Lecture 27
Tue. 12.2.2014

Parametric Resonance and Multi-particle Wave Modes

(Ch. 7-8 of Unit 4 12.02.14)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)
- Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)
- Schrodinger wave equation related to Parametric resonance dynamics
  - Electronic band theory and analogous mechanics

Wave resonance in cyclic symmetry
- Harmonic oscillator with cyclic C_2 symmetry
  - C_2 symmetric (B-type) modes
- Harmonic oscillator with cyclic C_3 symmetry
  - C_3 symmetric spectral decomposition by 3rd roots of unity
  - Resolving C_3 projectors and moving wave modes
  - Dispersion functions and standing waves
- C_6 symmetric mode model:Distant neighbor coupling
  - C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ...)
- C_N symmetric mode models: Made-to order dispersion functions
  - Quadratic dispersion models: Super-beats and fractional revivals
  - Phase arithmetic

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

Relating C_N symmetric H and K matrices to differential wave operators

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

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Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Nonlinear or *multiplicative resonance.*
Example: oscillating magnetic **B**-field is applied to a cyclotron orbit.

$$\ddot{x}+\left(\omega_0^2+B\cos\left(\omega_s t\right)\right)x=0$$

Chapter 4.7

Also called *parametric resonance.*

(Frequency parameter or spring constant $k=m\omega^2$ is being stimulated.)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

→ *Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)*

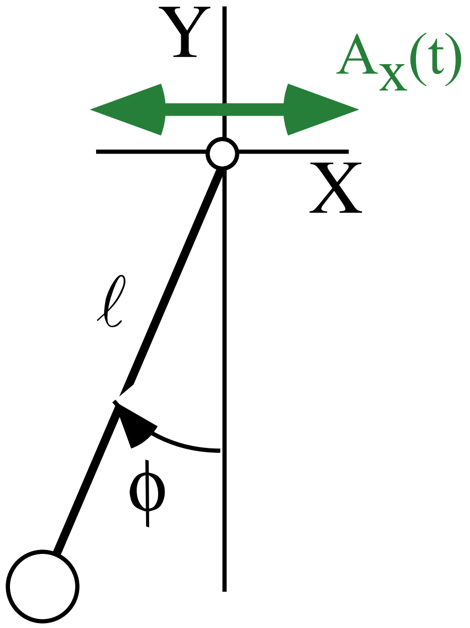
Schrodinger wave equation related to Parametric resonance dynamics

Electronic band theory and analogous mechanics

Coupled Rotation and Translation (Throwing)

Early non-human (or in-human) machines: trebuchets, whips.. (3000 BCE-1542 CE)

X-stimulated pendulum:
(Quasi-Linear Resonance)



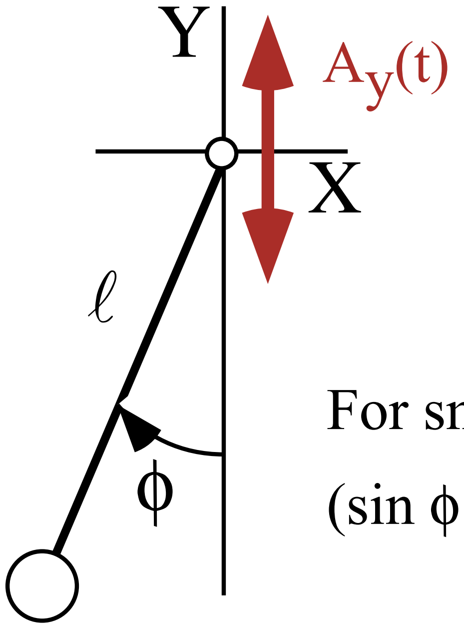
For small ϕ
($\cos\phi \sim 1$) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2}+\frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with Γ=0)

Y-stimulated pendulum:
(Non-Linear Resonance)

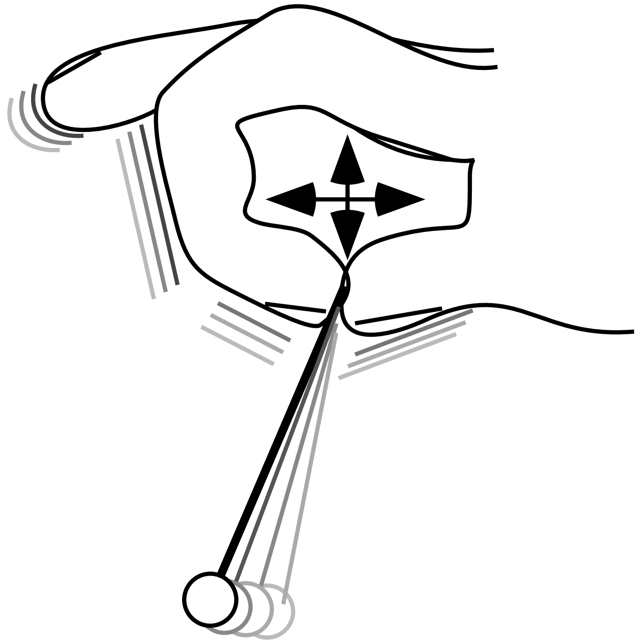


For small ϕ
($\sin\phi \sim \phi$) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{A_y(t)}{\ell}\right)\phi = 0$$

A Schrodinger-like equation
(Time *t* replaces coord. *x*)

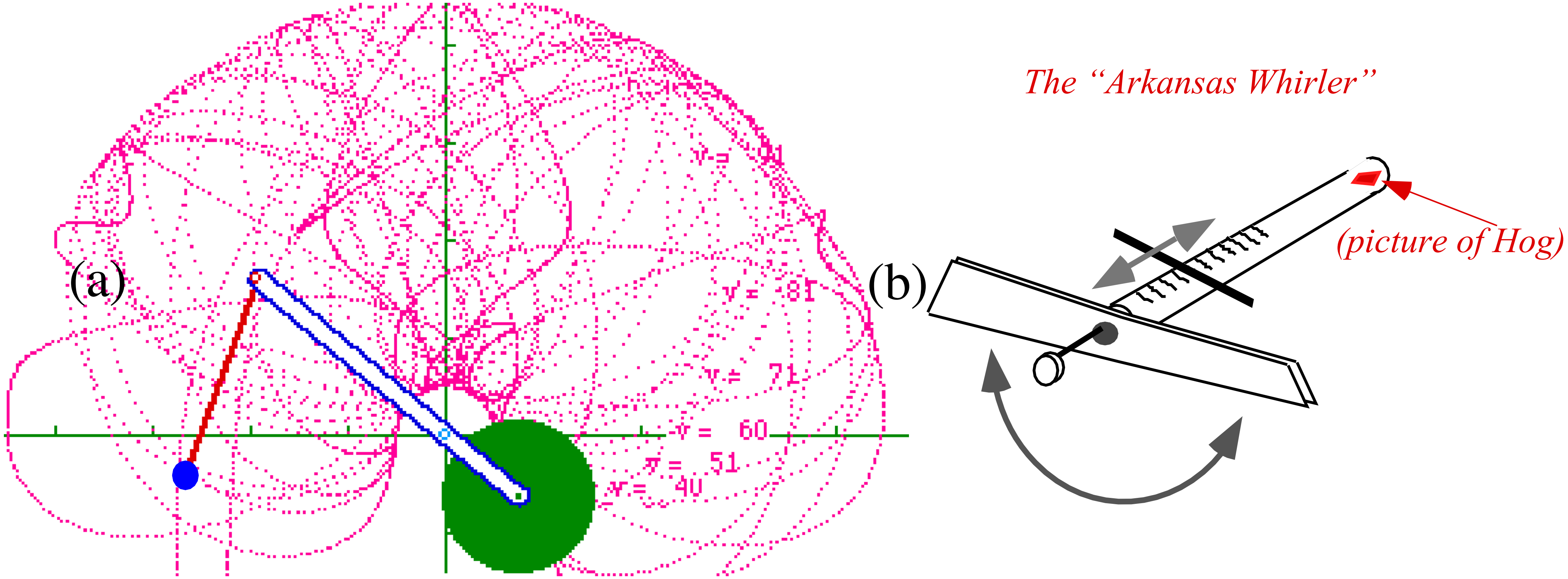


General ϕ :

(1542-2012 CE)

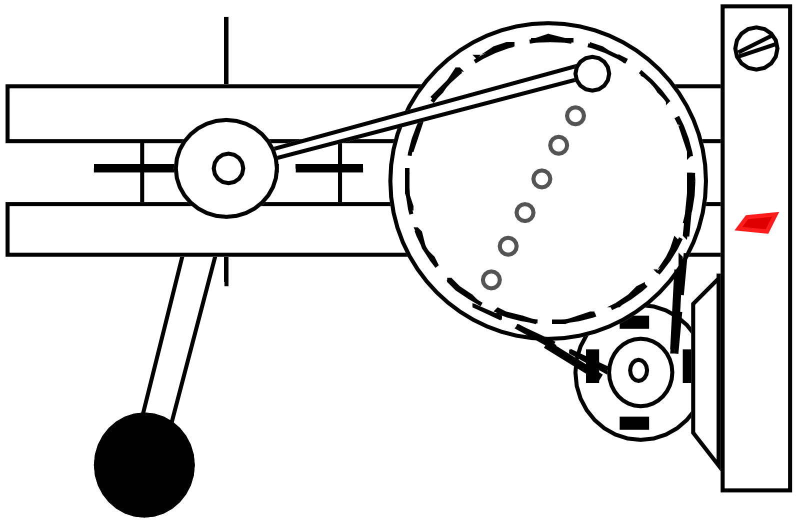
General case: A Nasty equation! $$\frac{d^2\phi}{dt^2}+\frac{g+A_y(t)}{\ell}\sin\phi+\frac{A_x(t)}{\ell}\cos\phi = 0$$

Coupled Rotation and Translation (Throwing)

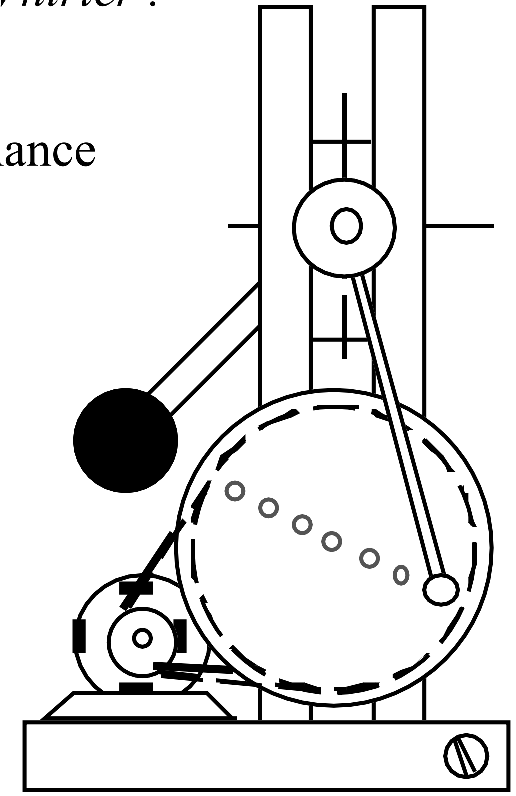


Chaotic motion from both linear and non-linear resonance (a) Trebuchet, (b) Whirler .

Positioned for linear resonance



Positioned for nonlinear resonance



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Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

→ *Schrodinger wave equation related to Parametric resonance dynamics*

Electronic band theory and analogous mechanics

Schrodinger Equation Parametric Resonance

Related to

Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y\left(t\right)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y\left(t\right) = \omega_y{}^2 A_y \cos(\omega_y t)$$

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Mathieu Equation

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$$Nx=\omega_y t$$

Connection Relations

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{\omega_y{}^2 A_y}{\ell}\cos(\omega_y t)\right)\phi=0$$

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$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{\omega_y{}^2A_y}{\ell}\cos(\omega_y t)\right)\phi=0$$

Connection Relations

$$Nx=\omega_y t$$

$$\frac{N}{\omega_y}dx=dt \longrightarrow \frac{N^2}{\omega_y{}^2}dx^2=dt^2$$

Schrodinger Equation Parametric Resonance ⇔ Related to ⇔ Jerked-Pendulum Trebuchet Dynamics

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$$\frac{N}{\omega_y}dx=dt\longrightarrow\frac{N^2}{\omega_y^{\,2}}dx^2=dt^2$$

$$\frac{d^2\phi}{dx^2}+\frac{N^2}{\omega_y^{\,2}}\left(\frac{g}{\ell}+\frac{\omega_y^{\,2}A_y}{\ell}\cos(Nx)\right)\phi=0$$

Let N=2 to get Band-edge modes

Schrodinger Equation Parametric Resonance ⟷ *Related to* ⟷ *Jerked-Pendulum Trebuchet Dynamics*

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Connection Relations

$$\frac{N}{\omega_y}dx = dt \longrightarrow \frac{N^2}{\omega_y^2}dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\omega_y^2}\left(\frac{g}{\ell} + \frac{\omega_y^2 A_y}{\ell}\cos(Nx)\right)\phi = 0$$

Let N=2 to get Band-edge modes

$$\boxed{E = \frac{N^2}{\omega_y^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

Schrodinger Equation Parametric Resonance ⟺ Related to ⟺ Jerked-Pendulum Trebuchet Dynamics

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Connection Relations

$$\frac{N}{\omega_y}dx = dt \rightarrow \frac{N^2}{\omega_y{}^2}dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\cancel{\omega_y{}^2}}\left(\frac{g}{\ell} + \frac{\cancel{\omega_y{}^2} A_y}{\ell}\cos(Nx)\right)\phi = 0$$

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$$\boxed{E = \frac{N^2}{\omega_y{}^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

$$\boxed{V_0 = \frac{N^2 A_y}{\ell}}$$

QM Potential V_0-A_y Amplitude Connection

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Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi \qquad\qquad -\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Electronic band theory and analogous mechanics

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Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E=k^2 \qquad\qquad \langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0=\sqrt{\frac{g}{\ell}}$$

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Bohr has *periodic boundary conditions* x between *0* and L Pendulum repeats perfectly after a time T.

$$\phi(0) = \phi(L) \Rightarrow e^{ikL} = 1, \text{ or: } k = \frac{2\pi m}{L} \qquad\qquad \phi(0) = \phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0 = \frac{2\pi m}{T}$$

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Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0,1,4,9,16... \qquad\qquad \omega_0 = m = 0,\pm1,\pm2,\pm3,\pm4,...$$

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Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2} + V_0\cos(Nx)\phi = E\phi\ , \qquad (\mathbf{D}+\mathbf{V})|\phi\rangle = E|\phi\rangle$$

Fourier representation: $\langle j|\mathbf{D}|k\rangle = j^2\delta_j^k$

$$\Sigma\langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle$$

Matrix eigenvalue equation

Electronic band theory and analogous mechanics

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Matrix eigenvalue equation

(Move Fourier reps. to top)

Electronic band theory and analogous mechanics

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$$\Sigma\left\langle j\right|\left(\mathbf{D}+\mathbf{V}\right)\left|k\right\rangle\left\langle k\middle|\phi\right\rangle = E\left\langle j\middle|\phi\right\rangle$$

Matrix eigenvalue equation

$\left\langle j\right|\left(\mathbf{D}+\mathbf{V}\right)\left|k\right\rangle =$ (for j and k *even*)

$$\cdots\ \left|-6\right\rangle,\left|-4\right\rangle,\left|-2\right\rangle,\left|0\right\rangle,\left|2\right\rangle,\ \left|4\right\rangle,\ \left|6\right\rangle,\cdots$$

$$\begin{array}{cc}\begin{array}{c}\vdots\\ \left\langle -6\right| \\ \left\langle -4\right| \\ \left\langle -2\right| \\ \left\langle -0\right| \\ \left\langle +2\right| \\ \left\langle +4\right| \\ \left\langle +6\right| \\ \vdots\end{array} & \left(\begin{array}{ccccccc}\ddots &&&&&& \\ 6^2 & v &&&&& \\ v & 4^2 & v &&&& \\ & v & 2^2 & v &&& \\ && v & 0 & v && \\ &&& v & 2^2 & v & \\ &&&& v & 4^2 & v \\ &&&&& v & 6^2 \\ &&&&&& \ddots\end{array}\right)\end{array},$$

Here: $v=\frac{V_0}{2}=\frac{2^2A_y}{2\ell}=\frac{2A_y}{\ell}$ for: $N=2$

$\left\langle j\right|\left(\mathbf{D}+\mathbf{V}\right)\left|k\right\rangle =$ (for j and k *odd*)

$$\cdots\ \left|-7\right\rangle,\left|-5\right\rangle,\left|-3\right\rangle,\left|-1\right\rangle,\left|1\right\rangle,\left|3\right\rangle,\ \left|5\right\rangle,\cdots$$

$$\begin{array}{cc}\begin{array}{c}\vdots\\ \left\langle -7\right| \\ \left\langle -5\right| \\ \left\langle -3\right| \\ \left\langle -1\right| \\ \left\langle +1\right| \\ \left\langle +3\right| \\ \left\langle +5\right| \\ \vdots\end{array} & \left(\begin{array}{ccccccc}\ddots &&&&&& \\ 7^2 & v &&&&& \\ v & 5^2 & v &&&& \\ & v & 3^2 & v &&& \\ && v & 1^2 & v && \\ &&& v & 1^2 & v & \\ &&&& v & 3^2 & v \\ &&&&& v & 5^2 \\ &&&&&& \ddots\end{array}\right)\end{array}$$

E_m-values vary with amplitude V_0 or wiggle amplitude $A_y=V_0\ell/N^2=2v/N^2=v/2$. *(N=2 and ℓ=1 here)*

Eigenvalues for $V_0=0.2$ or $v=0.1$ and $V_0=2.0$ or $v=1.0$.

$E_0 =$	-0.0050	← *inverted*
$E_{1-} =$	0.8988	
$E_{1+} =$	1.0987	
$E_{2-} =$	3.9992	
$E_{2+} =$	4.0042	
$E_{3-} =$	9.0006	
$E_{3+} =$	9.0006	

$E_0 =$	-0.4551	← *inverted*
$E_{1-} =$	-0.1102	← *inverted*
$E_{1+} =$	1.8591	
$E_{2-} =$	3.9170	
$E_{2+} =$	4.3713	
$E_{3-} =$	9.0477	
$E_{3+} =$	9.0784	

When pendulum is "normal" and near its lowest point (ϕ~0) then $\cos\phi\sim 1$ and $\sin\phi\sim\phi$

$$\frac{d^2\phi}{dx^2}+\frac{N^2}{\omega_y{}^2}\left(\frac{g}{\ell}-\frac{\omega_y{}^2A_y}{\ell}\cos(Nx)\right)\phi=0=\frac{d^2\phi}{dx^2}+\left(\frac{N^2}{\omega_y{}^2}\frac{g}{\ell}-\frac{N^2A_y}{\ell}\cos(Nx)\right)\phi,\quad \text{(where: }\phi\sim 0)$$

When pendulum is "inverted" near highest point ($\phi\sim\pi$) then $\cos\phi\sim -1$ and $\sin\phi\sim\pi-\phi$.

$$\frac{d^2\phi}{dt^2}-\left(\frac{g}{\ell}-\frac{\omega_y{}^2A_y}{\ell}\cos(\omega_y t)\right)(\phi-\pi)=0\ ,\qquad \text{(where: }\phi\sim\pi)$$

E_m-eigenvalue determines pendulum Y-wiggle frequency $\omega_{y(m)}$.

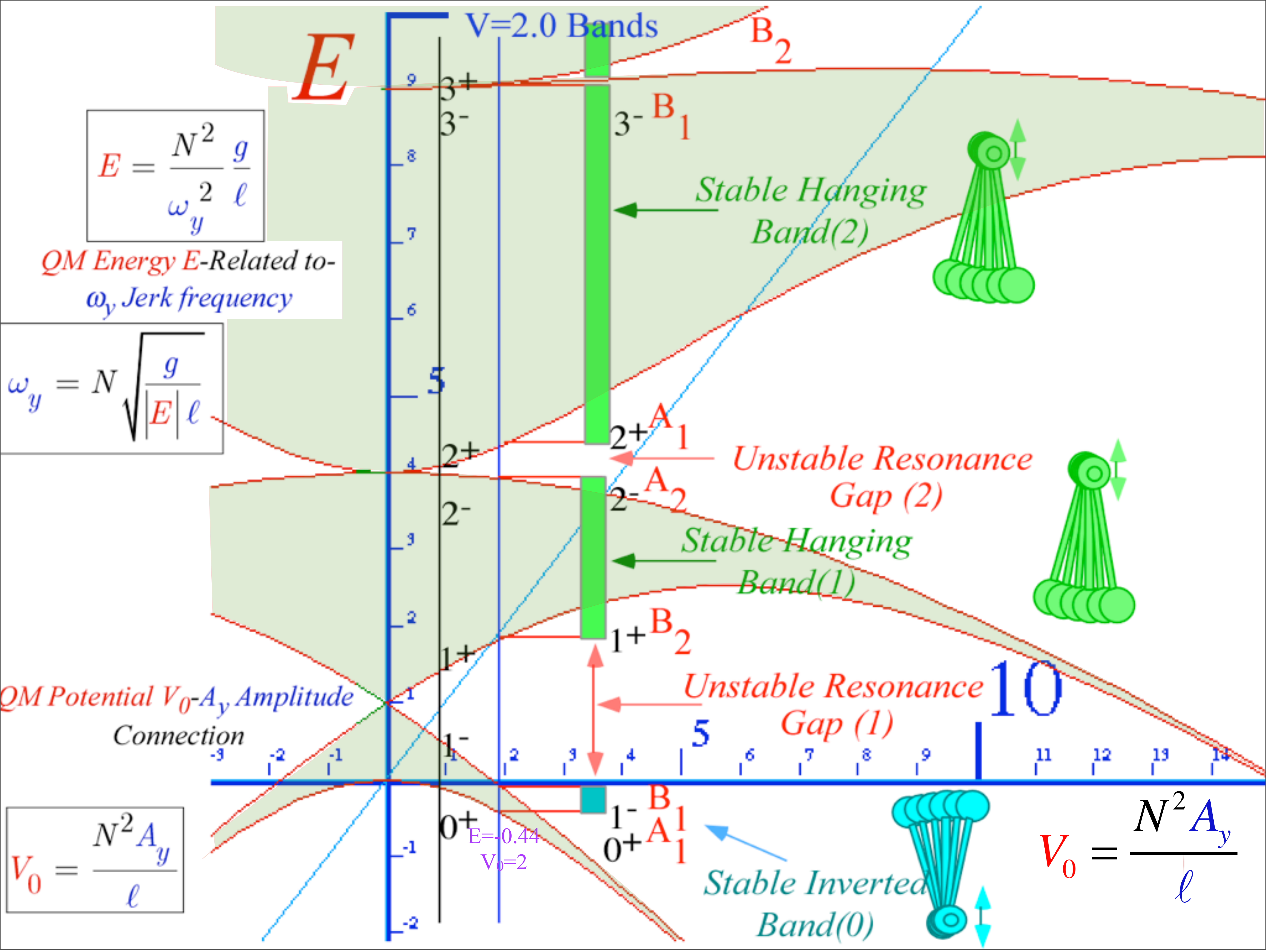
$$E_m=\frac{N^2}{\omega^2_{y(m)}}\frac{g}{\ell}\qquad \text{implies:}\qquad \omega_{y(m)}=\frac{N}{\sqrt{E_m}}\sqrt{\frac{g}{\ell}}=\frac{2}{\sqrt{E_m}}$$

(g=1, too)

Pendulum Y-wiggle frequency $\omega_{y(m)}$ for $V_0=0.2$ and for $V_0=2.0$.

$\omega_{y(0)}=2/\sqrt{.0050}$	$=28.2843$	← *inverted*
$\omega_{y(1^-)}=2/\sqrt{.8988}$	$=2.10959$	
$\omega_{y(1^+)}=2/\sqrt{1.0987}$	$=1.90805$	
$\omega_{y(2^-)}=2/\sqrt{3.9992}$	$=1.00010$	
$\omega_{y(2^+)}=2/\sqrt{4.0042}$	$=0.99948$	

$\omega_{y(0)}=2/\sqrt{.4551}$	$=2.9646$	← *inverted*
$\omega_{y(1^-)}=2/\sqrt{.1102}$	$=6.02475$	← *inverted*
$\omega_{y(1^+)}=2/\sqrt{1.8591}$	$=1.4668$	
$\omega_{y(2^-)}=2/\sqrt{3.9170}$	$=1.0105$	
$\omega_{y(2^+)}=2/\sqrt{4.3713}$	$=0.9566$	



$$E = \frac{N^2}{\omega_y^2}\frac{g}{\ell}$$

QM Energy E-Related to-
ω_y Jerk frequency

$$\omega_y = N\sqrt{\frac{g}{|E|\,\ell}}$$

QM Potential V_0-A_y Amplitude
Connection

$$V_0 = \frac{N^2 A_y}{\ell}$$

Stable Hanging Band(2)

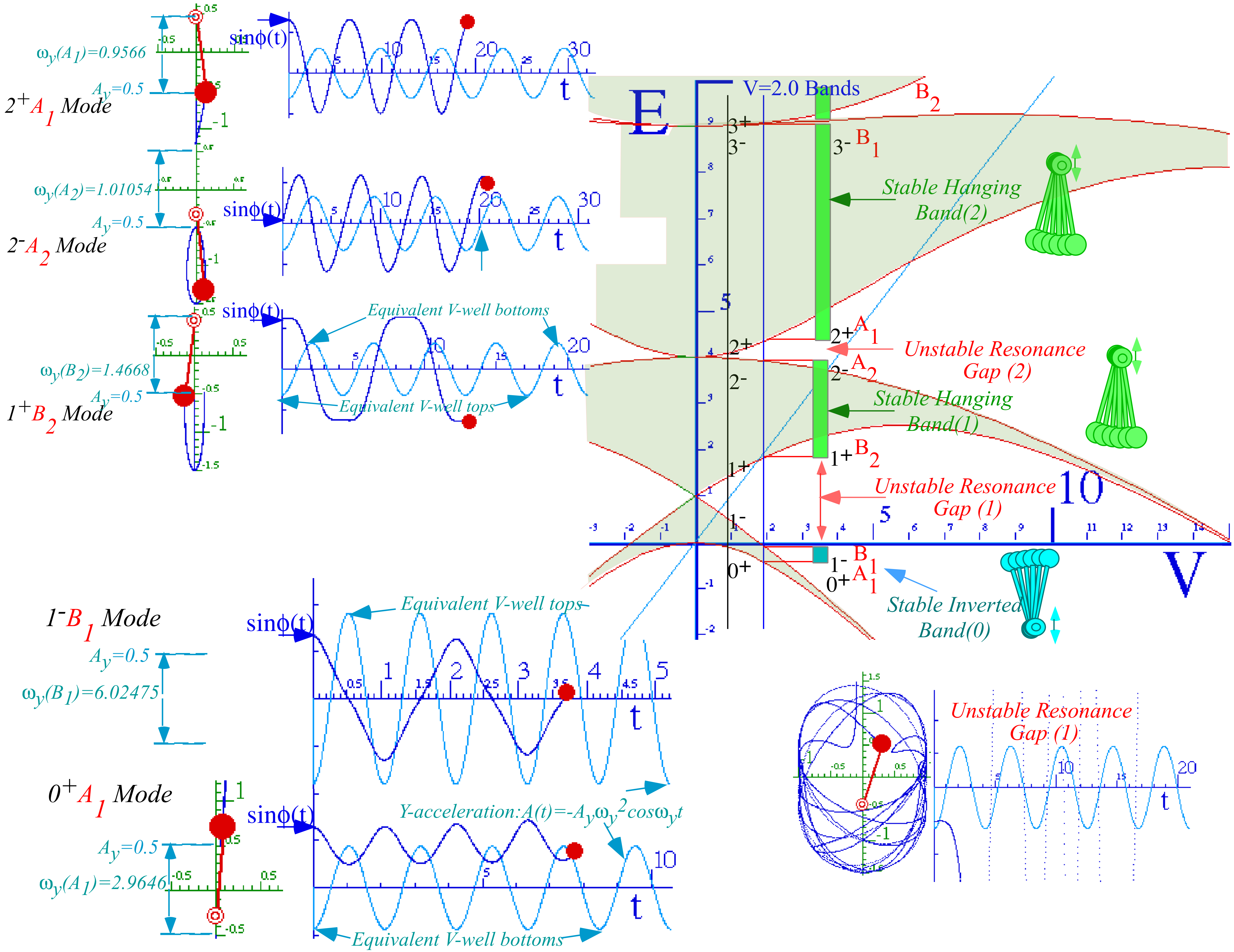
Unstable Resonance Gap (2)

Stable Hanging Band(1)

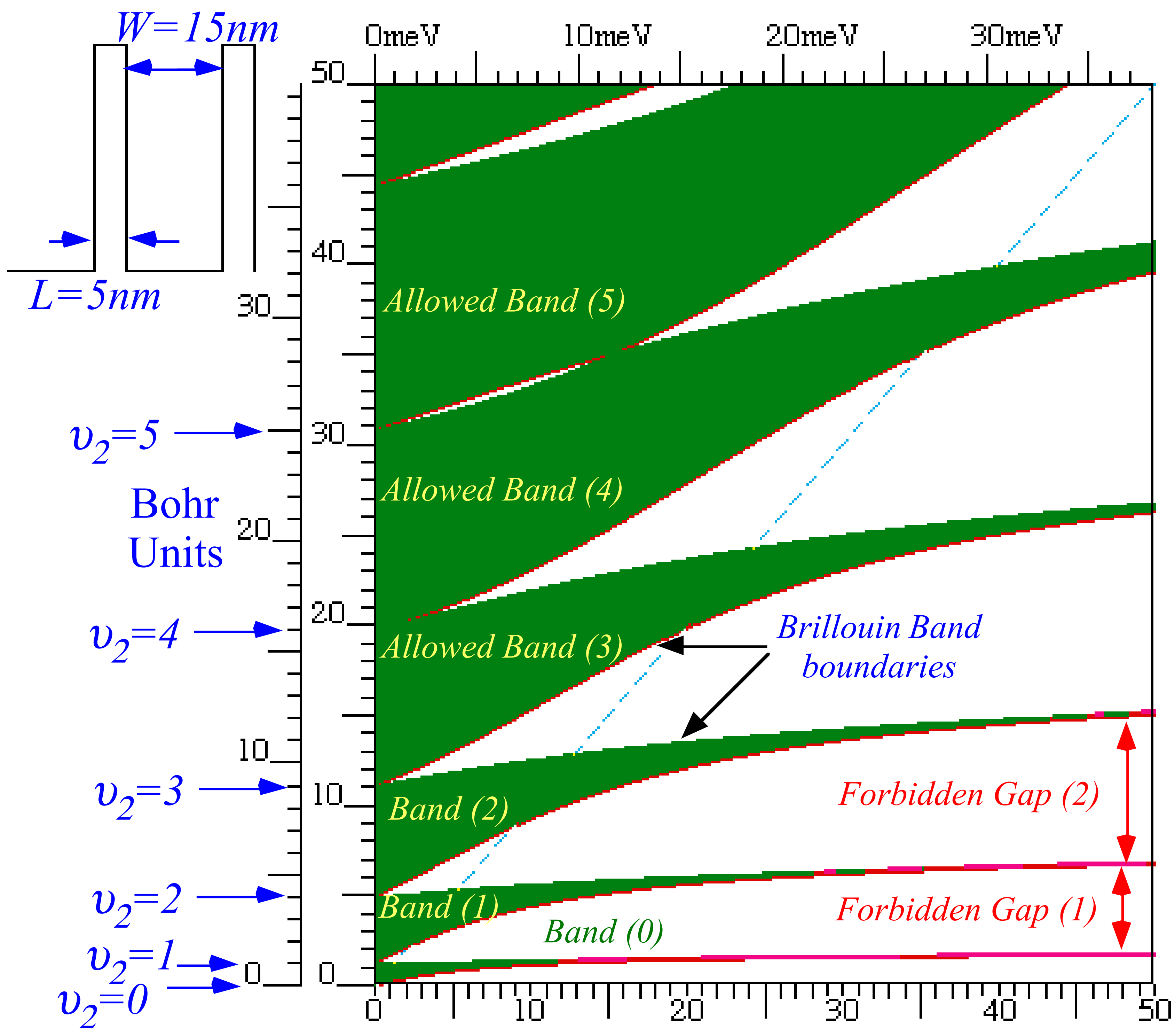
Unstable Resonance Gap (1)

Stable Inverted Band(0)

$$V_0 = \frac{N^2 A_y}{\ell}$$


$\omega_y(A_1)=0.9566$
$A_y=0.5$
2^+A_1 Mode
sinφ(t)
t
$\omega_y(A_2)=1.01054$
$A_y=0.5$
2^-A_2 Mode
$\omega_y(B_2)=1.4668$
$A_y=0.5$
1^+B_2 Mode
Equivalent V-well bottoms
Equivalent V-well tops
E
V=2.0 Bands
B_2
3^+
3^-
$3^-\ B_1$
Stable Hanging Band(2)
2^+A_1
Unstable Resonance Gap (2)
2^+
2^-A_2
2^-
Stable Hanging Band(1)
1^+B_2
1^+
Unstable Resonance Gap (1)
1^-
$1^-\ B_1$
0^+A_1
0^+
Stable Inverted Band(0)
V
1^-B_1 Mode
$A_y=0.5$
$\omega_y(B_1)=6.02475$
Equivalent V-well tops
0^+A_1 Mode
$A_y=0.5$
$\omega_y(A_1)=2.9646$
Y-acceleration:$A(t)=-A_y\omega_y^2\cos\omega_y t$
Equivalent V-well bottoms
Unstable Resonance Gap (1)

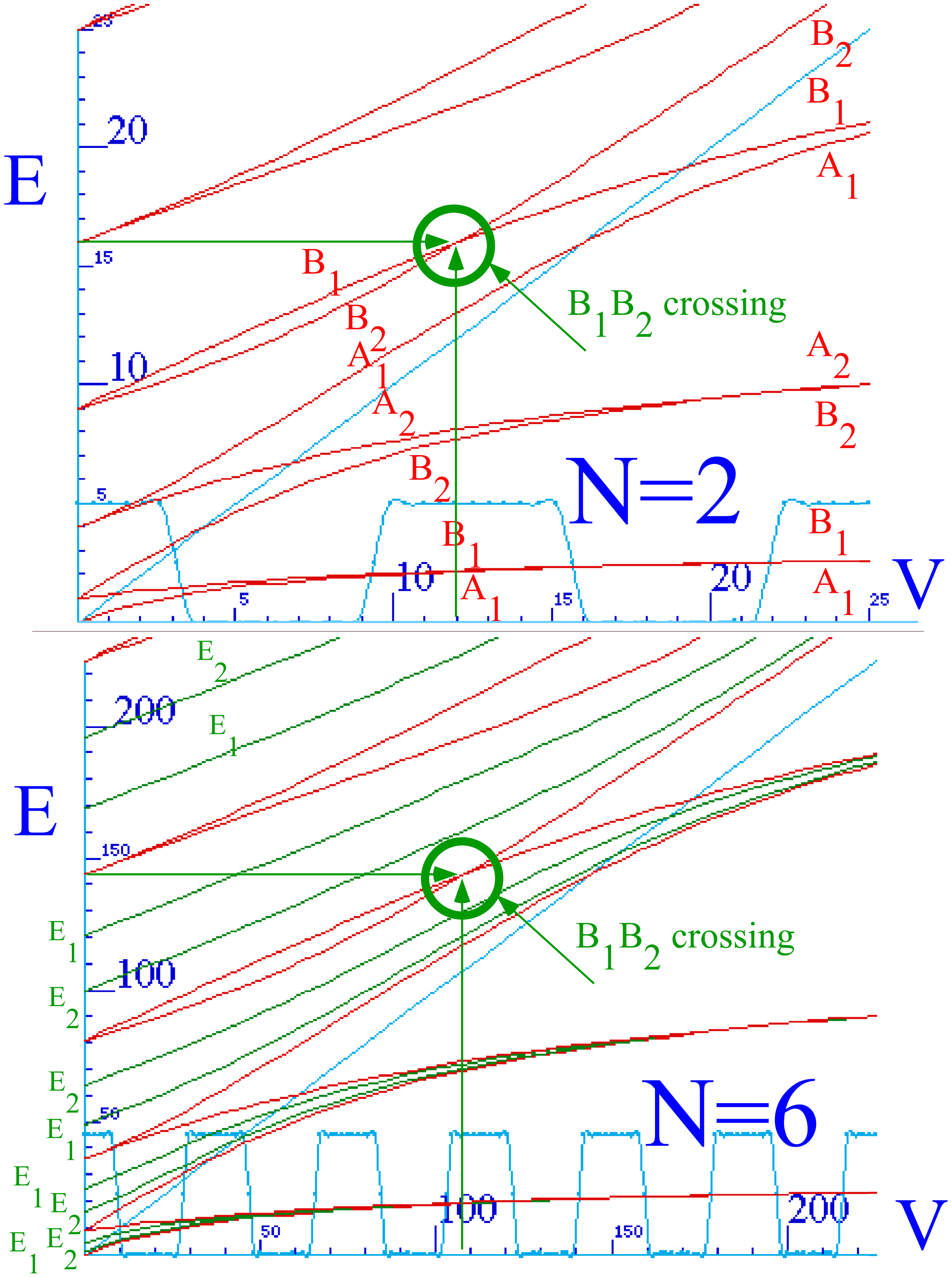
A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5 Quantum Theory for the Computer Age (QTftCA)

Fig. 14.2.7 Bands vs. V.(W=15nm well ,L=5nm barrier) showing Bohr splitting for (N=2)-ring.

