Lecture 19
Tue. 10.28.2014

Hamilton Equations for Trebuchet and Other Things
(Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
- *Hamilton's equations and Poincare invariant relations*
- *Hamiltonian expression and contravariant γ^{mn} tensor*

Hamiltonian energy and momentum conservation and symmetry coordinates
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
    - *Algebraic approach*
    - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
    - *Many approaches to Mechanics*

Chapter 1. The Trebuchet: A dream problem for Galileo?

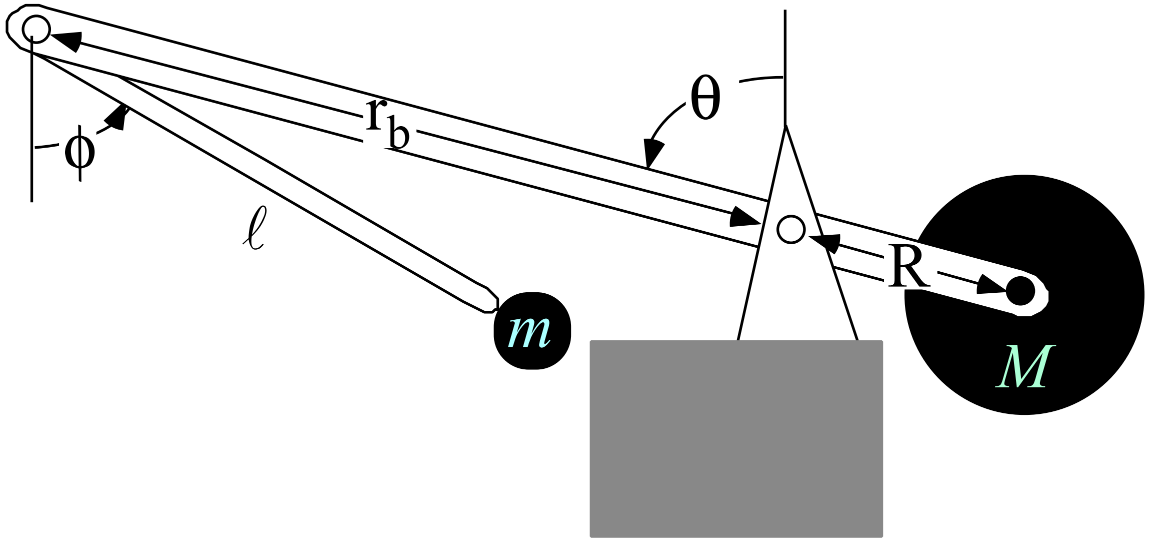
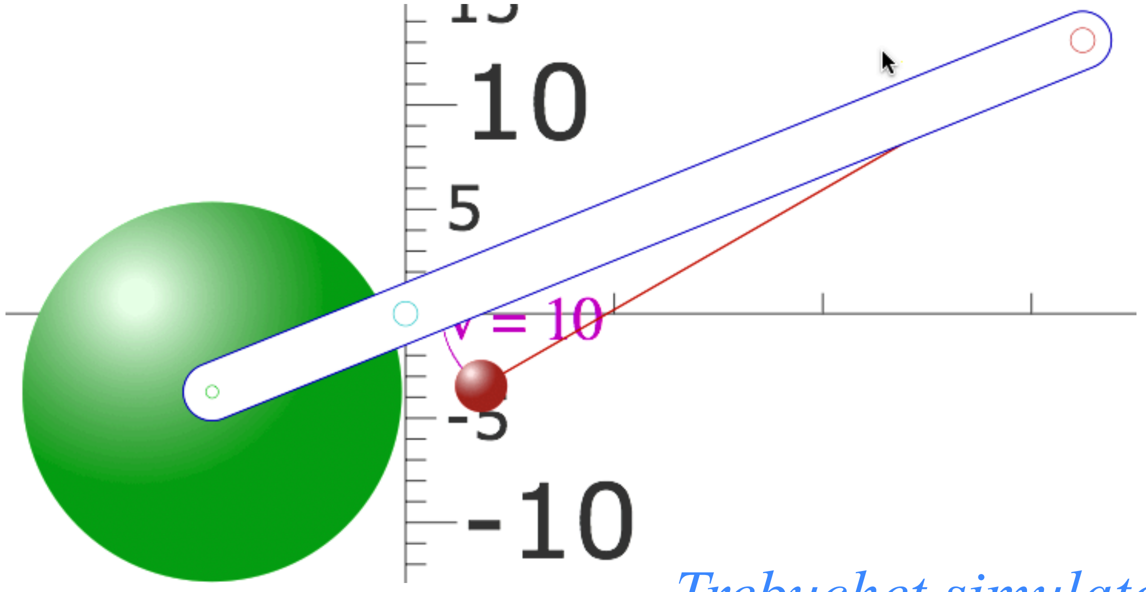


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/rso/modphys/testing/markup/TrebuchetWeb.html

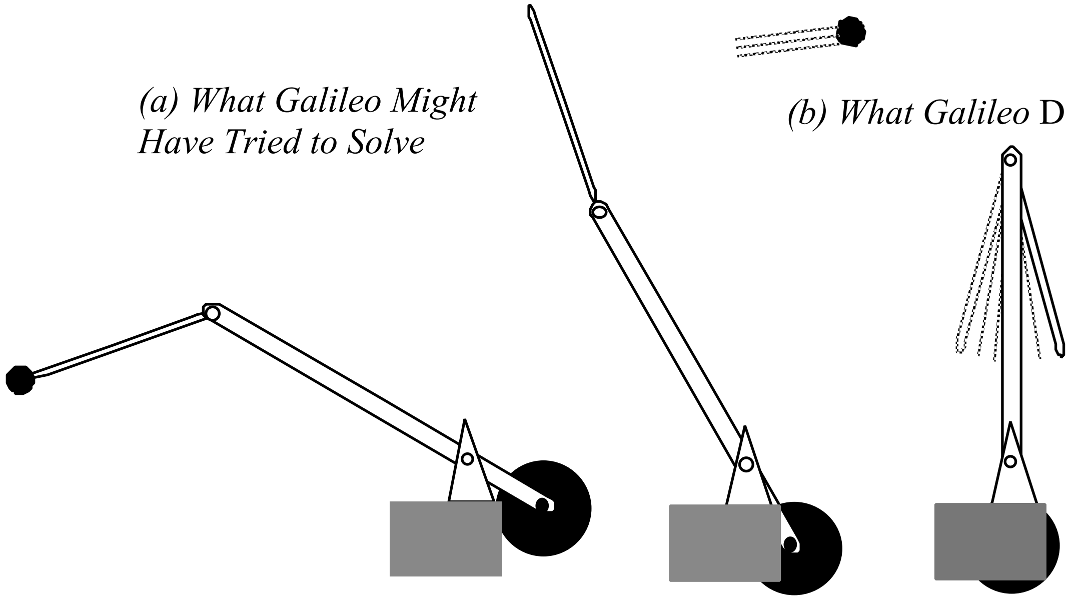
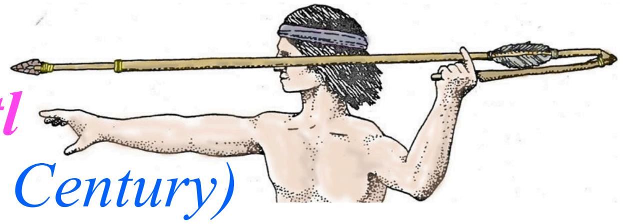


Fig. 2.1.2 Galileo's (supposed) problem

(Simple pendulum dynamics)



The Atlatl
(Cahokia, IL 12th Century)

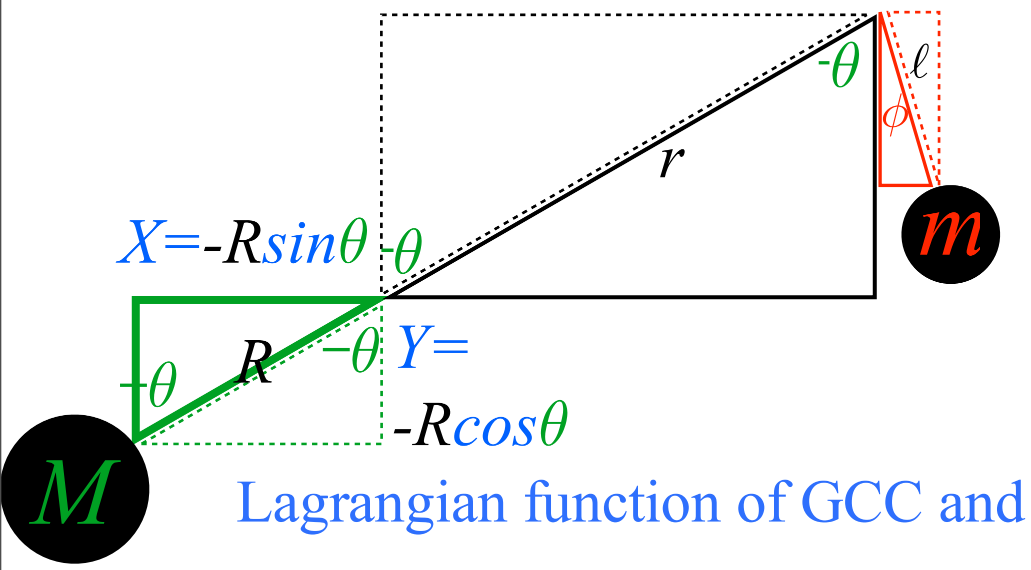
Review of Hamiltonian equation derivation (Elementary trebuchet)

→ *Hamiltonian definition from Lagrangian and γ_{mn} tensor*

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} tensor

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

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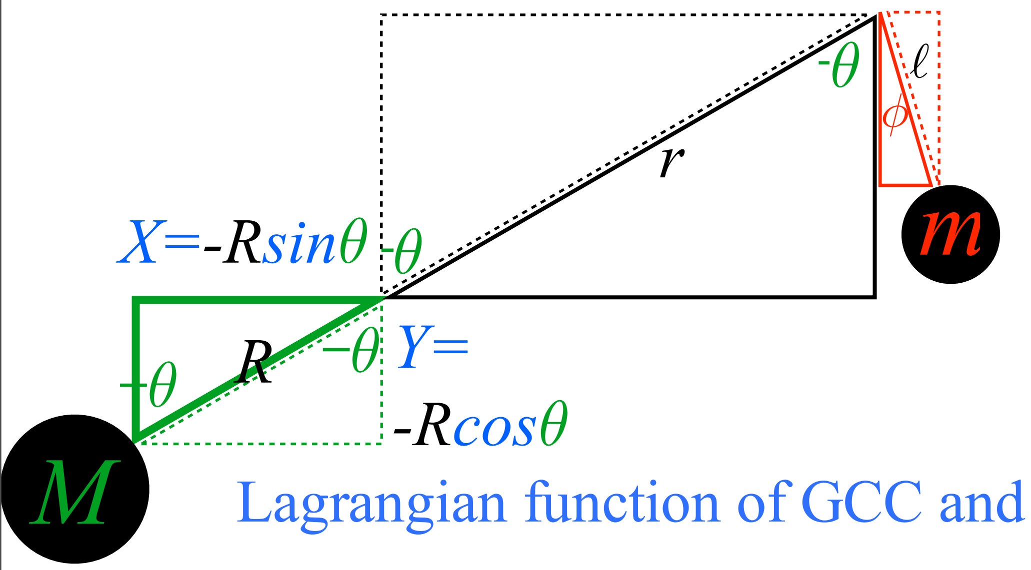
1st differential chain

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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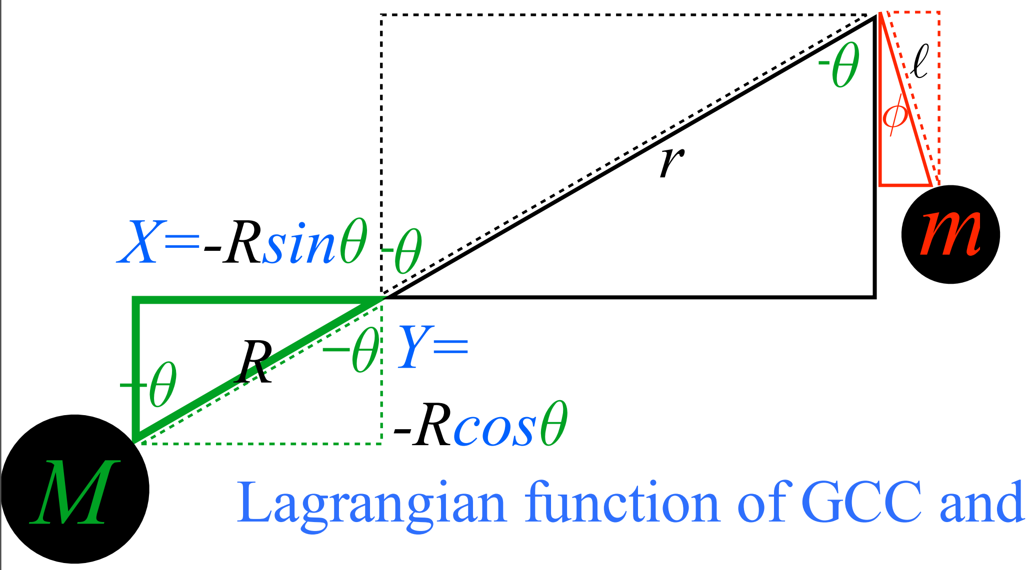

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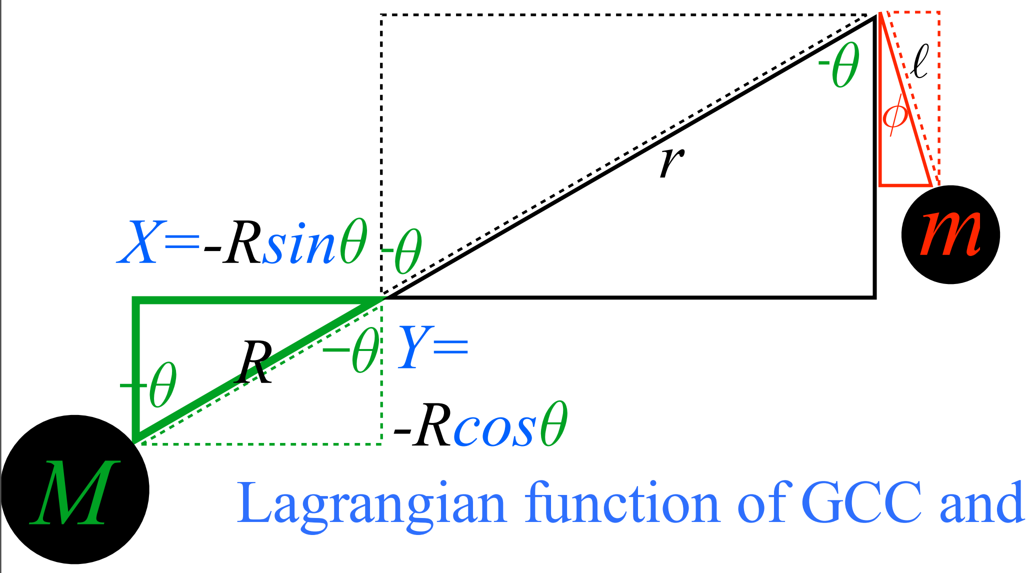

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(Consolidating)

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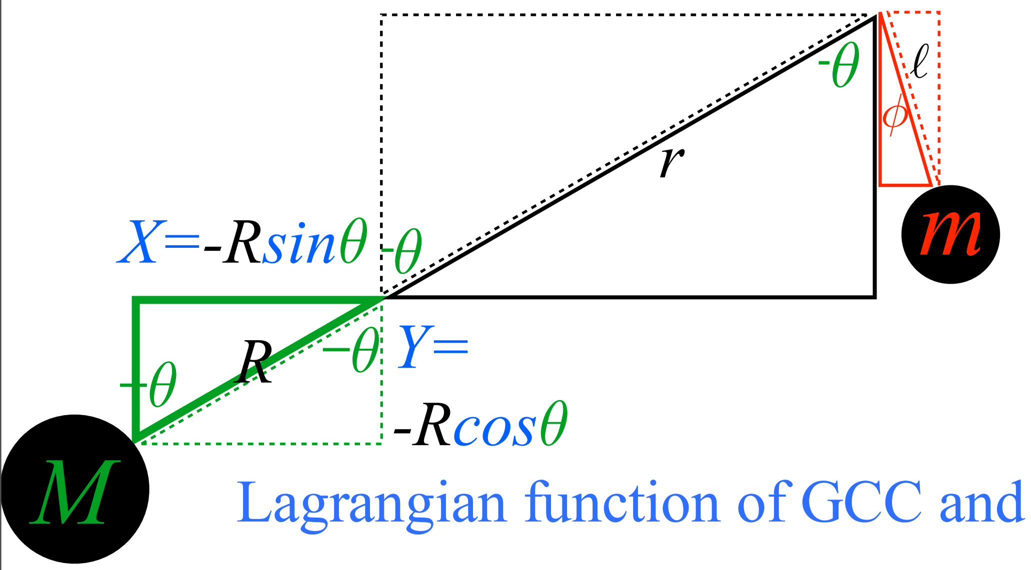

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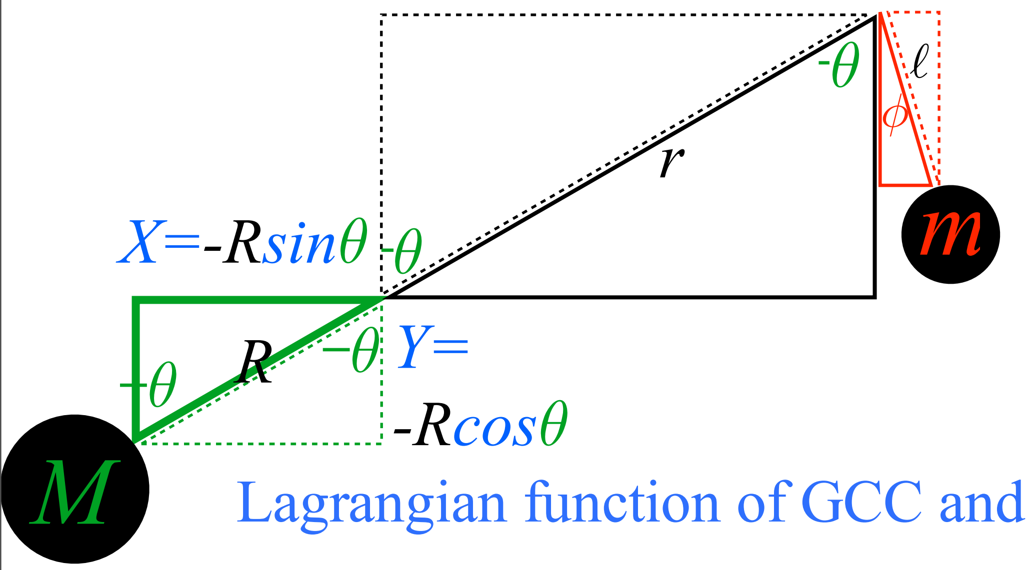

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(Consolidating)

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(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

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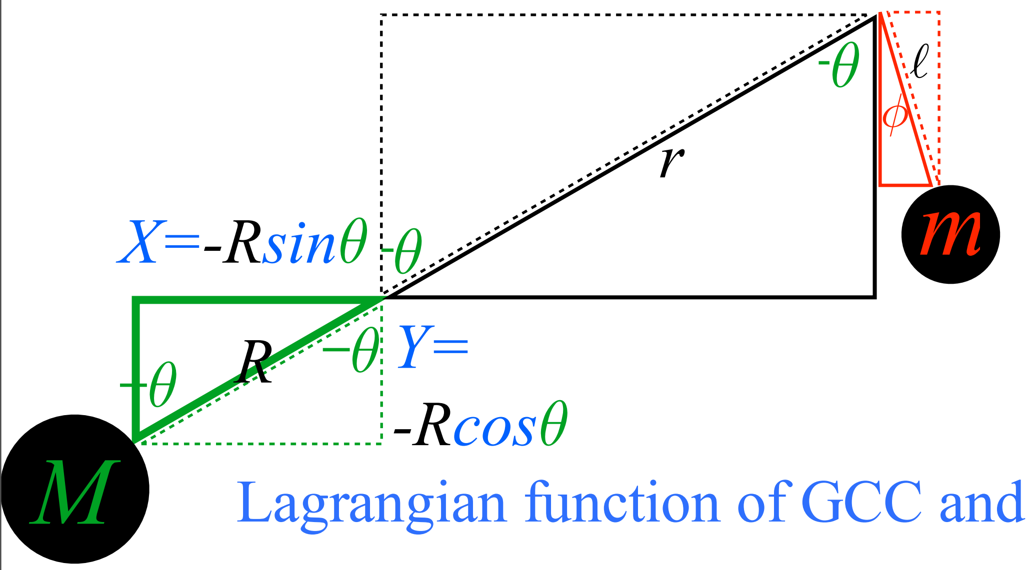

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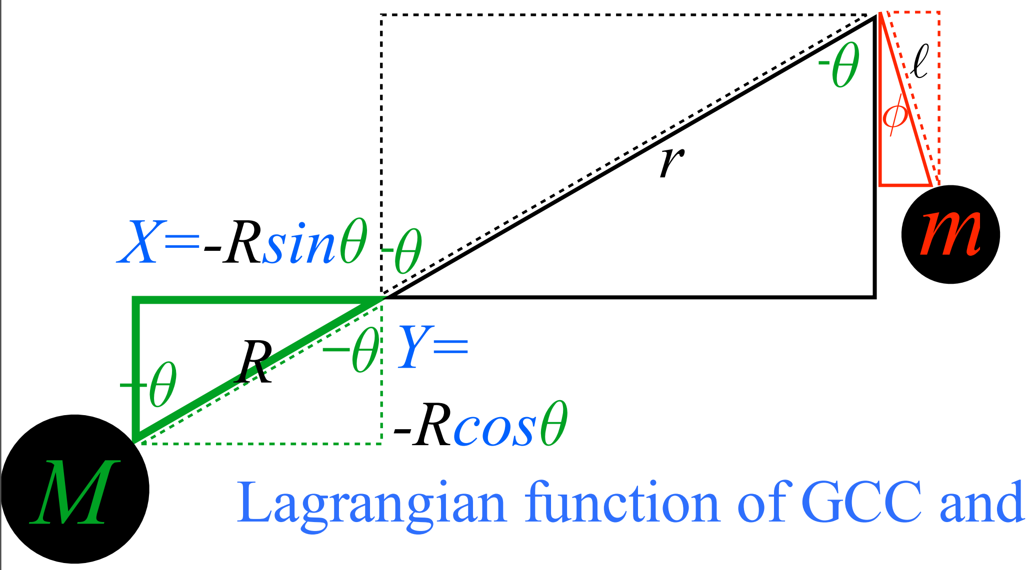

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by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial \theta} \underset{\uparrow}{=} -\dot{p}_\theta$$

by Lagrange equations

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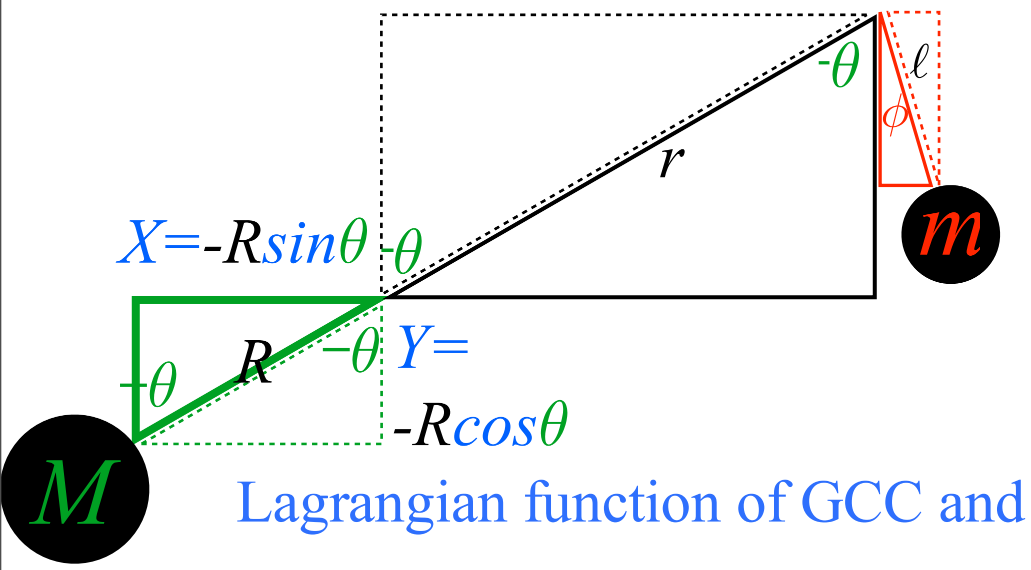

$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

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velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial \theta} \underset{\uparrow}{=} -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} \underset{\uparrow}{=} \dot{\theta}$$

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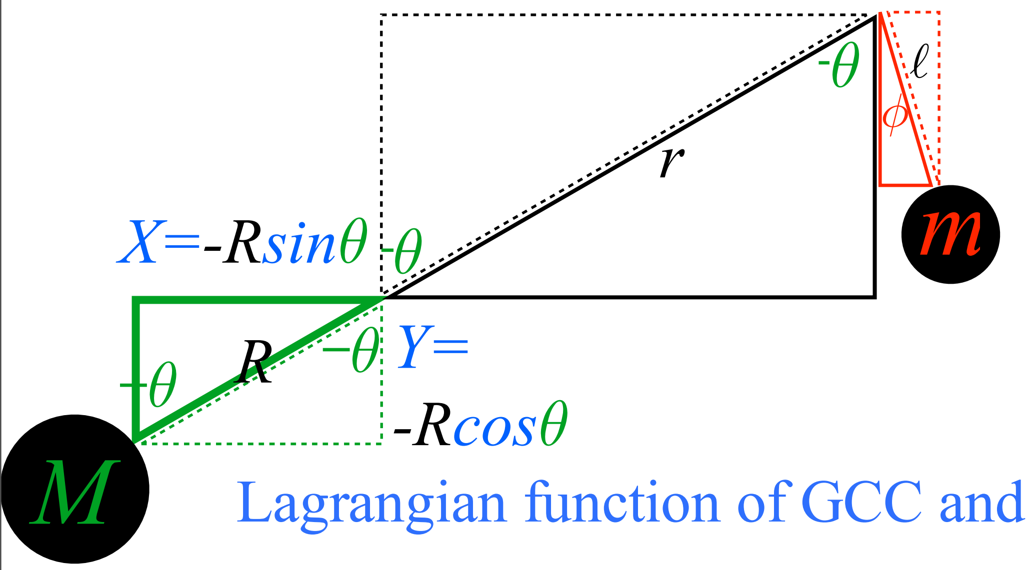
$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



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Lagrange equations

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(Consolidating)

$$\frac{d}{dt}\left(\underbrace{p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}\right) = -\frac{\partial L}{\partial t}$$

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$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

