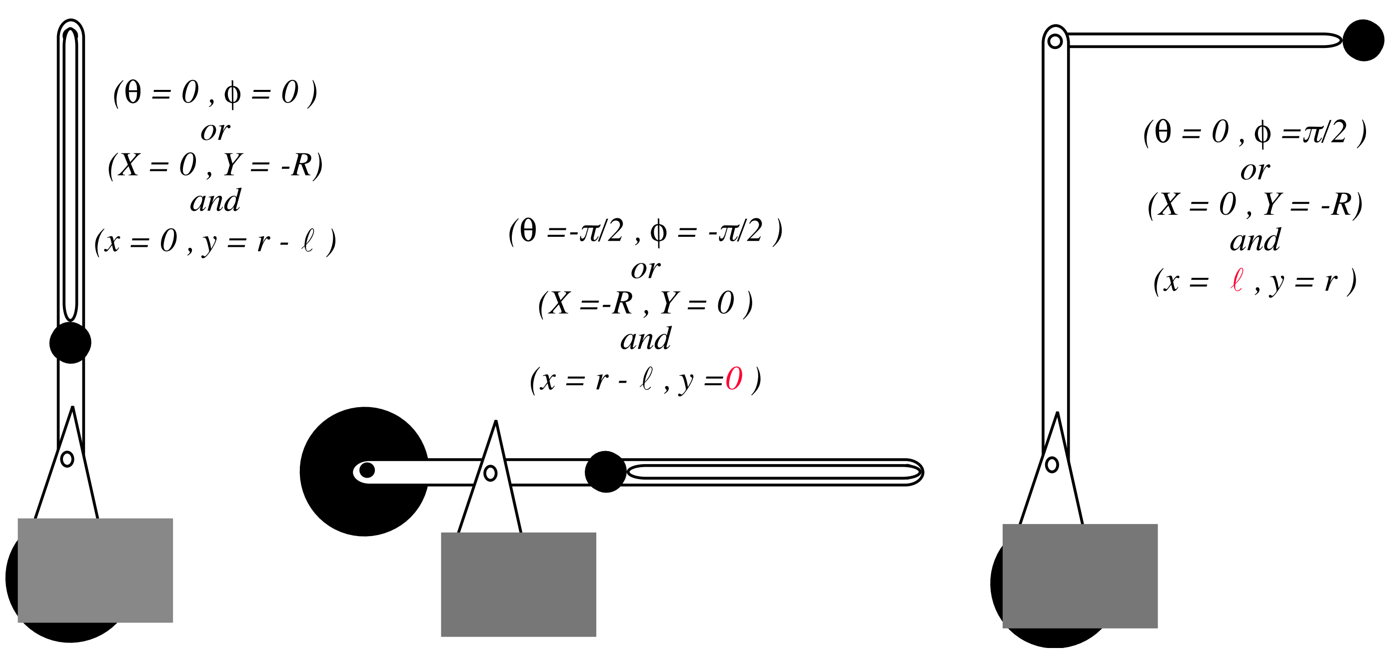

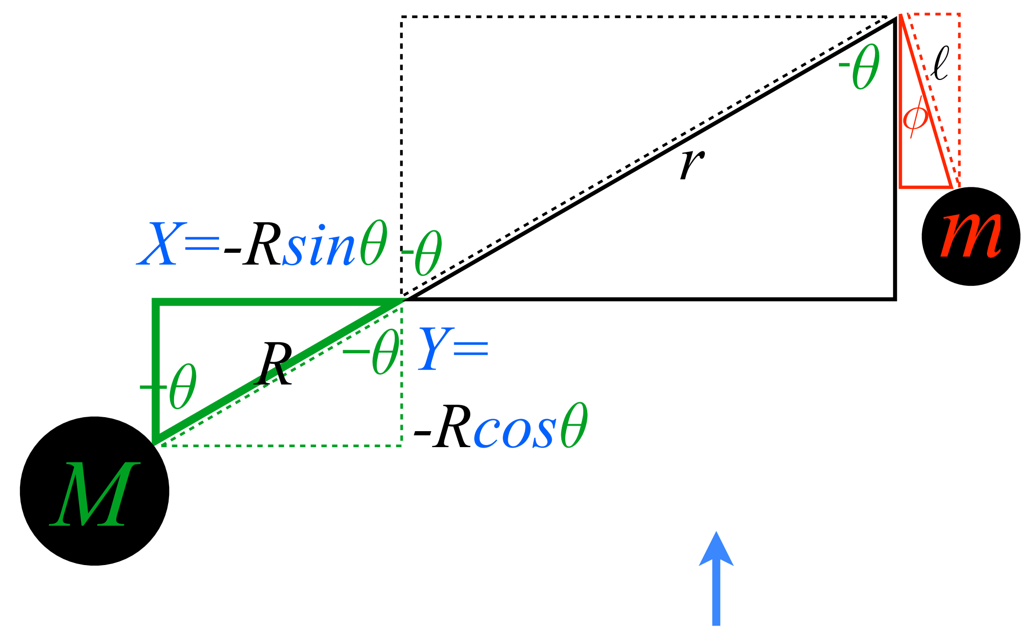

geometry of trebuchet simplified somewhat...

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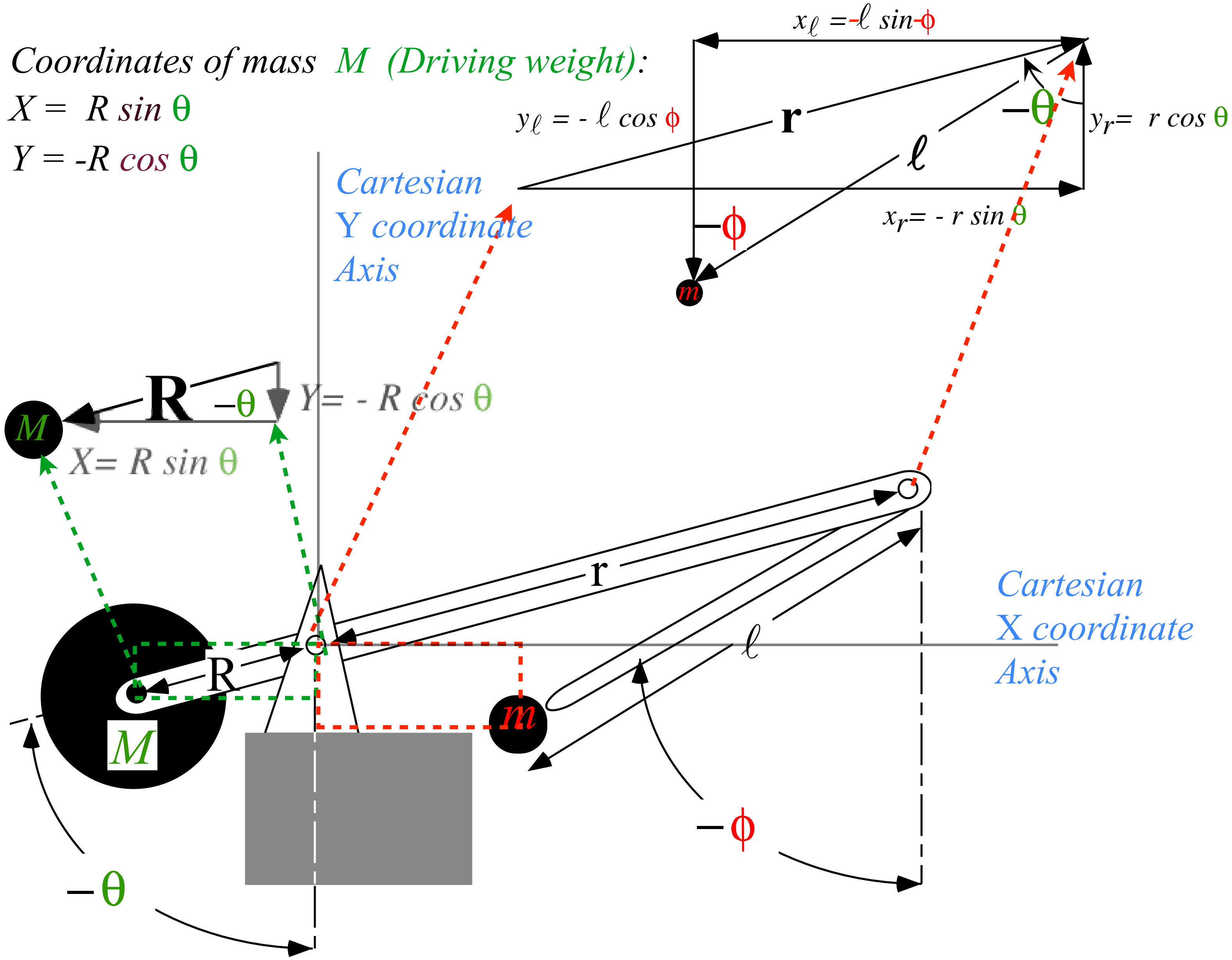

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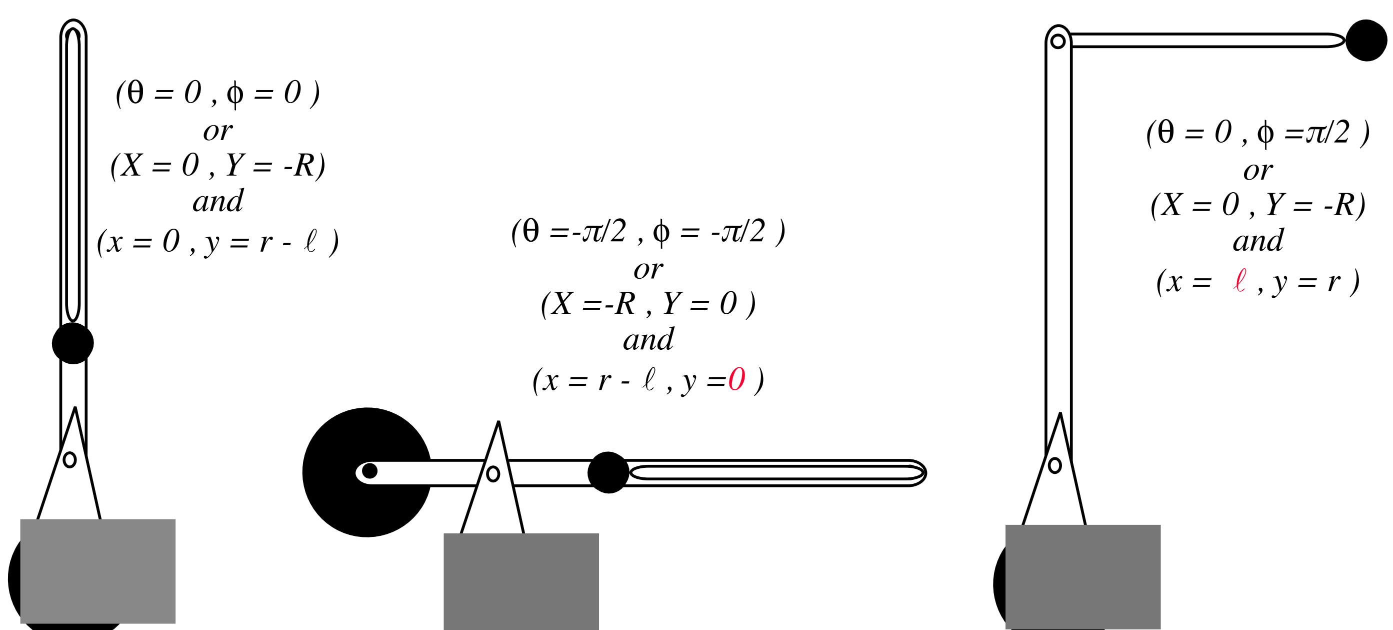

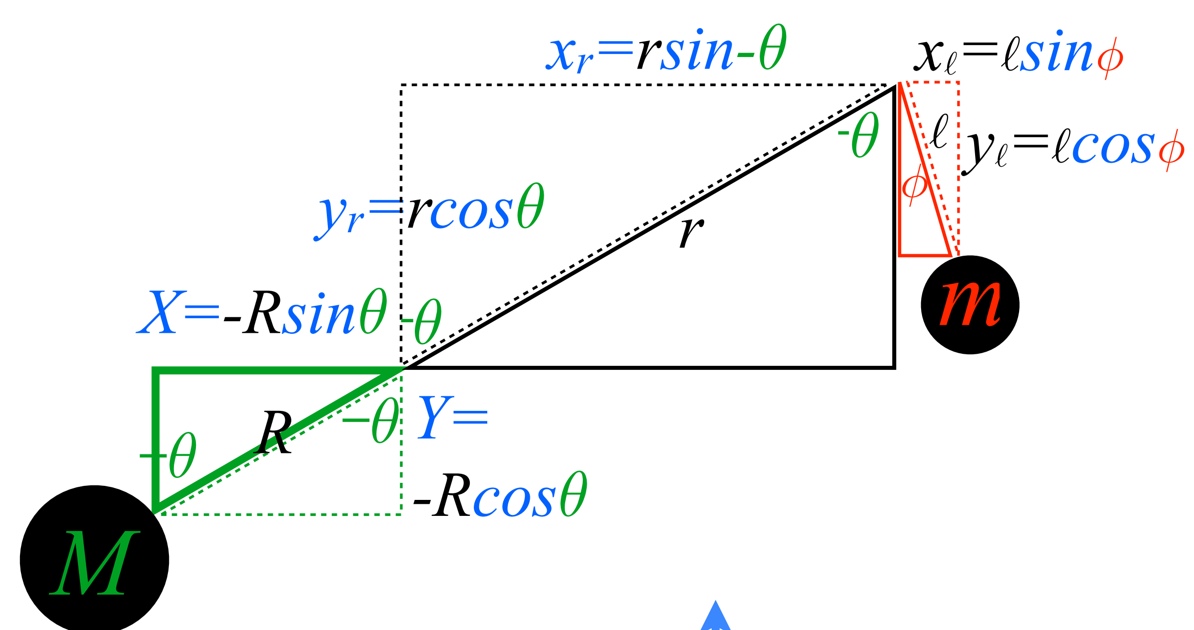

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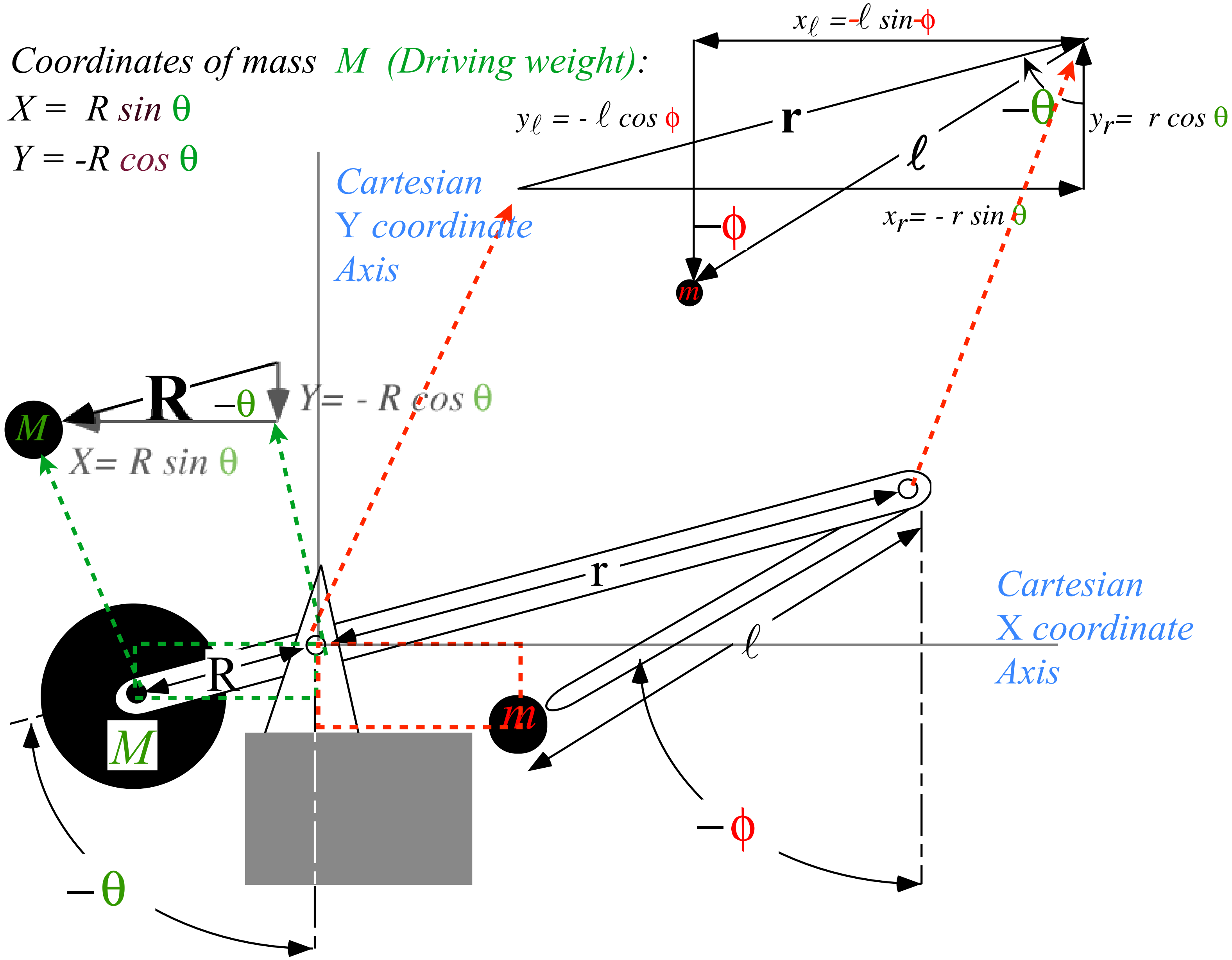

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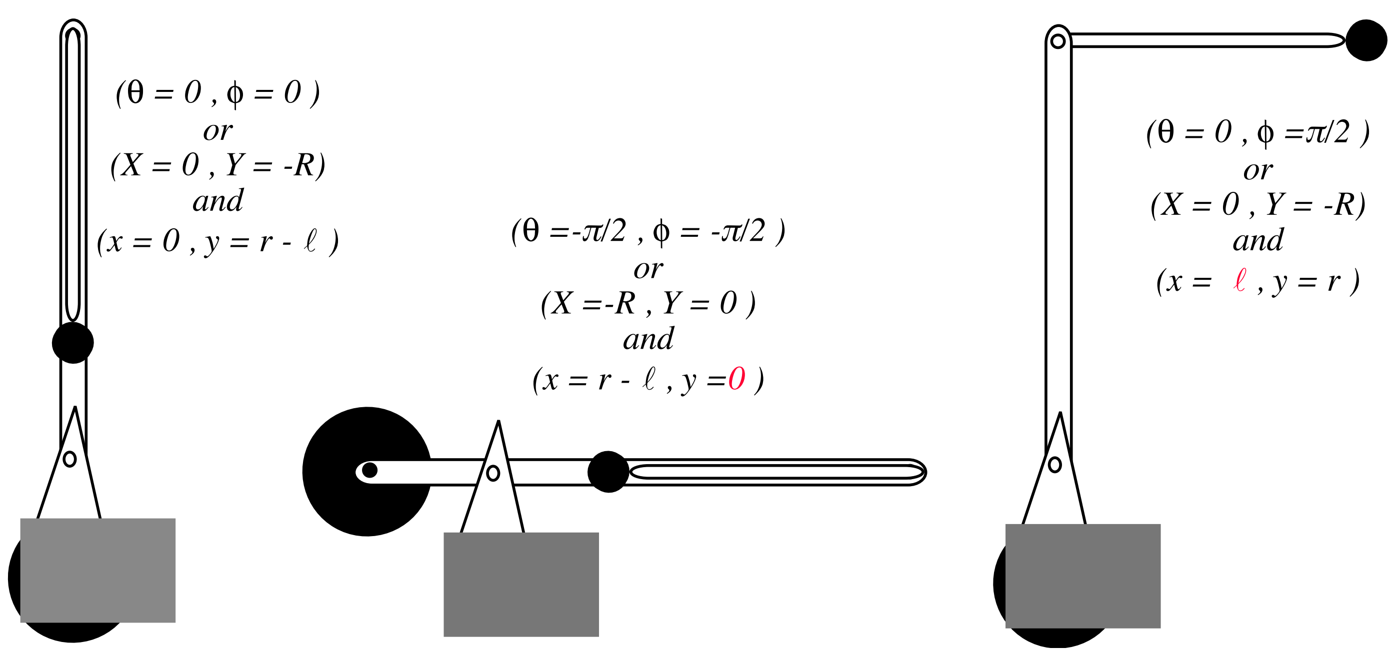

