Lecture 23
Tue. 11.11.2014

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

Some Ways to do constraint analysis

Way 1. Simple constraint insertion
Way 2. GCC constraint webs
- *Find covariant force equations*
- *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

Way 3. OCC constraint webs
- *Preview of atomic-Stark orbits*
  - *Classical Hamiltonian separability*

Way 4. Lagrange multipliers
- *Lagrange multiplier as eigenvalues*
- *Multiple multipliers*
- *"Non-Holonomic" multipliers*

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

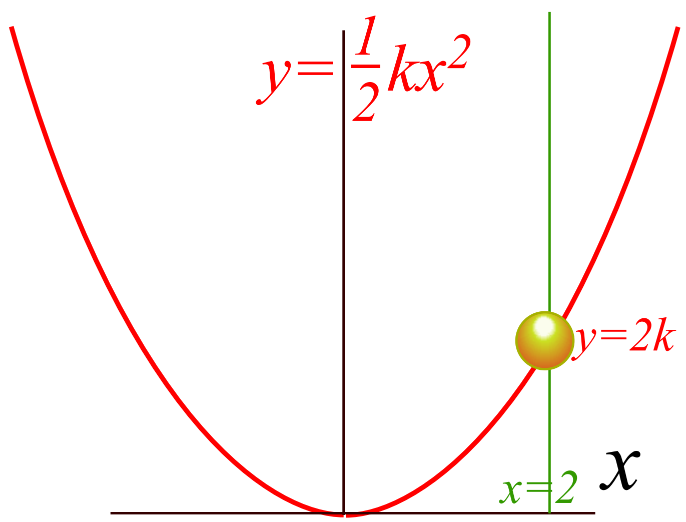
Way 2. GCC constraint webs

Find covariant force equations

Compare covariant vs. contravariant forces

Ways to analyze a particle m constrained to parabola $y=1/2kx^2$
on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

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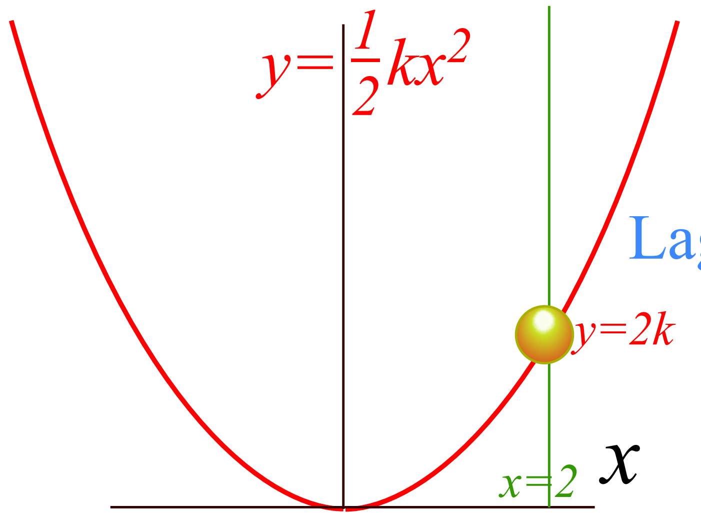

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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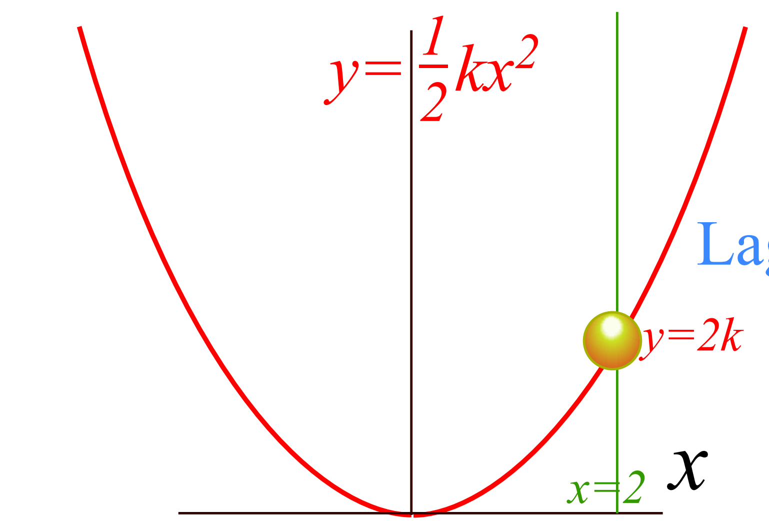

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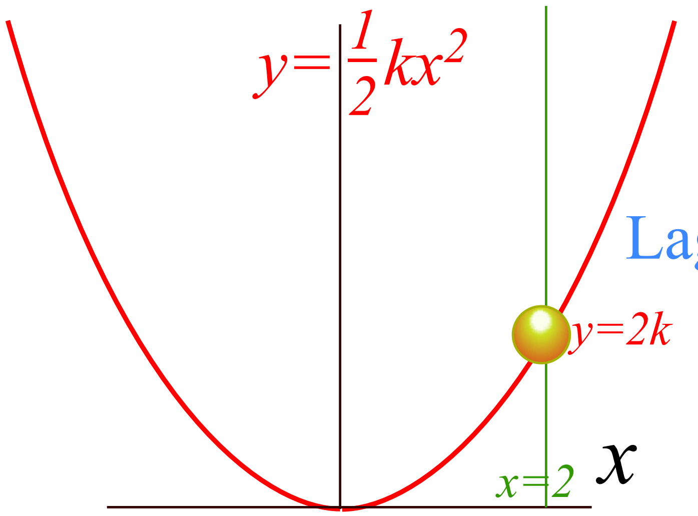

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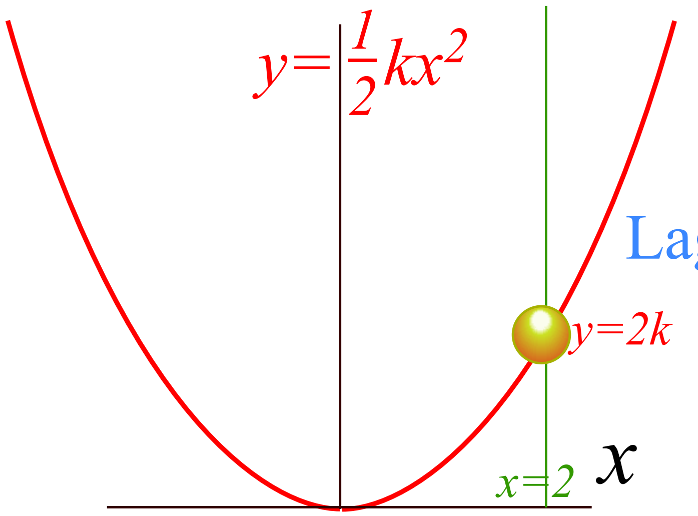

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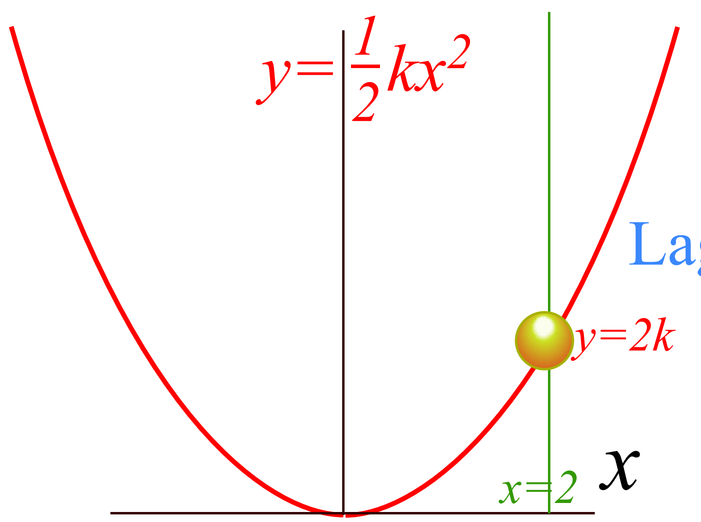

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}$$

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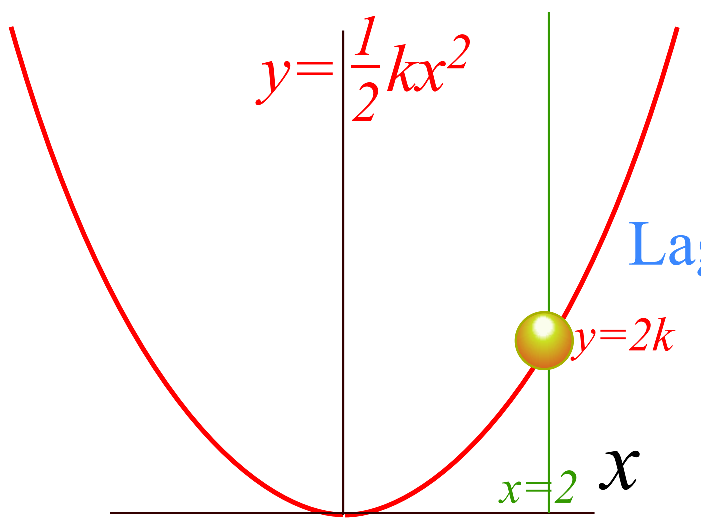

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$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + k^2x^2\dot{x})$$

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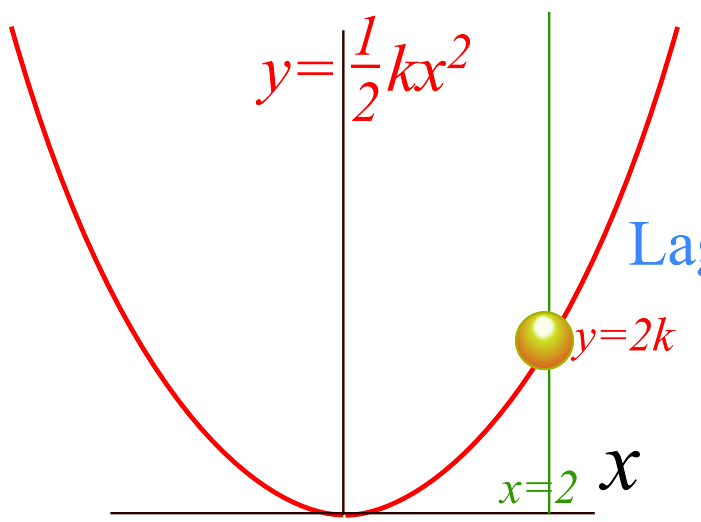

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$$=\frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2) \qquad =m(\dot{x}+k^2x^2\dot{x}) \qquad =m(k^2x\dot{x}^2-gkx)$$

Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$\dot{p}_x=m(1+k^2x^2)\ddot{x} \qquad =-mk^2x\dot{x}^2-mgkx$$

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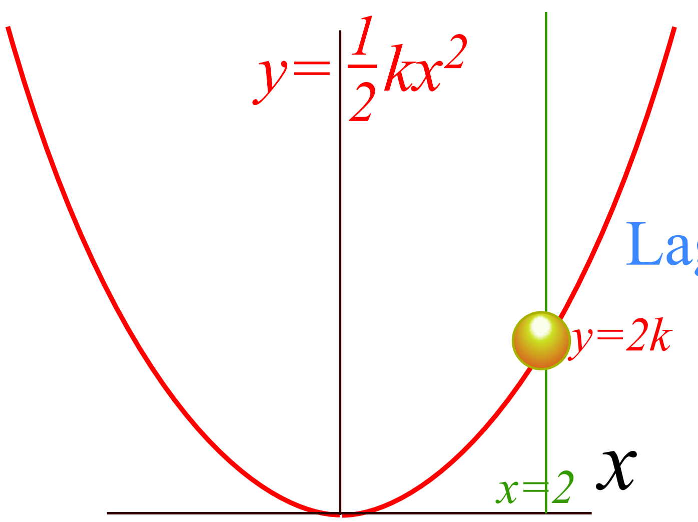

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x}=-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

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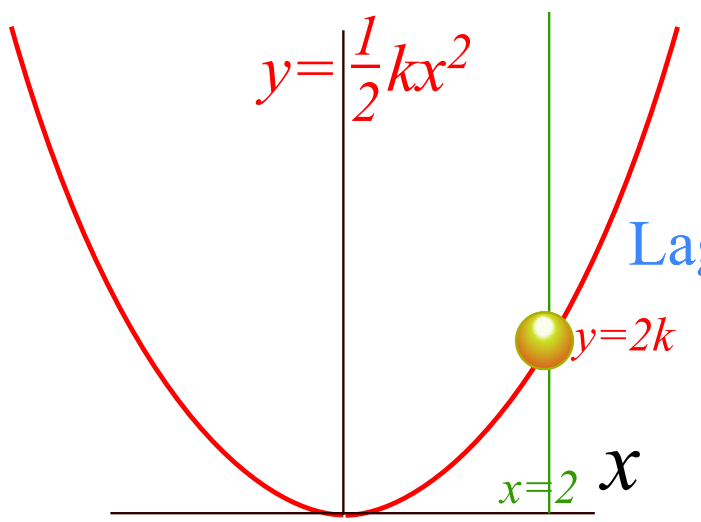

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$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 = \frac{m}{2}(\dot{x}^2+k^2x^2\dot{x}^2-gkx^2)$$

$$p_x=\frac{\partial L}{\partial \dot{x}} = m(\dot{x}+k^2x^2\dot{x}) \qquad f_x=\frac{\partial L}{\partial x} = m(k^2x\dot{x}^2-gkx)$$

Lagrange equation $\dot{p}_x = f_x = \frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x = m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2) = \frac{\partial L}{\partial x} = m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x} \qquad = -mk^2x\dot{x}^2-mgkx = -m(k\dot{x}^2-g)kx$$

$$\boxed{\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx}$$

Some Ways to do constraint analysis

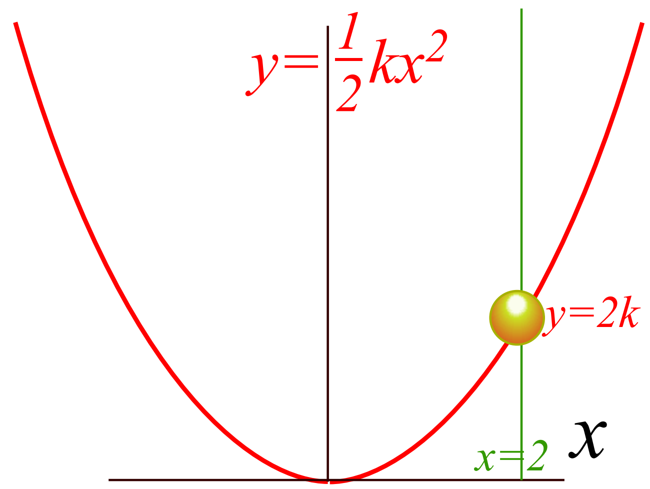
Way 1. Simple constraint insertion

→ *Way 2. GCC constraint webs*

Find covariant force equations

Compare covariant vs. contravariant forces

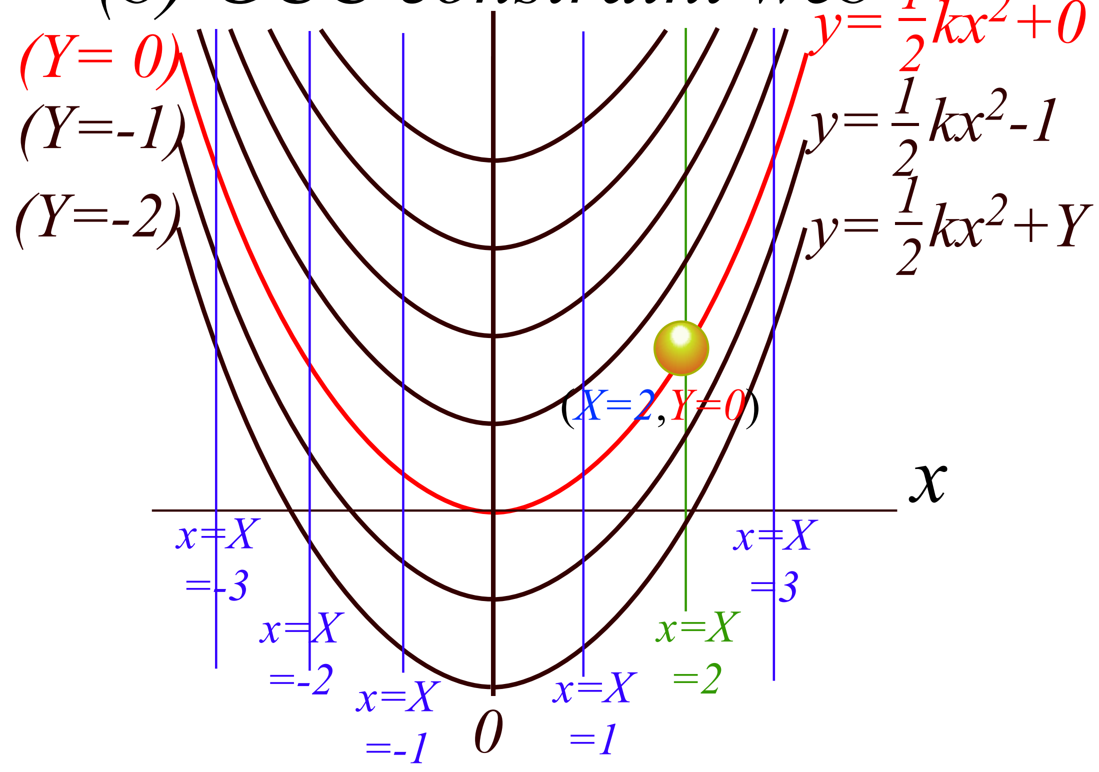
(a) Constrained motion



$$x = X$$
$$y = \frac{1}{2}kx^2 + Y$$

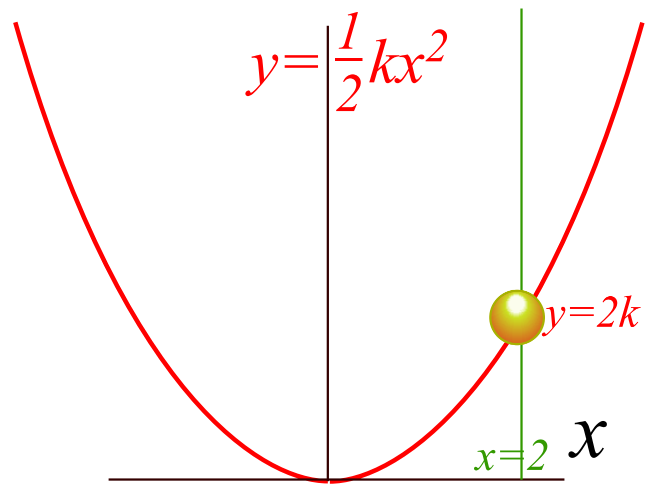
Cartesian (x,y) transform to GCC (X,Y)

(b) GCC constraint web



Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

(a) Constrained motion



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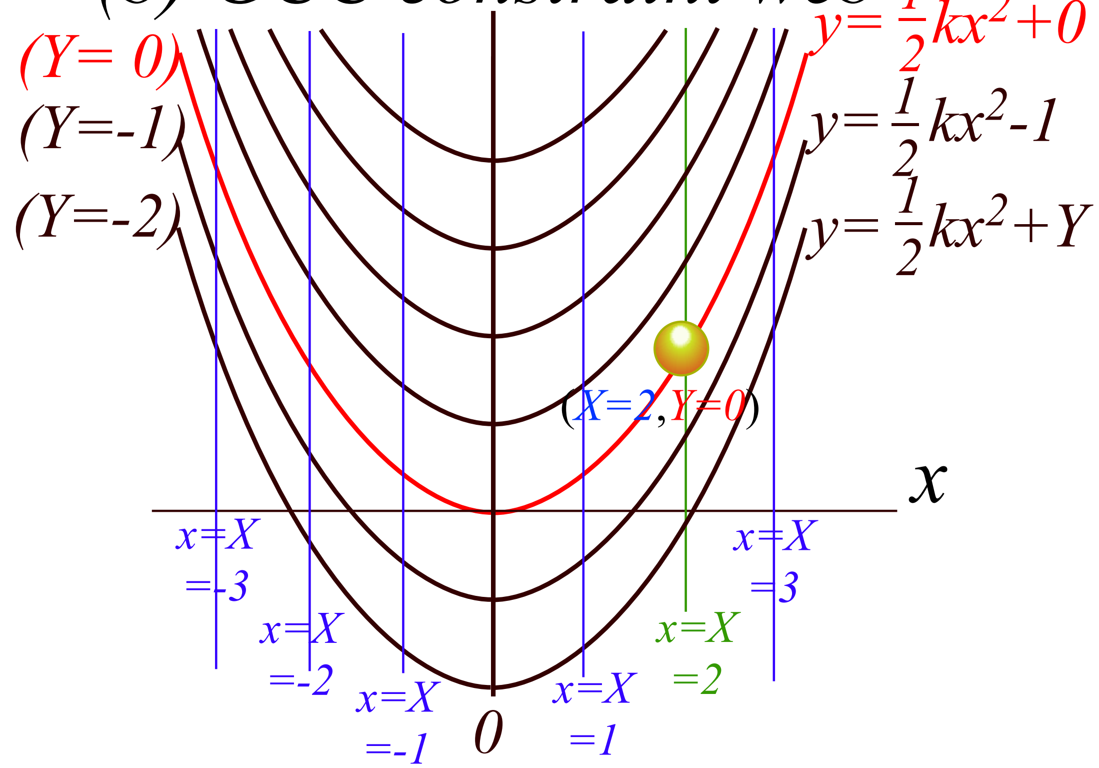