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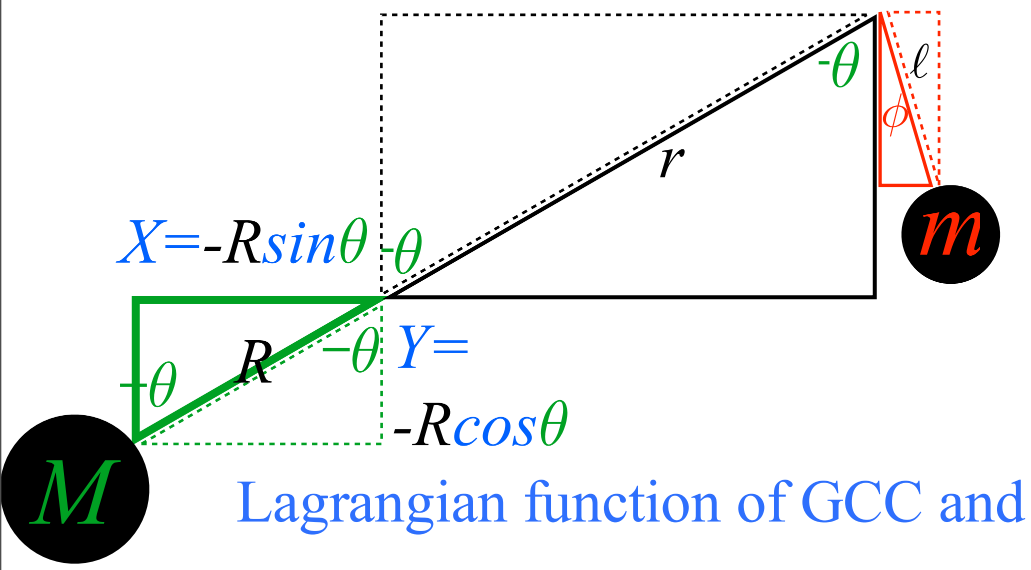
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial\theta} \underset{\uparrow}{=} -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} \underset{\uparrow}{=} \dot{\theta} \qquad \frac{\partial H}{\partial\dot{\theta}} \overset{\downarrow}{=} p_\theta - \frac{\partial L}{\partial\dot{\theta}} \underset{\uparrow}{=} 0$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



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velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and **momenta**: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

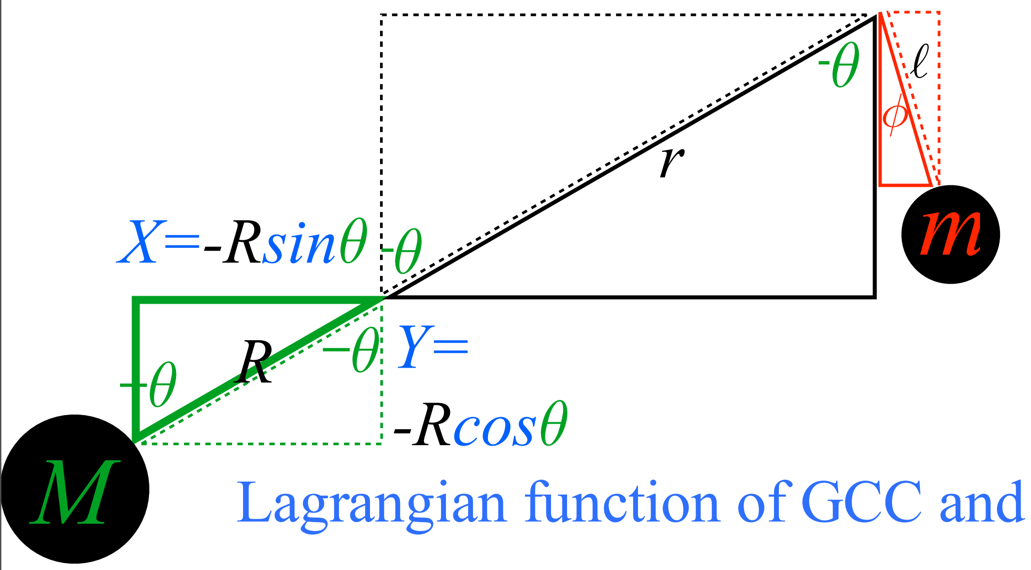
by assumed Lagrange functionality

$$\frac{\partial H}{\partial \theta} \overset{\downarrow}{\equiv} -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{\equiv} \dot{\theta} \qquad \cancel{\frac{\partial H}{\partial \dot{\theta}} \overset{\downarrow}{\equiv} 0} \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \cancel{\frac{\partial H}{\partial \dot{\phi}} \equiv 0}$$

Hamilton's equations

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

$$dL\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = \frac{\partial L}{\partial \theta}d\theta + \frac{\partial L}{\partial \phi}d\phi + \frac{\partial L}{\partial \dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

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Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

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(Rearranging)

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $\boxed{H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}$ *Poincare-Legendre relation*

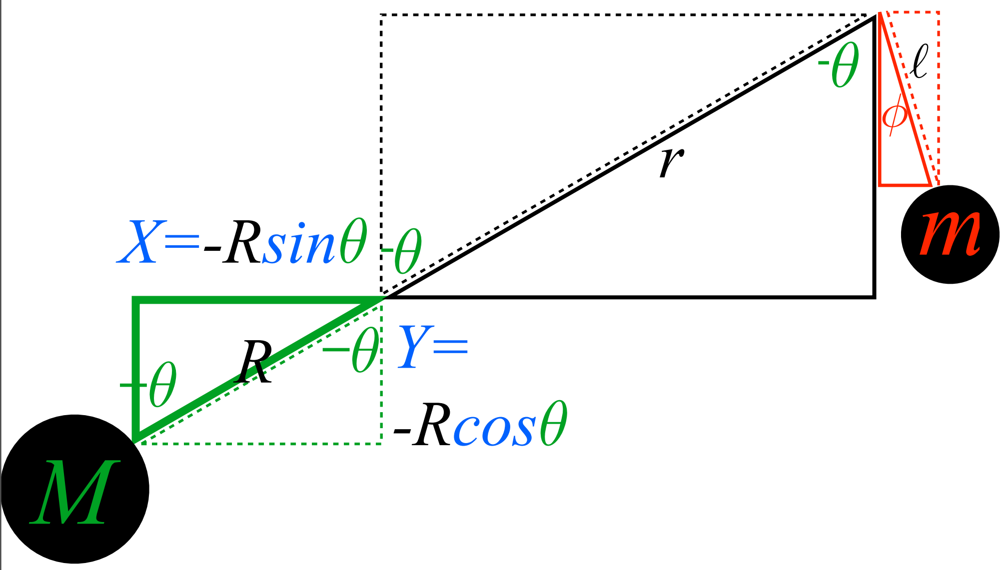
$$\boxed{\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \cancel{\frac{\partial H}{\partial \dot{\theta}} = 0} \quad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0}$$

Hamilton's equations

Review of Hamiltonian equation derivation (Elementary trebuchet)

- *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
- *Hamilton's equations and Poincare invariant relations*
- → *Hamiltonian expression and contravariant γ^{mn} tensor*

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

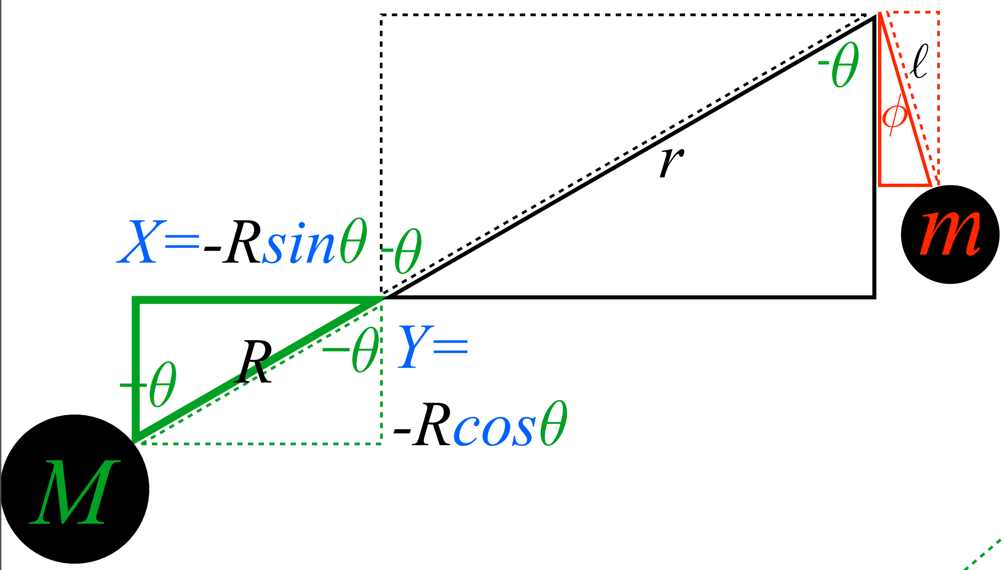
Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \qquad + \qquad p_\phi\dot{\phi} \qquad\qquad - T \qquad\qquad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



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Covariant metric tensor

γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

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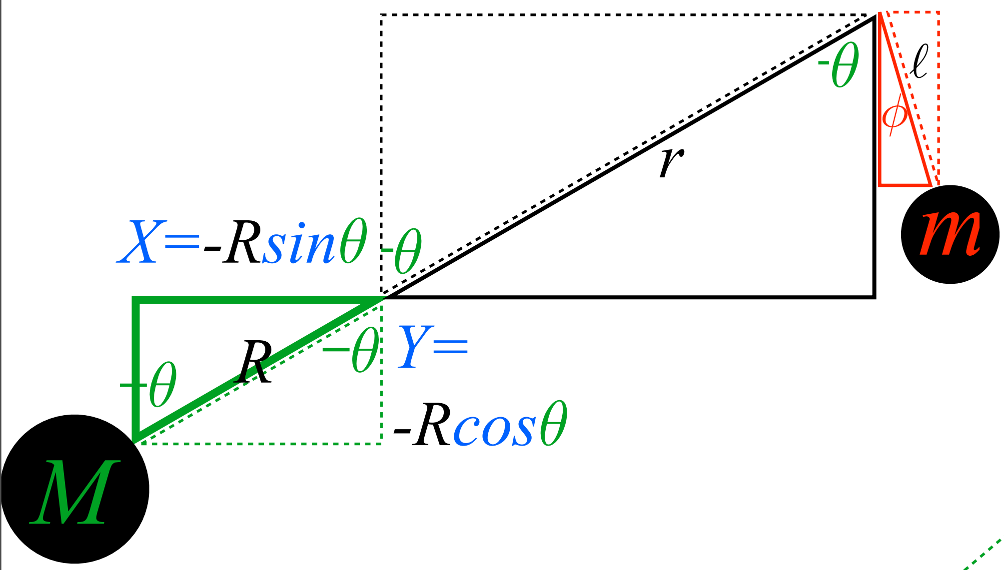
$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V$$

(Only correct numerically!)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Covariant metric tensor γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \underbrace{\begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi\end{pmatrix}\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2\end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

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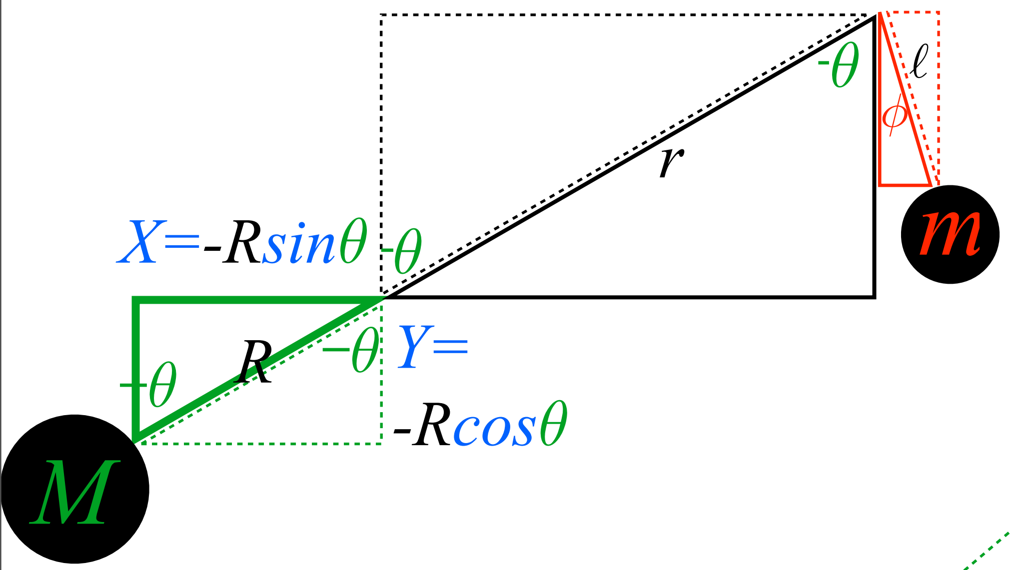
$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit
in momenta p_m

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix}\overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

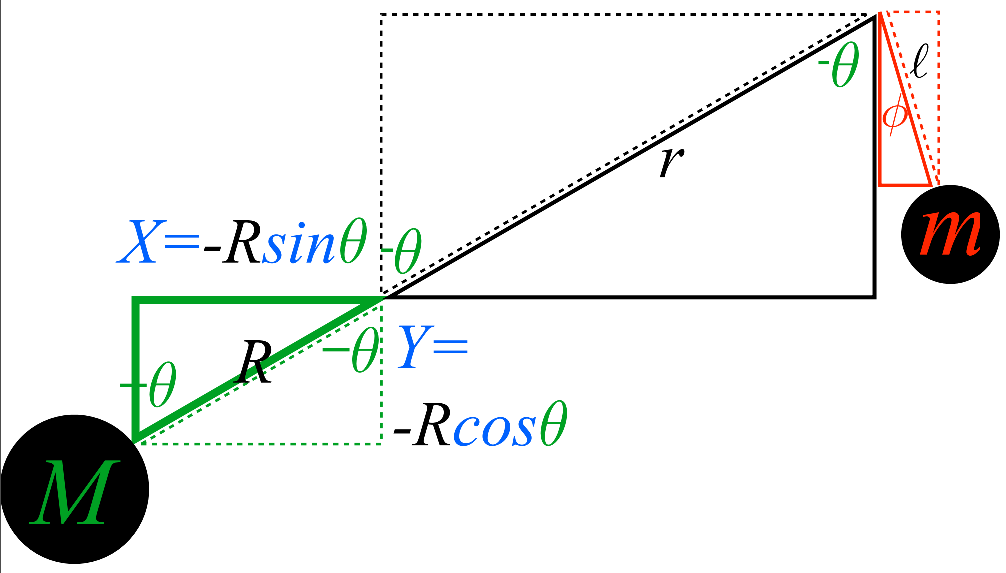
(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V = T + V$$

(Correct formally and numerically)

$$Total\ KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \quad - T \quad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

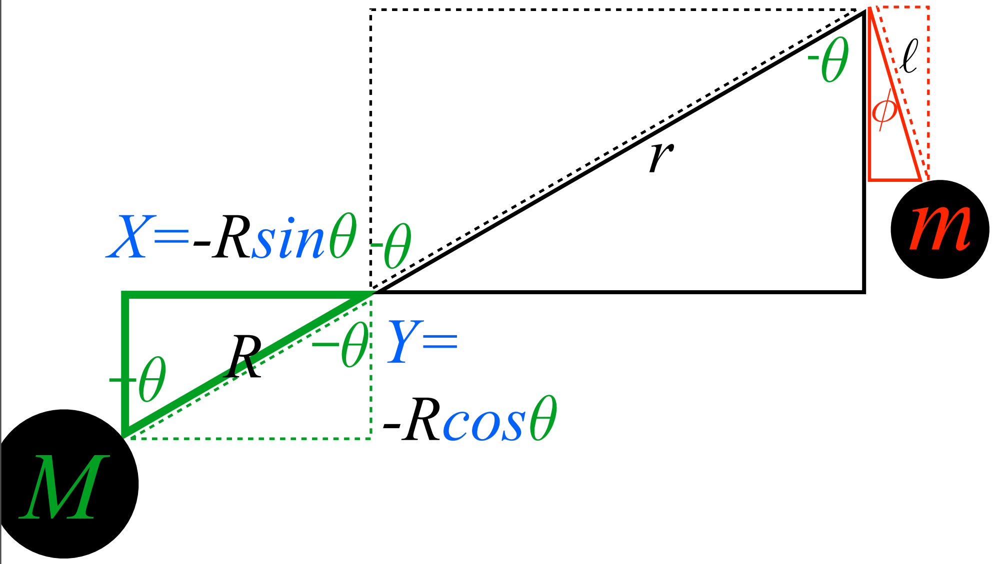
Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V \quad = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta-\phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot{\theta}=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

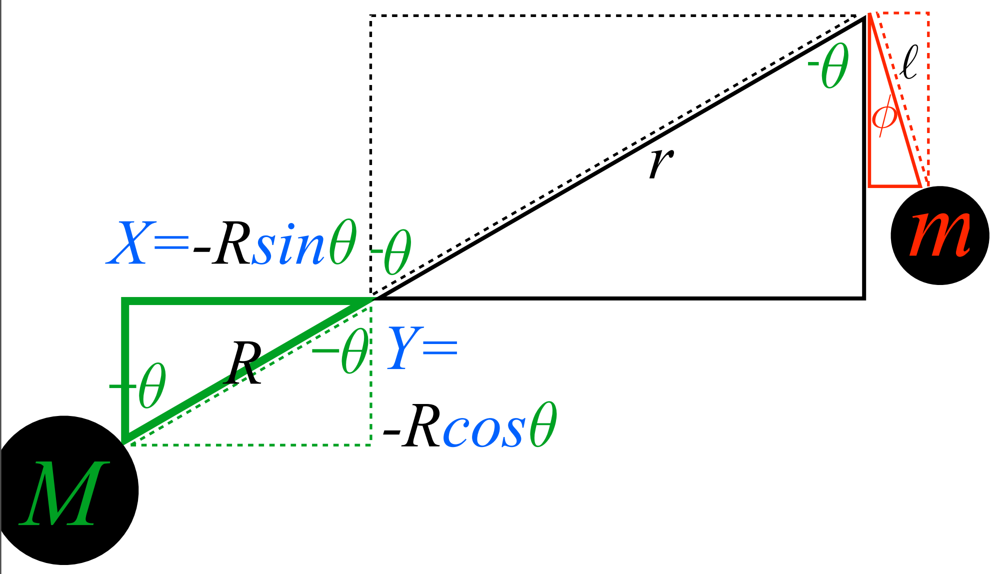
$$\dot{\phi}=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot\theta=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

$$\dot\phi=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

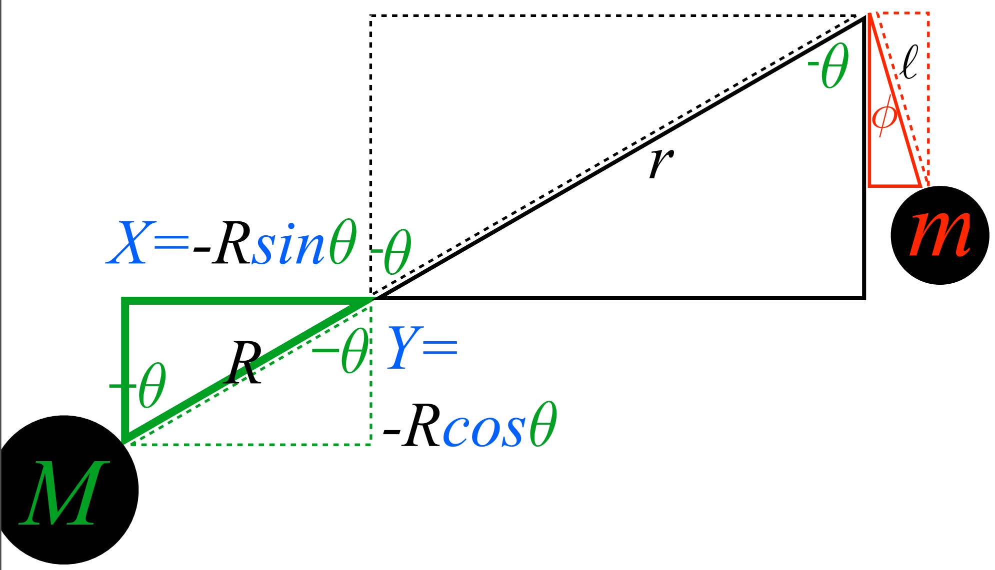
Momentum/force equations

$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$
$$=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$
$$=\ \ \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot\theta = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

$$\dot\phi = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{\partial L}{\partial\theta} = \frac{\partial T}{\partial\theta} - \frac{\partial V}{\partial\theta}$$

$$= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta$$

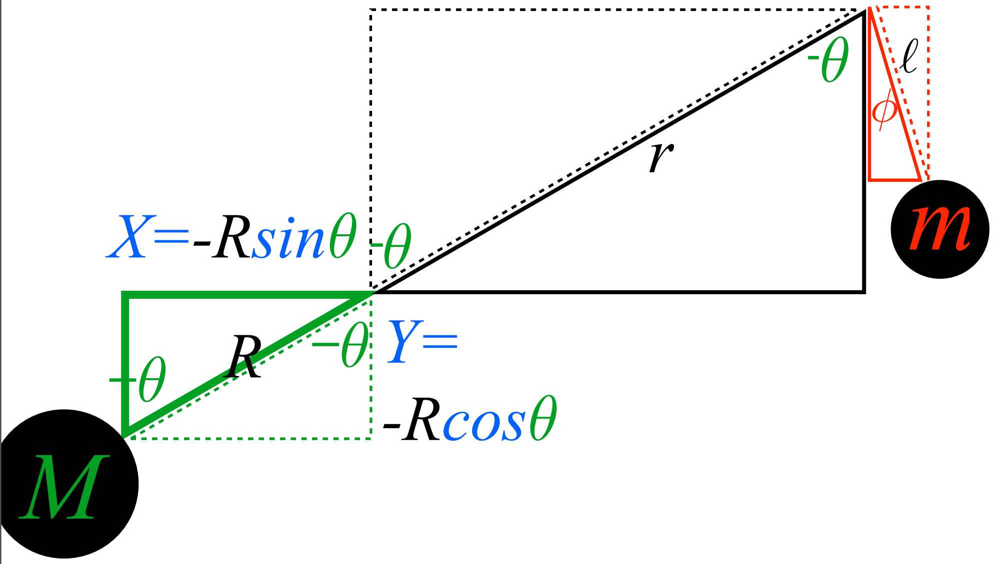
(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi = -\frac{\partial H}{\partial\phi} = \frac{\partial L}{\partial\phi} = \frac{\partial T}{\partial\phi} - \frac{\partial V}{\partial\phi}$$

$$= \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi$$

$$= \text{-}mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

$$\gamma^{mn}$$

$$T = \frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot\theta = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

$$\dot\phi = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Momentum/force equations

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{\partial L}{\partial\theta} = \frac{\partial T}{\partial\theta} - \frac{\partial V}{\partial\theta}$$

$$= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta$$

$$= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2]p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi) + F_\theta$$

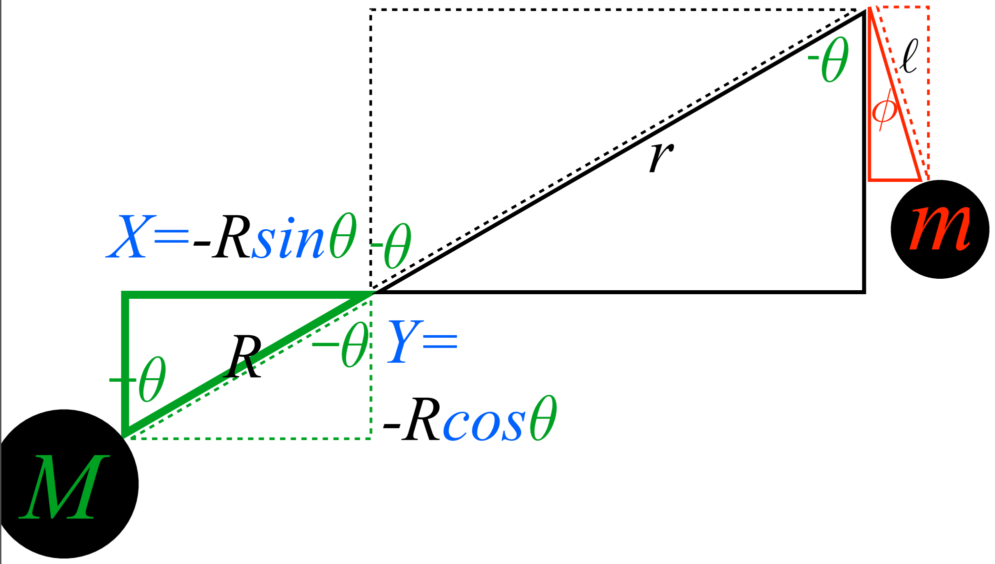
$$\dot p_\phi = -\frac{\partial H}{\partial\phi} = \frac{\partial L}{\partial\phi} = \frac{\partial T}{\partial\phi} - \frac{\partial V}{\partial\phi}$$

$$= -mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi$$

$$= -mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi$$

$$= -[\text{messy factor}]\sin(\theta-\phi) + F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

$$\dot\theta = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

$$\dot\phi = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Momentum/force equations

$$\dot p_\theta = -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta}$$
$$= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta$$
$$= mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\theta$$
$$= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2]p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi) + F_\theta$$

$$\dot p_\phi = -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi}$$
$$= \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi$$
$$= \text{-}mr\ell(\gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi)\sin(\theta-\phi) + F_\phi$$
$$= \text{-[messy factor]}\sin(\theta-\phi) + F_\phi$$

A lesson on Hamiltonian "elegance"...
...may be very elegant formally...but may <u>not</u> be so elegant computationally!