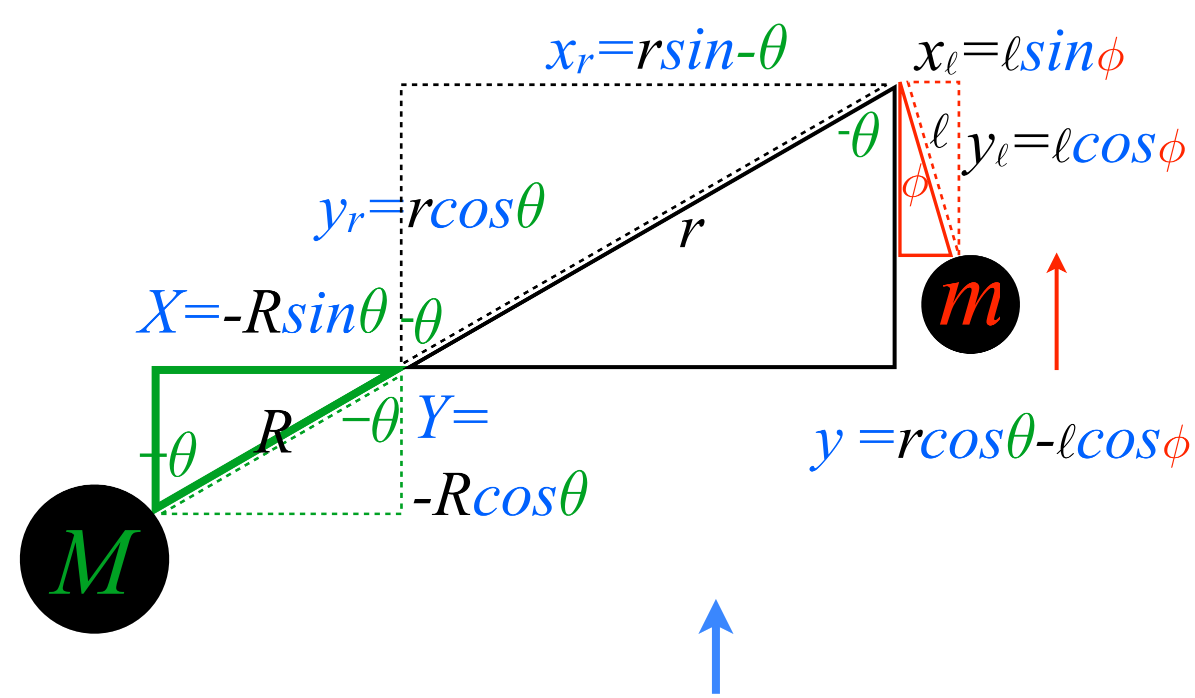

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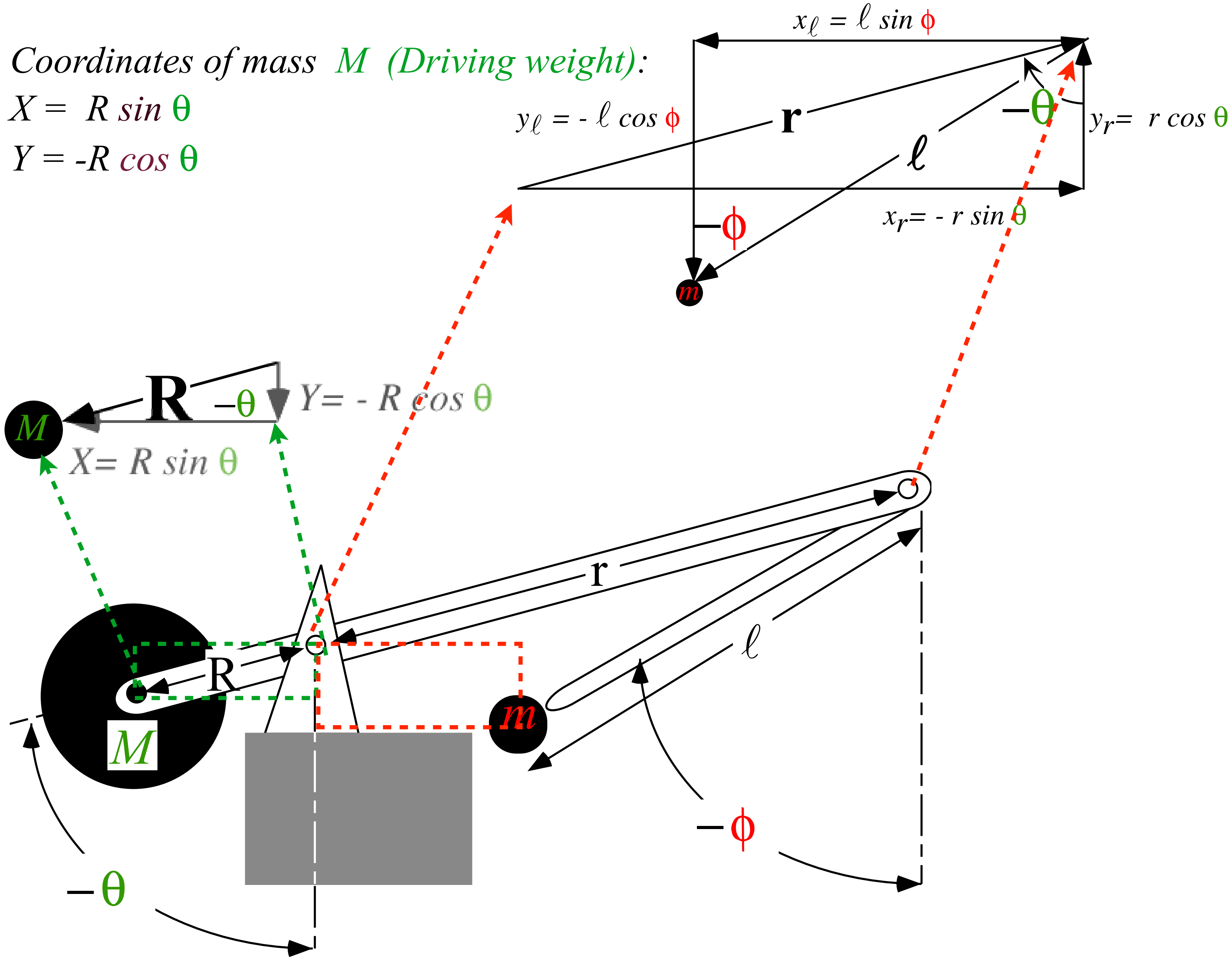

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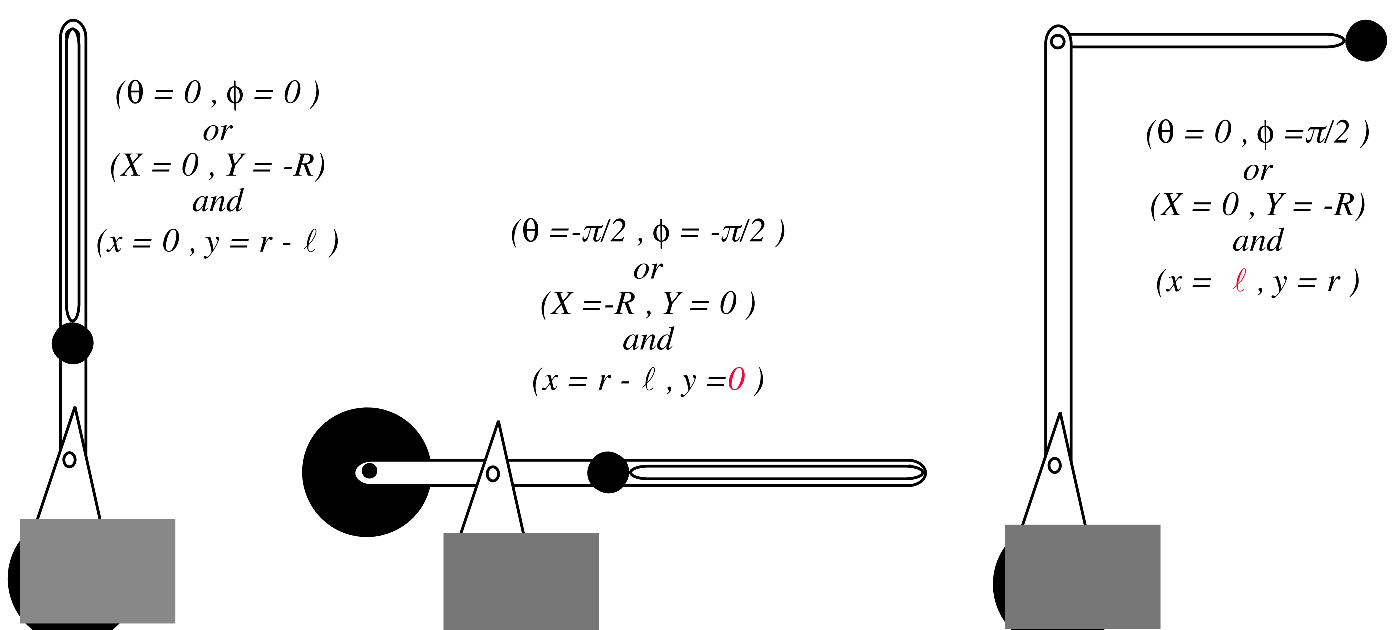

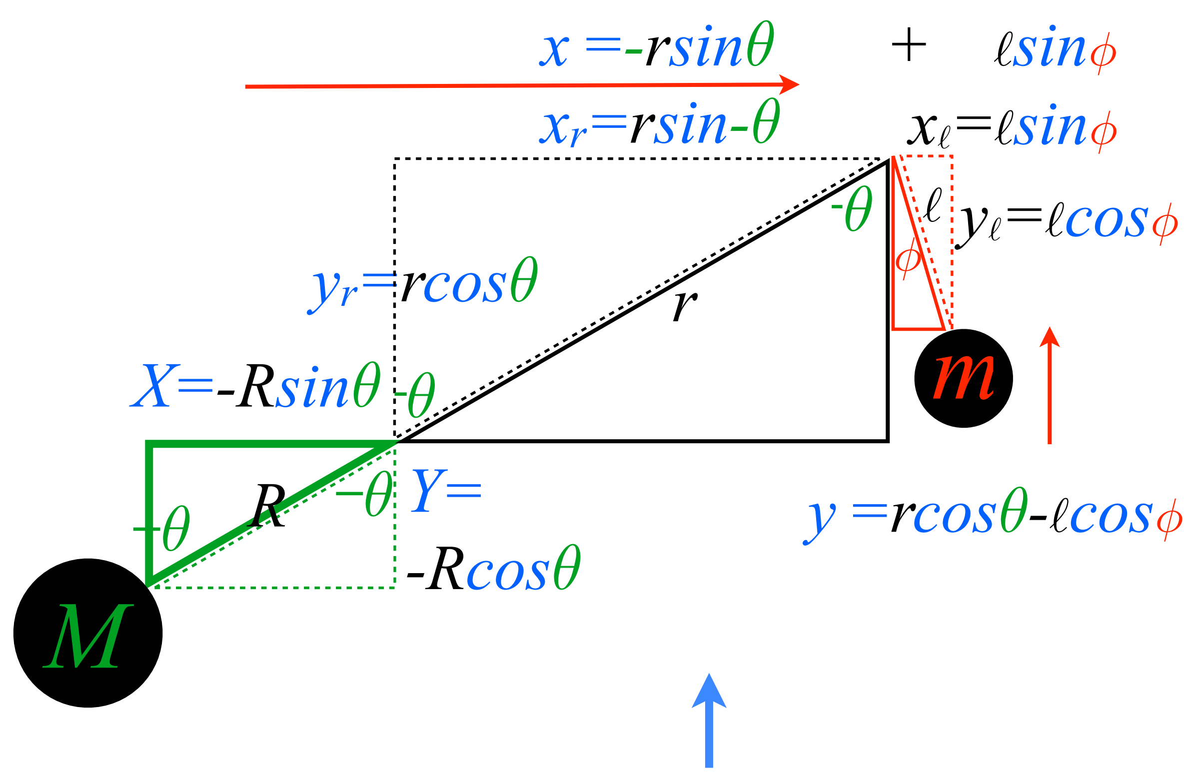

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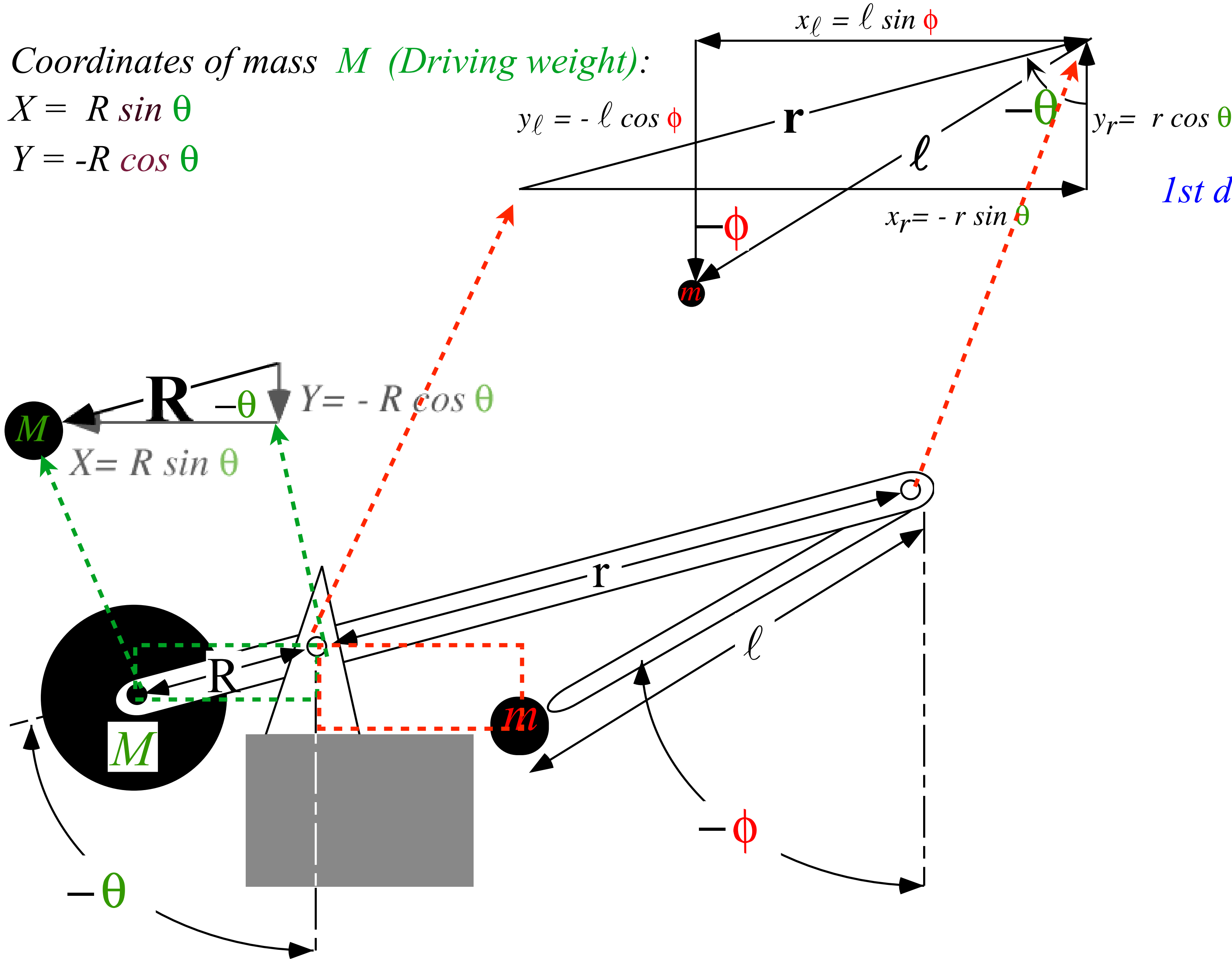

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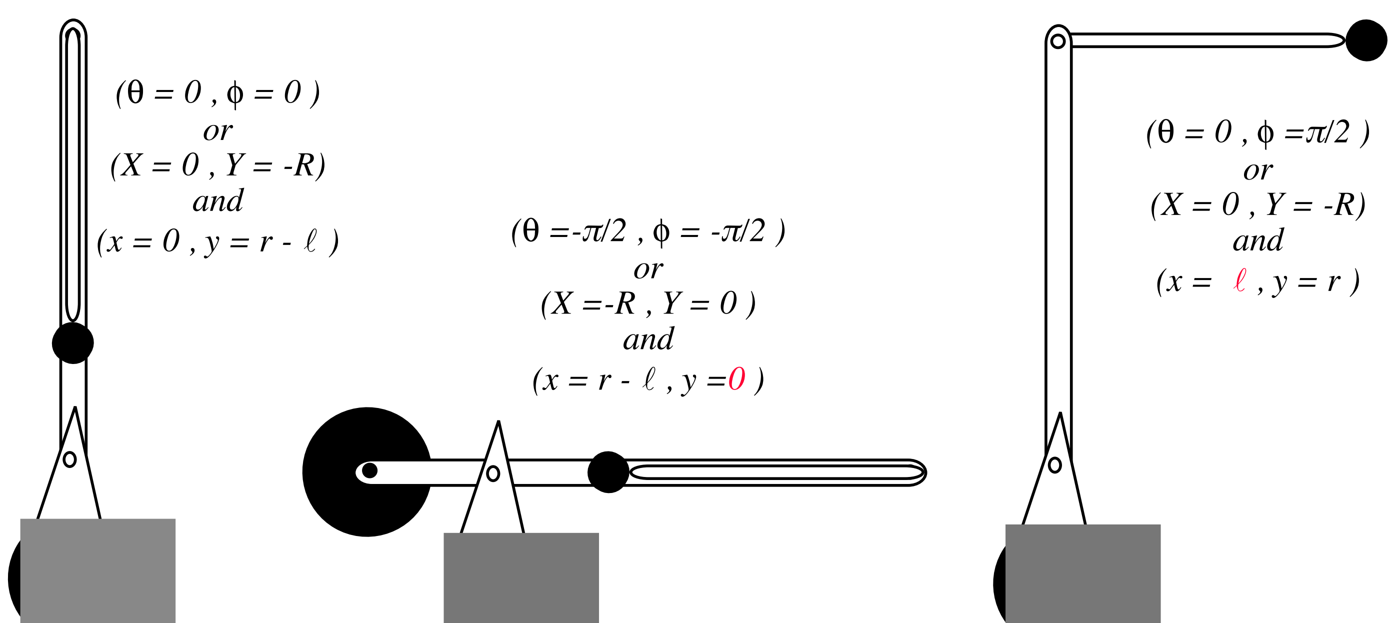

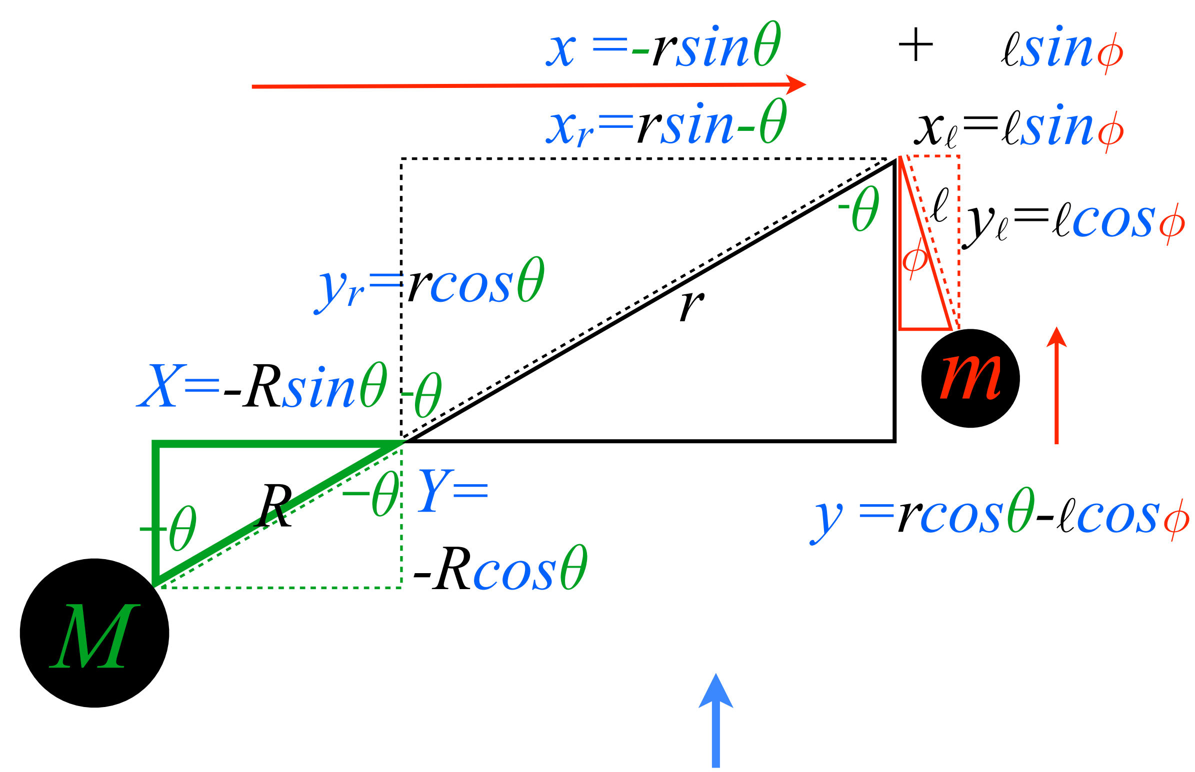

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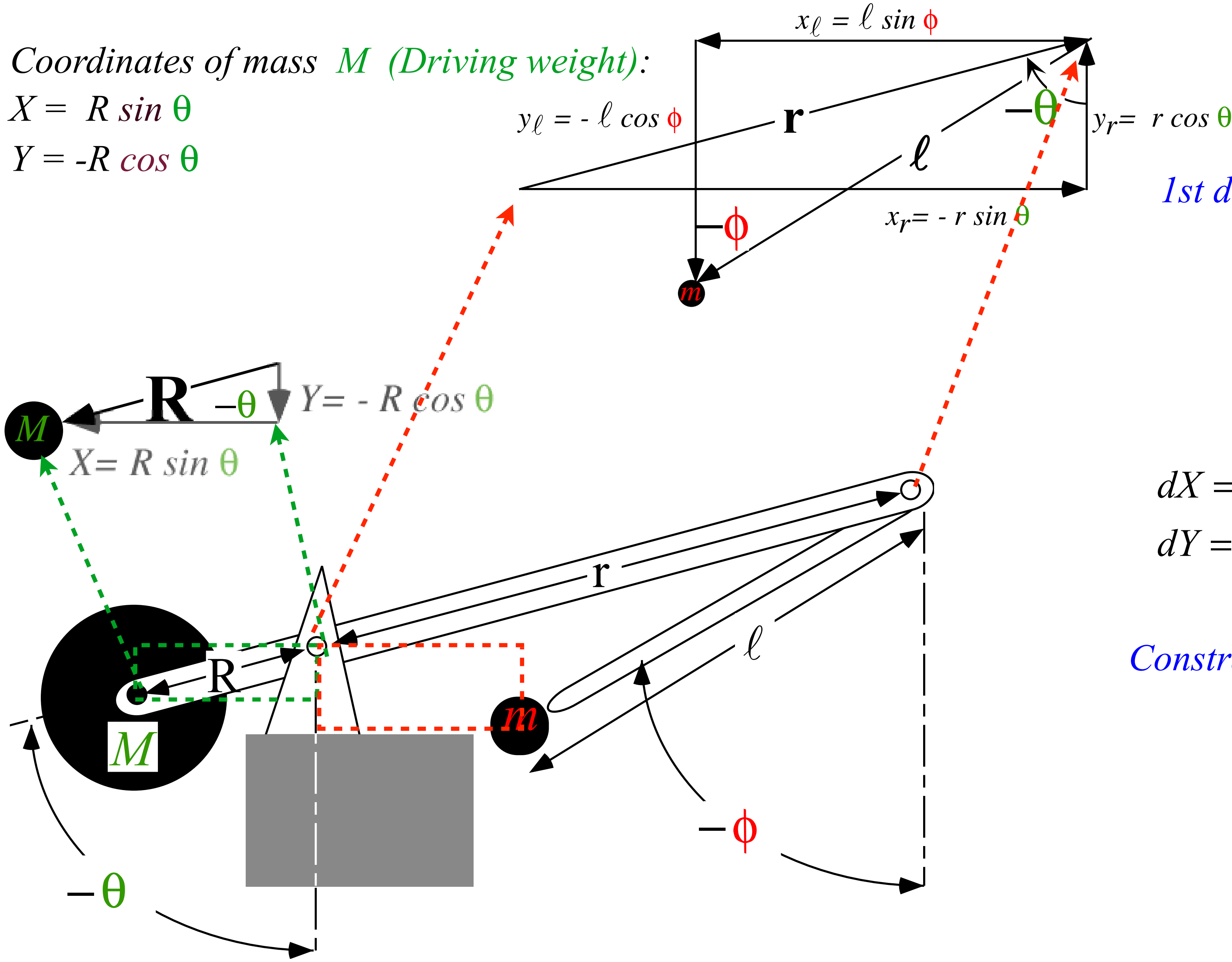