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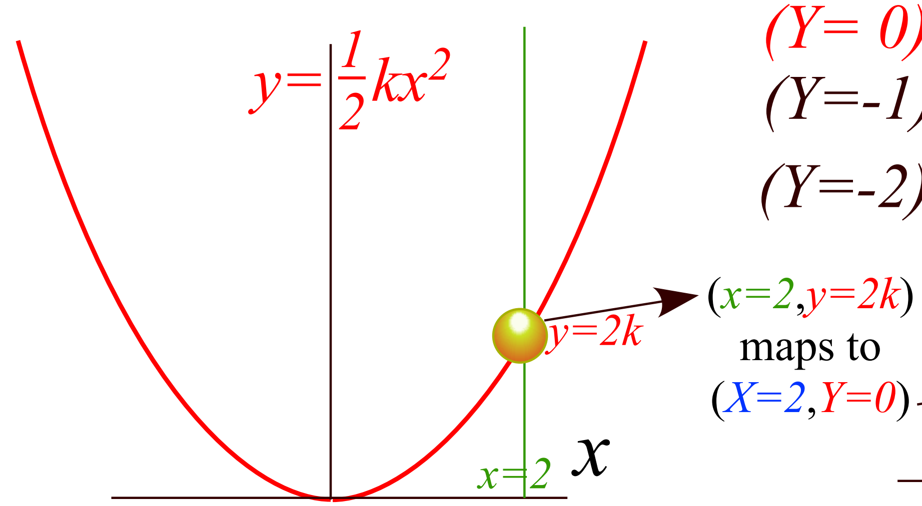
$x=q^1=X$ $\quad$ $y=1/2kx^2 + q^2 = kX^2/2+Y$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Way 2. GCC constraint webs.

(a) Constrained motion



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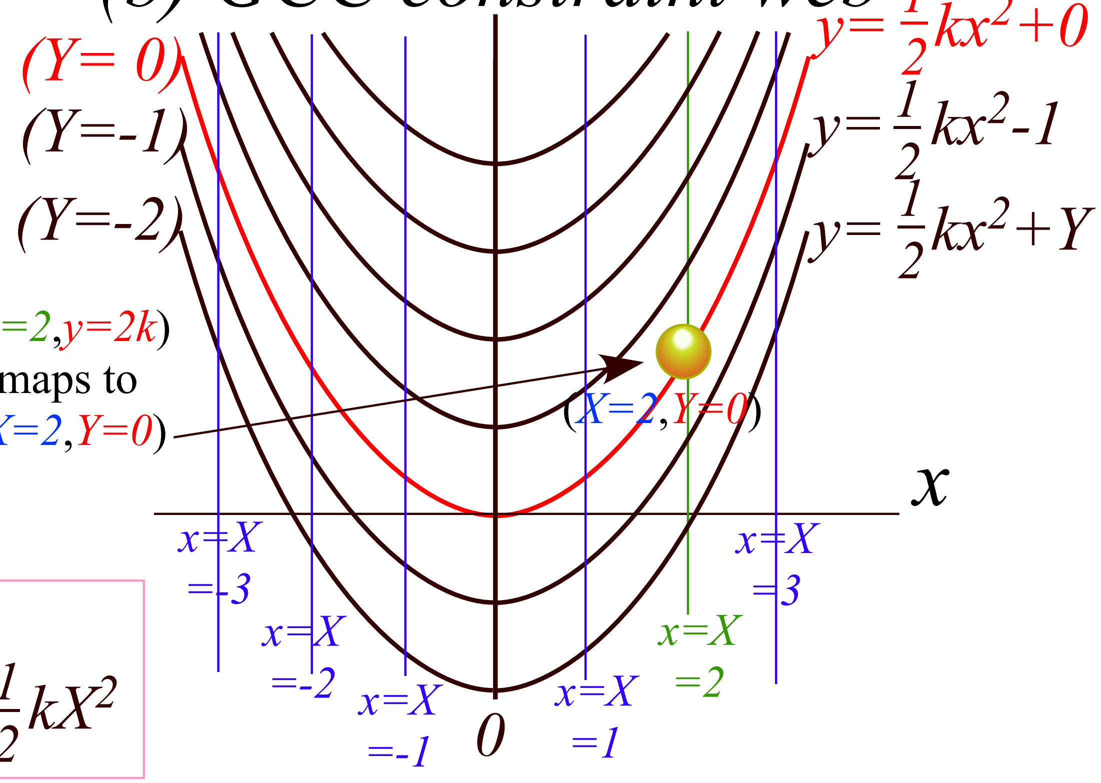
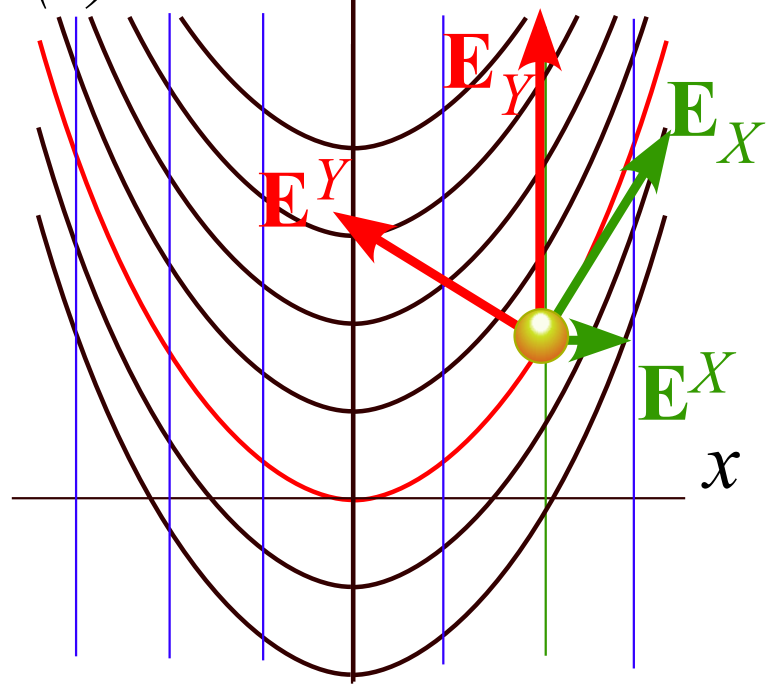


(c) GCC **E**-*vectors*



$$x = X$$
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Cartesian (x,y) transform to GCC (X,Y)

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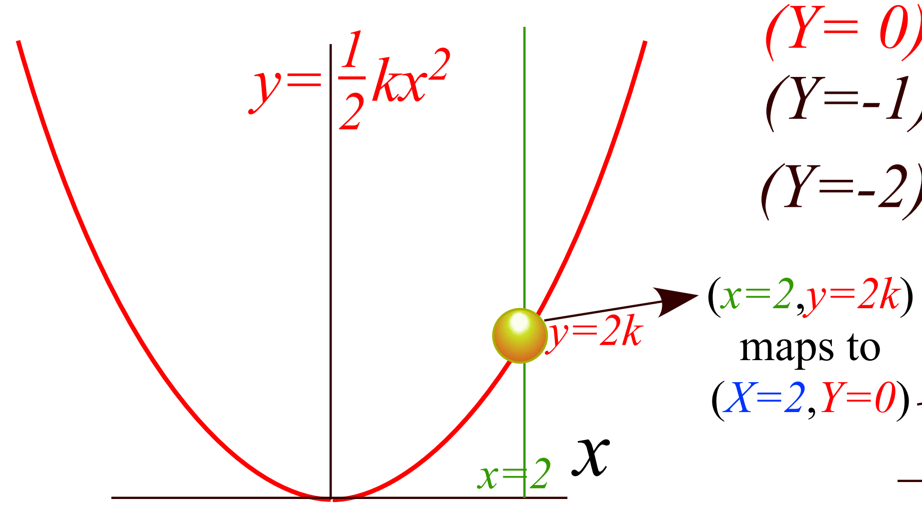
Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$$x = q^1 = X \qquad y = 1/2kx^2 + q^2 = kX^2/2 + Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \ \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Constrained motion



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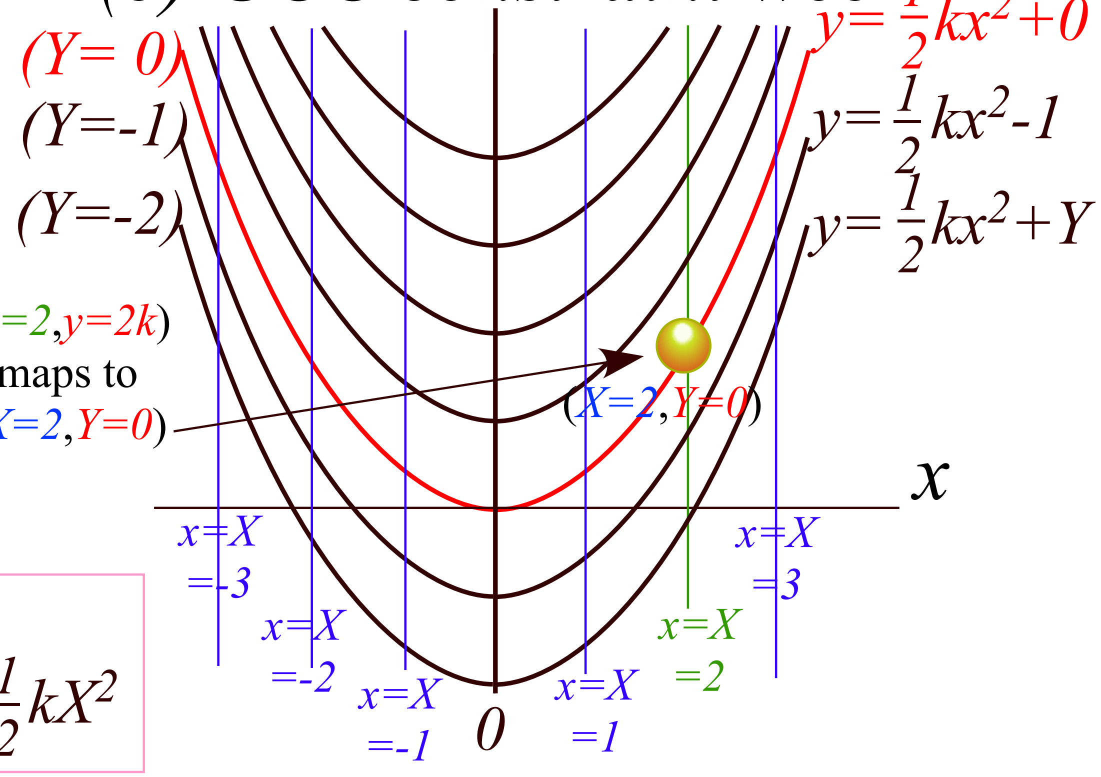

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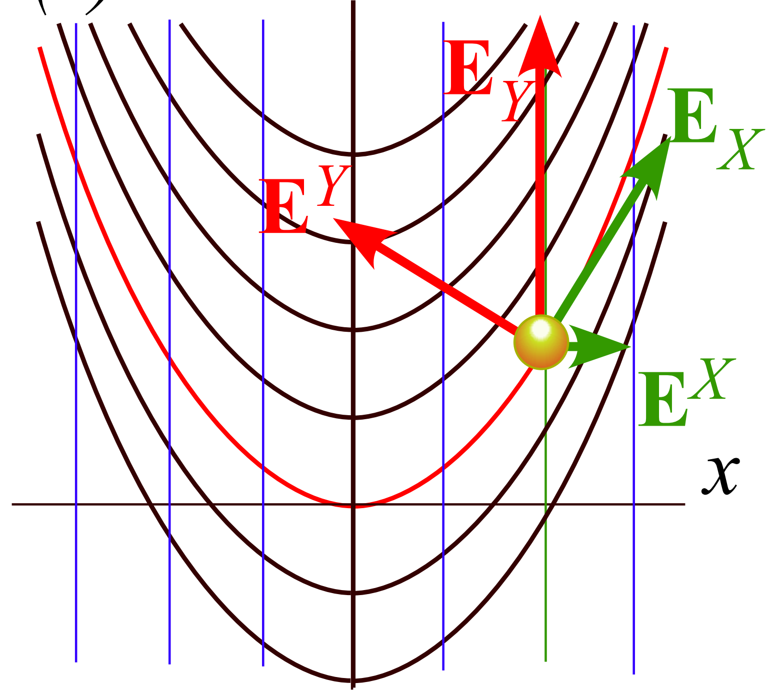

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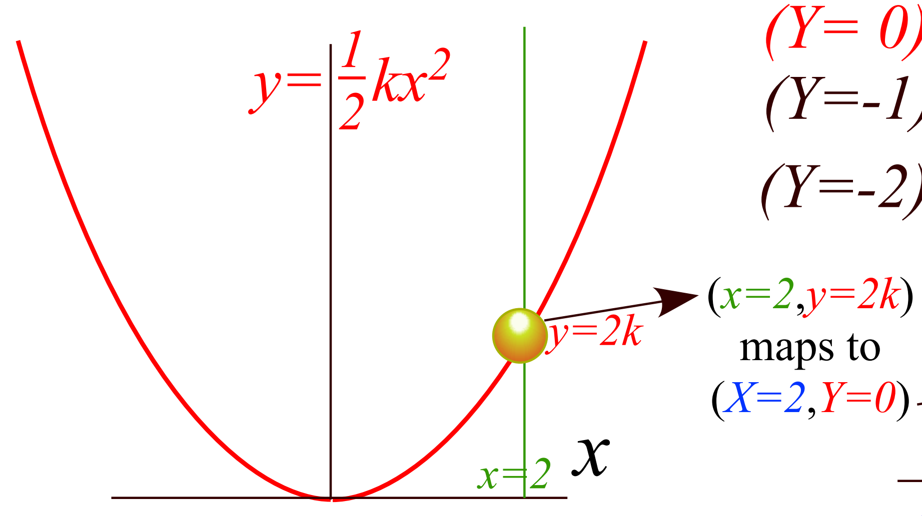
Contravariant $\mathbf{E}^k$ in rows of Kajobian K

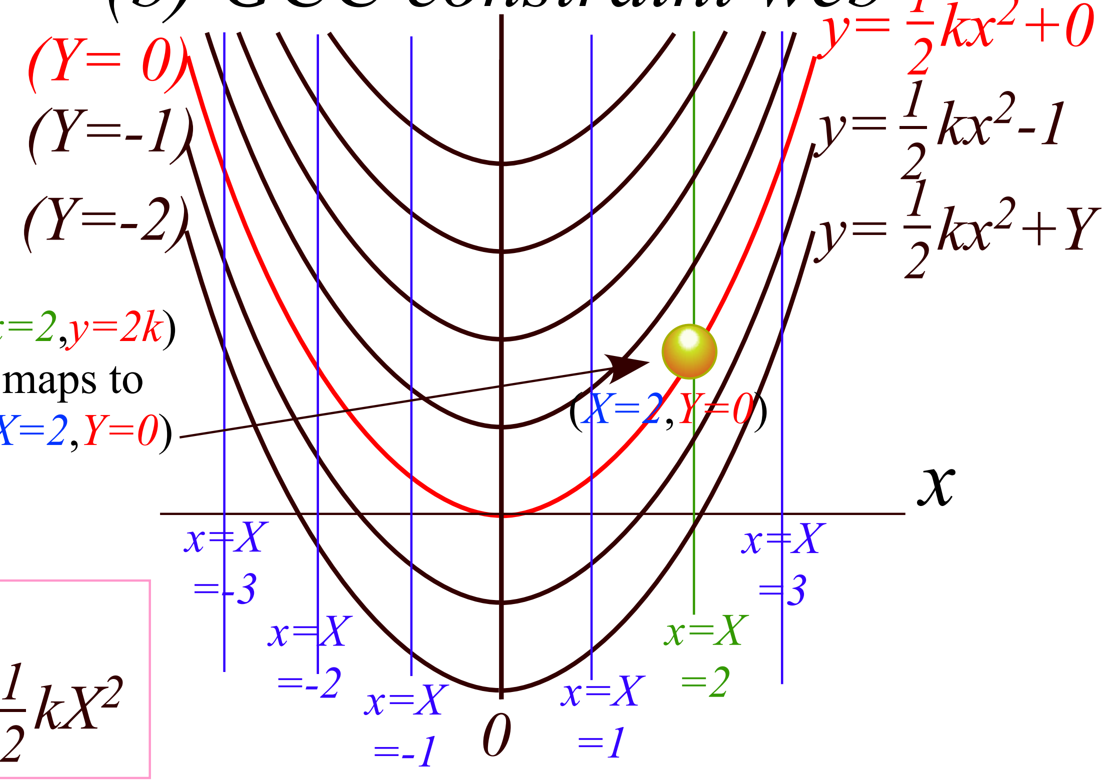
$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix}$$

Way 2. *GCC constraint webs.*

(a) Constrained motion



(b) GCC constraint web



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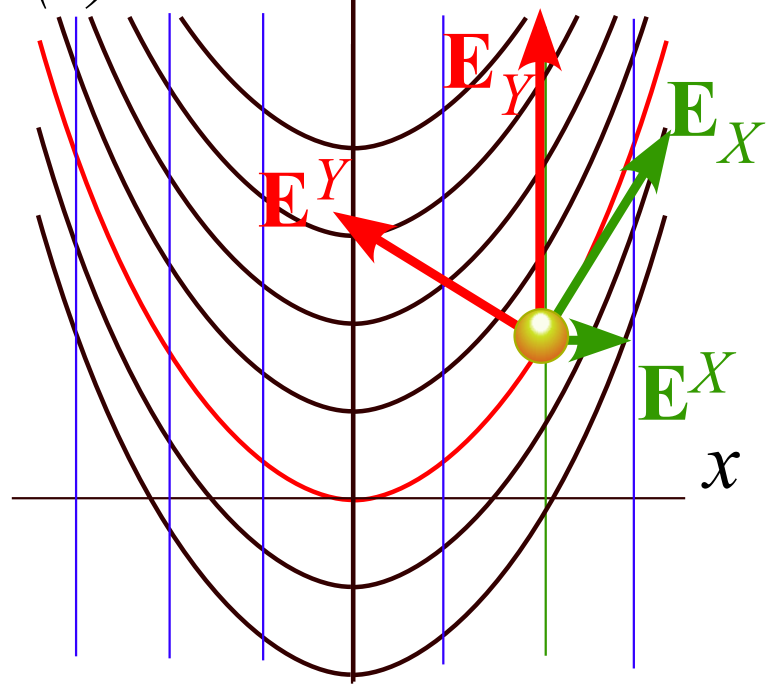

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$y= \frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$

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we define shorthand:

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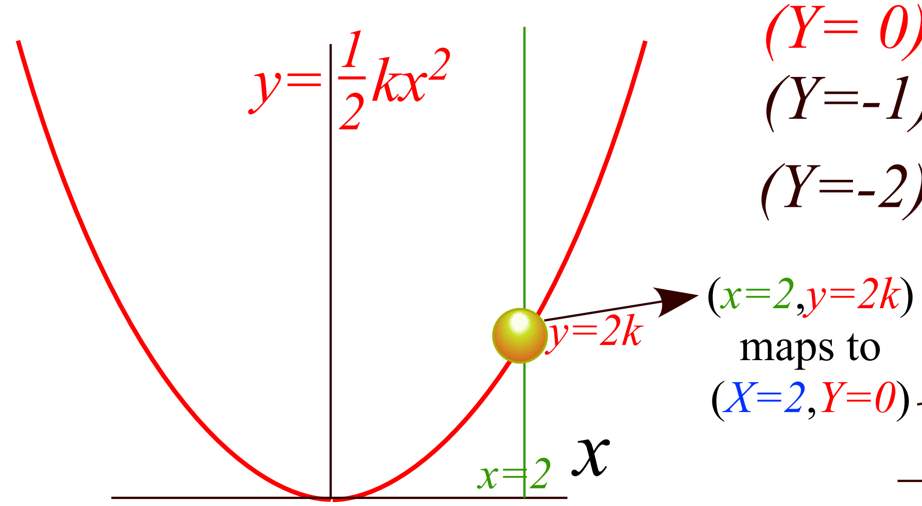
Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \begin{matrix} \mathbf{E}^X=\begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y=\begin{pmatrix} -kx & 1 \end{pmatrix} \end{matrix}$$

Find: 1st coordinate differentials and velocity relations: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}$ $\qquad$ $\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

Way 2. *GCC constraint webs.*

(a) Constrained motion



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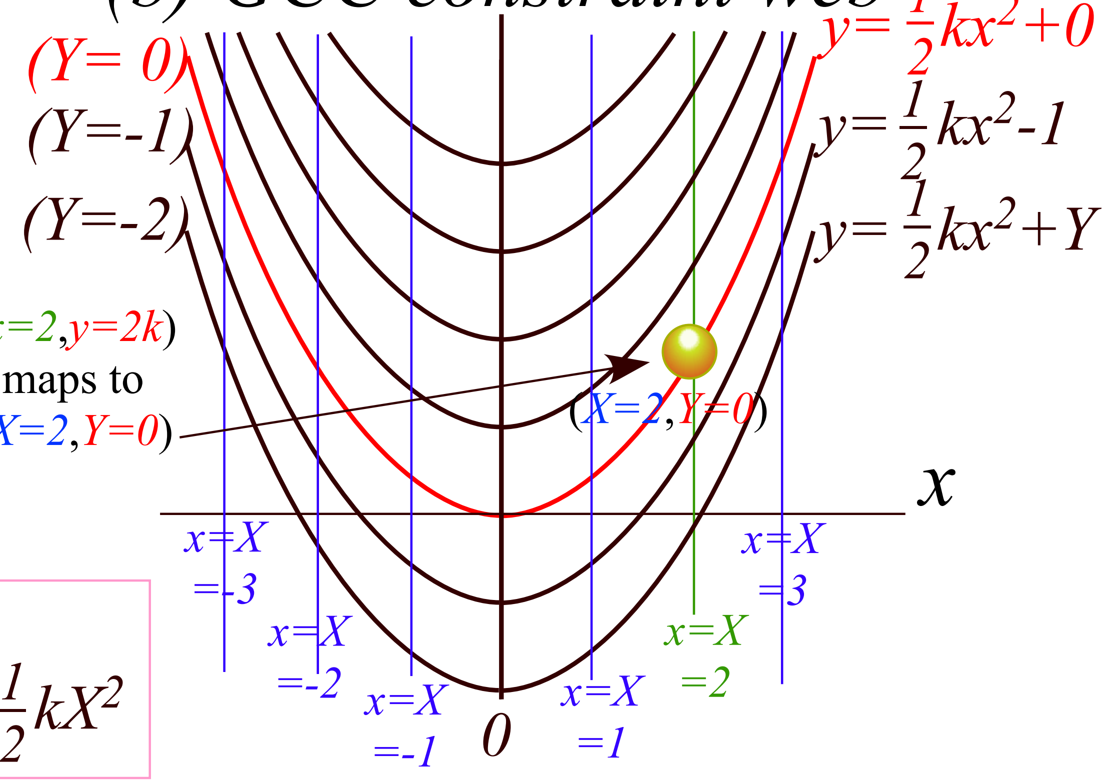

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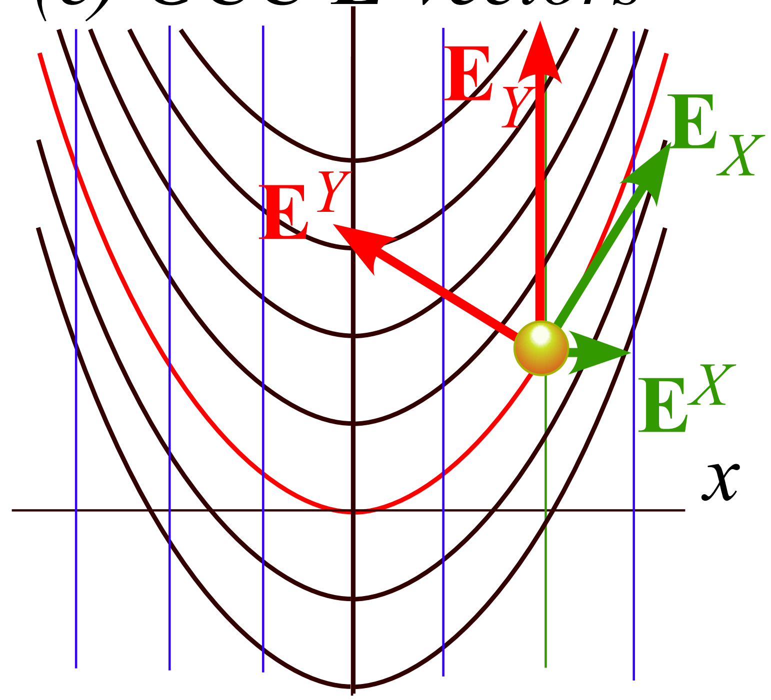

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Incorporate the constraint curve $y=1/2kx^2$ into any matching GCC web.

$$x=q^1=X \qquad y=1/2kx^2+q^2=kX^2/2+Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \begin{matrix}\mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix}\\ \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}\end{matrix}$$

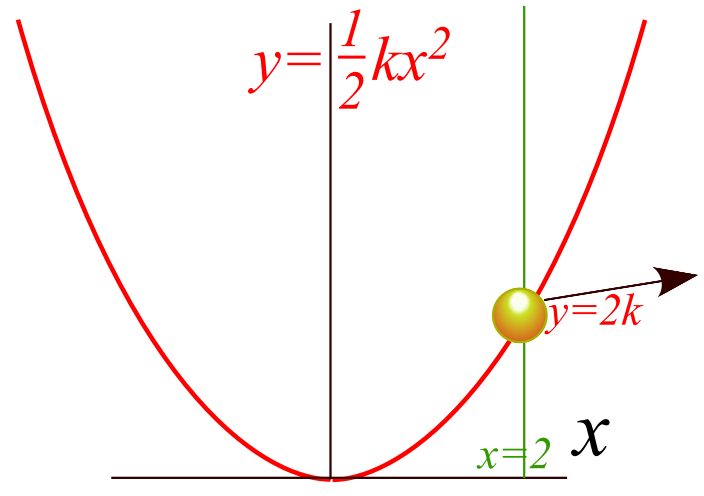
Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} \qquad \begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}=(JJ^\dagger)_{AB}$

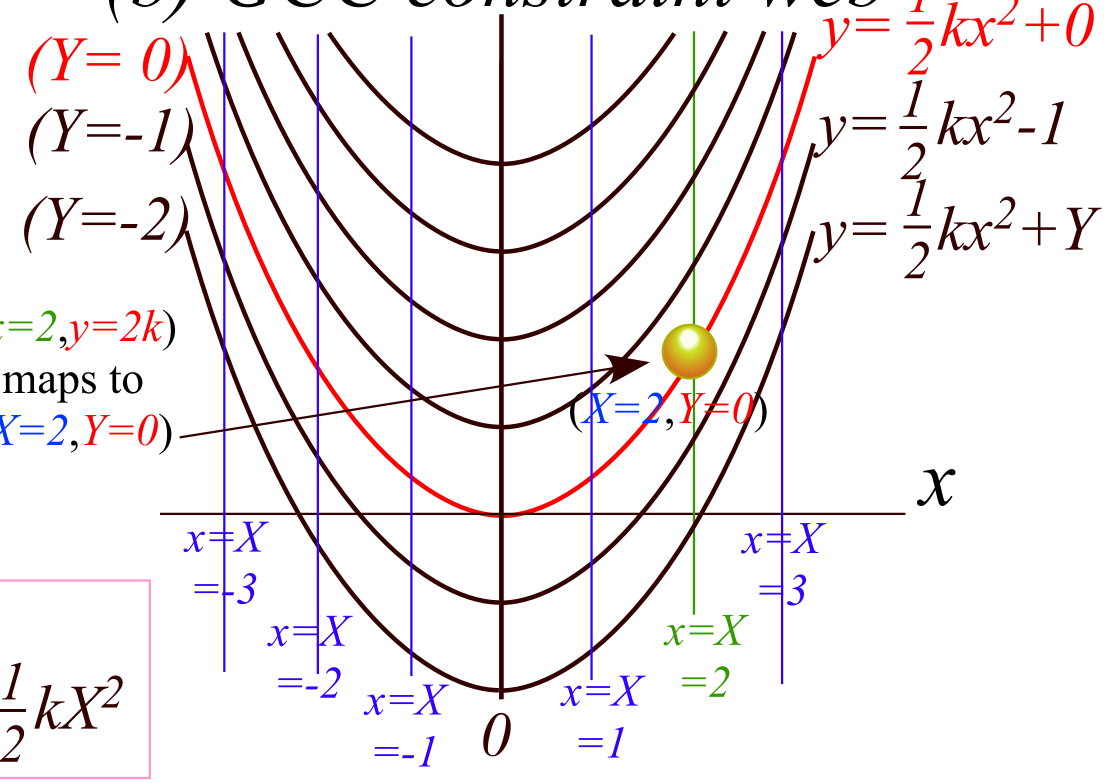
$$m\begin{pmatrix}\mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y\\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y\end{pmatrix}=\begin{pmatrix}\gamma_{XX} & \gamma_{XY}\\ \gamma_{YX} & \gamma_{YY}\end{pmatrix}=m\begin{pmatrix}1+k^2x^2 & kx\\ kx & 1\end{pmatrix}$$

(a) Constrained motion

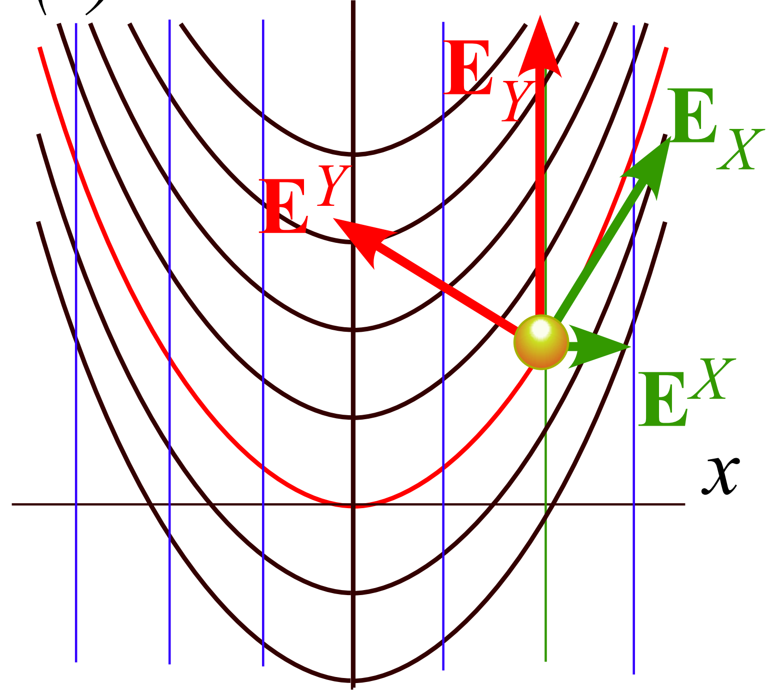


(x=2,y=2k) maps to (X=2,Y=0)

(b) GCC constraint web



(c) GCC **E**-vectors



$x = X$ $y=\frac{1}{2}kx^2+Y$	Cartesian (x,y) transform to GCC (X,Y)	$X=x$ $Y=y-\frac{1}{2}kX^2$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{ueer}^{Indices}$

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Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \begin{matrix}\mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix}\\ \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}\end{matrix}$$

Find: 1st coordinate differentials and velocity relations:

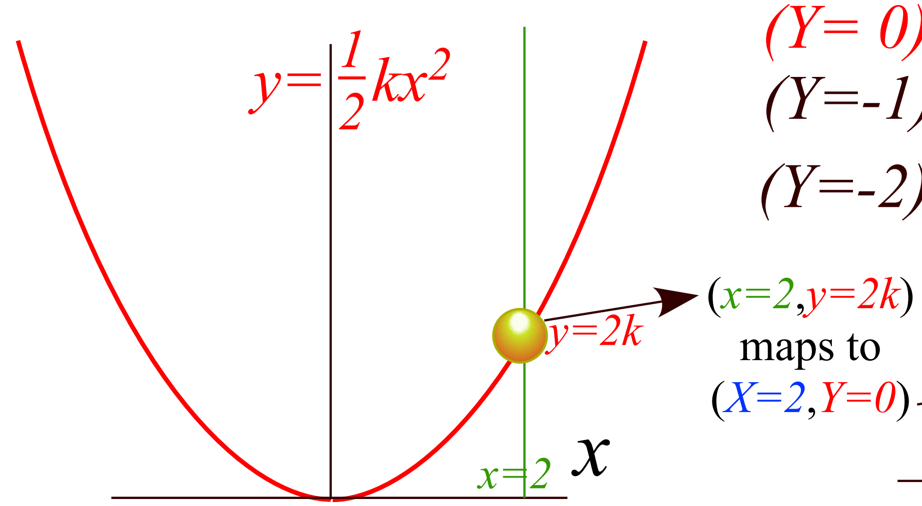
$$\begin{pmatrix}\dot{x}\\\dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\\dot{Y}\end{pmatrix} \qquad \begin{pmatrix}\dot{X}\\\dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\\dot{y}\end{pmatrix}$$

Find: Kinetic coefficients $\gamma_{AB}=mg_{AB}$ from metric tensor g_{AB} or Jacobian square $g_{AB}=J_{AC}J_{BC}=(JJ^{\dagger})_{AB}$

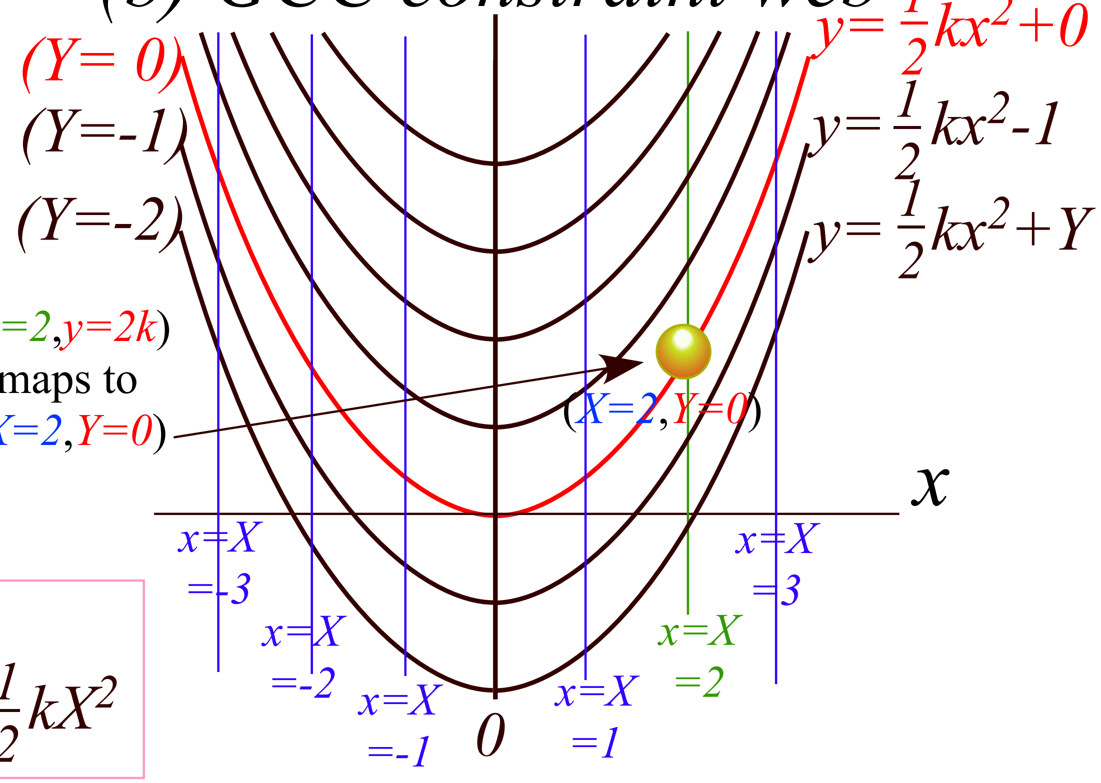
$$m\begin{pmatrix}\mathbf{E}_X\bullet\mathbf{E}_X & \mathbf{E}_X\bullet\mathbf{E}_Y\\ \mathbf{E}_Y\bullet\mathbf{E}_X & \mathbf{E}_Y\bullet\mathbf{E}_Y\end{pmatrix}=\begin{pmatrix}\gamma_{XX} & \gamma_{XY}\\ \gamma_{YX} & \gamma_{YY}\end{pmatrix}=m\begin{pmatrix}1+k^2x^2 & kx\\ kx & 1\end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix}\mathbf{E}^X\bullet\mathbf{E}^X & \mathbf{E}^X\bullet\mathbf{E}^Y\\ \mathbf{E}^Y\bullet\mathbf{E}^X & \mathbf{E}^Y\bullet\mathbf{E}^Y\end{pmatrix}=\begin{pmatrix}\gamma^{XX} & \gamma^{XY}\\ \gamma^{YX} & \gamma^{YY}\end{pmatrix}=\frac{1}{m}\begin{pmatrix}1 & -kx\\ -kx & 1+k^2x^2\end{pmatrix}$$

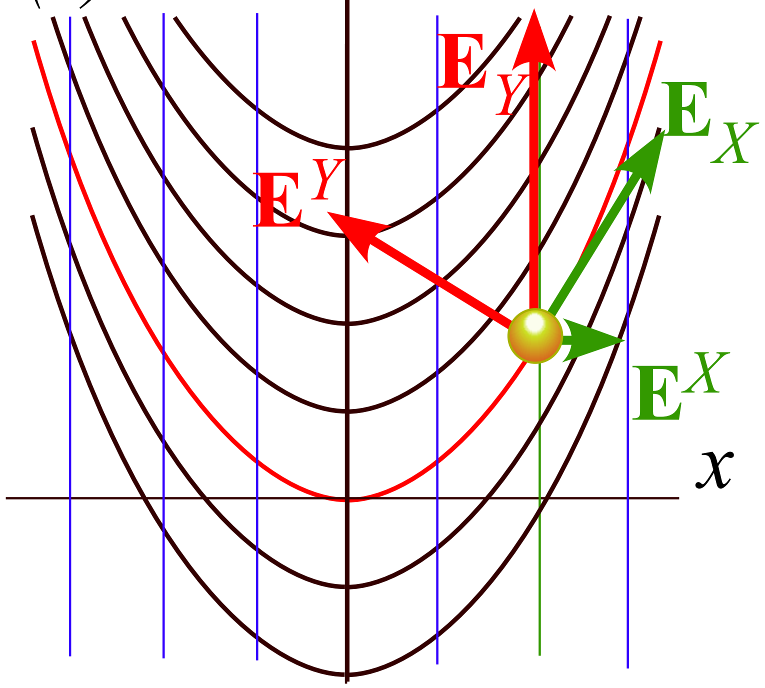
(Need contra-γ for Hamilton or Riemann equations) *(a) Constrained motion*