A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5
Quantum Theory for the
Computer Age (QT$_{ft}$CA)

Fig. 14.2.13 (B_1, B_2) crossing for:(N=2) at V=12 and E=16, and (N=6) at V=144 and E=108.

Wave resonance in cyclic symmetry

- ➔ *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix $\mathbf{K}=\mathbf{H}^2$ with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$=A\cdot\mathbf{1} \qquad +B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

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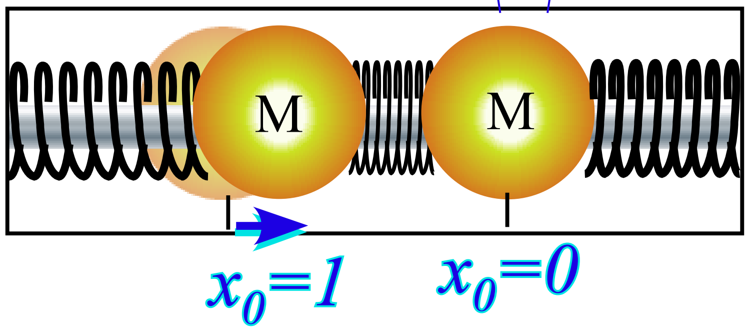
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry $\boldsymbol{\sigma}_B$ defined by $(\boldsymbol{\sigma}_B)^2=\mathbf{1}$ in C_2 group product table.

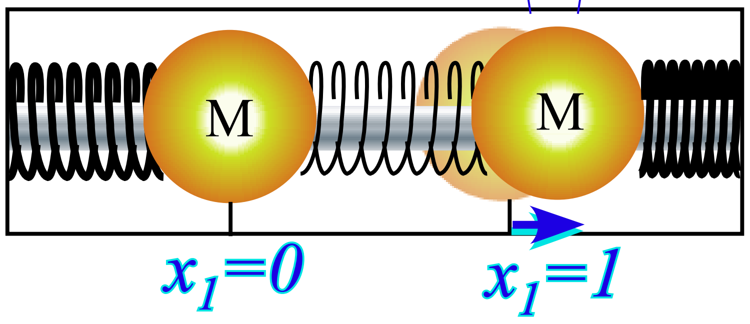
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

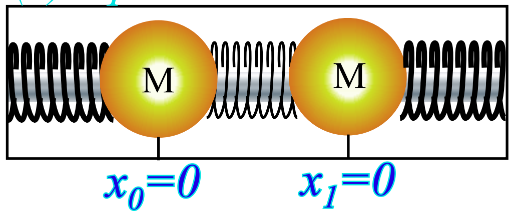


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

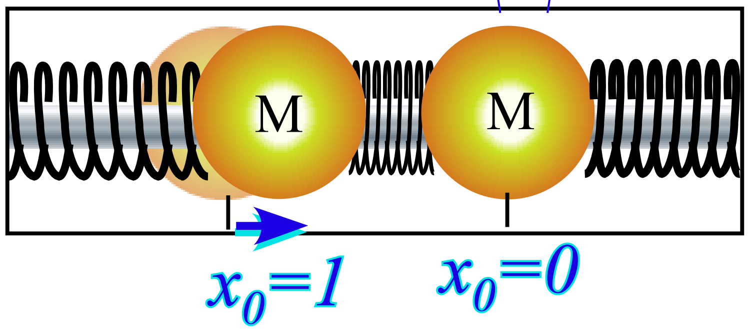
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

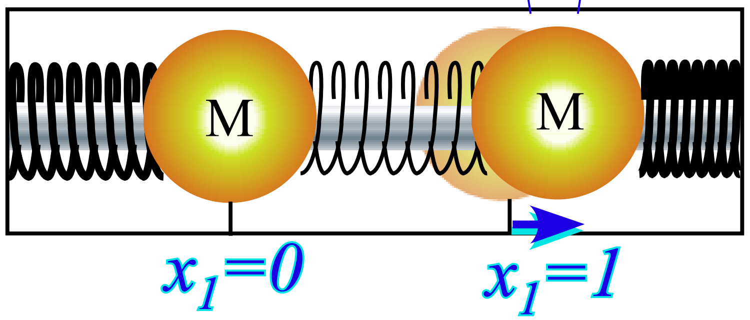
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

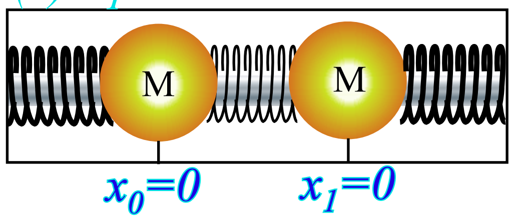


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|\text{-}1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2$-$\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix}A&B\\B&A\end{pmatrix}=A\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$=A\cdot\mathbf{1}\qquad+B\cdot\boldsymbol{\sigma}_B$$

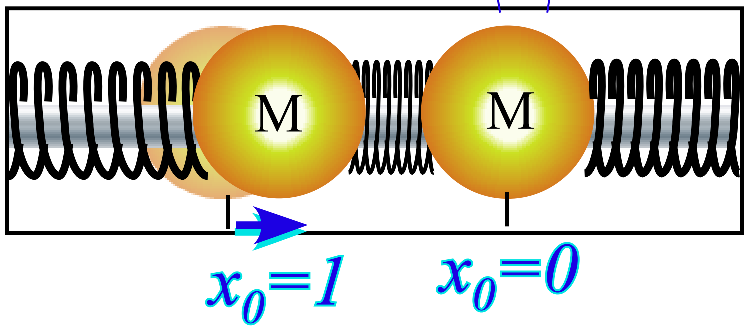
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix}A^2+B^2&2AB\\2AB&A^2+B^2\end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1}\qquad+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

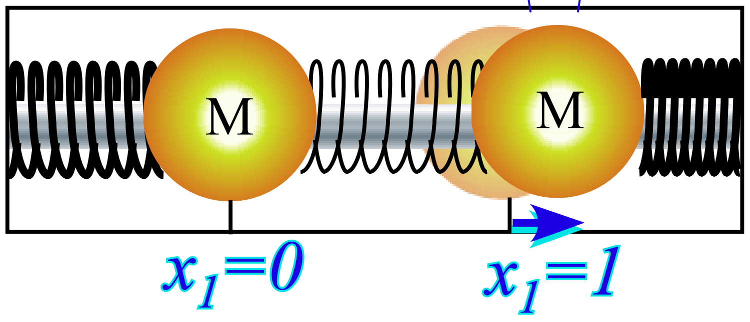
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$

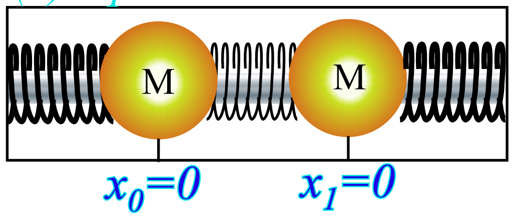


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix}A & B\\ B & A\end{pmatrix}=A\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+B\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$=A\cdot\mathbf{1}\qquad +B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix}A^2+B^2 & 2AB\\ 2AB & A^2+B^2\end{pmatrix}$$

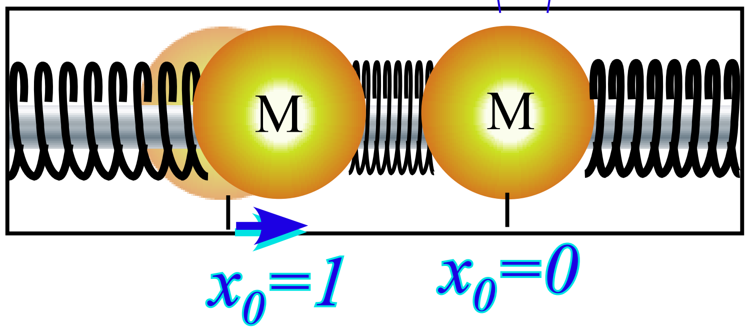
$$=(A^2+B^2)\cdot\mathbf{1}\qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

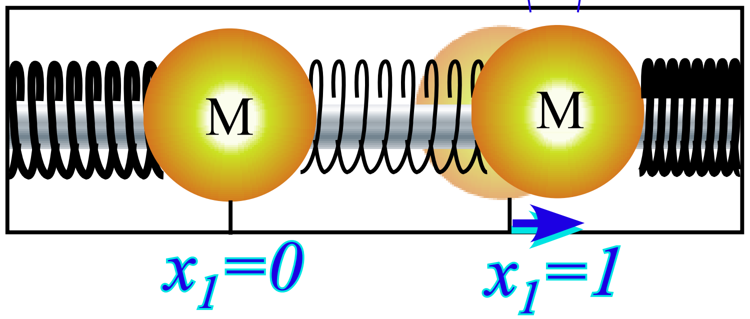
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$$

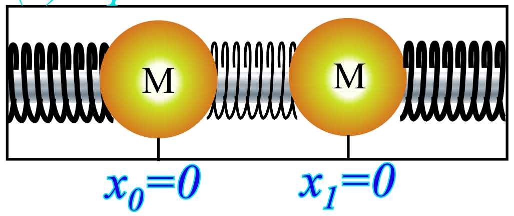


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$$|1\rangle=|y\rangle=|{-1}\rangle=\begin{pmatrix}0\\1\end{pmatrix}$$

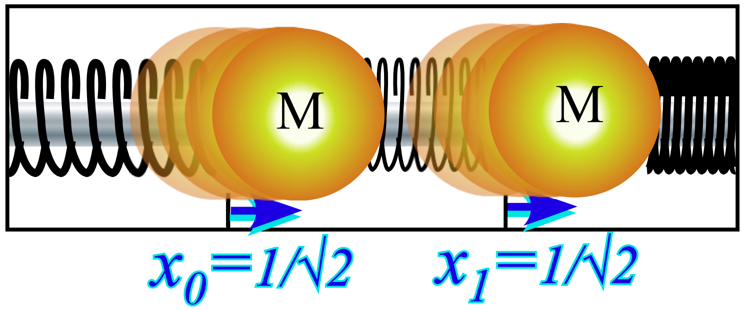


(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$

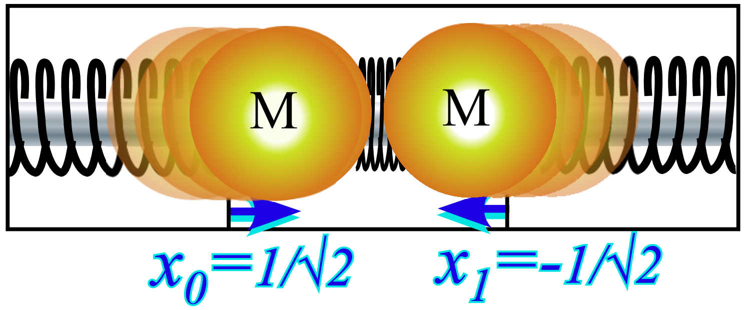


C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix}A&B\\B&A\end{pmatrix}=A\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$=A\cdot\mathbf{1}\qquad+B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix}A^2+B^2&2AB\\2AB&A^2+B^2\end{pmatrix}$$

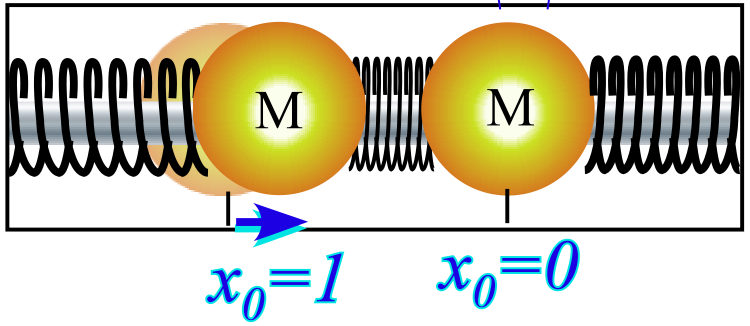
$$=(A^2+B^2)\cdot\mathbf{1}\qquad+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

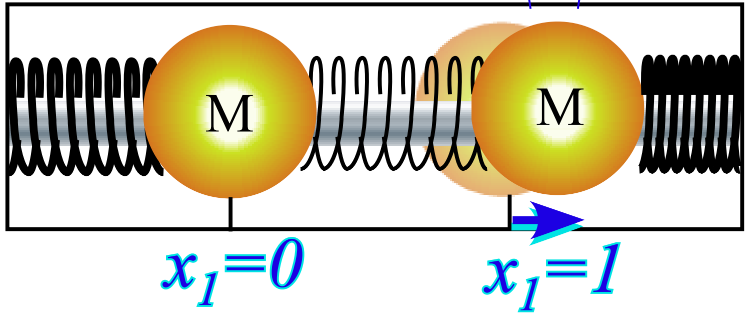
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$

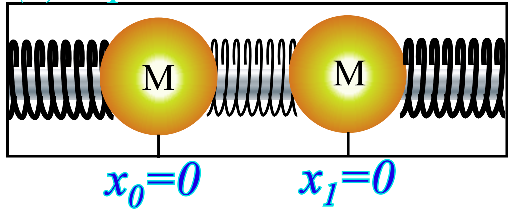


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

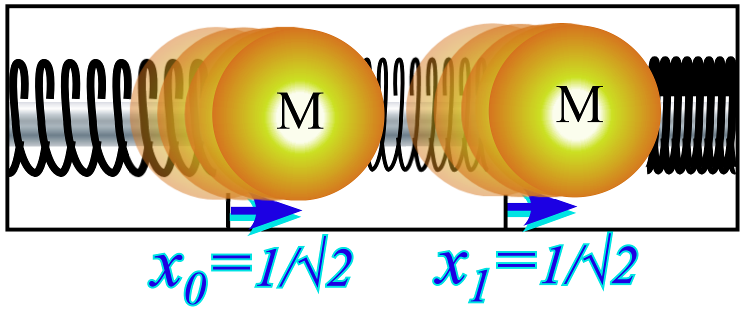
$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

C_2 symmetry (B-type) modes

Mode state projection:

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$

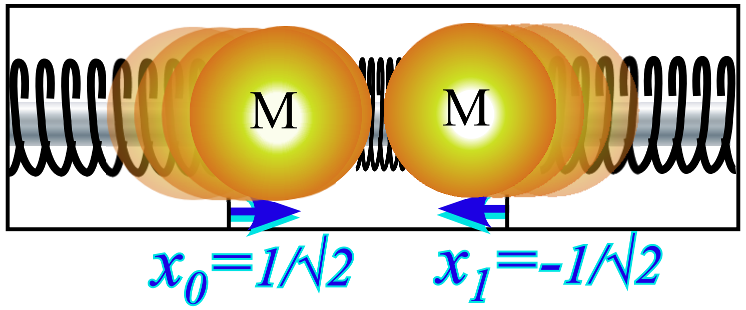


$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$

$=(|0\rangle+|2\rangle)/\sqrt{2}$

$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$

$=(|0\rangle-|2\rangle)/\sqrt{2}$

$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= A\cdot\mathbf{1} \qquad + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$

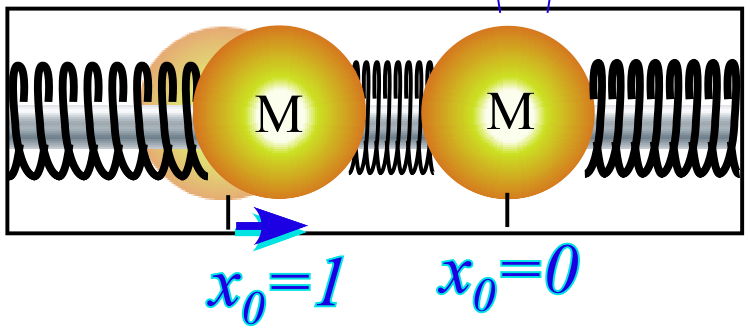
$$=(A^2+B^2)\cdot\mathbf{1} \qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

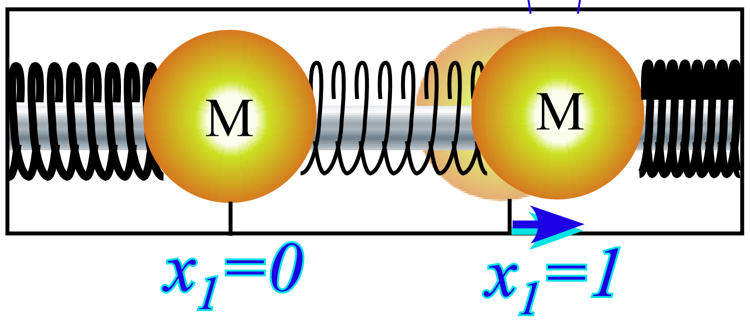
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

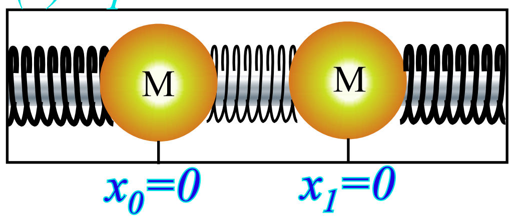


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

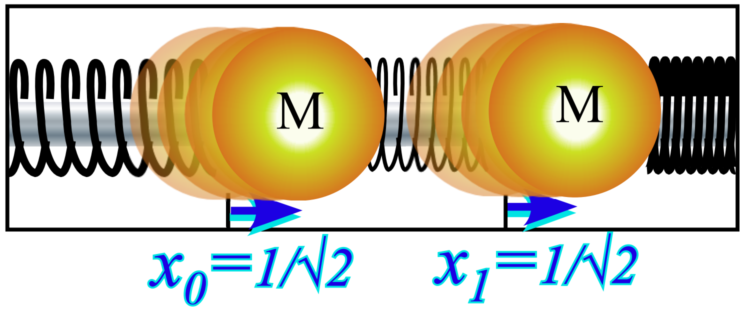
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\,\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

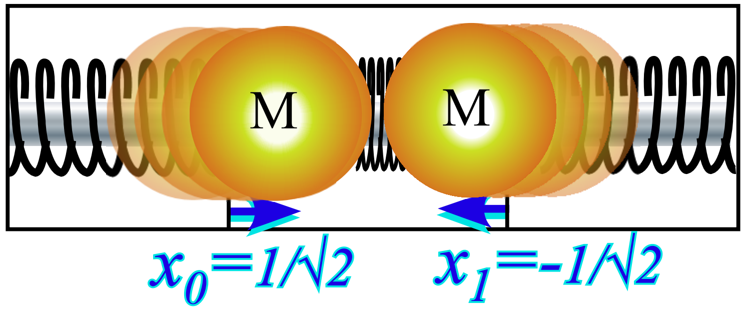
(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



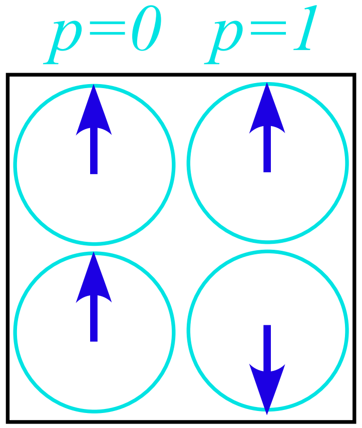
Mode state projection:

$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

C_2 *mode phase & character tables*

p= position point (modulo-2)



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

Wave resonance in cyclic symmetry

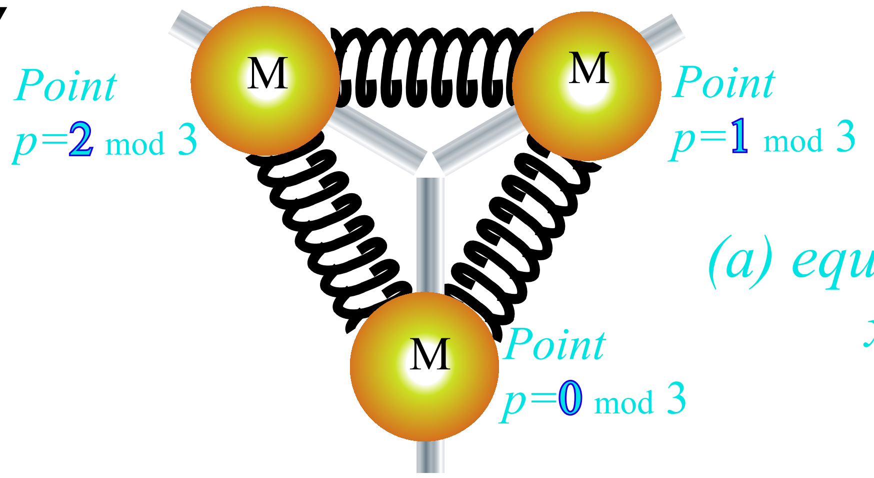
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- ➔ *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$$

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row
then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} \quad = \quad r_0 \cdot \mathbf{1} \quad + r_1 \cdot \mathbf{r}^1 \quad + r_2 \cdot \mathbf{r}^2$$

$$\mathbf{r}^0=\mathbf{1}$$

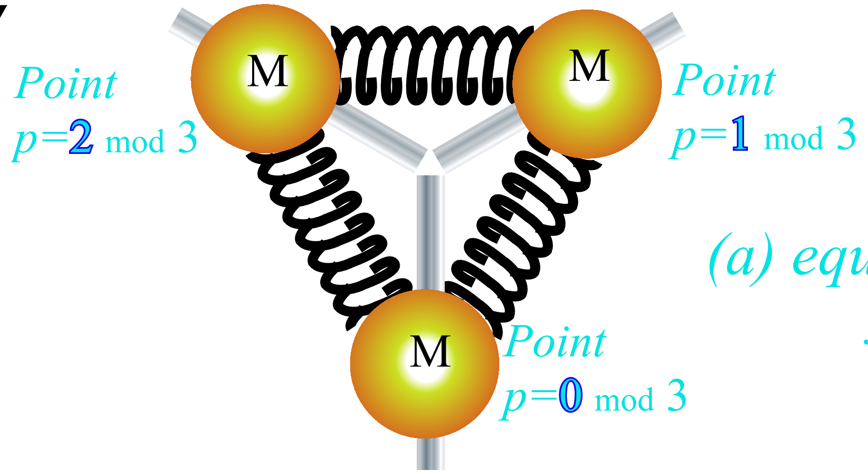
Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C₃ symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C₃ **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$$

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + r_1\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} + r_2\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\mathbf{H} \quad = \quad r_0\cdot\mathbf{1} \quad + r_1\cdot\mathbf{r}^1 \quad + r_2\cdot\mathbf{r}^2$$

$$\mathbf{r}^0=\mathbf{1}$$

C₃ unit base states

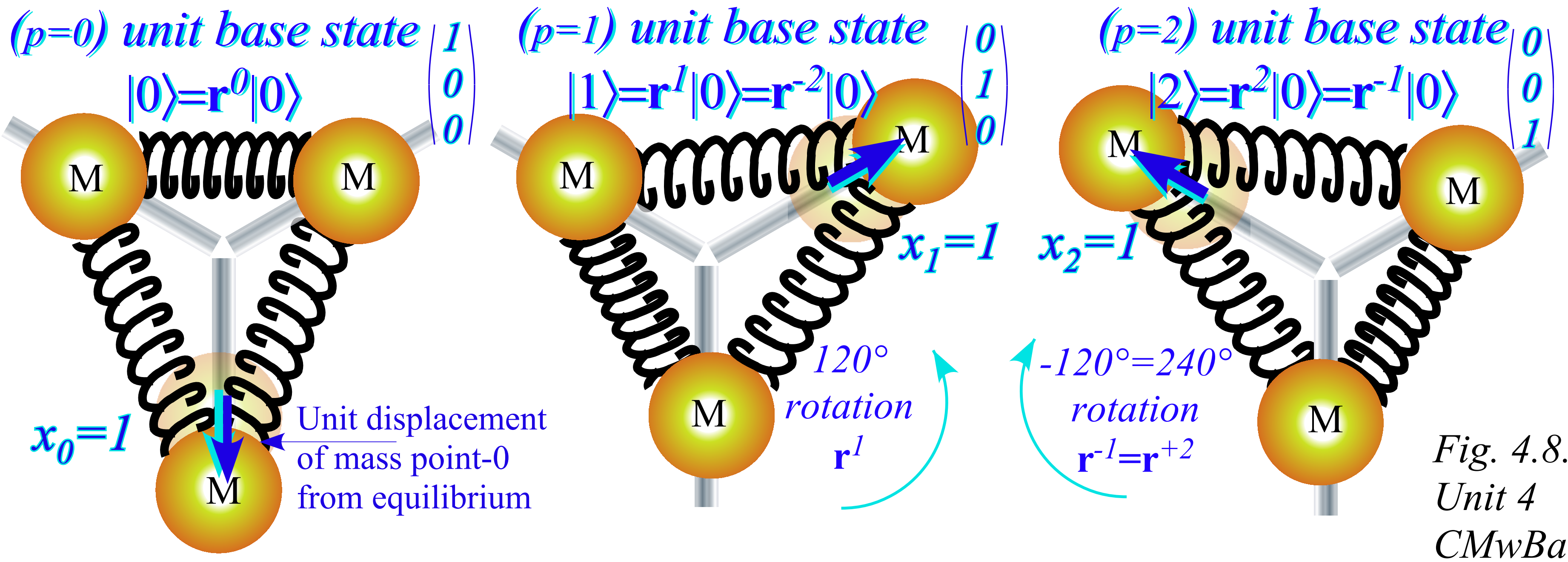


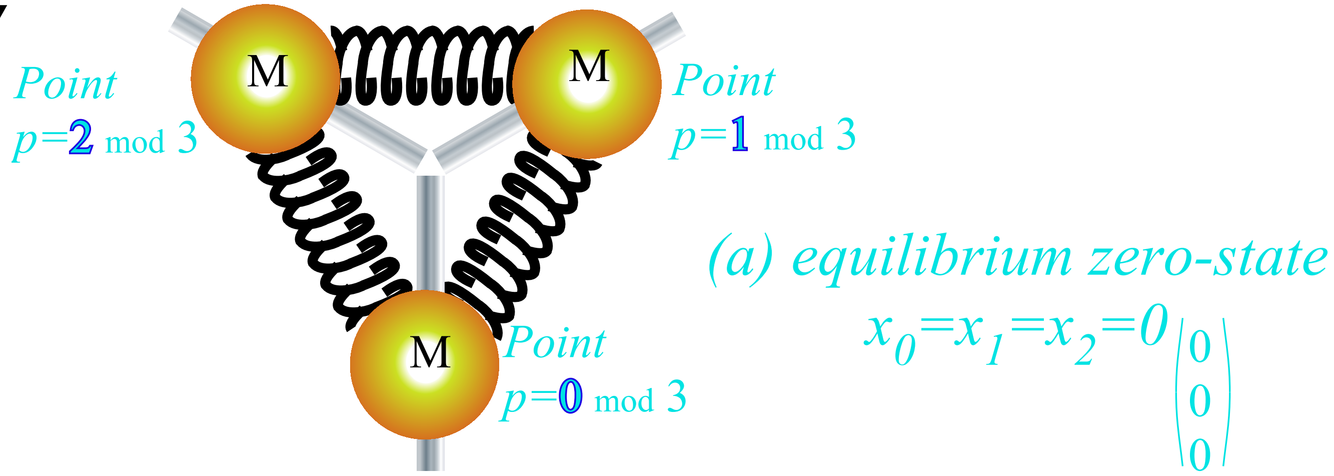
Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\ \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row
then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} \quad = \quad r_0\cdot\mathbf{1} \qquad + r_1\cdot\mathbf{r}^1 \qquad + r_2\cdot\mathbf{r}^2$$

$$\mathbf{r}^0=\mathbf{1}$$

C_3 unit base states

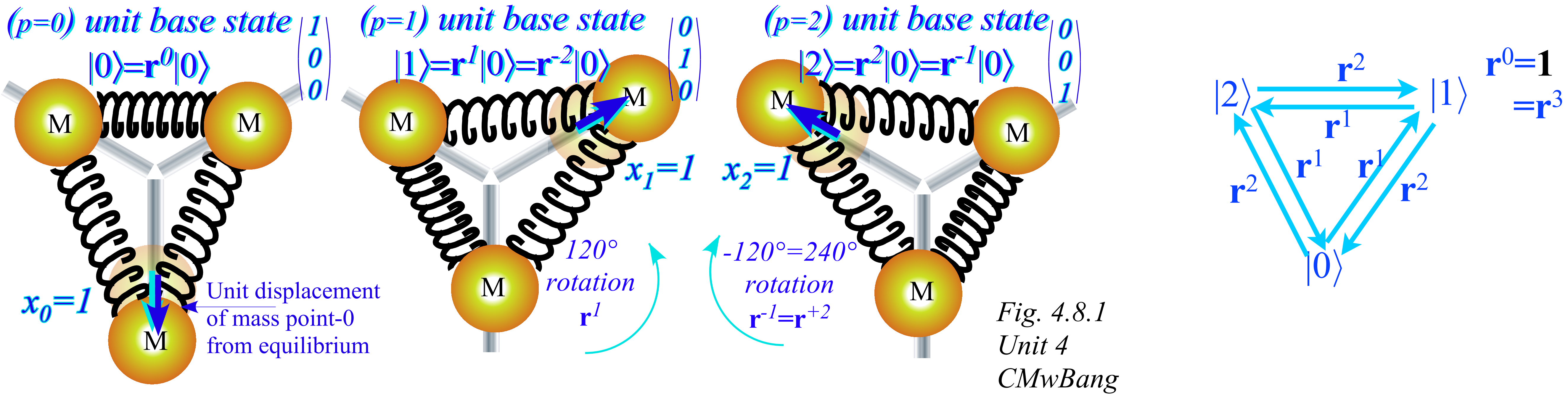


Fig. 4.8.1
Unit 4
CMwBang

Each **H**-matrix coupling constant $r_p=\{r_0, r_1, r_2\}$ is amplitude of its operator power $\mathbf{r}^p=\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - ➤ *C_3 symmetric spectral decomposition by 3rd roots of unity*
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  - *Phase arithmetic*

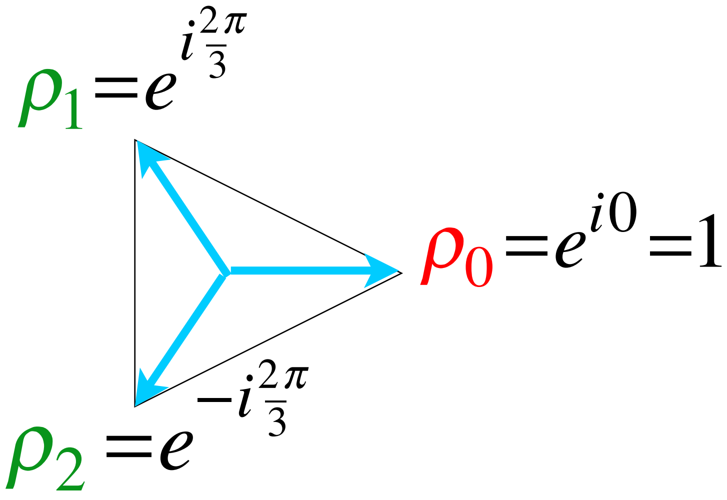
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3\text{-}\mathbf{1}=\mathbf{0}$ and resolves to factors of *3^{rd} roots of unity* $\rho_m=e^{im2\pi/3}$.



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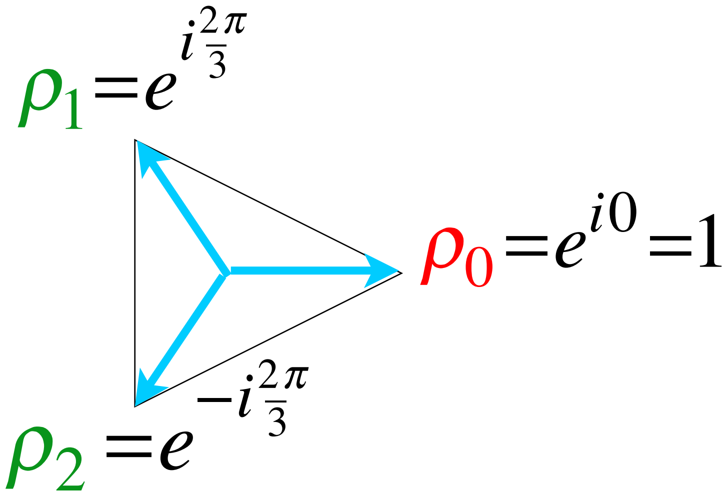
$$\mathbf{1}=\mathbf{r}^3 \ \textit{implies:}\ \ \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1})\ \textit{where:}\ \ \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.