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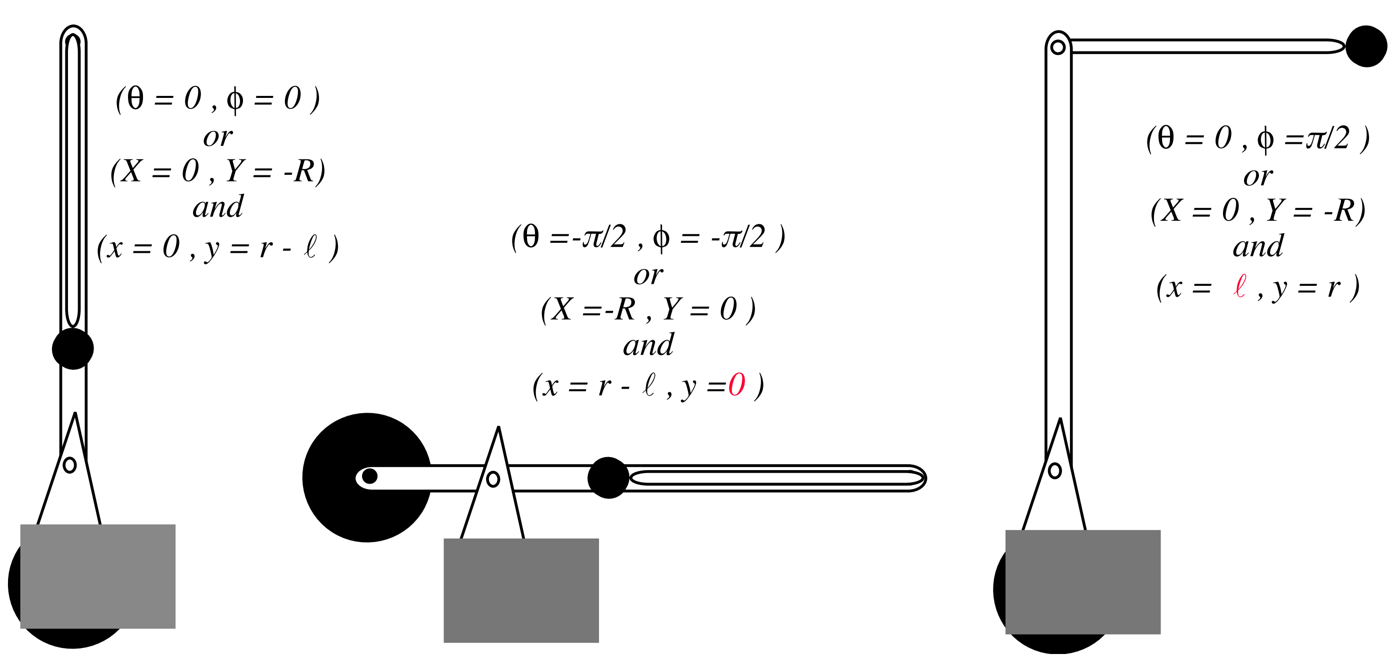

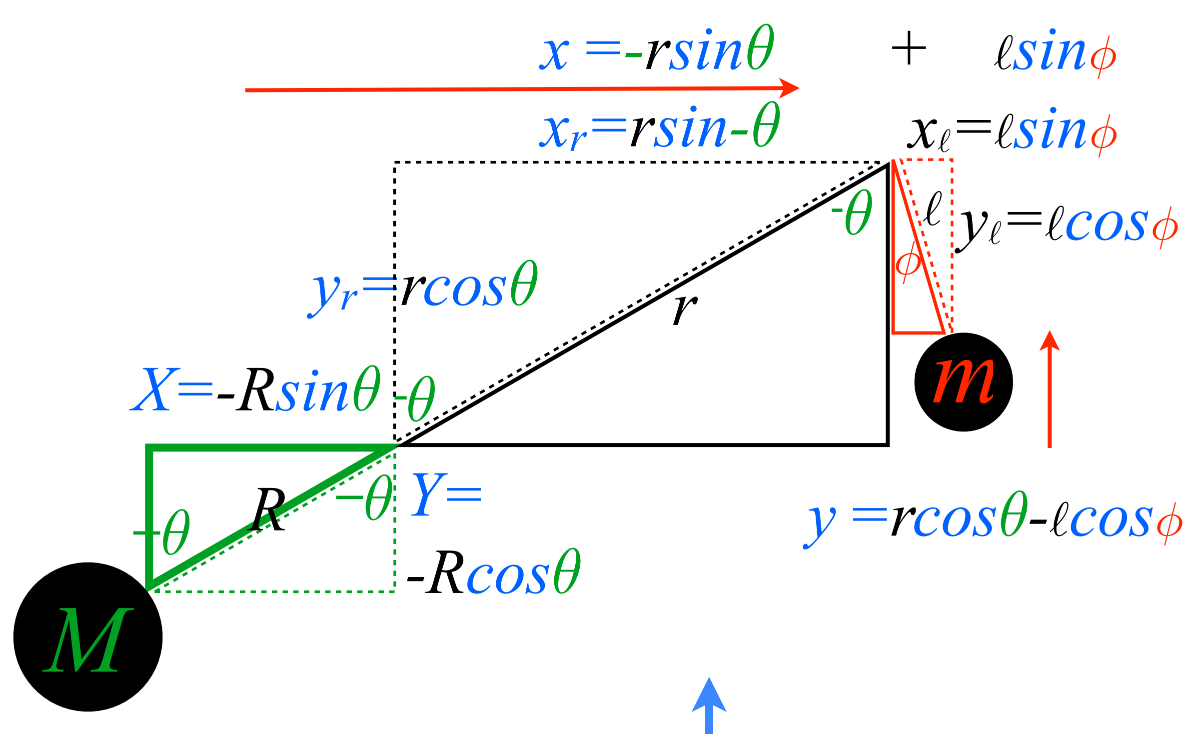

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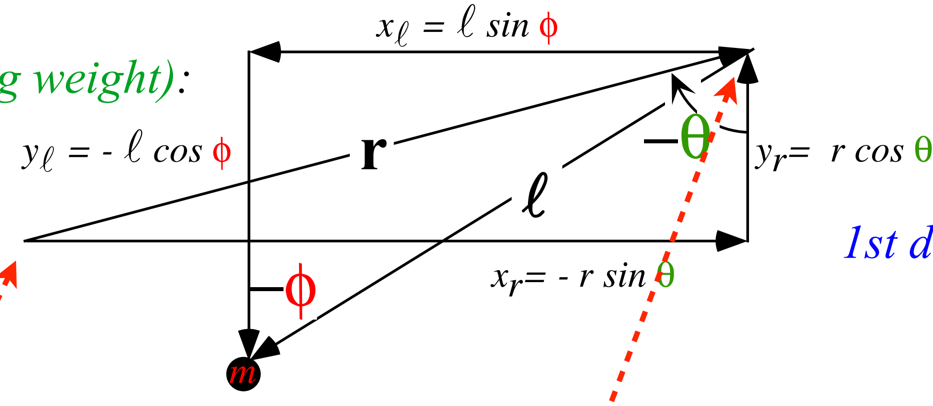

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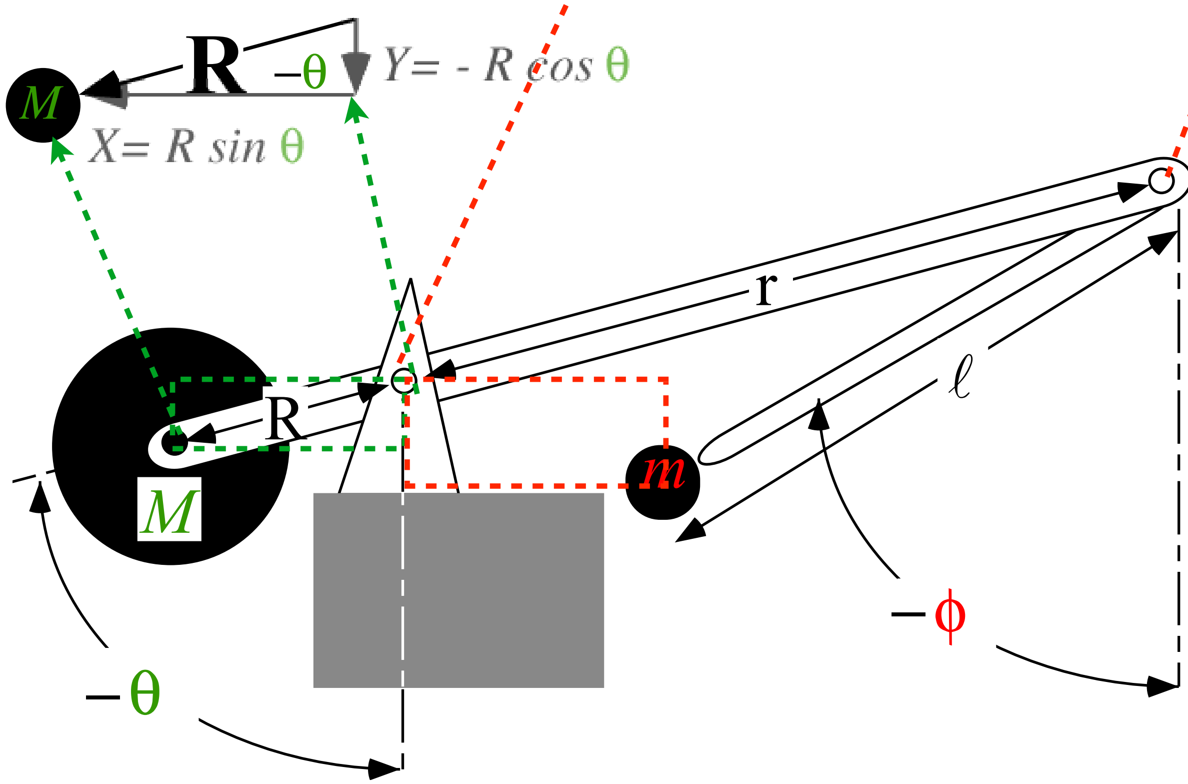

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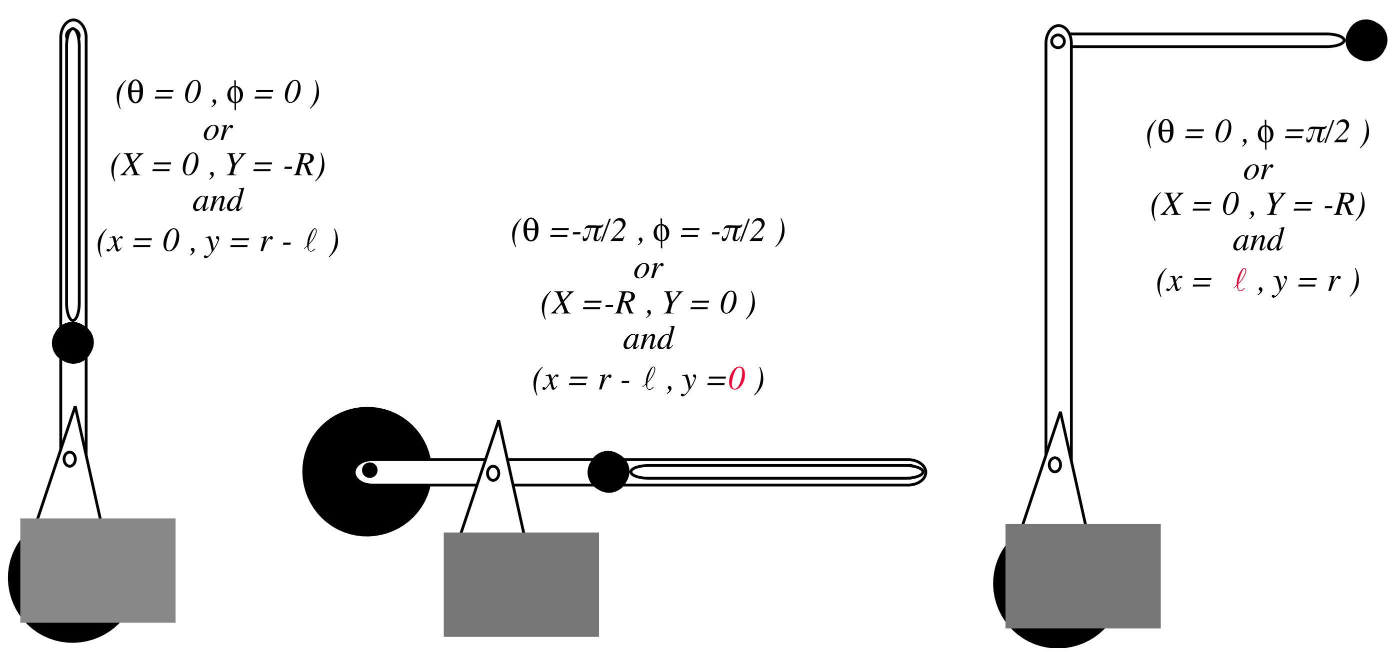
Raw Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Fig. 2.2.1 Cartesian coordinates related to trebuchet angles θ and φ.

Fig. 2.2.2 Singular positions of the trebuchet



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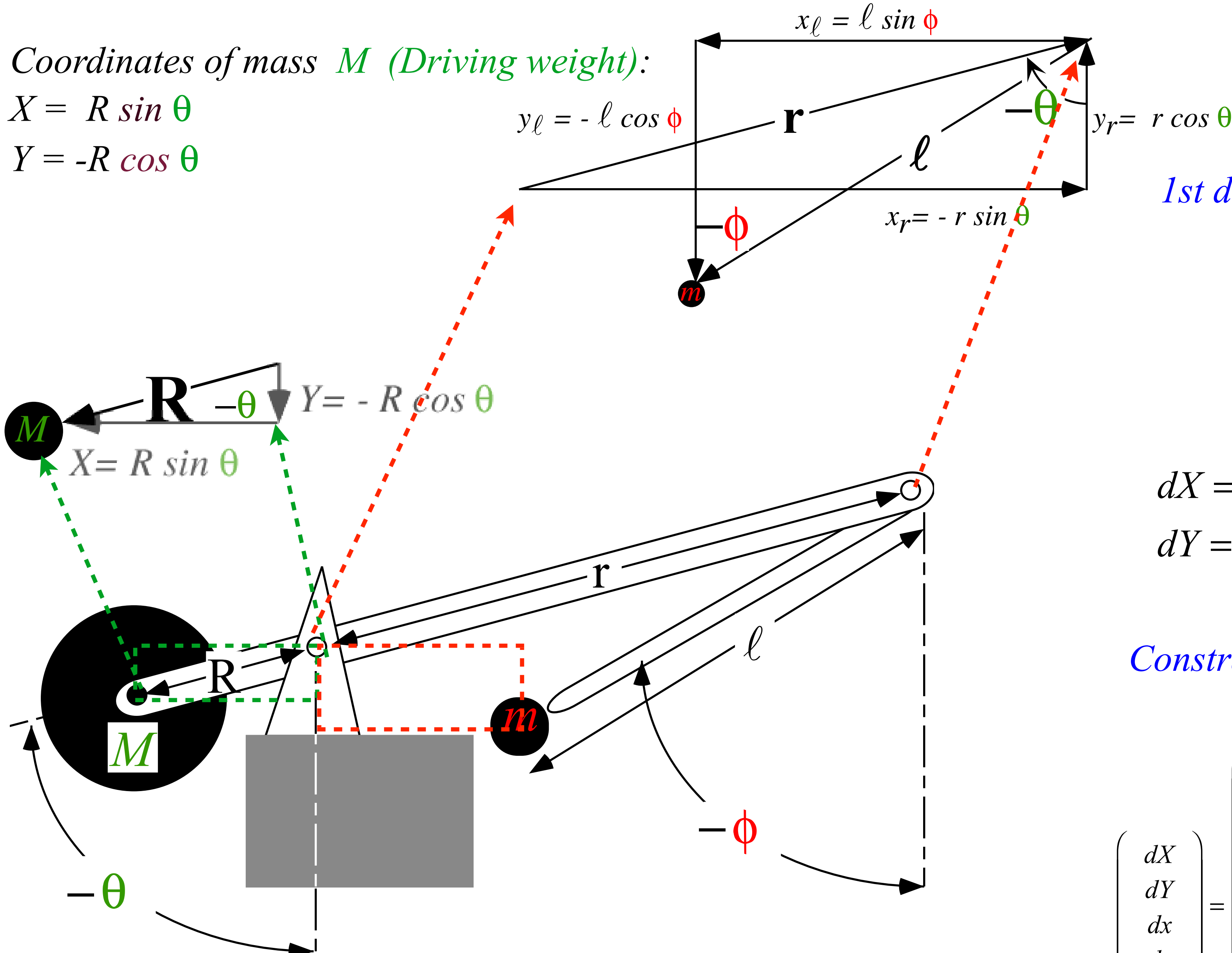

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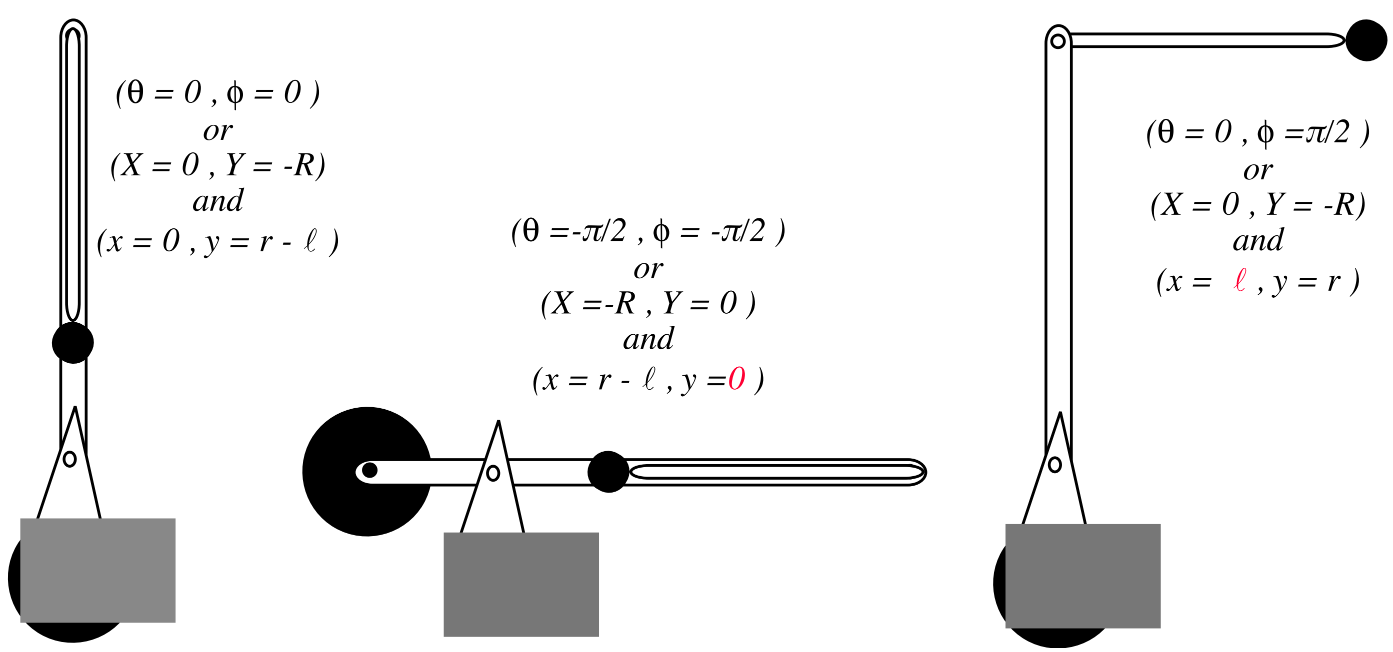

Coordinates of mass m (Payload or projectile):

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Raw Jacobian form

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Finding a reduced Jacobian form