(b) GCC constraint web



(c) GCC **E**-*vectors*



$$x = X$$
$$y = \tfrac{1}{2}kx^2 + Y$$

Cartesian (x,y) transform to GCC (X,Y)

$$X = x$$
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Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix} = K \qquad \mathbf{E}^X = \begin{pmatrix} 1 & 0 \end{pmatrix},\quad \mathbf{E}^Y = \begin{pmatrix} -kx & 1 \end{pmatrix}$$

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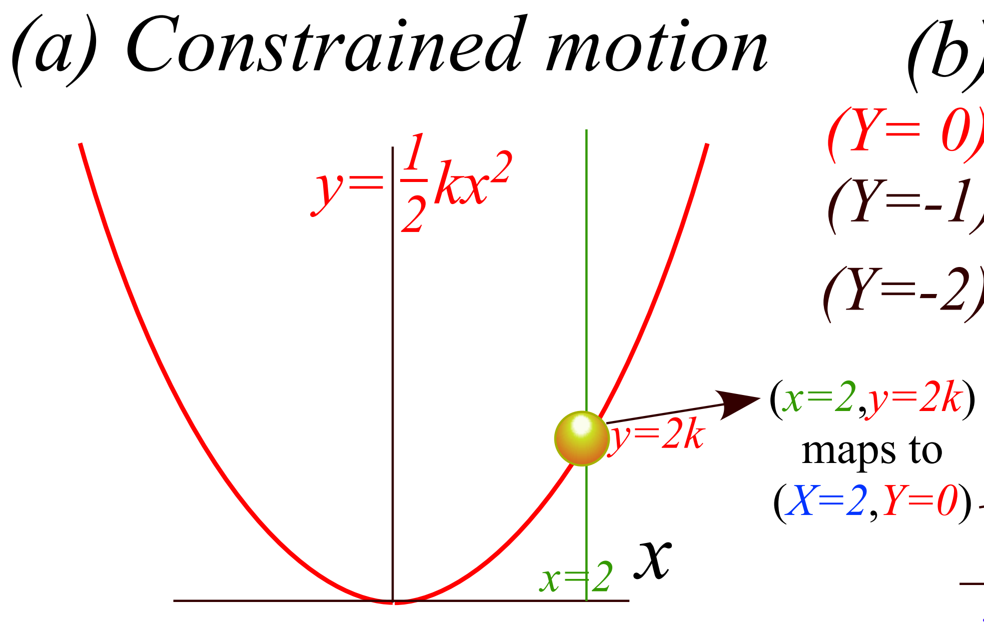
$$\frac{1}{m}\begin{pmatrix} \mathbf{E}^X \bullet \mathbf{E}^X & \mathbf{E}^X \bullet \mathbf{E}^Y \\ \mathbf{E}^Y \bullet \mathbf{E}^X & \mathbf{E}^Y \bullet \mathbf{E}^Y \end{pmatrix} = \begin{pmatrix} \gamma^{XX} & \gamma^{XY} \\ \gamma^{YX} & \gamma^{YY} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & -kx \\ -kx & 1+k^2x^2 \end{pmatrix}$$

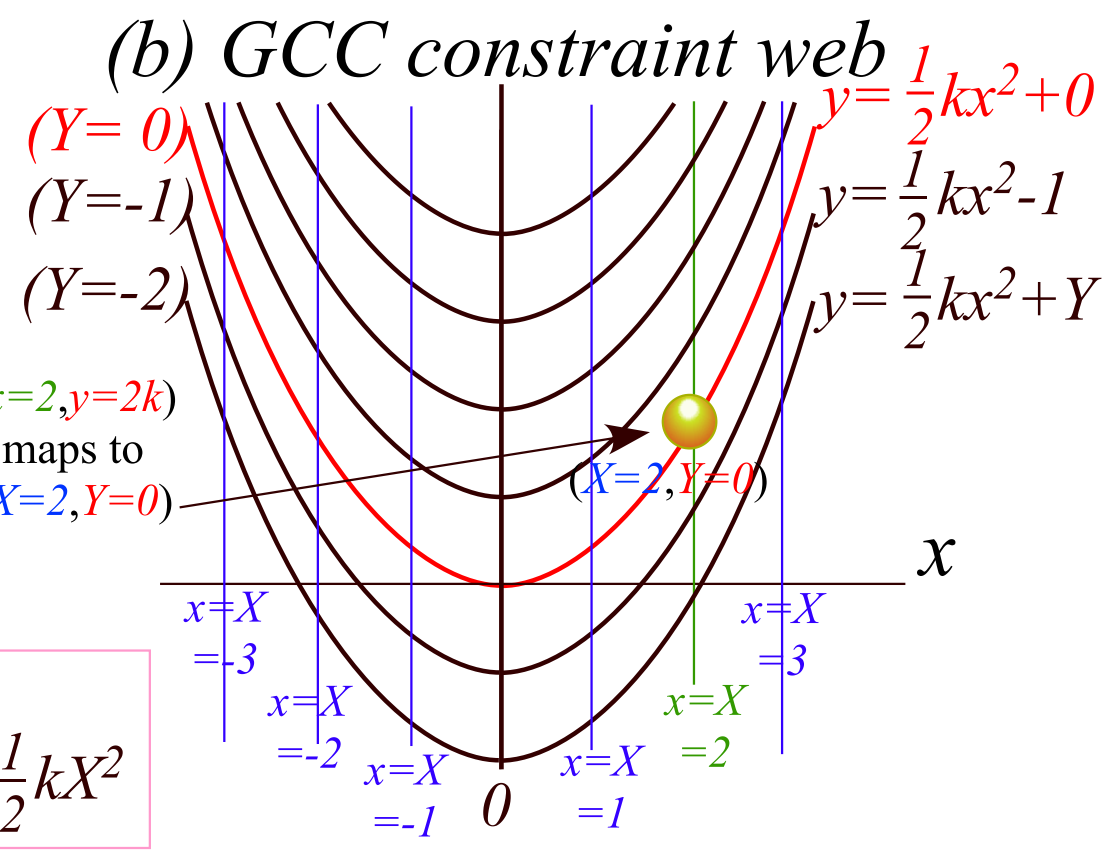
(Need contra-γ for Hamilton or Riemann equations)

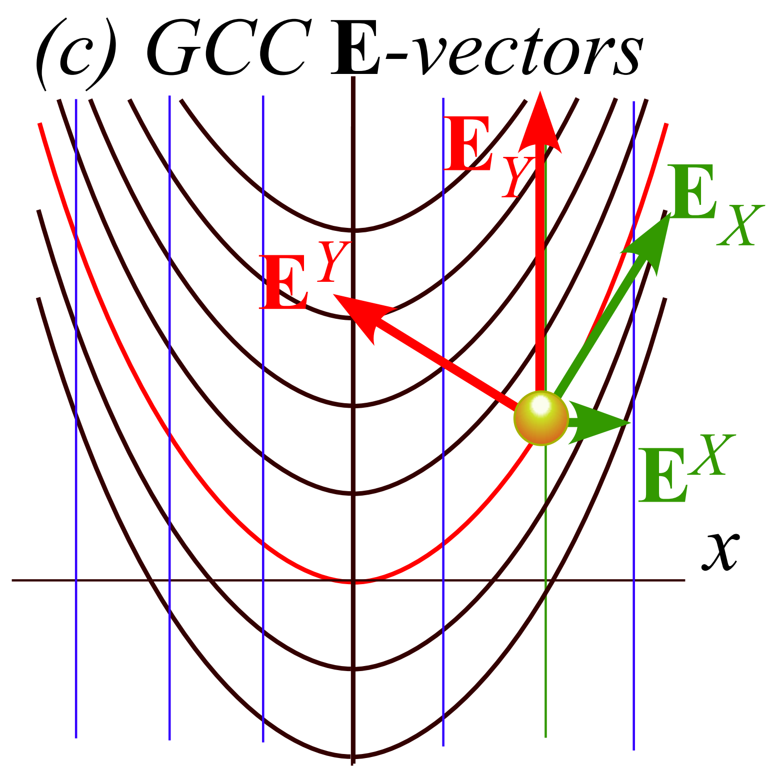
Find: Kinetic energy:

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}(\gamma_{XX}\dot{X}^2 + 2\gamma_{XY}\dot{X}\dot{Y} + \gamma_{YY}\dot{Y}^2) = m\left[\tfrac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \tfrac{1}{2}\dot{Y}^2\right]$$

Way 2. GCC constraint webs.







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Contravariant $\mathbf{E}^k$ in rows of Kajobian K

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Find: Kinetic energy: $T=\frac{1}{2}m(\dot x^2+\dot y^2)=\frac{1}{2}(\gamma_{XX}\dot X^2+2\gamma_{XY}\dot X\dot Y+\gamma_{YY}\dot Y^2)=m\left[\frac{1}{2}(1+k^2X^2)\dot X^2+kX\dot X\dot Y+\frac{1}{2}\dot Y^2\right]$

...and Lagrangian: $L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot X^2+kX\dot X\dot Y+\frac{1}{2}\dot Y^2-gY-\frac{gk}{2}X^2\right]$ $\qquad V=mgy=mg(Y+kX^2/2)$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

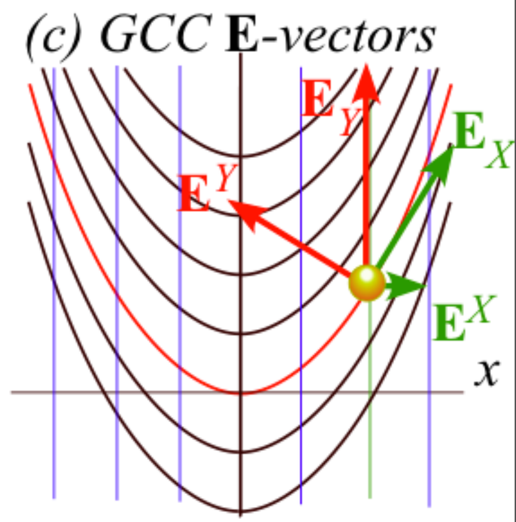
→ *Find covariant force equations*

Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$



Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

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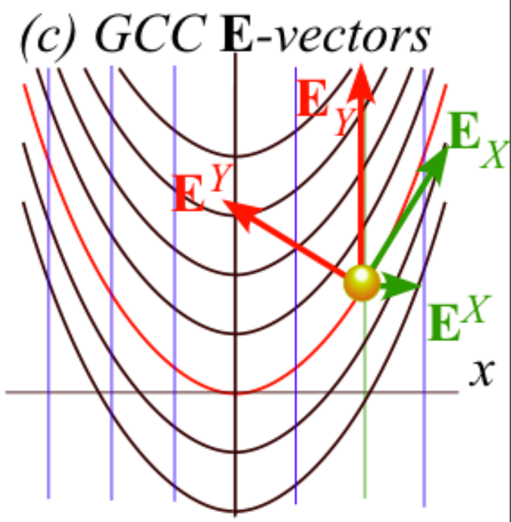
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m \overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$



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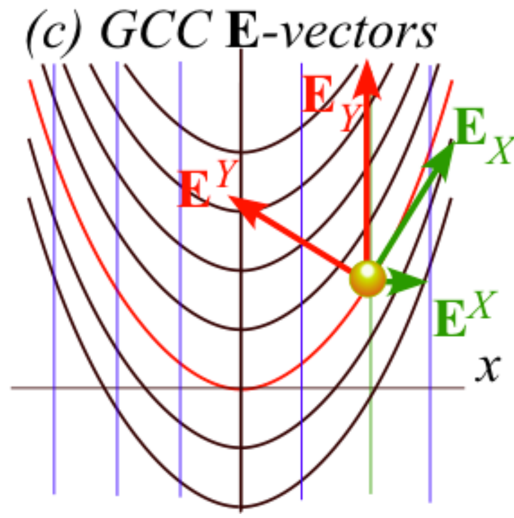

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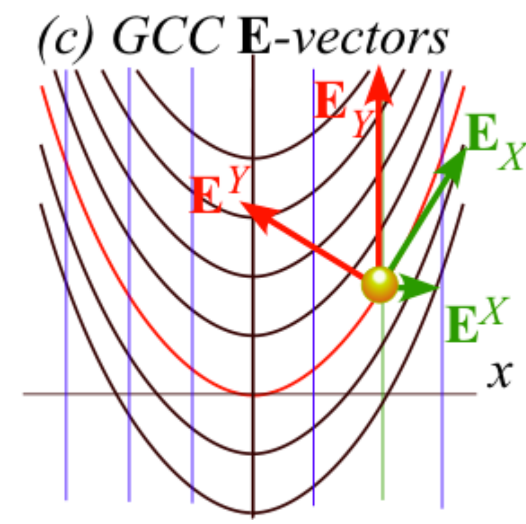

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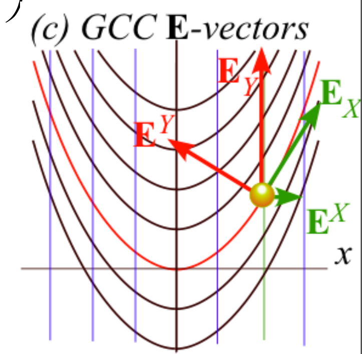
$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



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$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

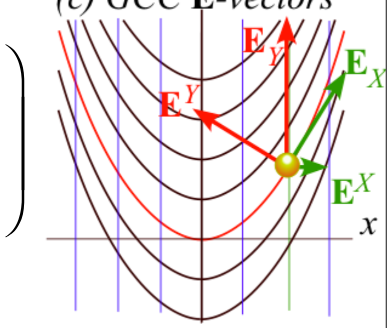
No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero.($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



Find: Lagrange equations from Lagrangian $L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2+kX\dot{X}\dot{Y}+\frac{1}{2}\dot{Y}^2-gY-\frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix}=m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}=\begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1^{st} Lagrange equations) $p_m=\frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix}=\frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right]=\begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2^{nd} Lagrange equations) $\dot{p}_m=\frac{\partial L}{\partial q^m}+F_m^{cov}$

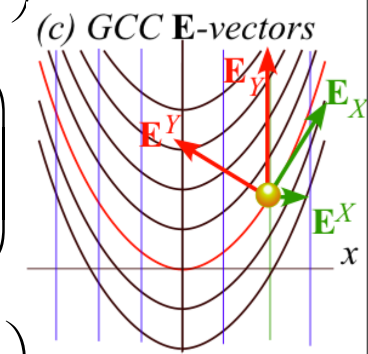
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix}=m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}+m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}=\begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}=m\begin{pmatrix} k^2X\dot{X}^2+k\dot{X}\dot{Y}-gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero.($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X-\frac{\partial L}{\partial X} \\ \dot{p}_Y-\frac{\partial L}{\partial Y} \end{pmatrix}=m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix}+m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}-m\begin{pmatrix} k^2X\dot{X}^2+k\dot{X}\dot{Y}-gkX \\ -g \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X-\frac{\partial L}{\partial X} \\ \dot{p}_Y-\frac{\partial L}{\partial Y} \end{pmatrix}=m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix}+m\begin{pmatrix} k^2X\dot{X}^2+gkX \\ k\dot{X}^2+g \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X-\frac{\partial L}{\partial X} \\ \dot{p}_Y-\frac{\partial L}{\partial Y} \end{pmatrix}=m\begin{pmatrix} (1+k^2X^2)\ddot{X}+kX\ddot{Y}+k^2X\dot{X}^2+gkX \\ kX\ddot{X}+\ddot{Y}+k\dot{X}^2+g \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\overset{(metric\ \gamma_{AB})}{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F^{cov}_m$

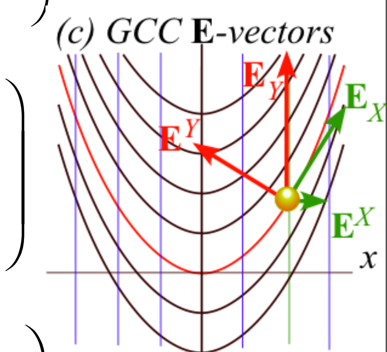
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F^{cov}_m is zero. ($F^{cov}_X = 0 = F^{cov}_Y$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^{cov}_X \\ F^{cov}_Y \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint Y=$_{const.}$)

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

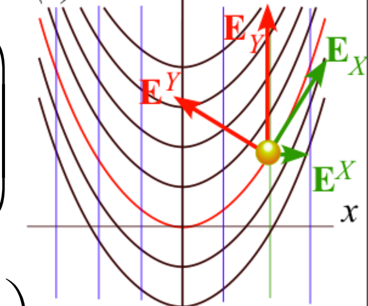
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB})

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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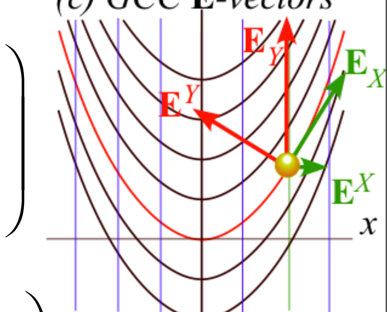
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



(c) GCC **E**-vectors

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB})

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} \cancel{1} & \cancel{-kX} \\ \cancel{-kX} & \cancel{1+k^2X^2} \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $\quad p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\quad \dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. $(F_X^{cov}=0=F_Y^{cov})$

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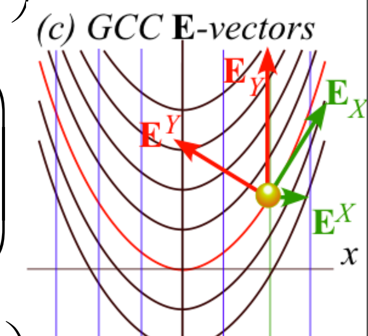
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

$$\frac{1}{m}\underbrace{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}_{\text{(inverse of } \gamma_{AB})}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \underbrace{\cancel{\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}}}_{\text{(inverse of } \gamma_{AB})\text{ (cancelations)}}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F^X_{con} \\ F^Y_{con} \end{pmatrix}$$

$$\ddot{x} = 0 = \ddot{X}$$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

Find covariant force equations

→ *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

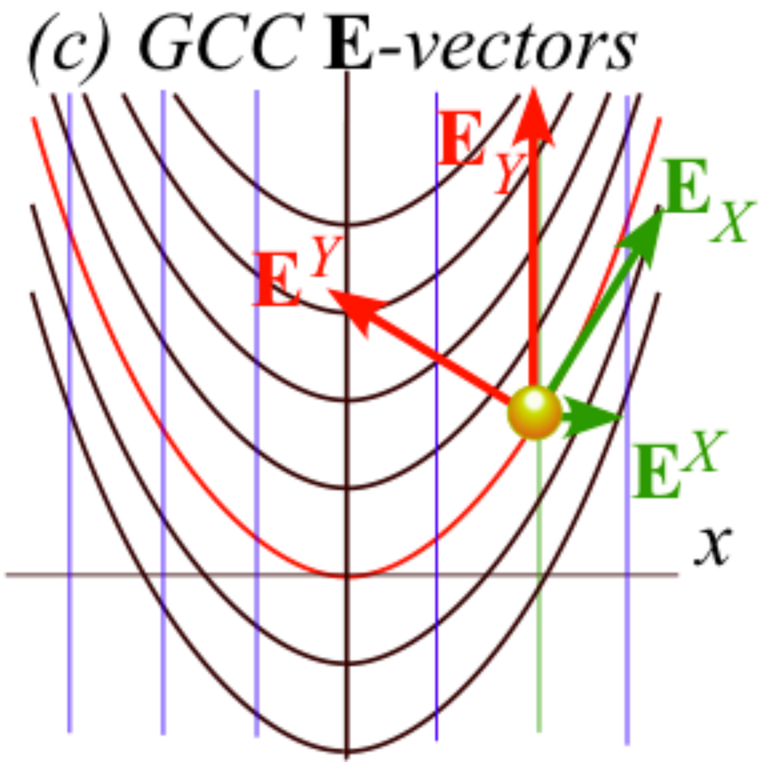
*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*