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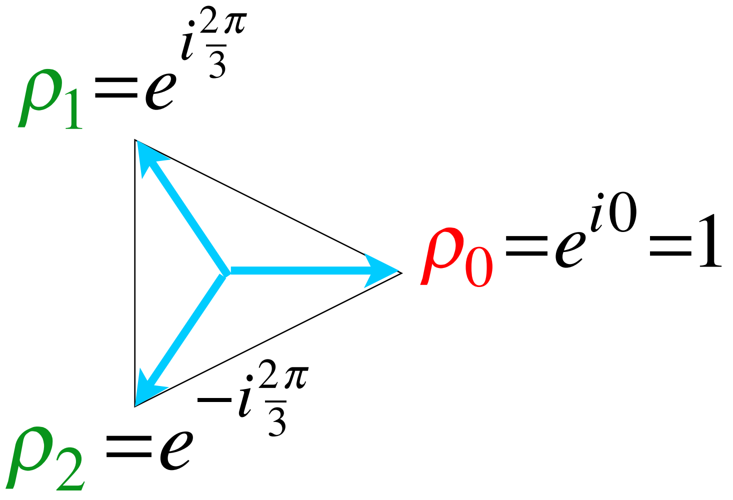
Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.

All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$$\mathbf{1} \;=\; \mathbf{P}^{(0)} \;+\; \mathbf{P}^{(1)} \;+\; \mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

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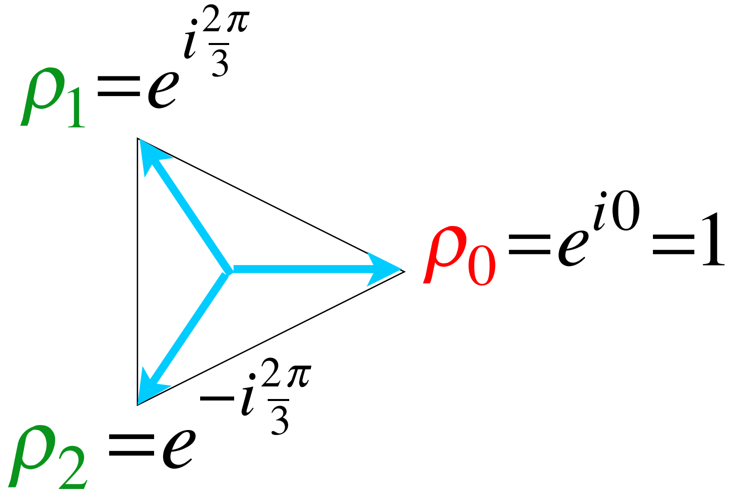
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3^{rd} roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3^{rd} roots of unity* $\rho_m=e^{im2\pi/3}$.

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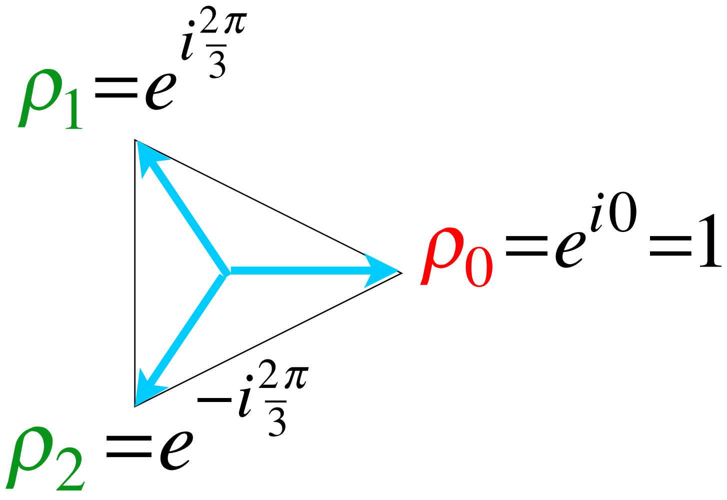
Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.

All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

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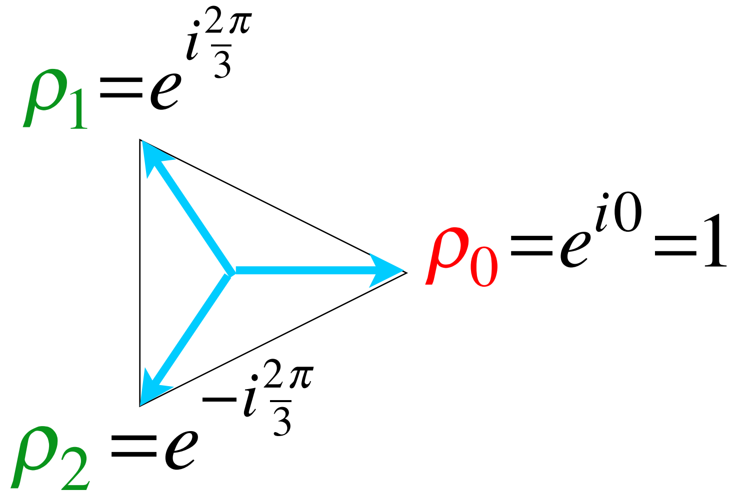
Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.

All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$$\mathbf{1}=\mathbf{P}^{(0)}+\mathbf{P}^{(1)}+\mathbf{P}^{(2)}$$

$$\mathbf{r}=\rho_0\,\mathbf{P}^{(0)}+\rho_1\,\mathbf{P}^{(1)}+\rho_2\,\mathbf{P}^{(2)}$$

$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

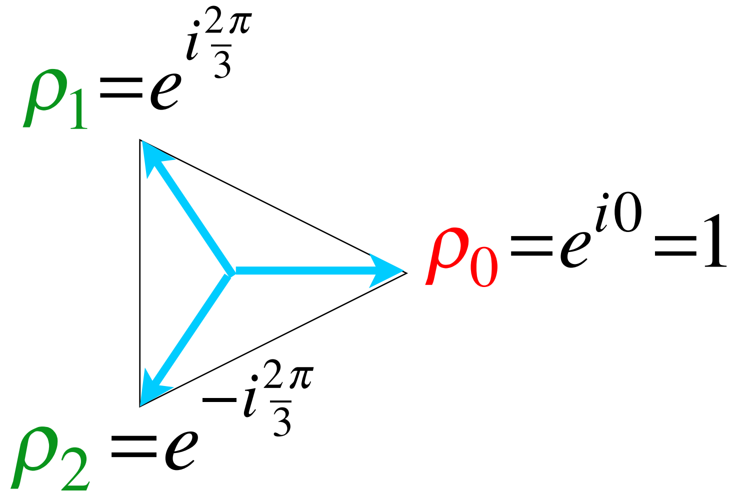
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \textit{ implies: } \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \textit{ where: } \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\,\mathbf{P}^{(n)}=\delta_{mn}\,\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).



$\rho_1=e^{i\frac{2\pi}{3}}$

$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

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(m_3) means: *m-modulo-3* (Details follow)

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    - *Harmonic oscillator with cyclic C_2 symmetry*
        - *C_2 symmetric (B-type) modes*
    - *Harmonic oscillator with cyclic C_3 symmetry*
        - *C_3 symmetric spectral decomposition by 3rd roots of unity*
        - ➡ *Resolving C_3 projectors and moving wave modes*
        - *Dispersion functions and standing waves*
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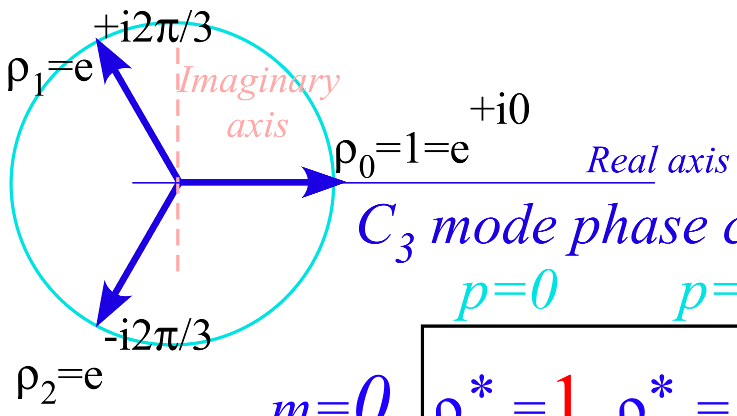

C_3 mode phase character tables

wave-number
m=
“momentum”

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
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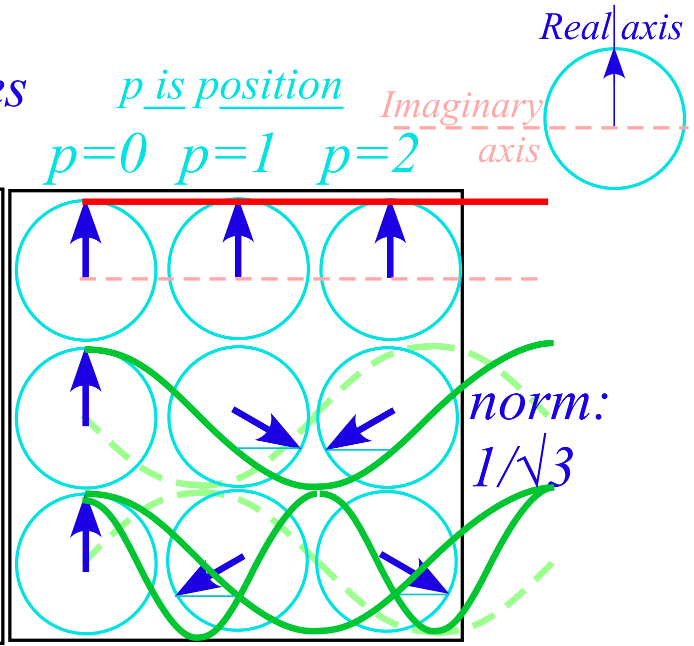

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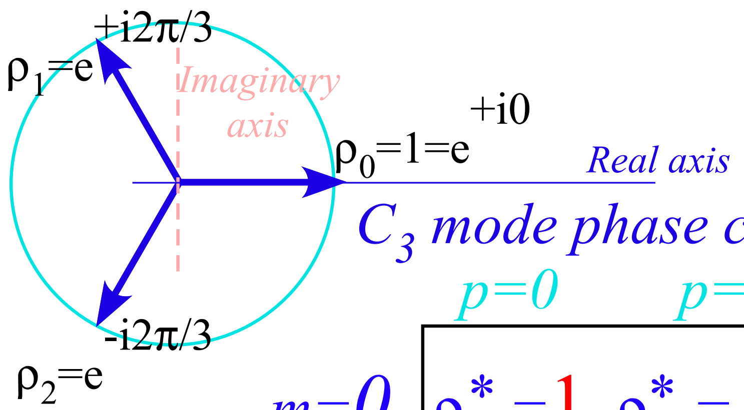

C_3 mode phase character tables

wave-number *m*= "momentum"

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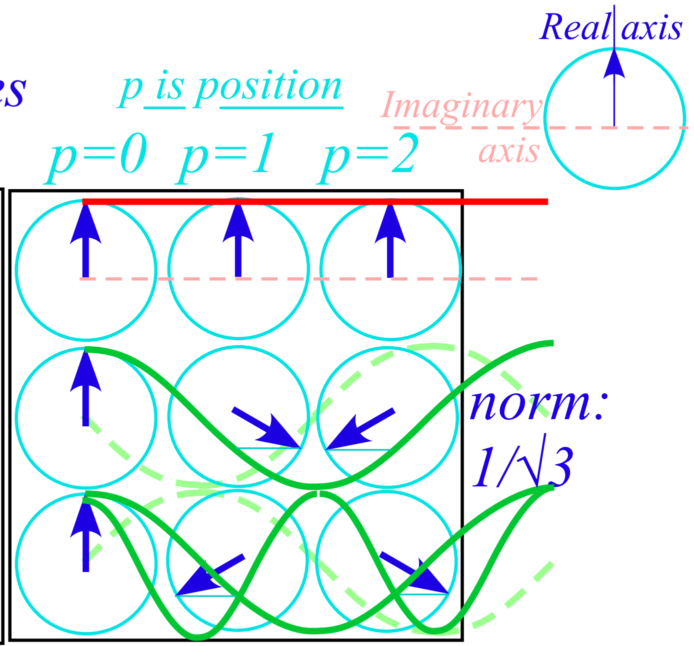

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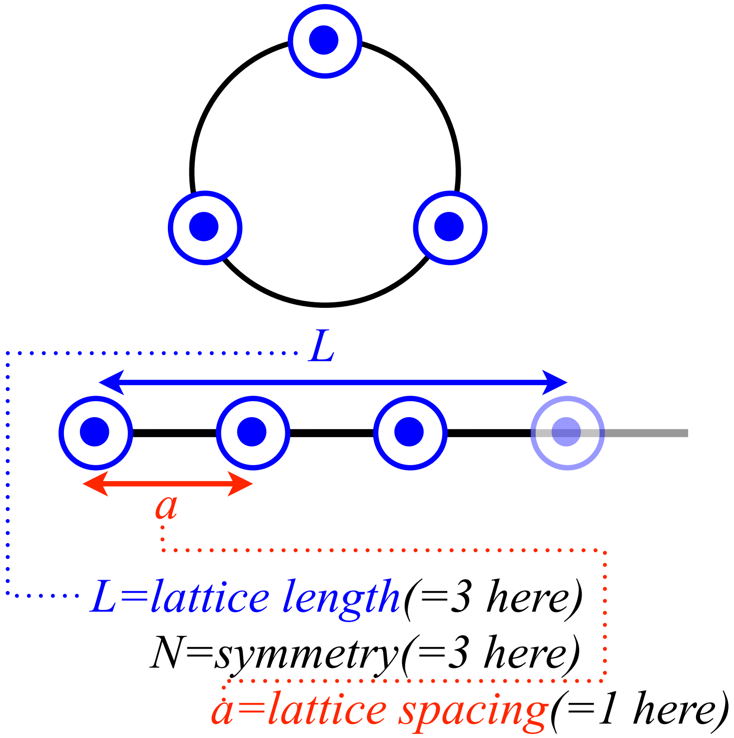

L=lattice length(=3 here)

N=symmetry(=3 here)

a=lattice spacing(=1 here)

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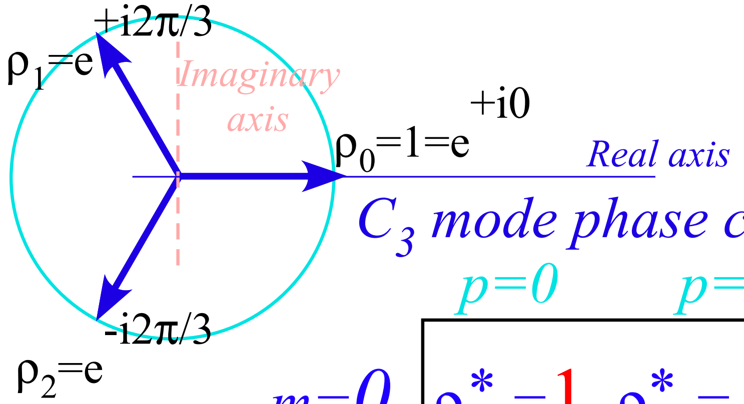

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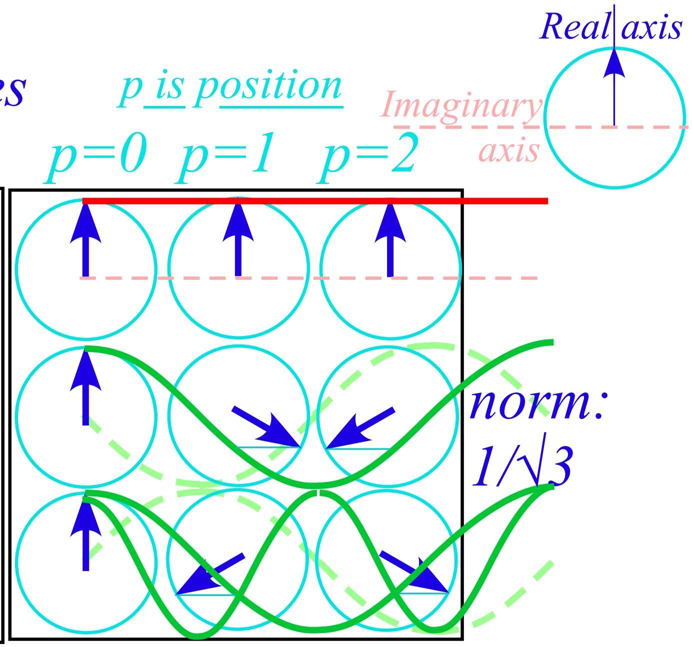

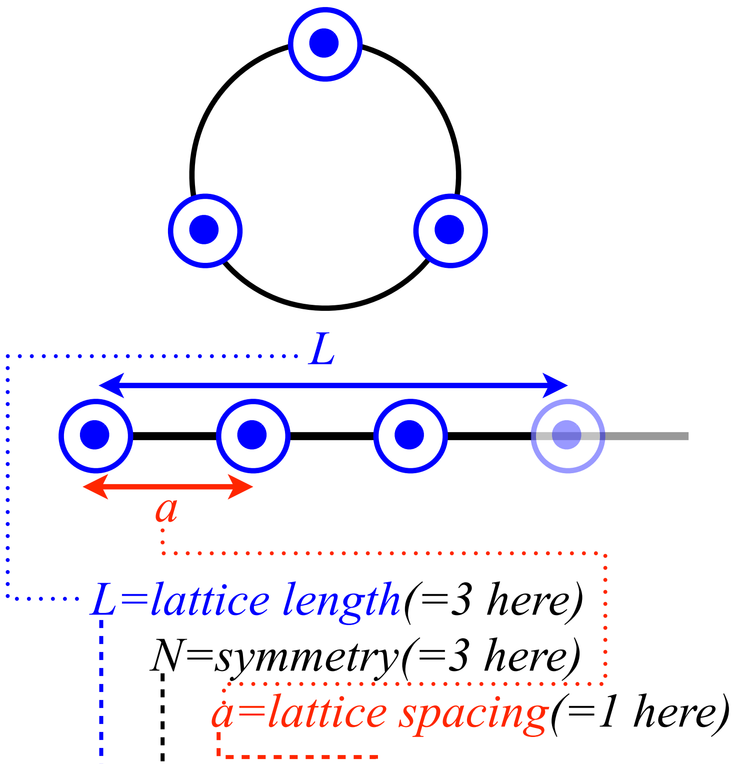

Two distinct types of "quantum" numbers.

$p=0,1$,or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator's *position point p*.

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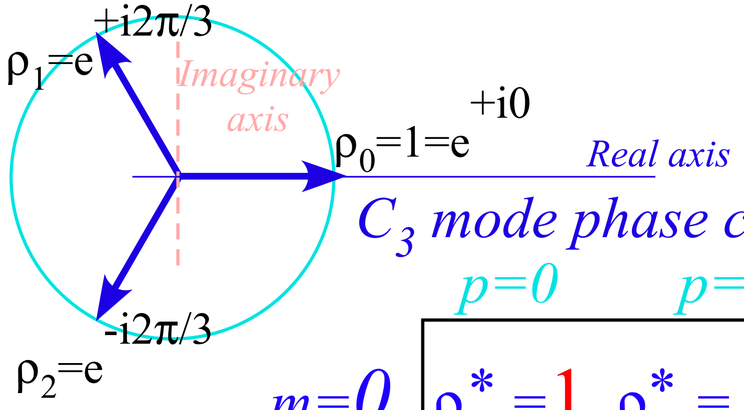

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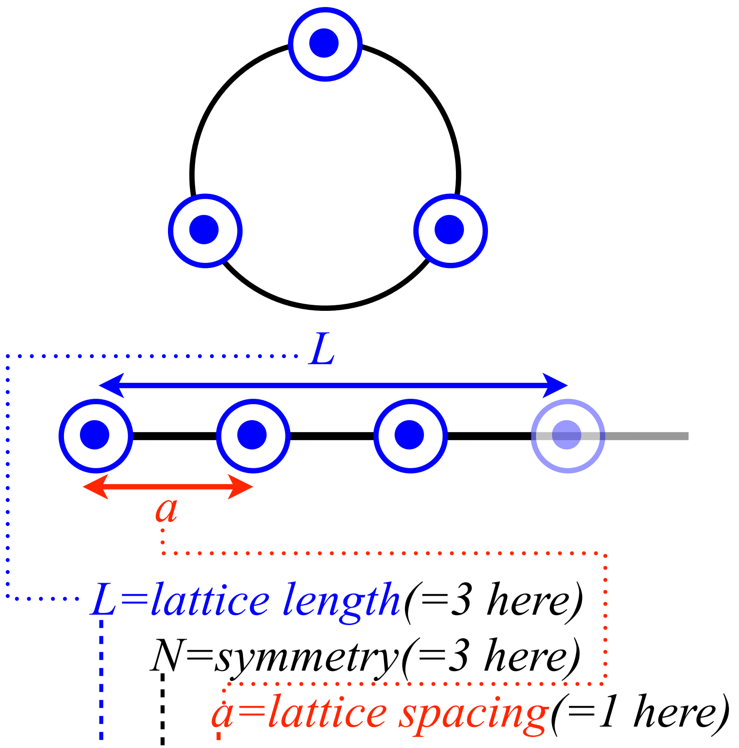

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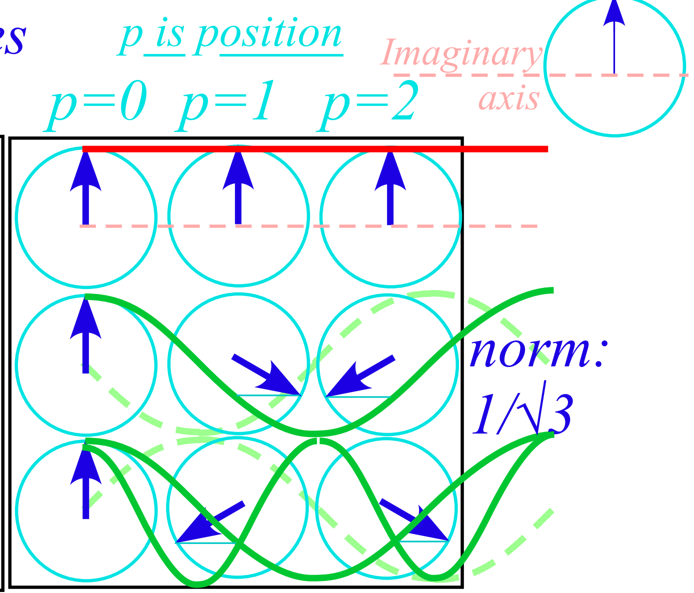

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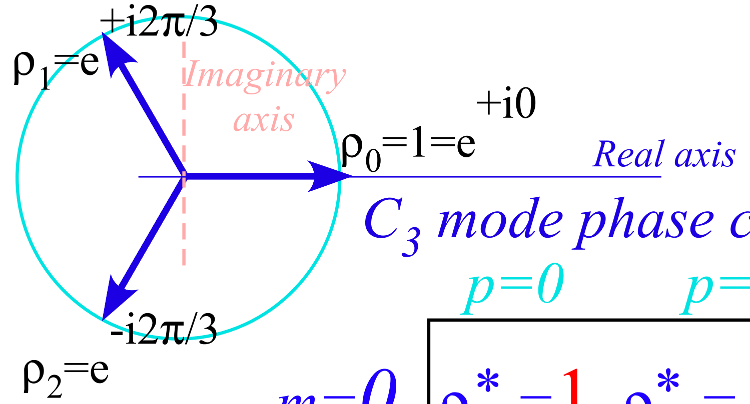

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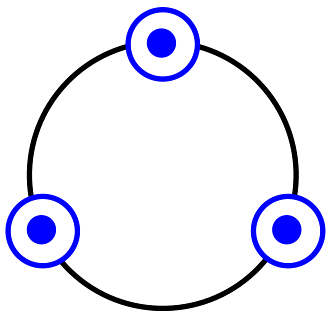

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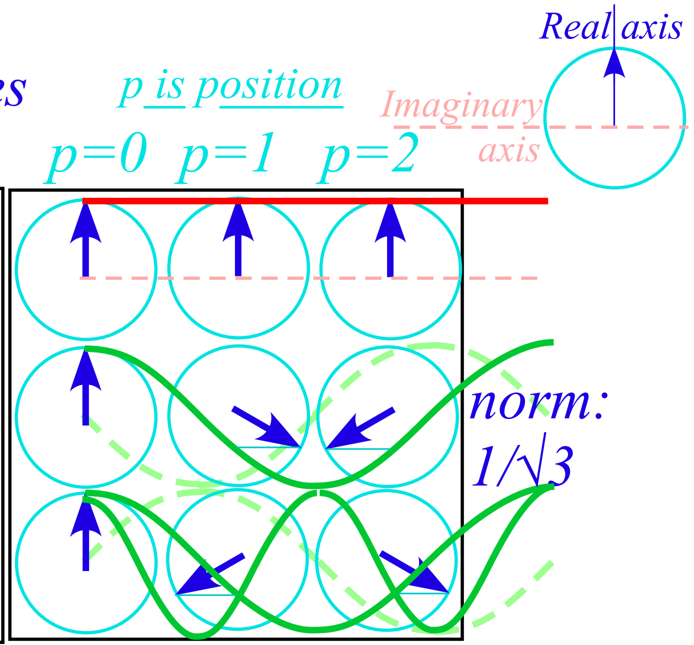

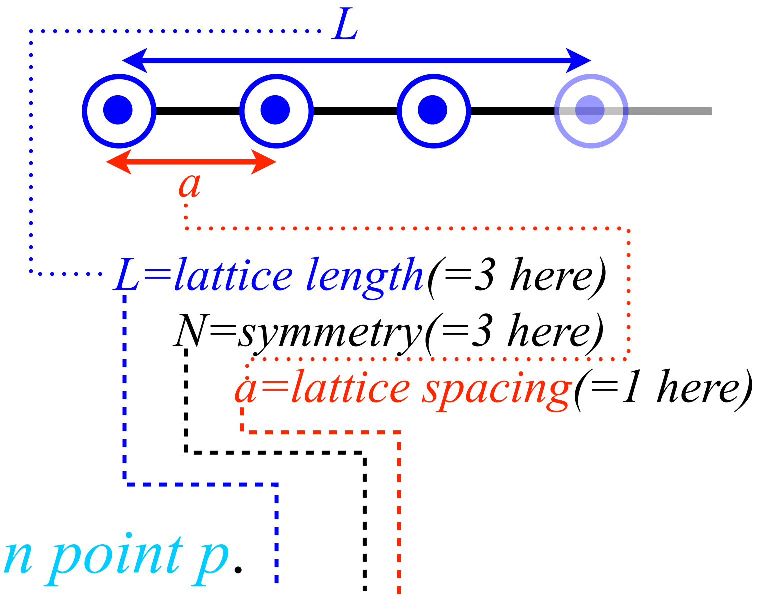

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For example, for *m*=*2* and *p*=*2* the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}=e^{i4\cdot 2\pi/3}=e^{i1\cdot 2\pi/3}\,e^{i2\pi}=e^{i2\pi/3}=\rho_1$.
That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3*=*1*, the remainder of *4* divided by *3*.)

Wave resonance in cyclic symmetry

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Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m| r_0\mathbf{r}^{0} + r_1\mathbf{r}^{1} + r_2\mathbf{r}^{2} |m\rangle = r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\, \frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\, \frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^{p}$

$\langle m|\ \mathbf{r}^{p}\ |m\rangle = e^{\,i\, m\cdot p\ 2\pi/3}$

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$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}) = r_0 + 2r\cos(\tfrac{2\pi m}{3}) = \begin{cases} r_0+2r & (\text{for } m=0) \\ r_0-r & (\text{for } m=\pm 1) \end{cases}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

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H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

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K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\, \frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\, \frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\, m\cdot p\ 2\pi/3}$

$$= r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}) = r_0 + 2r\cos(\tfrac{2\pi m}{3}) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\|(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\|c_3\rangle = \frac{\|(+1)_3\rangle + \|(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\|(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\|s_3\rangle = \frac{\|(+1)_3\rangle - \|(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\|(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle=\langle m|r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2|m\rangle=r_0e^{i\,m\cdot 0\,\frac{2\pi}{3}}+r_1e^{i\,m\cdot 1\,\frac{2\pi}{3}}+r_2e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th}\ Eigenvalue\ of\ \mathbf{r}^p \\ \langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}\end{array}}$$

$$=r_0e^{i\,m\cdot 0\,\frac{2\pi}{3}}+r(e^{i\frac{2\pi m}{3}}+e^{-i\frac{2\pi m}{3}})=r_0+2r\cos(\tfrac{2\pi m}{3})=\begin{cases} r_0+2r\ \ (\text{for } m=0) \\ r_0-r\ \ \ (\text{for } m=\pm 1)\end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}=\left(r_0+2r\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

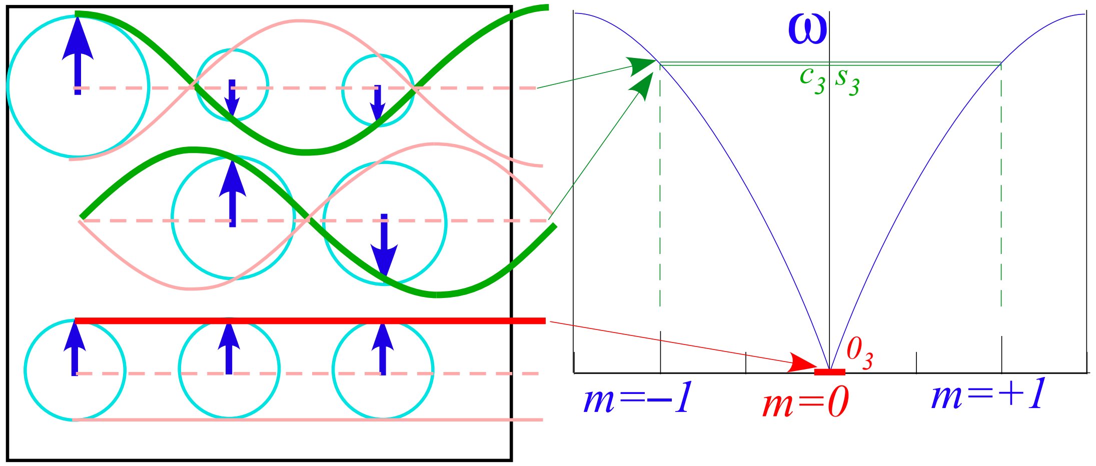
K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}=\left(K-2k\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\left\|(+1)_3\right\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3}\end{pmatrix}$	$\left\|c_3\right\rangle=\frac{\left\|(+1)_3\right\rangle+\left\|(-1)_3\right\rangle}{\sqrt{2}}=\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1\end{pmatrix}$	$r_0+2r\cos(\frac{2m\pi}{3})$ $=r_0-r$	$\sqrt{k_0-2k\cos(\frac{2m\pi}{3})}$ $=\sqrt{k_0+k}$
$\left\|(-1)_3\right\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3}\end{pmatrix}$	$\left\|s_3\right\rangle=\frac{\left\|(+1)_3\right\rangle-\left\|(-1)_3\right\rangle}{i\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1\end{pmatrix}$	$r_0+2r\cos(\frac{-2m\pi}{3})$ $=r_0-r$	$\sqrt{k_0-2k\cos(\frac{2m\pi}{3})}$ $=\sqrt{k_0+k}$
	$\left\|(0)_3\right\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1\end{pmatrix}$	r_0+2r	$\sqrt{k_0-2k}$

C_3 standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
$m=0_3$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th}\ \textit{Eigenvalue of}\ \mathbf{r}^p \\ \langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\ 2\pi/3} \end{array}}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}) = r_0 + 2r\cos(\tfrac{2\pi m}{3}) = \begin{cases} r_0 + 2r\ \text{(for } m=0) \\ r_0 - r\ \ \text{(for } m=\pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

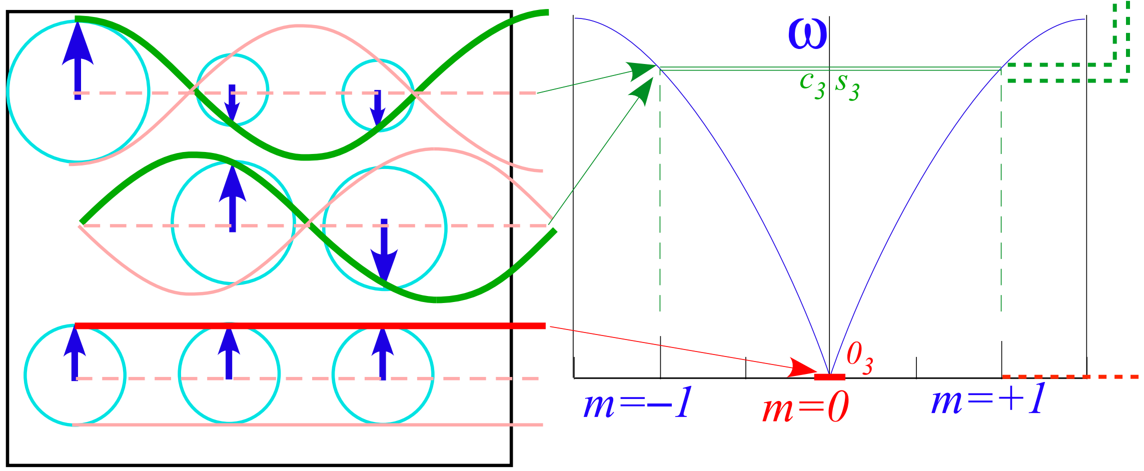
K-eigenvalues:

$$\begin{pmatrix} K & \text{-}k & \text{-}k \\ \text{-}k & K & \text{-}k \\ \text{-}k & \text{-}k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos(\tfrac{2m\pi}{3})\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