Hamiltonian energy and momentum conservation and symmetry coordinates

- → *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

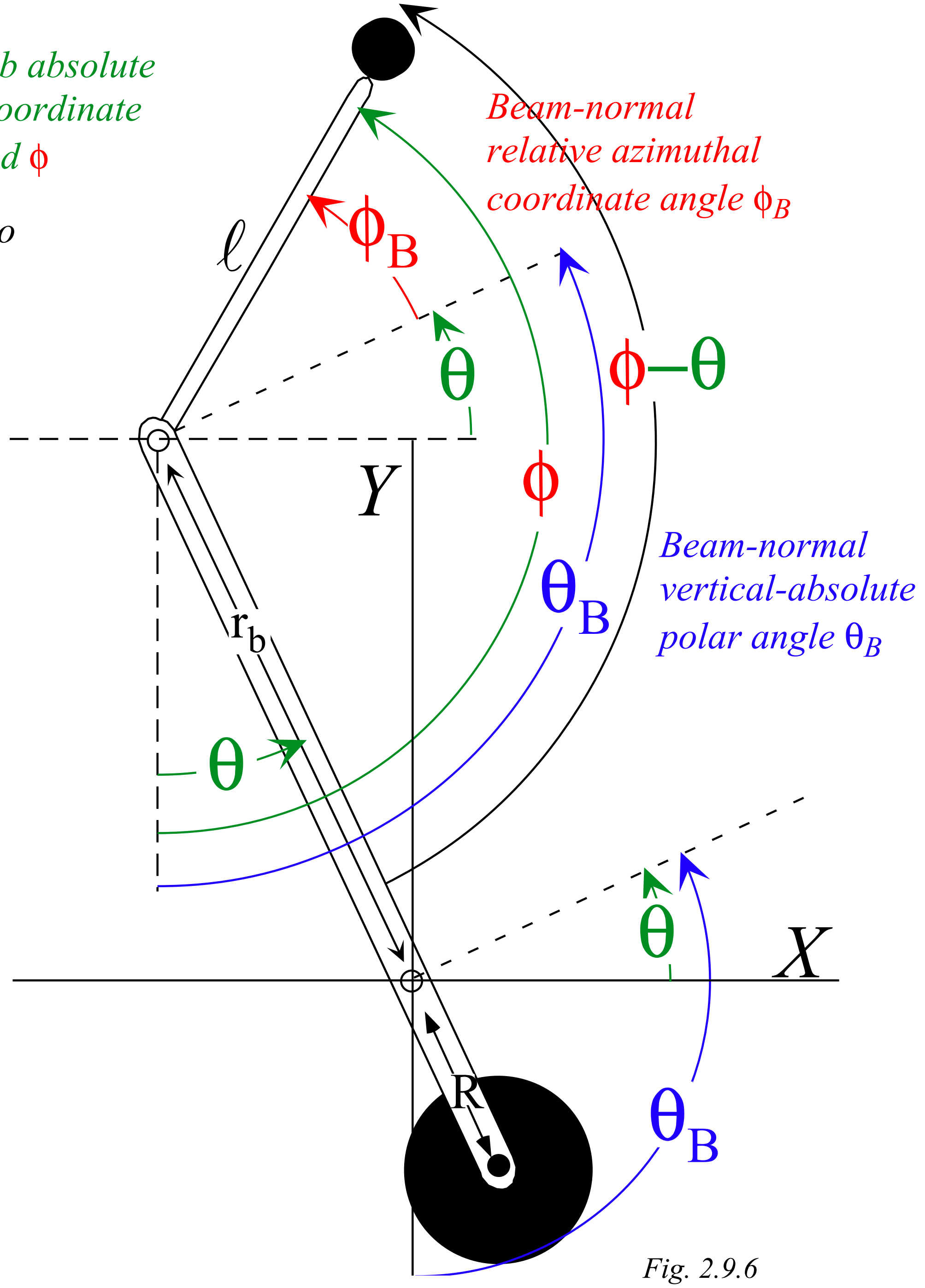


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \quad -\pi/2$

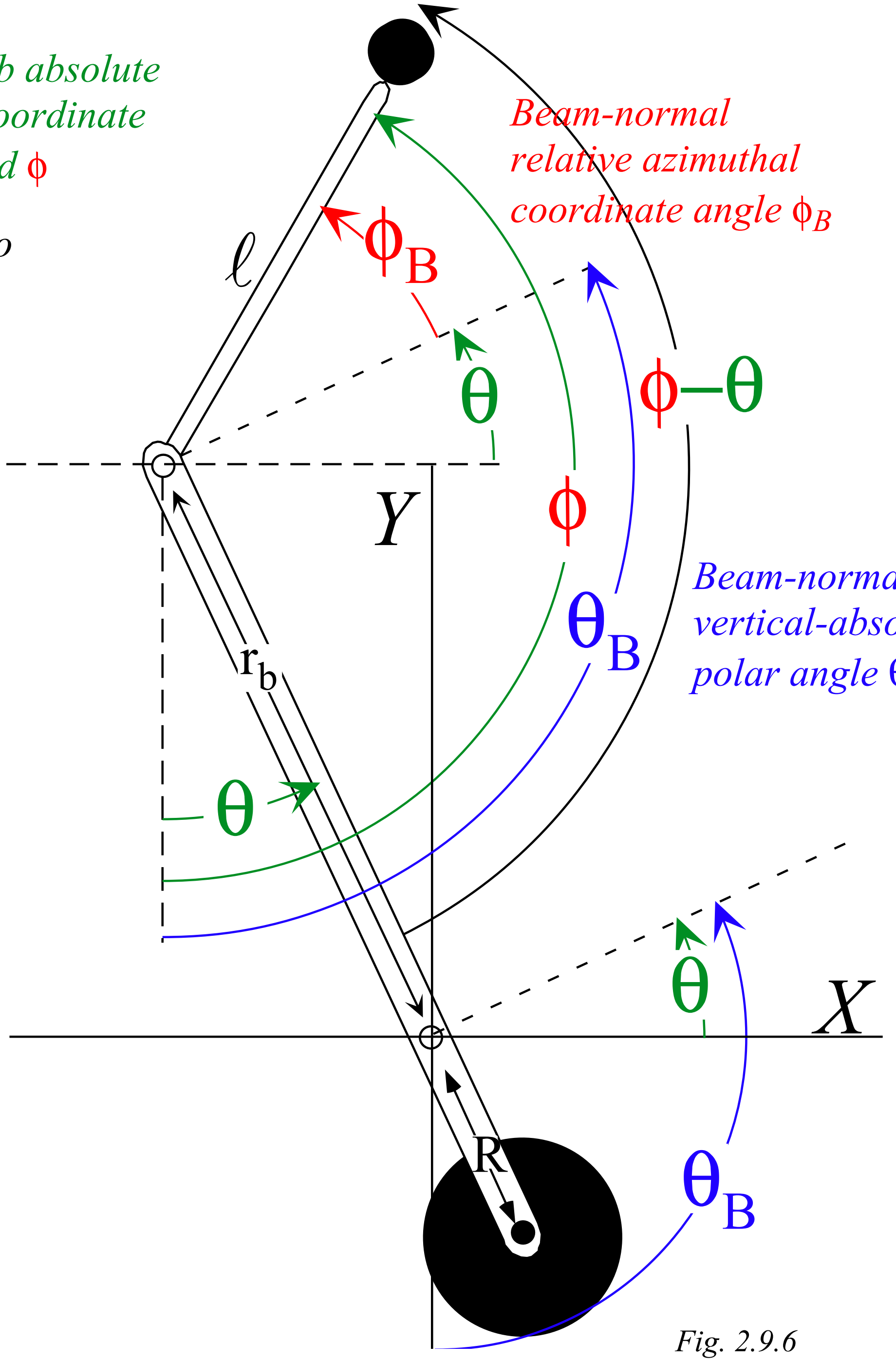
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \quad -\pi/2$

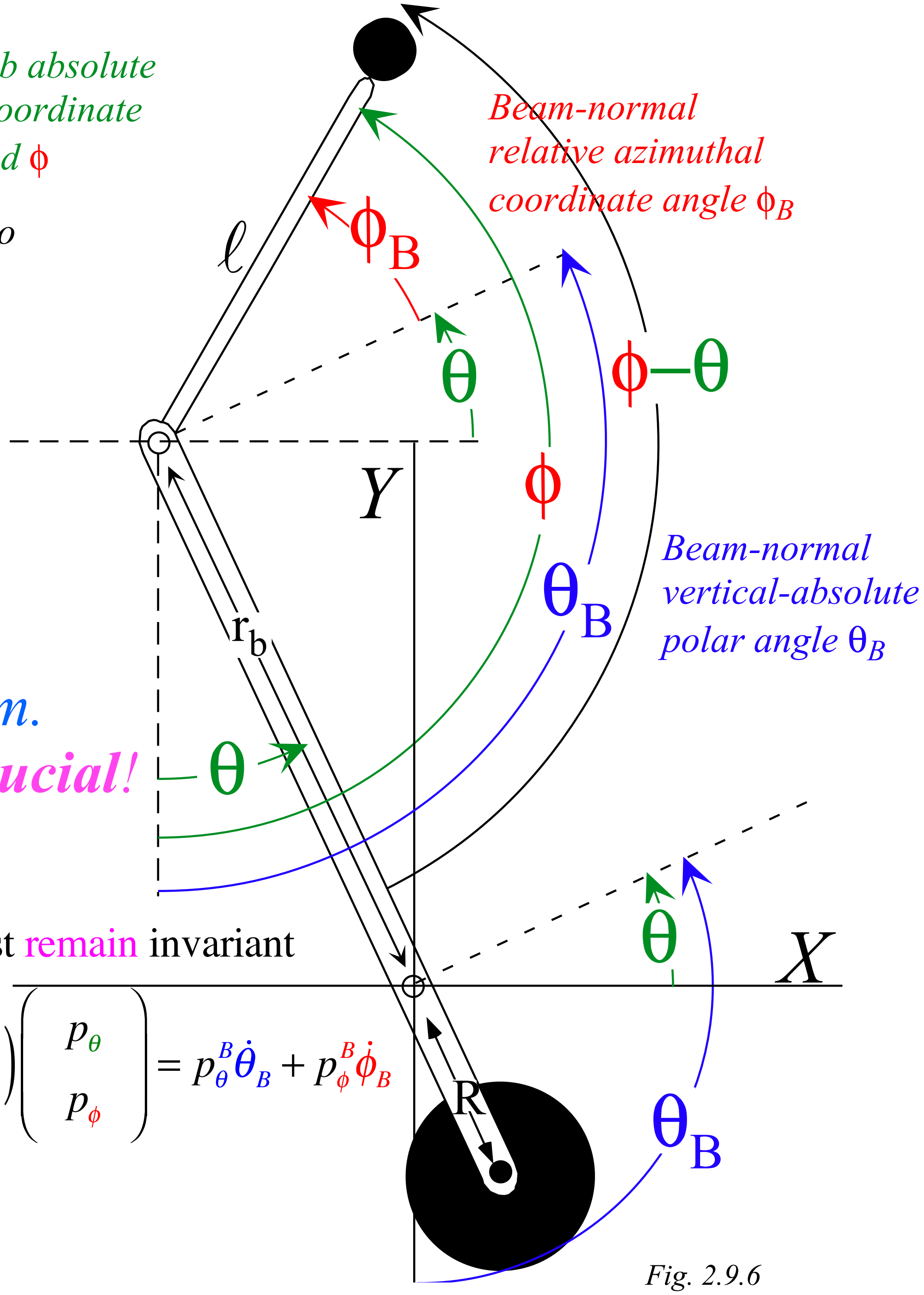
$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Be careful with momentum.
Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot{\theta} + p_\phi\dot{\phi} = \begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p_\theta^B\dot{\theta}_B + p_\phi^B\dot{\phi}_B$$

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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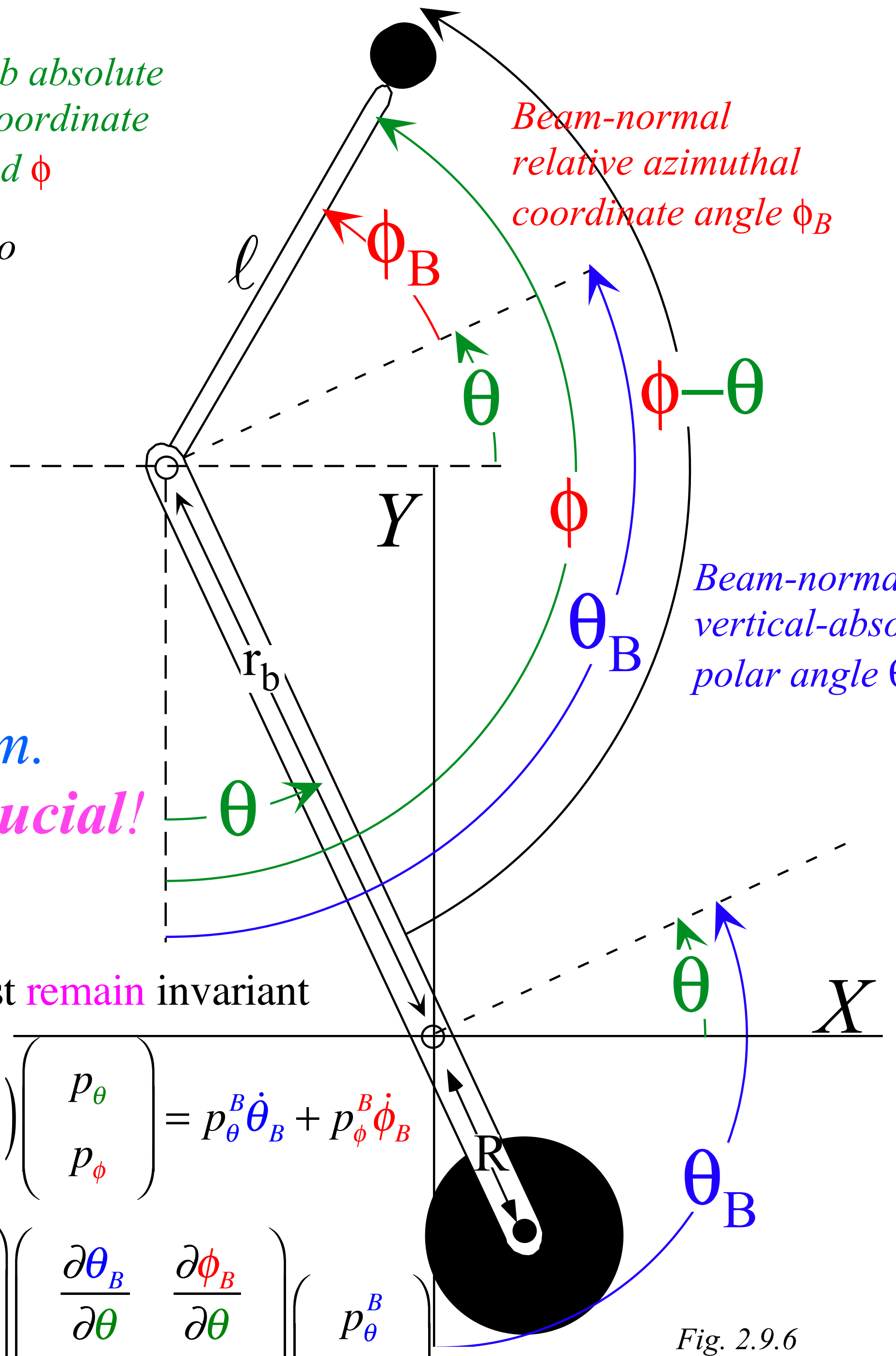
Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Be careful with momentum.
*Poincare invariance is **crucial**!*

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

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Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi B = \phi - \theta - \pi/2$ and $\theta B = \theta \quad +\pi/2$

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$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

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$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

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Previous lab absolute trebuchet coordinate angles θ and φ

compared to

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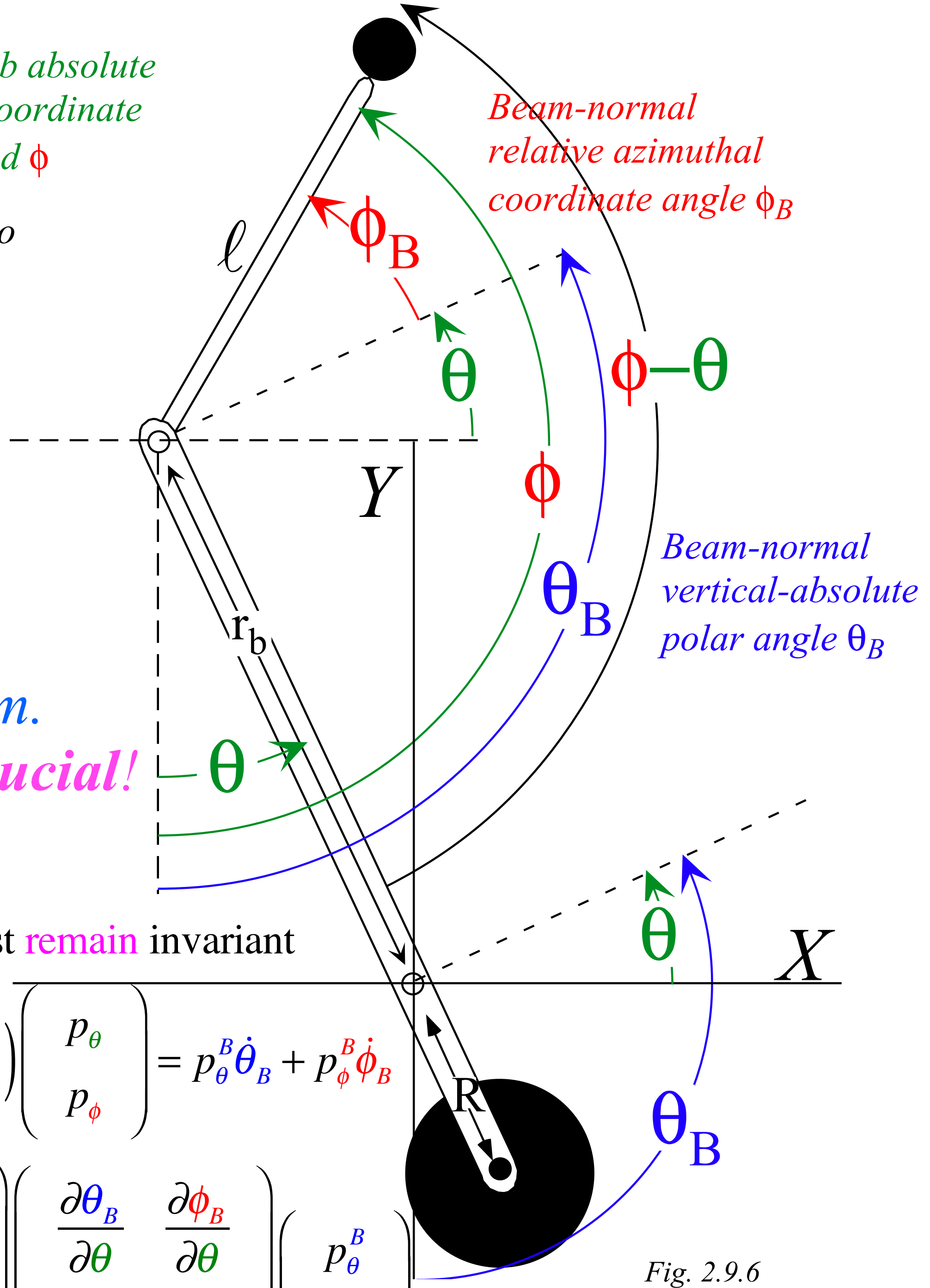

Beam-normal relative azimuthal coordinate angle ϕ_B

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p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1&1\\ 0&1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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Be careful with momentum.
Poincare invariance is ***crucial!***

Previous lab absolute trebuchet coordinate angles θ and φ

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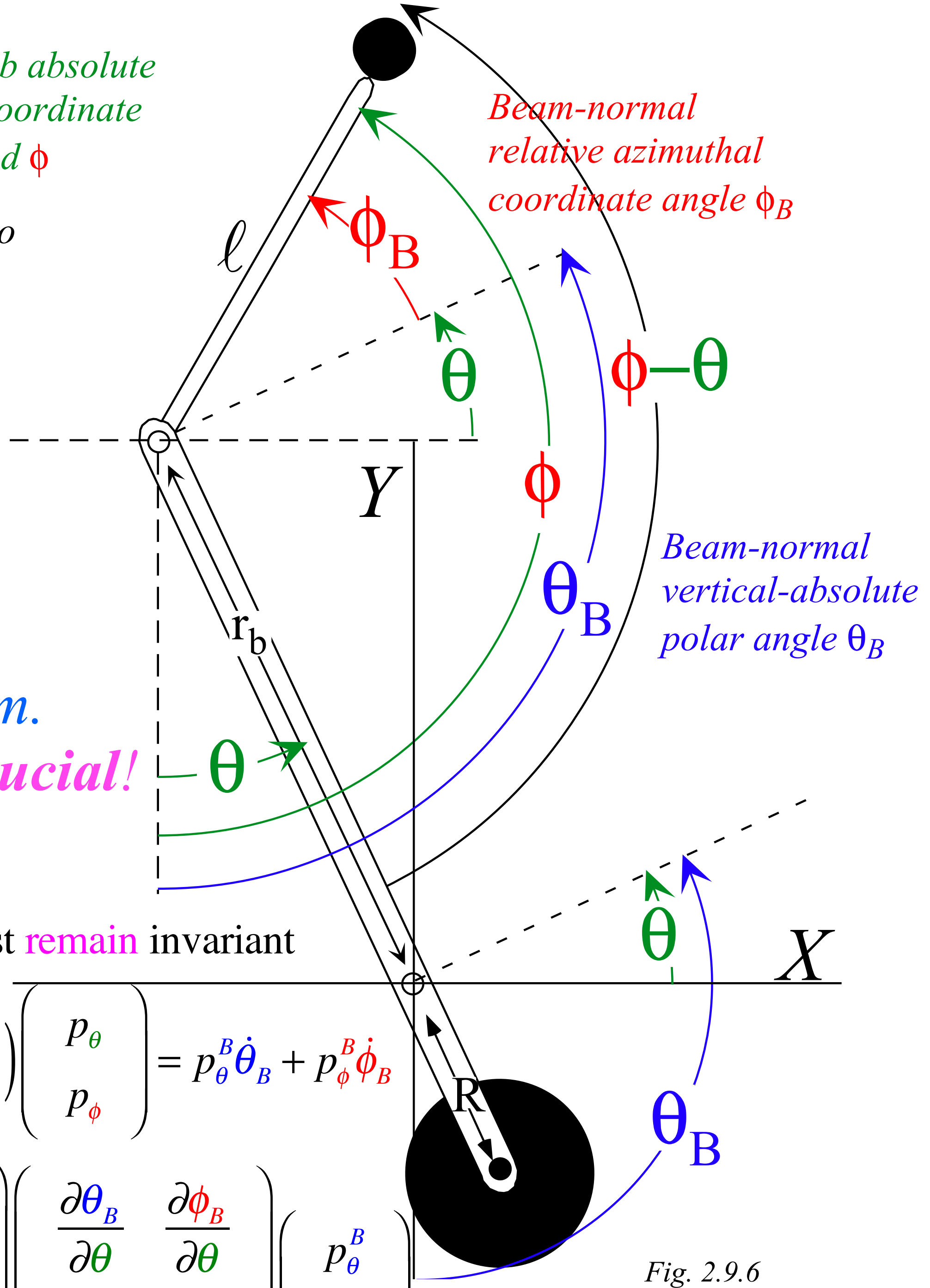

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

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Original (φ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi B = \phi - \theta - \pi/2$ and $\theta B = \theta$ $+\pi/2$

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$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

p_m transform is **TRANSPOSE INVERSE** *to q^m*

$$\begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

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$p_\phi = p^B_\phi$

Previous lab absolute trebuchet coordinate angles θ and φ

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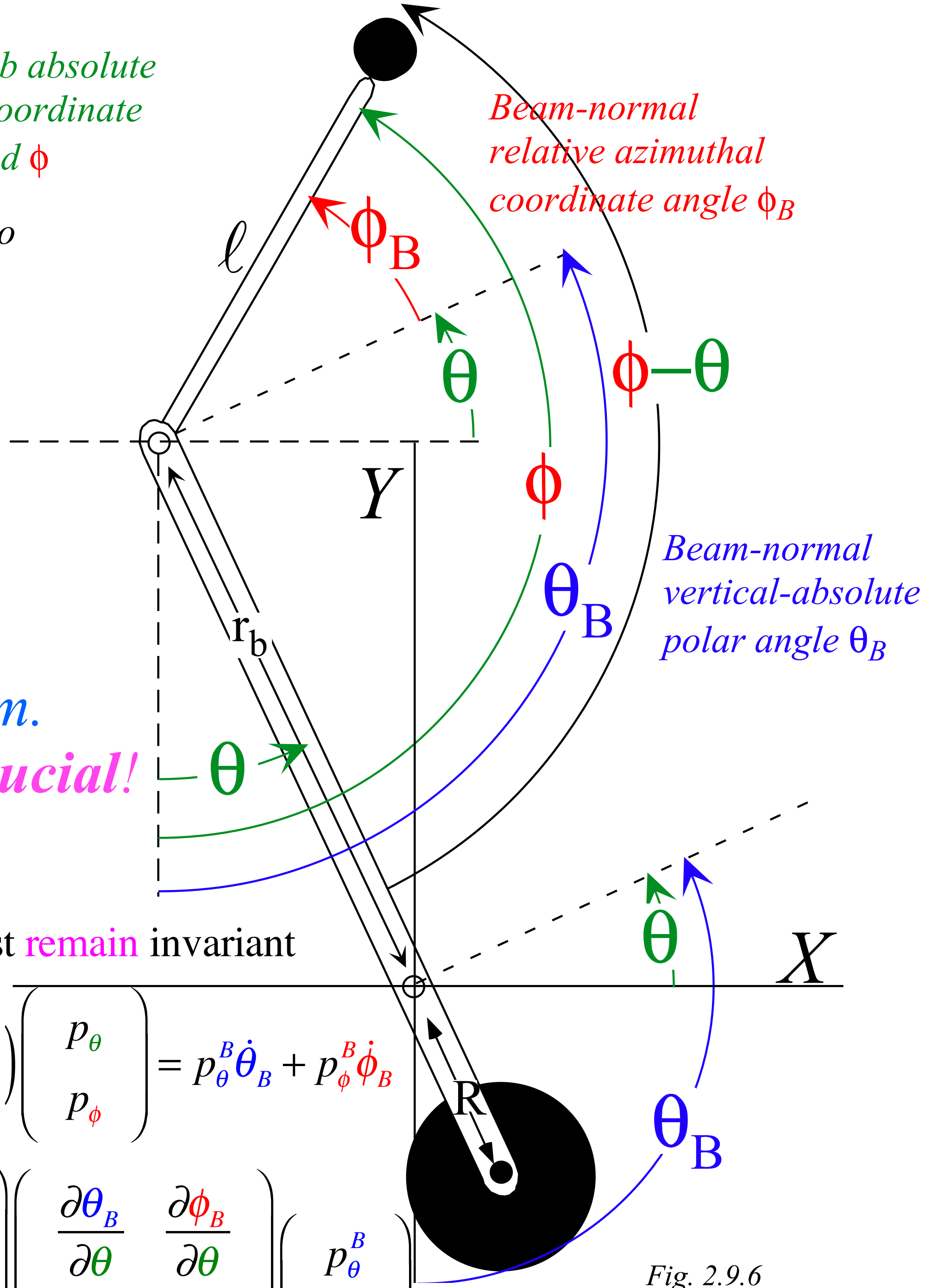

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Be careful with momentum.
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$$\begin{pmatrix} \dot\theta_B & \dot\phi_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p^B_\theta \\ p^B_\phi \end{pmatrix}$$

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

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Original (φ,θ) Hamiltonian

$$H = \frac{m\ell^2\left(p^B_\theta - p^B_\phi\right)^2 + \left(MR^2 + mr^2\right)\left(p^B_\phi\right)^2 - 2mr\ell p^B_\phi\left(p^B_\theta - p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} + V$$

Transformed (ϕ_B,θ_B) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi B = \phi - \theta - \pi/2$ and $\theta B = \theta \quad +\pi/2$

Jacobian: $\qquad \phi B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \theta_B}{\partial \phi} \\ \frac{\partial \phi_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi B = \phi - \theta - \pi/2$ and $\theta = \theta B \quad -\pi/2$

$$\phi = \theta B + \phi B$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \theta}{\partial \phi_B} \\ \frac{\partial \phi}{\partial \theta_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE to q^m

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p_\theta^B - p_\phi^B$

$$p_\phi = \quad p_\phi^B$$

Be careful with momentum.
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Previous lab absolute trebuchet coordinate angles θ and ϕ

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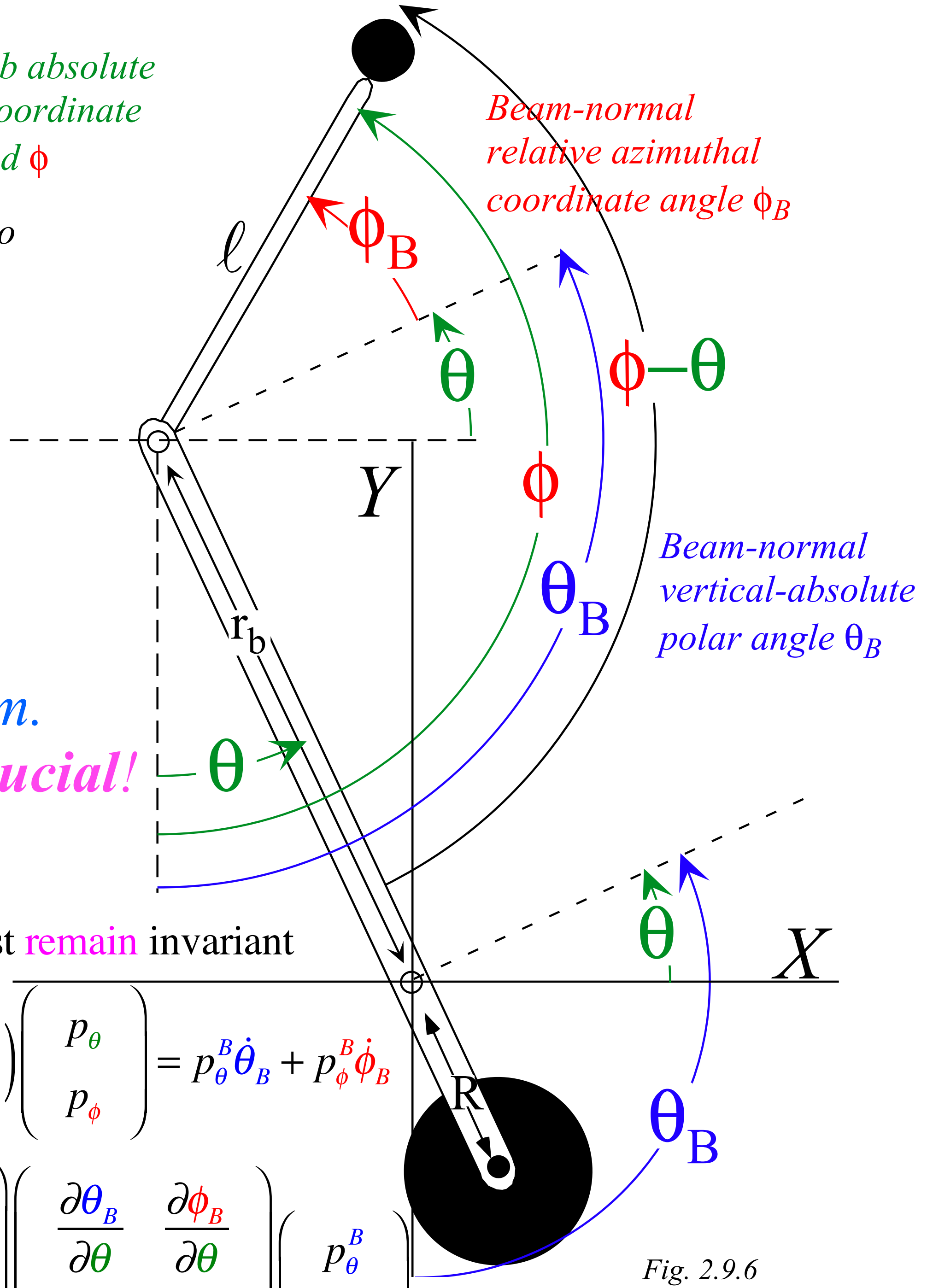