General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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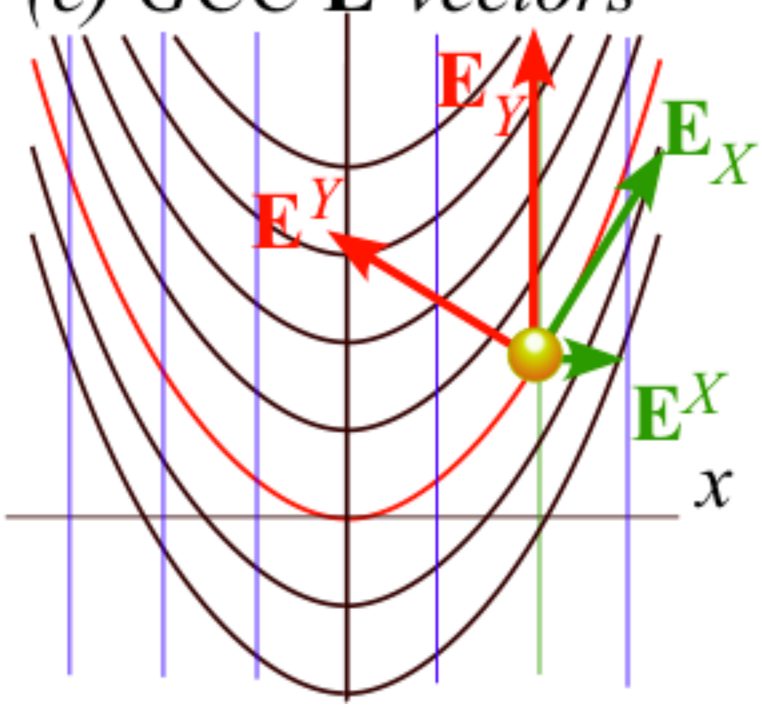
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

(c) GCC **E**-*vectors*



General case repeated from p.34

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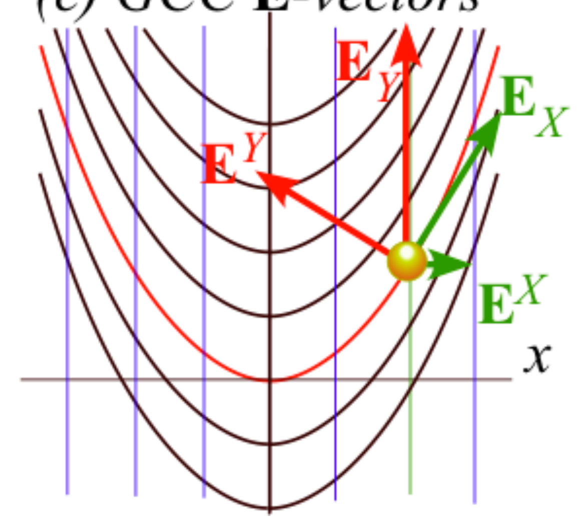

$\dot{Y} = 0 = \ddot{Y}$

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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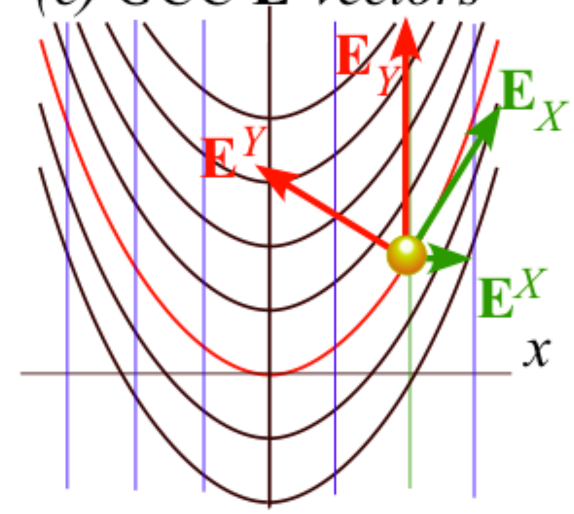
*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E**-vectors



$\dot{Y} = 0 = \ddot{Y}$

FINALLY ! We get the Way 1. solution of p.12

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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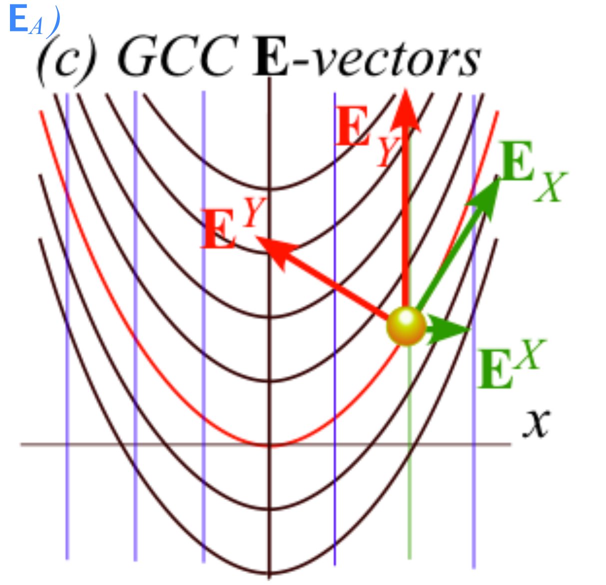

$$\dot{Y} = 0 = \ddot{Y}$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2+g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

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Frictional or driving forces have *contravariant* components F_{con}^A

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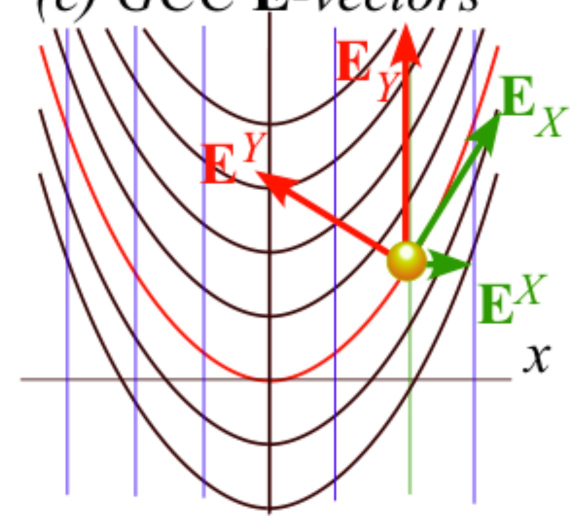
(F^A are coefficients of ***tangent*** *vectors $\mathbf{E}_A$)*

(c) GCC **E**-vectors



$\dot{Y} = 0 = \ddot{Y}$

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$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

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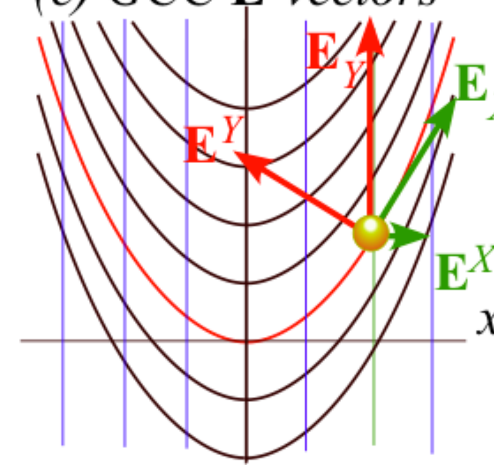
Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F^A_{con}

$$\mathbf{F} = F^X_{con}\mathbf{E}_X + F^Y_{con}\mathbf{E}_Y = F^X_{con}\frac{\partial \mathbf{r}}{\partial X} + F^Y_{con}\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E***-vectors*



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$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$\begin{pmatrix}F_x \\ F_y\end{pmatrix} = \left(=\begin{pmatrix}0 \\ mk\dot{X}^2 + mg\end{pmatrix}\right)_{at:X=0}$$

Centripetal force mkv^2+mg *(what roller-coaster rider feels at bottom)*

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

$$-g = \ddot{y} = \tfrac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right)$$
$$= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\,(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0)$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

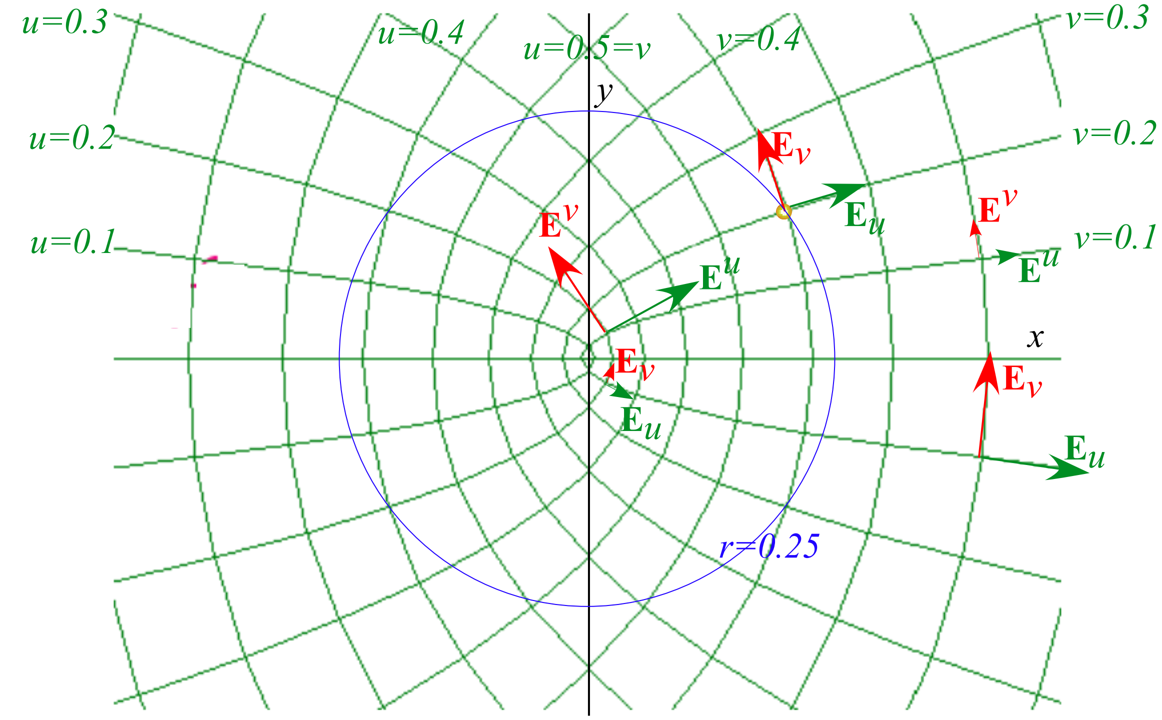
"Non-Holonomic" multipliers

Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$



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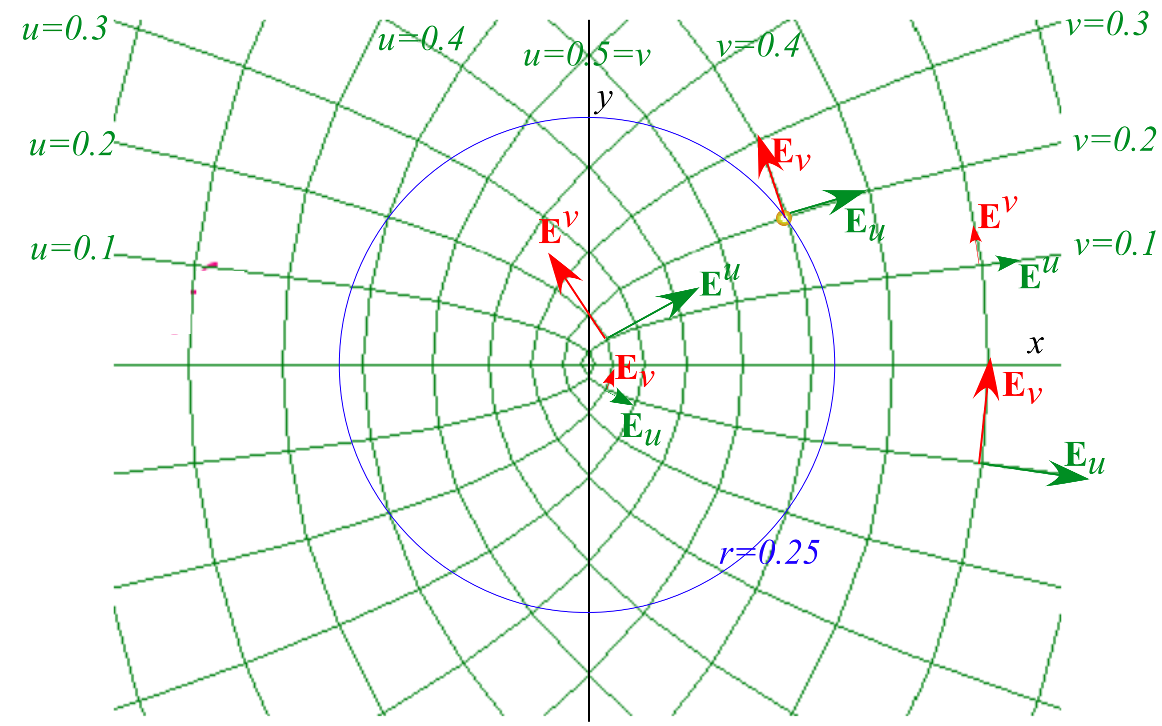
$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$



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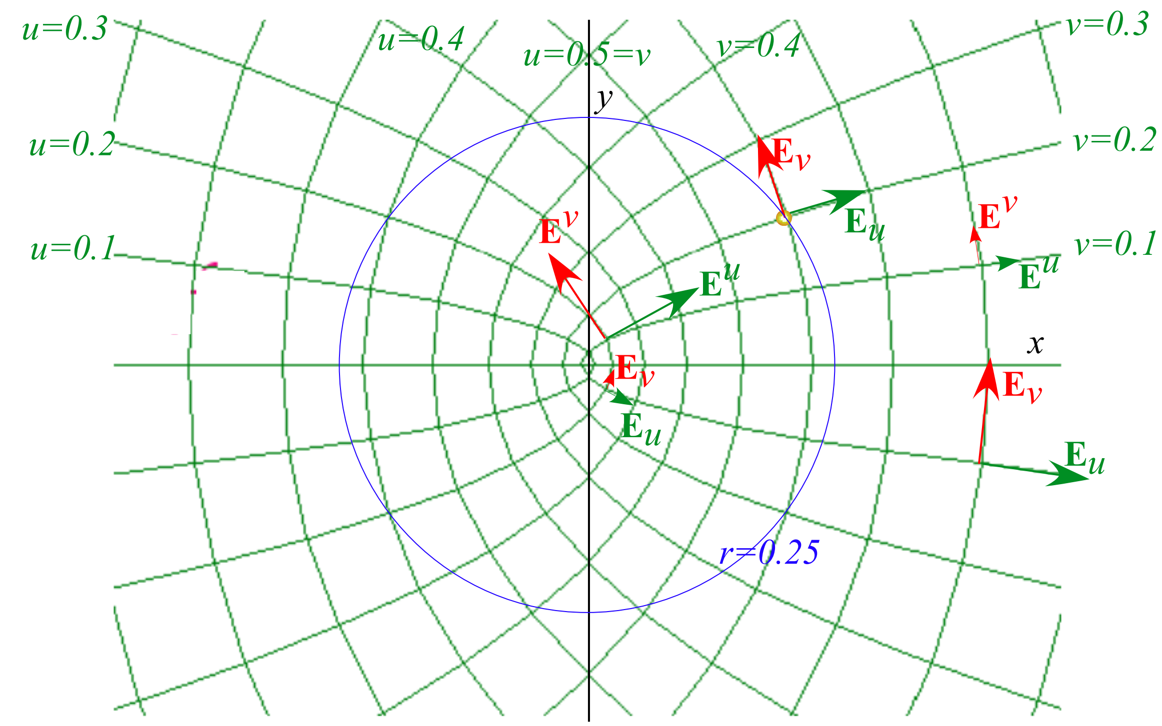
$$x = u^2 - v^2$$

$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2 + y^2} + x$$

$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2 + y^2} - x$$



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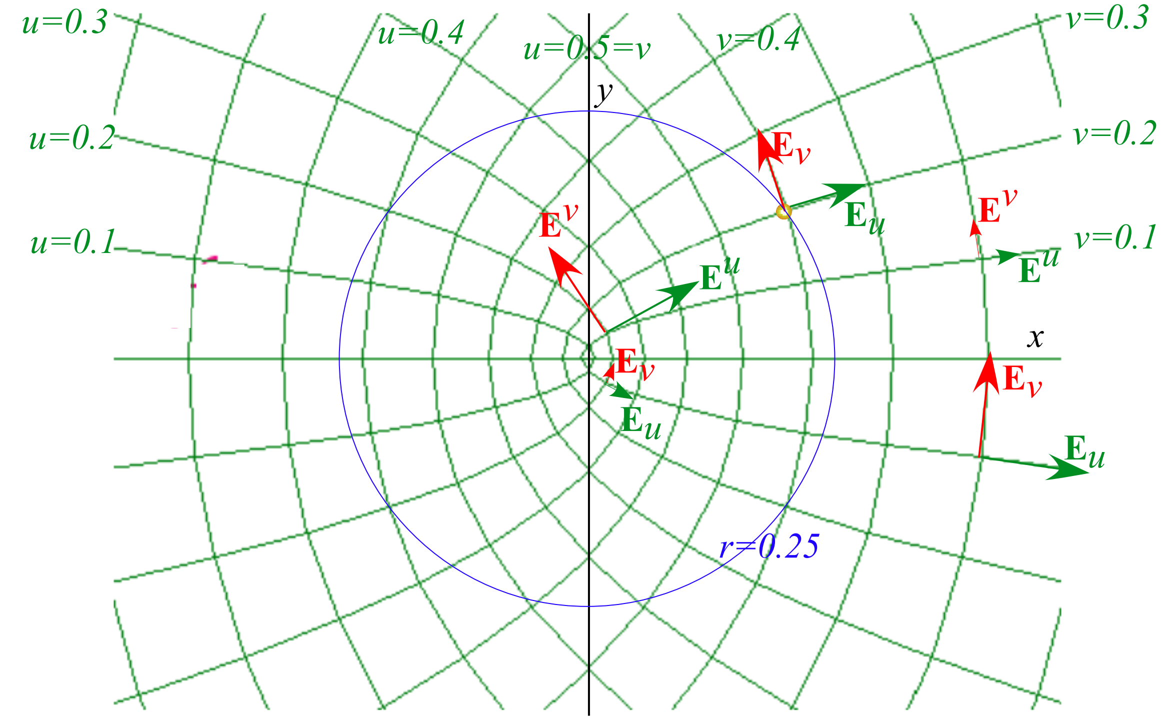
$$y^2 = 4u^2v^2 = 4u^2(u^2 - x)$$

$$y^2 = 4v^2u^2 = 4v^2(v^2 + x)$$

Gives confocal parabolics



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$$x = u^2 - v^2$$

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$$r = u^2 + v^2 \qquad 2v^2 = r - x = \sqrt{x^2+y^2} - x$$

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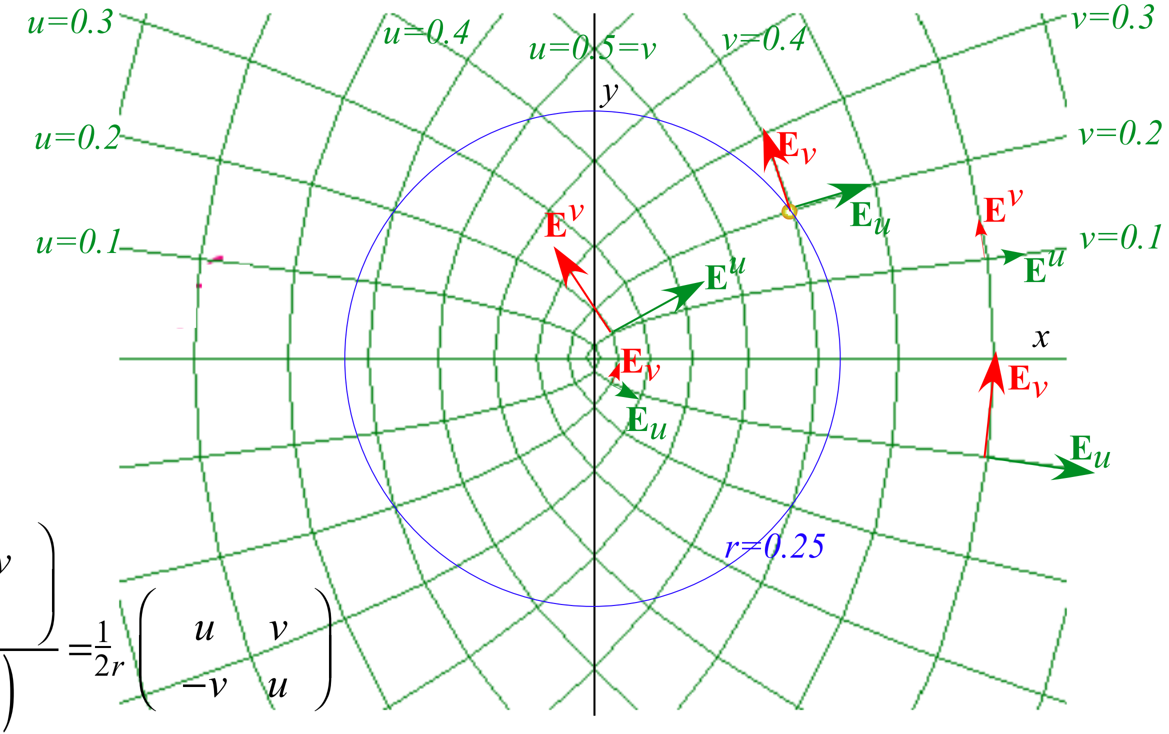