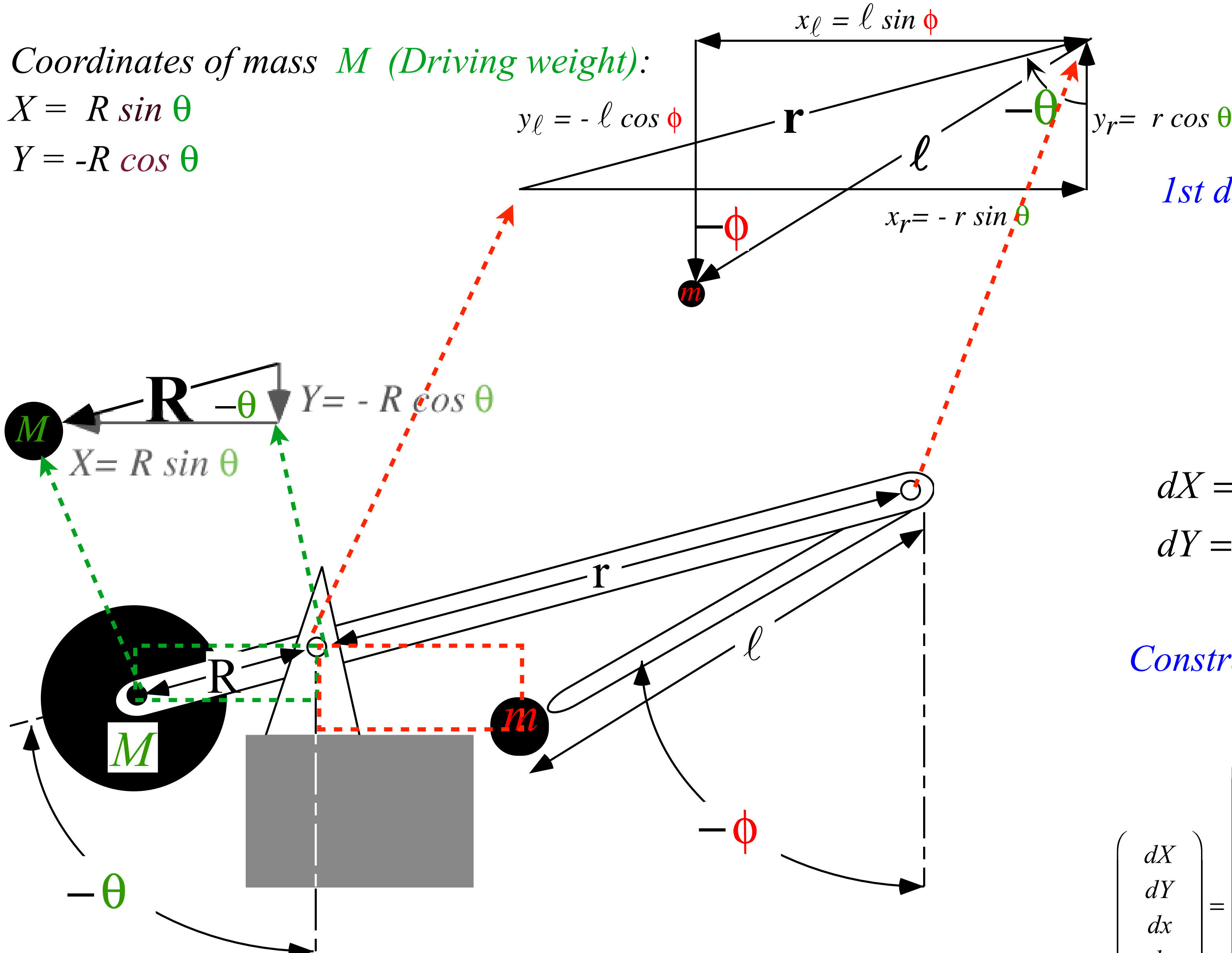
$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: } \det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

Coordinates of mass *M* (Driving weight):

$X = R \sin\theta$

$Y = -R \cos\theta$



Coordinates of mass *m* (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

Constraint relations:

$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

$c_\ell(x_\ell, y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$

$c_r(x_r, y_r) = x_r^2 + y_r^2 = r^2 = const.$

Raw Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} \text{ FAILS since: det} \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$$

FAILS! (Always singular)

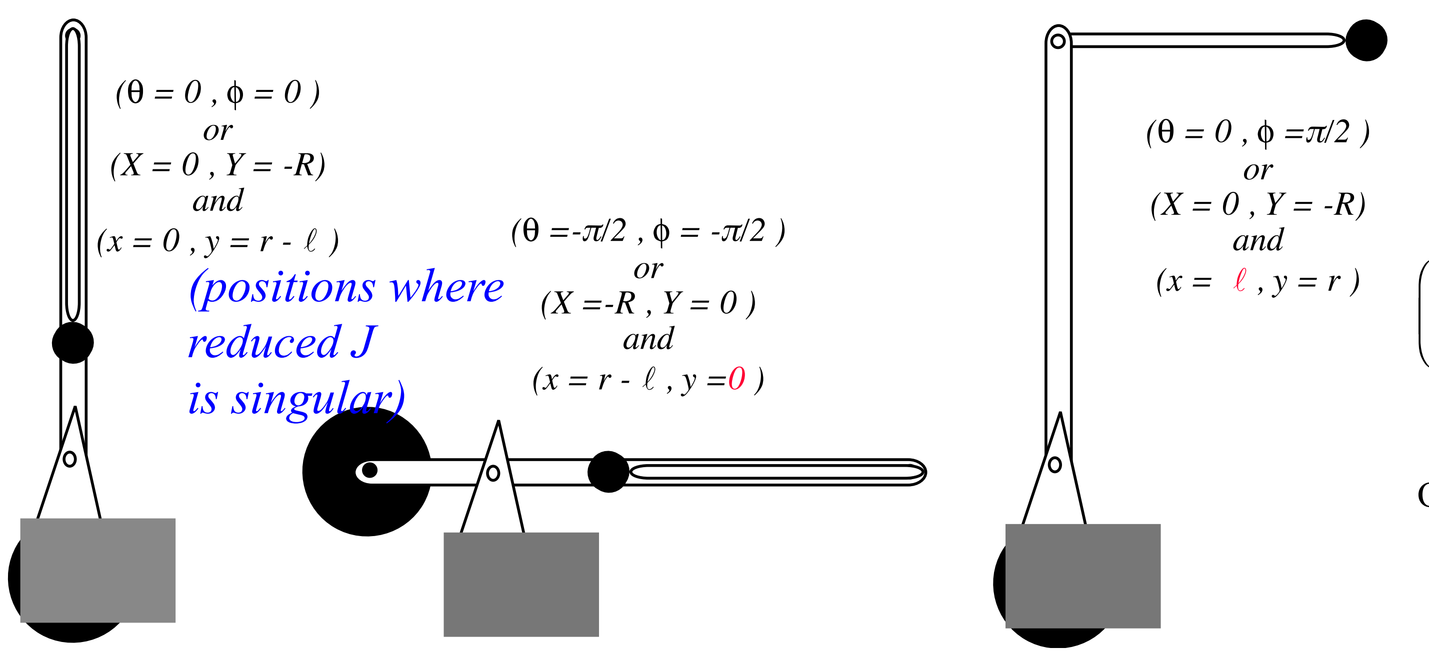
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\text{OK: det}\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = r\ell\sin(\theta - \phi)$$

SUCCESS! (Usually non-singular)

Fig. 2.2.1 Cartesian coordinates related to trebuchet angles θ and φ.

Fig. 2.2.2 Singular positions of the trebuchet



Cartesian to GCC transformations

Jacobian relations

→ *Kinetic energy calculation*

Dynamic metric tensor γ_{mn}

Coordinates of mass M (Driving weight):

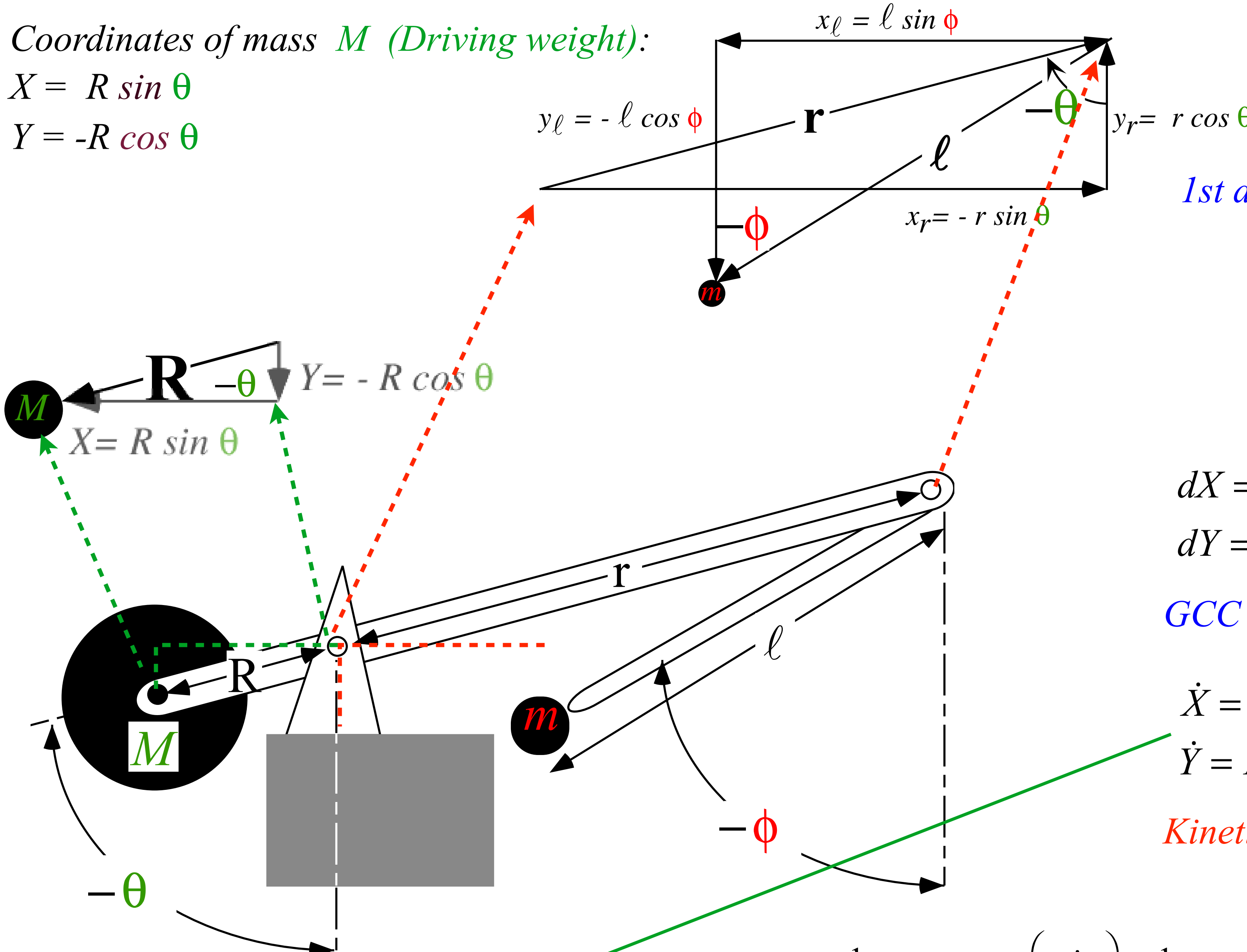
$X = R \sin\theta$

$Y = -R \cos\theta$

Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$



1st differential relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta\ \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta\ \dot{\theta} + \ell\cos\phi\ \dot{\phi}$$

$$\dot{Y} = R\sin\theta\ \dot{\theta} + 0, \qquad \dot{y} = -r\sin\theta\ \dot{\theta} + \ell\sin\phi\ \dot{\phi}$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

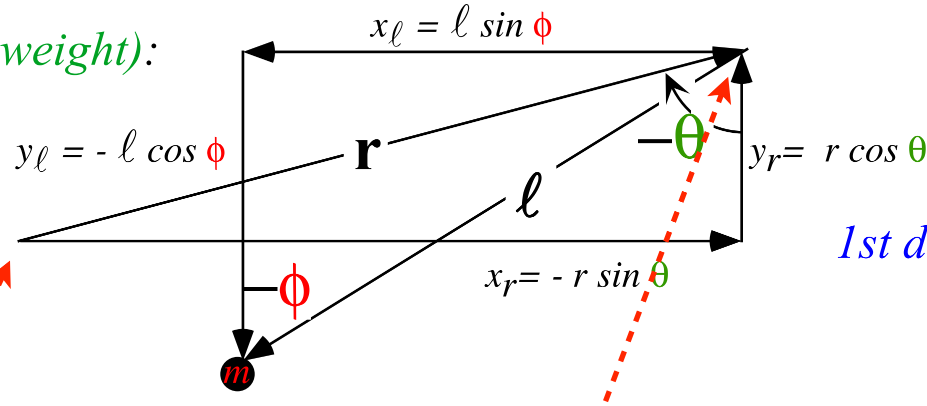
Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

Coordinates of mass M (Driving weight):

$X = R \sin \theta$

$Y = -R \cos \theta$



Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r \sin \theta + \ell \sin \phi$

$y = y_r + y_\ell = r \cos \theta - \ell \cos \phi$

1st differential relations:

$$dX = \frac{\partial X}{\partial \theta} d\theta + \frac{\partial X}{\partial \phi} d\phi, \qquad dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi,$$

$$dY = \frac{\partial Y}{\partial \theta} d\theta + \frac{\partial Y}{\partial \phi} d\phi, \qquad dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi.$$

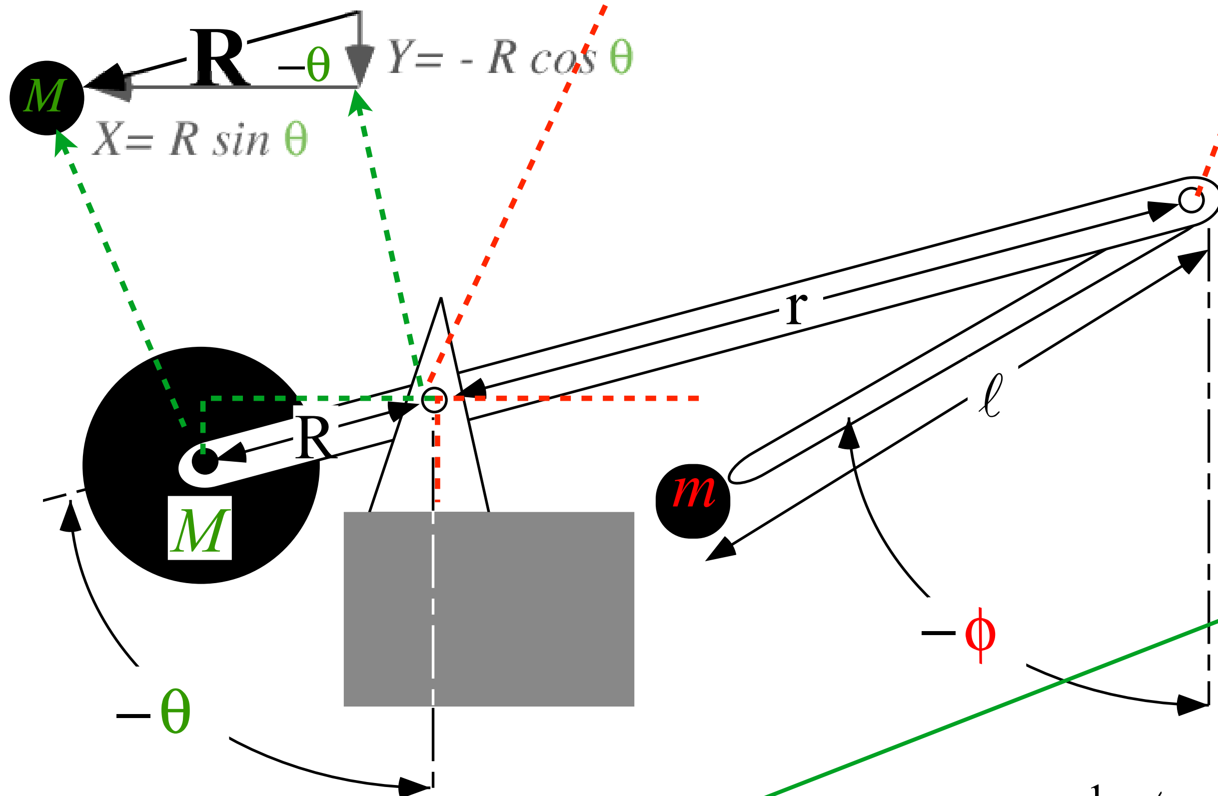
$$dX = R\cos\theta \; d\theta + 0, \qquad dx = -r\cos\theta \; d\theta + \ell\cos\phi \; d\phi$$

$$dY = R\sin\theta \; d\theta + 0, \qquad dy = -r\sin\theta \; d\theta + \ell\sin\phi \; d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta \; \dot{\theta} + 0, \qquad \dot{x} = -r\cos\theta \; \dot{\theta} + \ell\cos\phi \; \dot{\phi}$$

$$\dot{Y} = R\sin\theta \; \dot{\theta} + 0, \qquad \dot{y} = -r\sin\theta \; \dot{\theta} + \ell\sin\phi \; \dot{\phi}$$



Kinetic energy of projectile m

$$T(m) = \frac{1}{2} m \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix}^T \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$= \frac{1}{2} m \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kinetic energy of driver M

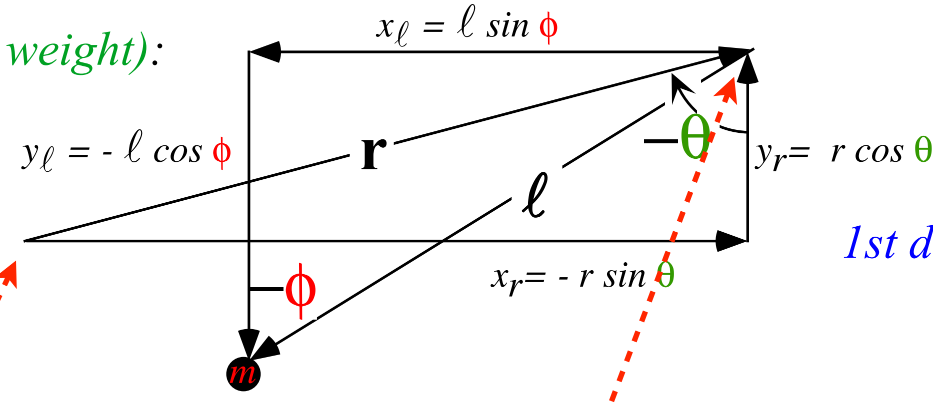
$$T(M) = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} M \dot{Y}^2$$

$$= \frac{1}{2} M \left(R\cos\theta \; \dot{\theta} \right)^2 + \frac{1}{2} M \left(R\sin\theta \; \dot{\theta} \right)^2$$

Coordinates of mass M (Driving weight):

$X = R \sin\theta$

$Y = -R \cos\theta$



Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

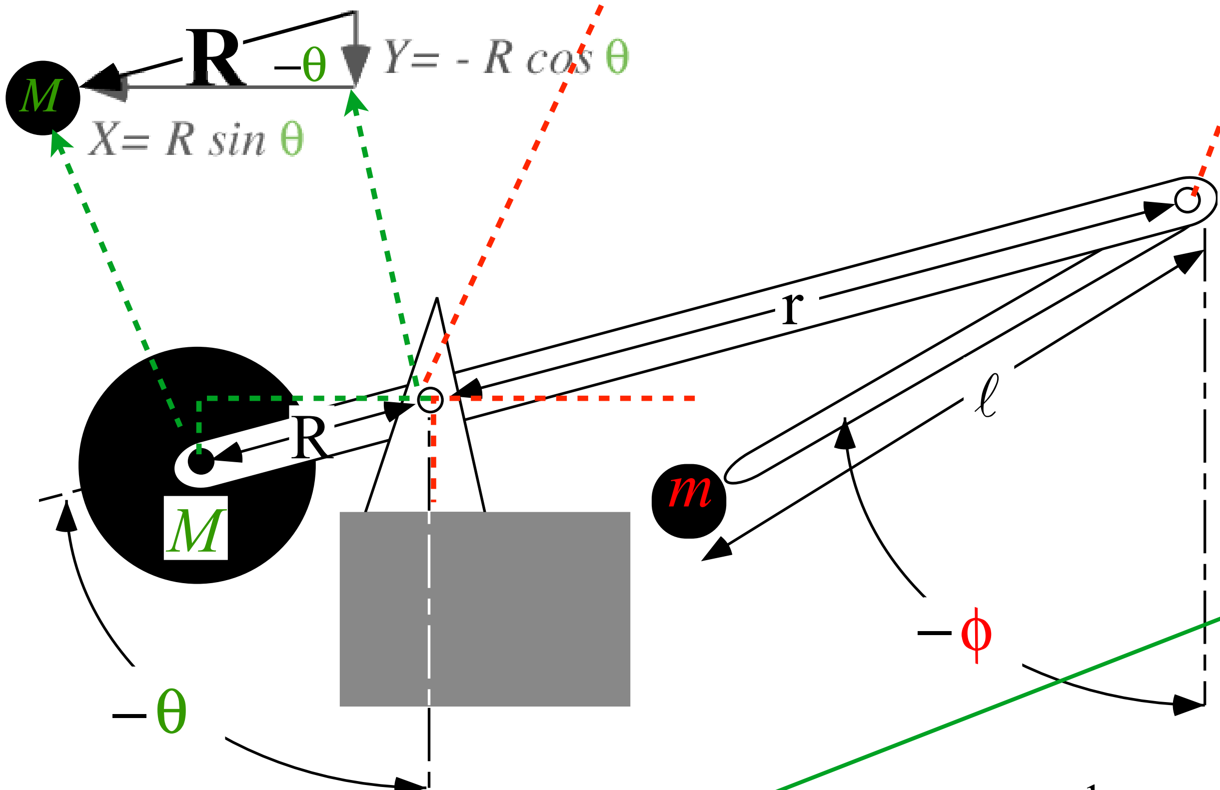
1st differential relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$



GCC Velocity relations:

$$\dot X = R\cos\theta\ \dot\theta + 0, \qquad \dot x = -r\cos\theta\ \dot\theta + \ell\cos\phi\ \dot\phi$$

$$\dot Y = R\sin\theta\ \dot\theta + 0, \qquad \dot y = -r\sin\theta\ \dot\theta + \ell\sin\phi\ \dot\phi$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}m\begin{pmatrix}\dot x & \dot y\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta + r^2\sin^2\theta & -r\ell\cos\theta\cos\phi - r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta - r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi + \ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot X^2 + \frac{1}{2}M\dot Y^2$$