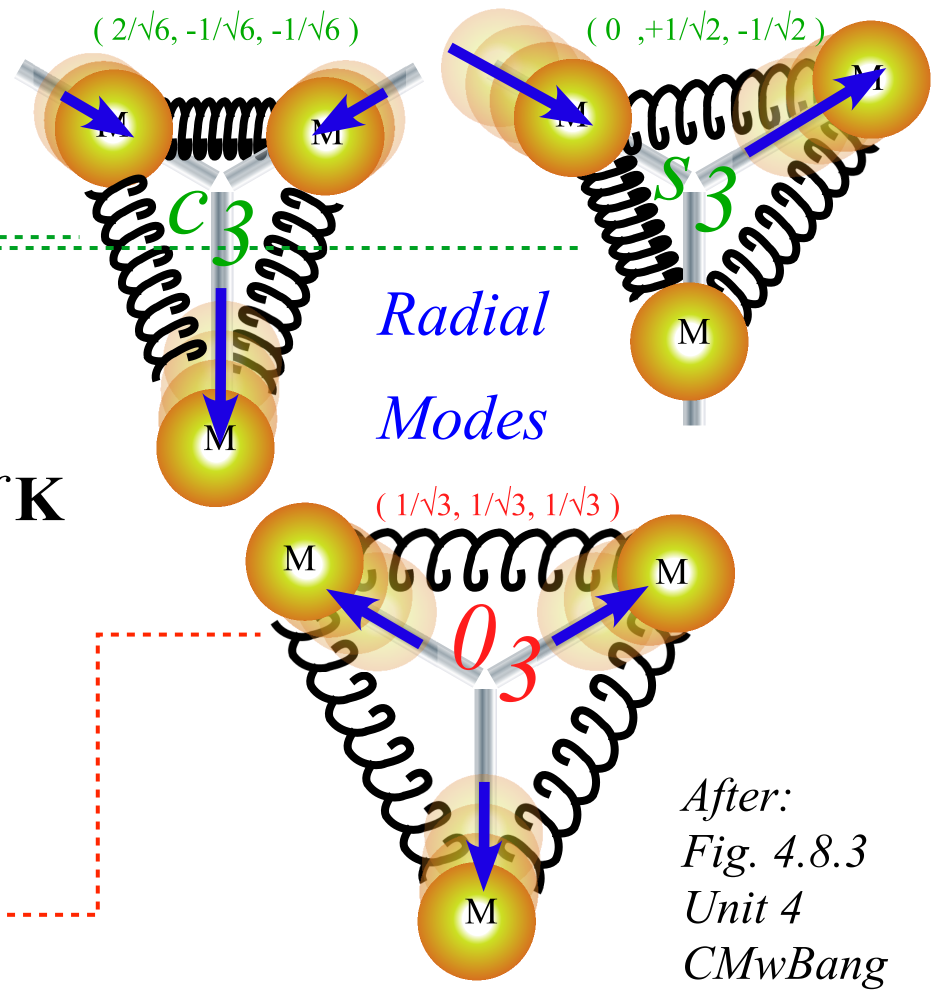
Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\|(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\|c_3\rangle = \frac{\|(+1)_3\rangle + \|(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\|(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\|s_3\rangle = \frac{\|(+1)_3\rangle - \|(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\|(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

	p=0	p=1	p=2
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3

C_3 standing wave modes and eigenfrequencies of **K**



Transverse (to k) Waves



After:
Fig. 4.8.3
Unit 4
CMwBang

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} Eigenvalue of $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

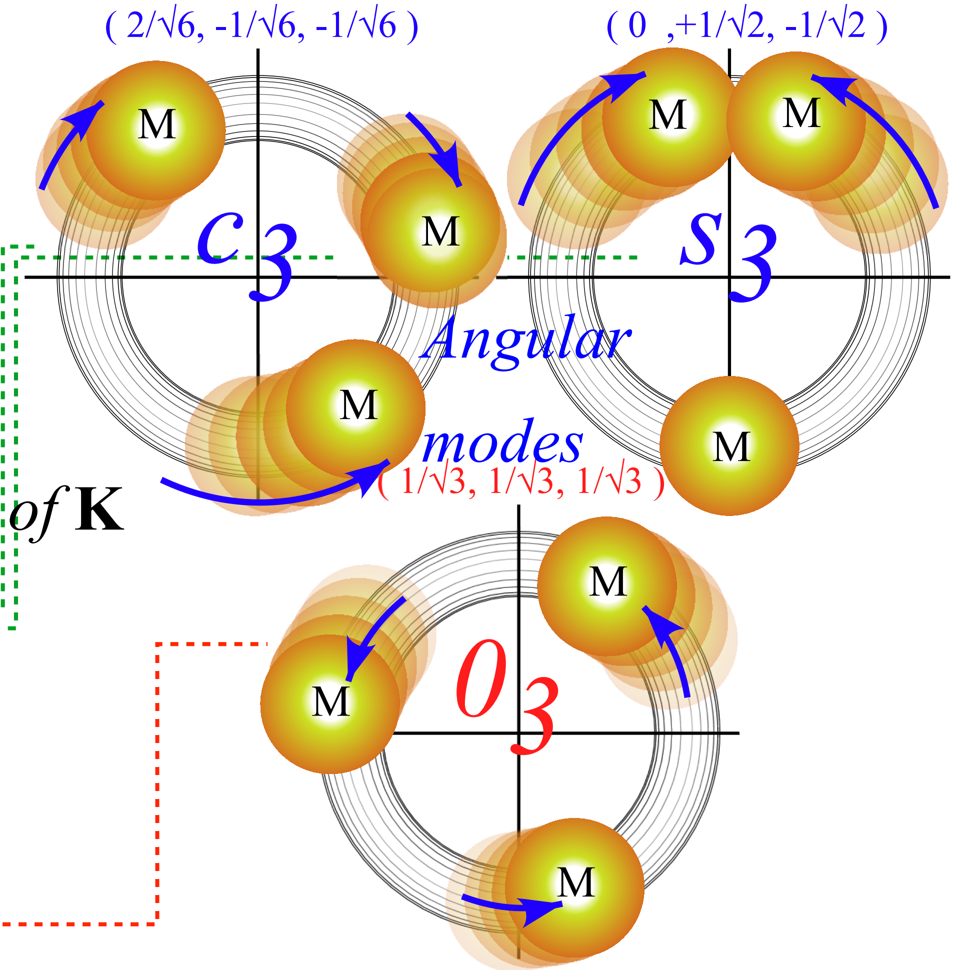
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

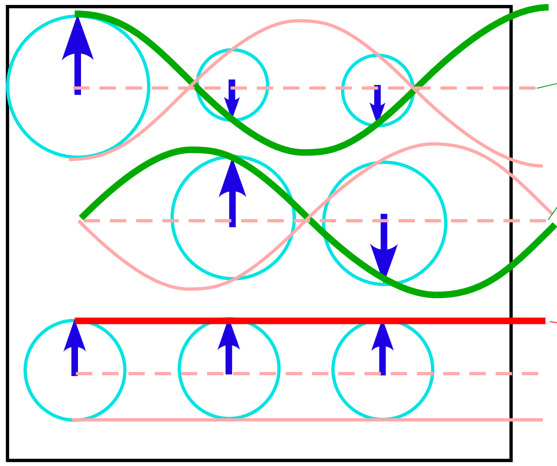
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

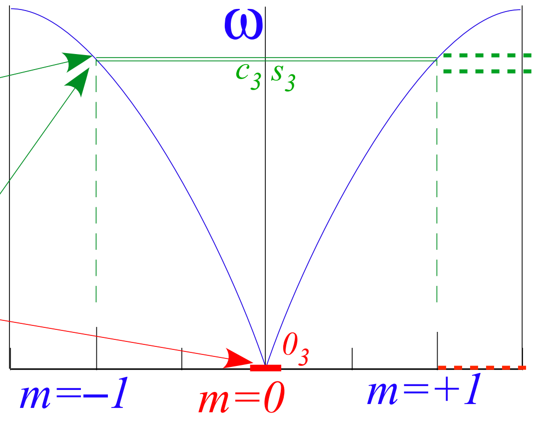
Longitudinal (to k) Waves



C_3 standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3

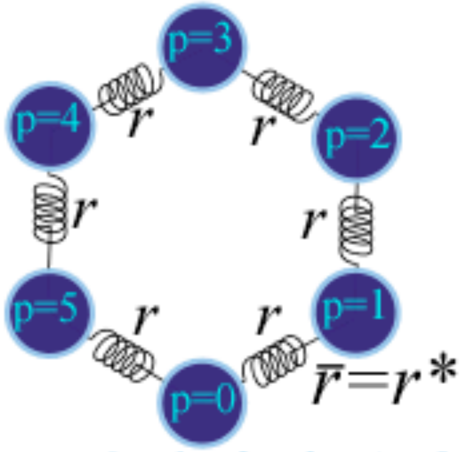




Wave resonance in cyclic symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- ➔ *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

C_6 Symmetric Mode Model: Distant neighbor coupling

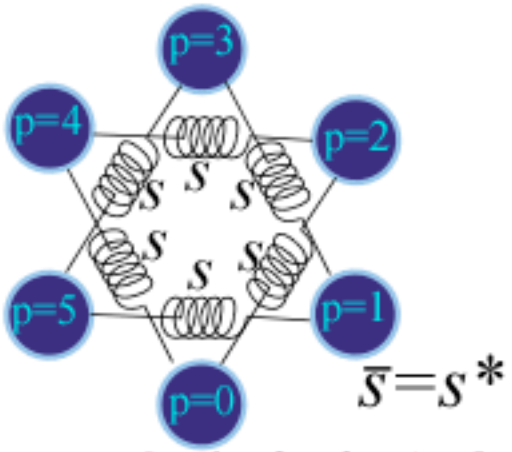
(a) 1st Neighbor C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} & 0 \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r & 4 \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 & 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

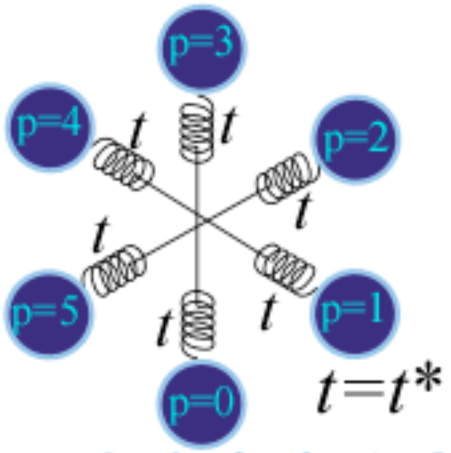
(b) 2nd Neighbor C_6



$$\mathbf{H}^{B2(6)}=\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot & 0 \\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s} & 1 \\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot & 2 \\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s & 3 \\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot & 4 \\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 & 5 \end{array}$$

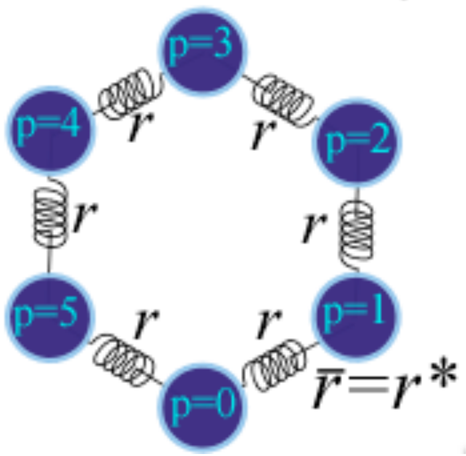
$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$

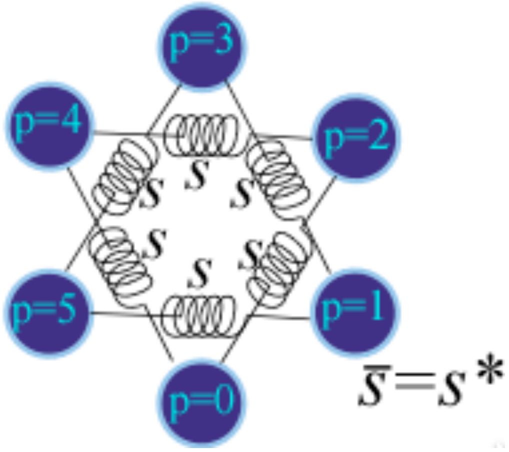
(c) 3rd Neighbor C_6



$$\mathbf{H}^{B3(6)}=\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_3 & \cdot & \cdot & -t & \cdot & \cdot & 0 \\ \cdot & H_3 & \cdot & \cdot & -t & \cdot & 1 \\ \cdot & \cdot & H_3 & \cdot & \cdot & -t & 2 \\ -t & \cdot & \cdot & H_3 & \cdot & \cdot & 3 \\ \cdot & -t & \cdot & \cdot & H_3 & \cdot & 4 \\ \cdot & \cdot & -t & \cdot & \cdot & H_3 & 5 \end{array}$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$





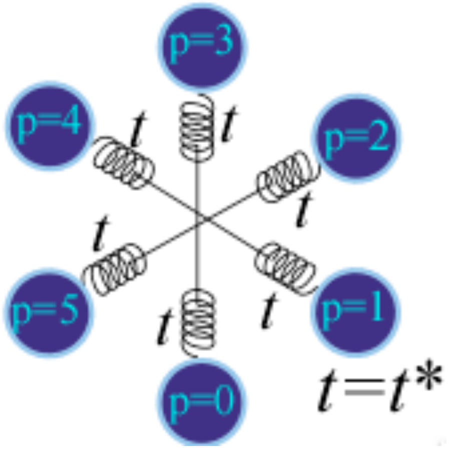
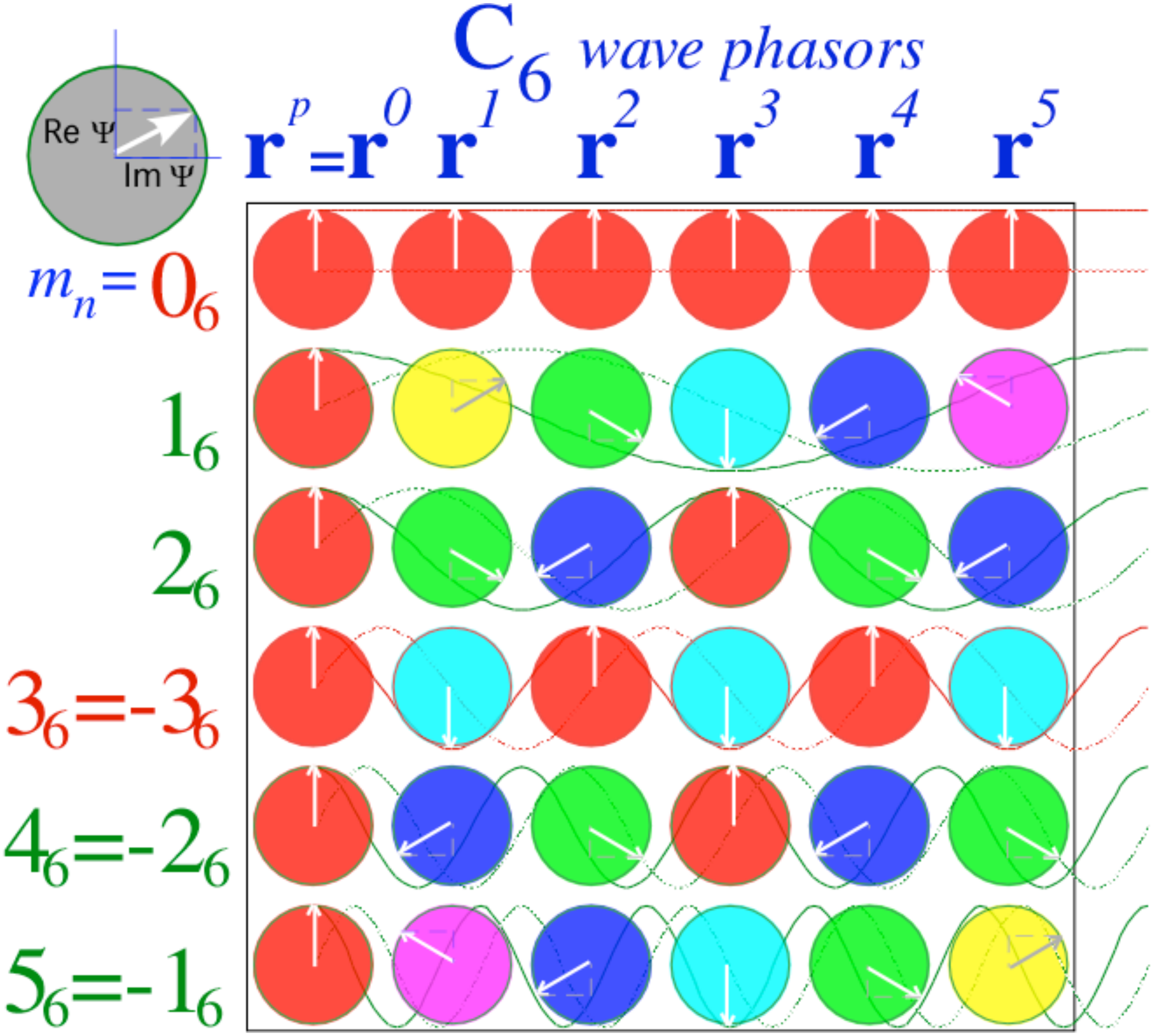


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Spectral resolution: 6th roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

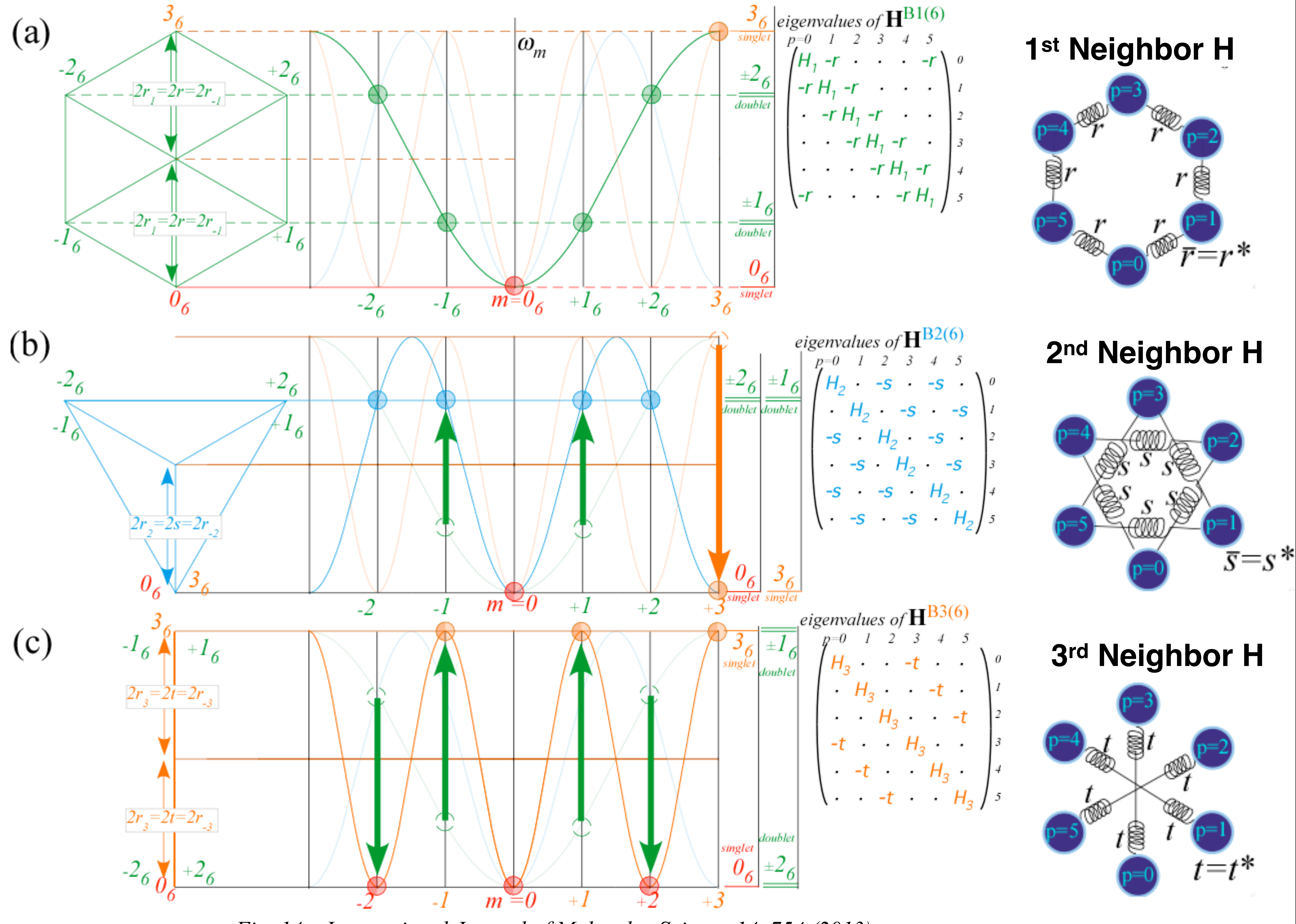
$$Wavefunction: \ \Psi^m(x_p)=\chi_p^{m*}=D^{m*}(\mathbf{r}^p)$$



$$\chi_p^m=e^{ik_m r^p}=e^{\frac{2\pi imp}{6}}$$

Fig. 13 International Journal of Molecular Science 14, 752 (2013)

C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion



$$\begin{array}{cccccc|c}
p=0 & 1 & 2 & 3 & 4 & 5 & \\
H_1 & -r & \cdot & \cdot & \cdot & -r & 0\\
-r & H_1 & -r & \cdot & \cdot & \cdot & 1\\
\cdot & -r & H_1 & -r & \cdot & \cdot & 2\\
\cdot & \cdot & -r & H_1 & -r & \cdot & 3\\
\cdot & \cdot & \cdot & -r & H_1 & -r & 4\\
-r & \cdot & \cdot & \cdot & -r & H_1 & 5
\end{array}$$

$$\begin{array}{cccccc|c}
p=0 & 1 & 2 & 3 & 4 & 5 & \\
H_2 & \cdot & -s & \cdot & -s & \cdot & 0\\
\cdot & H_2 & \cdot & -s & \cdot & -s & 1\\
-s & \cdot & H_2 & \cdot & -s & \cdot & 2\\
\cdot & -s & \cdot & H_2 & \cdot & -s & 3\\
-s & \cdot & -s & \cdot & H_2 & \cdot & 4\\
\cdot & -s & \cdot & -s & \cdot & H_2 & 5
\end{array}$$

$$\begin{array}{cccccc|c}
p=0 & 1 & 2 & 3 & 4 & 5 & \\
H_3 & \cdot & \cdot & -t & \cdot & \cdot & 0\\
\cdot & H_3 & \cdot & \cdot & -t & \cdot & 1\\
\cdot & \cdot & H_3 & \cdot & \cdot & -t & 2\\
-t & \cdot & \cdot & H_3 & \cdot & \cdot & 3\\
\cdot & -t & \cdot & \cdot & H_3 & \cdot & 4\\
\cdot & \cdot & -t & \cdot & \cdot & H_3 & 5
\end{array}$$

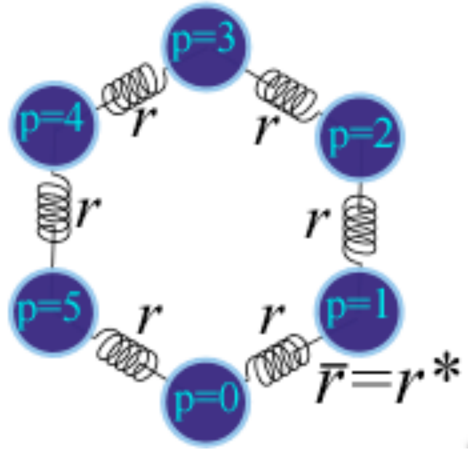
Fig. 14 International Journal of Molecular Science 14, 754 (2013)

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - ➔ *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current,*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,

1st Neighbor H



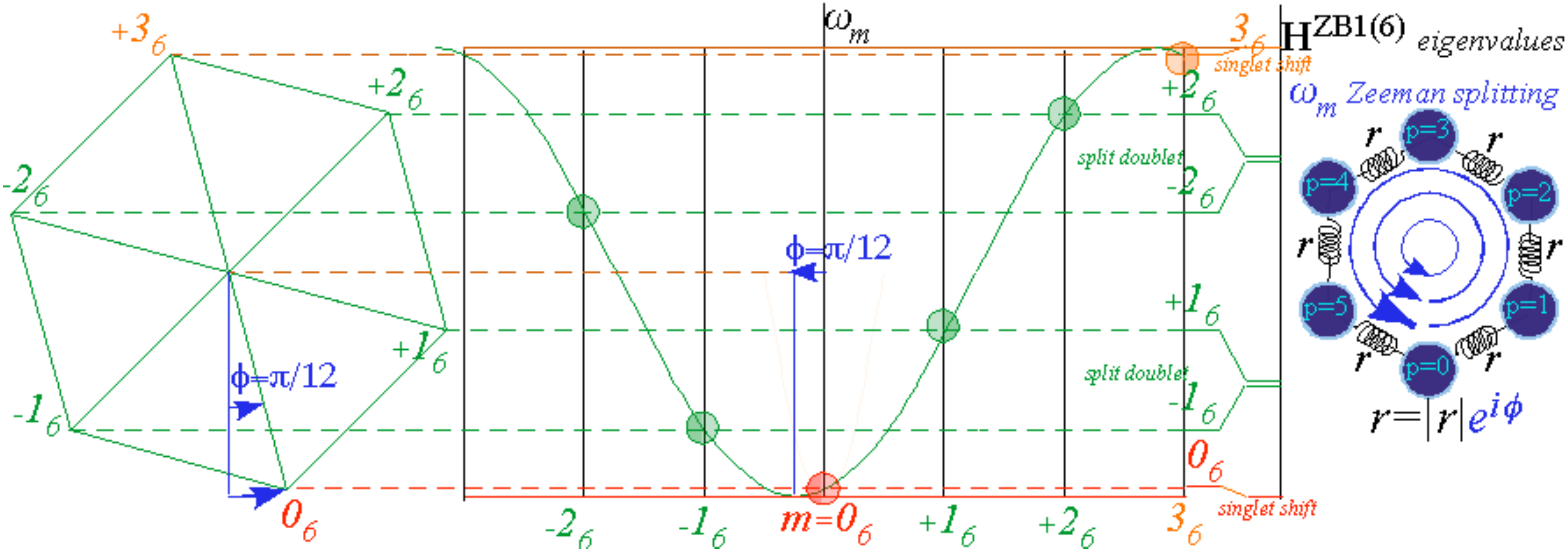


Fig. 15 International Journal of Molecular Science 14, 755 (2013)

Wave resonance in cyclic symmetry
Harmonic oscillator with cyclic C_2 *symmetry*
C_2 *symmetric (B-type) modes*
Harmonic oscillator with cyclic C_3 *symmetry*
C_3 *symmetric spectral decomposition by 3rd roots of unity*
Resolving C_3 *projectors and moving wave modes*
Dispersion functions and standing waves
C_6 *symmetric mode model:Distant neighbor coupling*
C_6 *spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
➔ C_N *symmetric mode models: Made-to order dispersion functions*
Quadratic dispersion models: Super-beats and fractional revivals
Phase arithmetic

C_N Symmetric Mode Models:

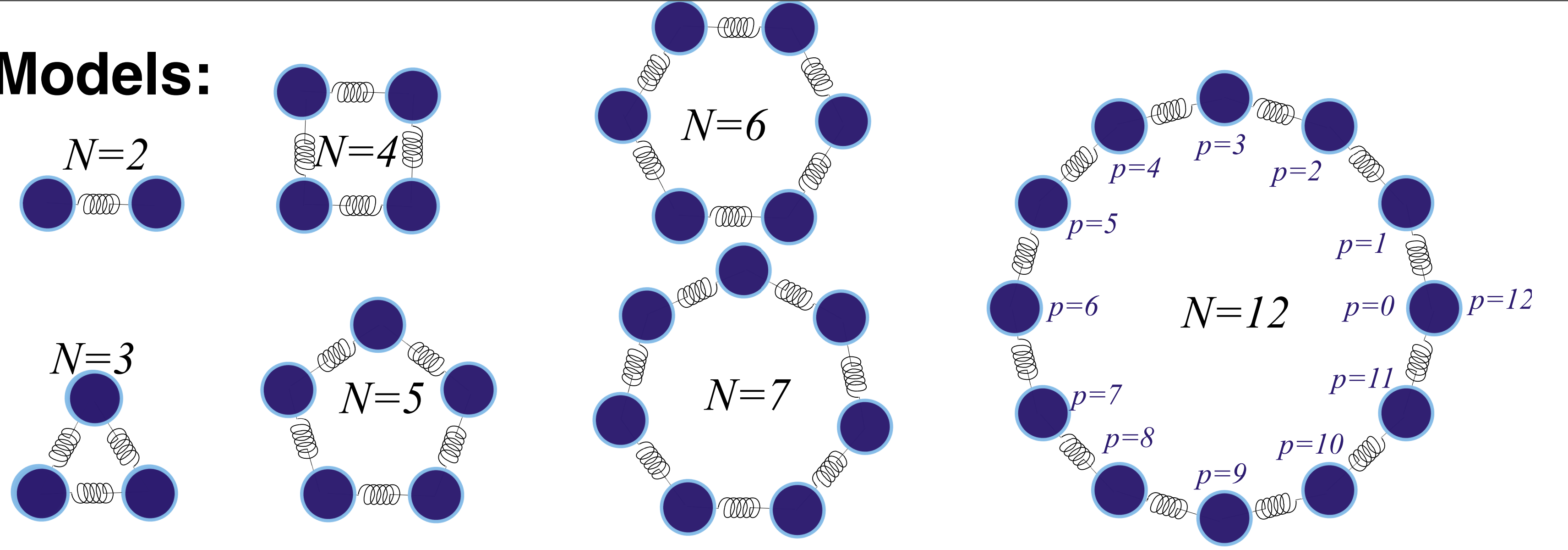


Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

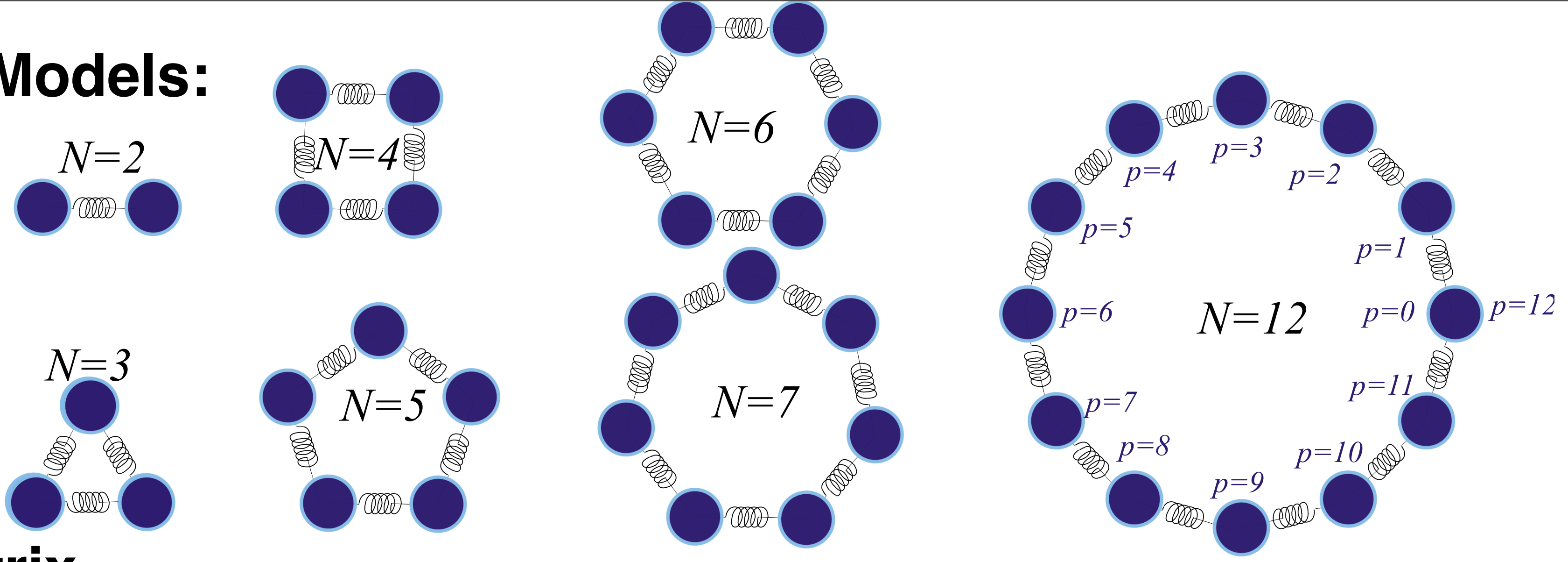


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix} \quad \text{where:} \quad \begin{array}{c} K = k + 2k_{12} \\ k = \dfrac{Mg}{\ell} \\ (\cdot) = 0 \end{array}$$

C_N Symmetric Mode Models:

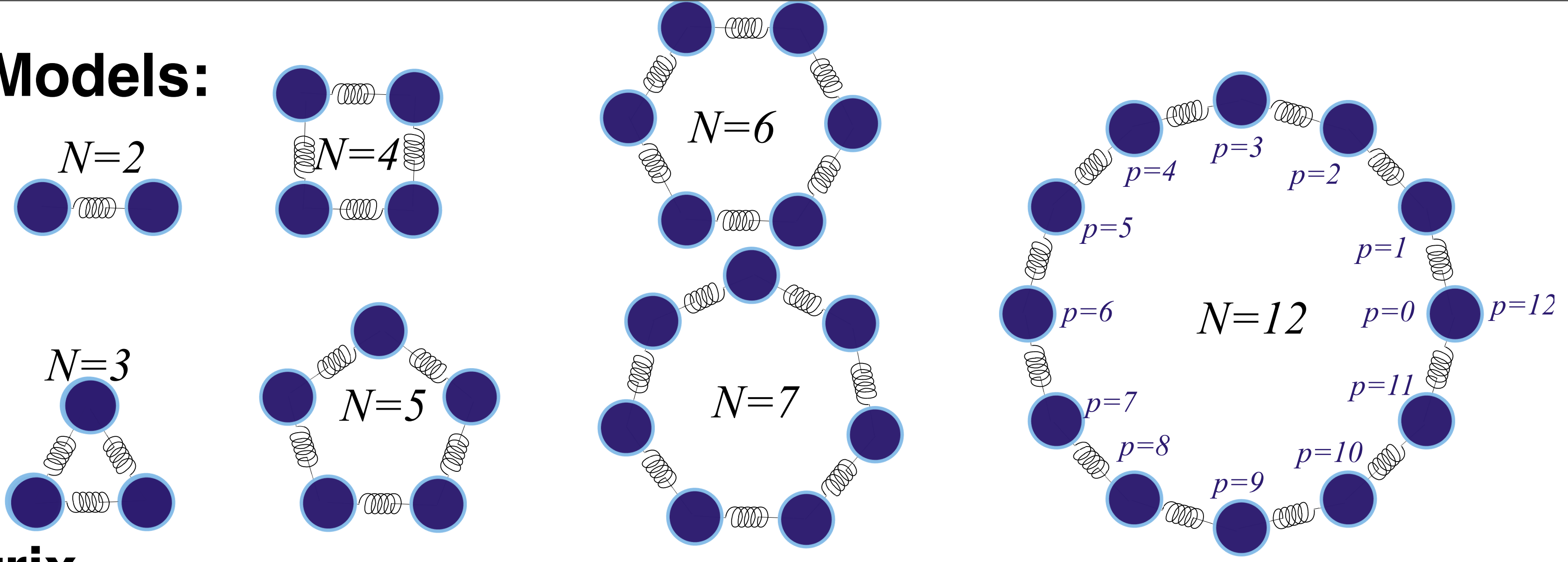


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix}F_0\\F_1\\F_2\\F_3\\F_4\\\vdots\\F_{N-1}\end{pmatrix}=\begin{pmatrix}K & -k_{12} & . & . & . & \cdots & -k_{12}\\-k_{12} & K & -k_{12} & . & . & \cdots & .\\. & -k_{12} & K & -k_{12} & . & \cdots & .\\. & . & -k_{12} & K & -k_{12} & \cdots & .\\. & . & . & -k_{12} & K & \cdots & .\\\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12}\\-k_{12} & . & . & . & . & -k_{12} & K\end{pmatrix}\bullet\begin{pmatrix}x_0\\x_1\\x_2\\x_3\\x_4\\\vdots\\x_{N-1}\end{pmatrix}$$

where: $K = k + 2k_{12}$

$k = \dfrac{Mg}{\ell}$

$(\cdot) = 0$

Nth roots of 1 $e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

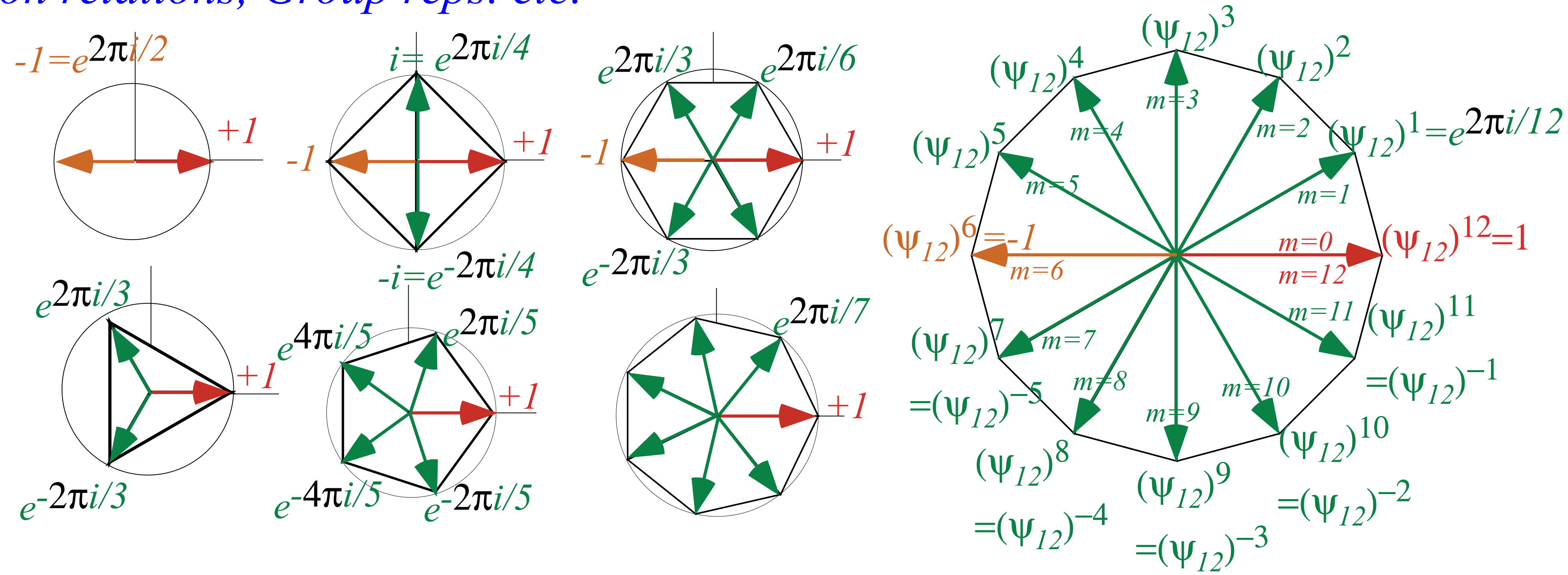
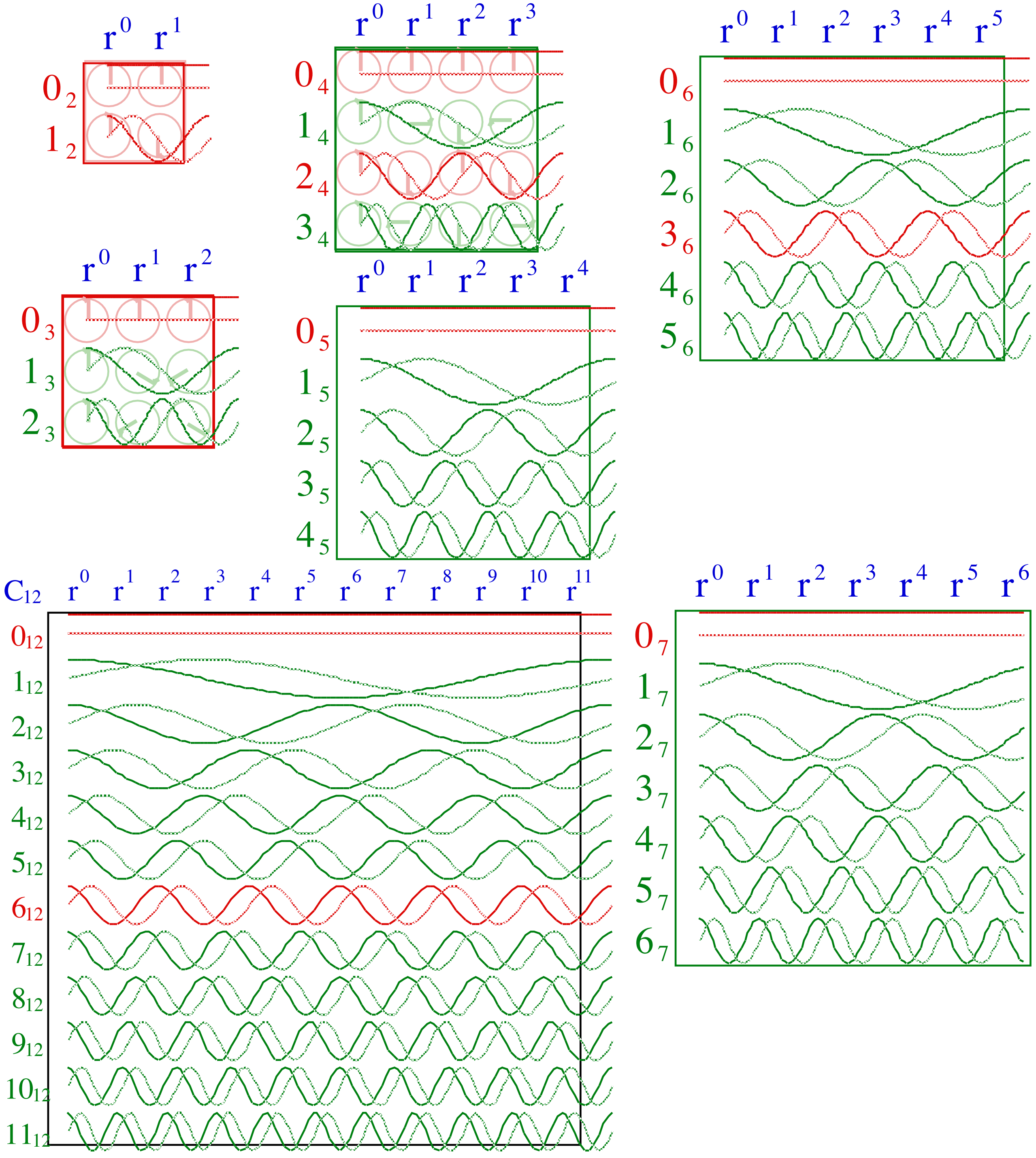
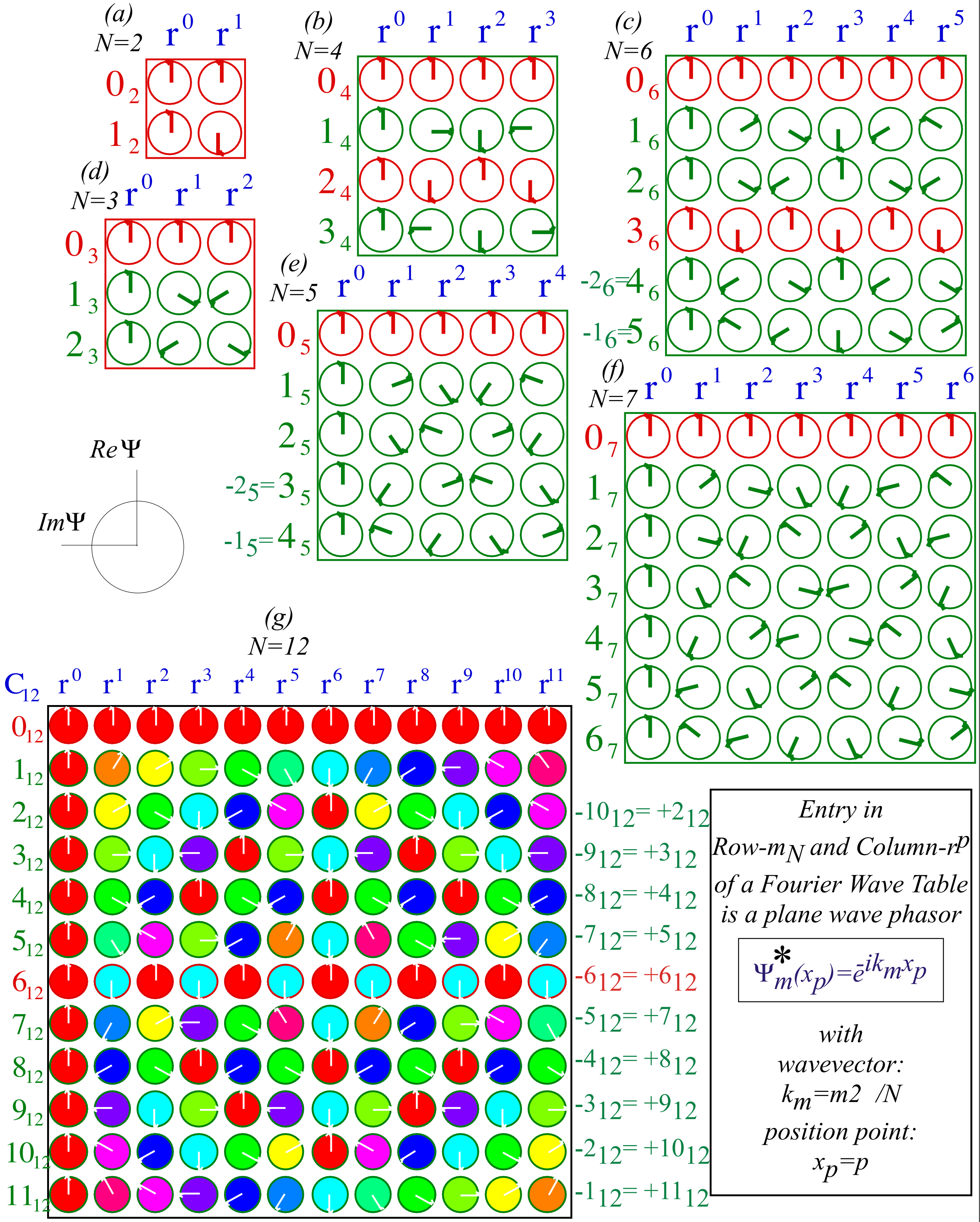


Fig. 4.8.5
Unit 4
CMwBang

C_N Symmetric Mode Models:

Nth roots of 1 *$e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*





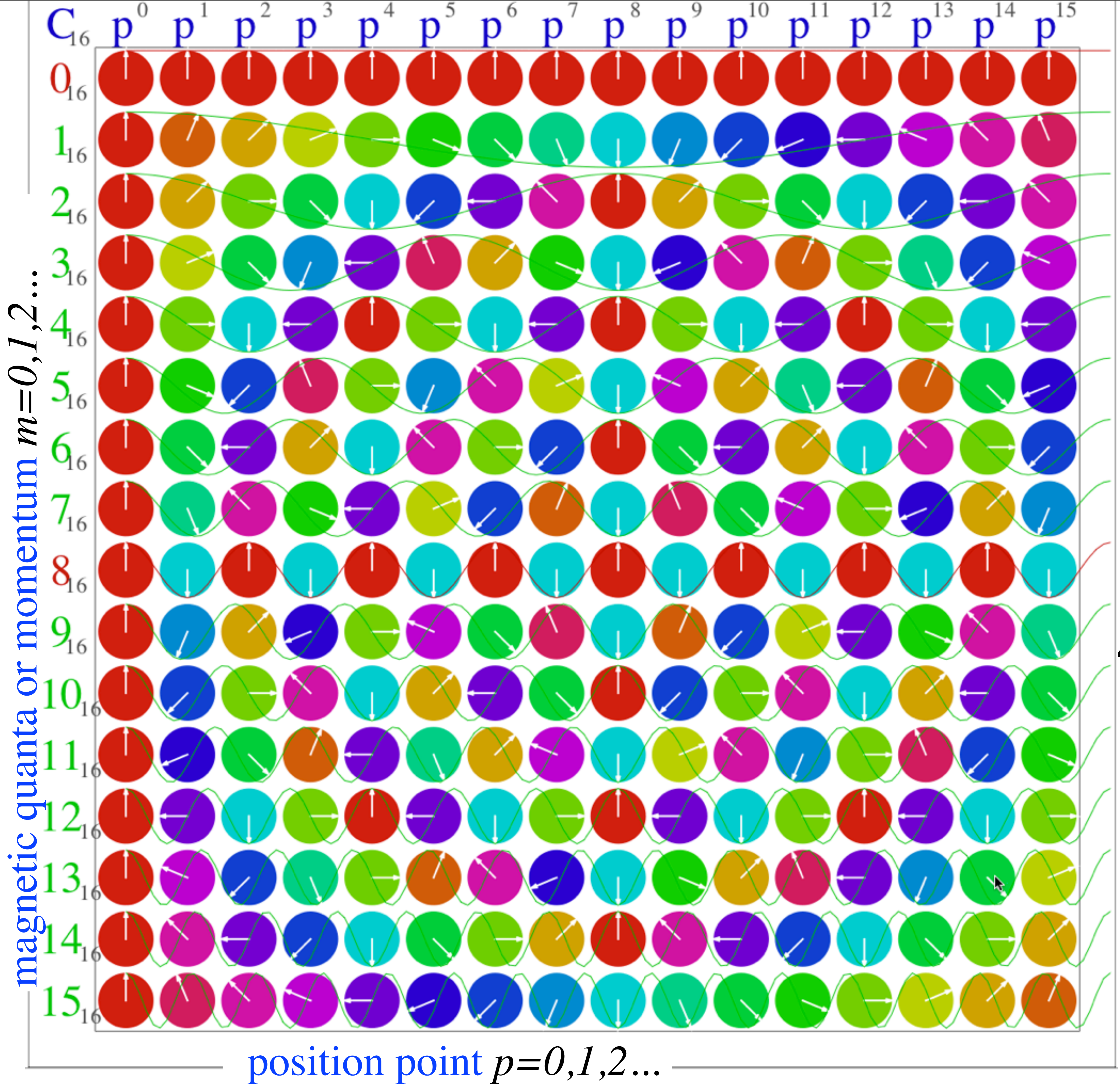
Entry in Row-m_N and Column-r^p of a Fourier Wave Table is a plane wave phasor

$$\Psi^*_m(x_p)=e^{-ik_m x_p}$$

with wavevector: $k_m=m2\ /N$ *position point:* $x_p=p$

Fig. 4.8.6-7
Unit 4
CMwBang

Fourier transformation matrices

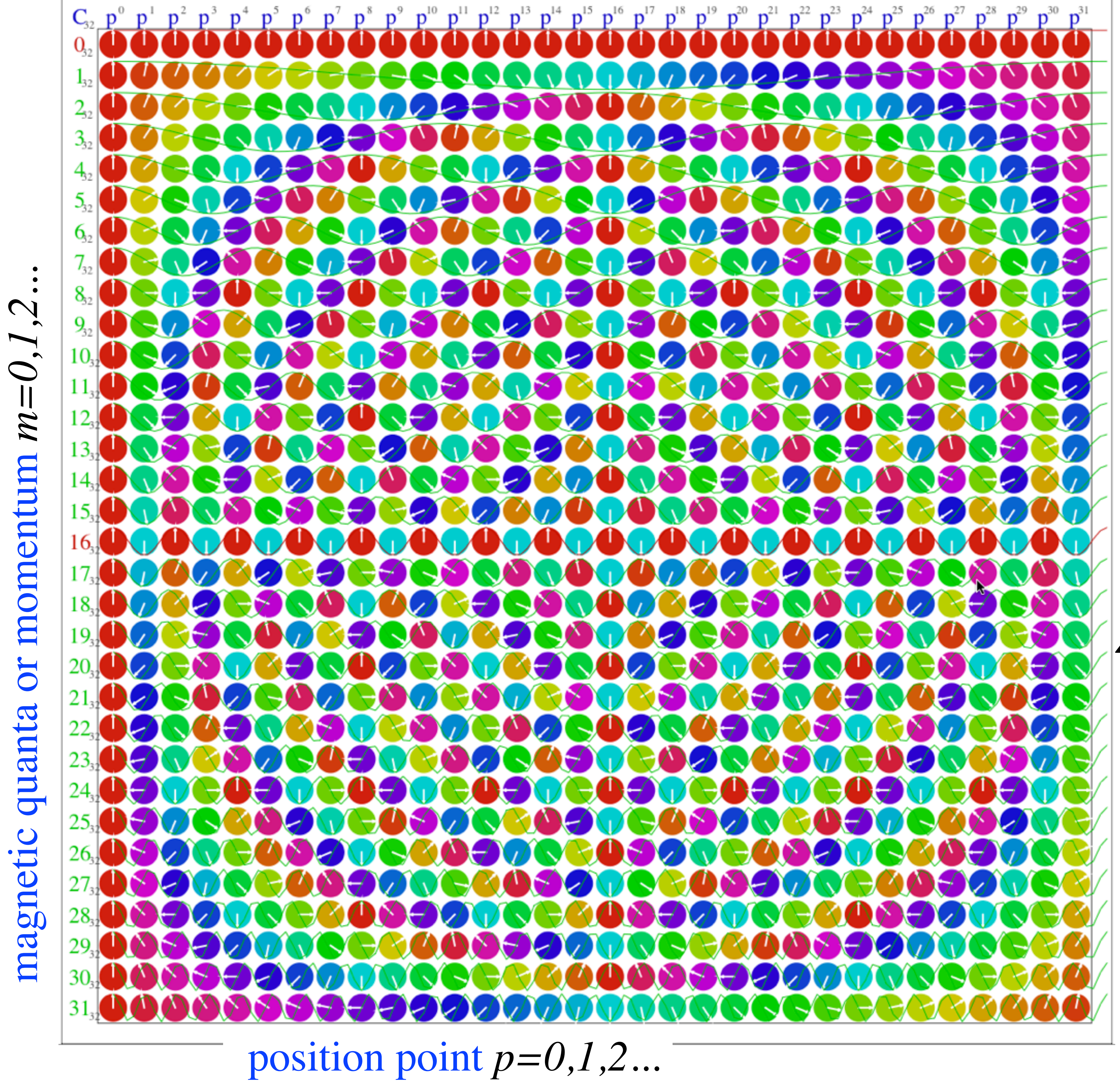


C_{16}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{16}}$$

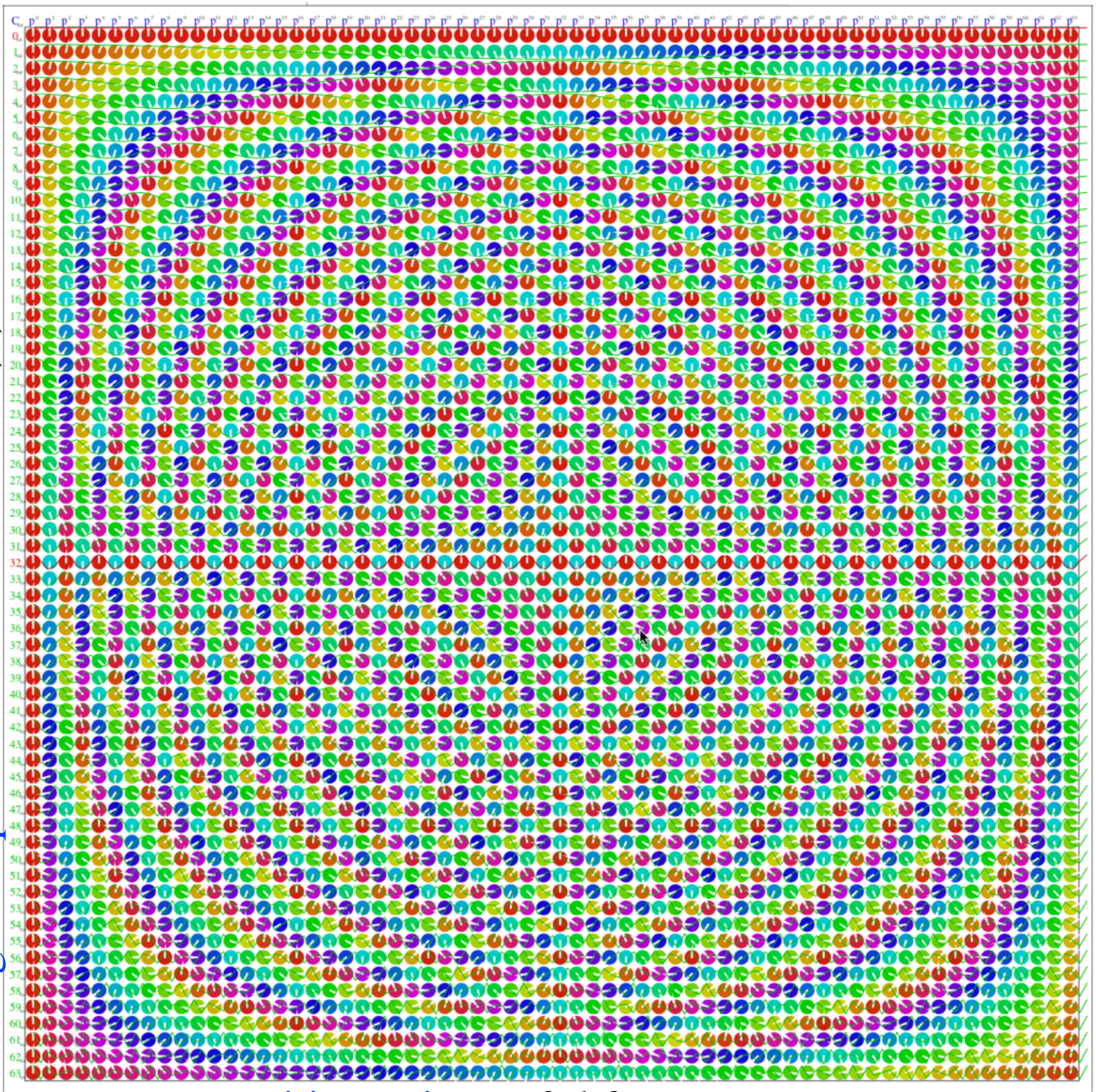


C_{32}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{32}}$$



magnetic quanta or momentum $m=0,1,2...$

position point $p=0,1,2...$

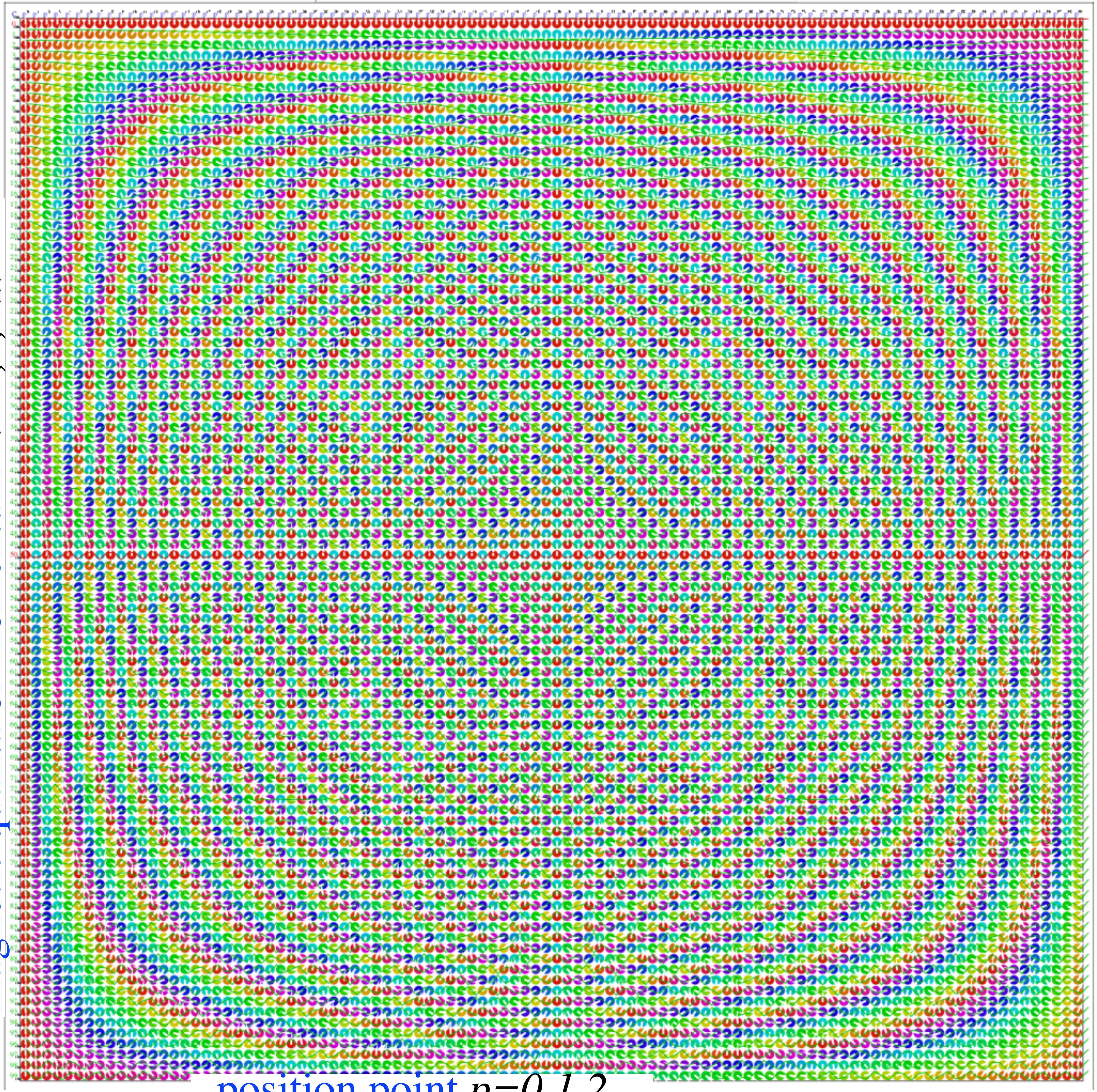
C_{64}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{64}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m\cdot p=const.$



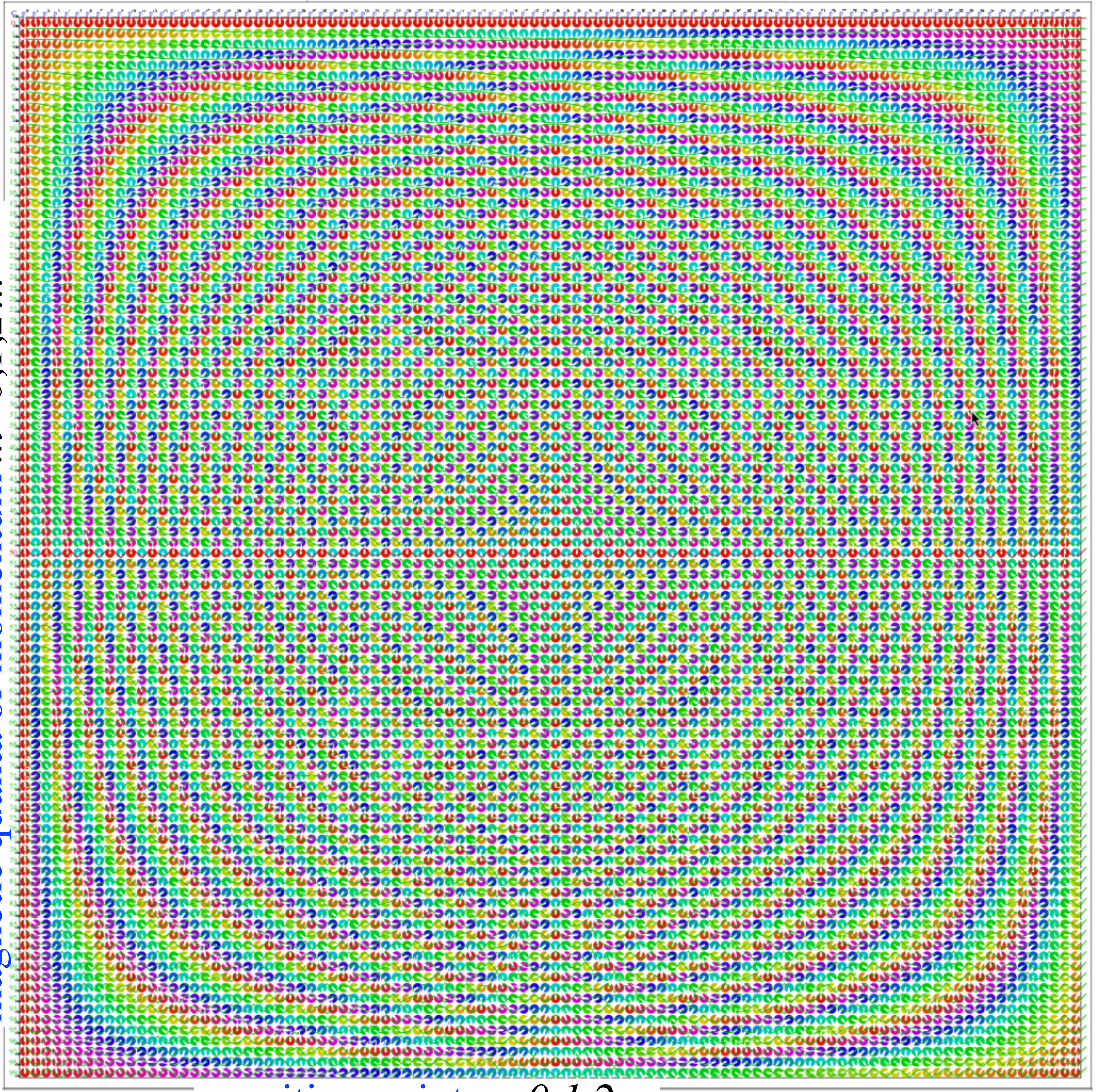
C_{100}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m\cdot p = const.$



C_{256}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{256}}$$

Invariant phase
“Uncertainty”
hyperbolas:
$m \cdot p = const.$

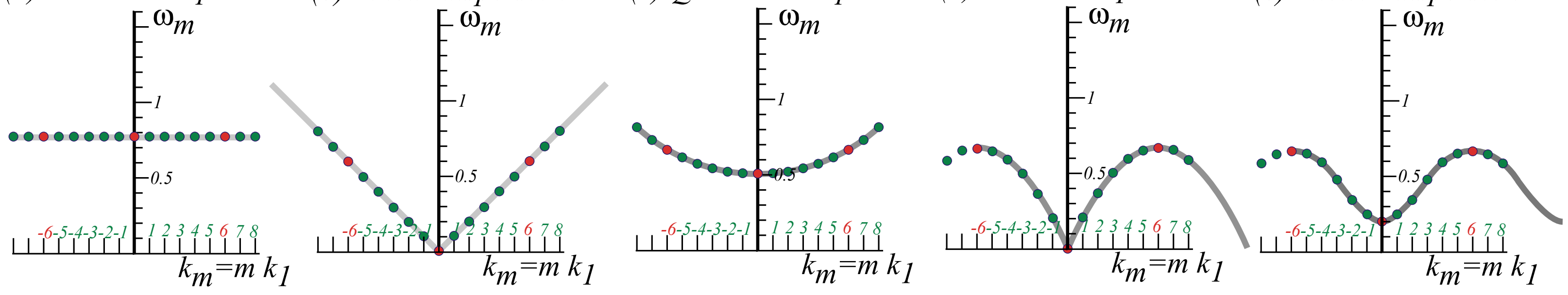
Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
    - ➔ *Quadratic dispersion models: Super-beats and fractional revivals*
    - *Phase arithmetic*

CN Symmetric Mode Models: Made-to-Order Dispersion *(and wave dynamics)*

(Making pure linear $\omega=ck$, quadratic $\omega=ck^2$, etc. ?)