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_u & \mathbf{E}_v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ +2v & 2u \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^u \\ \mathbf{E}^v \end{pmatrix} = \frac{\begin{pmatrix} 2u & +2v \\ -2v & 2u \end{pmatrix}}{4\left(u^2+v^2\right)} = \frac{1}{2r}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

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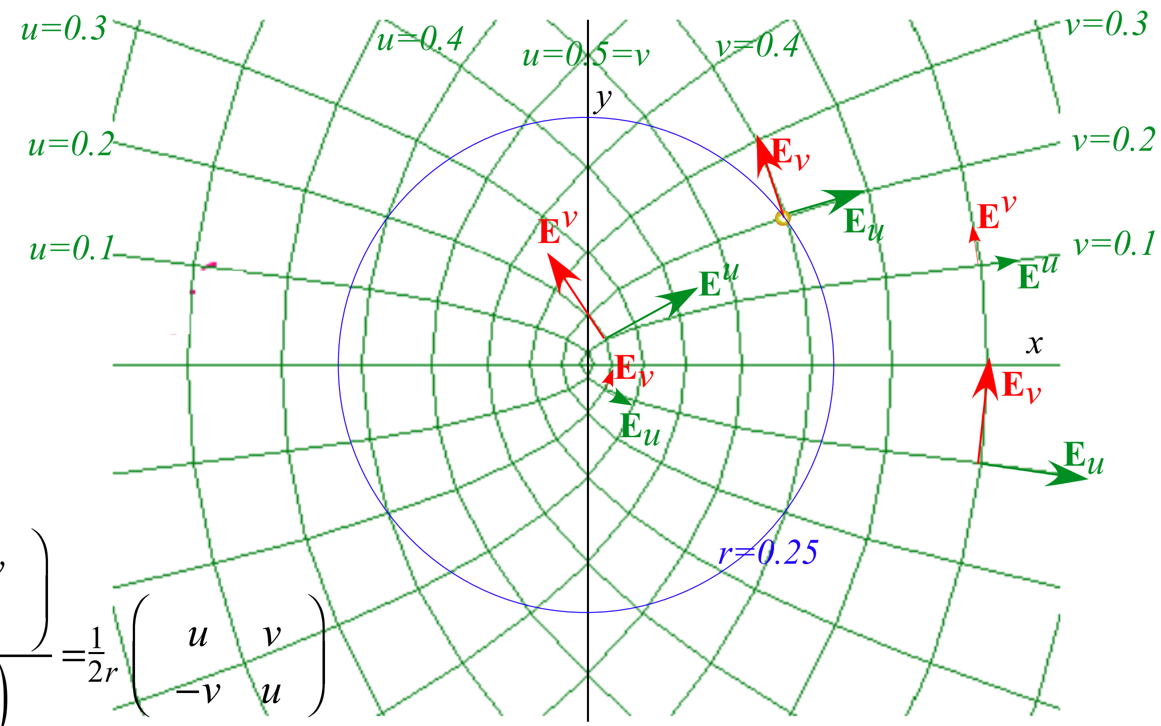

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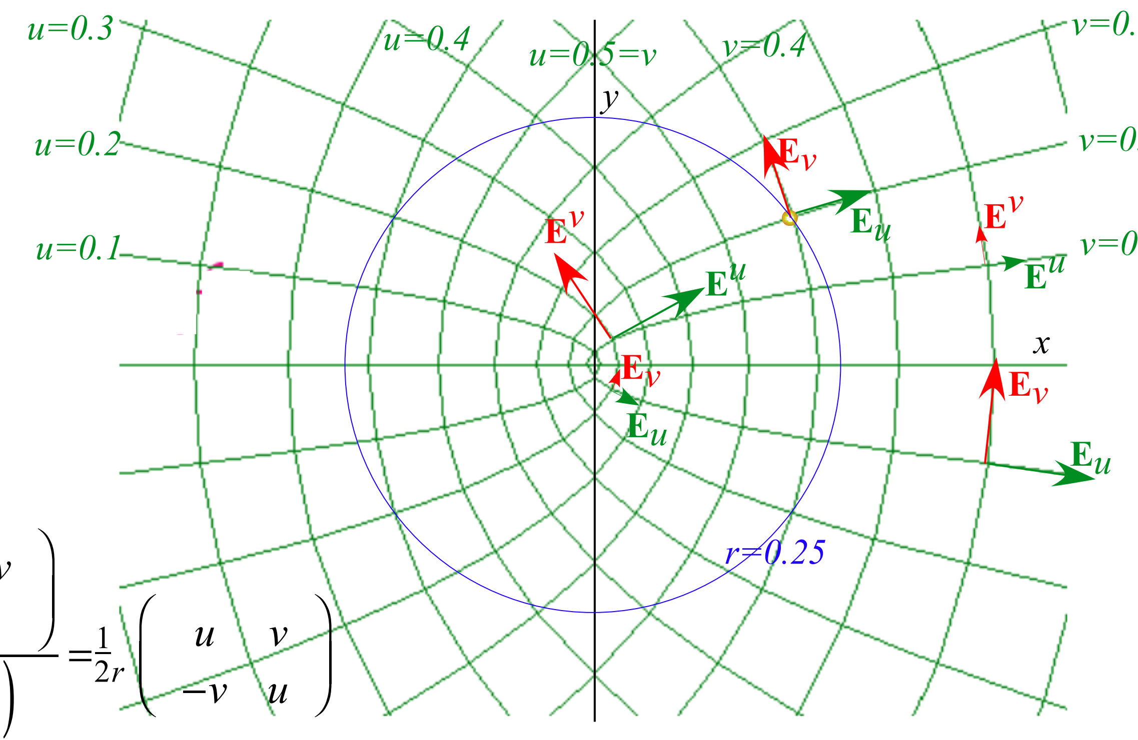

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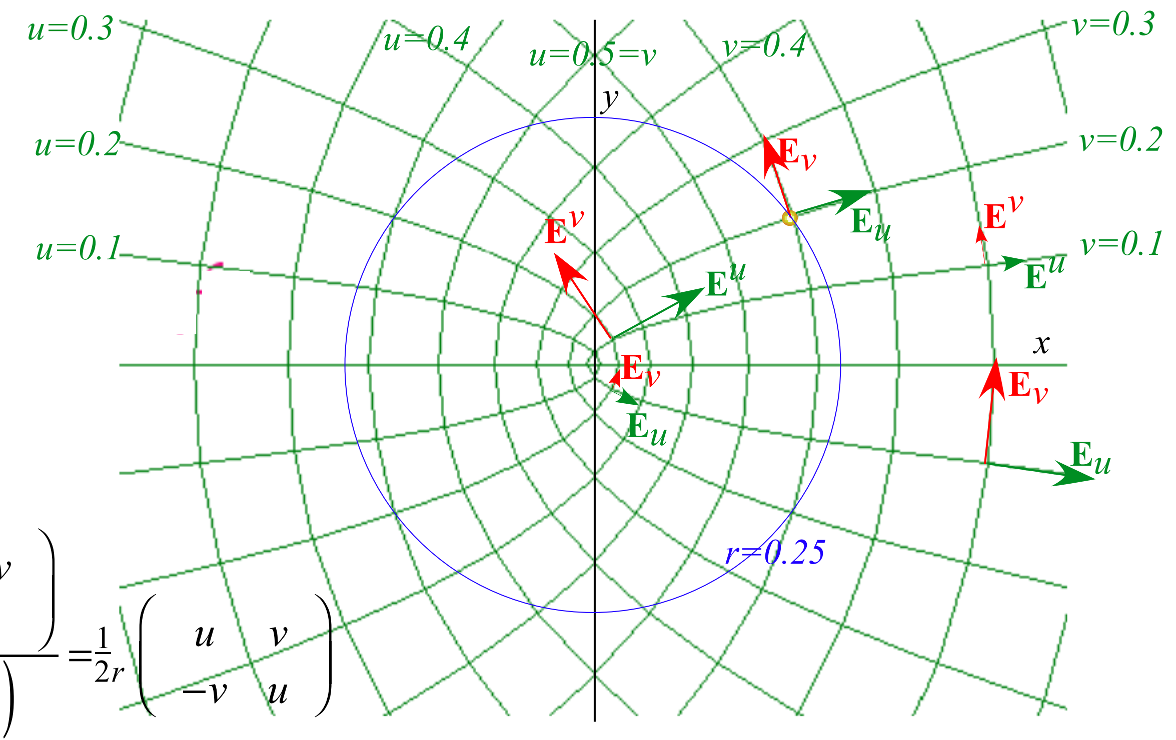

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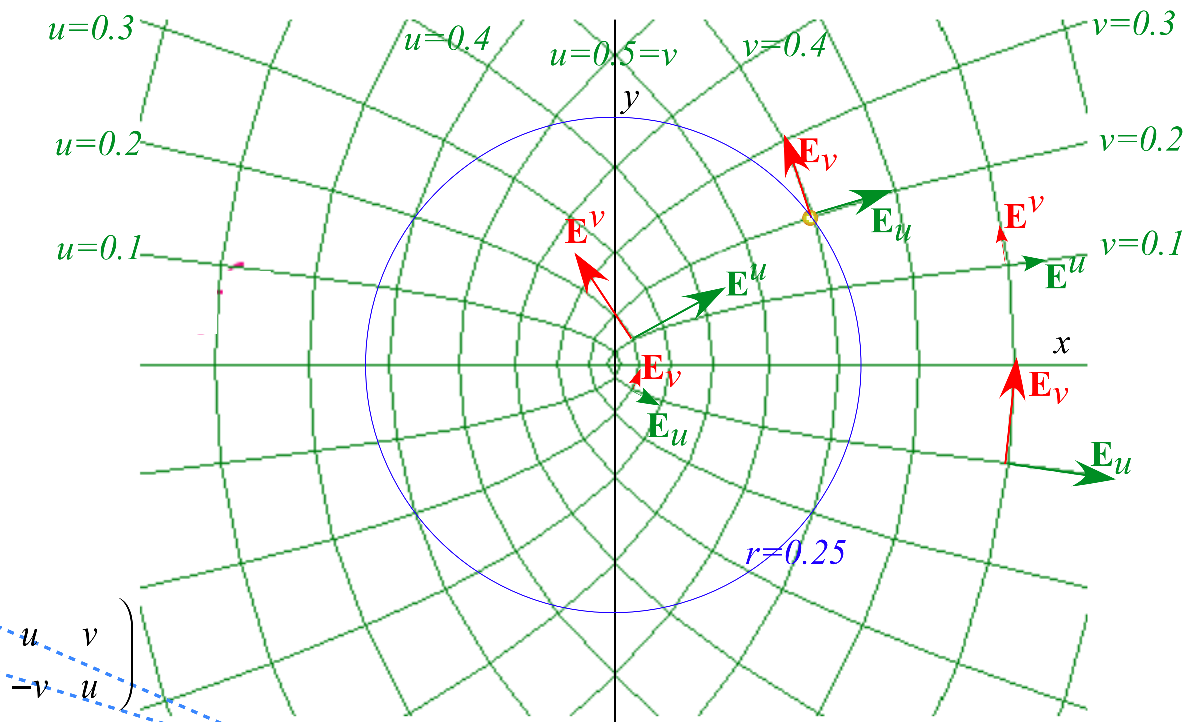

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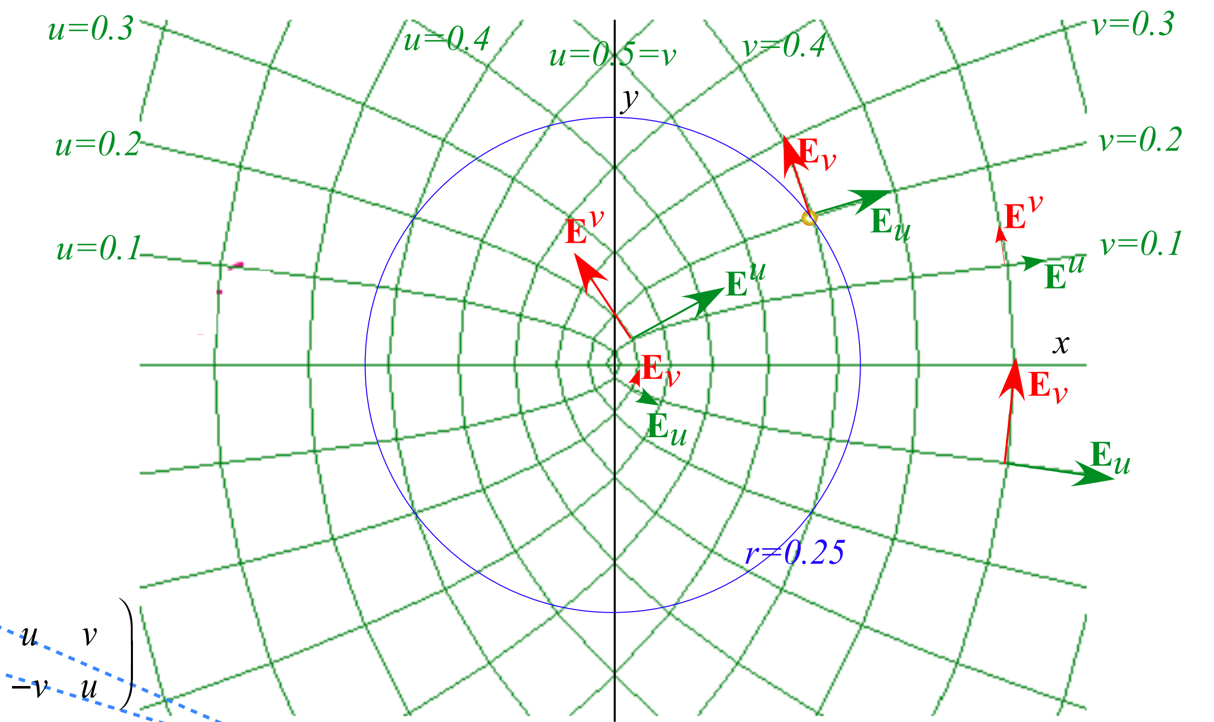

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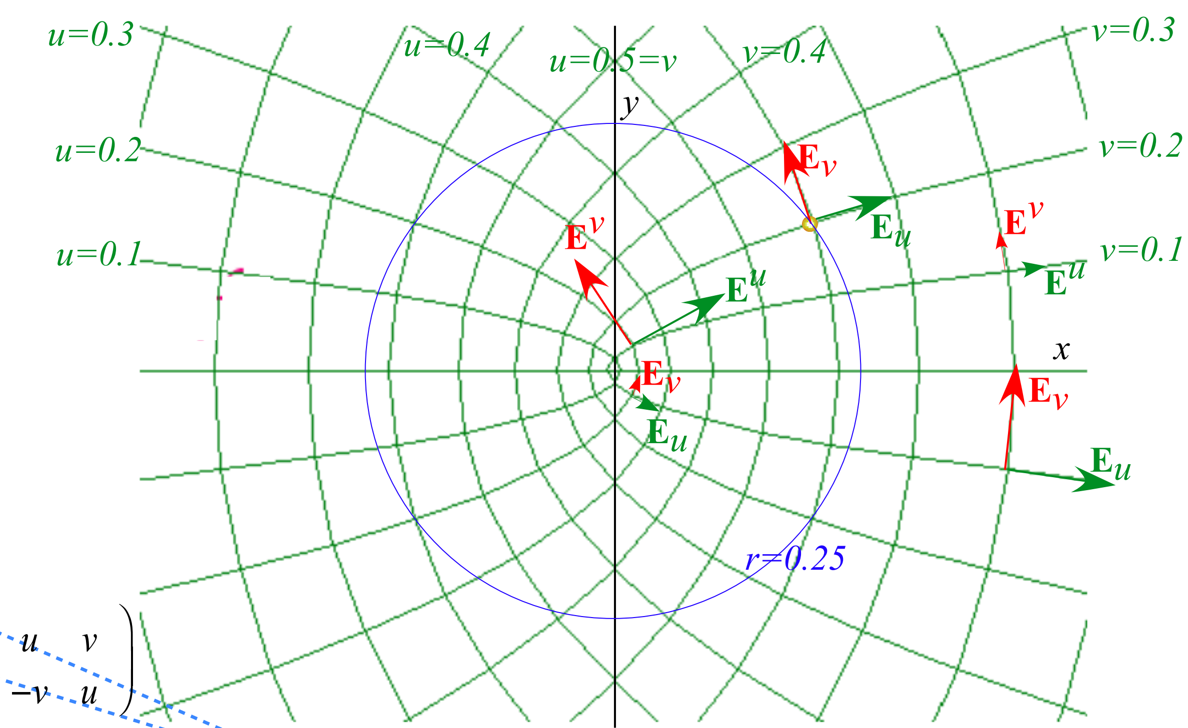

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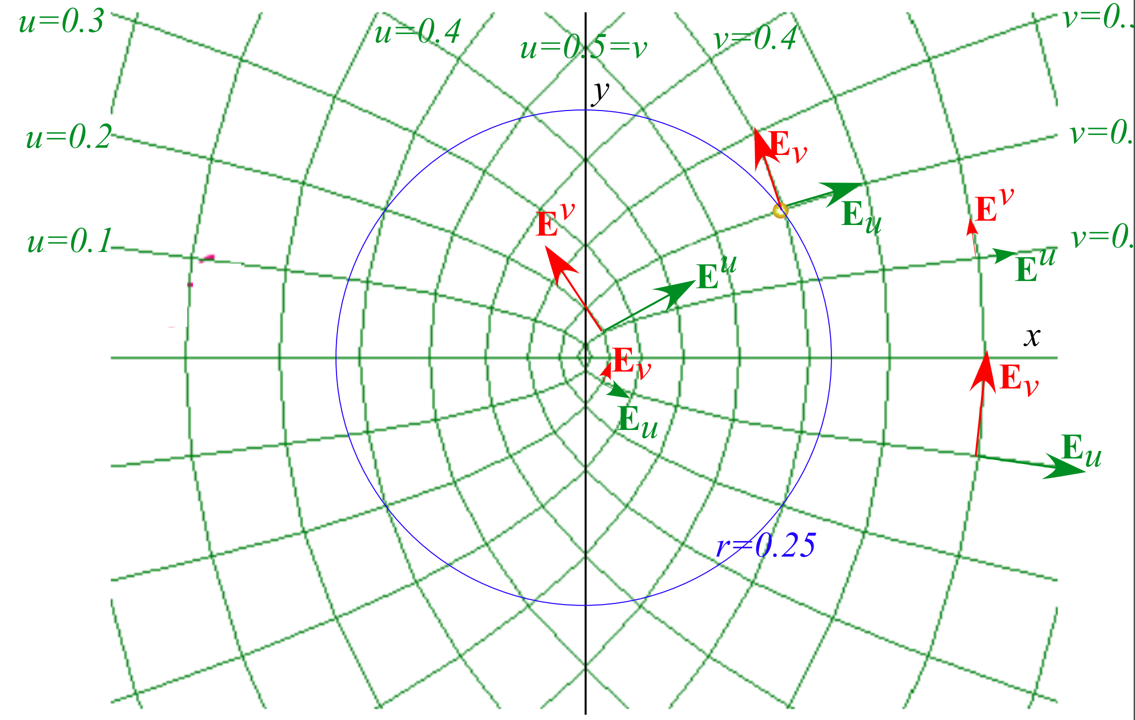