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Poincare invariant must remain invariant

$$p_\theta \dot{\theta} + p_\phi \dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B \dot{\theta}_B + p_\phi^B \dot{\phi}_B$$

$$\begin{pmatrix} \dot{\theta}_B & \dot{\phi}_B \end{pmatrix} \begin{pmatrix} \frac{\partial \theta}{\partial \theta_B} & \frac{\partial \phi}{\partial \theta_B} \\ \frac{\partial \theta}{\partial \phi_B} & \frac{\partial \phi}{\partial \phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial \theta_B}{\partial \theta} & \frac{\partial \phi_B}{\partial \theta} \\ \frac{\partial \theta_B}{\partial \phi} & \frac{\partial \phi_B}{\partial \phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2) p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2 \left[MR^2 + mr^2 \sin^2(\theta - \phi) \right]} + (MR - mr) g \cos\theta + mg\ell \cos\phi$$

$$F_\theta = -MgR \sin\theta + mgr \sin\theta$$
$$F_\phi = -mg\ell \sin\phi$$

$$H = \frac{m\ell^2 \left(p_\theta^B - p_\phi^B \right)^2 + \left(MR^2 + mr^2 \right) \left(p_\phi^B \right)^2 - 2mr\ell p_\phi^B \left(p_\theta^B - p_\phi^B \right) \sin\phi_B}{m\ell^2 \left[MR^2 + mr^2 \cos^2 \phi_B \right]} - \left(MR - mr \right) g \sin\theta_B \; - mg\ell \cos\left(\phi_B + \theta_B \right)$$

Hamiltonian energy and momentum conservation and symmetry coordinates
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - → *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not a function of t so : H=const.=E

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+I\right)\left(p_\phi^B\right)^2+2\Lambda\left(mr\ell\sin\phi_B-m\ell^2\right)p_\phi^B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]-m\ell^2\Lambda^2$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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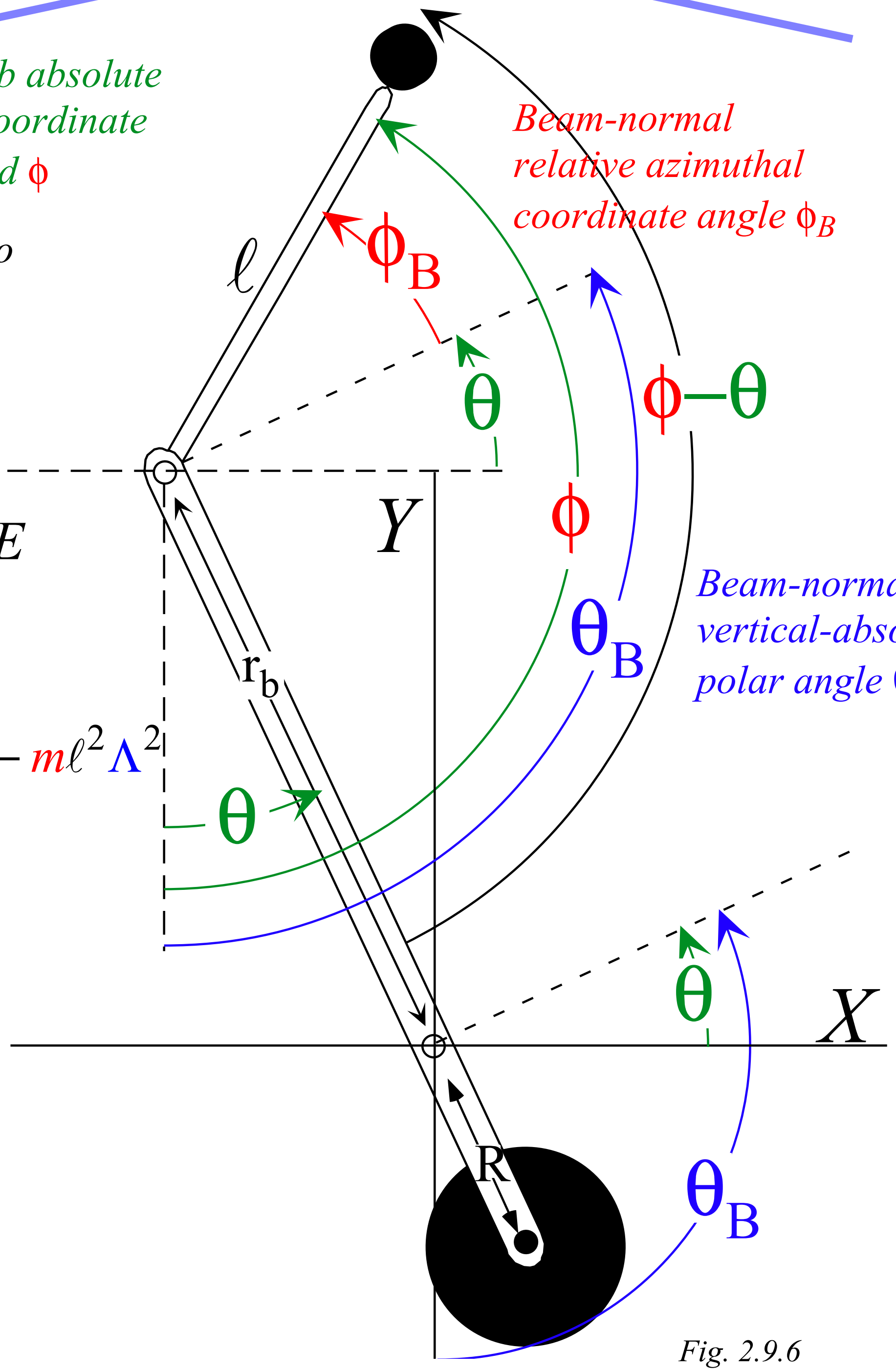

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(using last line of p. 36)

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$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} \xcancel{-\left(MR - mr\right)g\sin\theta_B - mg\ell\cos\left(\phi_B + \theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B = \frac{\partial H}{\partial \theta_B} = 0$ and : $p_\theta^B = \Lambda = const.$

H is not a function of t so : H=const.=E

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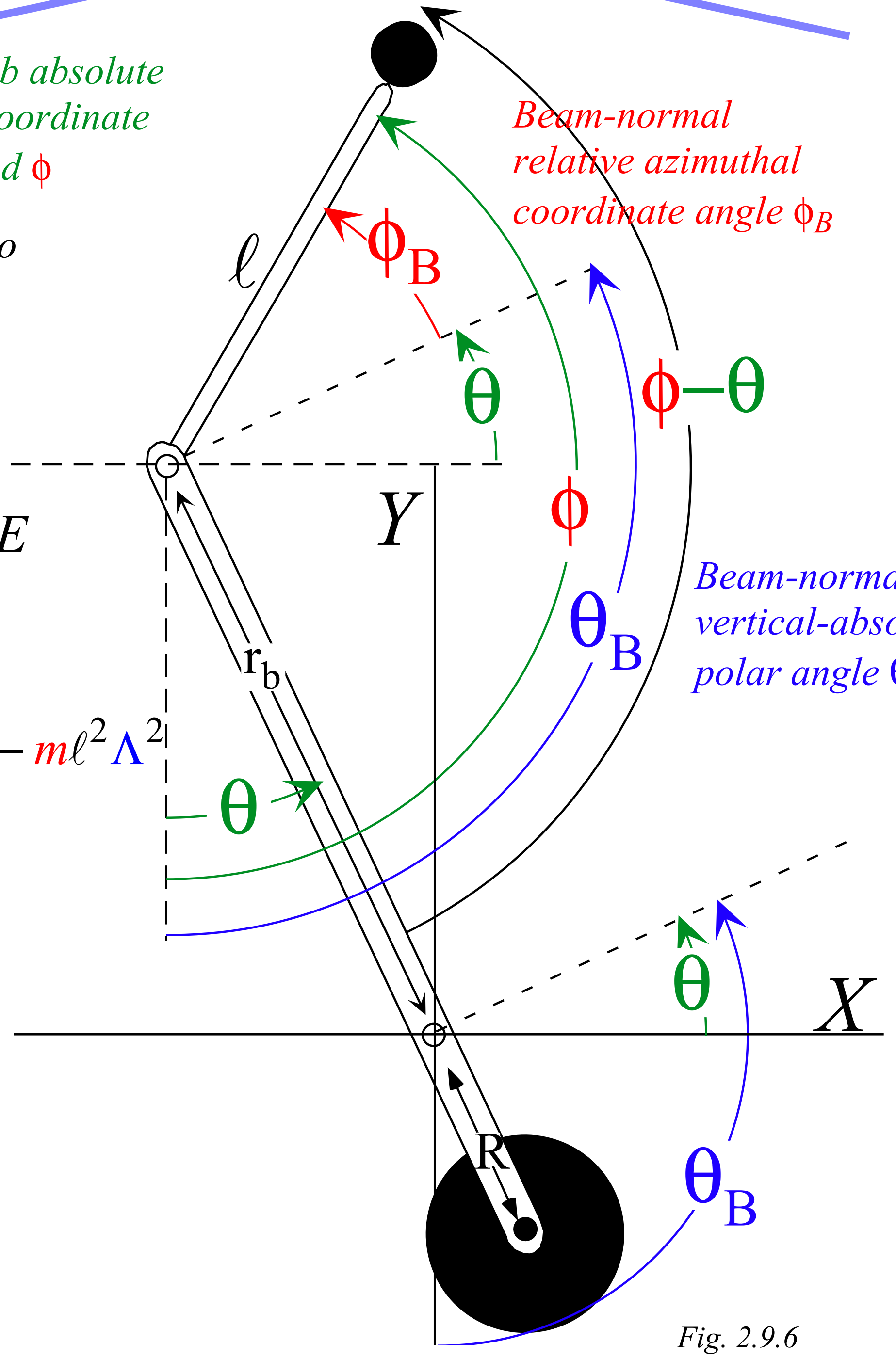
$$\left(m\ell^2 + 2mr\ell\sin\phi_B + I\right)\left(p_\phi^B\right)^2 + 2\Lambda\left(mr\ell\sin\phi_B - m\ell^2\right)p_\phi^B = Em\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right] - m\ell^2\Lambda^2$$

Divide thru by $m\ell^2$ and define I by: $MR^2 = I - mr^2$

$$\left(1 - 2\frac{r}{\ell}\sin\phi_B + J\right)\left(p_\phi^B\right)^2 + 2\Lambda\left(\frac{r}{\ell}\sin\phi_B - 1\right)p_\phi^B + \Lambda^2 - E\left[I - mr^2\sin^2\phi_B\right] = 0$$

where: $I = MR^2 + mr^2 = Jm\ell^2$ is defining J after dividing thru by $m\ell^2$



Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not a function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+I\right)\left(p_\phi^B\right)^2+2\Lambda\left(mr\ell\sin\phi_B-m\ell^2\right)p_\phi^B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]-m\ell^2\Lambda^2$$

Divide thru by $m\ell^2$ and define I by: $MR^2=I-mr^2$

$$\left(1-2\frac{r}{\ell}\sin\phi_B+J\right)\left(p_\phi^B\right)^2+2\Lambda\left(\frac{r}{\ell}\sin\phi_B-1\right)p_\phi^B+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

where: $I=MR^2+mr^2=Jm\ell^2$ is defining J after dividing thru by $m\ell^2$

Throwing-momentum p_ϕ^B is a function of beam-relative angle ϕ_B

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

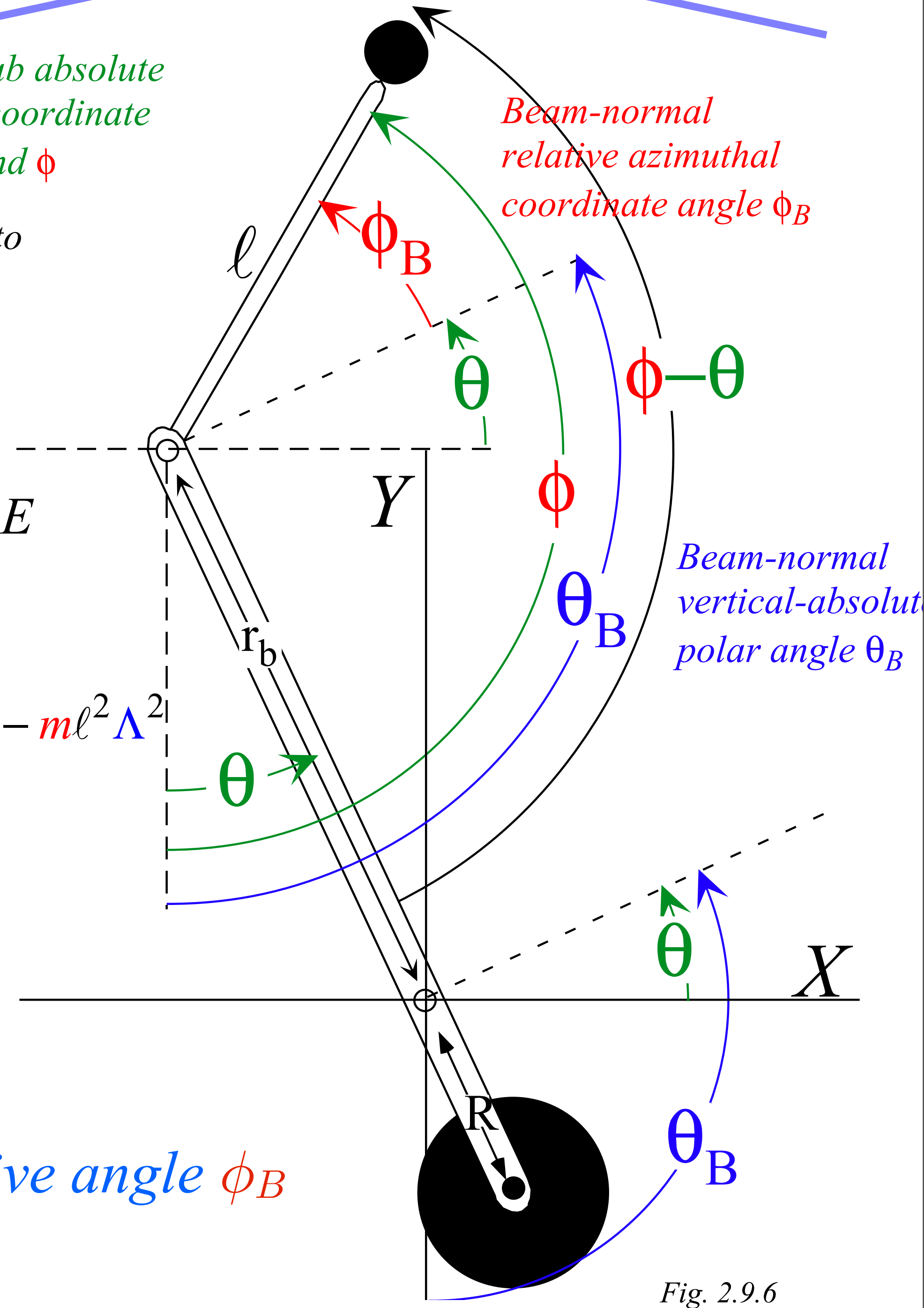


Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not a function of t so : H=const.=E

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+I\right)\left(p_\phi^B\right)^2+2\Lambda\left(mr\ell\sin\phi_B-m\ell^2\right)p_\phi^B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]-m\ell^2\Lambda^2$$

Divide thru by $m\ell^2$ and define I by: $MR^2=I-mr^2$

(to get function of $\sin^2\phi_B$)

$$\left(1-2\frac{r}{\ell}\sin\phi_B+J\right)\left(p_\phi^B\right)^2+2\Lambda\left(\frac{r}{\ell}\sin\phi_B-1\right)p_\phi^B+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

where: $I=MR^2+mr^2=Jm\ell^2$ is defining J after dividing thru by $m\ell^2$

Throwing-momentum p_ϕ^B *is a function of beam-relative angle* ϕ_B

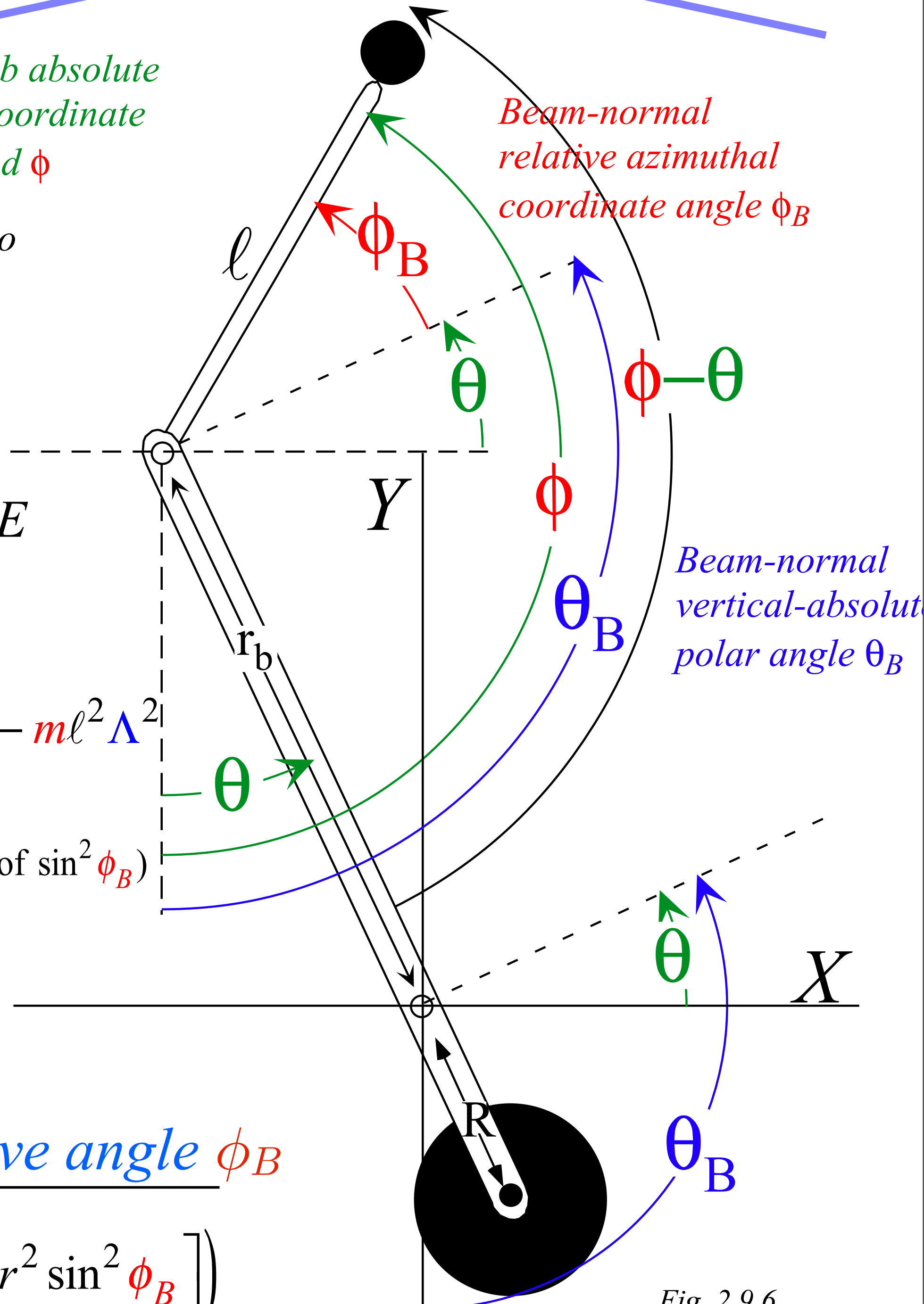
$$p_\phi^B=\frac{2\Lambda\left(1-\frac{r}{\ell}\sin\phi_B\right)\pm\sqrt{4\Lambda^2\left(1-\frac{r}{\ell}\sin\phi_B\right)^2-4\left(1-2\frac{r}{\ell}\sin\phi_B+J\right)\left(\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]\right)}}{2\left(1-2\frac{r}{\ell}\sin\phi_B+J\right)}$$

(using quadratic solution: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B *and* ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θB,φB) relative coordinates for trebuchet.

(Each value is positive.)

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - → *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

Transform to beam-relative coordinates and momenta

$\theta = \theta_B \quad -\pi/2$

$\phi = \theta_B + \phi_B$

$\theta_B = \theta \quad +\pi/2$

$\phi_B = -\theta + \phi - \pi/2$

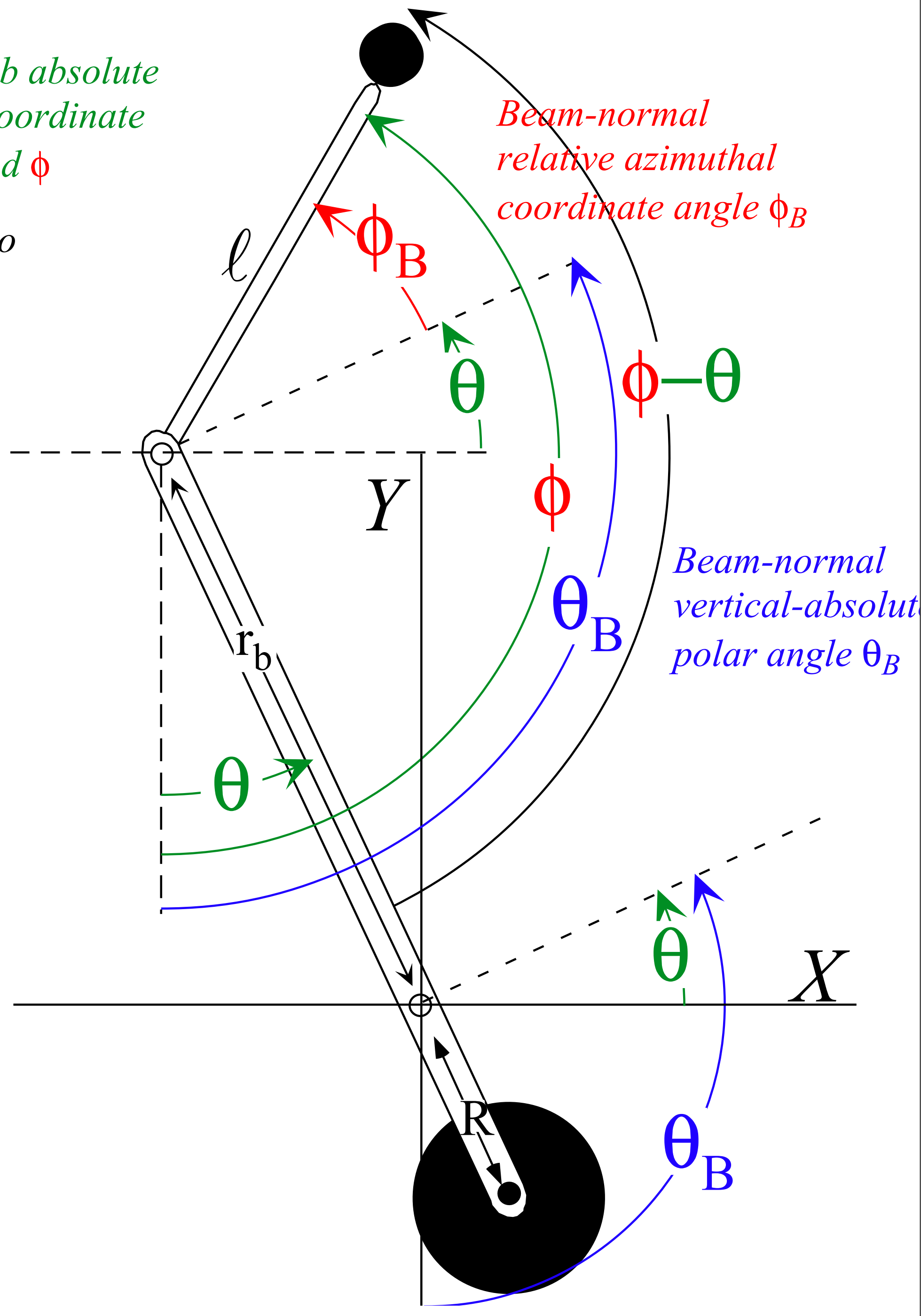
$p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$

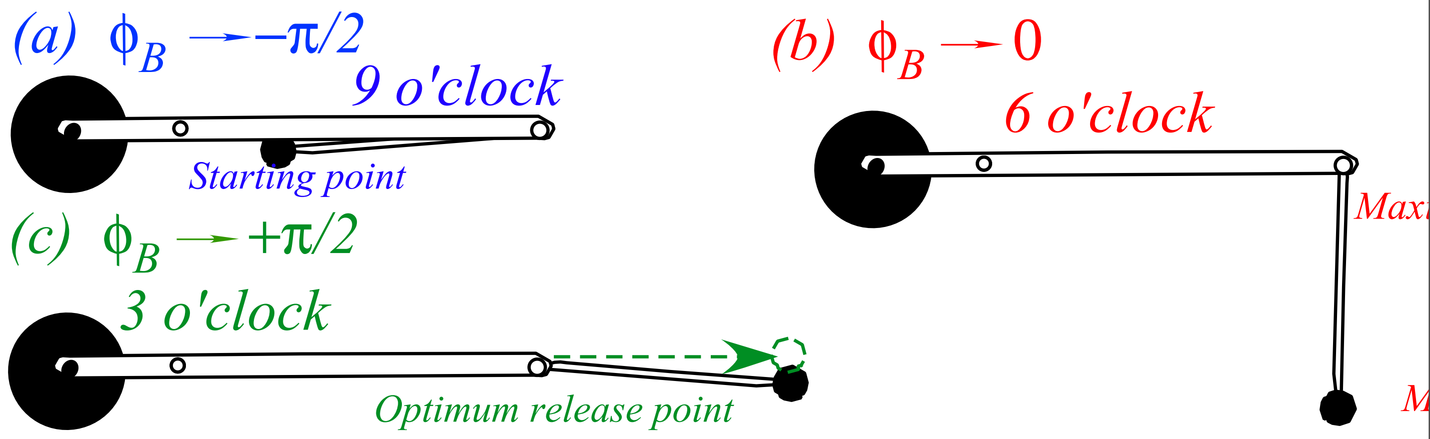
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