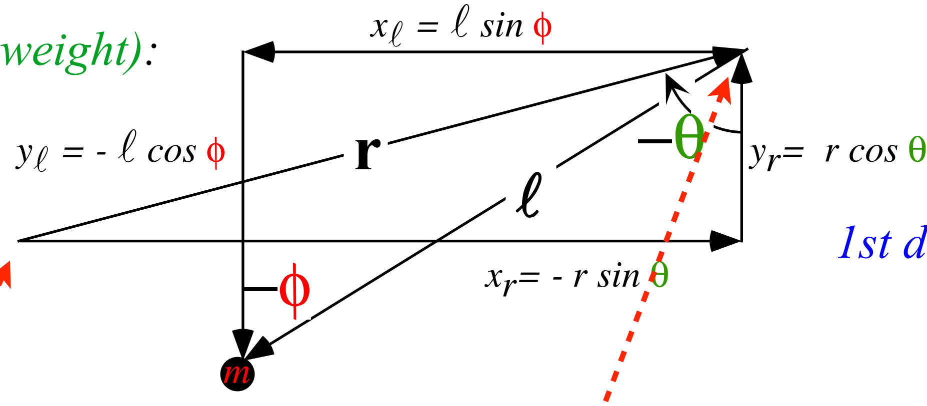
$$= \frac{1}{2}M\left(R\cos\theta\ \dot\theta\right)^2 + \frac{1}{2}M\left(R\sin\theta\ \dot\theta\right)^2$$

$$= \frac{1}{2}MR^2\dot\theta^2$$

Coordinates of mass M (Driving weight):

$X = R \sin\theta$

$Y = -R\cos\theta$



Coordinates of mass m (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

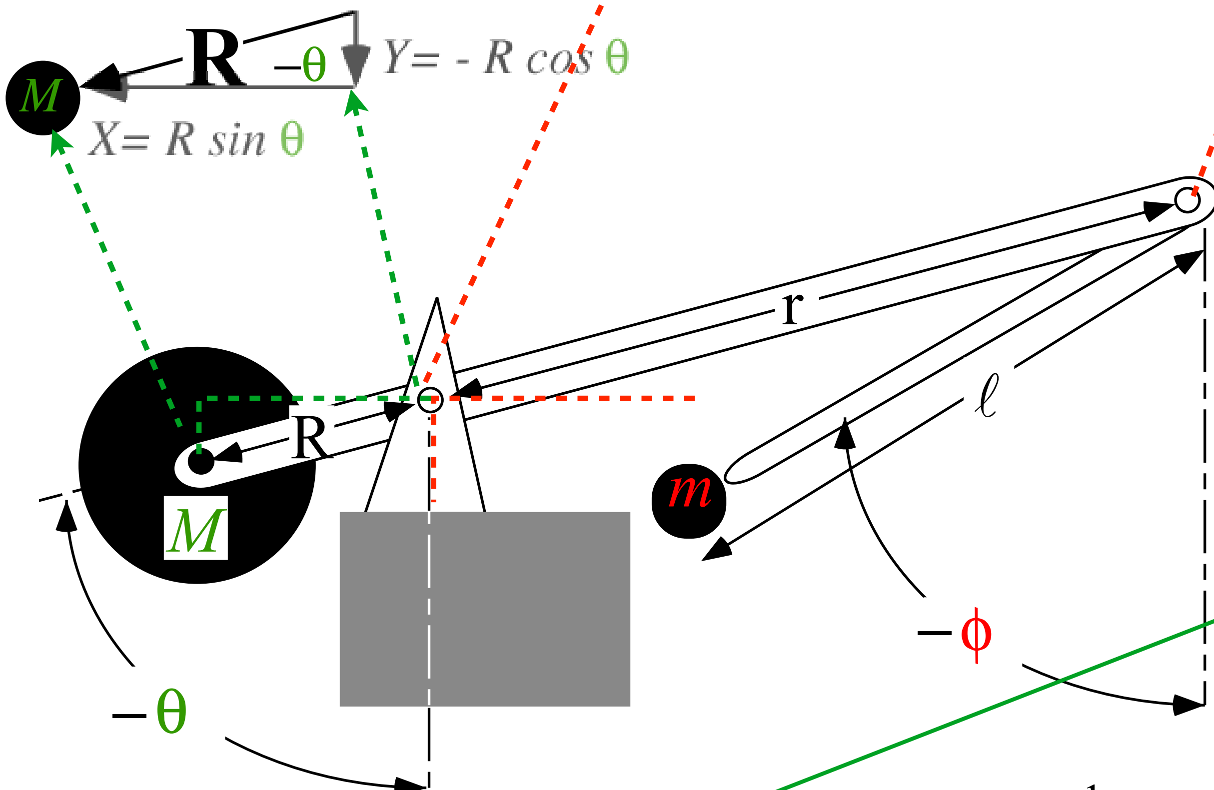
1st differential relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\; d\theta + 0, \qquad dx = -r\cos\theta\; d\theta + \ell\cos\phi\; d\phi$$

$$dY = R\sin\theta\; d\theta + 0, \qquad dy = -r\sin\theta\; d\theta + \ell\sin\phi\; d\phi$$



GCC Velocity relations:

$$\dot X = R\cos\theta\;\dot\theta + 0, \qquad \dot x = -r\cos\theta\;\dot\theta + \ell\cos\phi\;\dot\phi$$

$$\dot Y = R\sin\theta\;\dot\theta + 0, \qquad \dot y = -r\sin\theta\;\dot\theta + \ell\sin\phi\;\dot\phi$$

Kinetic energy of projectile m

$$T(m) = \frac{1}{2}m\begin{pmatrix}\dot x & \dot y\end{pmatrix}\begin{pmatrix}\dot x \\ \dot y\end{pmatrix} = \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\phi\end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\phi\end{pmatrix}$$

$$= \frac{1}{2}m\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta + r^2\sin^2\theta & -r\ell\cos\theta\cos\phi - r\ell\sin\theta\sin\phi \\ -\ell r\cos\phi\cos\theta - r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi + \ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\phi\end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot X^2 + \frac{1}{2}M\dot Y^2$$

$$= \frac{1}{2}M\left(R\cos\theta\;\dot\theta\right)^2 + \frac{1}{2}M\left(R\sin\theta\;\dot\theta\right)^2$$

$$= \frac{1}{2}MR^2\dot\theta^2$$

Total kinetic energy of M and m

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\phi\end{pmatrix} = \frac{1}{2}\left[(MR^2 + mr^2)\dot\theta^2 - 2mr\ell\cos(\theta-\phi)\,\dot\theta\dot\phi + m\ell^2\dot\phi^2\right]$$

Cartesian to GCC transformations

Jacobian relations

Kinetic energy calculation

→ *Dynamic metric tensor* γ_{mn}

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

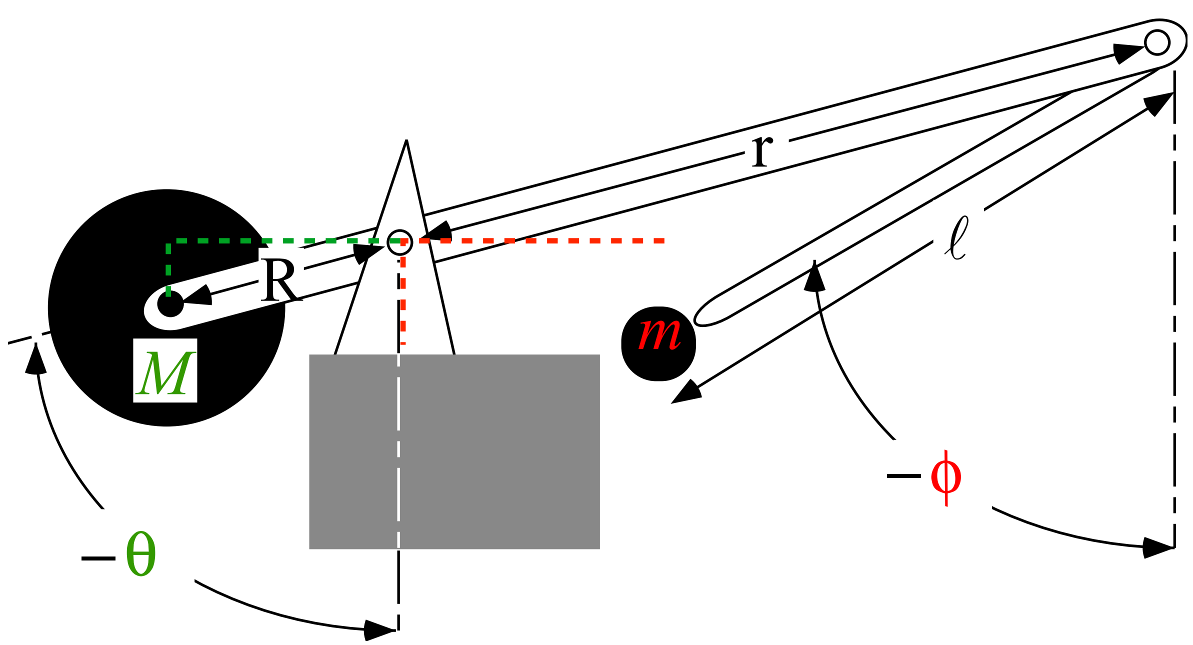
$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$



Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$Total\ KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

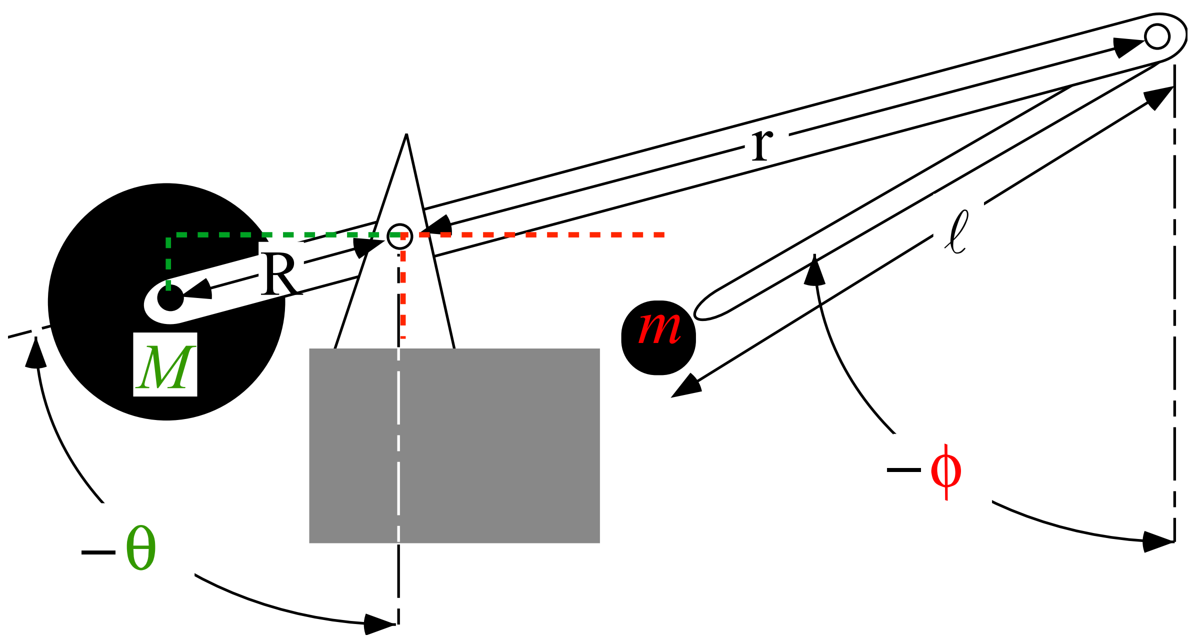
Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

$$\gamma_{mn}=\sum_{mass\ \mu}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\frac{\partial\mathbf{r}(\mu)}{\partial q^m}\bullet\frac{\partial\mathbf{r}(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\mathbf{E_m}(\mu)\bullet\mathbf{E_n}(\mu)$$



$$KE=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\dot{x}^j(\mu)\dot{x}^j(\mu)=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}\dot{q}^m\dot{q}^n$$

$$=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

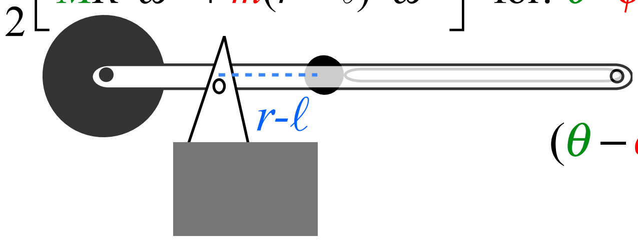
Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

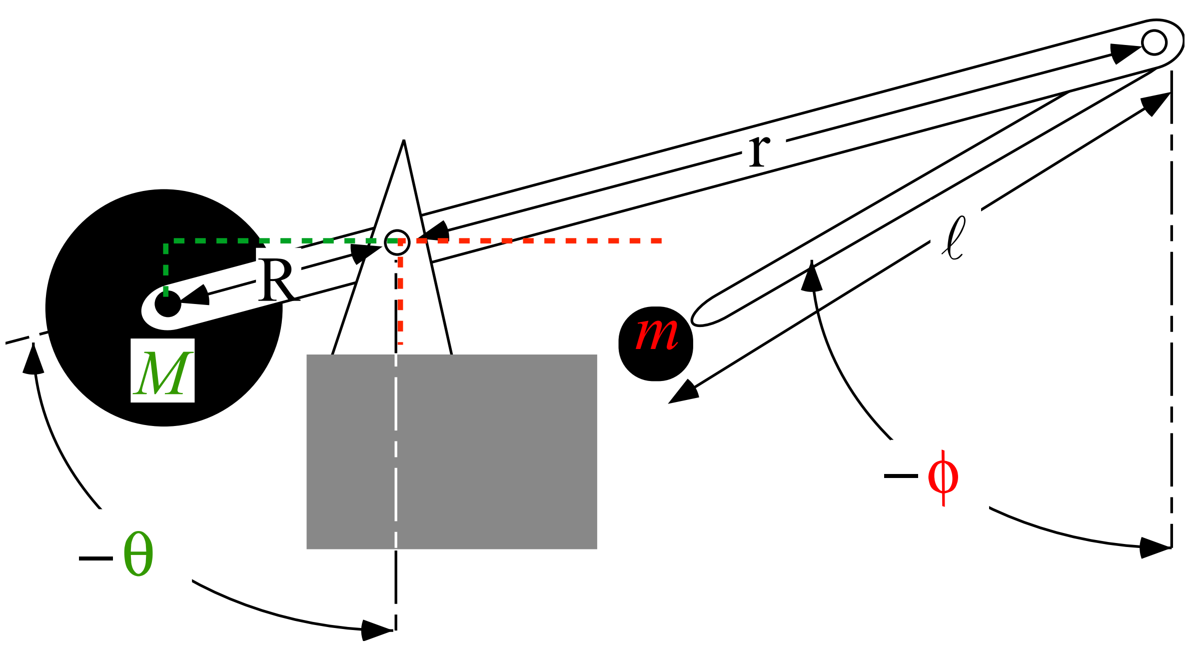
Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$

(J is Singular)



$(\theta-\phi)=0$



Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total }KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

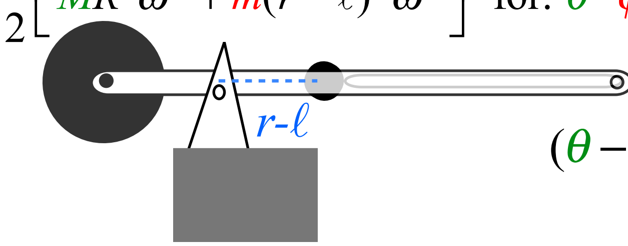
Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

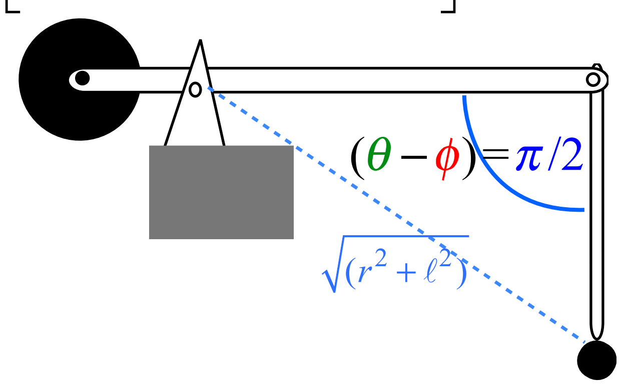
(J is Singular)

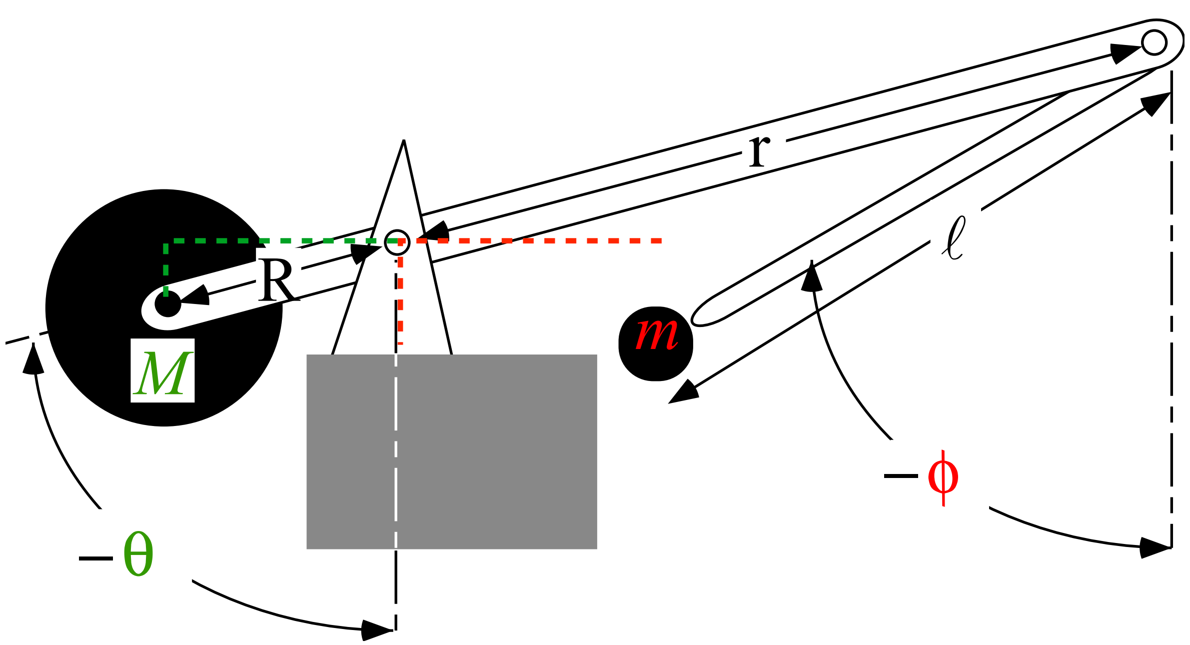


$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)





Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

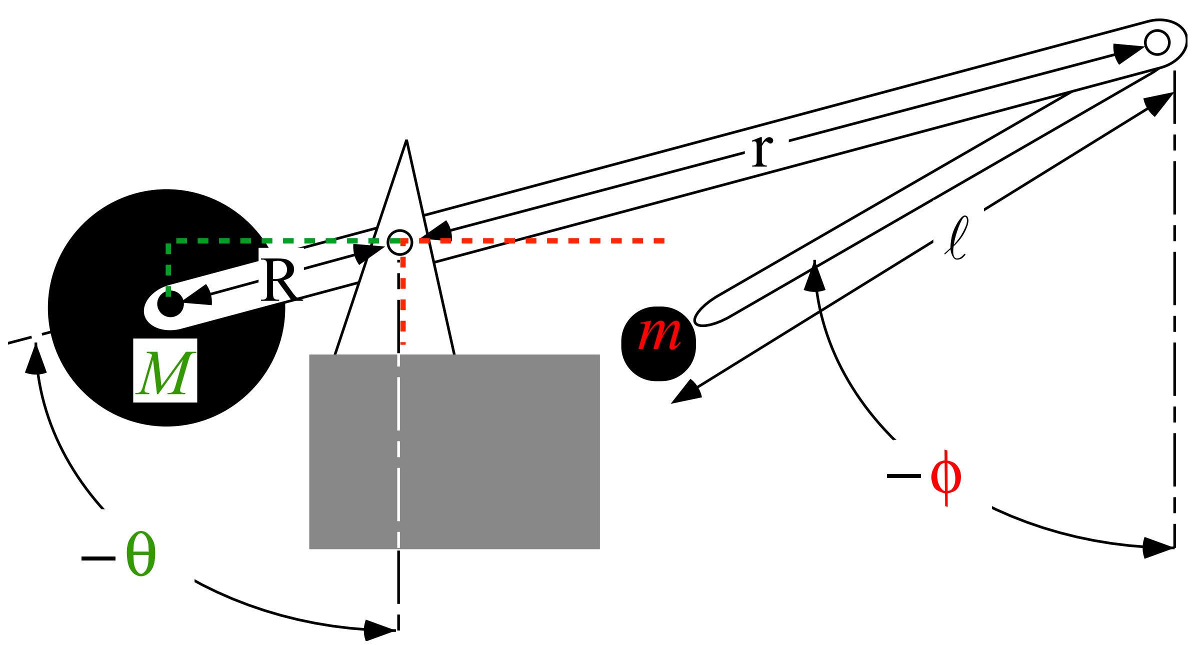
$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

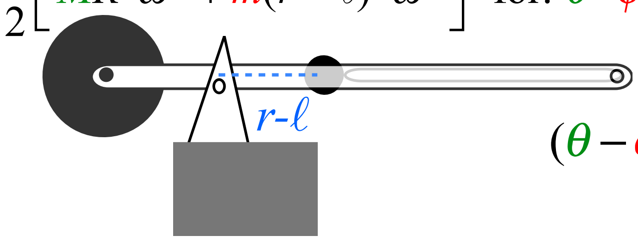
$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$



Special cases (rigid rotation)