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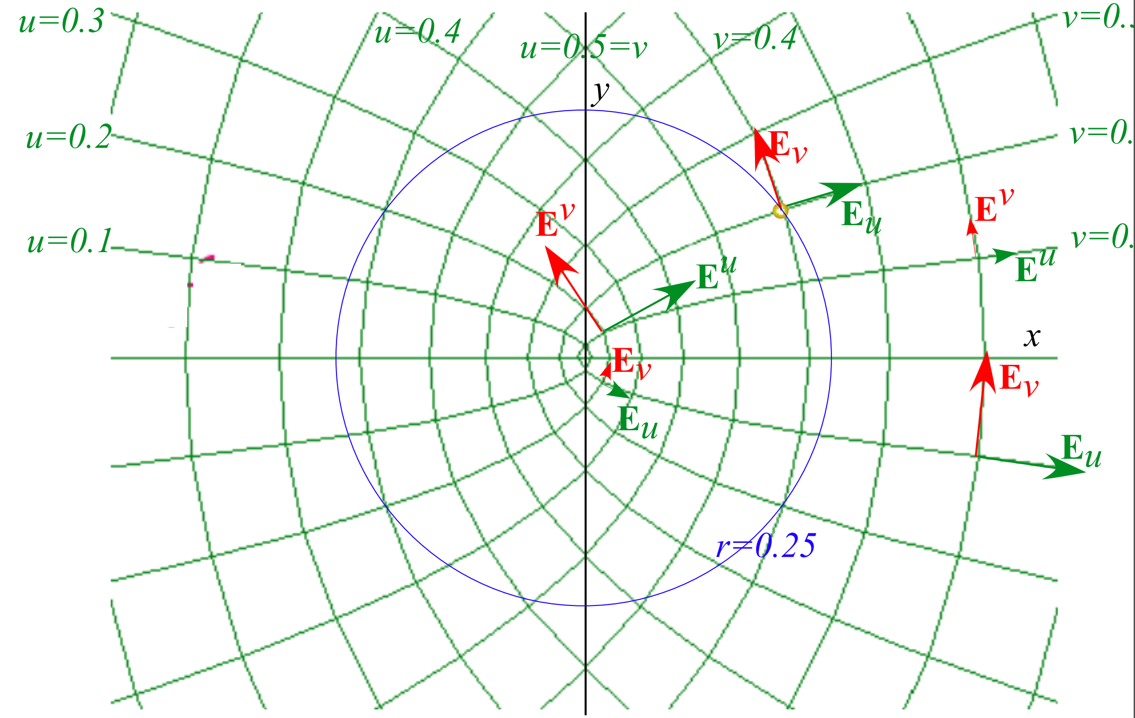

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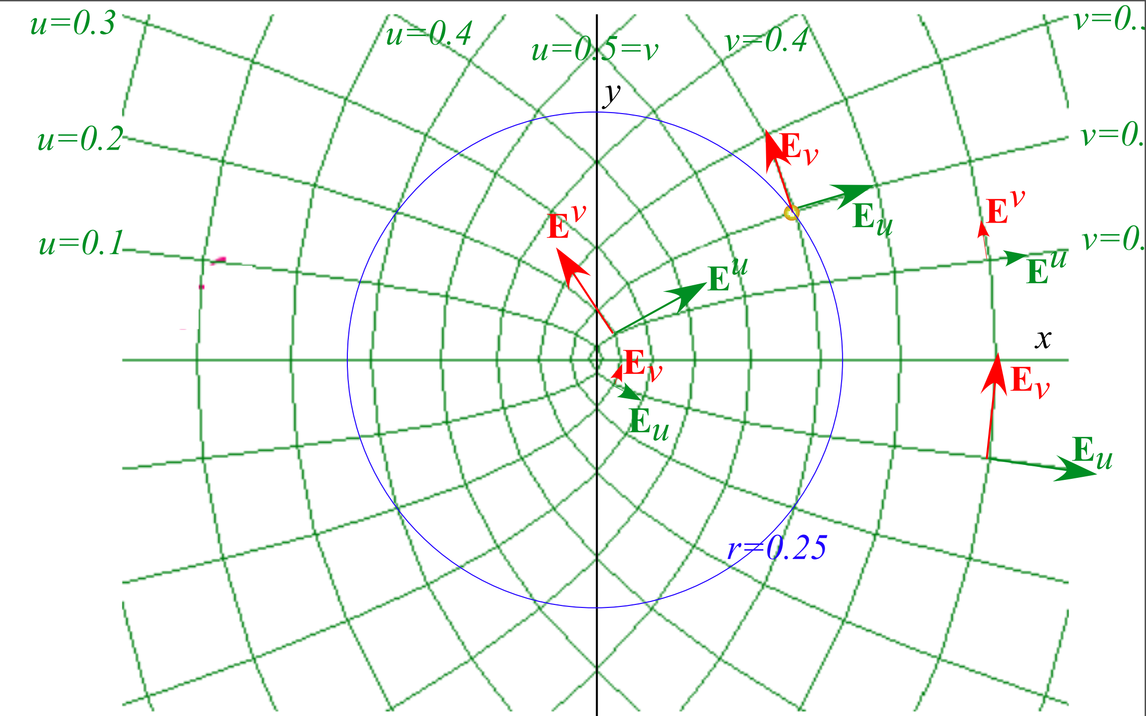

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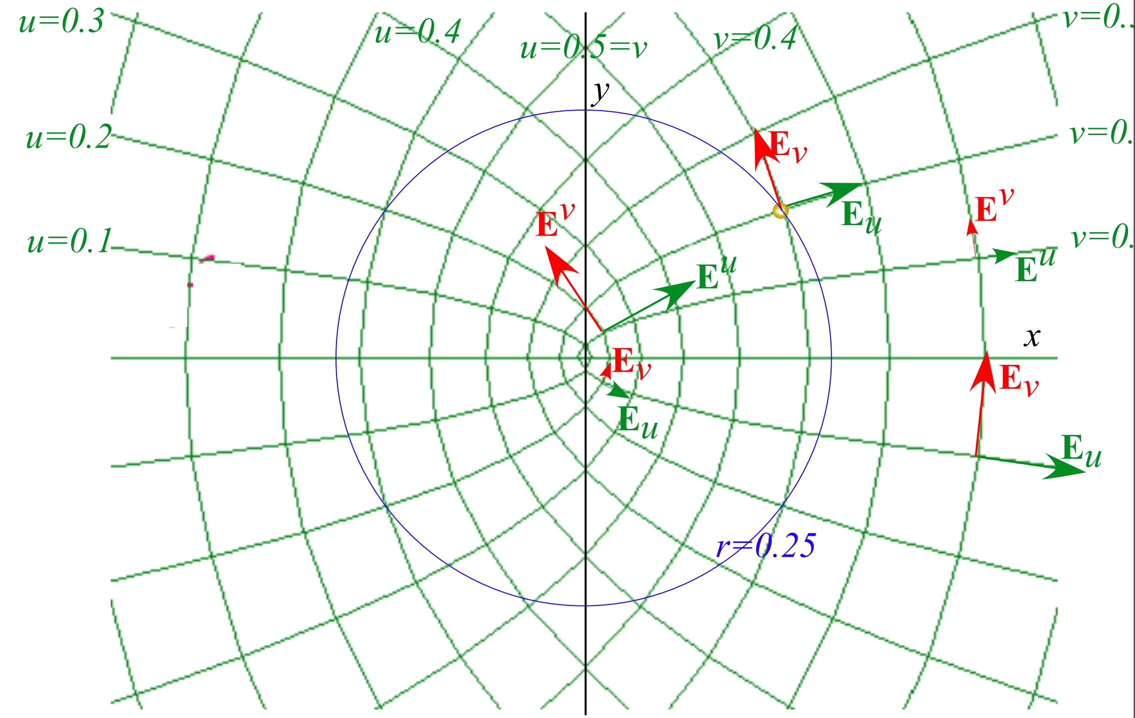

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Zero Stark-field *(ε=0)* gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u = -\frac{\partial h_u}{\partial u} = -8Eu + 16\varepsilon u^3 \qquad \dot{u} = \frac{\partial h_u}{\partial p_u} = p_u/m \qquad \dot{p}_v = -\frac{\partial h_v}{\partial v} = -8Ev - 16\varepsilon v^3 \qquad \dot{v} = \frac{\partial h_v}{\partial p_v} = p_v/m$$

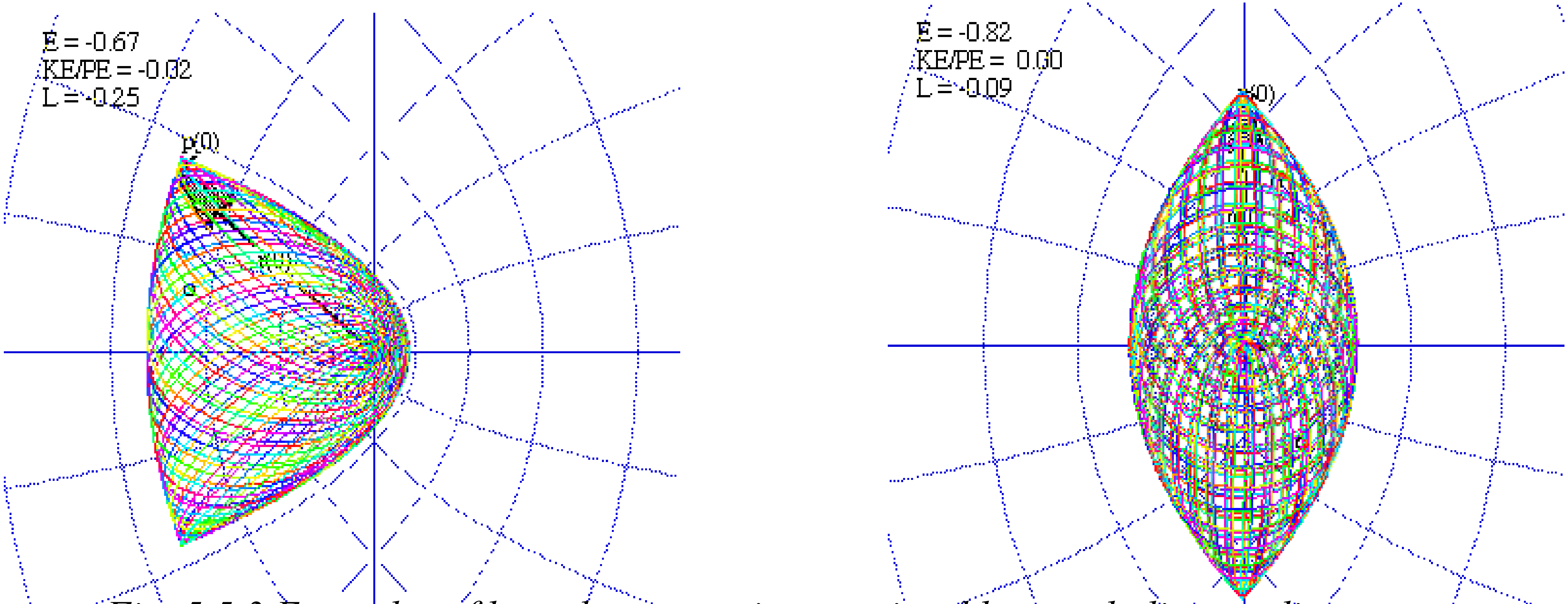


Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

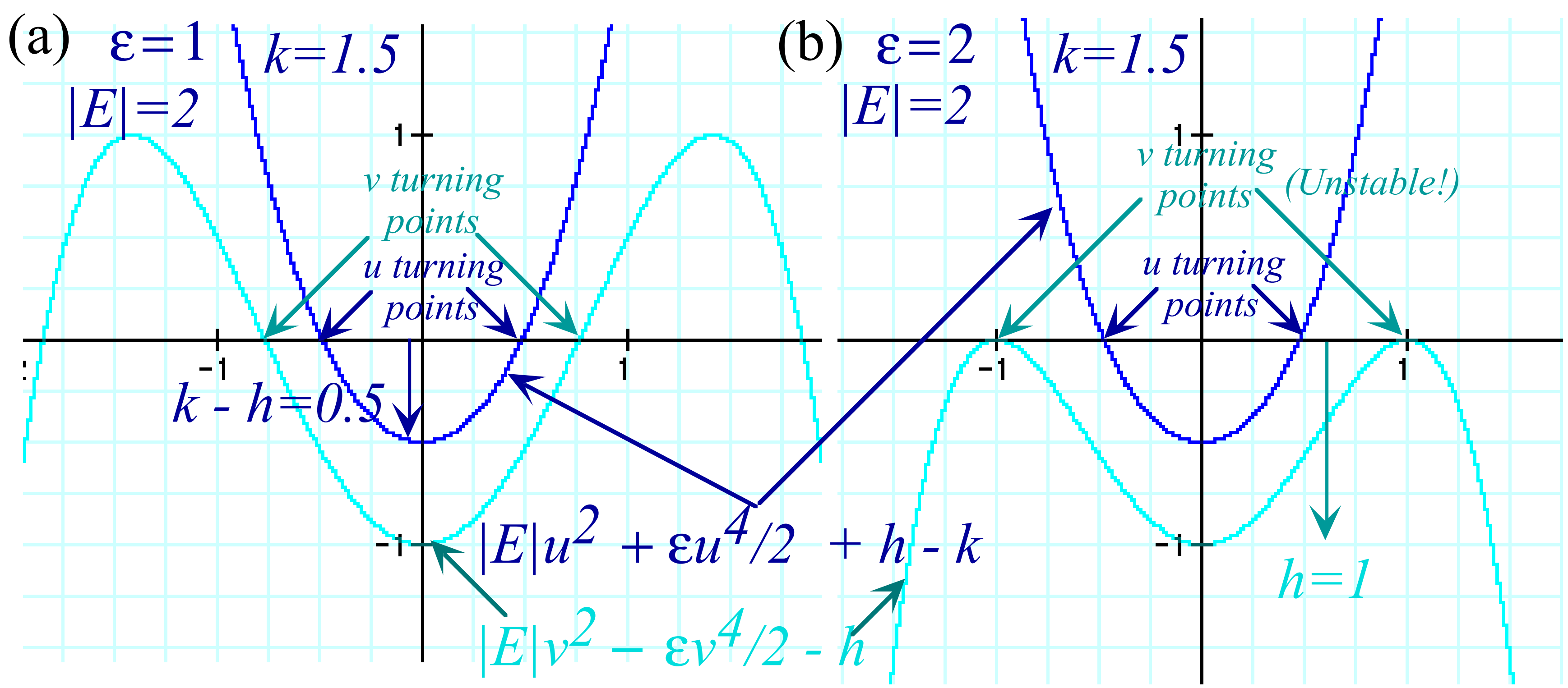


Fig. 5.5.2 Effective potentials for parabolic coordinates

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

→ *Way 4. Lagrange multipliers*

Lagrange multiplier as eigenvalues

Multiple multipliers

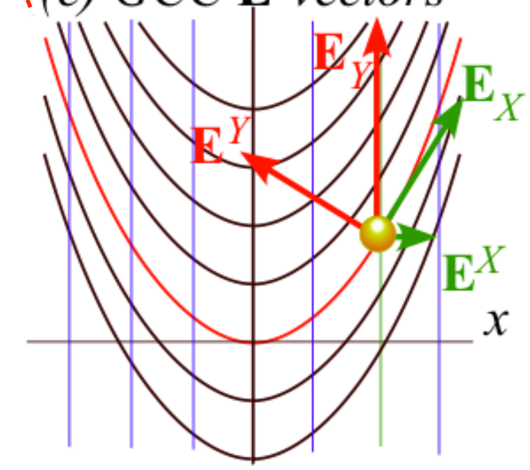
"Non-Holonomic" multipliers

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)



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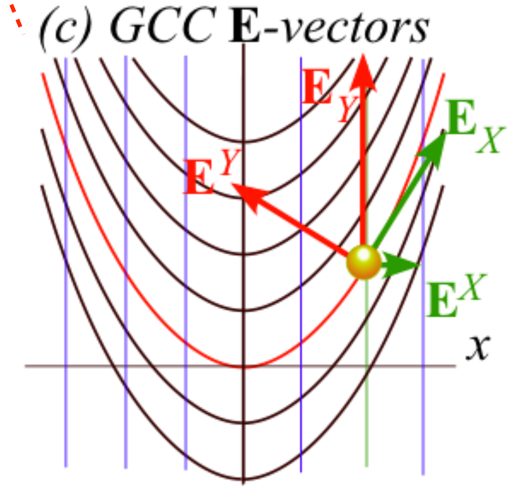

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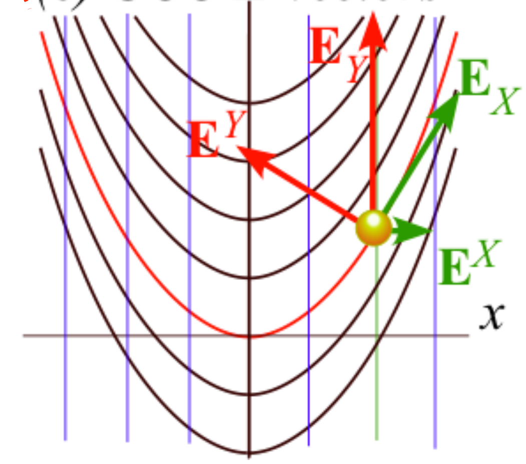
(c) GCC **E***-vectors*



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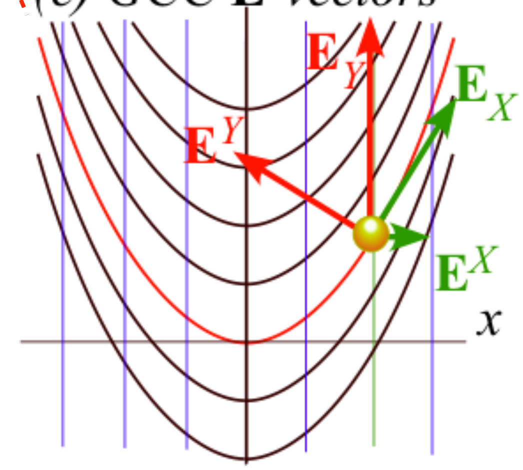

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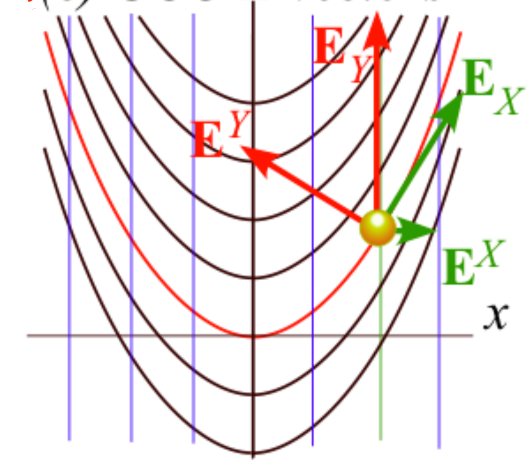

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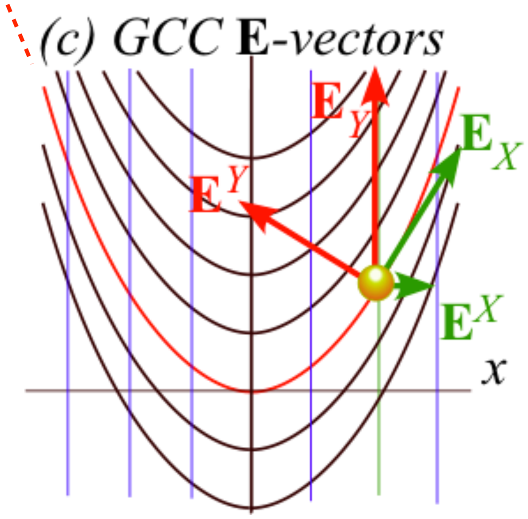