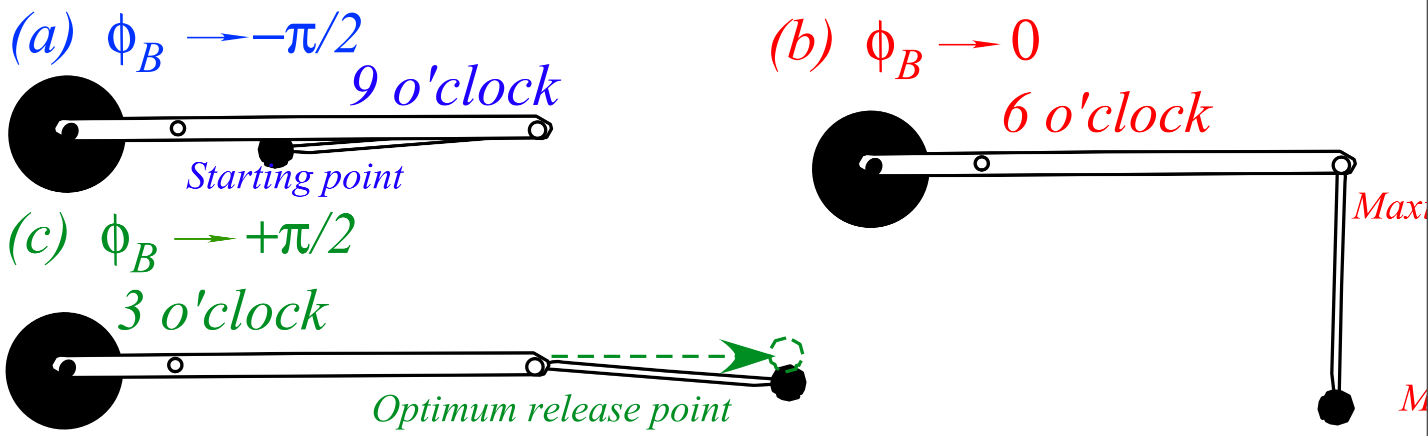
$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$



Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

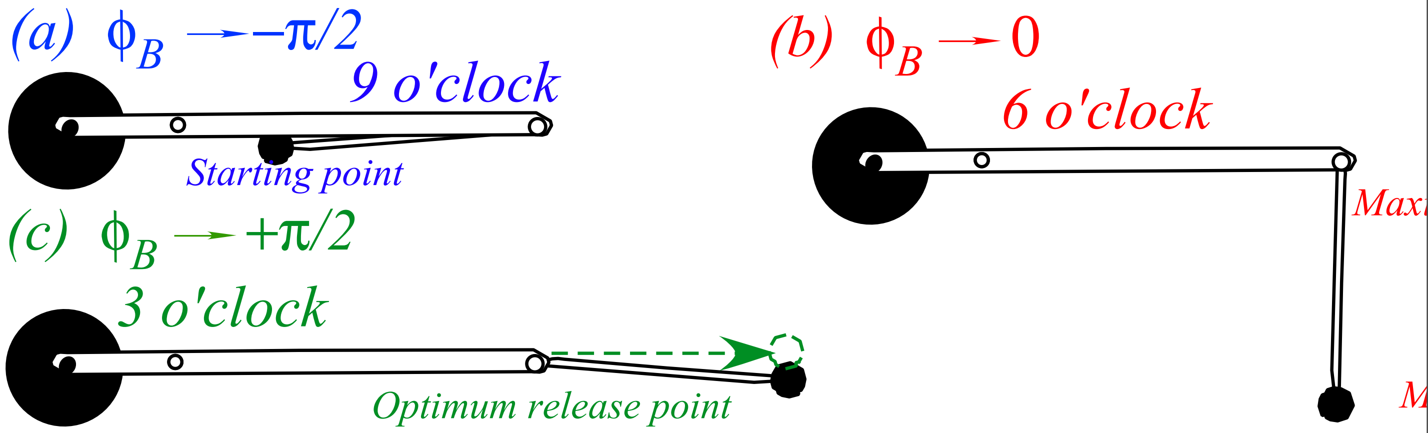
$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r = \ell\text{)}$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or : \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\left\{\begin{aligned} 2E &= MR^2\dot{\theta}_{-\pi/2}^2+mr^2\left(\dot{\phi}_{-\pi/2}-\dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or:\left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2}=\omega=\dot{\phi}_{-\pi/2}$$

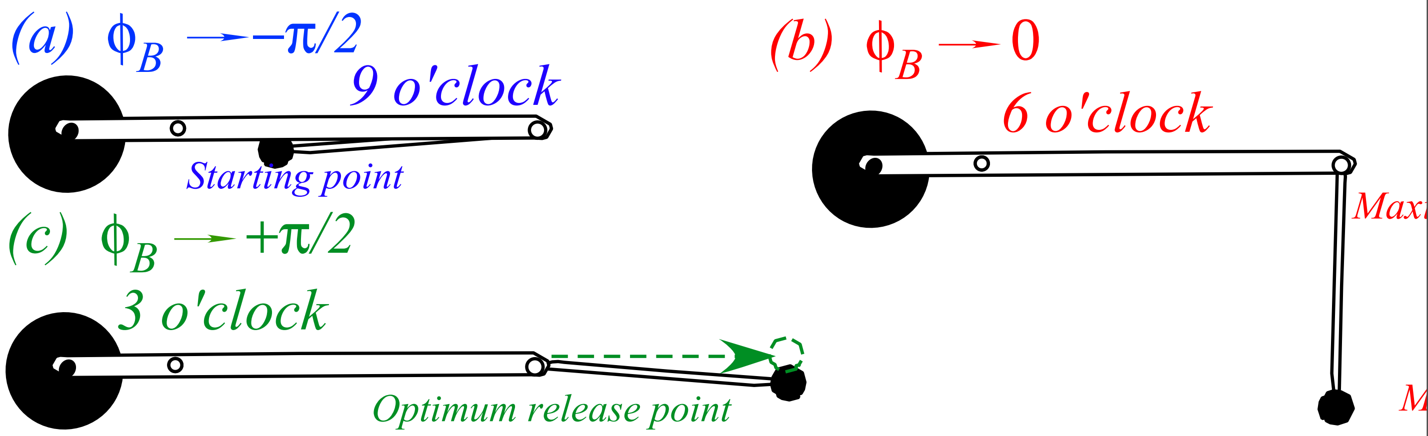
Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2+mr^2\left(\dot{\phi}_0^2+\dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0+mr^2\left(\dot{\phi}_0+\dot{\theta}_0\right) \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or : \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

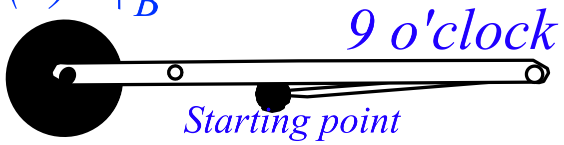
$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

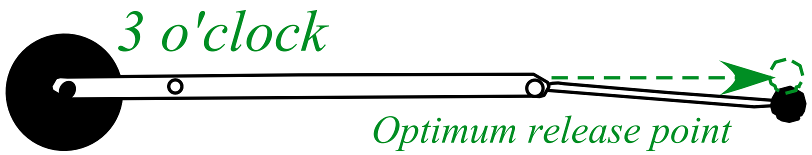
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r=\ell)$$

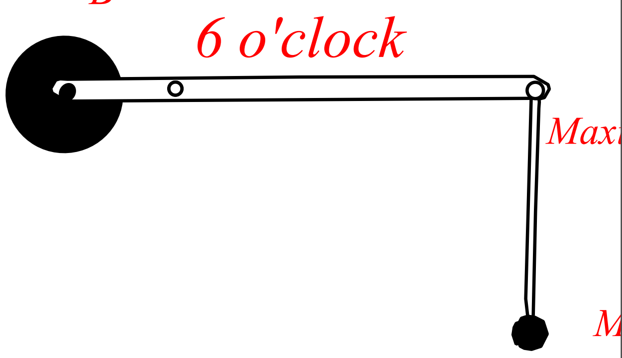
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or : \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 &= MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) &= MR^2\omega \end{aligned}\right.$$

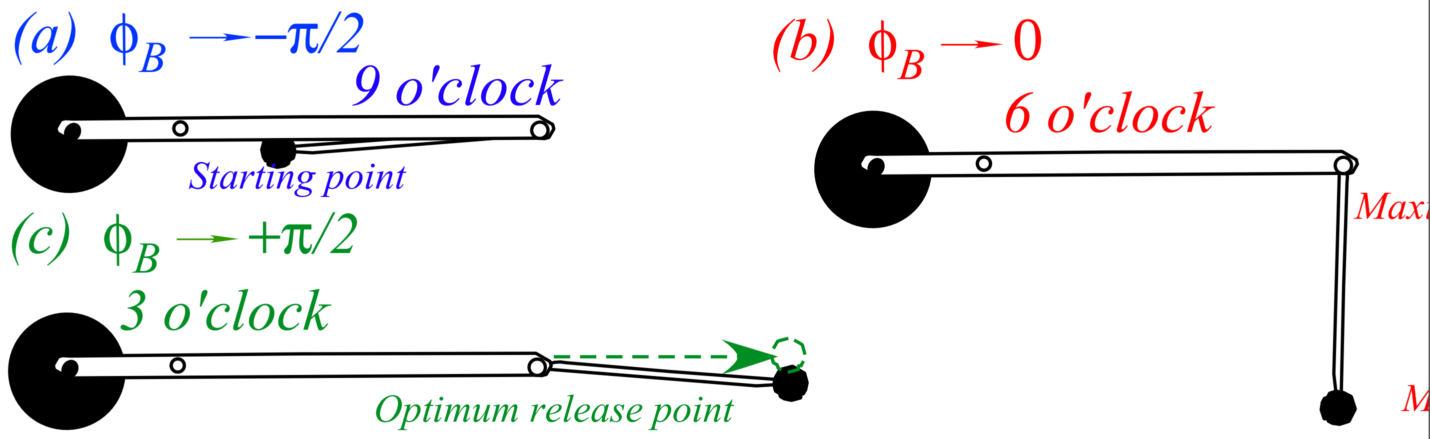
$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right)$$

$$= \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot{\phi} - \dot{\theta}\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot{\phi} + \dot{\theta}\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \begin{cases} 2E = MR^2\dot{\theta}_{-\pi/2}^2 + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{cases} \quad or: \begin{cases} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \begin{cases} 2E = MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{cases}$$

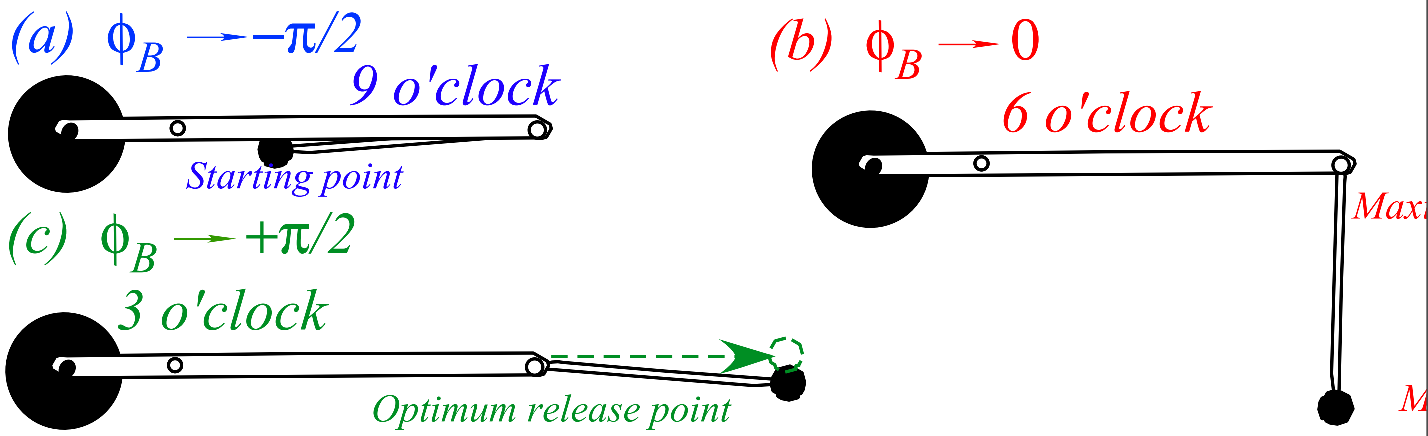
$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right)$$

$$= \begin{cases} \frac{mr^2}{2}\left(\dot{\phi} - \dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi} + \dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2: \begin{cases} 2E = MR^2\dot{\theta}_{\pi/2}^2 + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \rightarrow \left(\omega^2 - \dot{\theta}_{\pi/2}^2\right) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \rightarrow \left(\omega - \dot{\theta}_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \end{cases}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta + \dot\phi\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \begin{cases} 2E = MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot\theta_{-\pi/2} \end{cases} \quad or : \begin{cases} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{cases} \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \begin{cases} \dfrac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \begin{cases} 2E = MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \begin{cases} 2E = MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 & = MR^2\omega^2 \rightarrow \\ \Lambda = MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) & = MR^2\omega \rightarrow \end{cases}$$

$$(\omega^2 - \dot\theta^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$(\omega - \dot\theta_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\longrightarrow (\omega + \dot\theta_{\pi/2}) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

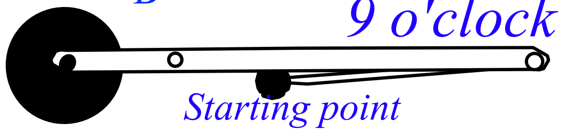
Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r = ℓ (easier algebra)

$$\left.\begin{aligned}2E &= MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda &= MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ (\text{For: } r=\ell)$$

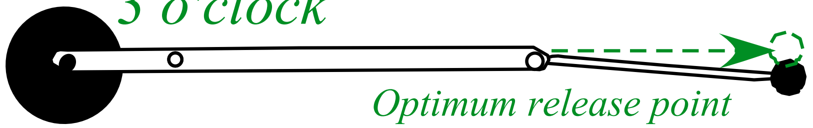
(a) $\phi_B \rightarrow -\pi/2$

9 o'clock



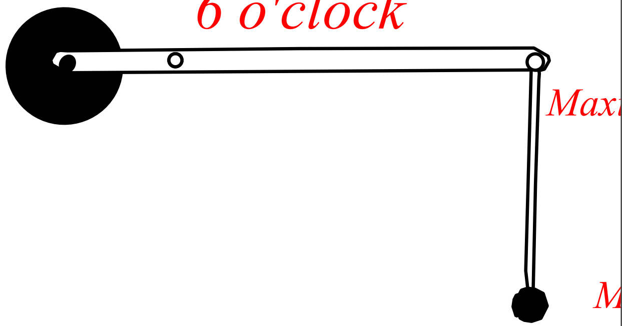
(c) $\phi_B \rightarrow +\pi/2$

3 o'clock



(b) $\phi_B \rightarrow 0$

6 o'clock



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\left\{\begin{array}{c}2E=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda=MR^2\dot\theta_{-\pi/2}\end{array}\right.\quad or:\left\{\begin{array}{c}2E=MR^2\omega^2\\ \Lambda=MR^2\omega\end{array}\right.\quad \text{For: }\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\left\{\begin{array}{c}2E=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{array}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)$$

$$=\begin{cases}\dfrac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 & \left(\text{For: }\phi_B=-\dfrac{\pi}{2}\right)\\ \dfrac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) & \left(\text{For: }\phi_B=0\right)\\ \dfrac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 & \left(\text{For: }\phi_B=\dfrac{\pi}{2}\right)\end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B=\pi/2:\left\{\begin{array}{l}2E=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2=MR^2\omega^2\\ \Lambda=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)=MR^2\omega\end{array}\right.$$

$$\left(\omega^2-\dot\theta^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

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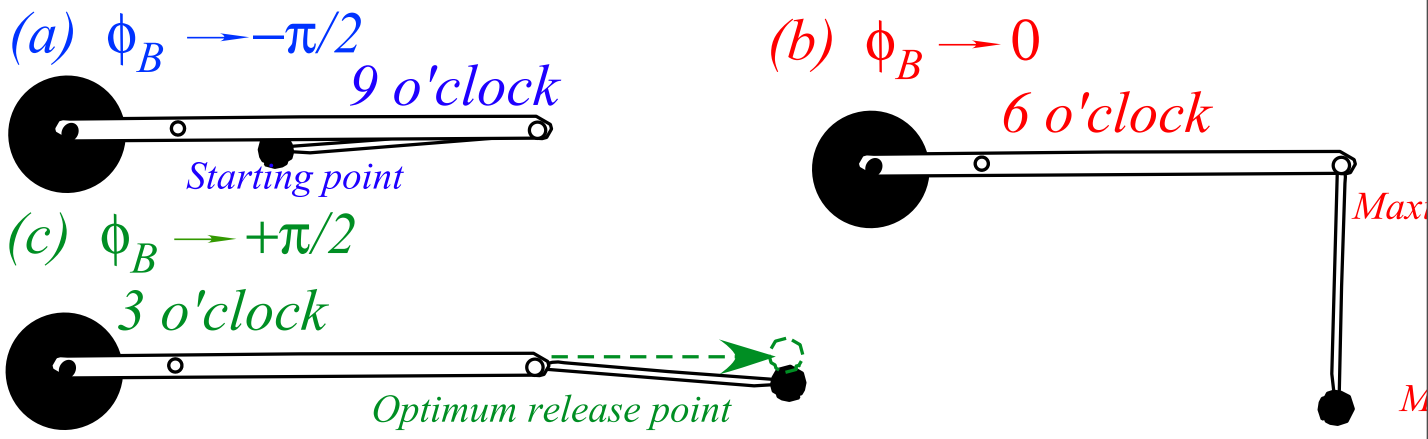
$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{l} 2E = MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda = MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{array}\right\} \text{(For: } r=\ell\text{)}$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{\begin{array}{l} 2E = MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{array}\right. \quad or : \left\{\begin{array}{l} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{array}\right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \left\{\begin{array}{ll} \frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{array}\right.$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{\begin{array}{l} 2E = MR^2\dot{\theta}^2_0 + mr^2\left(\dot{\phi}^2_0 + \dot{\theta}^2_0\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{array}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{\begin{array}{l} 2E = MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda = MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \end{array}\right.$$

$$\rightarrow \left(\omega^2 - \dot{\theta}^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2$$

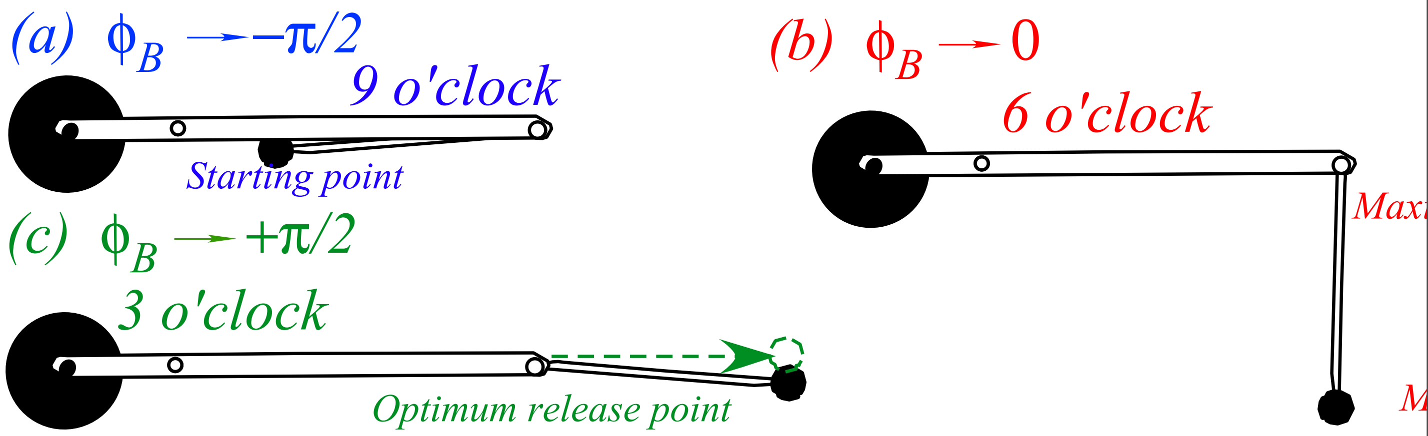
$$\rightarrow \left(\omega - \dot{\theta}_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\longrightarrow \left(\omega + \dot{\theta}_{\pi/2}\right) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

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Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

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$$= \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

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$$\phi_B = 0 : \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 &&= MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) &&= MR^2\omega \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

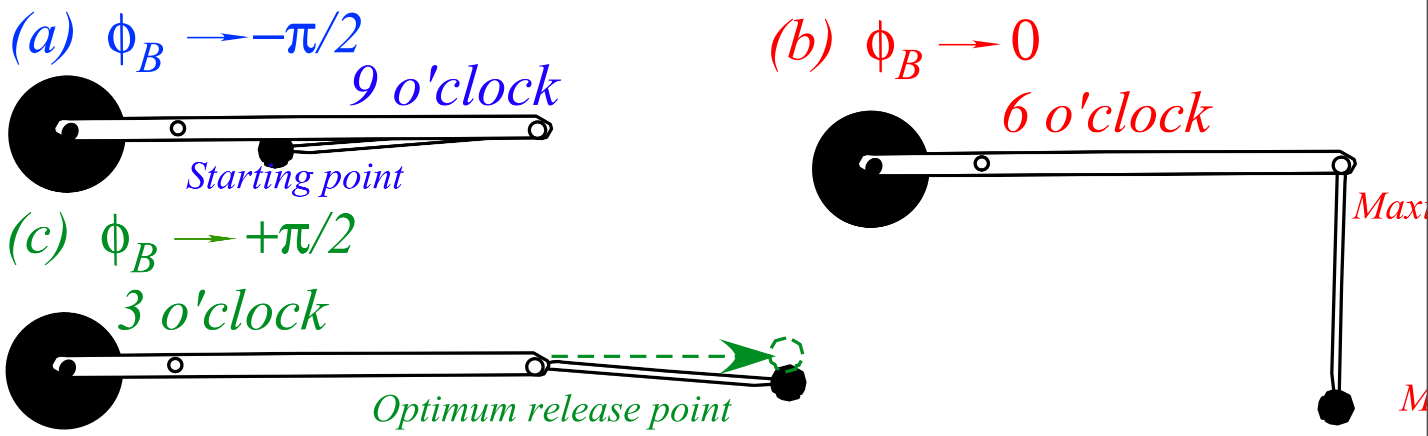
$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$



Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}: \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \left\{\begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0: \left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2: \left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$$\to \left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\to \left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\to \left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

$$\downarrow$$

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

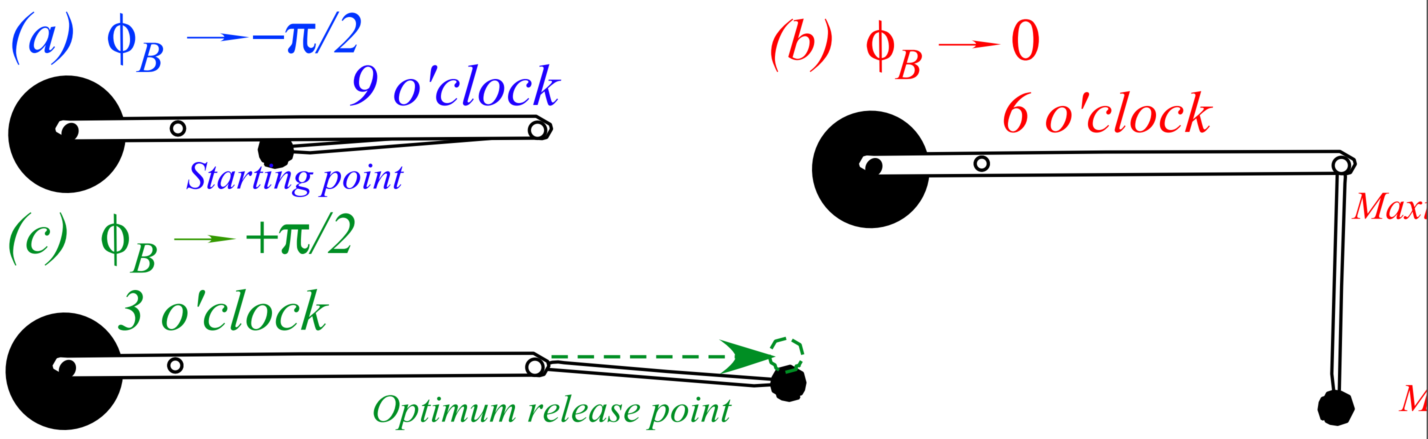
$$\downarrow$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ (\text{For: } r=\ell)$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2}:\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or:\left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega\end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0:\left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{aligned}\right.$$

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Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2:\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega\end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

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$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

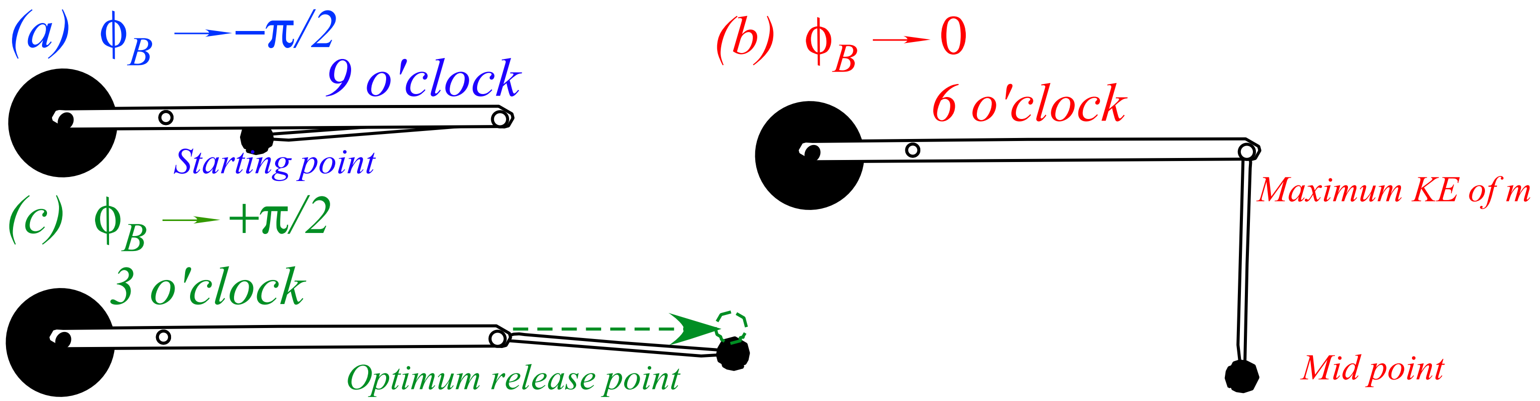
$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$