Archetypical Examples of Dispersion Functions



(a) Constant dispersion *(b) Linear dispersion* *(c) Quadratic dispersion* *(d) Phonon dispersion* *(e) Exciton dispersion*

Applications:

Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis *Xmas lights*	*Light in vacuum (Exactly)* *Sound (Approximately)*	*Light in fiber (Approx)* *Non-relativistic Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids* *Relativistic matter* *(If exact hyperbola)*

Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

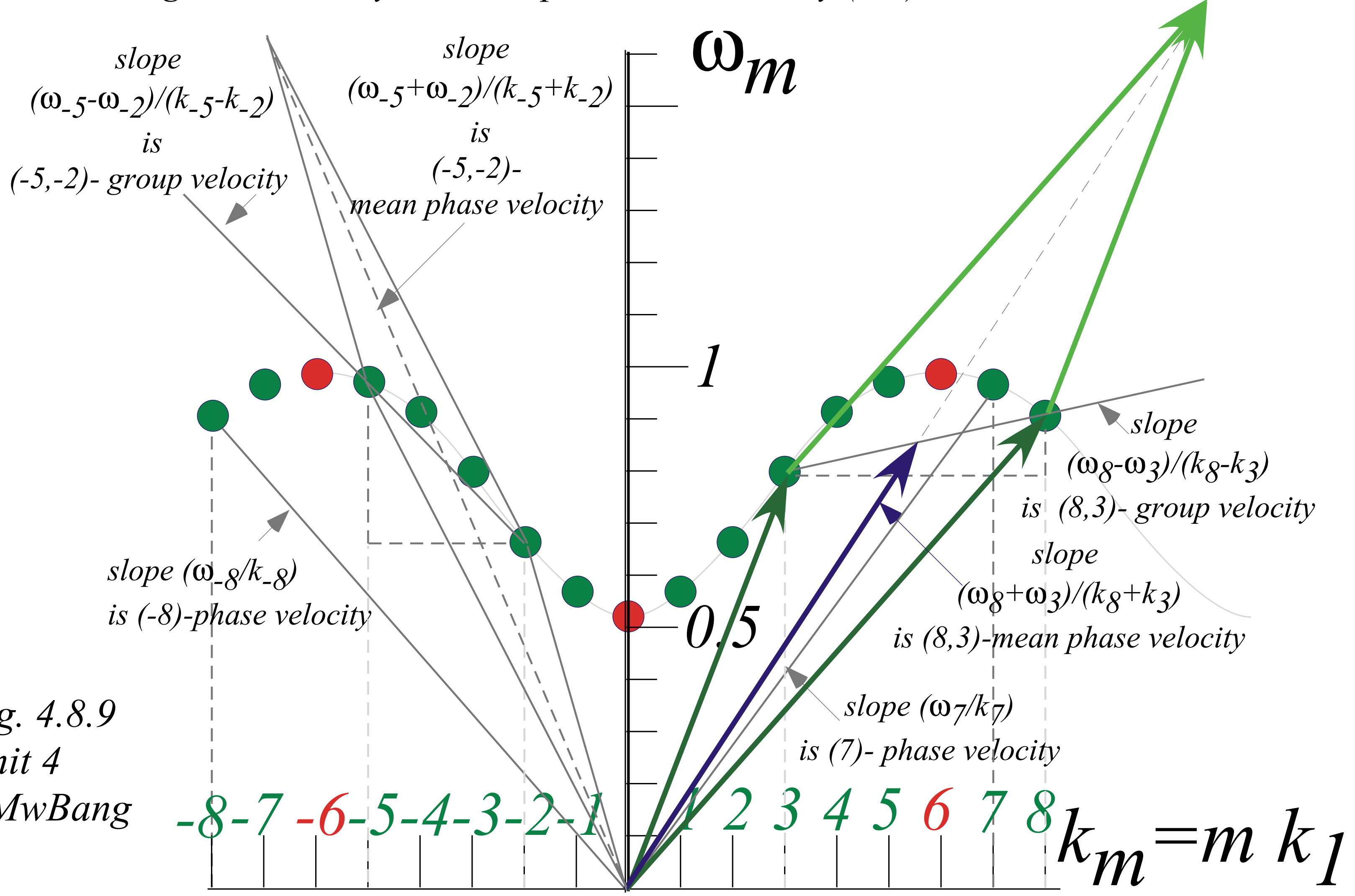


Fig. 4.8.9
Unit 4
CMwBang

$$a = k_a \cdot x - \omega_a \cdot t$$

$$b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right)$$

$$= e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

Things determined by

Dispersion $\omega = \omega(k)$

Individual phase velocity:

$$V_{phase-1} = \frac{\omega(k)}{k}$$

Pairwise phase velocity:

$$V_{phase-2} = \frac{\omega(k_a)+\omega(k_b)}{k_a+k_b}$$

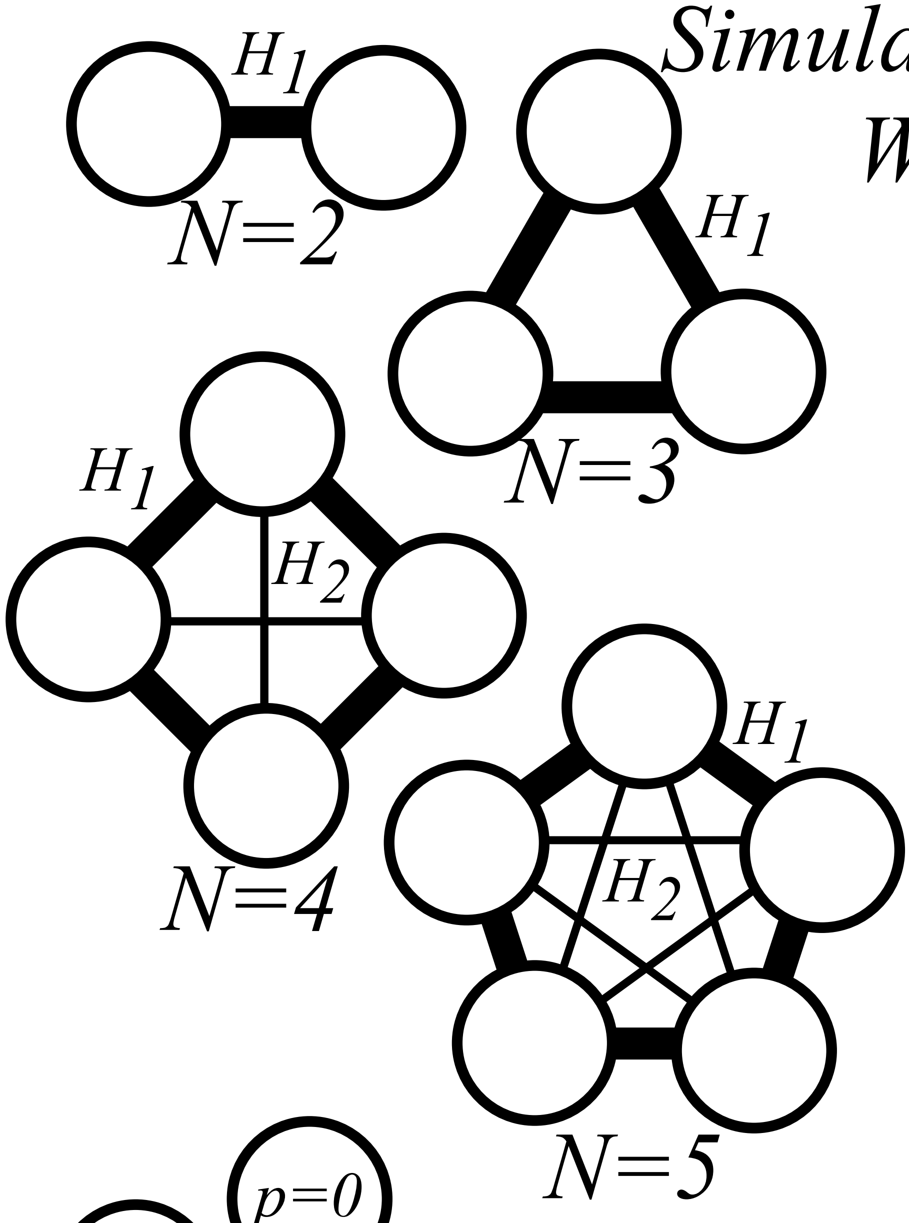
Pairwise group velocity:

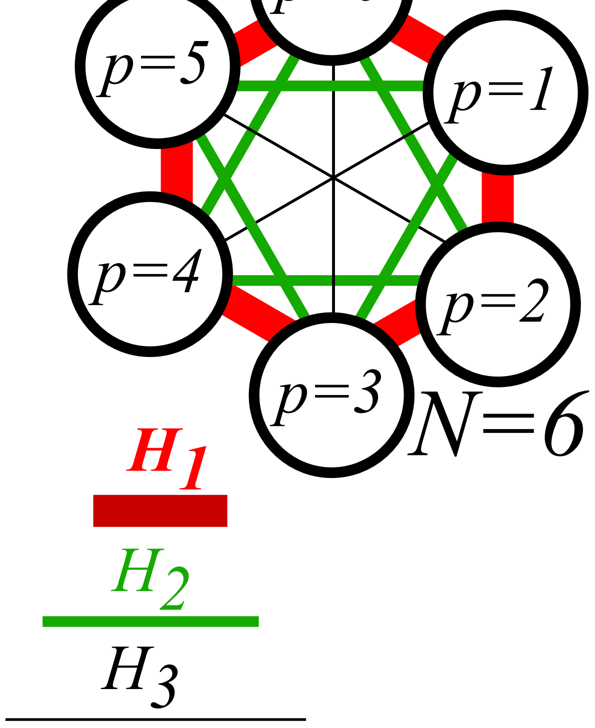
$$V_{group-2} = \frac{\omega(k_a)-\omega(k_b)}{k_a-k_b}$$

Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]





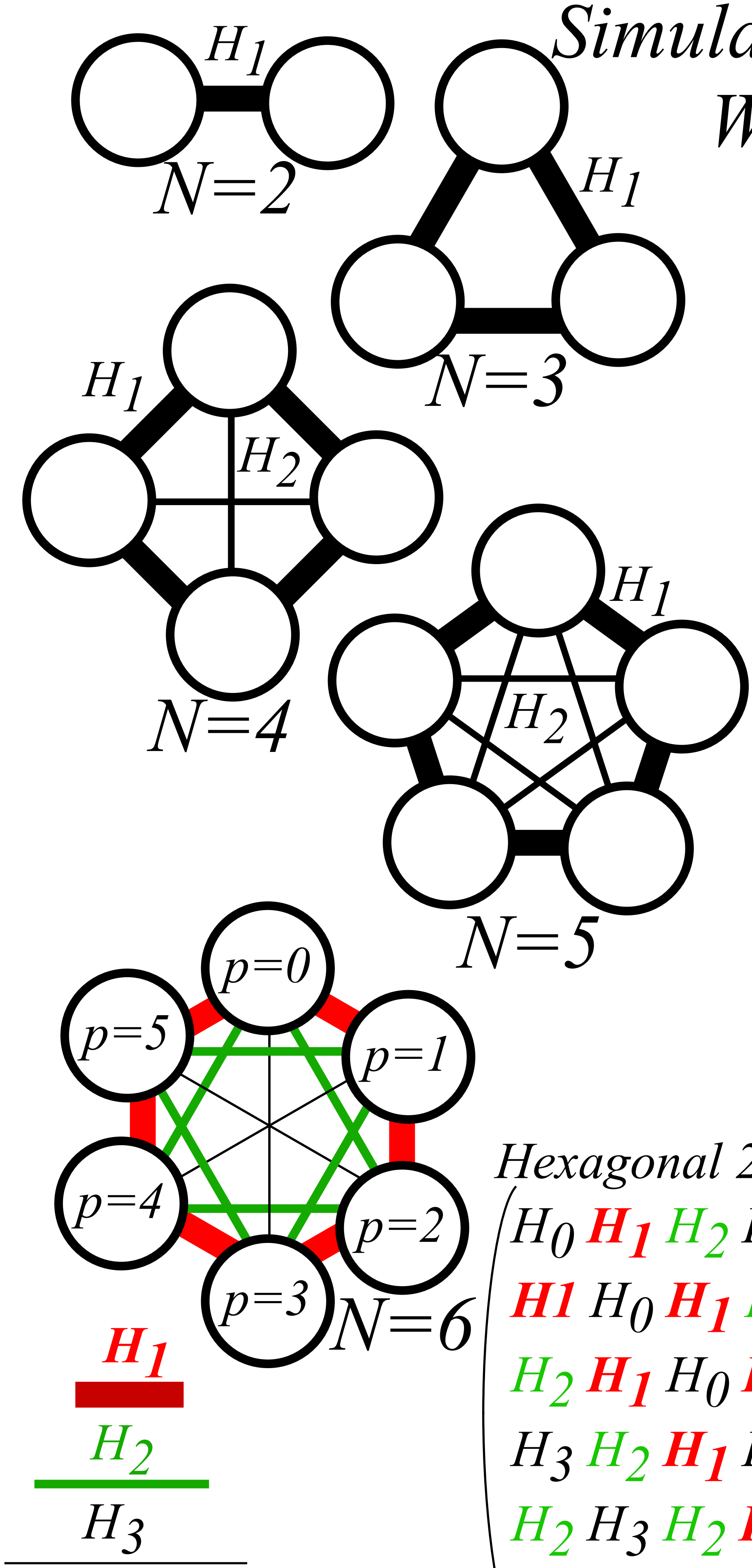
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

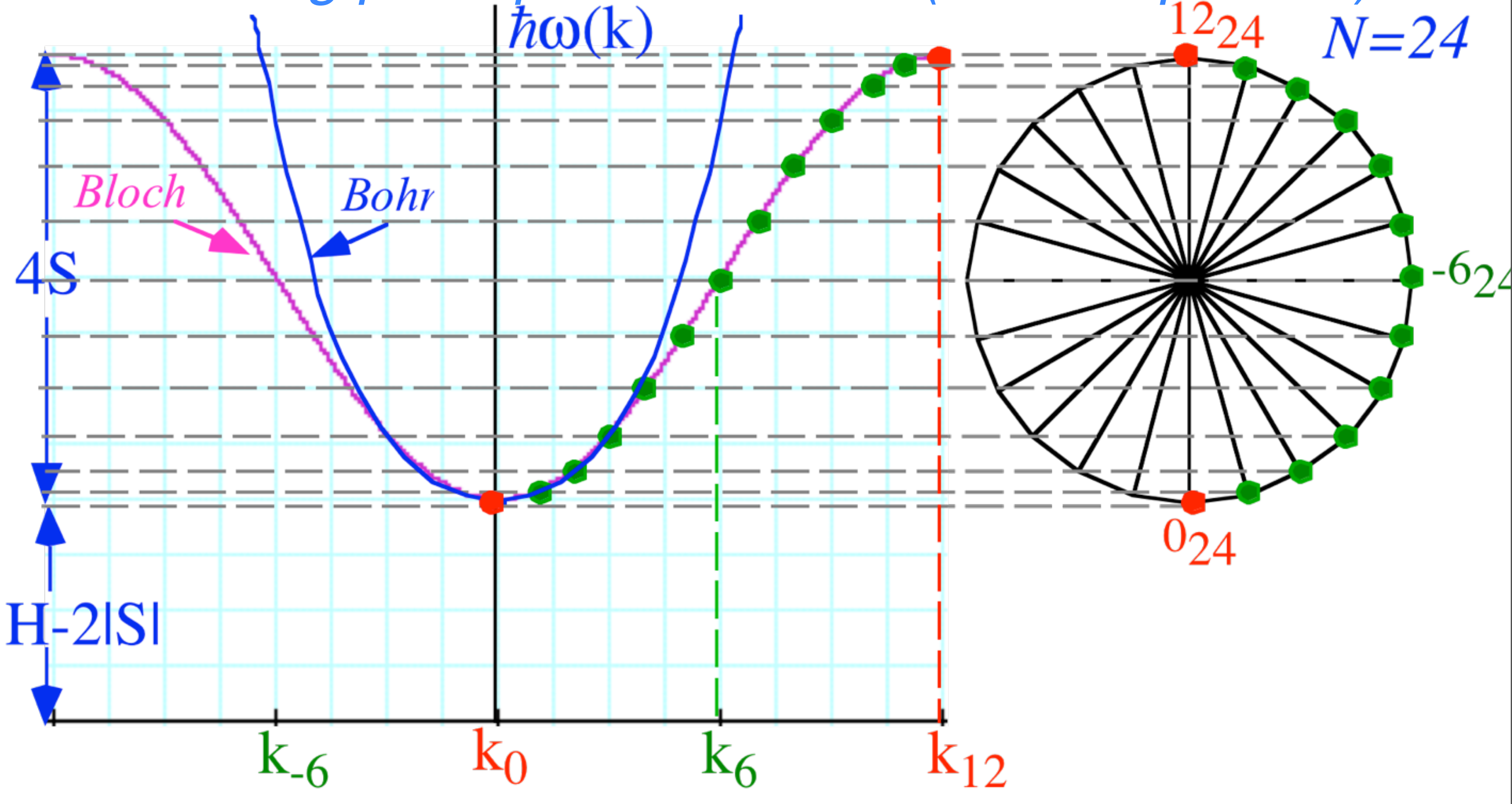
Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega=ck^2$ (Bohr dispersion)



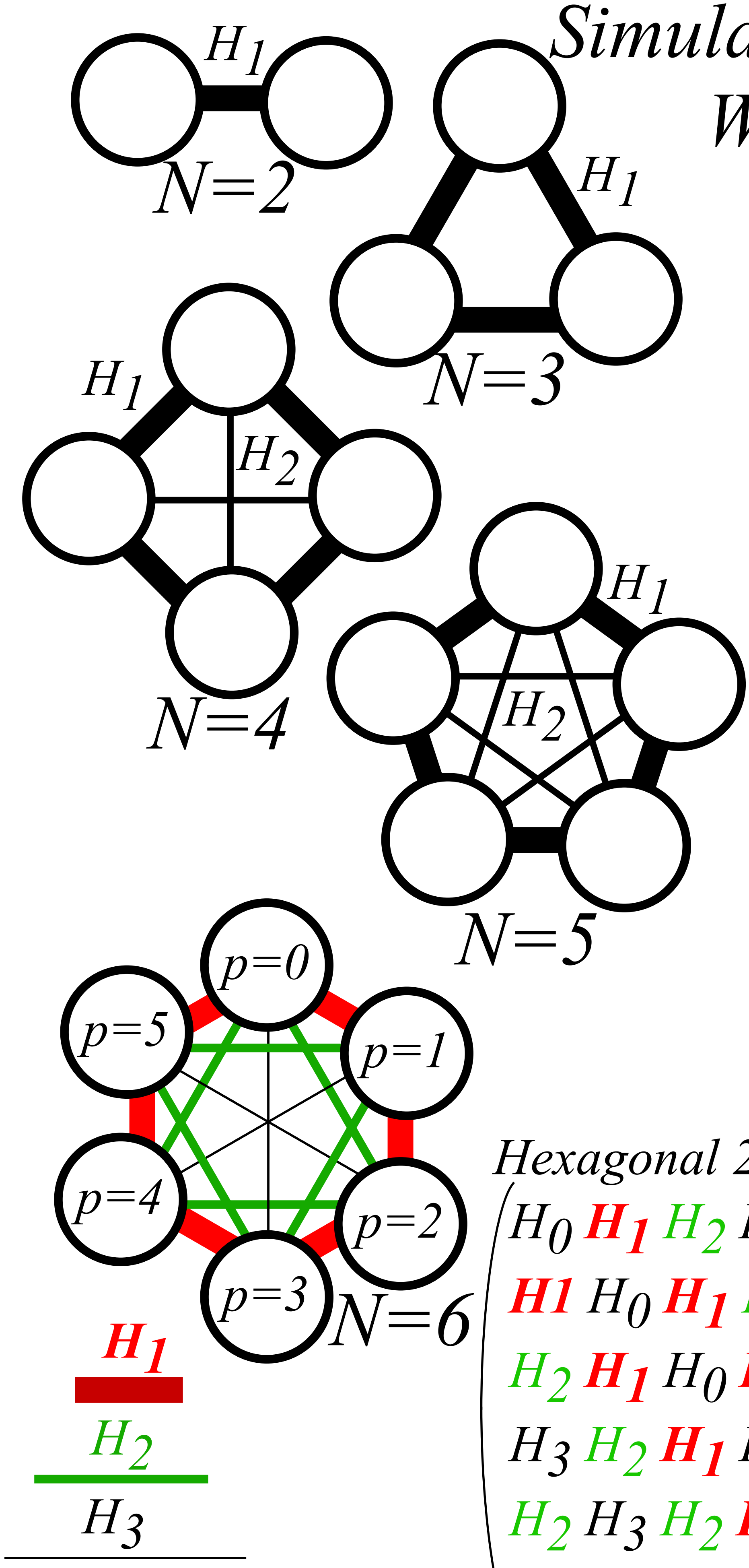
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

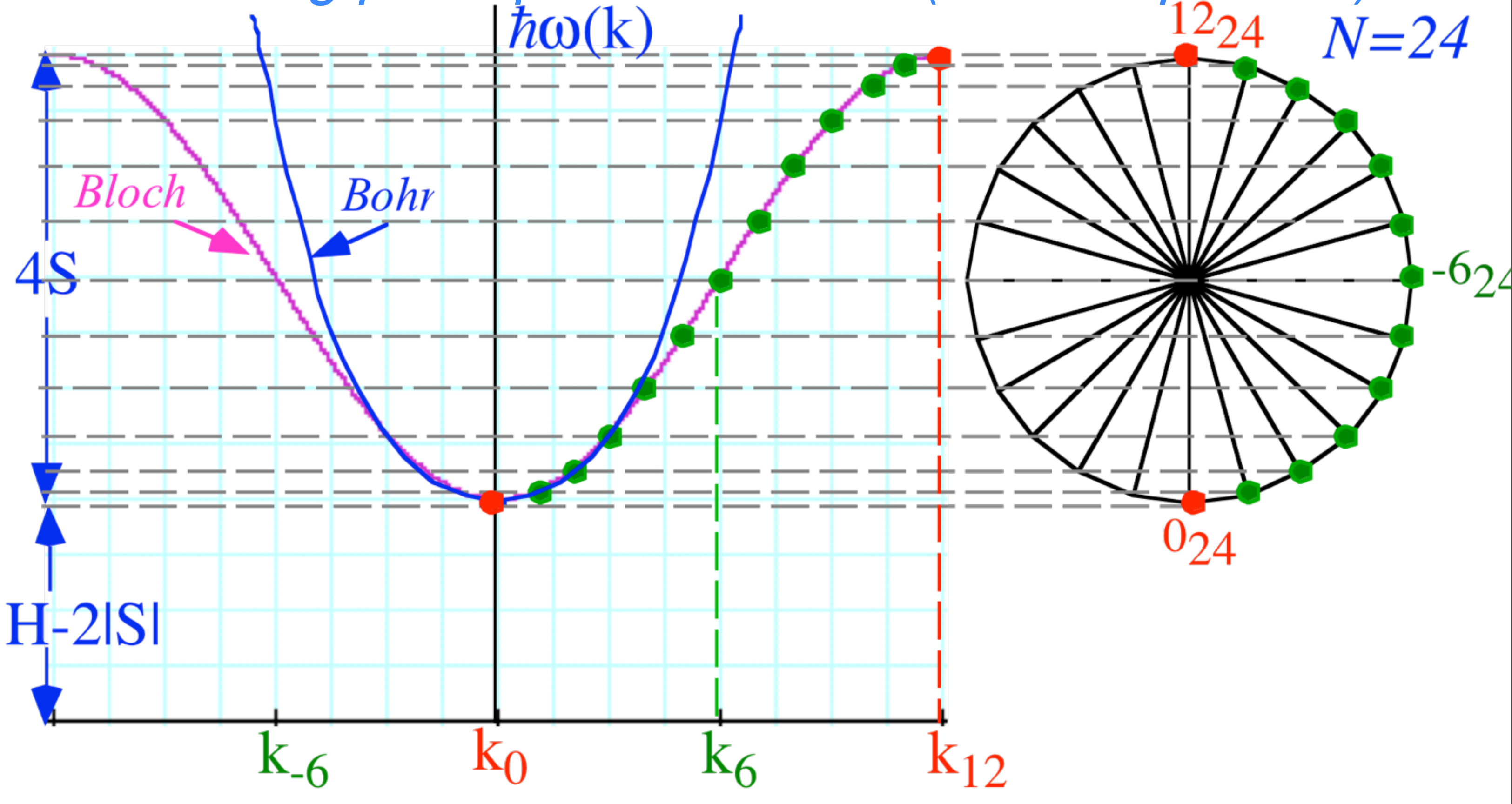
Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega=ck^2$ (Bohr dispersion)



Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
N=2	1/2	-1/2							
N=3	2/3	-1/3							
N=4	3/2	-1	1/2						
N=5	2	-1.1708	0.1708						
N=6	19/6	-2	2/3	-1/2					
N=7	4	-2.393	0.51	-0.1171					
N=8	11/2	-3.4142	1	-0.5858	1/2				
N=9	20/3	-4.0165	0.9270	-1/3	0.0895				
N=10	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
N=11	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
N=12	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
N=13	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
N=14	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
N=15	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
N=16	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
N=17	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

Wave resonance in cyclic symmetry

- Harmonic oscillator with cyclic C_2 symmetry
  - C_2 symmetric (B-type) modes
- Harmonic oscillator with cyclic C_3 symmetry
  - C_3 symmetric spectral decomposition by 3rd roots of unity
  - Resolving C_3 projectors and moving wave modes
  - Dispersion functions and standing waves
- C_6 symmetric mode model:Distant neighbor coupling
  - C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..
- C_N symmetric mode models: Made-to order dispersion functions
  - Quadratic dispersion models: Super-beats and fractional revivals
  - Phase arithmetic

2-level-system and C_2 symmetry phase dynamics

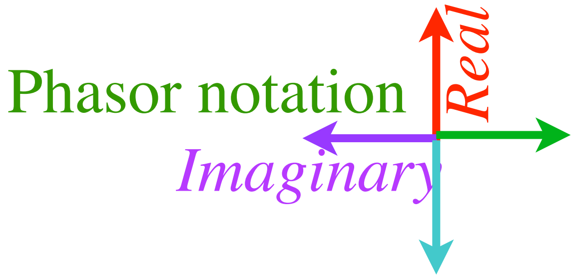
C_2 Character Table describes eigenstates

symmetric A_1

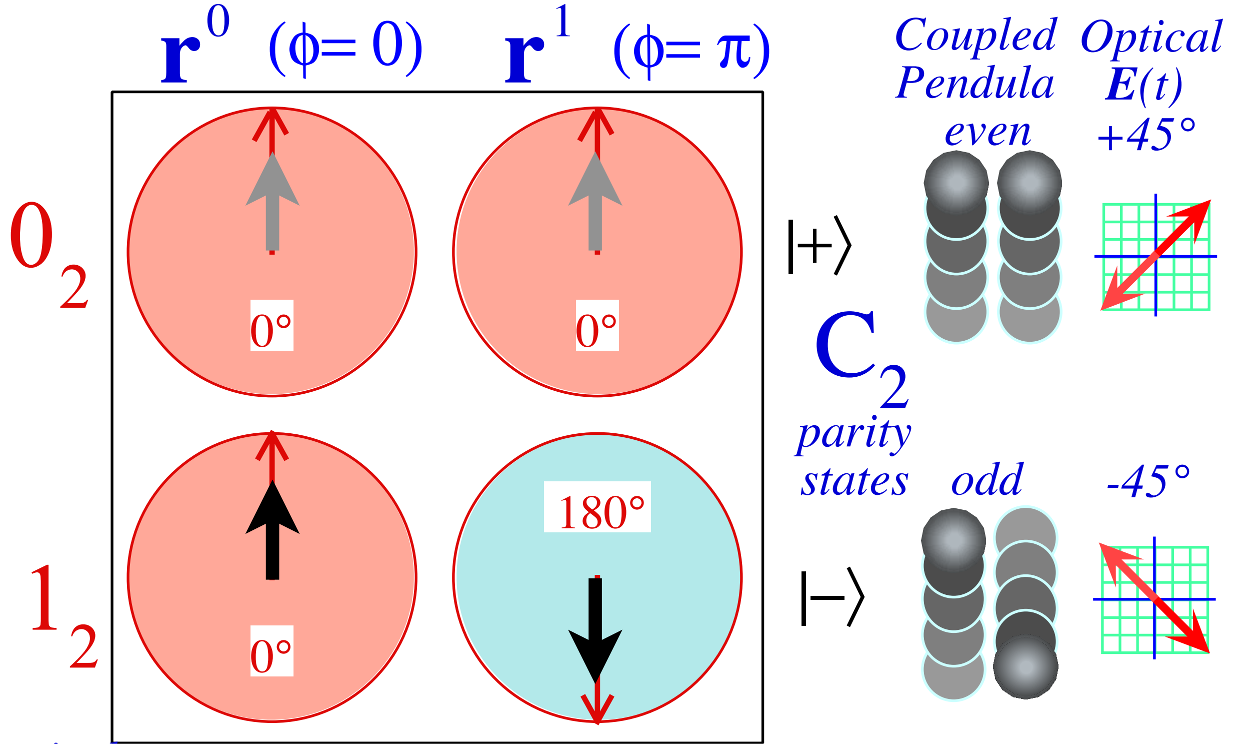
vs.

antisymmetric A_2

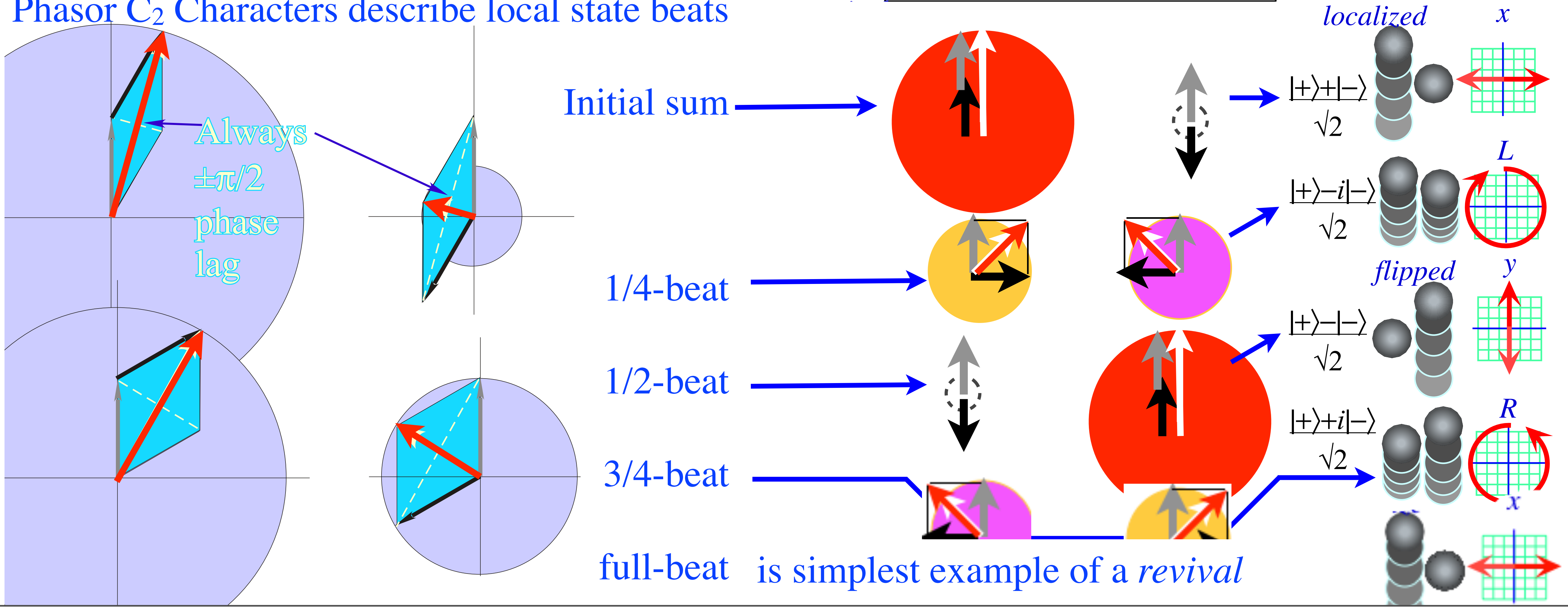
	$1 = r^0$	$r = r^1$
$0 \bmod 2$	1	1
$\pm 1 \bmod 2$	1	-1



C_2 *Phasor*-Character Table



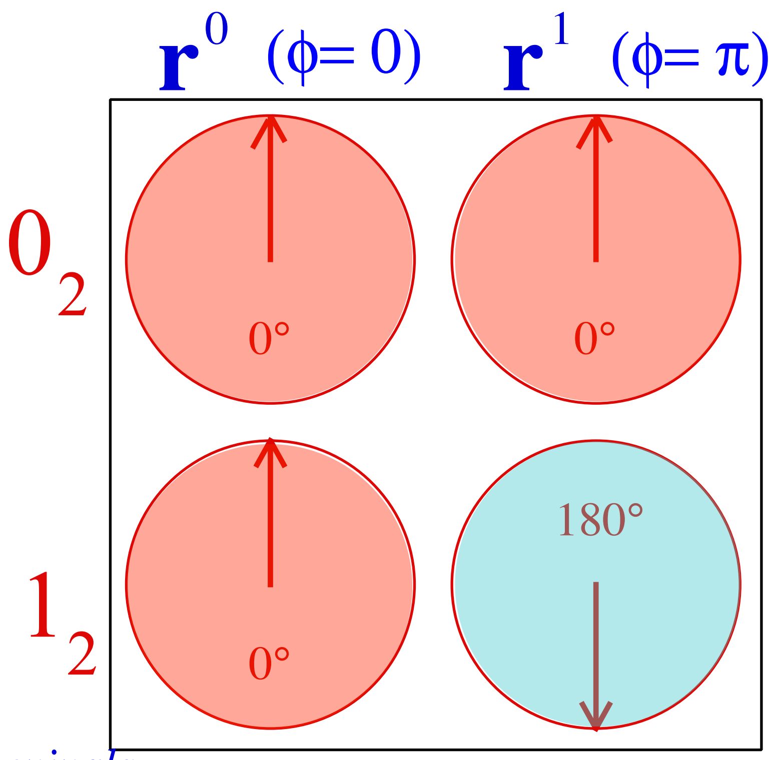
Phasor C_2 Characters describe local state beats

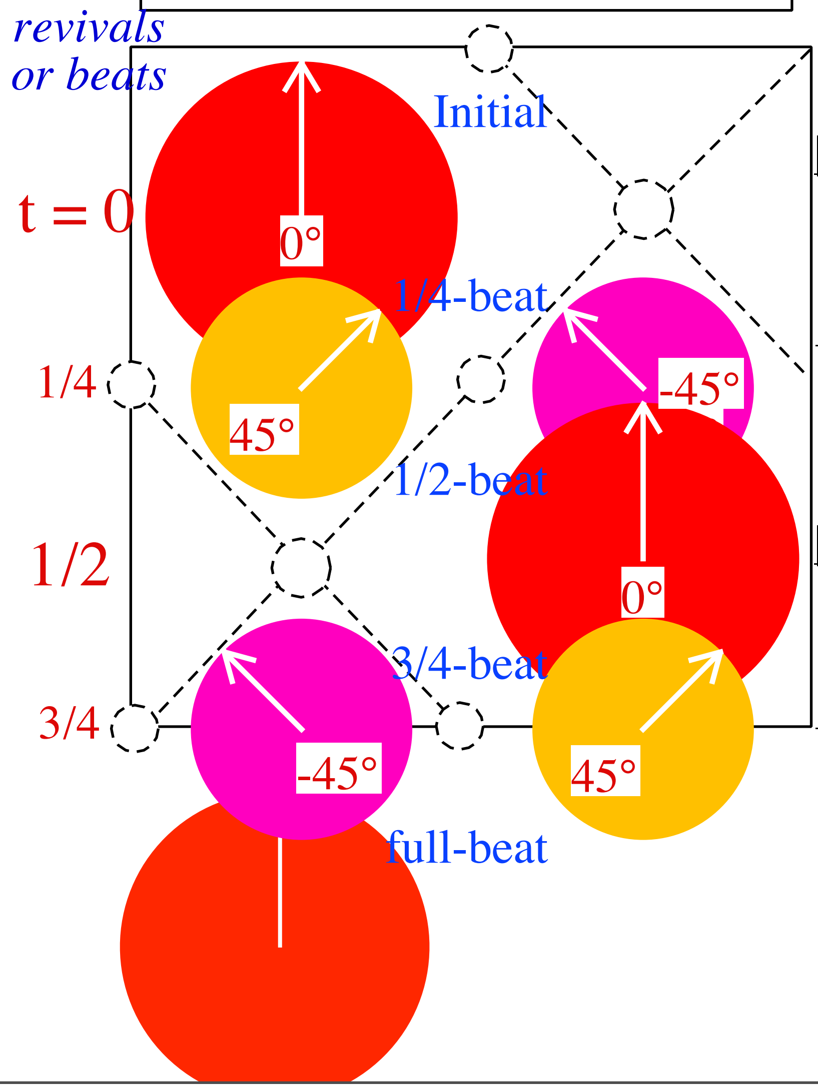


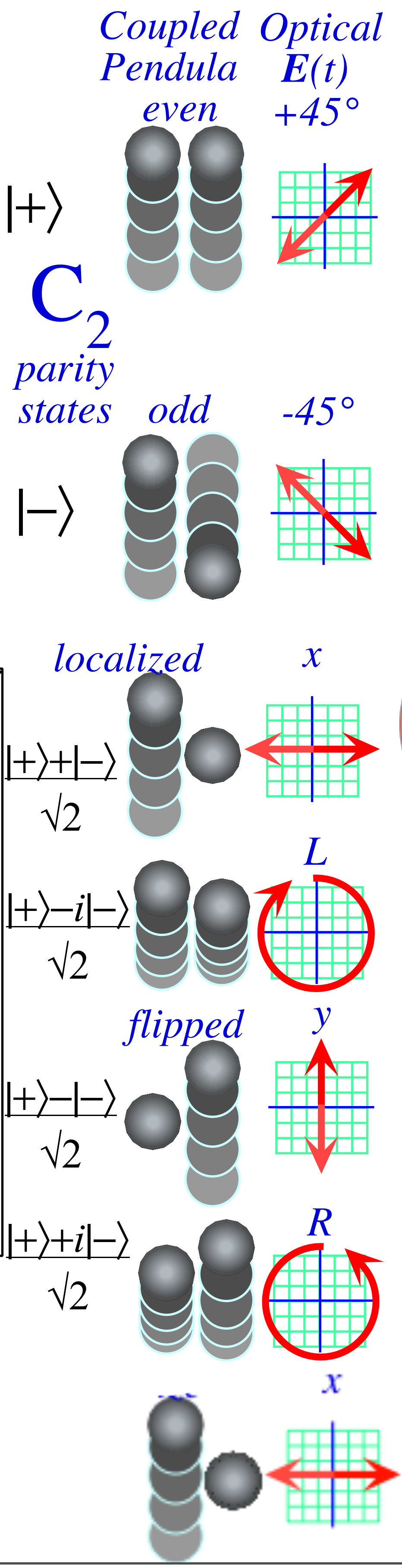
full-beat is simplest example of a *revival*