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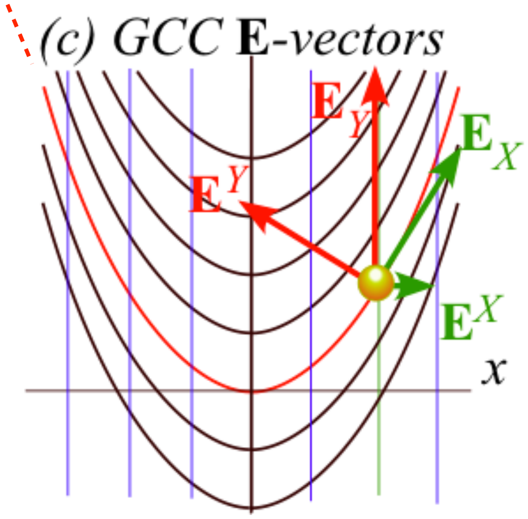

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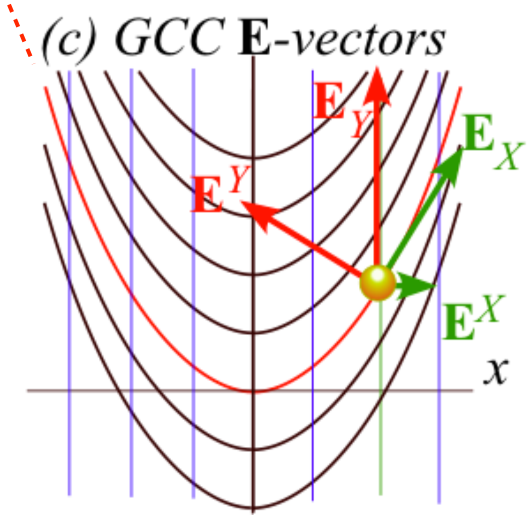

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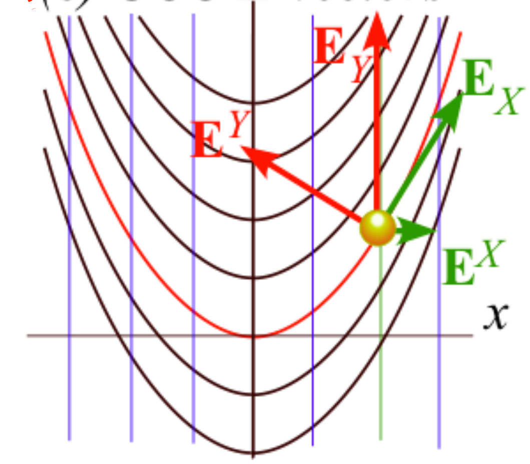

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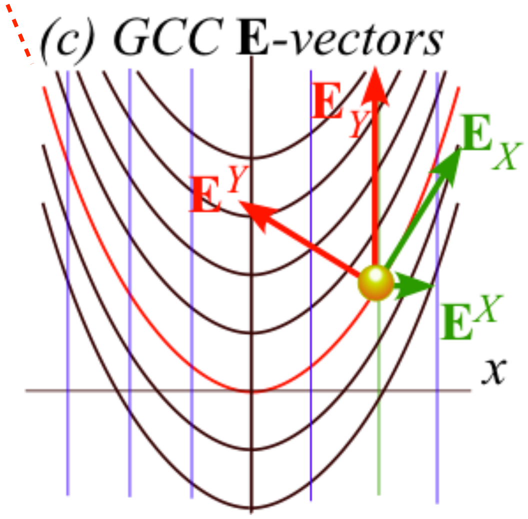

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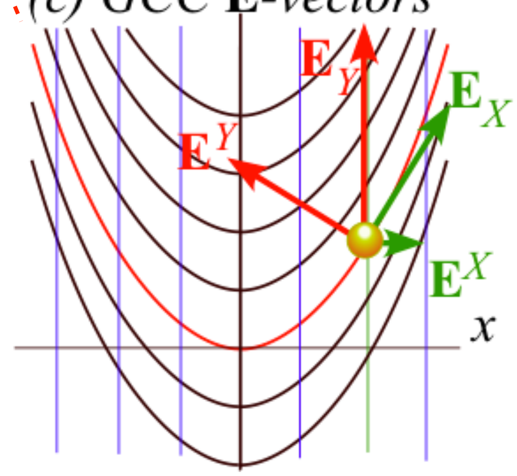

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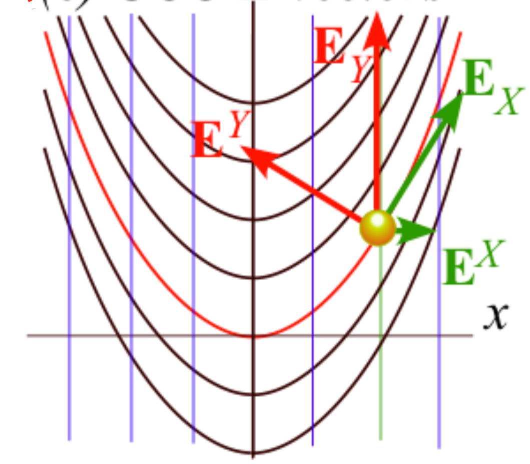

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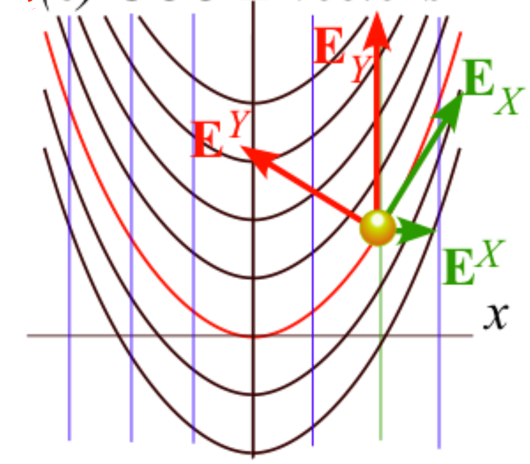

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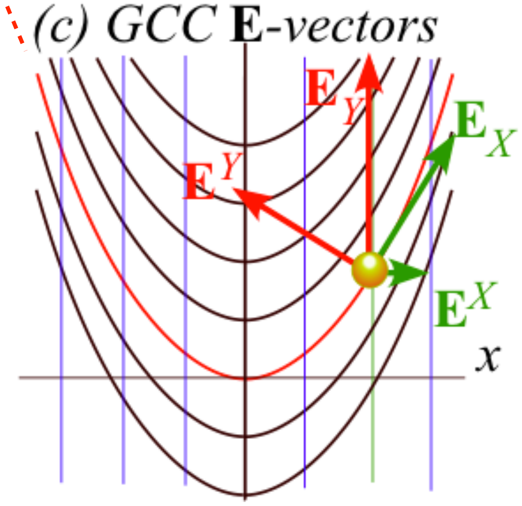

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$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

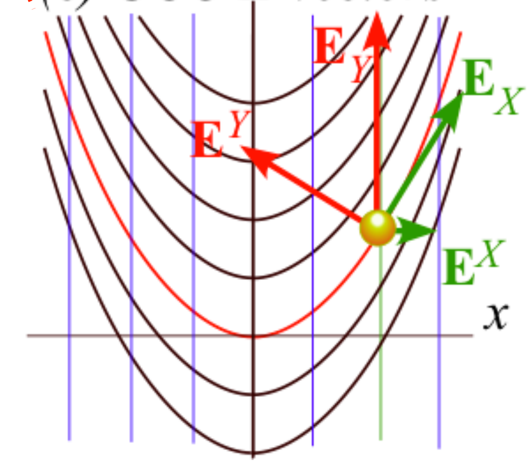
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=1/2kx^2$ is defined as follows.

$$c^1 = \tfrac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . $(\mathbf{F} \propto \nabla c^1)$



$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

The Newtonian-Cartesian equations $m\ddot{\mathbf{r}} = -m\mathbf{g}$ add constraint force **F** to become $m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{g} = \mathbf{F} - m\mathbf{g}$ with constraint : $\mathbf{F} = F_1^c \nabla c^1$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix} \qquad \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \begin{pmatrix} m\ddot{x} \\ mk(\dot{x}^2 + x\ddot{x}) \end{pmatrix} = \begin{pmatrix} \lambda kx \\ -\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=1/2kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained *x*-equation of motion.

$$m\ddot{x} = \lambda kx = -m(k\dot{x}^2 + kx\ddot{x} + g)kx = -m(k^2x\dot{x}^2 + k^2x^2\ddot{x} + kgx)$$

$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

(Same equation as on p.12)

$$\boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx}$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

→ *Lagrange multiplier as eigenvalues*

Multiple multipliers

"Non-Holonomic" multipliers

Lagrange multiplier basics

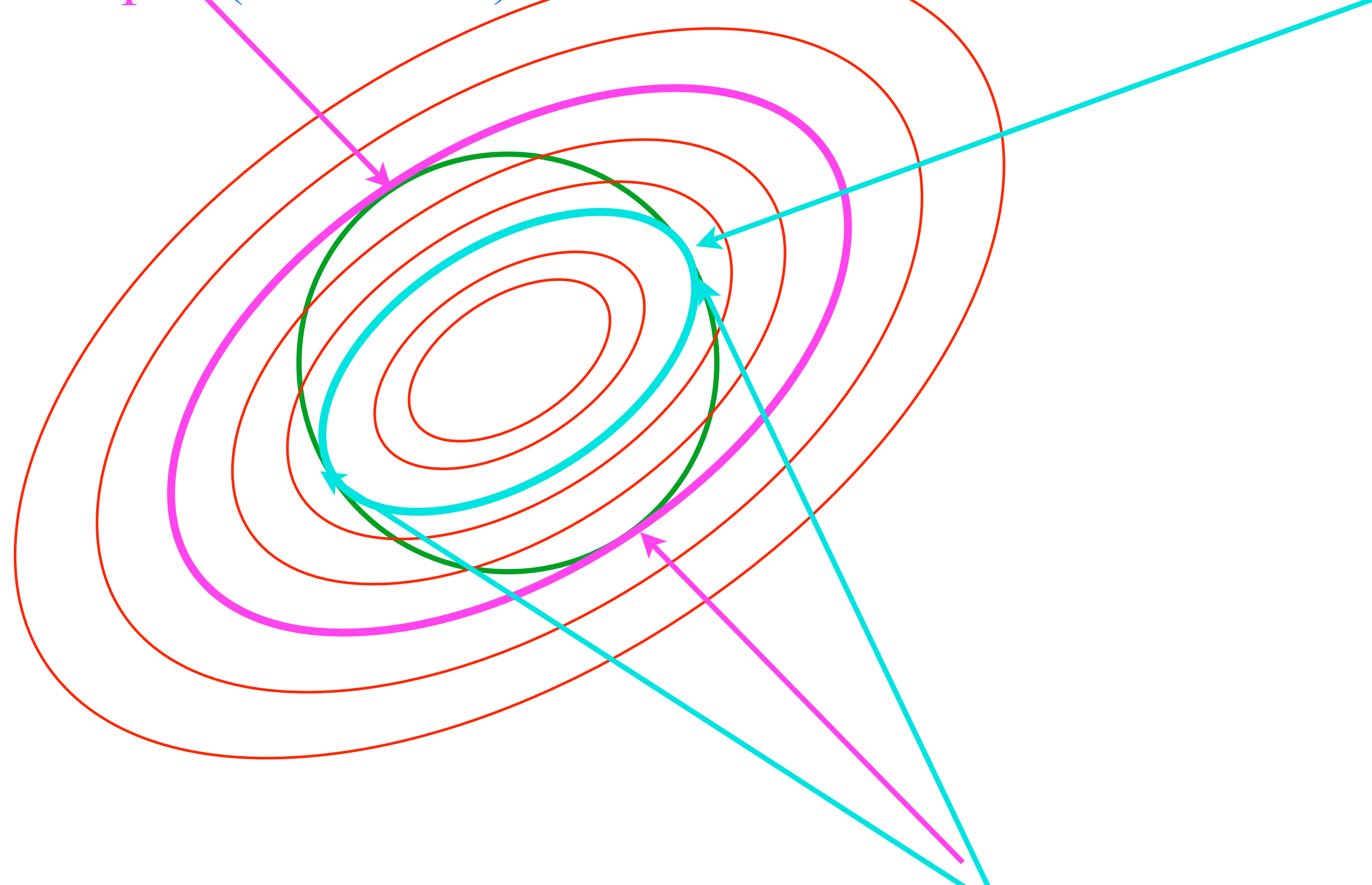
Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.

The contact points satisfy
gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



Extreme cases occur only at *contact points*

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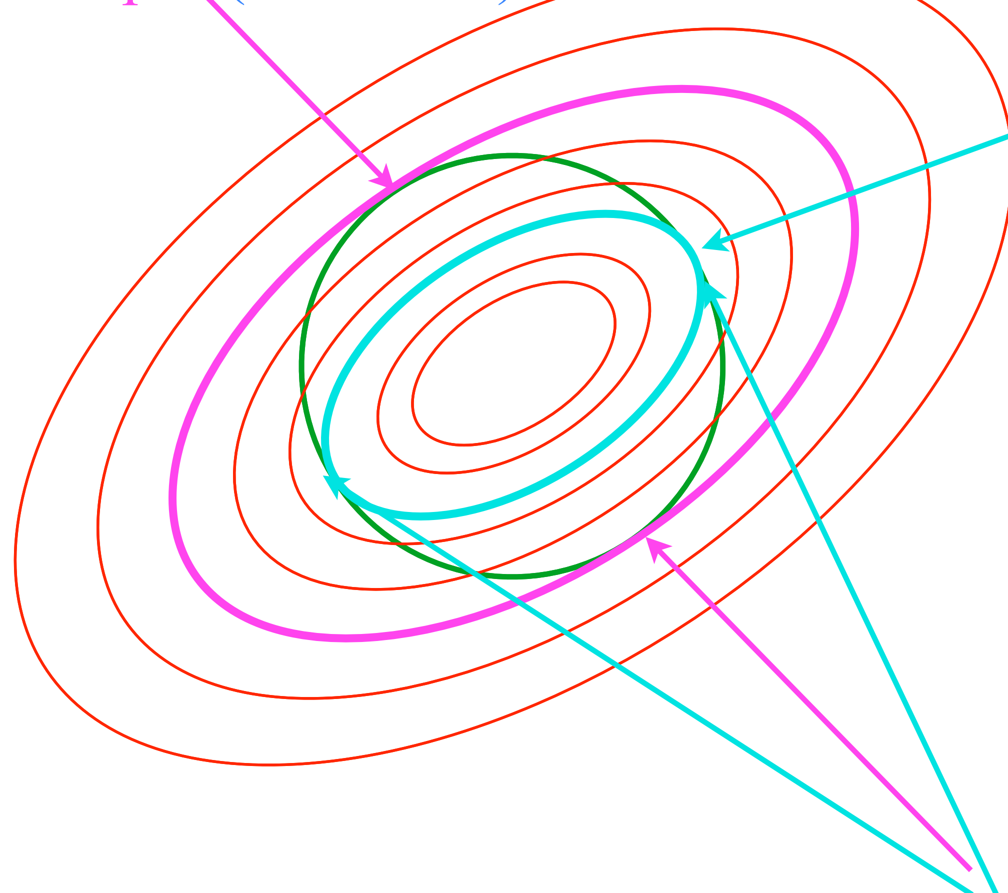

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This amounts to a λ-*eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek “L”)

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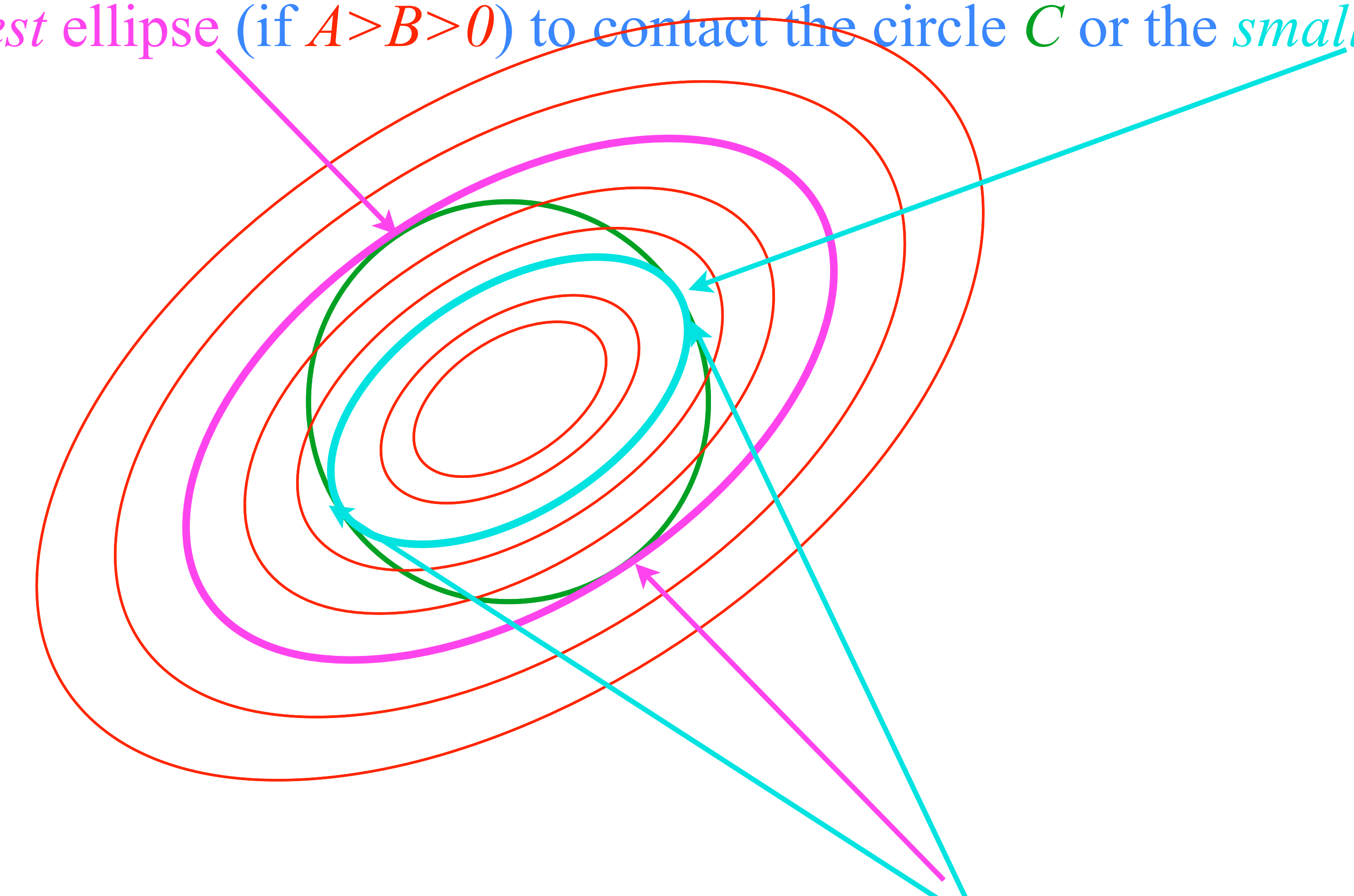

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(Perhaps, this is why we often label eigenvalues λ with a Greek “L”)

Eigenvalues λ are *extreme* matrix “own”-values $\langle\psi|M|\psi\rangle$ subject *Norm-constraint* $\langle\psi|\psi\rangle=1$

Other Ways to do constraint analysis

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"Non-Holonomic" multipliers

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{}{\partial q^k}\frac{\partial c}{} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

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Two or more constraints $c^1(q^k) = const.,\ c^2(q^k) = const.,\cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

Other Ways to do constraint analysis

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Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

$$0 = C^2_1 dq^1 + C^2_2 dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

Constrained equations of motion

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$$0 = C_1^1 dq^1 + C_2^1 dq^2 + \ldots$$

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If a differential can’t be integrated to give a constraint function it’s called a *non-holonomic constraint*.

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I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)

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Force components $F^\gamma_k = \frac{\partial c^\gamma}{\partial q^k} = C^\gamma_k$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

<u>*Integral constraint differentials*</u>

$$\frac{\partial F^\gamma_k}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F^\gamma_j}{\partial q^k}$$

<u>*General differential constraint relations*</u>

$$\frac{\partial C^\gamma_k}{\partial q^j} \begin{array}{c} \textit{may or} \\ \textit{may not be} \end{array} \frac{\partial C^\gamma_j}{\partial q^k}$$