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$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned} \right. \quad or : \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned} \right. \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned} \right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \left\{ \begin{aligned} &\frac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned} \right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) = MR^2\omega \end{aligned} \right.$$

$$(\omega^2 - \dot{\theta}^2_{\pi/2}) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \longrightarrow (\omega + \dot{\theta}_{\pi/2}) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\downarrow$$

$$\boxed{\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega}$$

$$(\omega - \dot{\theta}_{\pi/2}) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\downarrow$$

$$\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot{\theta}_{\pi/2}\right)$$

$$\downarrow$$

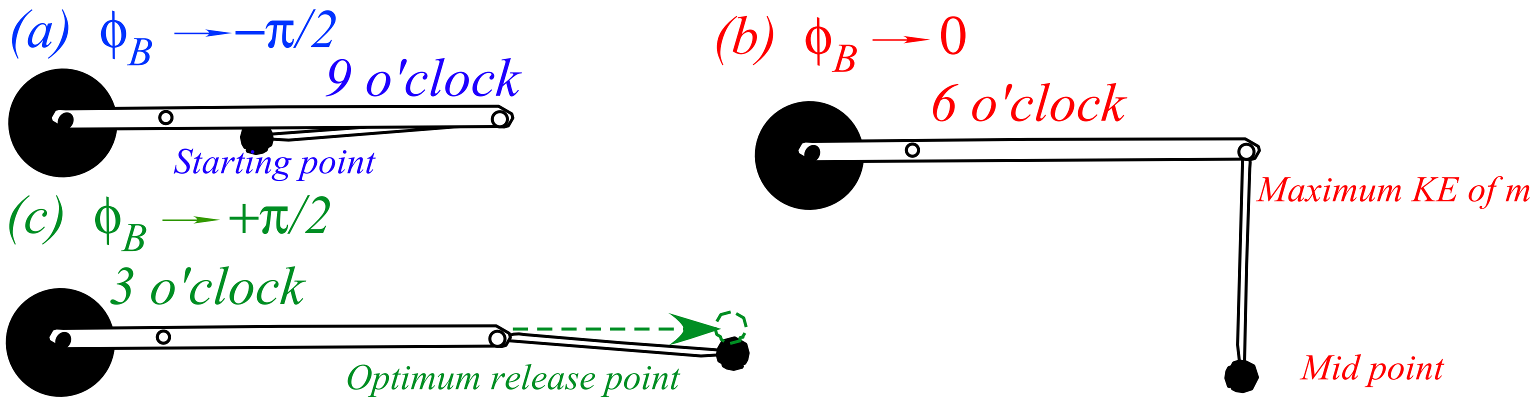
$$\omega - \frac{4mr^2}{MR^2}\omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2}\dot{\theta}_{\pi/2} \longrightarrow \boxed{\dot{\theta}_{\pi/2} = \frac{1 - \frac{4mr^2}{MR^2}}{1 + \frac{4mr^2}{MR^2}}\omega}$$

Large M≫m case

$$\boxed{\dot{\phi}_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot{\theta}_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{(For: } r=\ell\text{)}$$

Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^{\,2}_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or : \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right. \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \left\{ \begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 && \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) && \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 && \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta_{\pi/2}^2 + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta_{\pi/2}^2\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\omega}$$

Large M≫m case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot\phi_{\pi/2} = 0 + 2\omega = 2\omega}$$

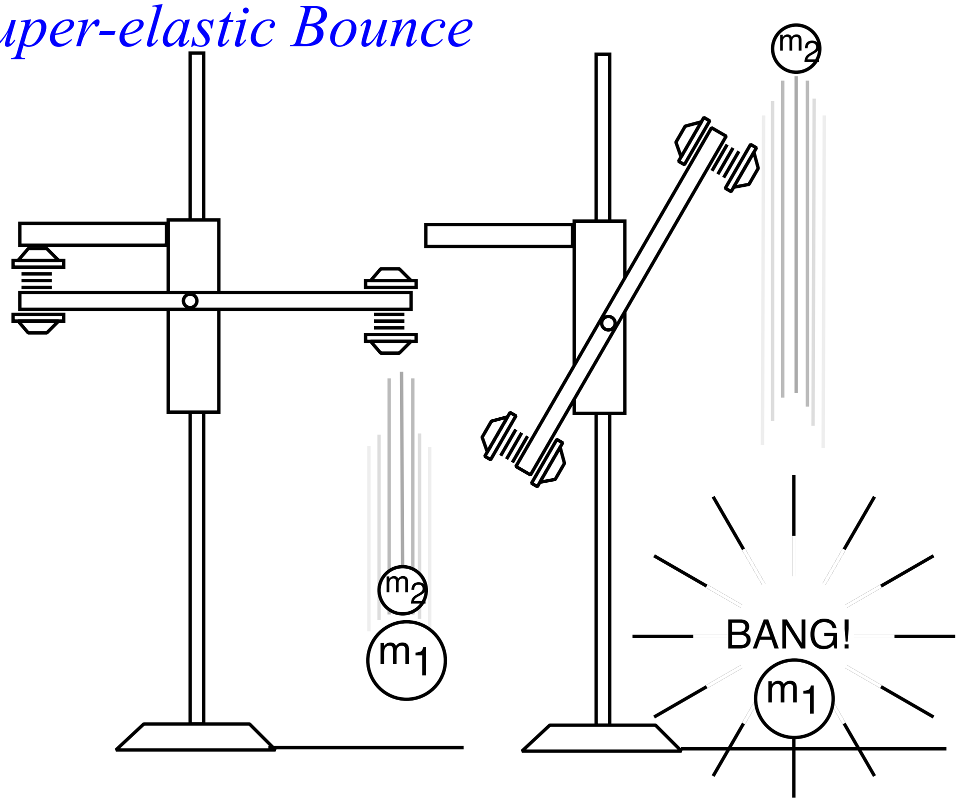
$$\boxed{\dot\theta_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Analogous Superball Models

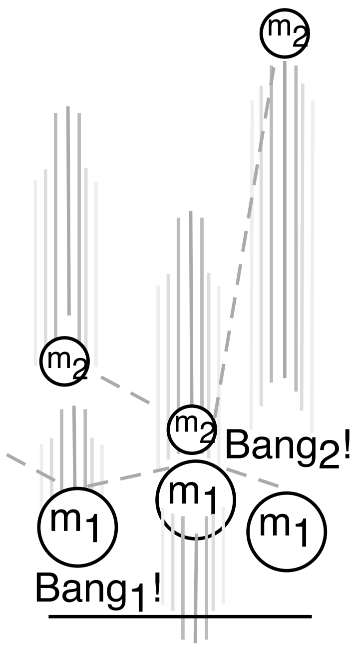
Similar in some ways to trebuchet models

Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. J. Phys. **39**, 656 (1971)
(A class project)

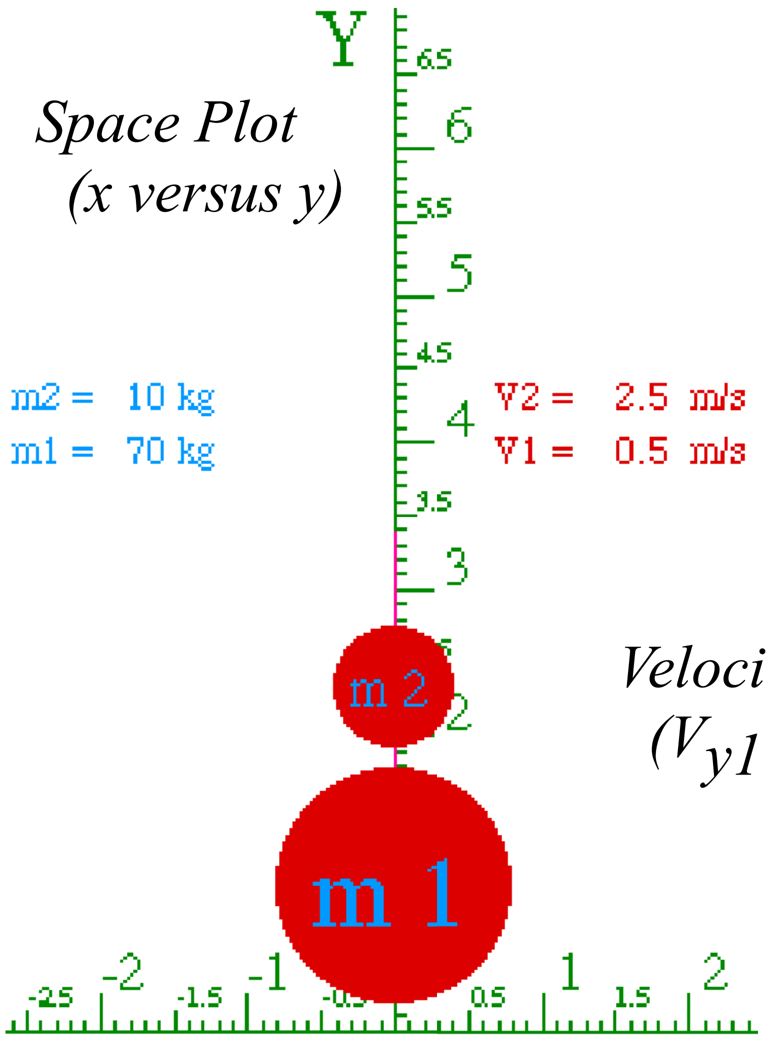
Super-elastic Bounce



2-Bang Model



Space Plot
(x versus y)



Velocity Plot →
(V_{y1} versus V_{y2})

Optimal Throw

100% Energy Transfer

m_1:m_2 = 3:1

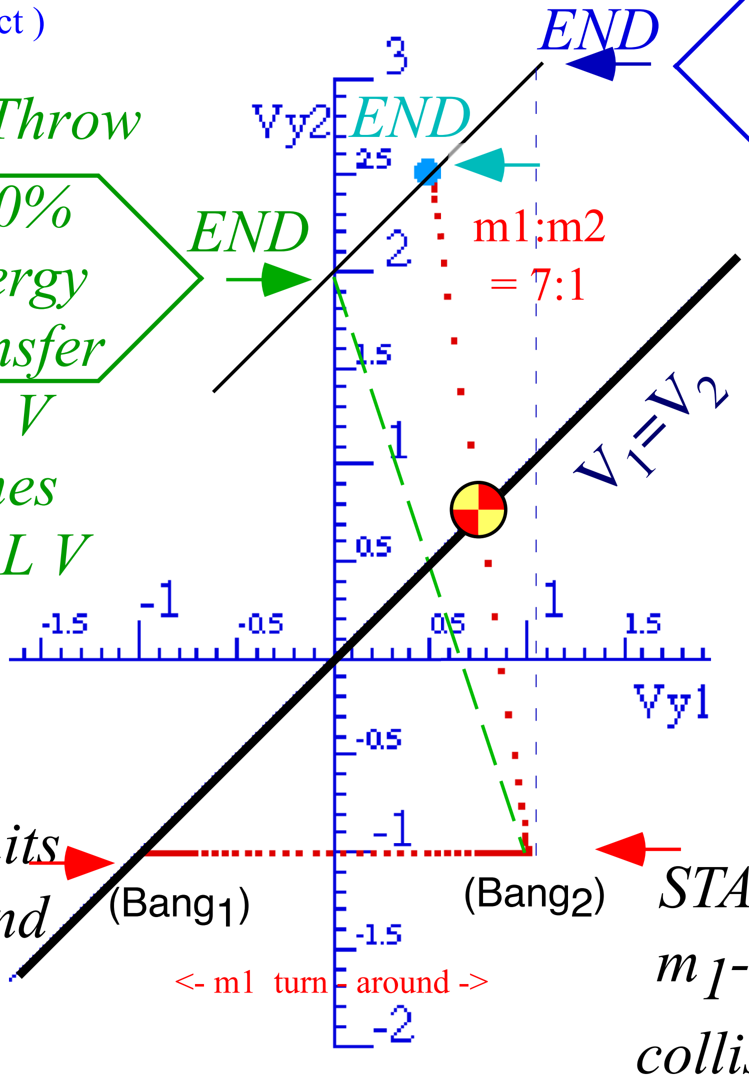
FINAL V is 2 times INITIAL V

Fastest Throw

0% Energy Transfer

m_1:m_2 = ∞:1

FINAL V is 3 times INITIAL V



m_1 first hits ground

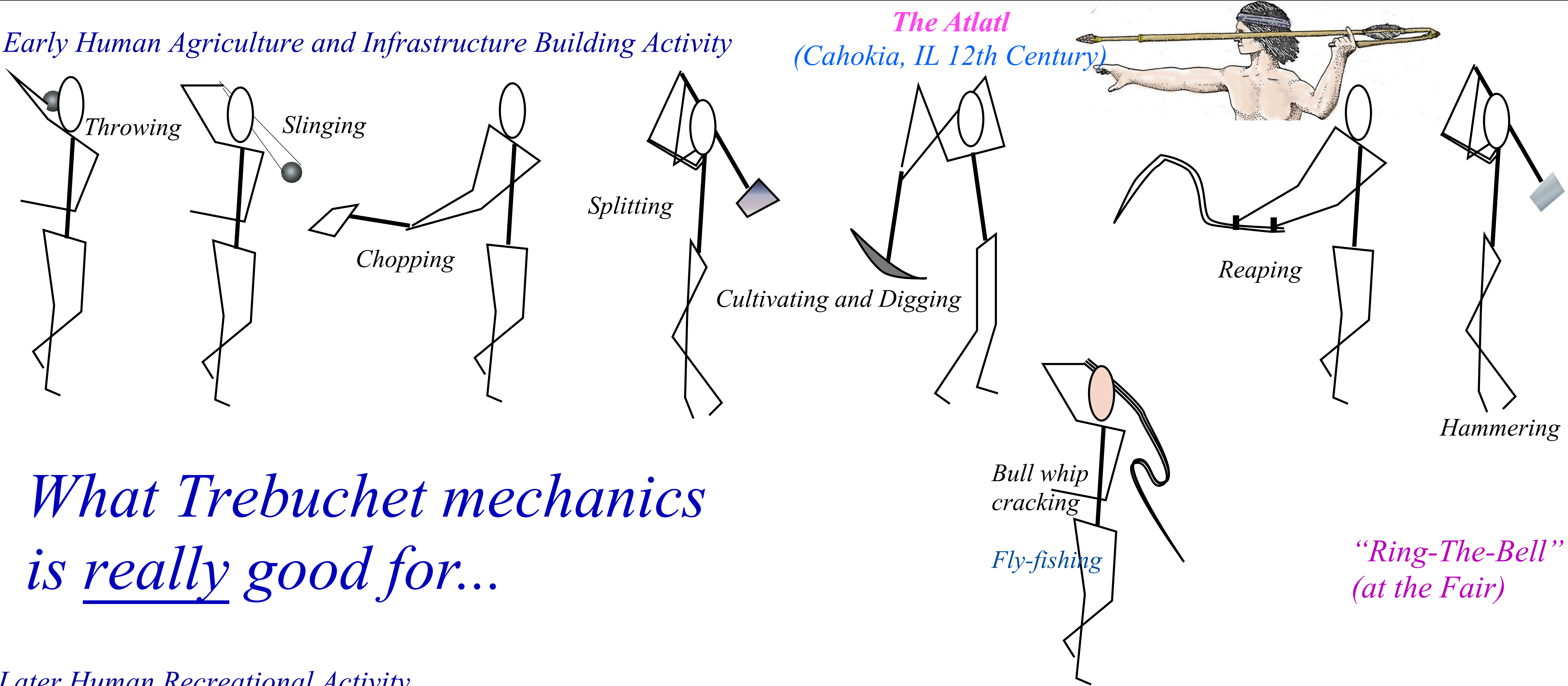
START m_1-m_2 collision

Graphic Solution

Hamiltonian energy and momentum conservation and symmetry coordinates

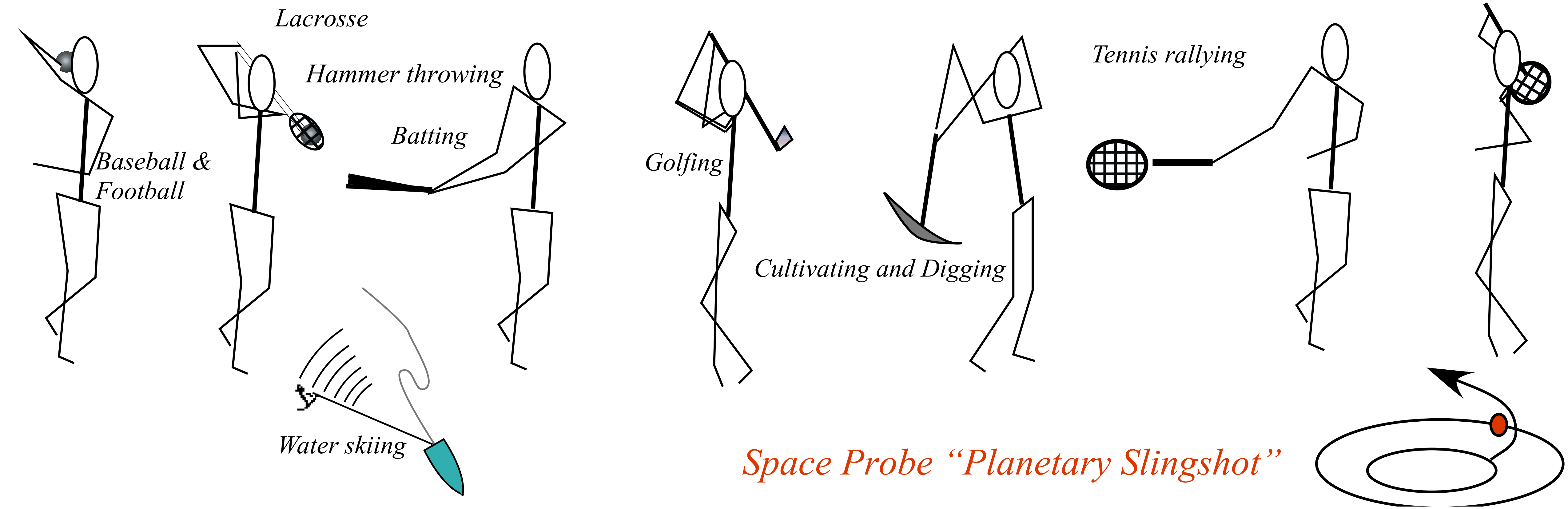
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
    - *Algebraic approach*
    - *Direct approach and Superball analogy*
- → *Trebuchet vs Flinger and sports kinematics*
    - *Many approaches to Mechanics*

Early Human Agriculture and Infrastructure Building Activity



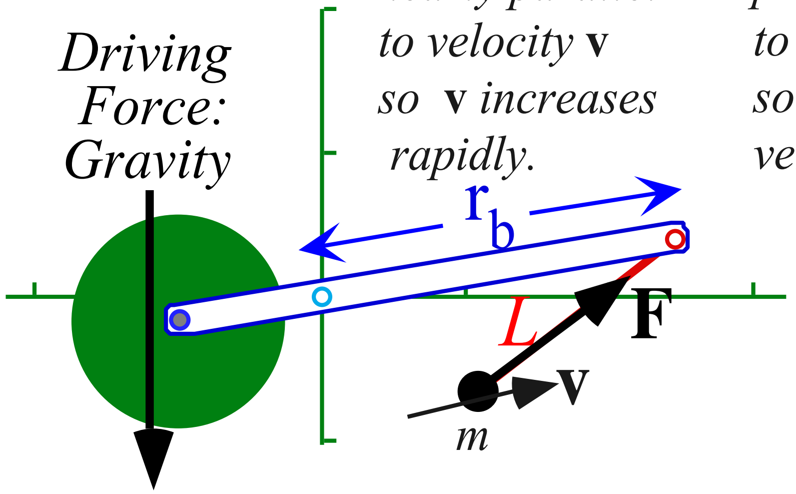
What Trebuchet mechanics is really good for...

Later Human Recreational Activity



Space Probe "Planetary Slingshot"

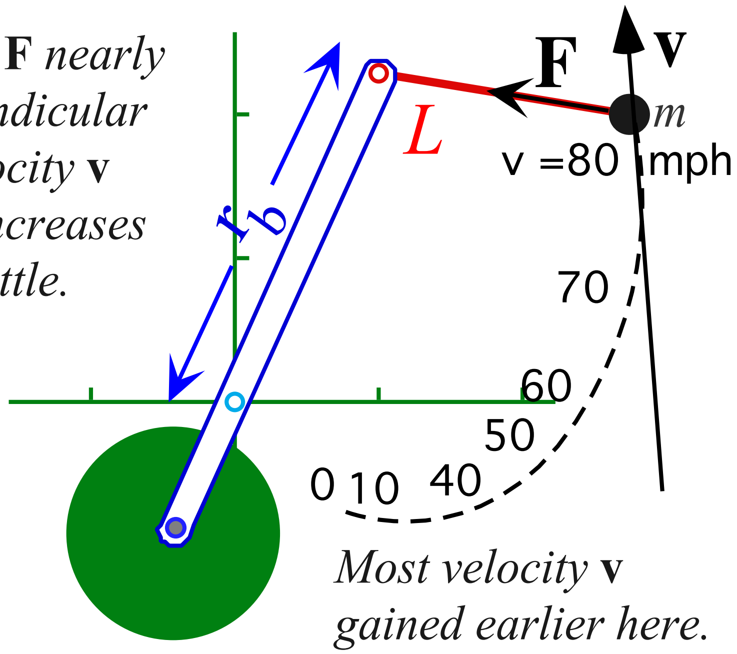
Trebuchet analogy with racquet swing - What we learn



Driving Force: Gravity

Large force **F** *nearly parallel to velocity* **v** *so* **v** *increases rapidly.*

Force **F** *nearly perpendicular to velocity* **v** *so* **v** *increases very little.*



F *mostly serves to steer m here.*

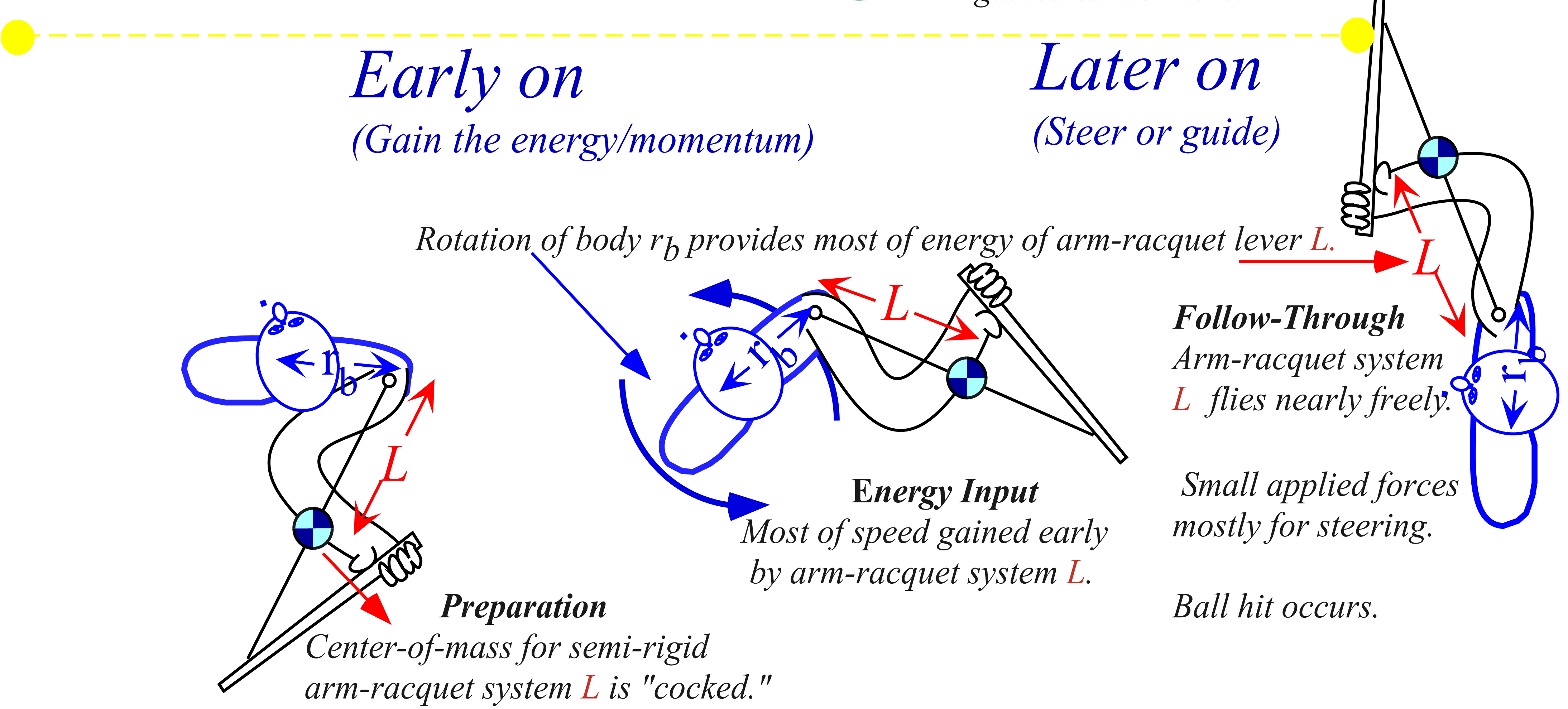
Most velocity **v** *gained earlier here.*

Early on

(Gain the energy/momentum)

Later on

(Steer or guide)



Rotation of body r_b provides most of energy of arm-racquet lever L.

Preparation

Center-of-mass for semi-rigid arm-racquet system L is "cocked."

Energy Input

Most of speed gained early by arm-racquet system L.

Follow-Through

Arm-racquet system L flies nearly freely.

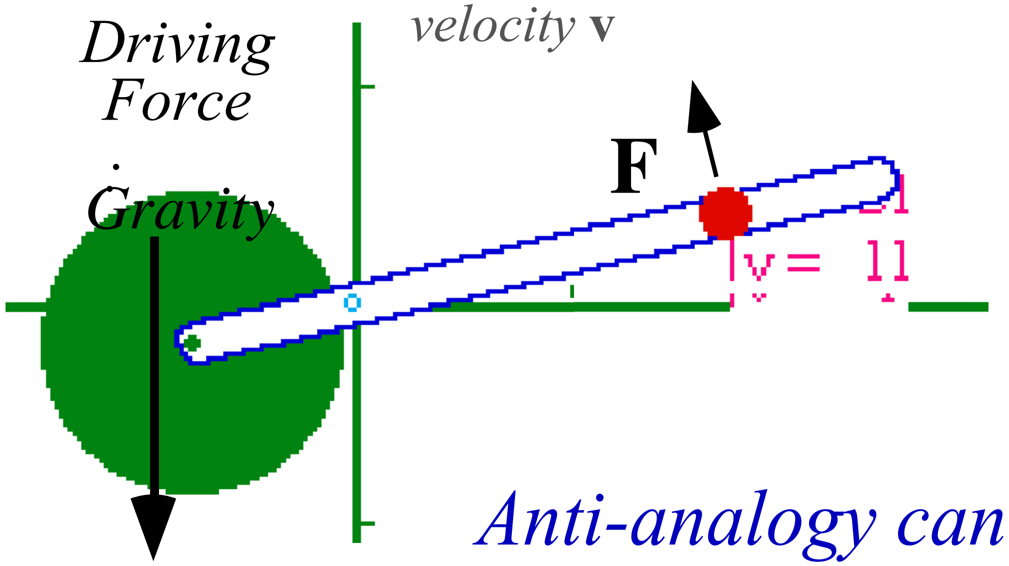
Small applied forces mostly for steering.

Ball hit occurs.