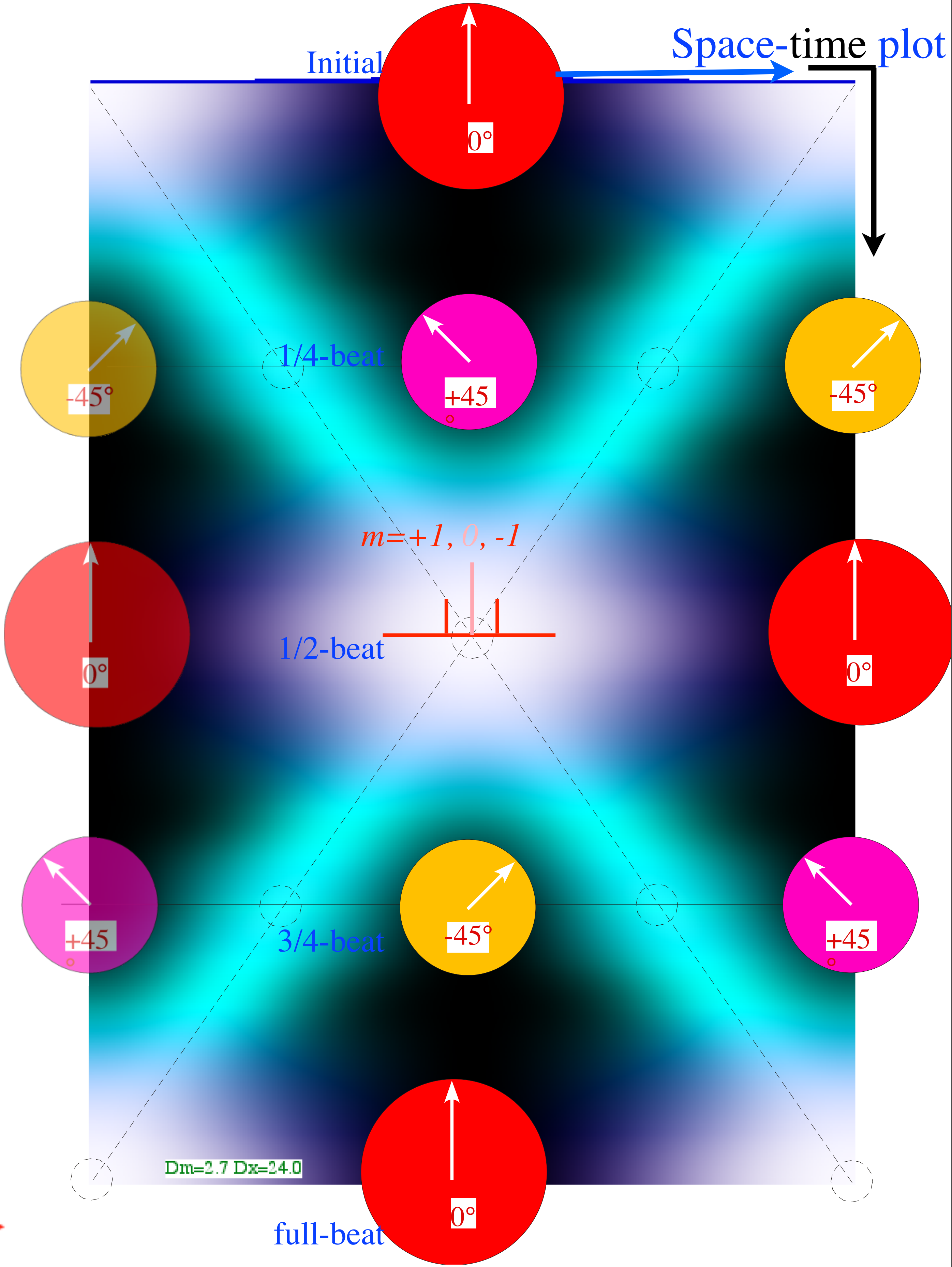
2-level-system and C_2 symmetry phase dynamics

C_2 *Phasor*-Character Table



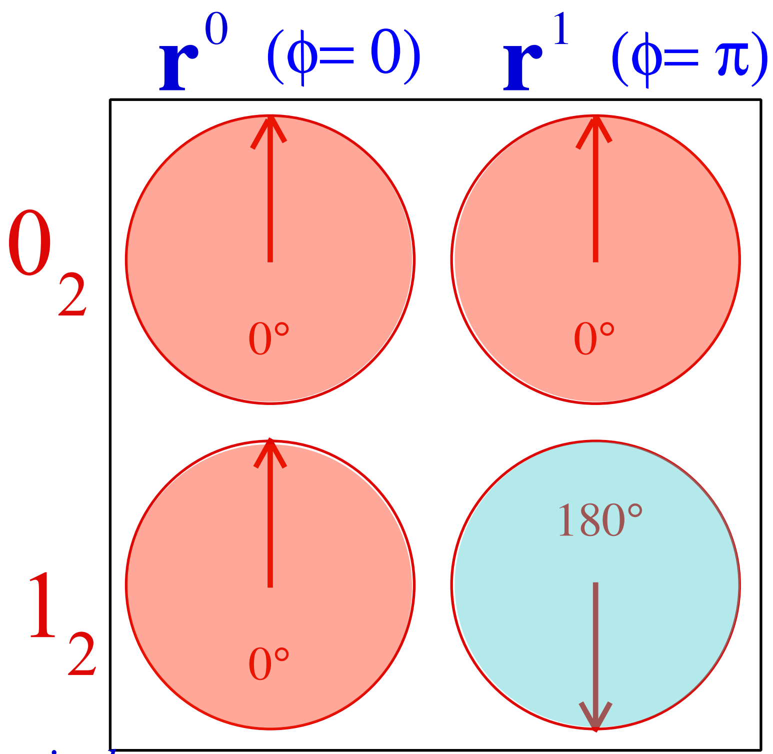




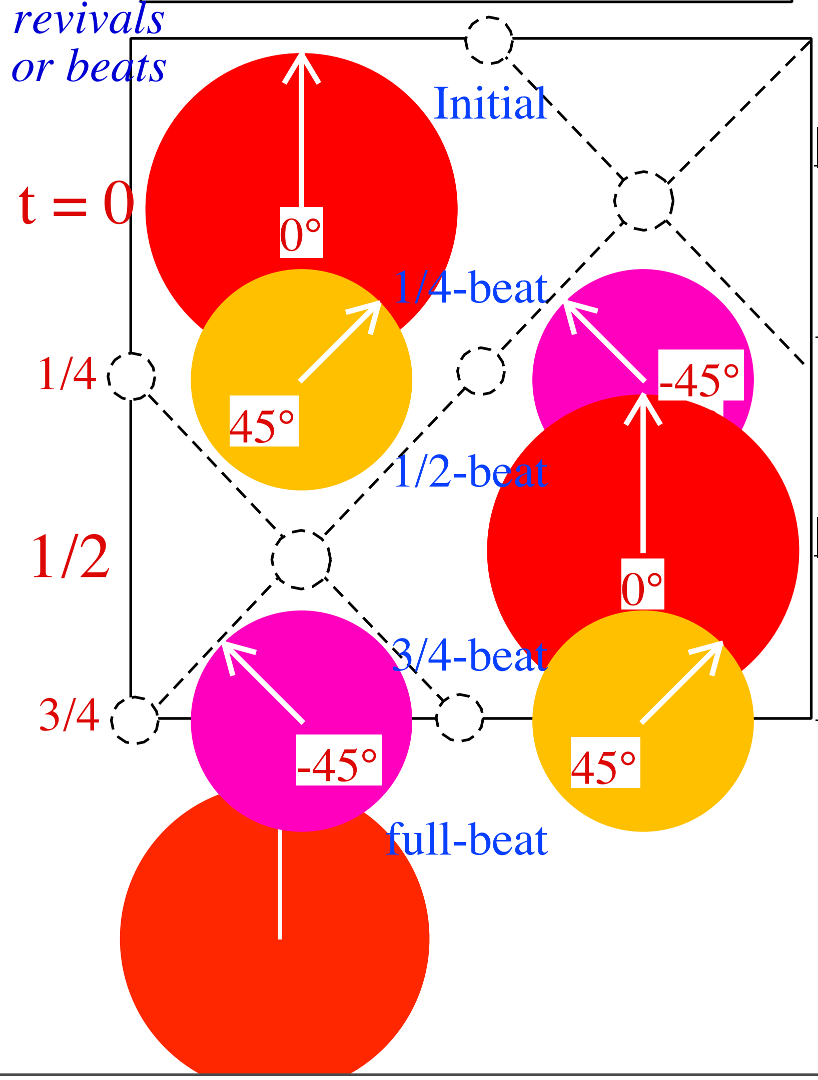


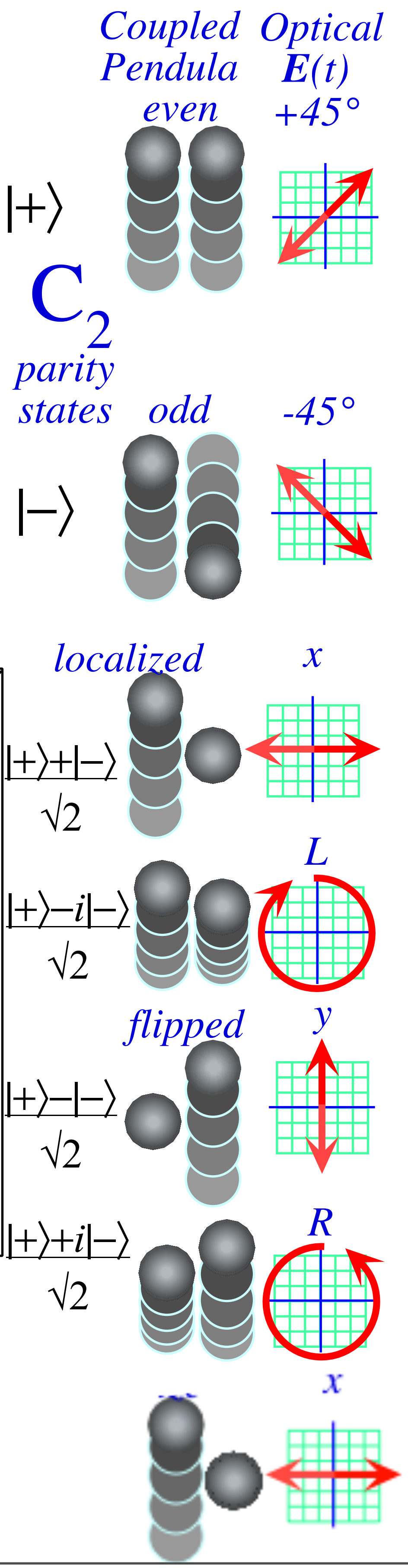
2-level-system and C_2 symmetry phase dynamics

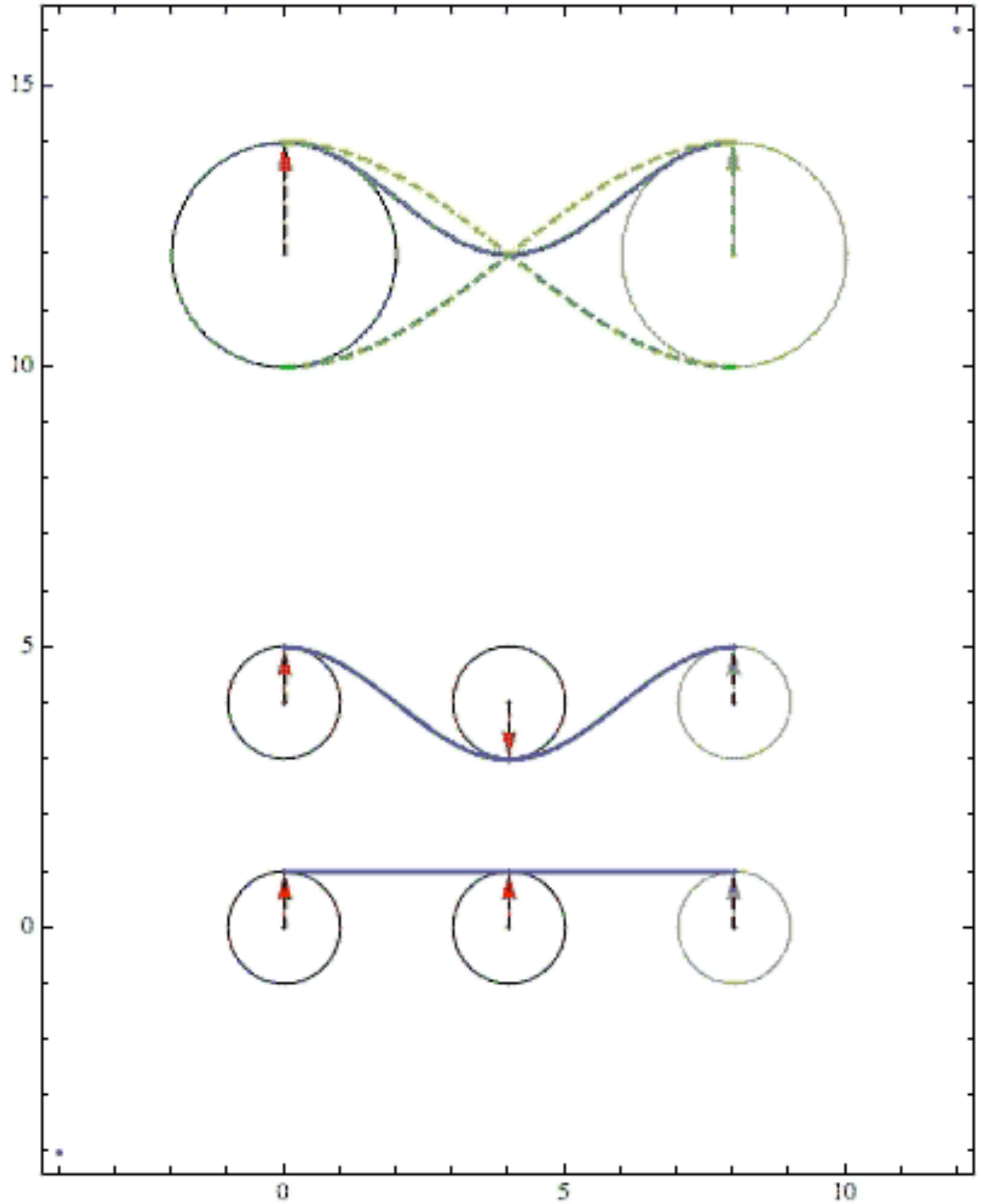
C_2 *Phasor*-Character Table



	$\mathbf{r}^0$ $(\phi = 0)$	$\mathbf{r}^1$ $(\phi = \pi)$
0_2	0°	0°
1_2	0°	180°





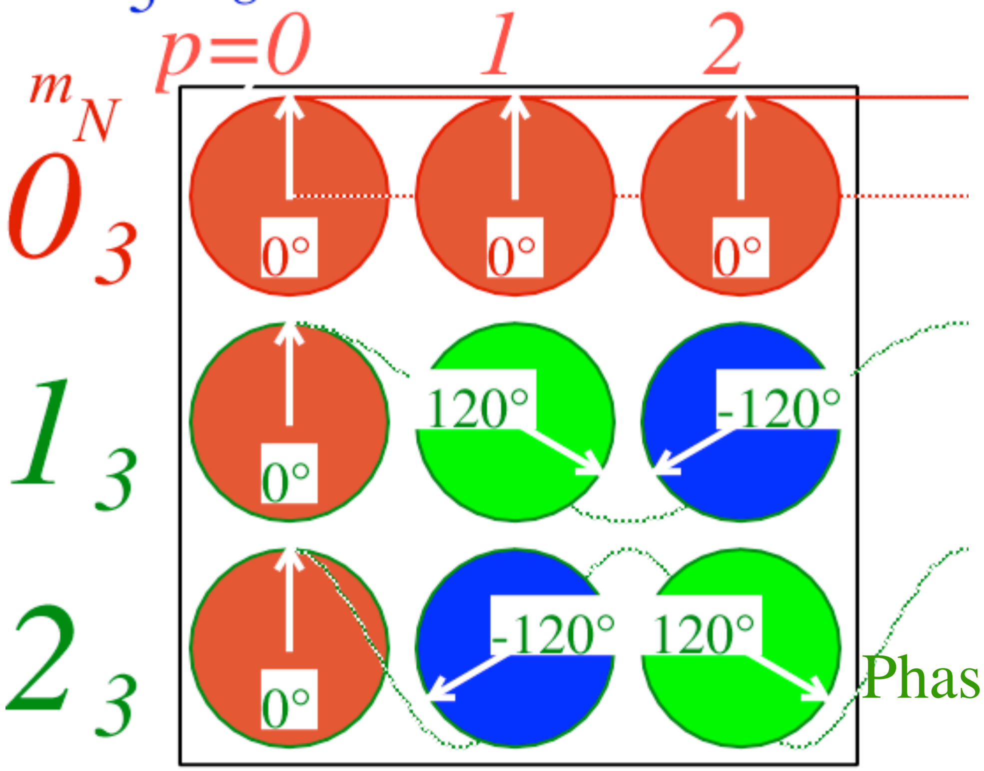


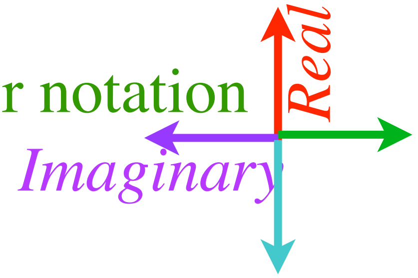
C_3 symmetry phase in *1*, ②, *or 3*-level-systems

Non - chiral
C_{3v} system

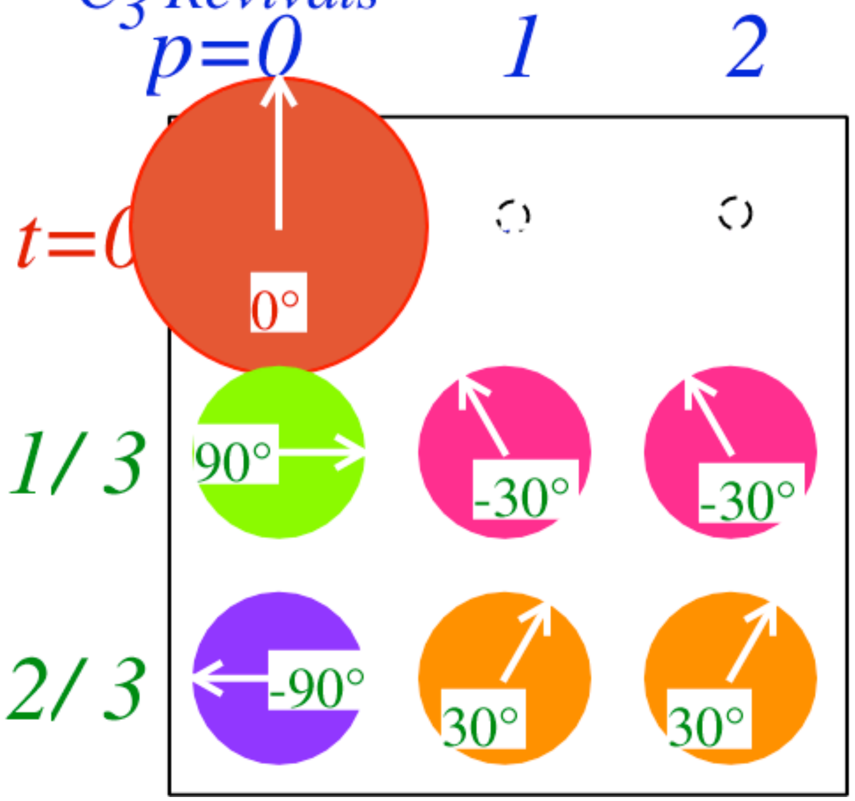
Chiral
“quantum-Hall-like”
systems
deserve special treatment

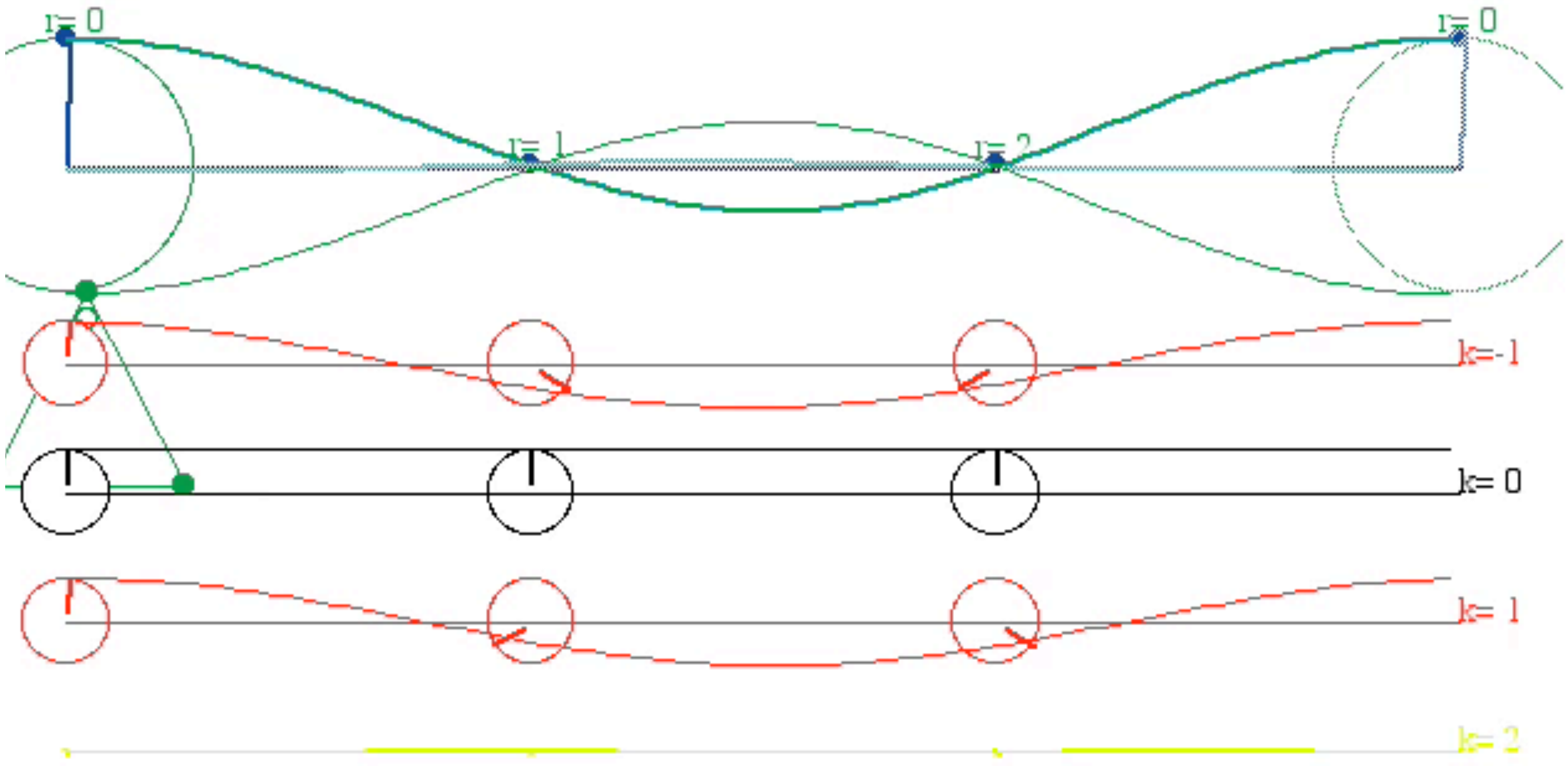
C_3 Eigenstate Characters



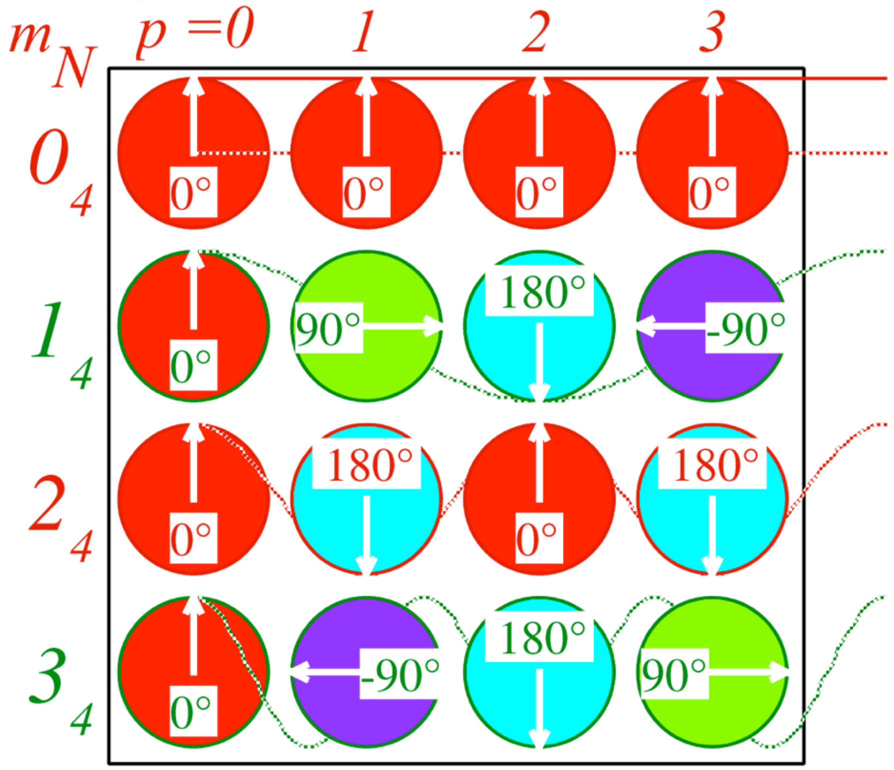


C_3 Revivals



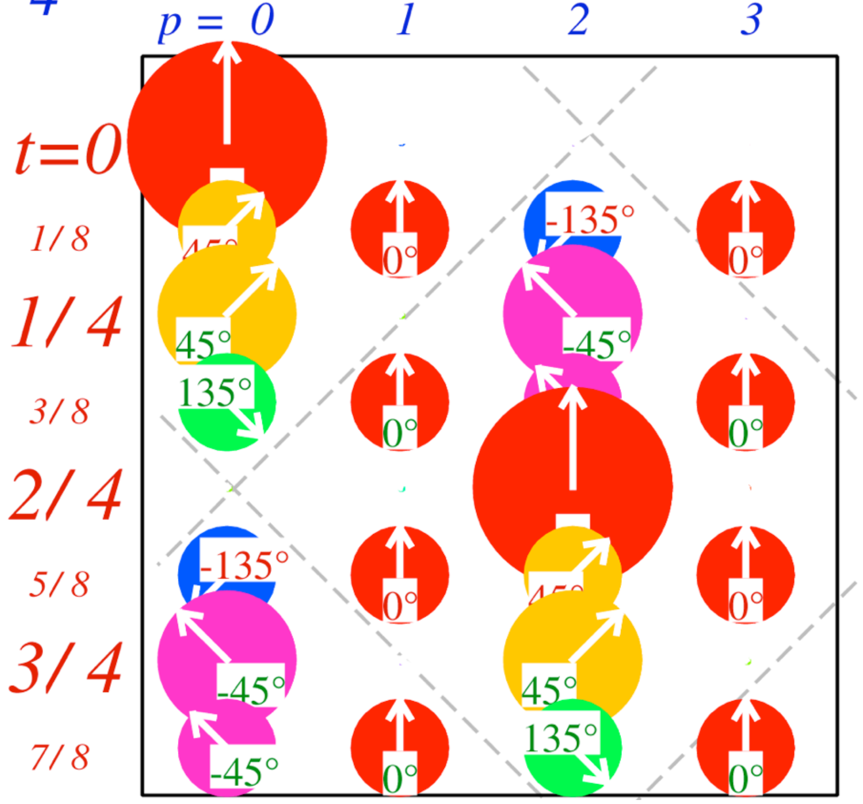


C_4 symmetry phase in 1, 2, 3 , or 4 level-systems

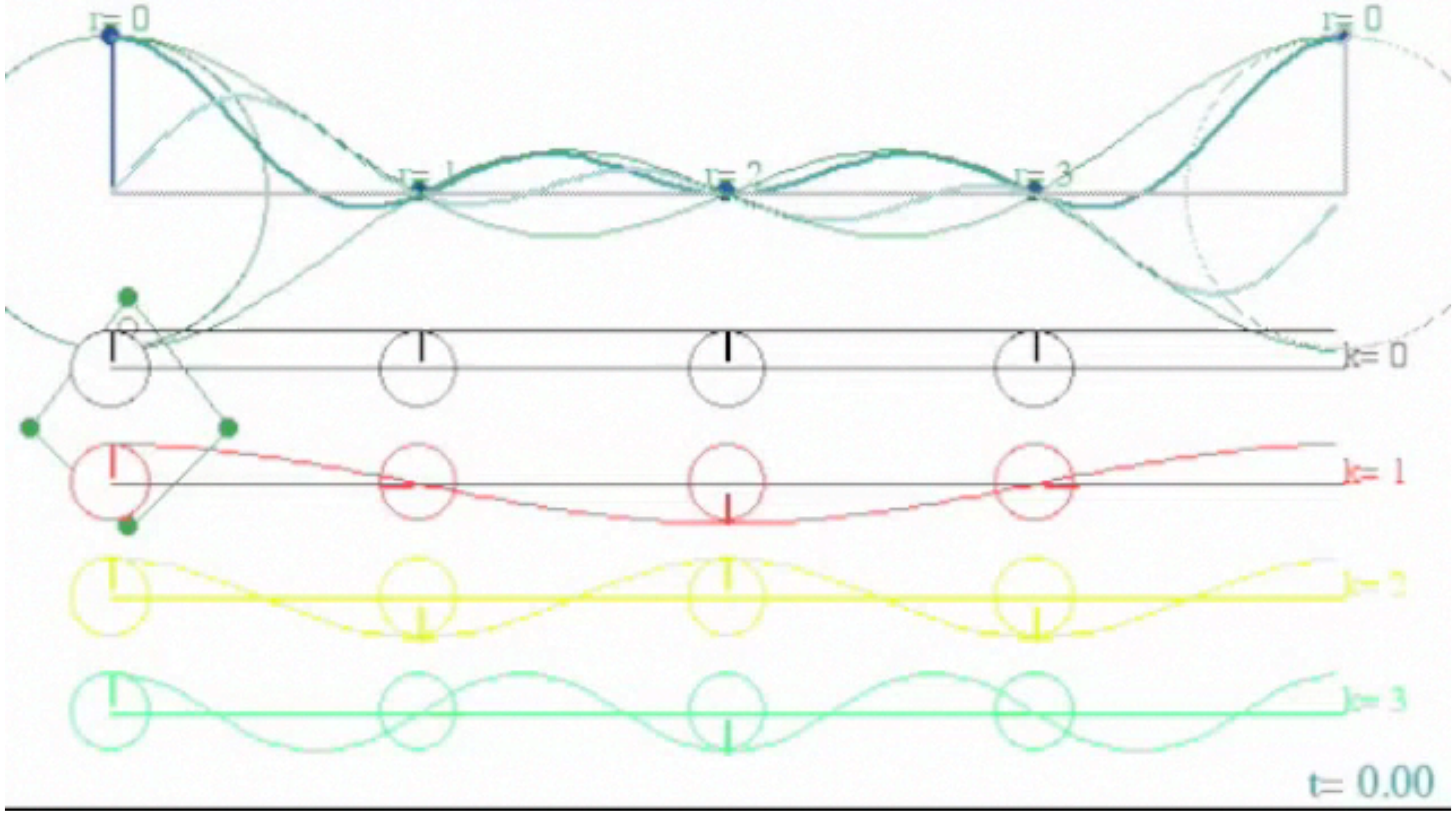


C_4 *Eigenstate Characters*

Non - chiral
C_{4v} *system*

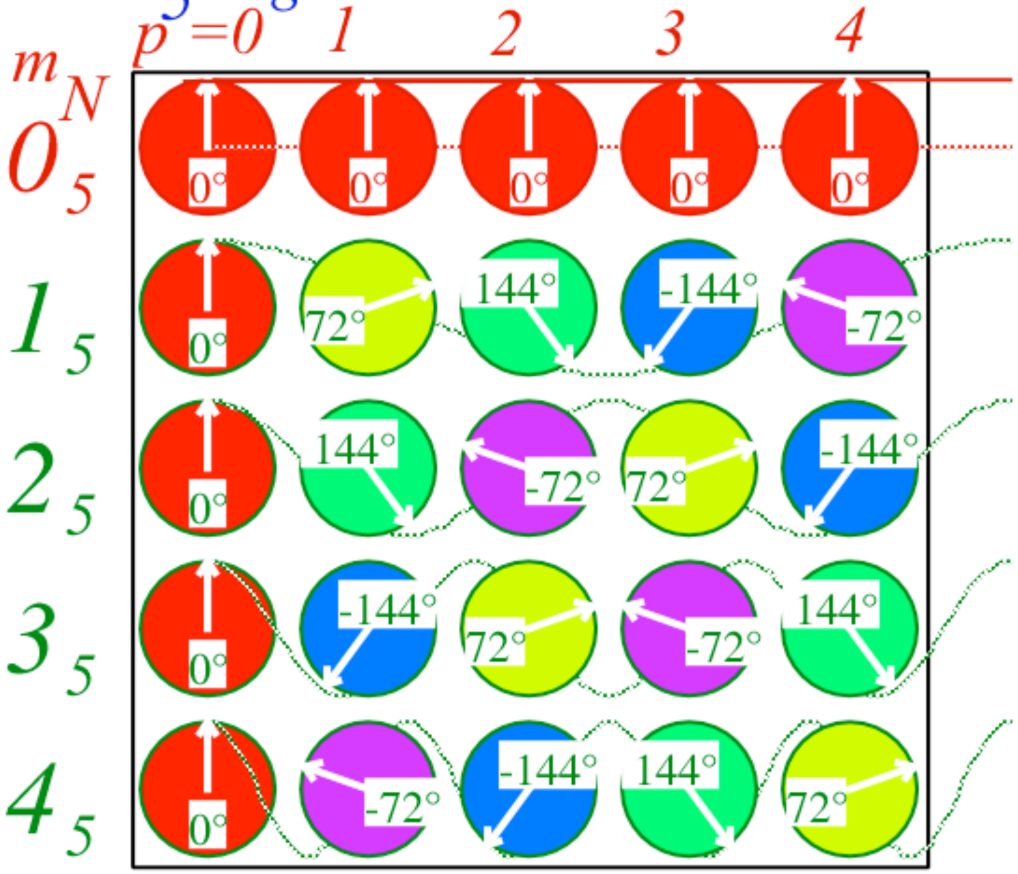


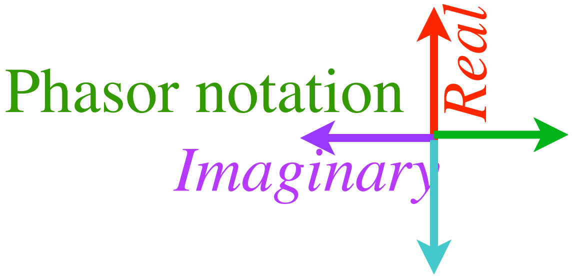
C_4 *Revivals*



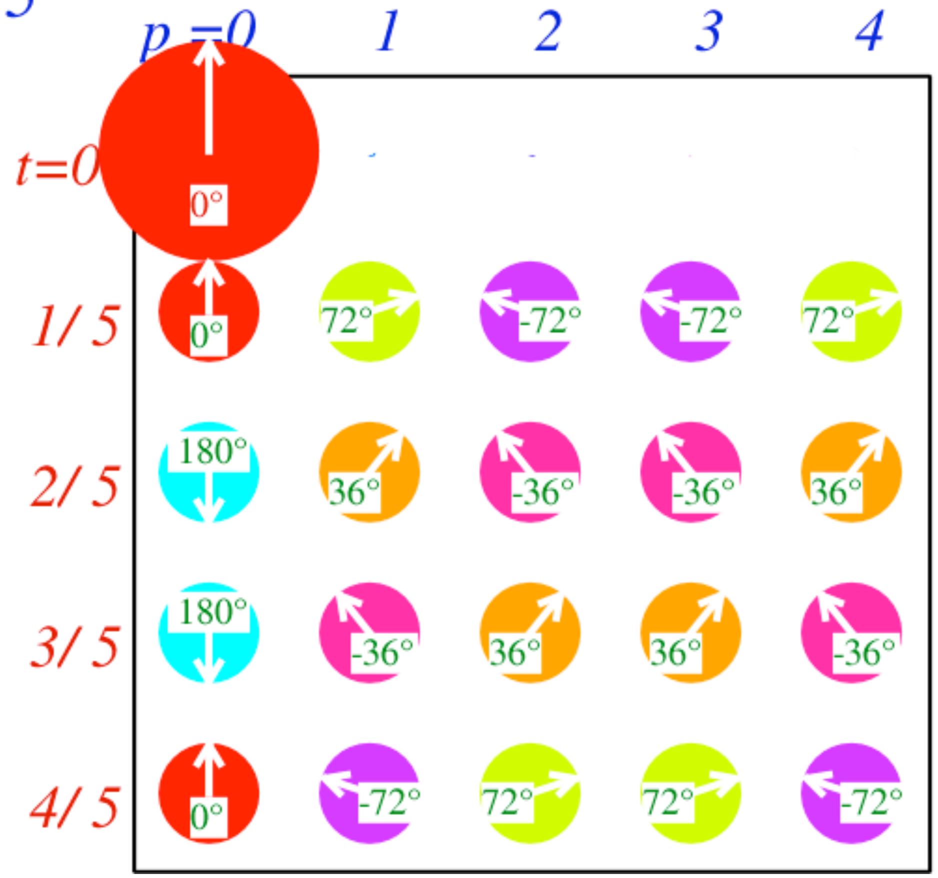
C_5 symmetry phase in *1*, 2,…*5* level-systems