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

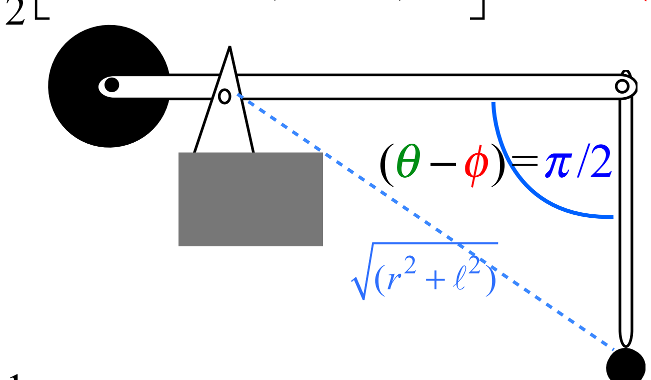
(J is Singular)



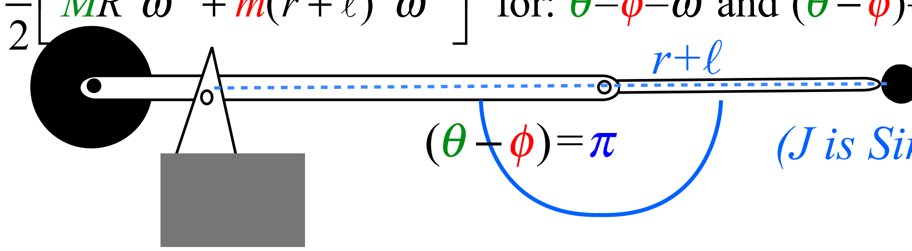
$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)



$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$



(J is Singular)

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Total kinetic energy of M and m

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

Dynamic metric tensor γ_{mn}

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

(a) When (θ,ϕ) coordinates are orthogonal

(b) When (θ,ϕ) coordinates are not orthogonal

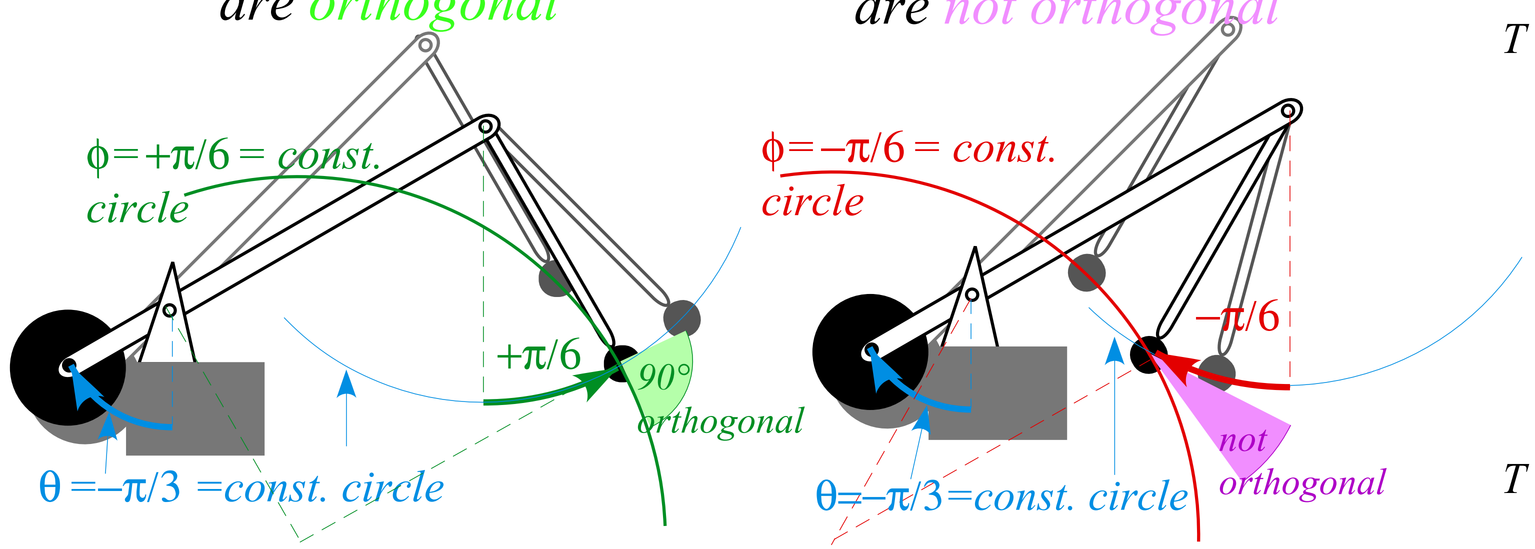
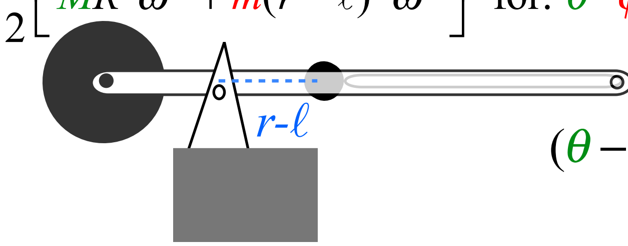


Fig. 2.3.1 Examples of (θ,ϕ) intersections (a) othogonal (special case), (b) non-orthogonal (typical).

Special cases (rigid rotation)

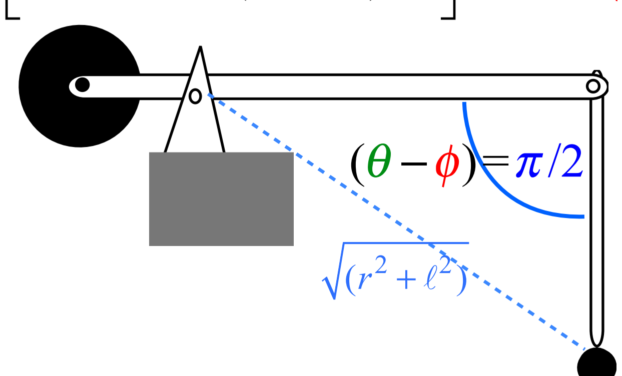
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

(J is Singular)



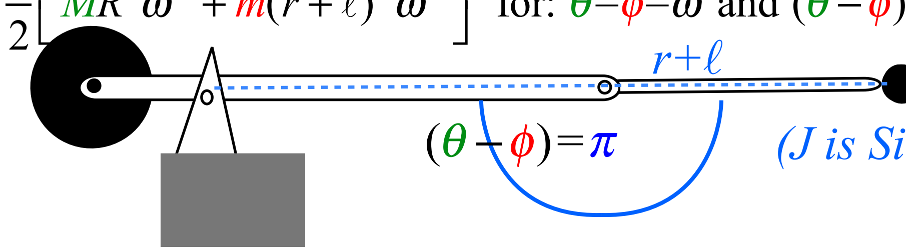
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)



$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$

(J is Singular)



Geometric and topological properties of GCC transformations (Mostly Unit 3.)

→ *Multivalued functionality and connections*

Covariant and contravariant relations

Metric tensors

Trebuchet Cartesian projectile coordinates are double-valued

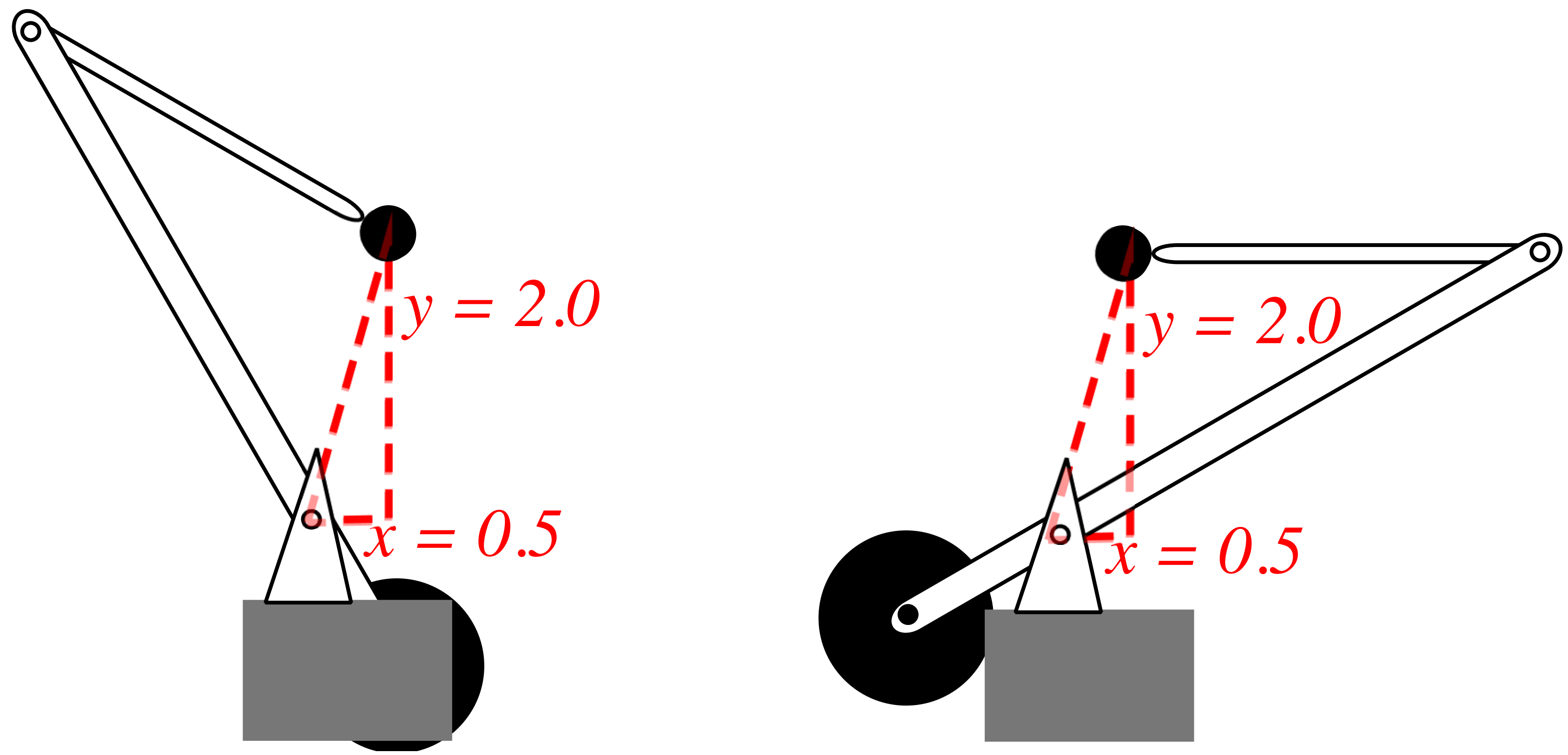


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)

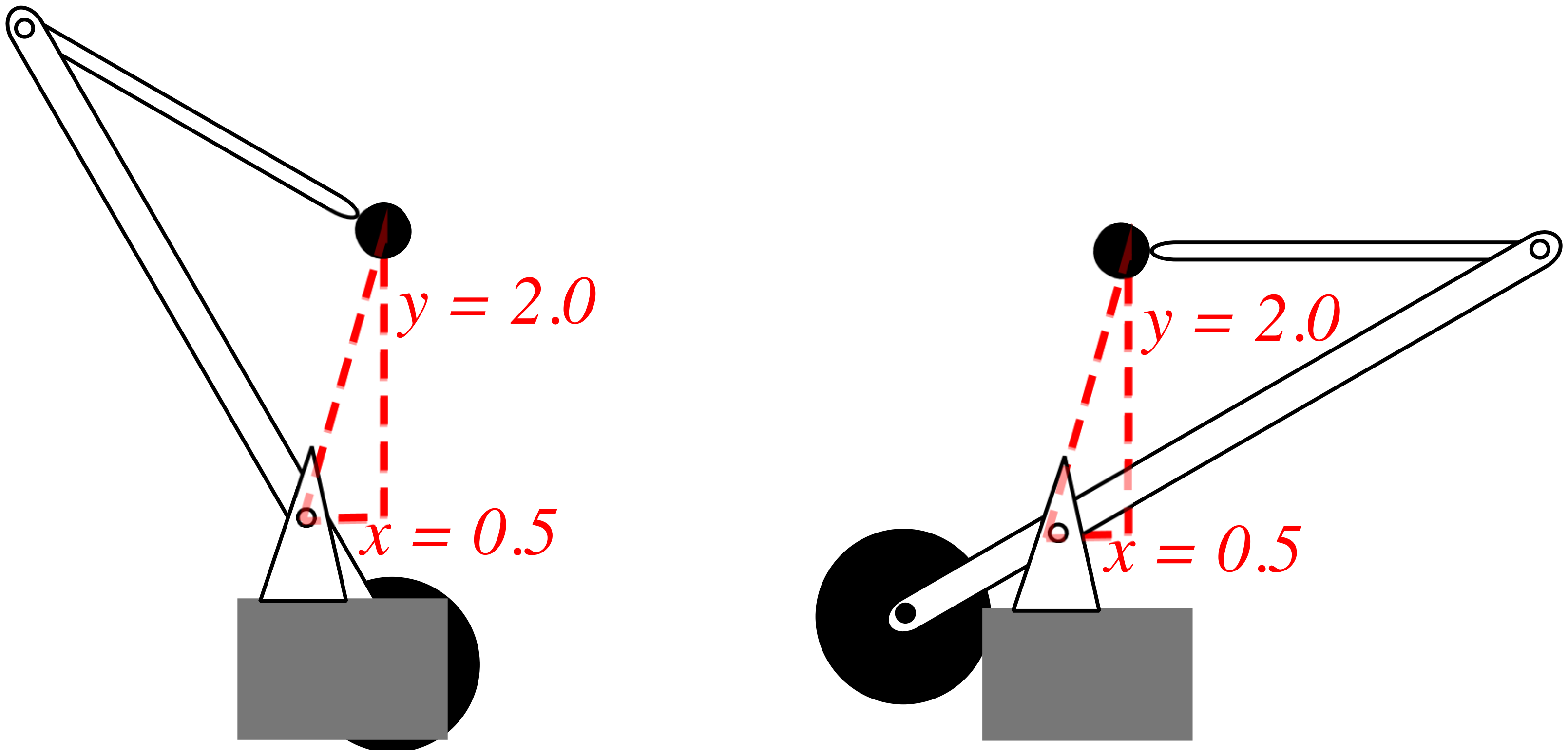


Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y of projectile *m*.

So, for example, are polar coordinates … (for each angle there are two r-values)

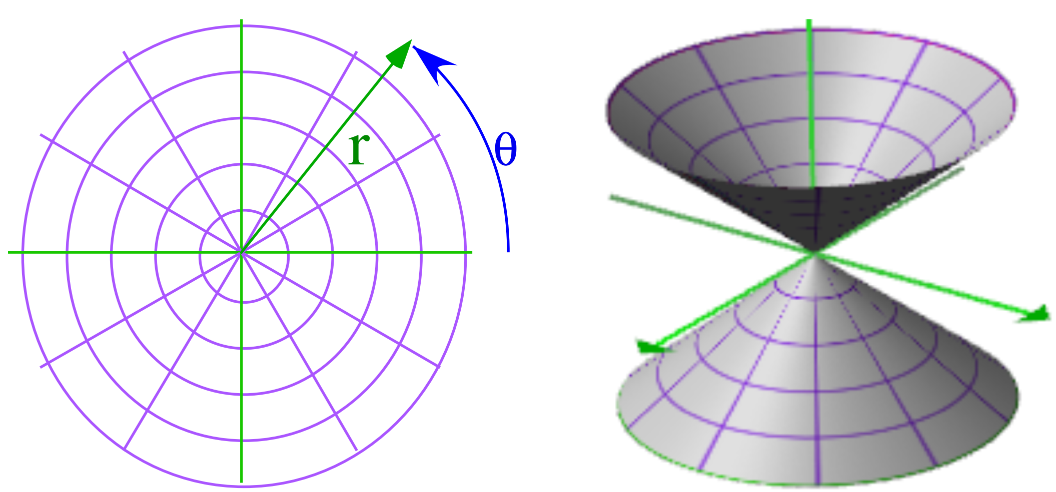
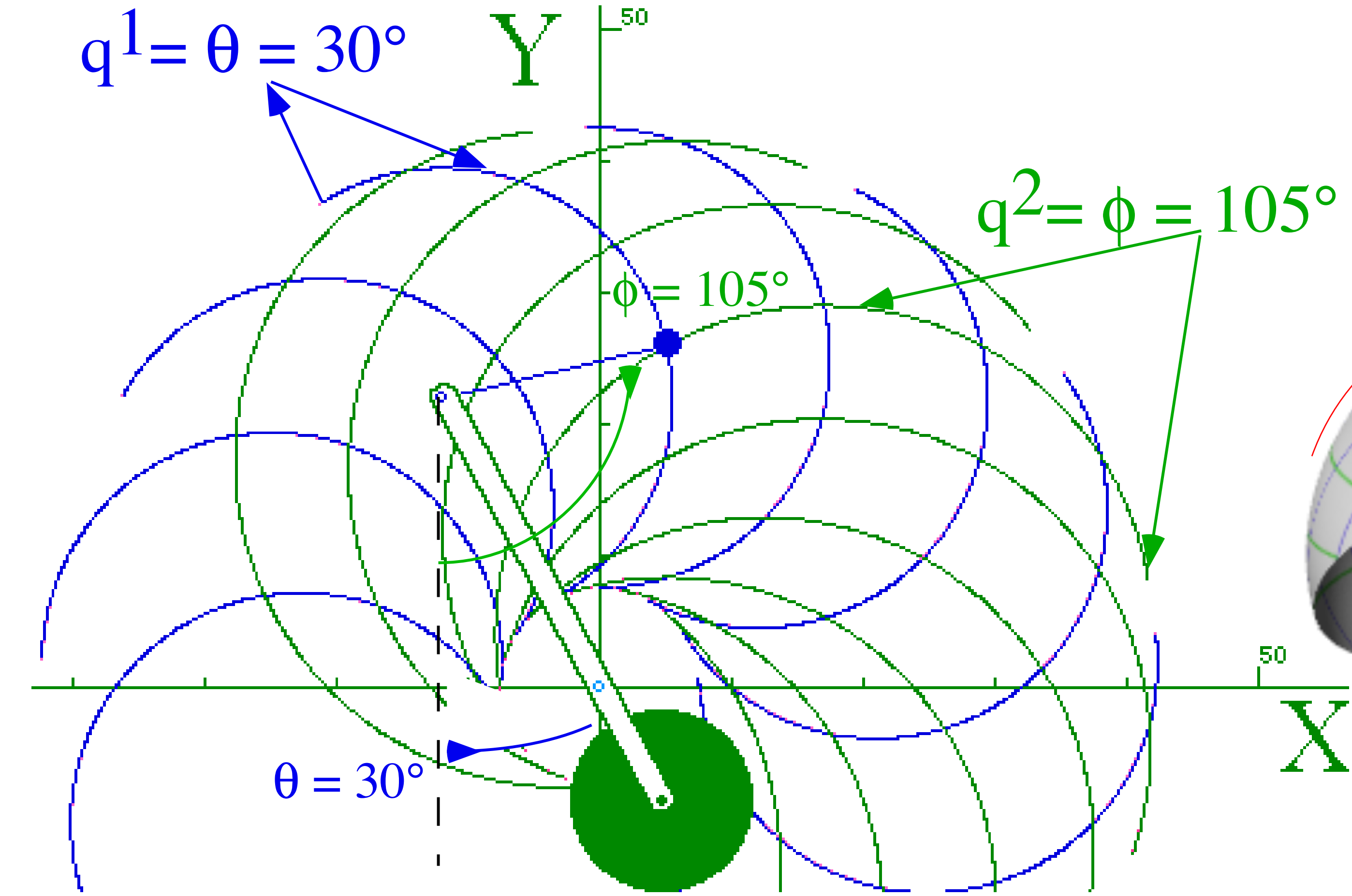


Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.



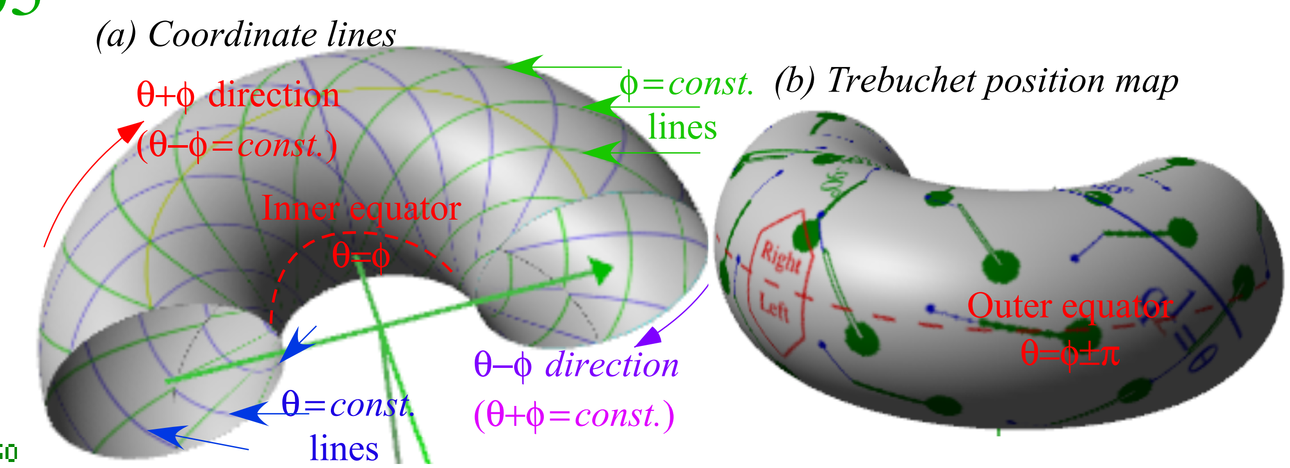


Fig. 3.1.2 Trebuchet torus.

(a) (q^1=θ, q^2=ϕ) coordinate lines. (b)Trebuchet position map and equators.

Fig. 3.1.1a (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Left handed sheet.)