An Opposite to Trebuchet Mechanics- The “Flinger”

Early on

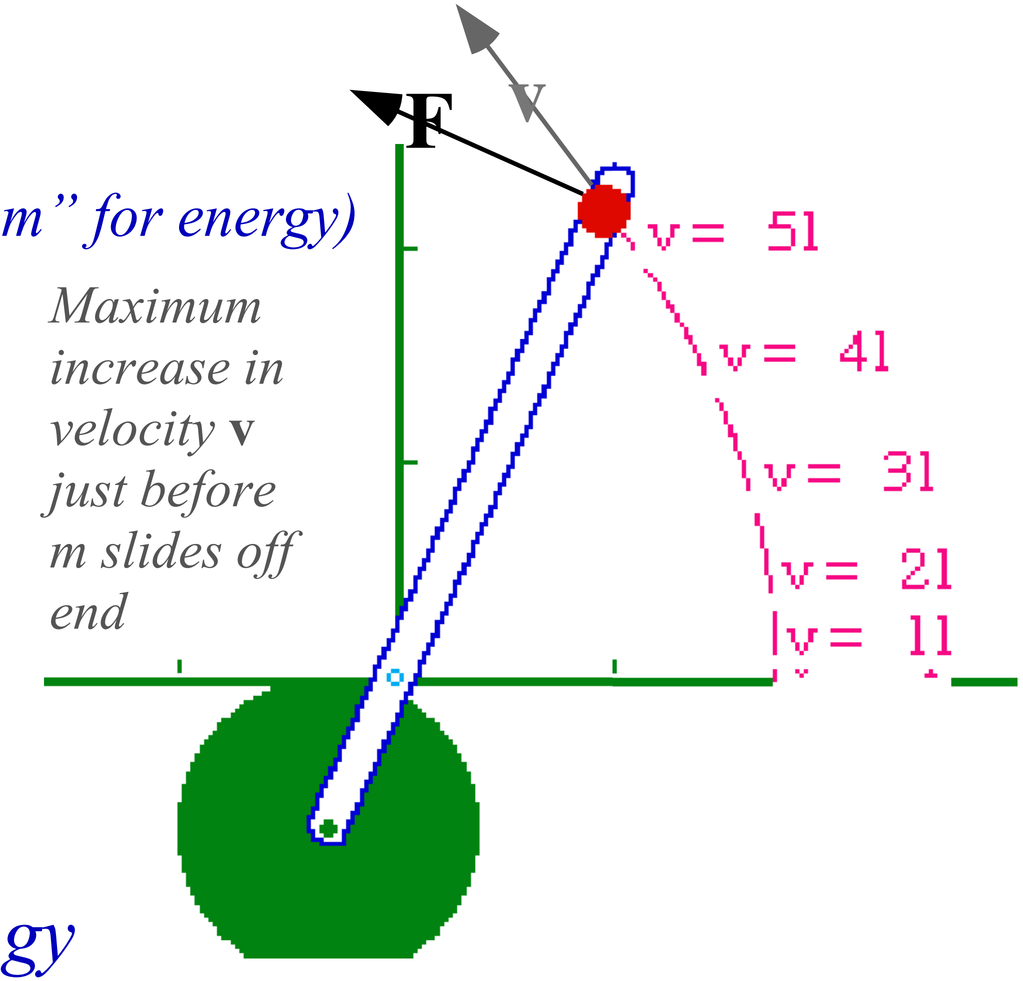
(Not much happening)



Not much increase in velocity **v**

Later on

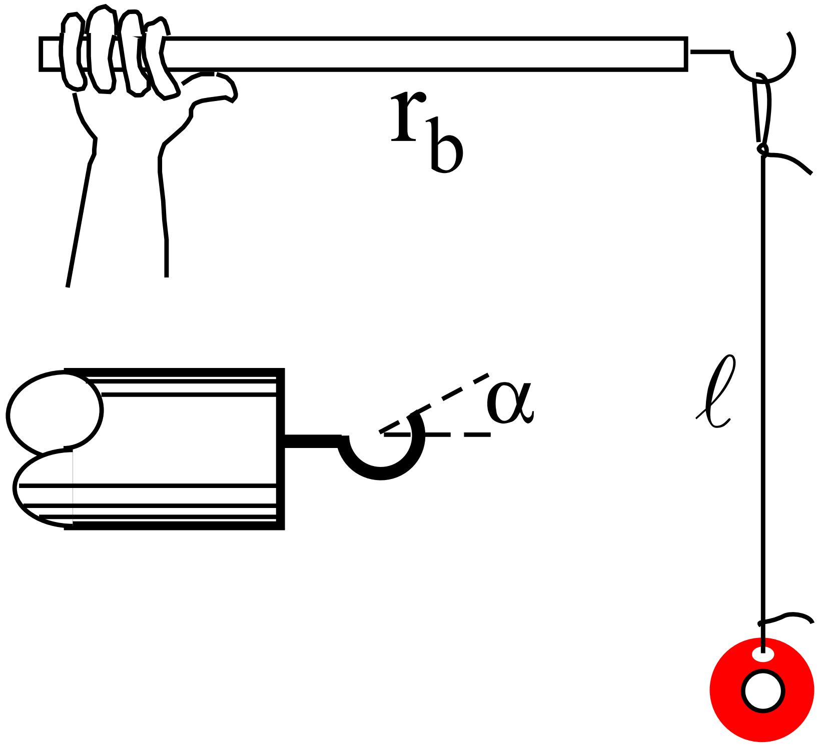
(Last-minute “cram” for energy)



Maximum increase in velocity **v** *just before m slides off end*

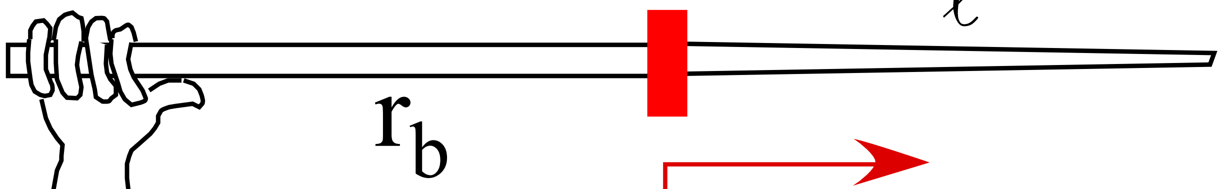
Anti-analogy can be useful pedagogy

Trebuchet-like experiment



skateboard wheel swings

Flinger experiment



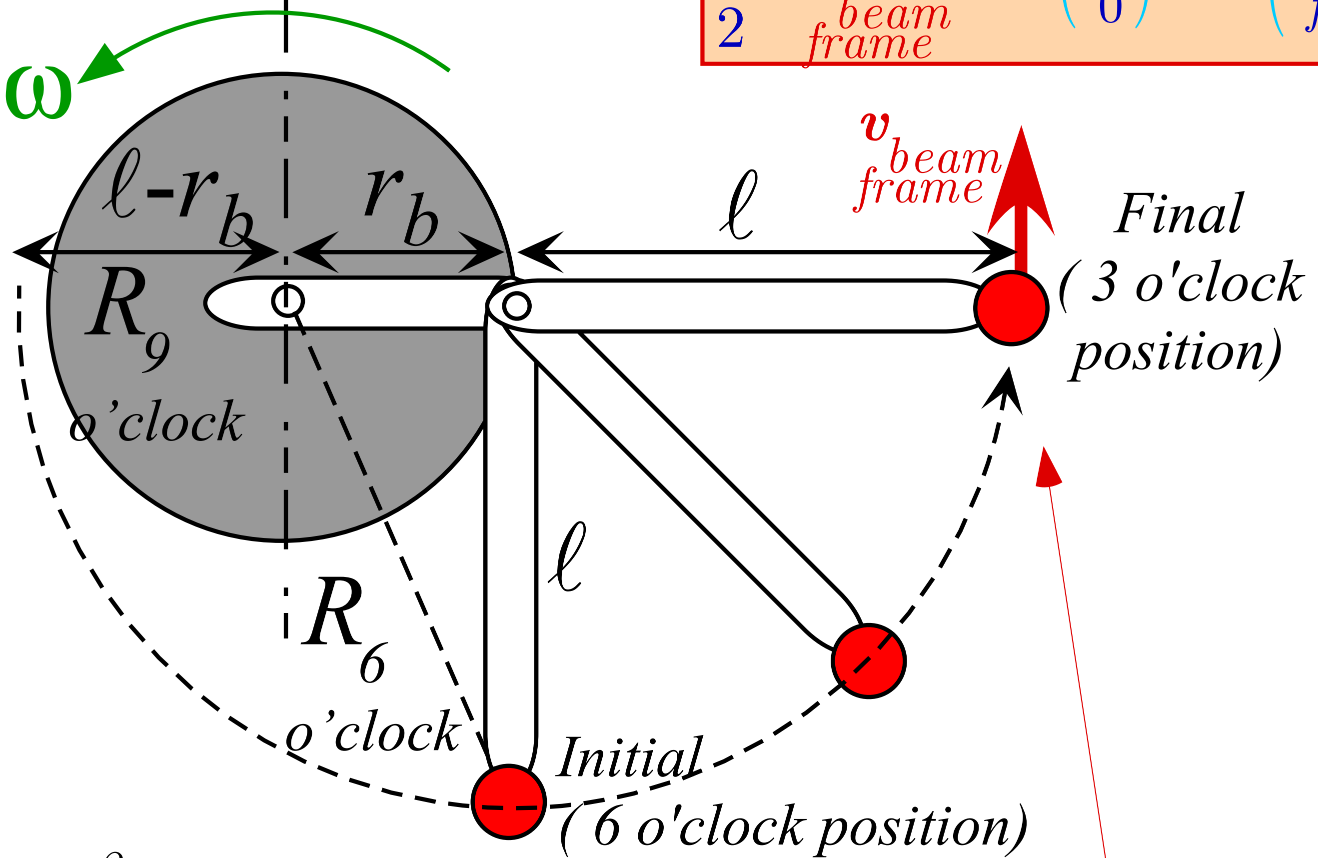
skateboard wheel slides on pool que-stick

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

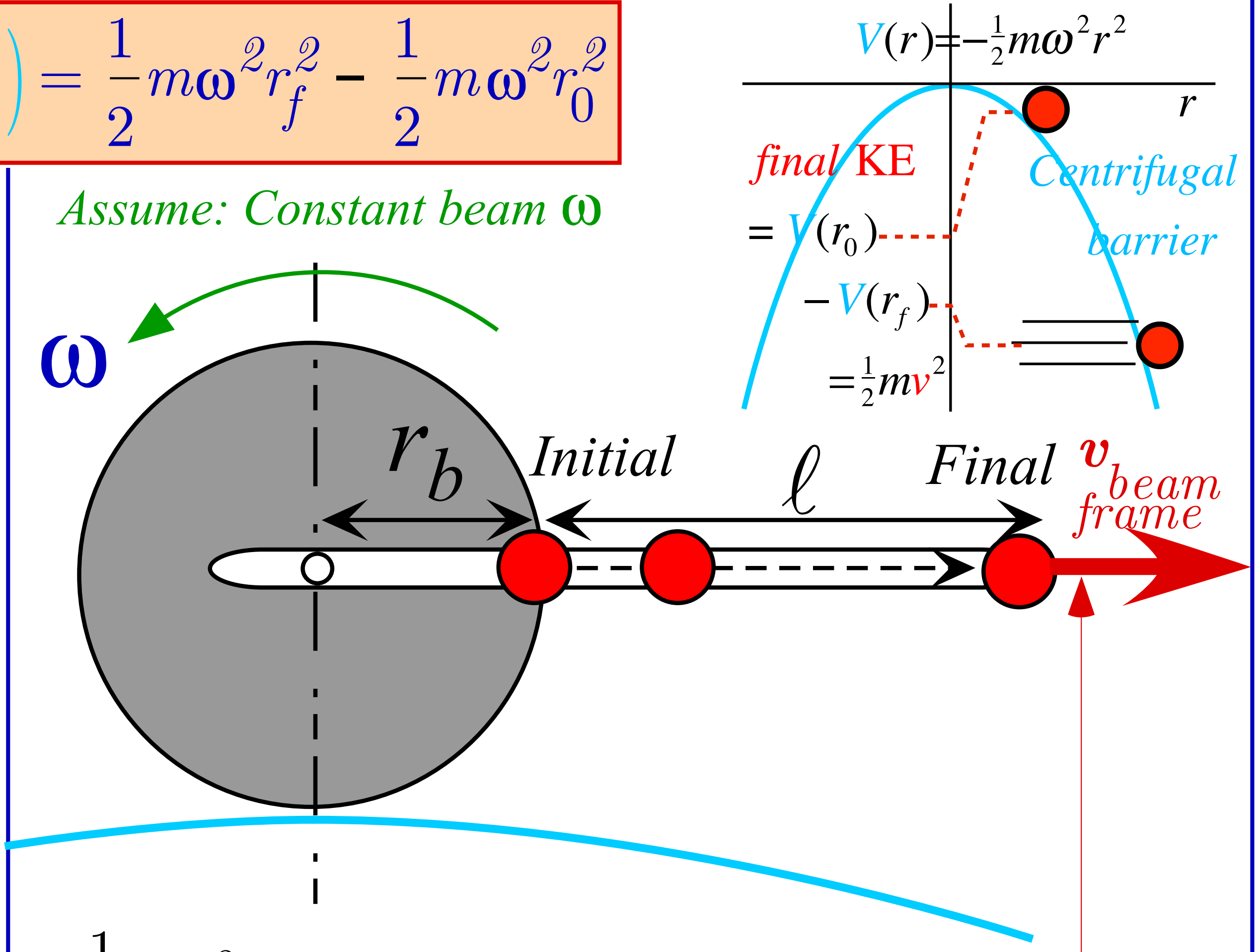
$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(trebuchet) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

Assume: Constant beam ω



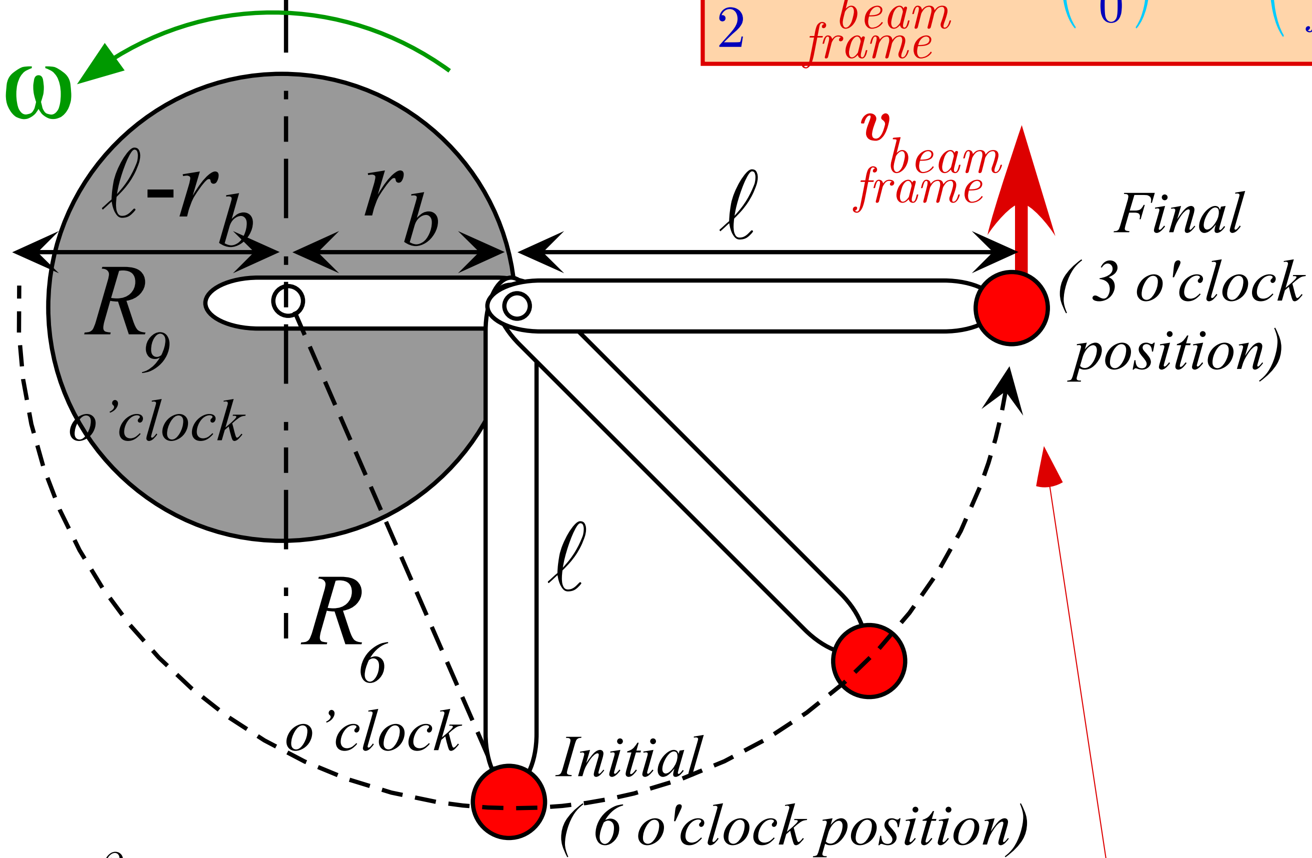
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(flinger) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

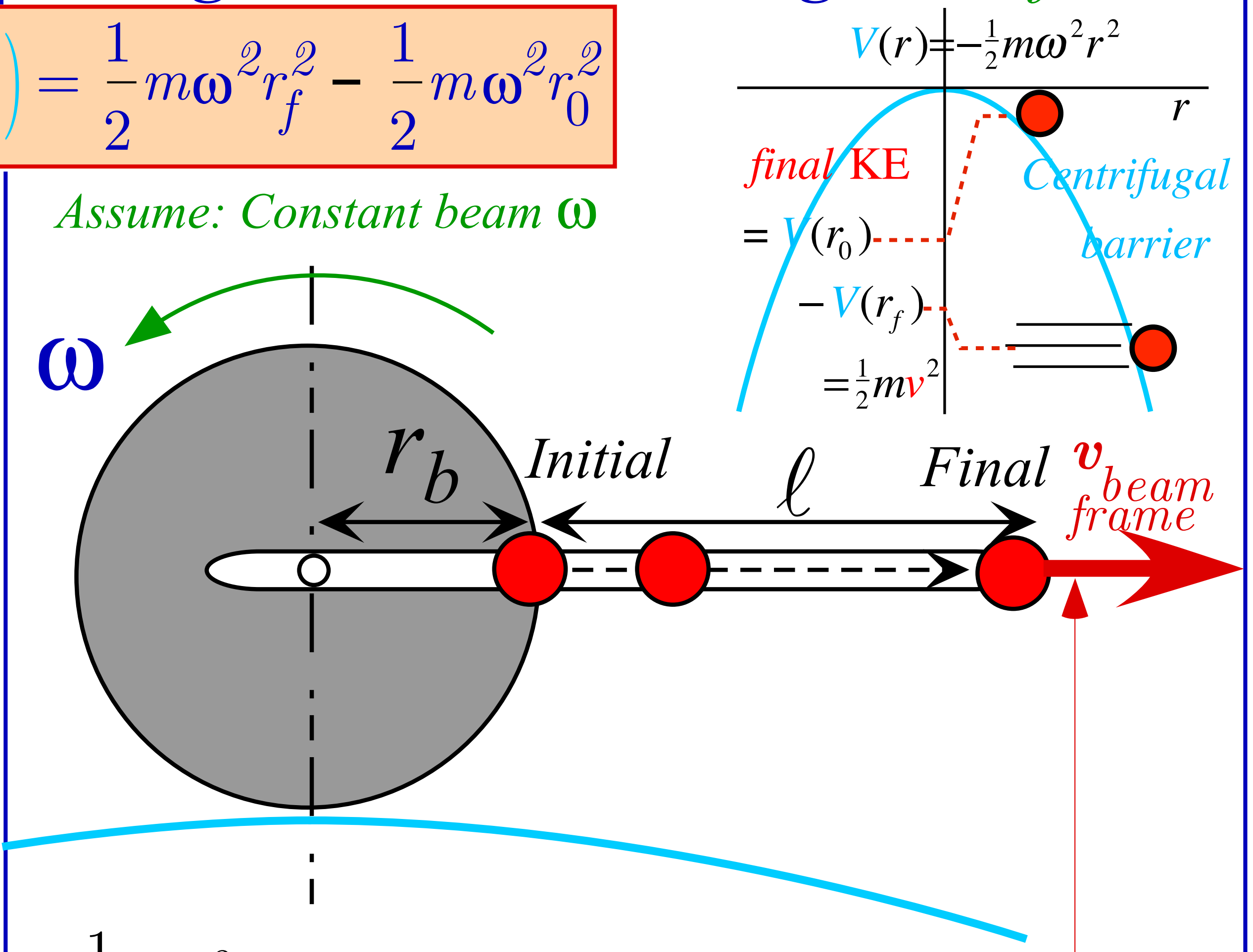
Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)}_{Initial} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

Assume: Constant beam ω



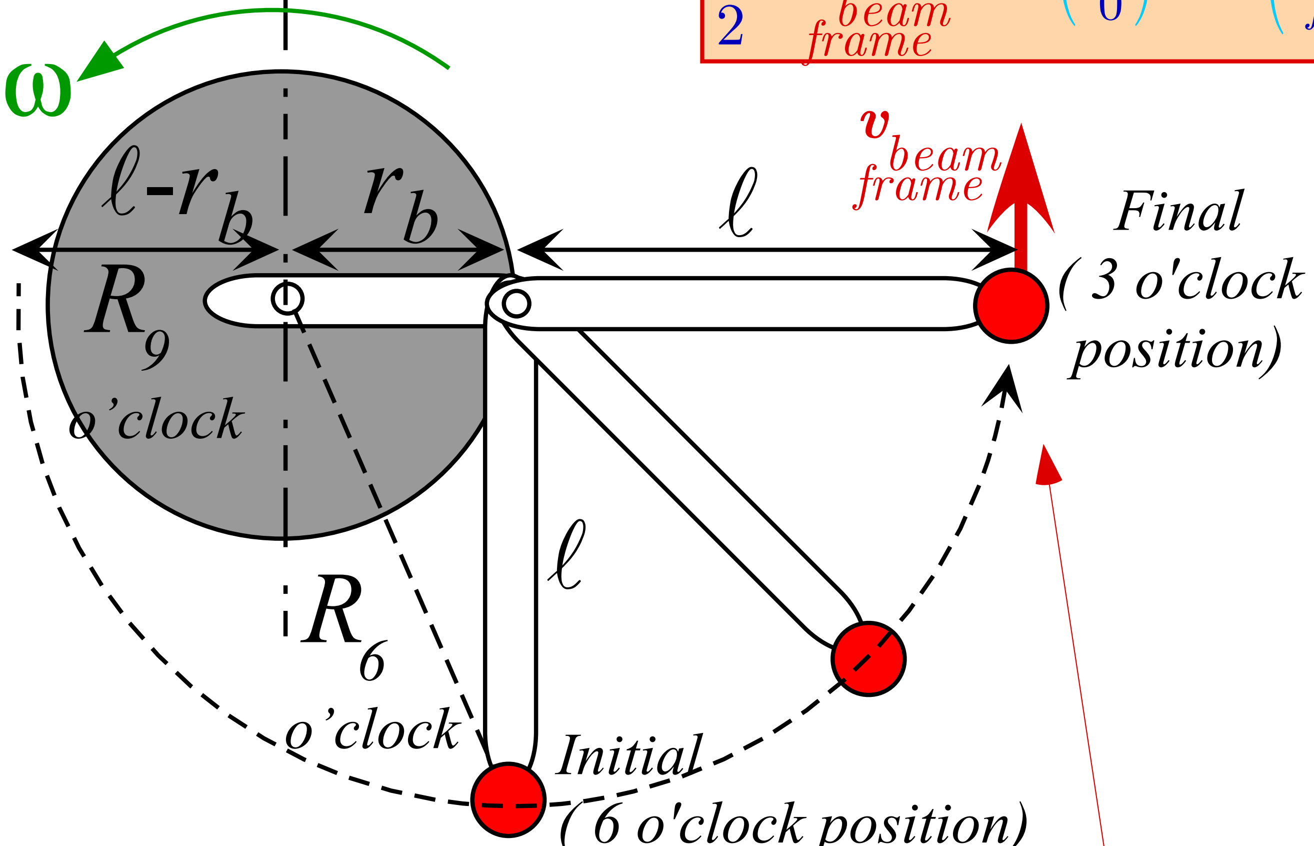
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{Final} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{Initial} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Trebuchet model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



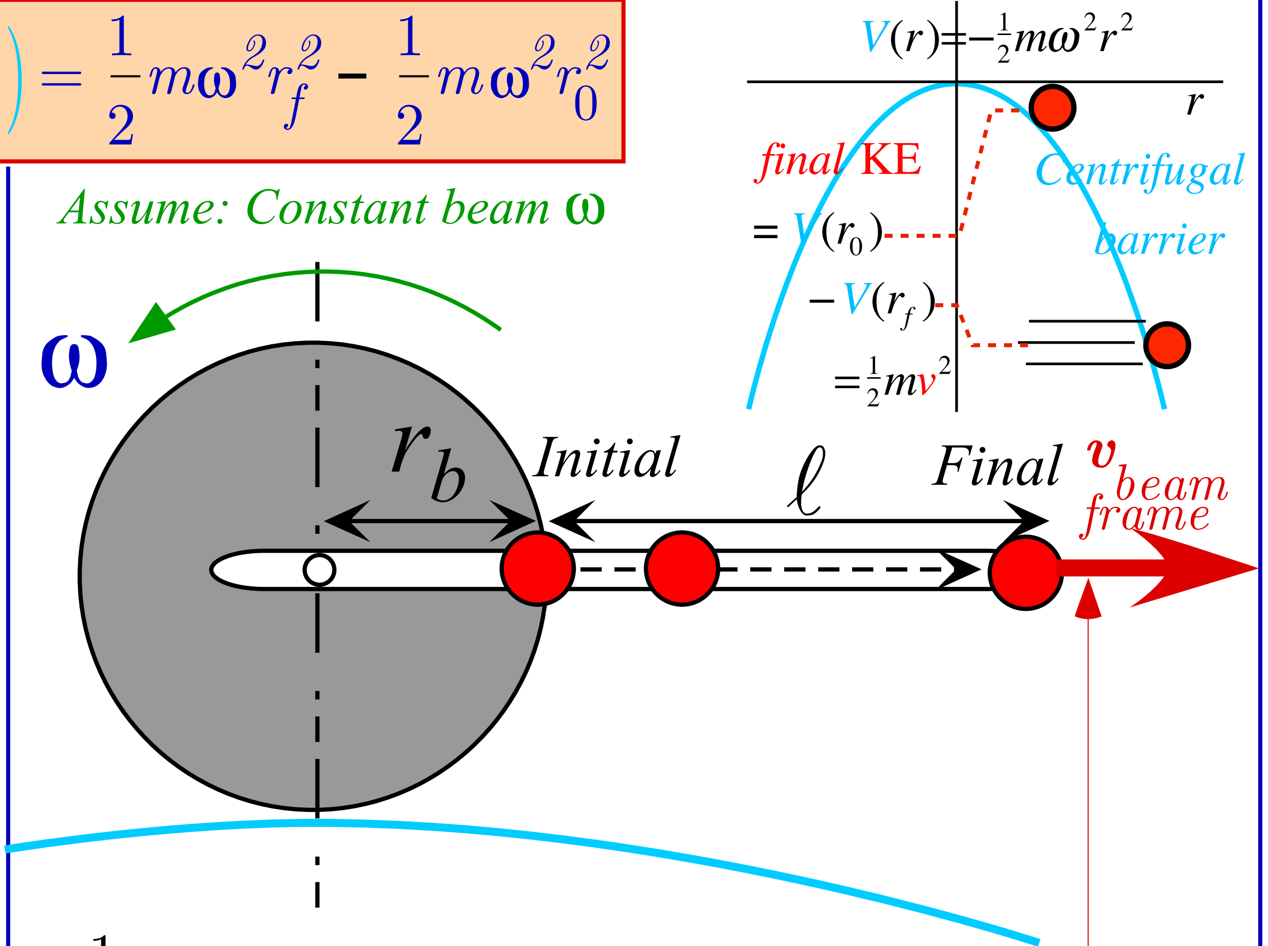
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{Initial\\6\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$

Flinger model in rotating beam frame

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{Initial\\3\ o'clock}}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