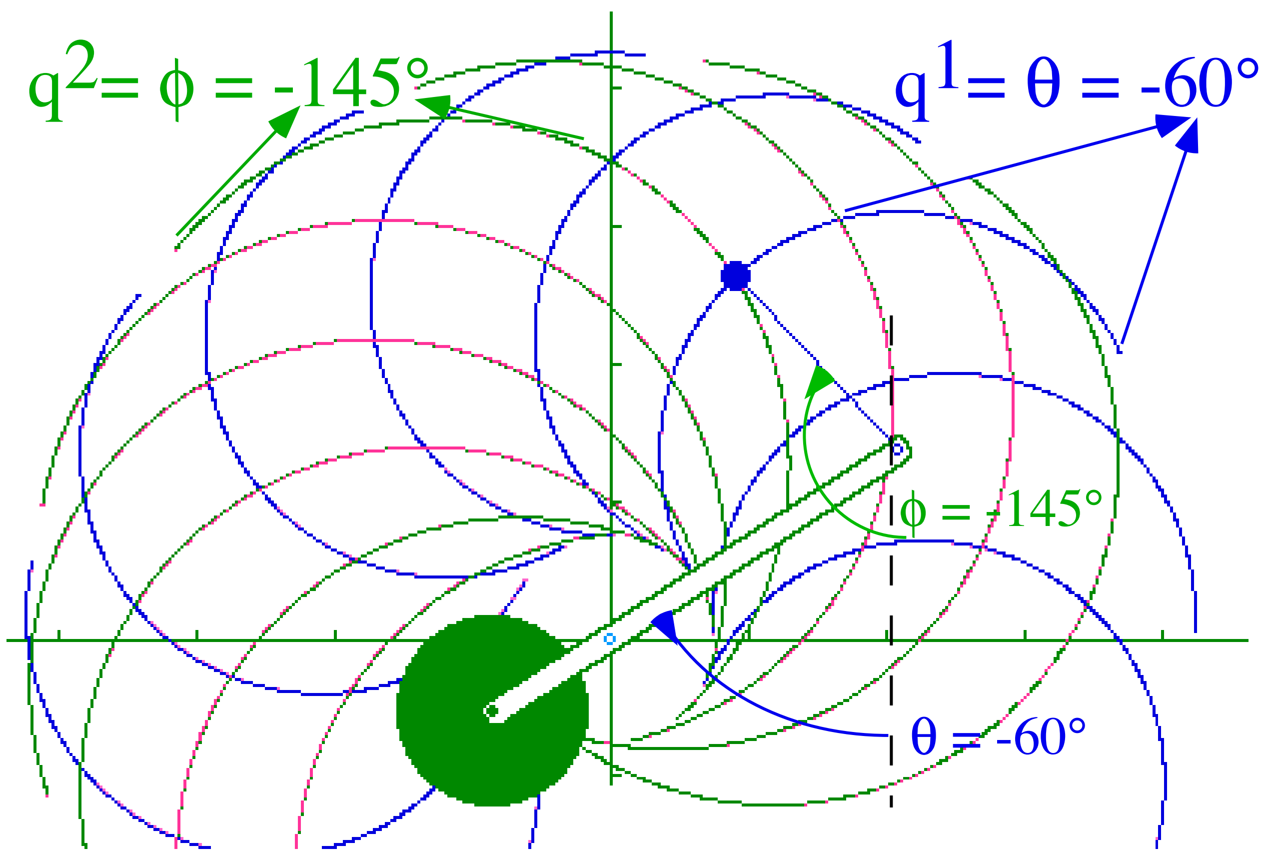
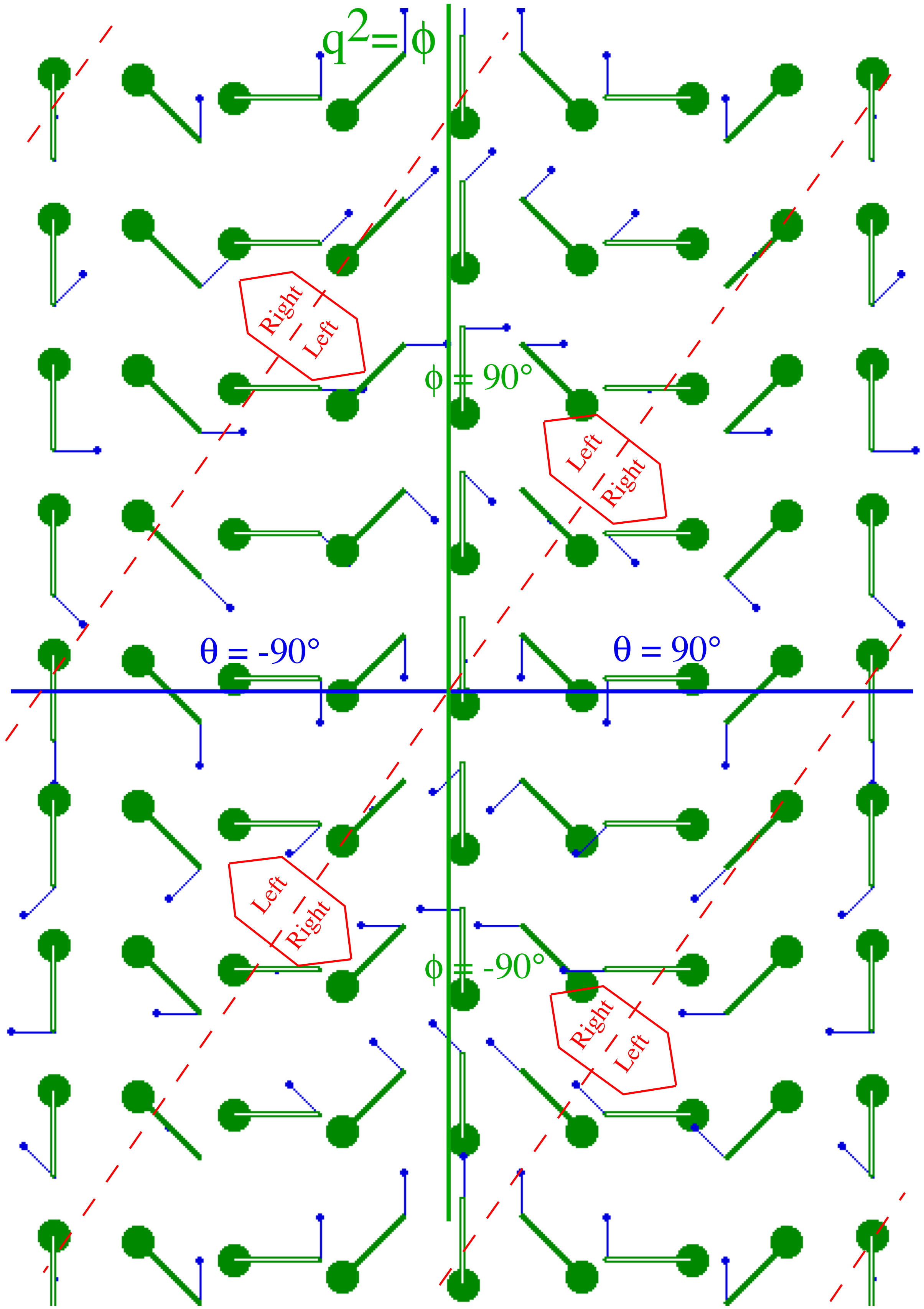


Fig. 3.1.1b (q^1=θ, q^2=ϕ)Coordinate manifold for trebuchet (Right handed sheet.)



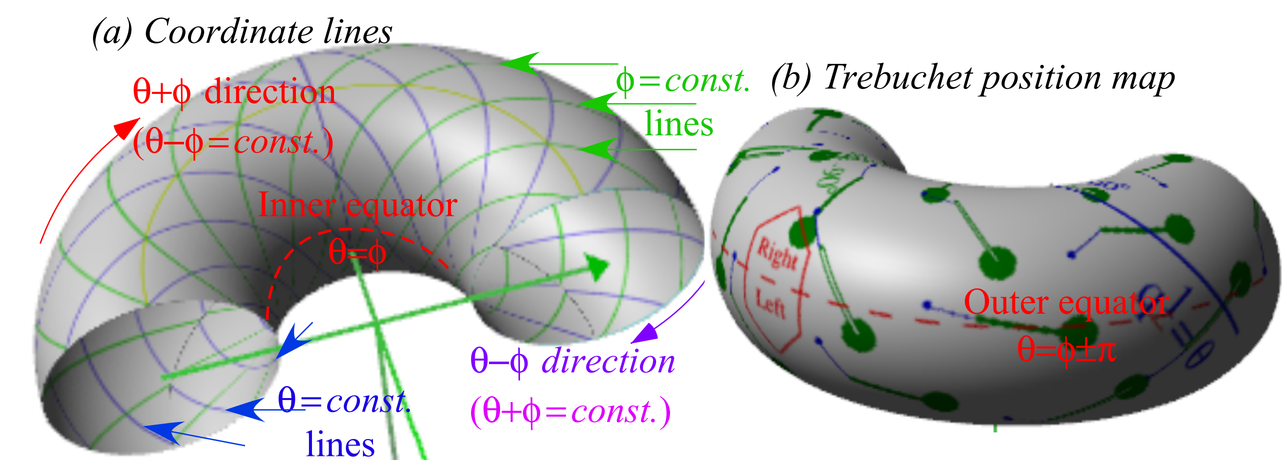


Fig. 3.1.2 Trebuchet torus.
(a) (q^1=θ, q^2=ϕ) coordinate lines.(b)Trebuchet position map and equators.

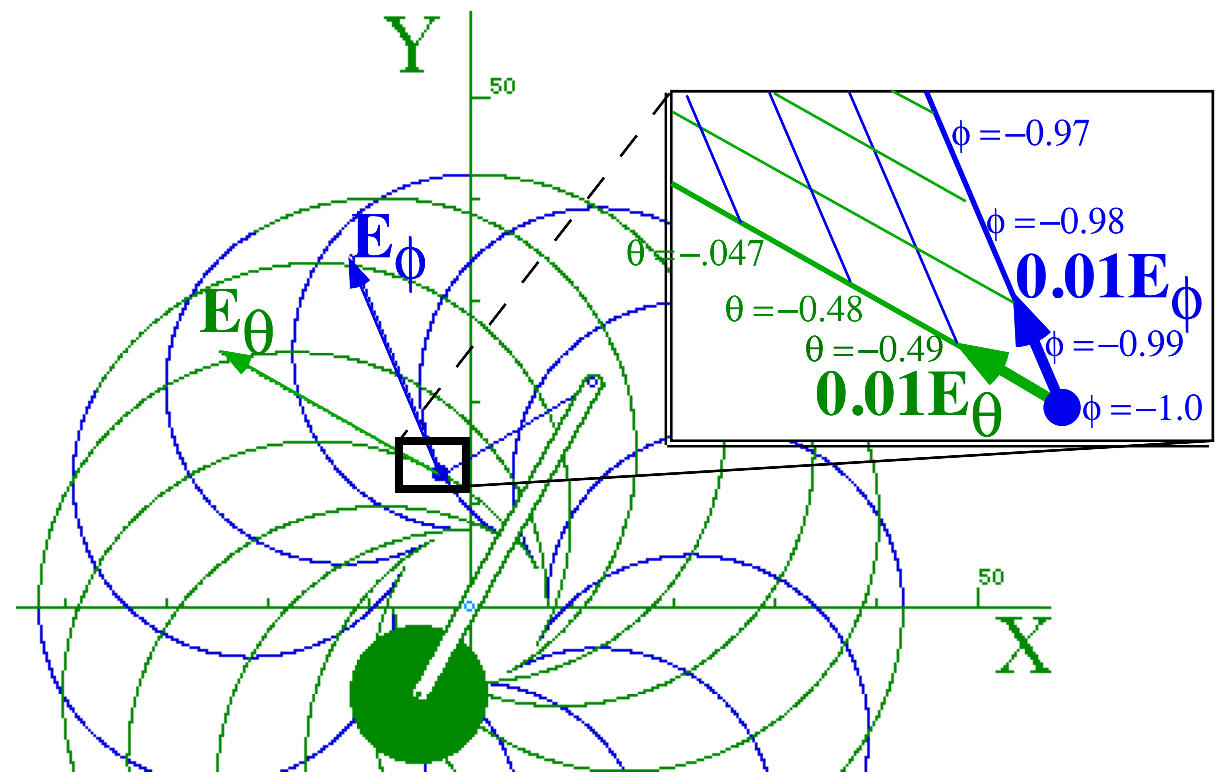


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Fig. 3.1.3 "Flattened" (q^1=θ, q^2=ϕ) coordinate manifold for trebuchet

Geometric and topological properties of GCC transformations (Mostly Unit 3.)

- *Multivalued functionality and connections*
- → *Covariant and contravariant relations*
- *Metric tensors*

Kajobian transfomation matrix *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle =$$

$$\begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \begin{matrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \vdots \end{matrix} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{vmatrix} \ell \sin\phi & -\ell\cos\phi \\ r\sin\theta & -r\cos\theta \end{vmatrix} \begin{matrix} \mathbf{E}^\theta \\ \mathbf{E}^\phi \end{matrix}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle =$$

$$\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{vmatrix} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix}$$

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / \, r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

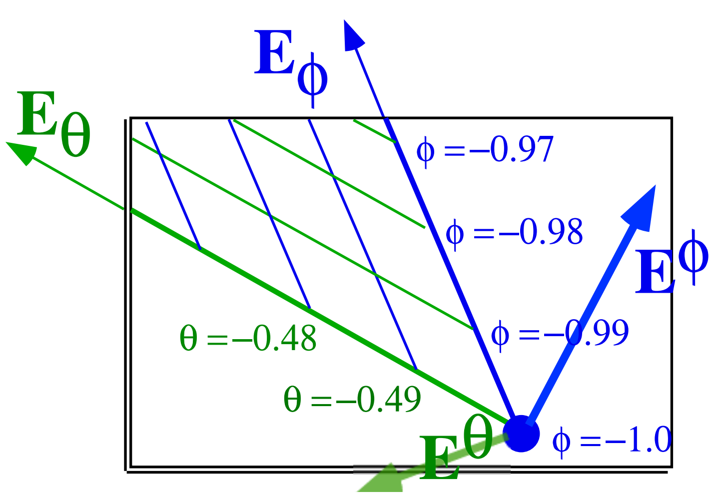


Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.

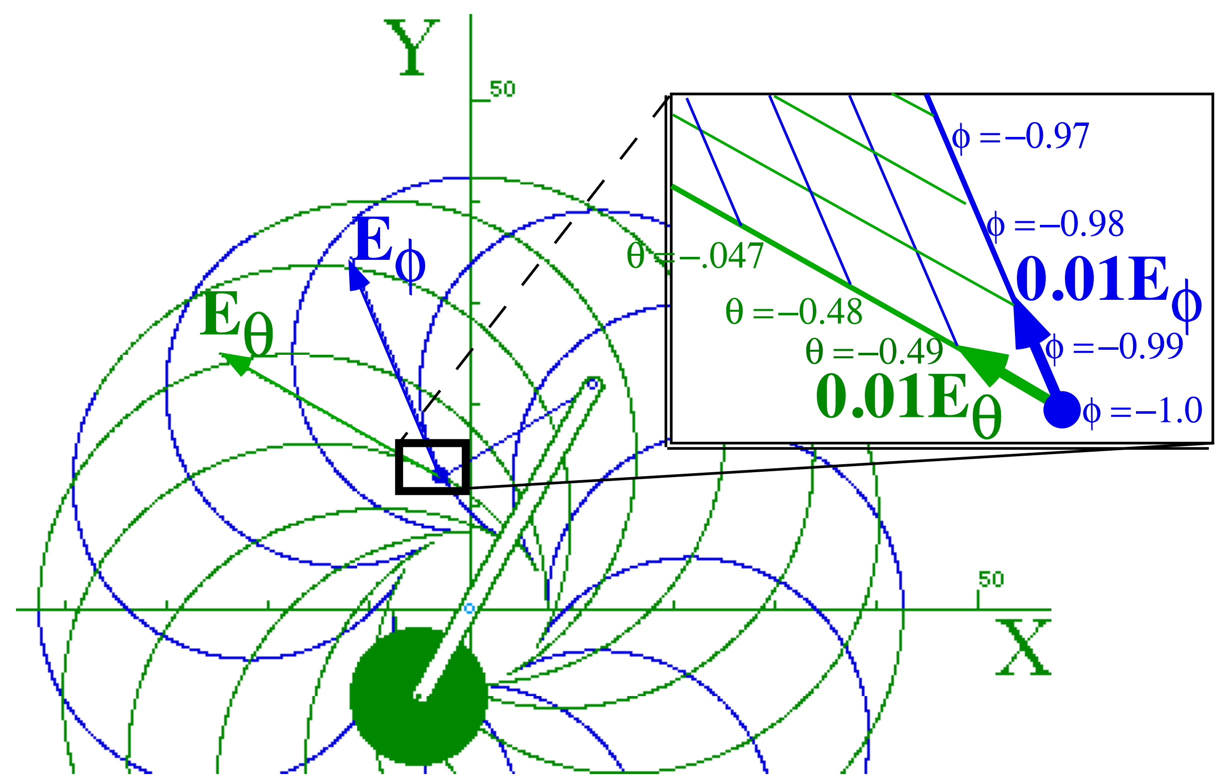


Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

$$\mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, ,

and the *Un , Vn ,...are covariant components*

$U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

Normal space (Contravariant)

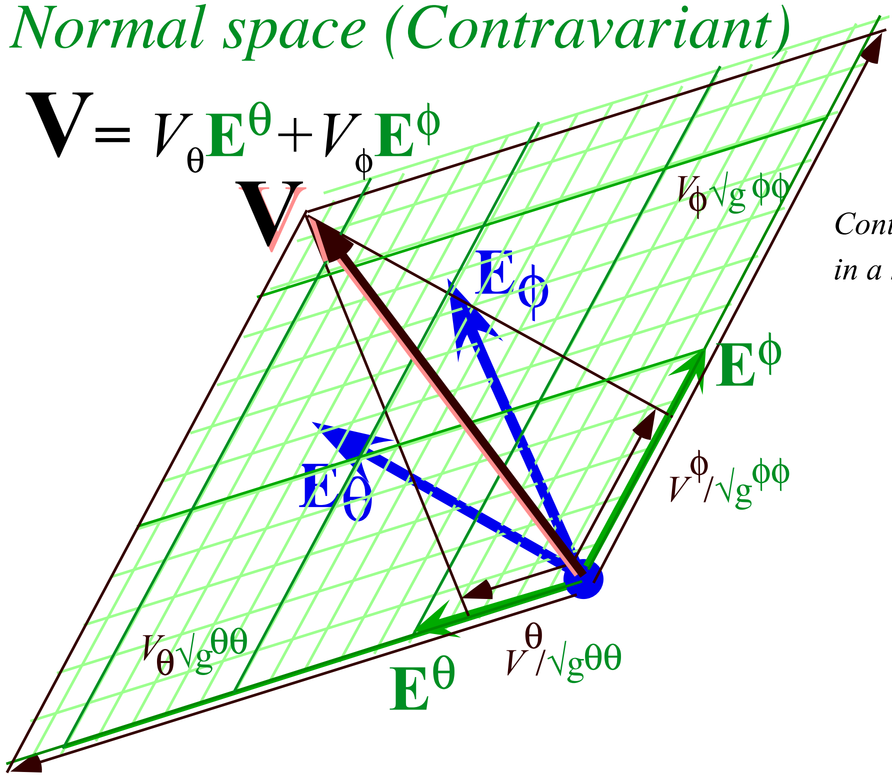


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)

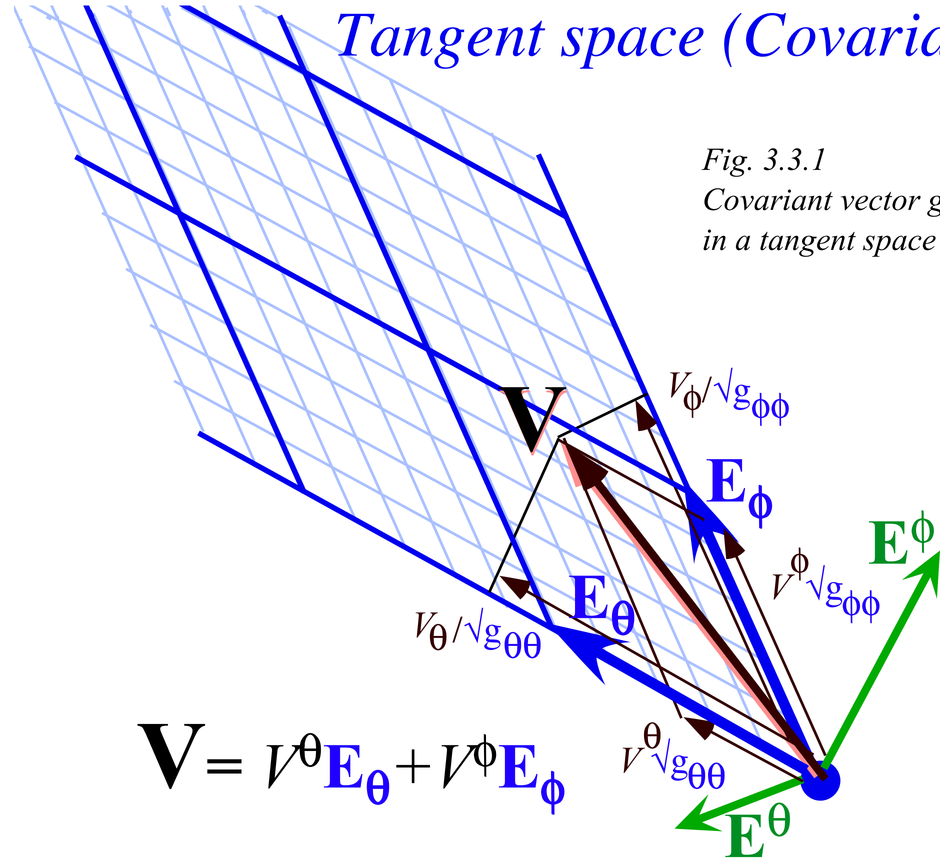


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vectors $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

Any vector **U**,**V**,**...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

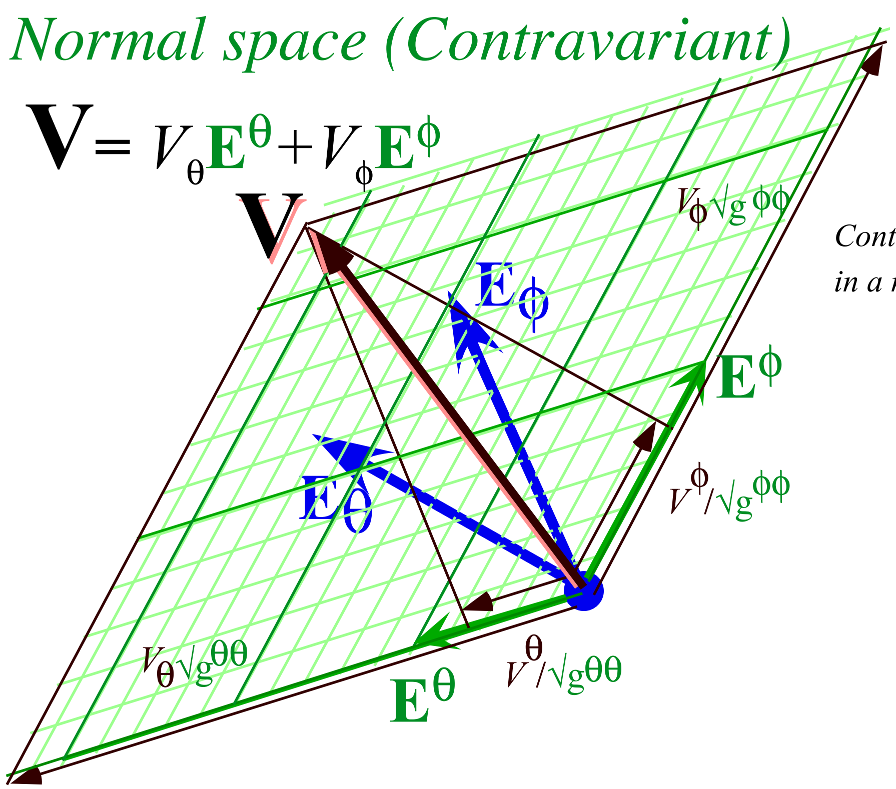


Fig. 3.3.2
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

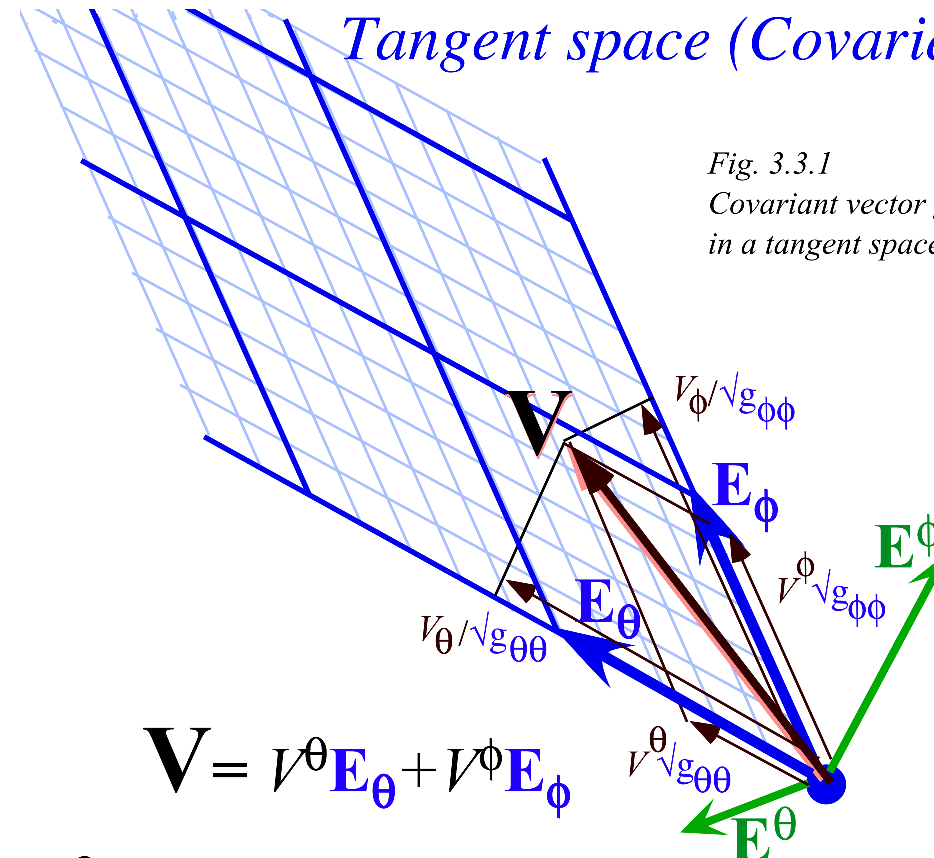


Fig. 3.3.1
Covariant vector geometry in a tangent space ($\mathbf{E}_\theta,\mathbf{E}_\phi$).

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^{\bar{1}}, \bar{q}^{\bar{2}}, \cdots\}$ using a *"chain-saw-sum rule"*....

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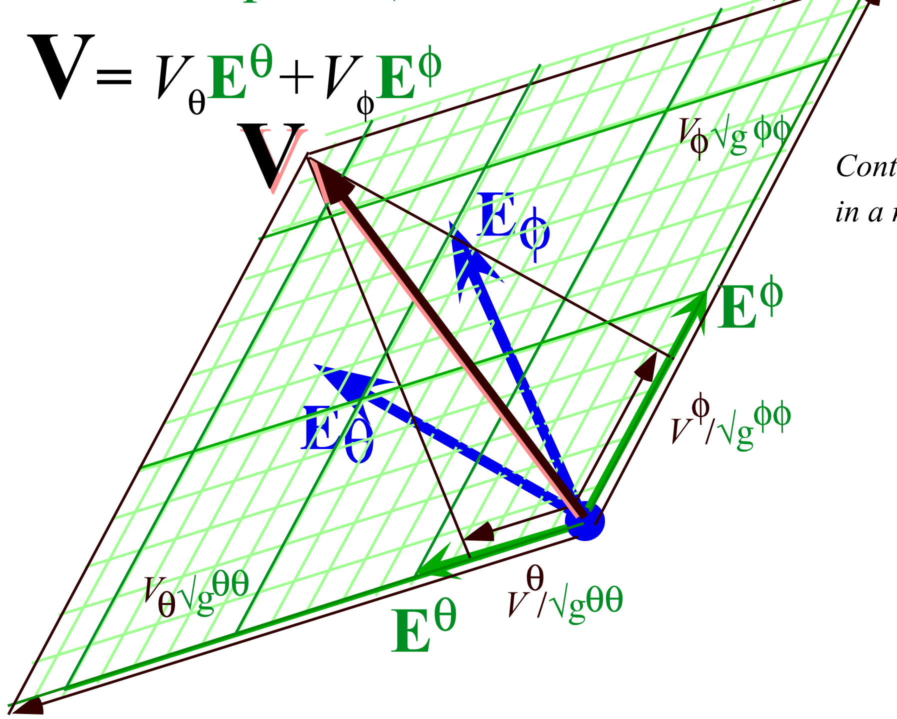
Normal space (Contravariant)



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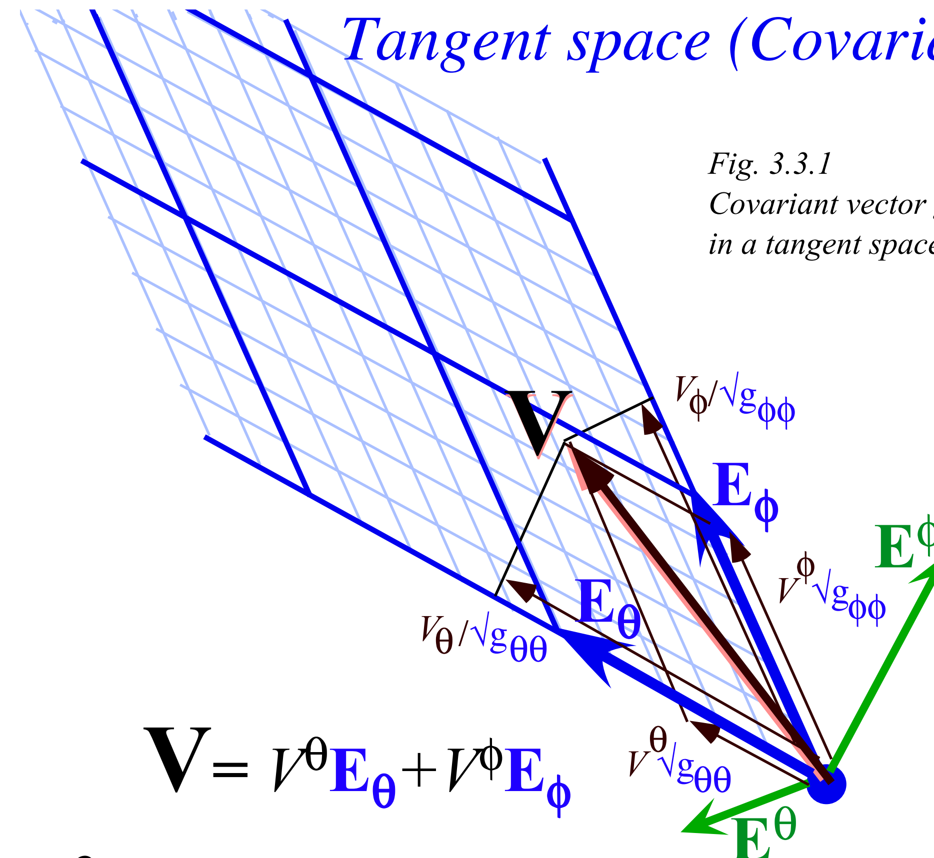
Tangent space (Covariant)



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Normal space (Contravariant)

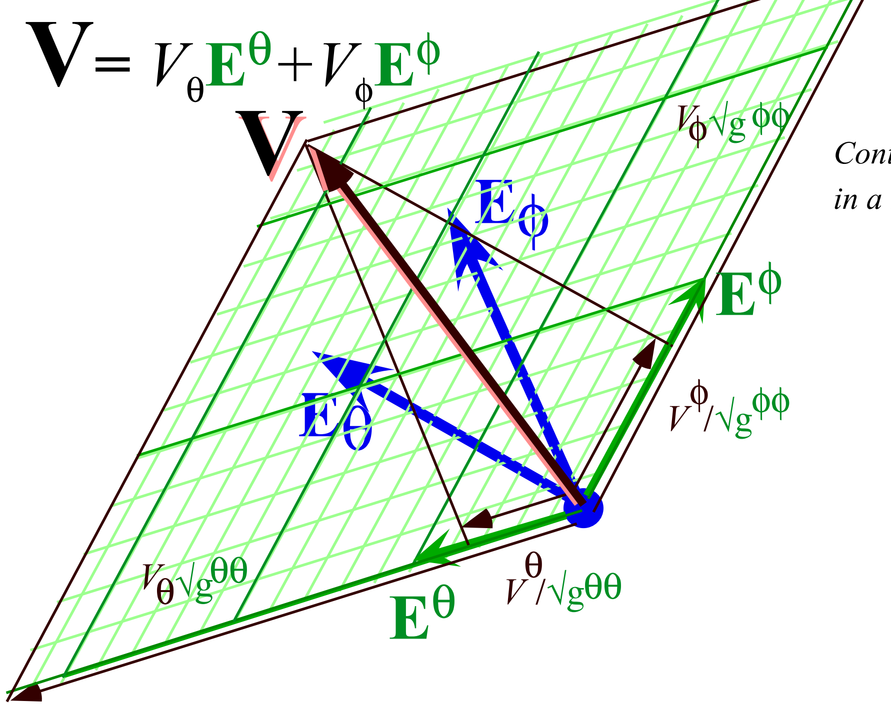
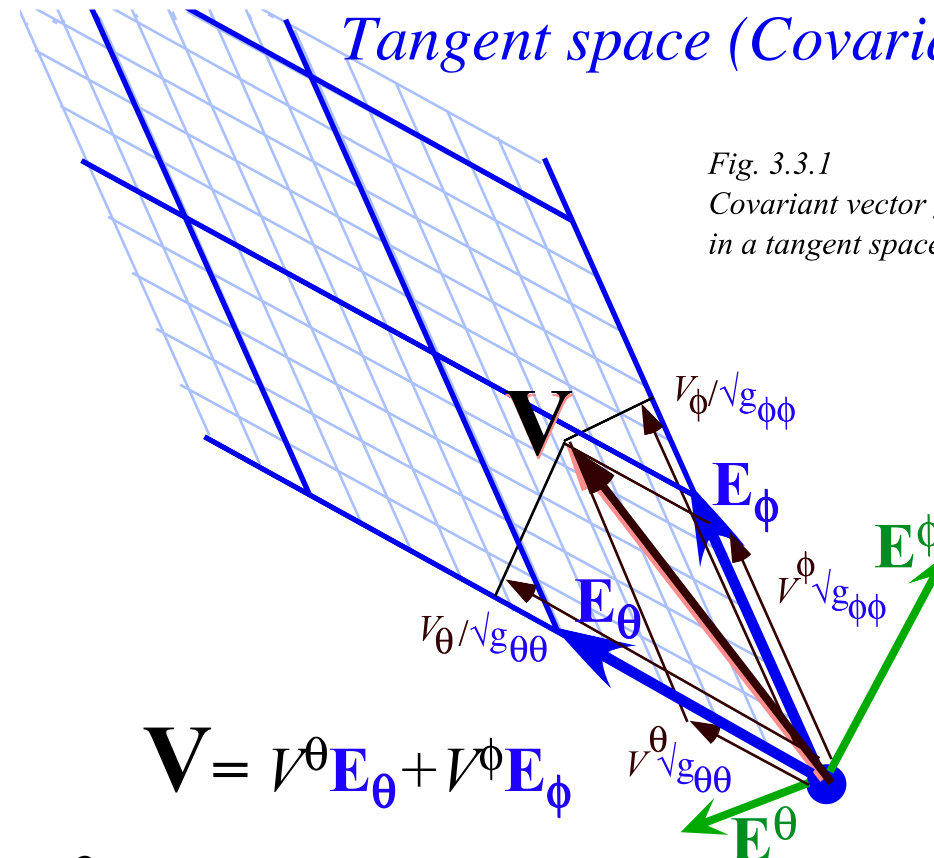


Fig. 3.3.2
Contravariant vector geometry
in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)



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implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

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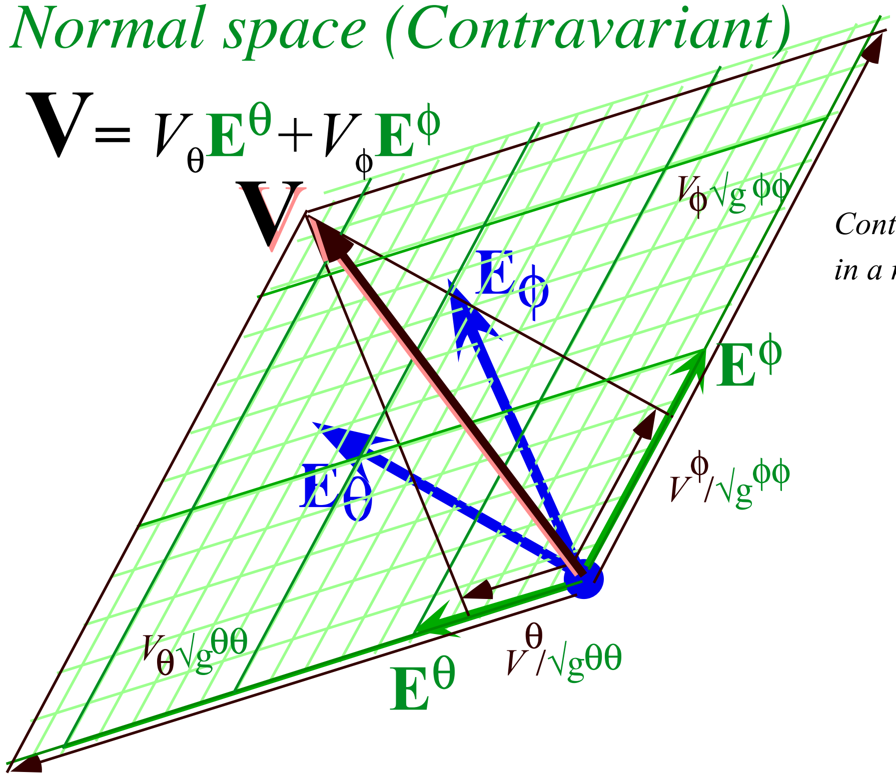
Normal space (Contravariant)



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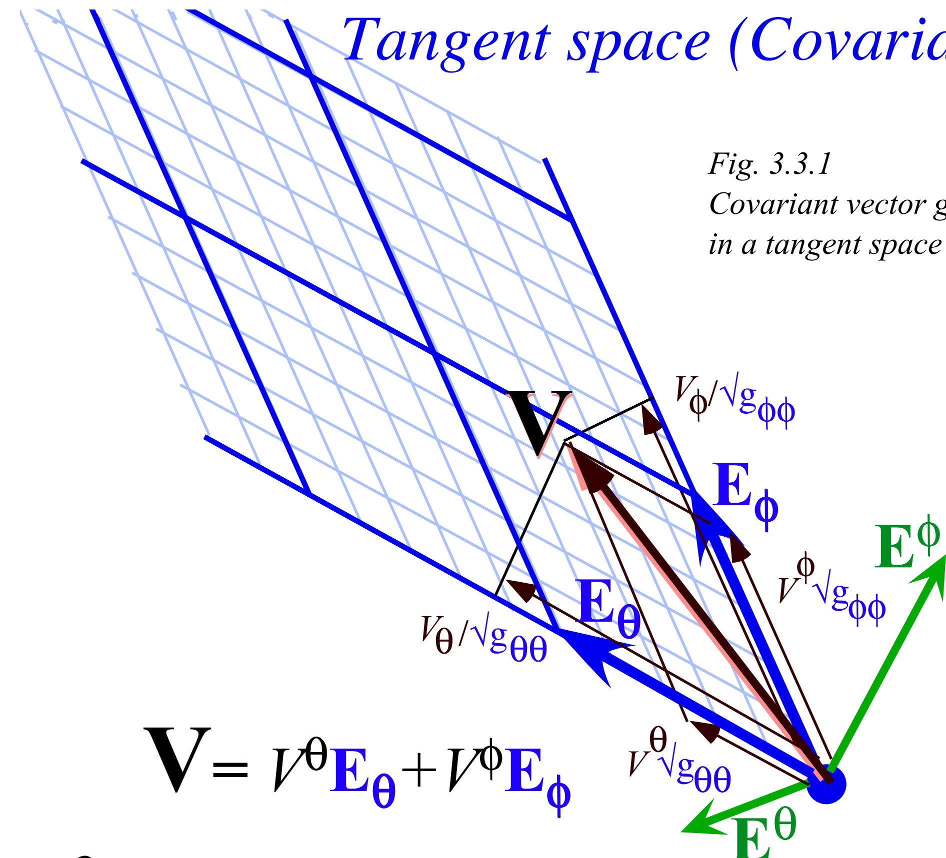
Contravariant vector geometry in a normal space ($\mathbf{E}^\theta,\mathbf{E}^\phi$).

Tangent space (Covariant)



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Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies:} \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

Geometric and topological properties of GCC transformations (Mostly Unit 3.)

Multivalued functionality and connections

Covariant and contravariant relations

→ *Metric tensors*

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ .$$

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \; , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \; .$$

"Mixed" covariant-contravariant metric components

$$g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = \delta_m^n = \begin{cases} 0 \text{ if: } m \neq n \\ 1 \text{ if: } m = n \end{cases}$$

Caution: δ_{mn} is g_{mn} and <u>not</u> δ_n^m in GCC.

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \,, \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \,.$$

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$$\mathbf{E_m} = g_{mn}\mathbf{E^n}\ , \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n}\ .$$

Co-and-Contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n\ , \quad T^{mm'} = g^{mn}g^{m'n'}T_{nn'}\ ,\ etc.$$

Metric tensor **g** *covariant (and contravariant) metric components* g_{mn} *(and* g^{mn} *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ .$$

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Co-and-Contra vector and tensor components are related by g-transformation. (So are g's themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \ , \qquad T^{mm'} = g^{mn}g^{m'n'}V_{nn'} \ , \ etc.$$

Diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors.

$$\left|\mathbf{E_m}\right| = \sqrt{\mathbf{E_m} \bullet \mathbf{E_m}} = \sqrt{g_{mm}}$$

$$\left|\mathbf{E^m}\right| = \sqrt{\mathbf{E^m} \bullet \mathbf{E^m}} = \sqrt{g^{mm}}$$

tangent space area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\left|\mathbf{E_1}\times\mathbf{E_2}\right|=V^1V^2\sqrt{\left(\mathbf{E_1}\times\mathbf{E_2}\right)\bullet\left(\mathbf{E_1}\times\mathbf{E_2}\right)}$$

$$Area\left(V^1E_1,V^2E_2\right)=V^1V^2\sqrt{\left(\mathbf{E_1}\bullet\mathbf{E_1}\right)\left(\mathbf{E_2}\bullet\mathbf{E_2}\right)-\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)\left(\mathbf{E_1}\bullet\mathbf{E_2}\right)}$$

$$=V^1V^2\sqrt{g_{11}g_{22}-g_{12}g_{12}}=V^1V^2\sqrt{\det\begin{vmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}}$$

3D Jacobian determinant J-columns are $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\left|\mathbf{E_1}\times\mathbf{E_2}\bullet\mathbf{E_3}\right|=V^1V^2V^3\det\begin{vmatrix}\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3}\\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3}\\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{vmatrix}$$

$$\begin{pmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}=\begin{pmatrix}\frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1}\\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2}\\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}\bullet\begin{pmatrix}\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3}\\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3}\\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3}\end{pmatrix}=J^T\bullet J$$

Determinant product ($det|A|\ det|B|\ =det|A\bullet B|$) and symmetry ($det|A^T|\ =det|A|$) gives

$$Volume\left(V^1\mathbf{E}_1,V^2\mathbf{E}_2,V^3\mathbf{E}_3\right)=V^1V^2V^3\det\left|J\right|=V^1V^2V^3\sqrt{\det\left|g\right|}$$