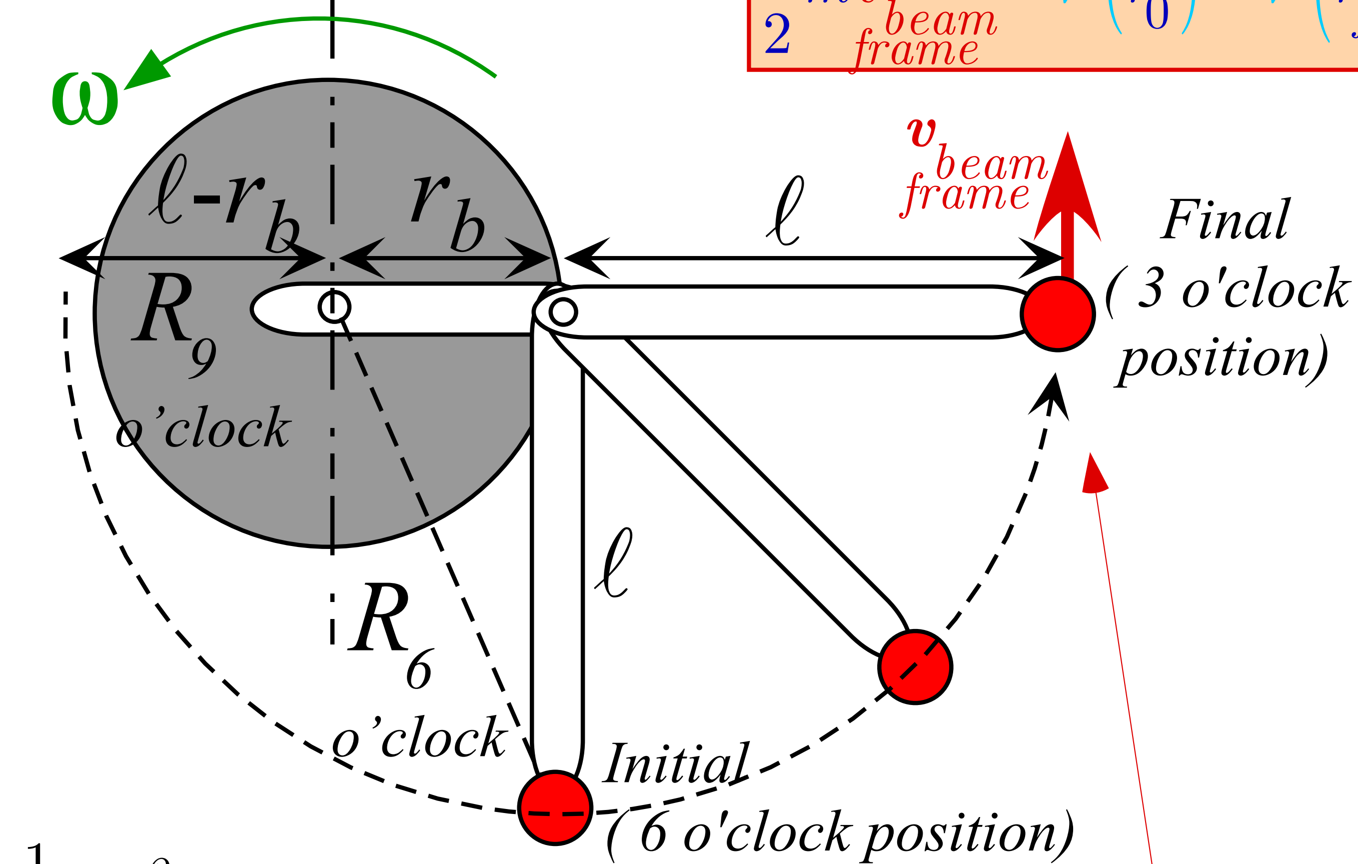
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet
but <u>misdirected</u>

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) \qquad V(r_f) = \frac{1}{2}m\omega^2 r_f^2 \qquad \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet KE}_{\substack{beam\\frame}}$$

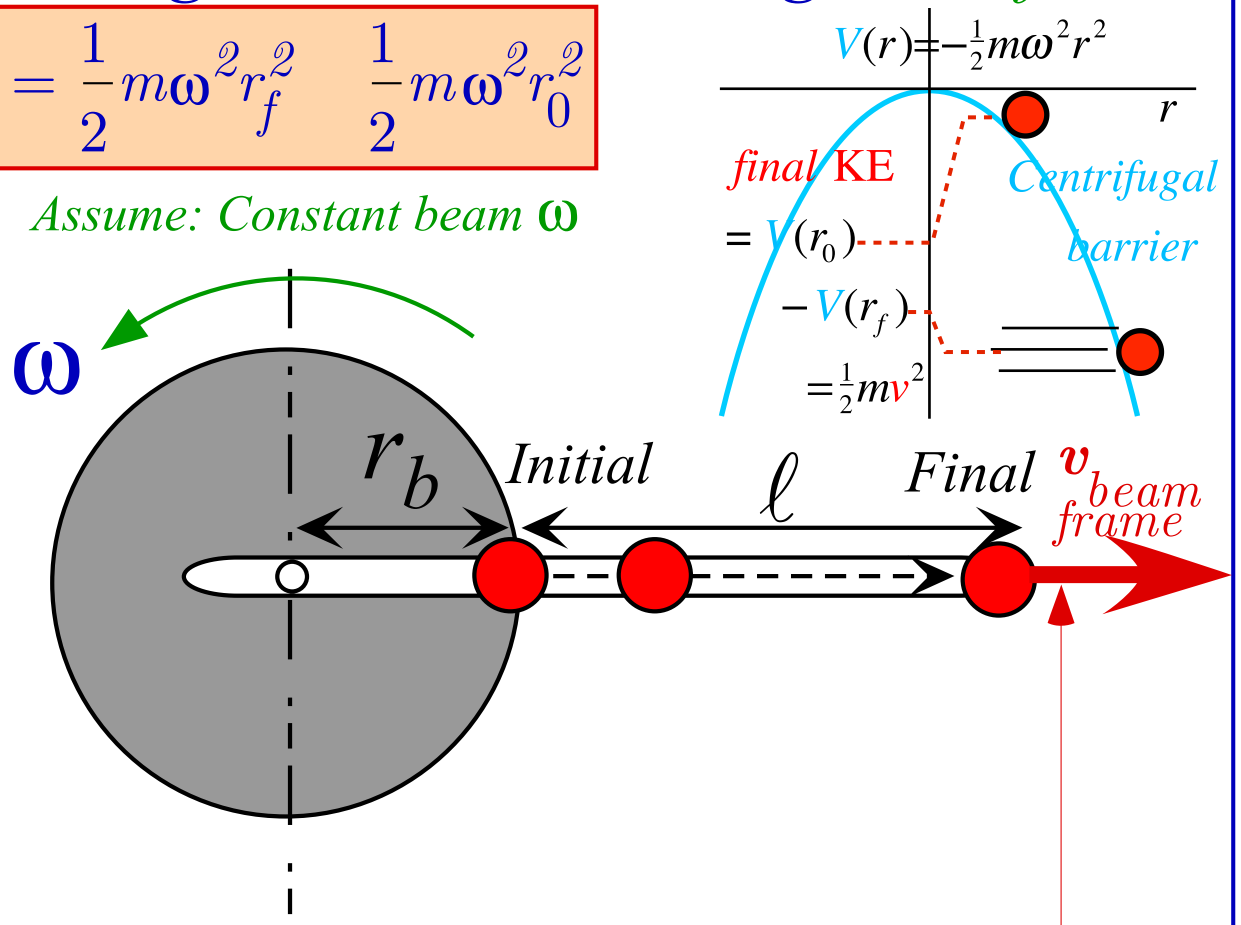
$$\underset{\substack{\text{Final}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{\text{Initial}\\ \text{6 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\ell\right)$$

$$R^2_{\substack{6\\ \text{o'clock}}} = r_b^2+\ell^2$$

$$\underset{\text{9 o'clock}}{\text{Initial}} \quad = \frac{1}{2}m\omega^2\left(4r_b\ell\right)$$

$$R^2_{\substack{9\\ \text{o'clock}}} = r_b^2+\ell^2-2r_b\ell$$

Assume: Constant beam ω



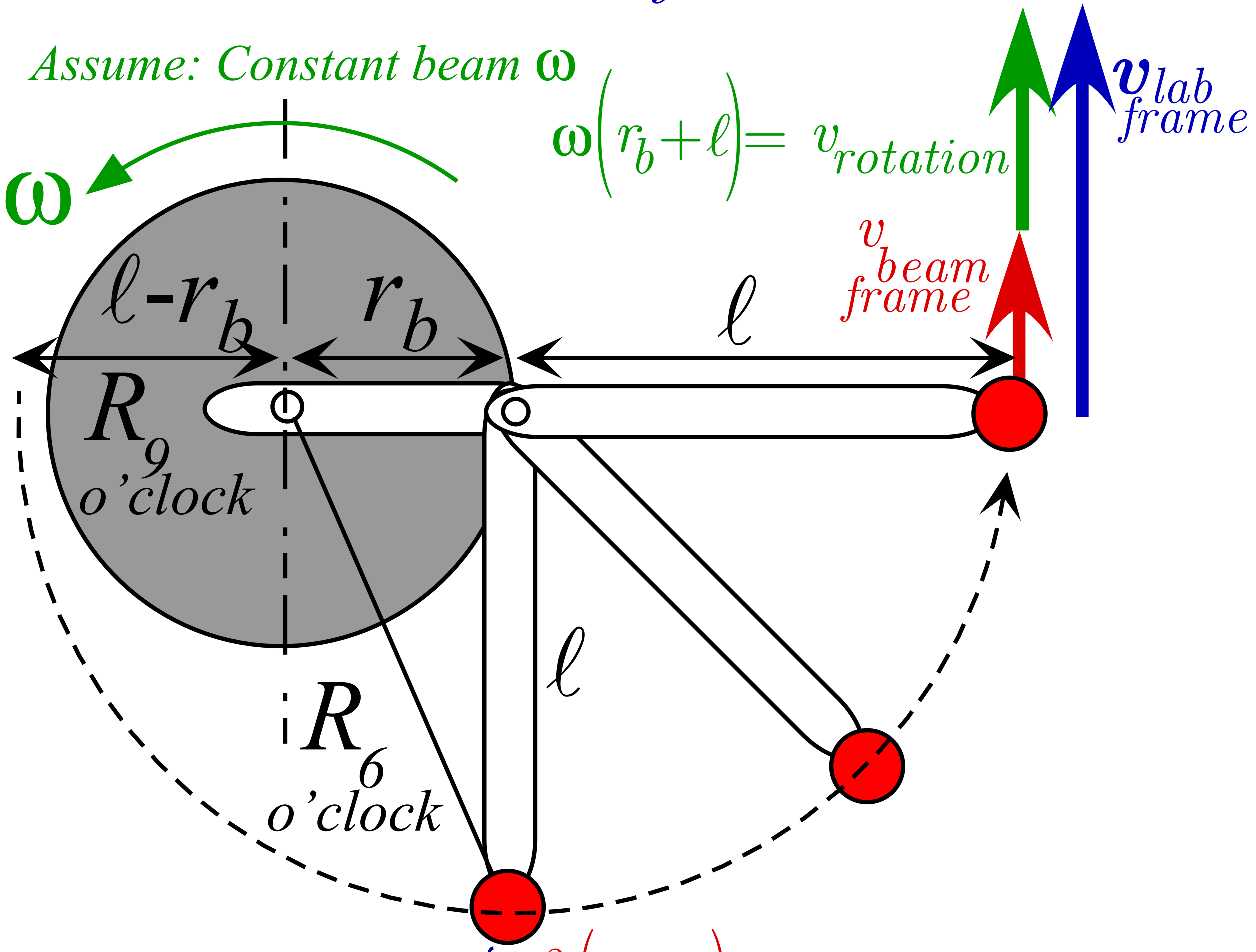
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger KE}_{\substack{beam\\frame}}$$

$$\underset{\substack{\text{Final}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{\text{Initial}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet but misdirected

Flinger KE is $\frac{m\omega^2}{2}\left(2r_b\ell-\ell^2\right)$ less than 9 o'clock trebuchet and misdirected

Trebuchet model in lab frame



Assume: Constant beam ω

$\omega\left(r_b+\ell\right)= v_{rotation}$

$$v^2_{\substack{beam\\frame}}(\text{trebuchet}) = \begin{cases} \omega^2\left(2r_b\ \ell\right) & \textit{half-cocked 6 o'clock}\\ \omega^2\left(4r_b\ \ell\right) & \textit{fully-cocked 9 o'clock}\end{cases}$$

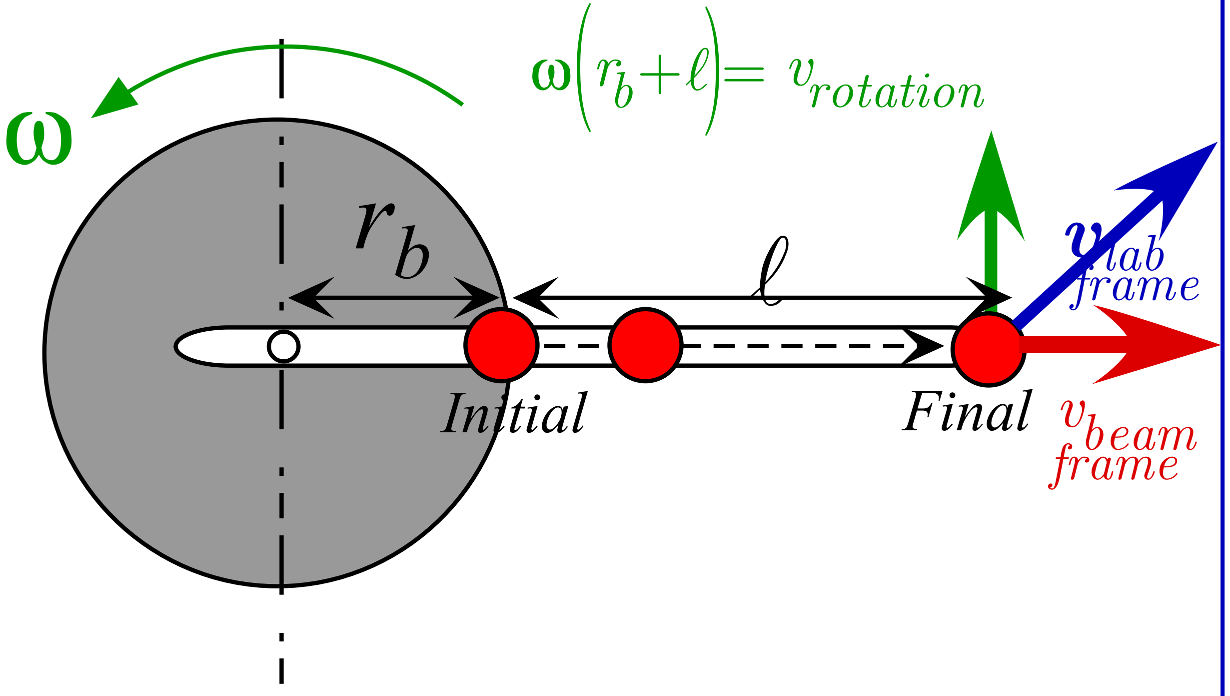
$$v_{lab\ \ frame}\left(trebuchet\right) =$$

$$\begin{cases} \omega\left(r_b+\ell+\sqrt{2\ell r_b}\right) & \textit{half-cocked 6 o'clock}\\ \omega\left(r_b+\ell+2\sqrt{\ell r_b}\right) & \textit{fully-cocked 9 o'clock}\end{cases}$$

$$=\begin{cases}5.00\omega\\5.82\omega\end{cases}\qquad =\begin{cases}5.16\omega\\6.00\omega\end{cases}\qquad =\begin{cases}5.00\omega\\5.82\omega\end{cases}$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Flinger model in lab frame



$\omega\left(r_b+\ell\right)= v_{rotation}$

$$v^2_{\substack{beam\\frame}}(\text{flinger}) = \omega^2\ell\left(2r_b+\ell\right)$$

$$v_{lab\ \ frame}\left(flinger\right) =$$

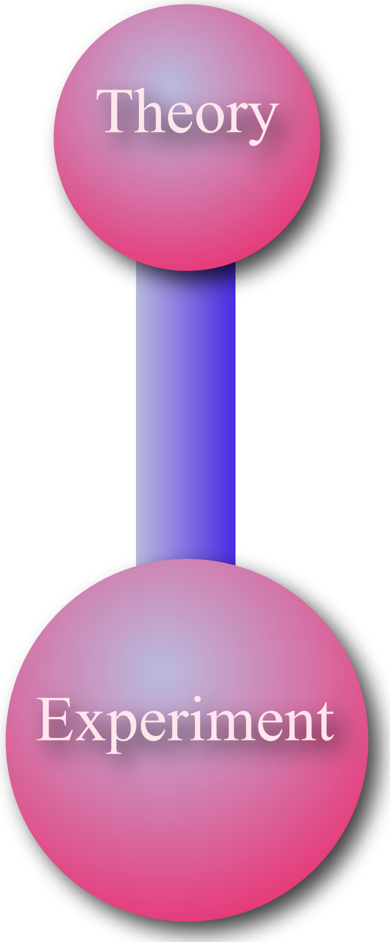
$$= \omega\sqrt{\left(r_b+\ell\right)^2 + \ell\left(2r_b+\ell\right)} = \omega\sqrt{2\left(r_b+\ell\right)^2 - r_b{}^2}$$

(compare)

$$= 3.74\omega \qquad = 3.96\omega \qquad = 4.12\omega$$

$$\left(r_b = 2\,,\ell = 1\right),\ \left(r_b = 1.5\,,\ell = 1.5\right),\left(r_b = 1\,,\ell = 2\right)$$

Physics used to be pretty much bi-polar...



Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

Quick'n dirty
Newton F=Ma Equations
Cartesian coordinates

- French Approach

Tres elegant
Lagrange Equations
in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} \qquad \frac{\partial T}{\partial q^\ell}$$

- German Approach

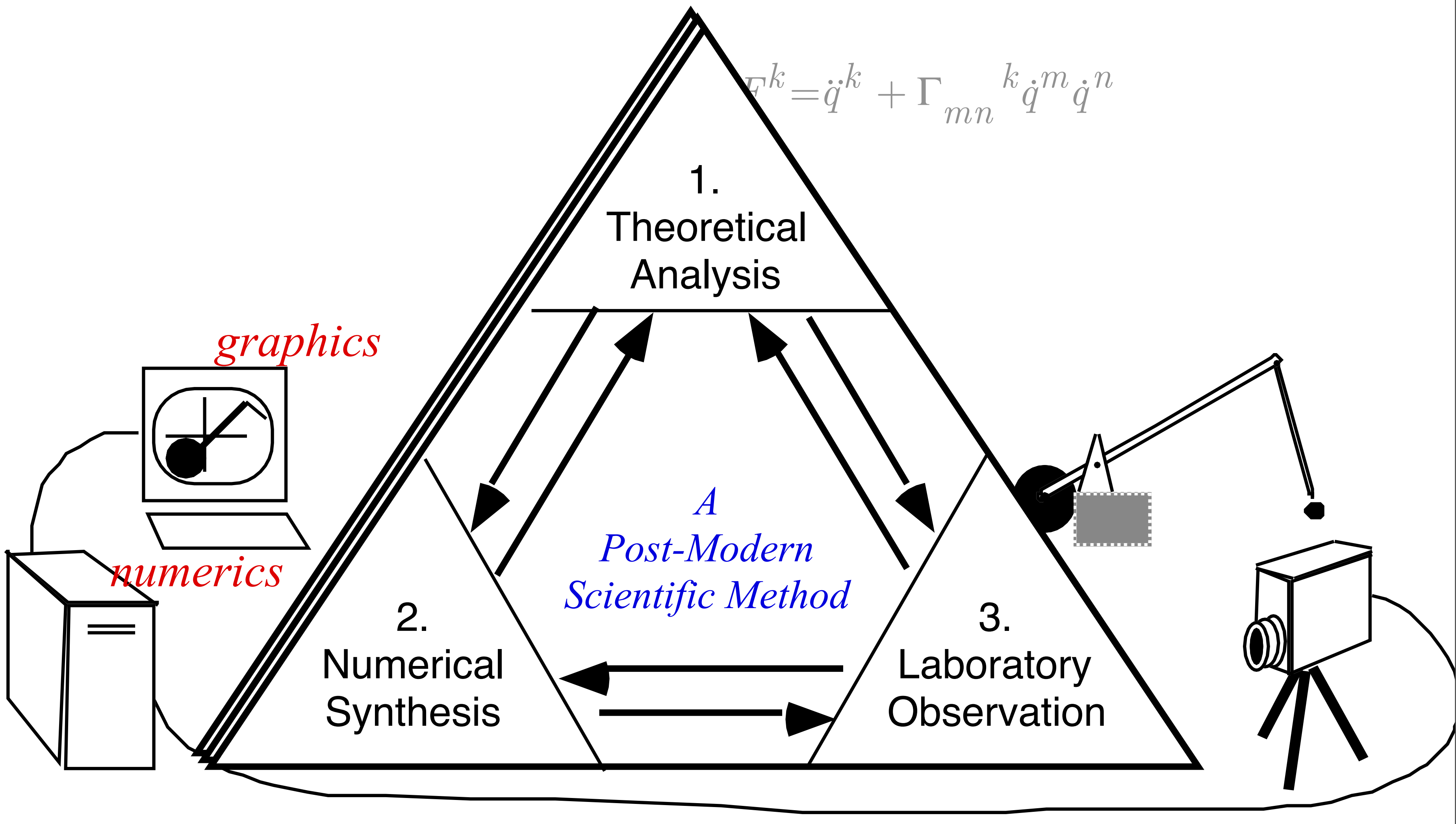
Pride and Precision
Riemann Christoffel Equations
in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

- Anglo-Irish Appproach

Powerfully Creative
Hamilton's Equations
Phase Space $\dot{p}_j = \frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach



$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$

1. Theoretical Analysis

2. Numerical Synthesis

3. Laboratory Observation

A Post-Modern Scientific Method

graphics

numerics

All approaches have one thing in common:
The *Art of Approximation*
Physics lives and dies by the art of
approximate models and analogs.

Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

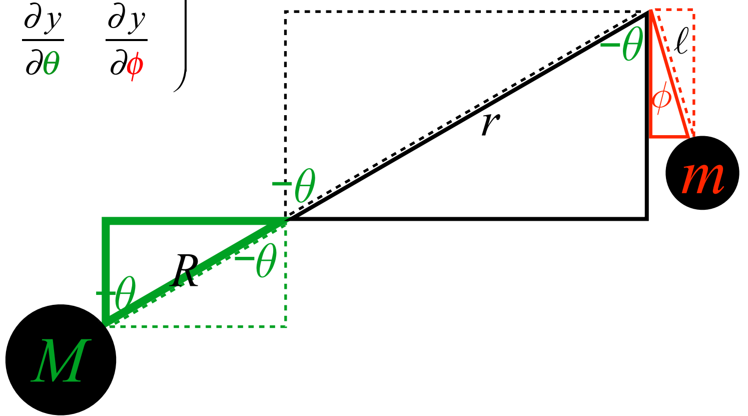
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{array}{lllll}
dW = F_X\ dX & = M\ddot{X}\,dX & = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi & = & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY & +M\ddot{Y}\,dY & +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi & & +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx & +m\ddot{x}\,dx & +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi & & +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy & +m\ddot{y}\,dy & +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi & & +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{array}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Lagrange trickery:

STEP **D**

Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

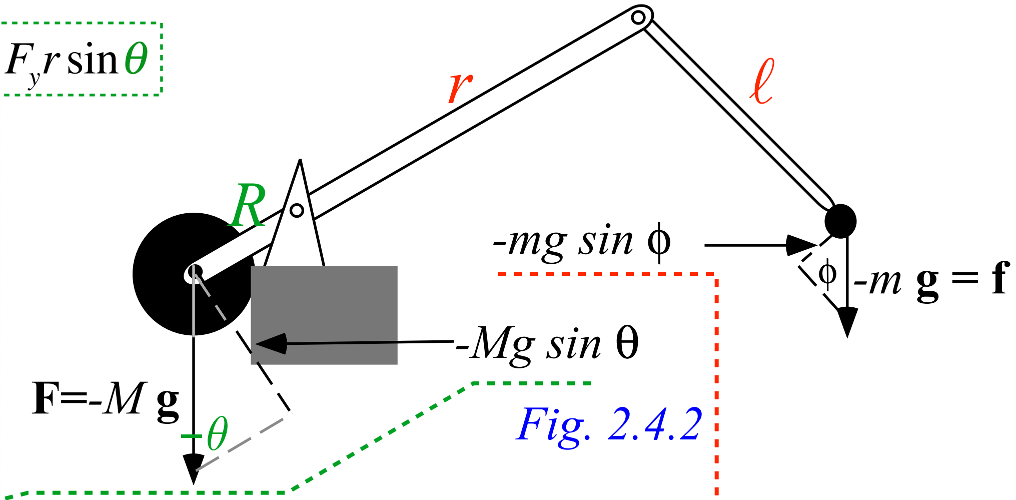


Fig. 2.4.2

Add F_θ gravity given

$(F_X=0\ ,\ F_Y=-Mg)$

$(F_x=0\ ,\ F_y=-mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

These are competing torques on main beam R...

Add F_ϕ gravity given

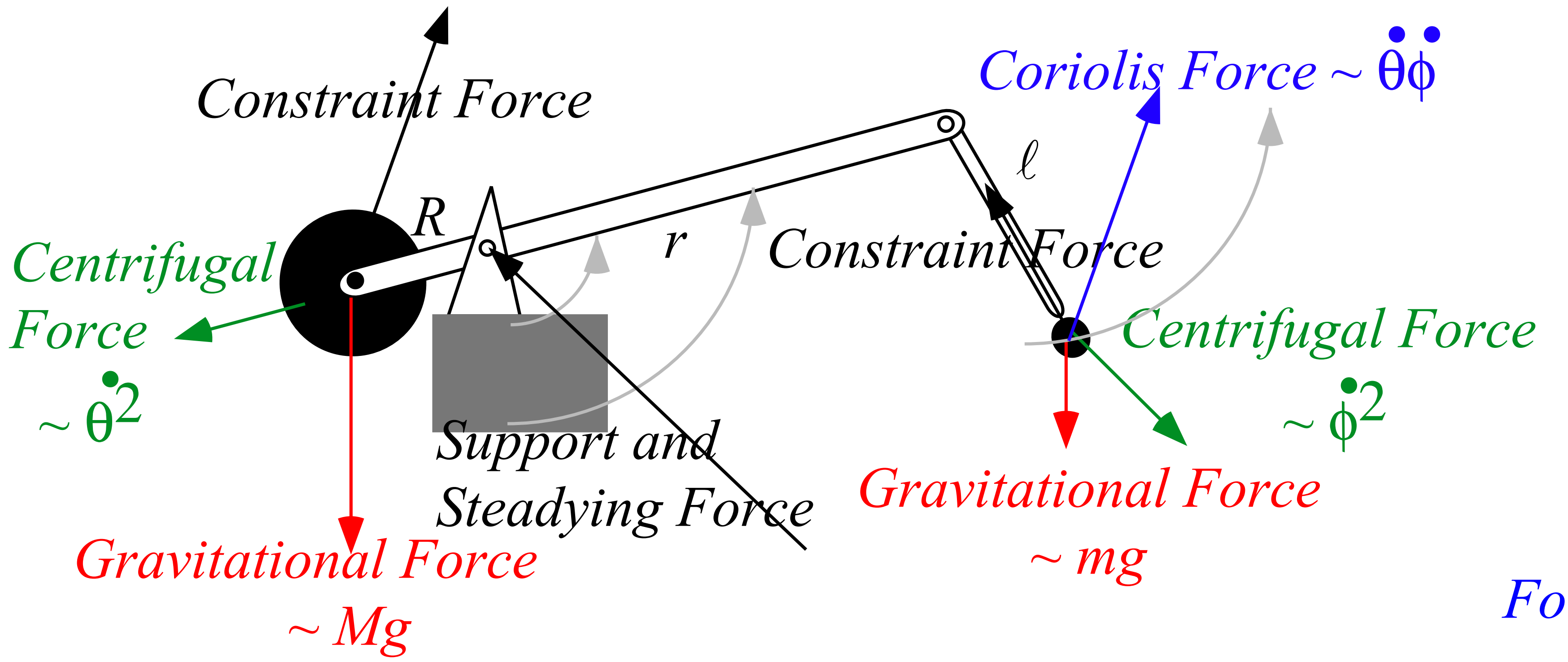
$(F_X=0\ ,\ F_Y=-Mg)$

$(F_x=0\ ,\ F_y=-mg)$

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

... and a torque on throwing lever ℓ

Forces: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot\theta} = 0$

$F_\phi = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot\phi} = 0$

Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot p_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot p_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

$$p_\theta = \frac{\partial L}{\partial \dot\theta} \qquad \dot p_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} \qquad \dot p_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
$L=T-V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.