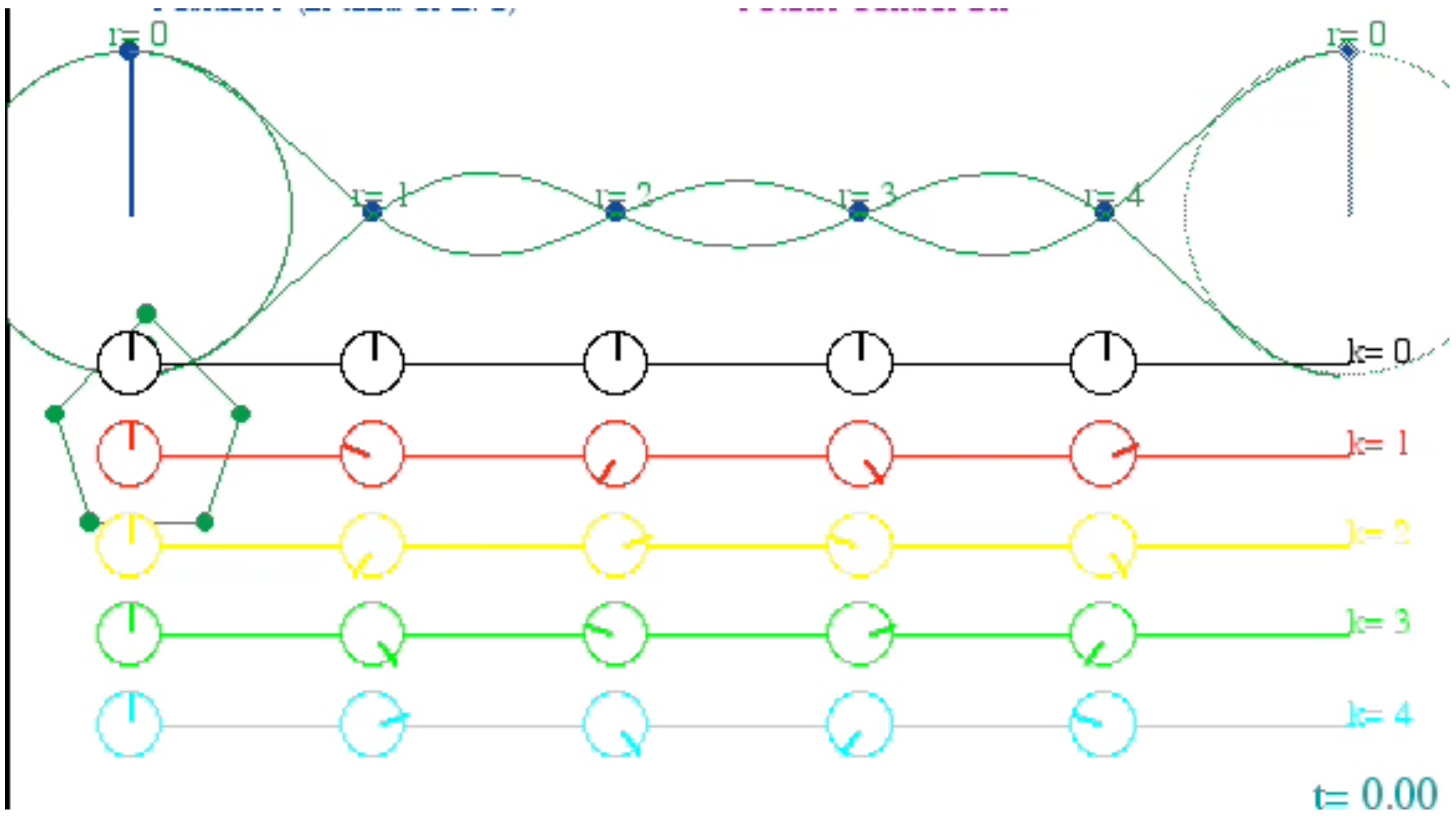
C_5 Eigenstate Characters



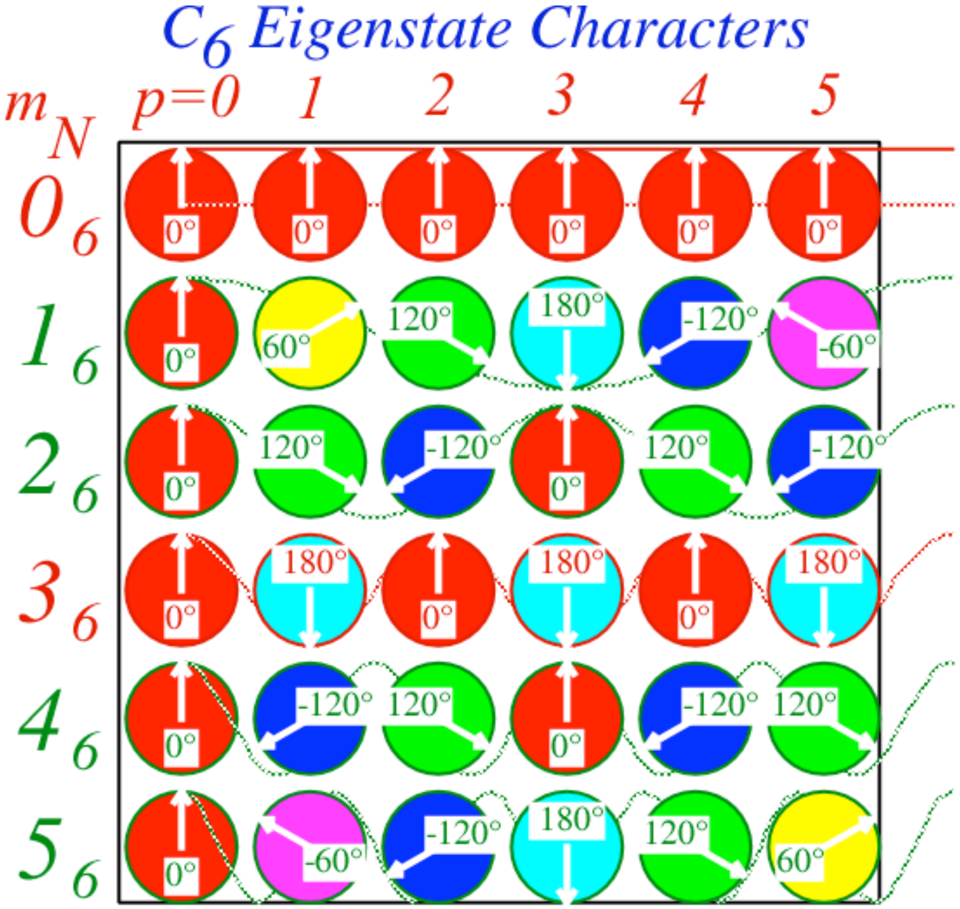


C_5 Revivals

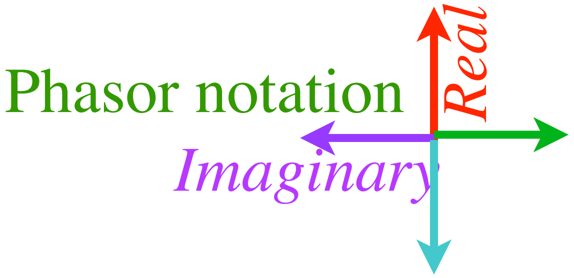


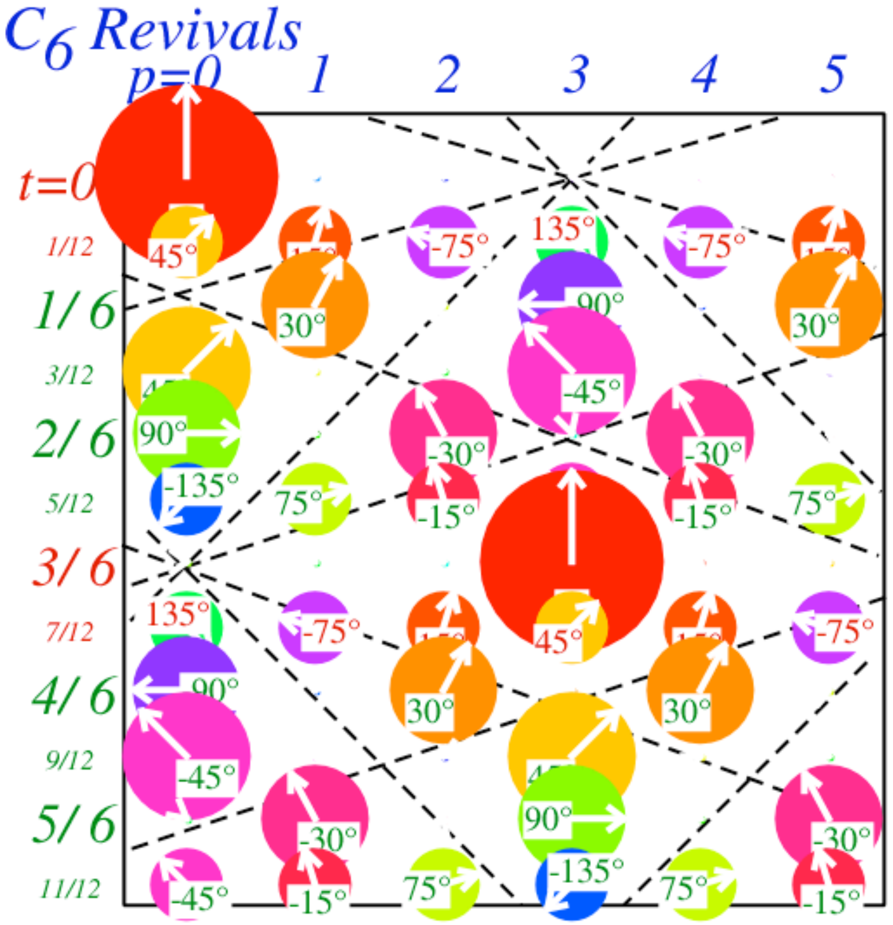


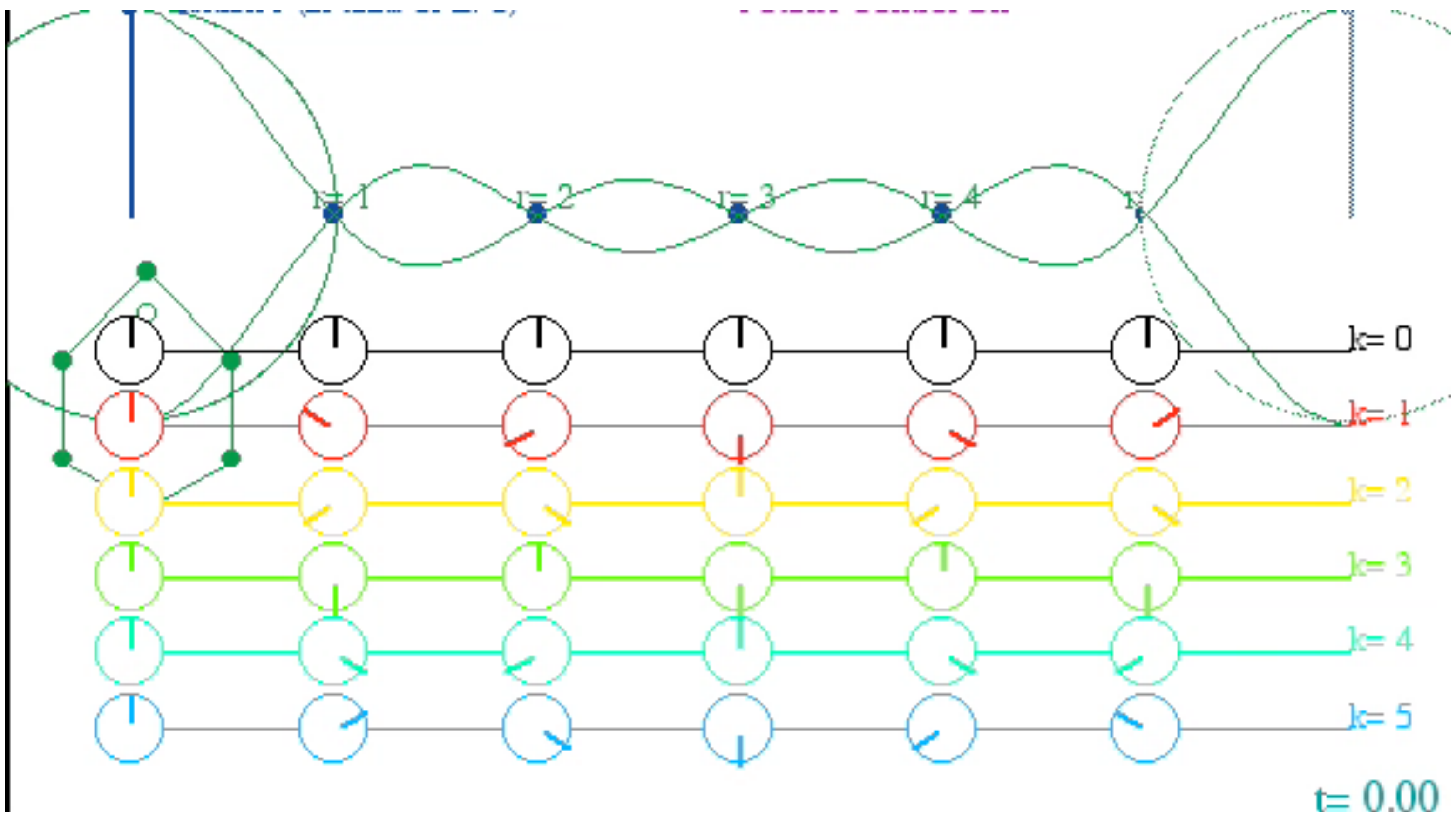
C_6 symmetry phase in *1*, ...*6* level-systems



m_N	p=0	1	2	3	4	5
0_6	0°	0°	0°	0°	0°	0°
1_6	0°	60°	120°	180°	-120°	-60°
2_6	0°	120°	-120°	0°	120°	-120°
3_6	0°	180°	0°	180°	0°	180°
4_6	0°	-120°	120°	0°	-120°	120°
5_6	0°	-60°	-120°	180°	120°	60°





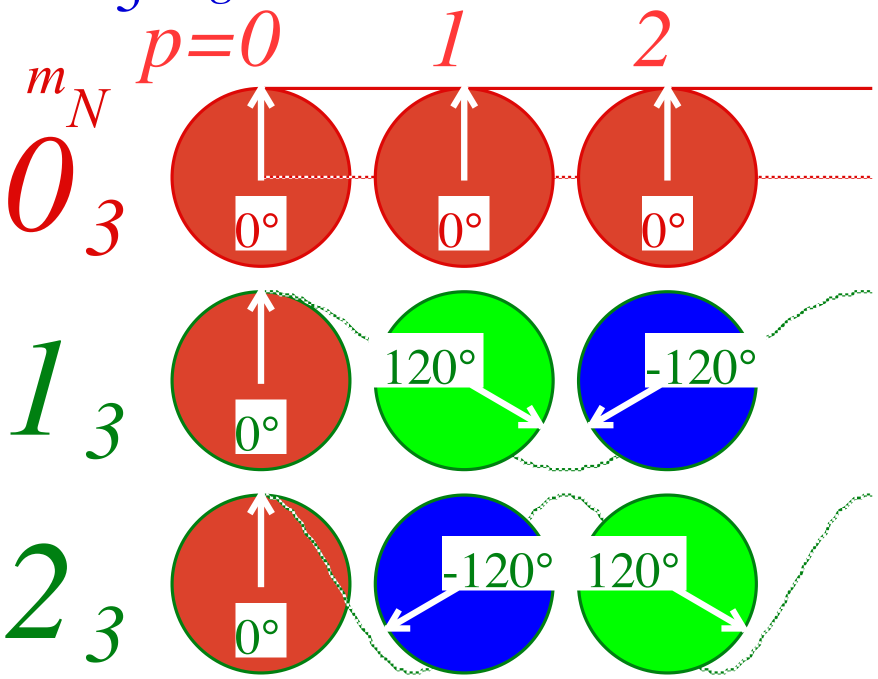


C_m algebra of revival-phase dynamics

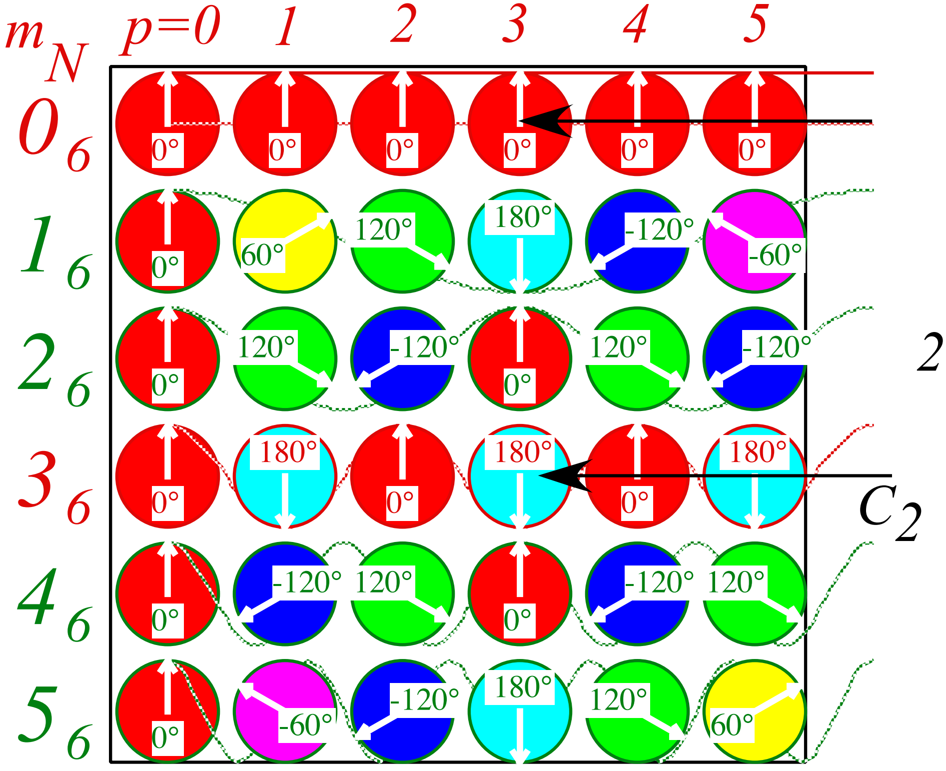
Discrete 3-State or Trigonal System
(Tesla's 3-Phase AC)

Discrete 6-State or Hexagonal System
(6-Phase AC)

C_3 Eigenstate Characters

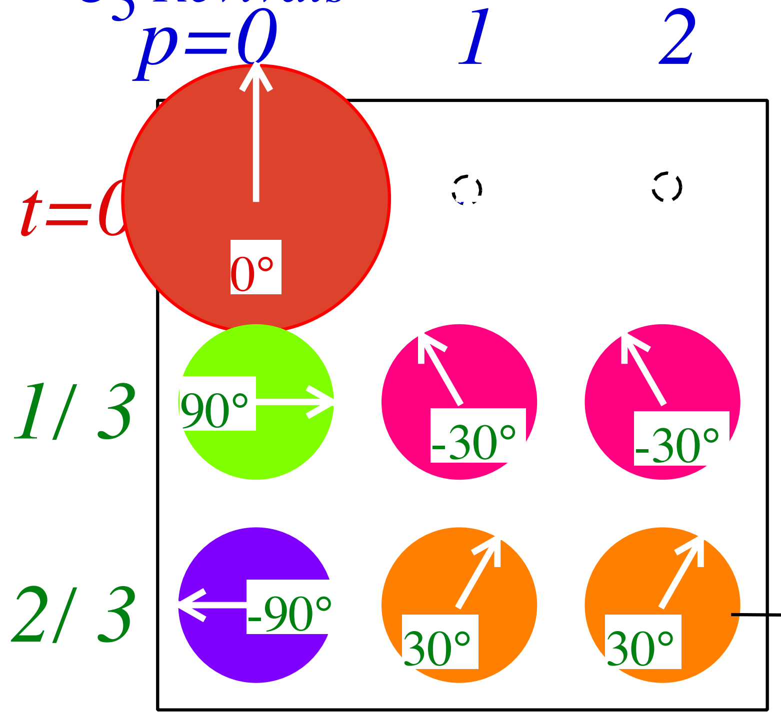


C_6 Eigenstate Characters



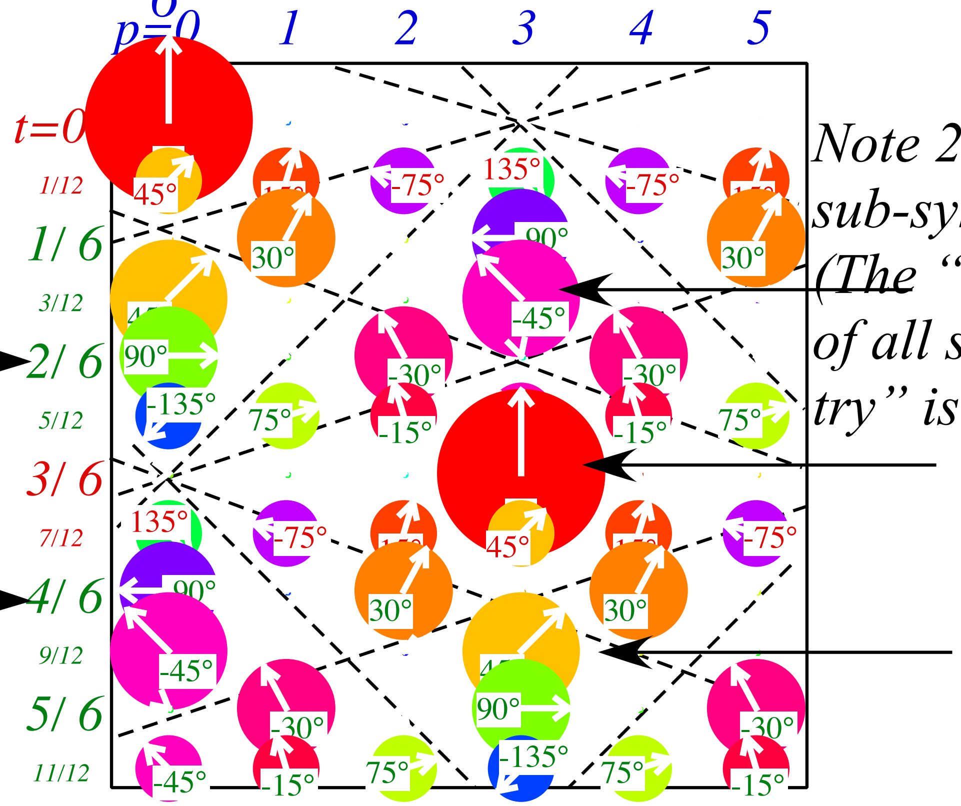
Note
2-phase
AC

C_3 Revivals



Note 3-phase
sub-symmetry

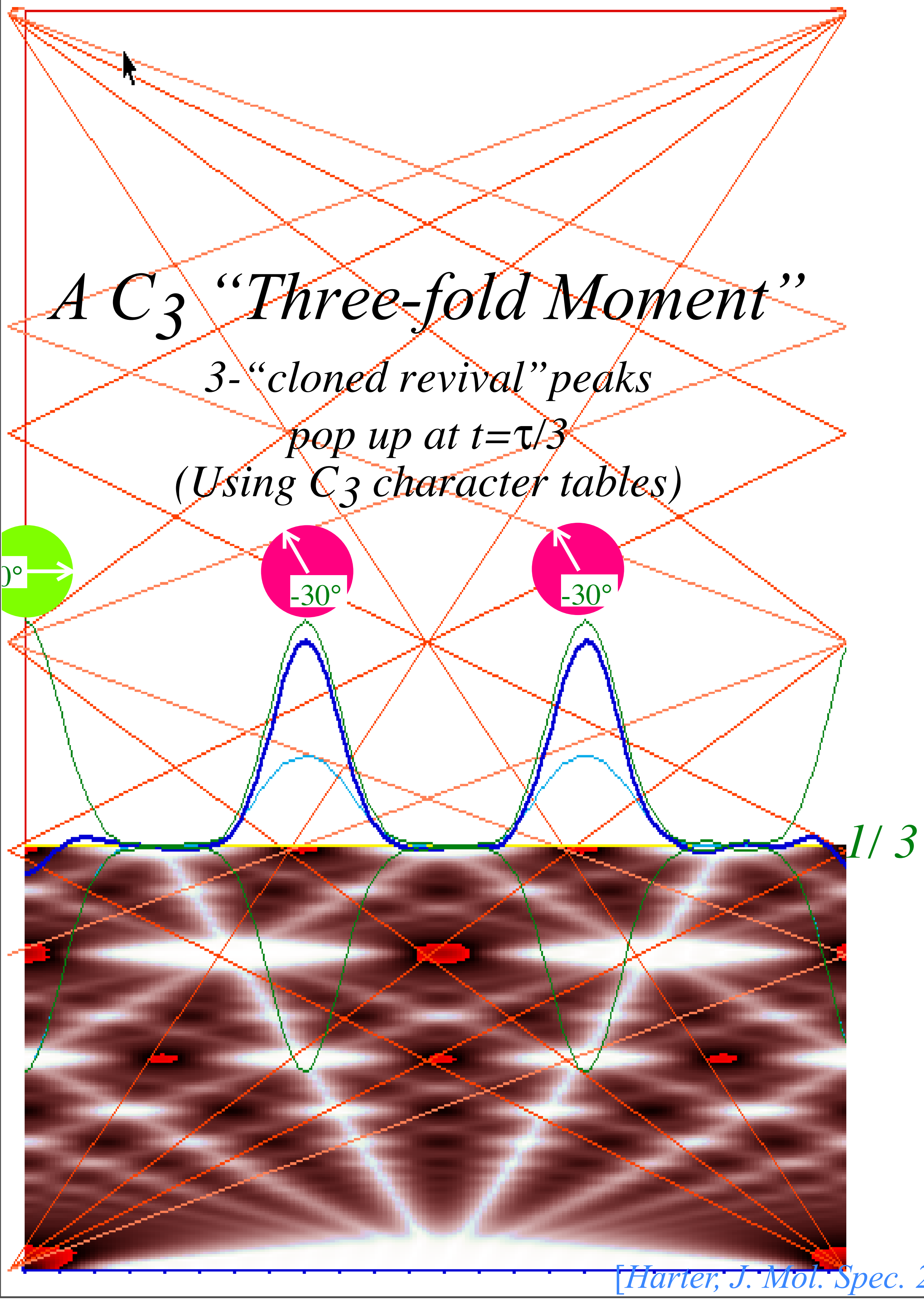
C_6 Revivals



Note 2-phase sub-symmetry (The "Mother of all symmetry" is C_2)

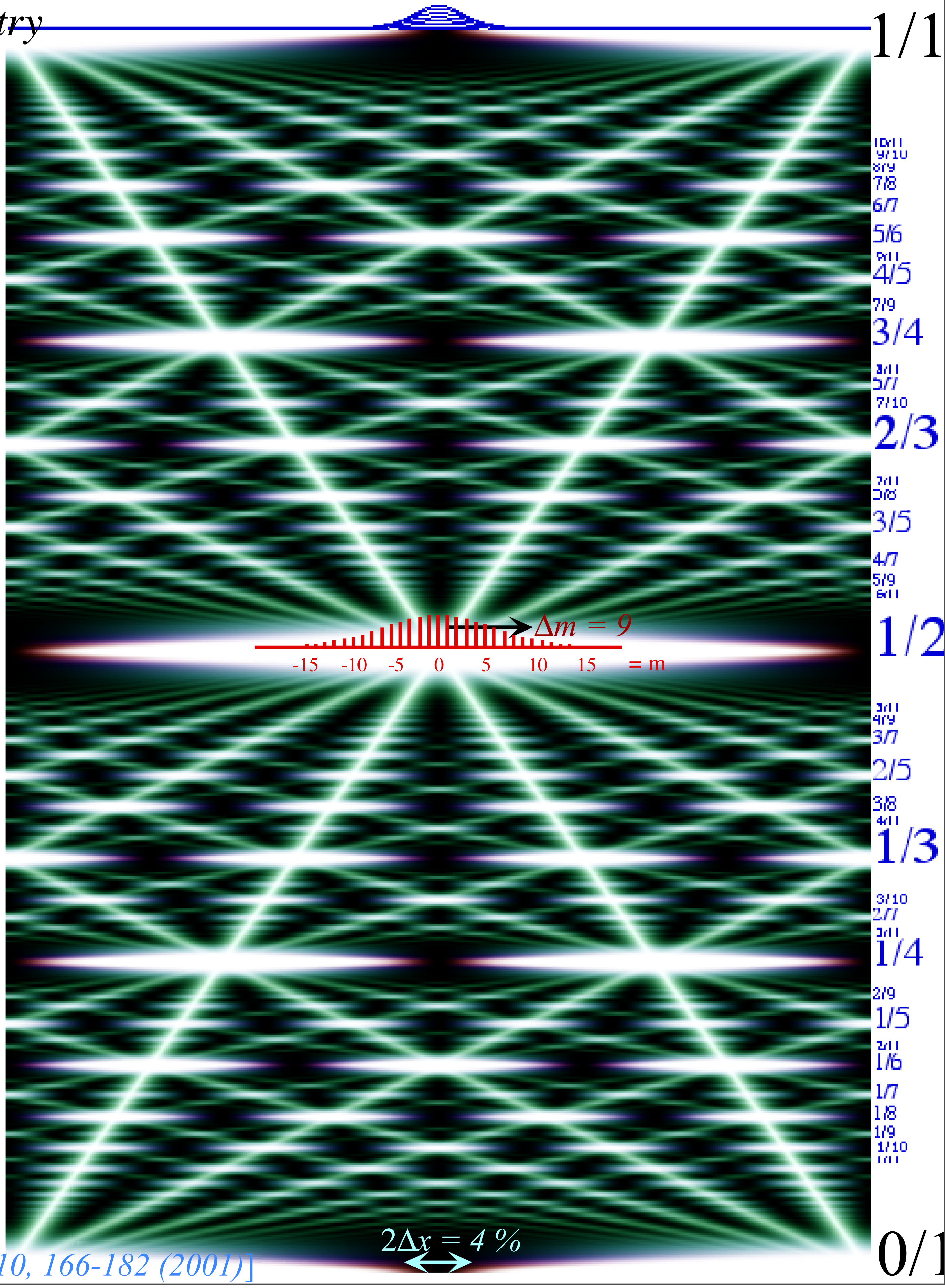
C_m algebra of revival-phase dynamics

Quantum rotor fractional take turns at Cn symmetry



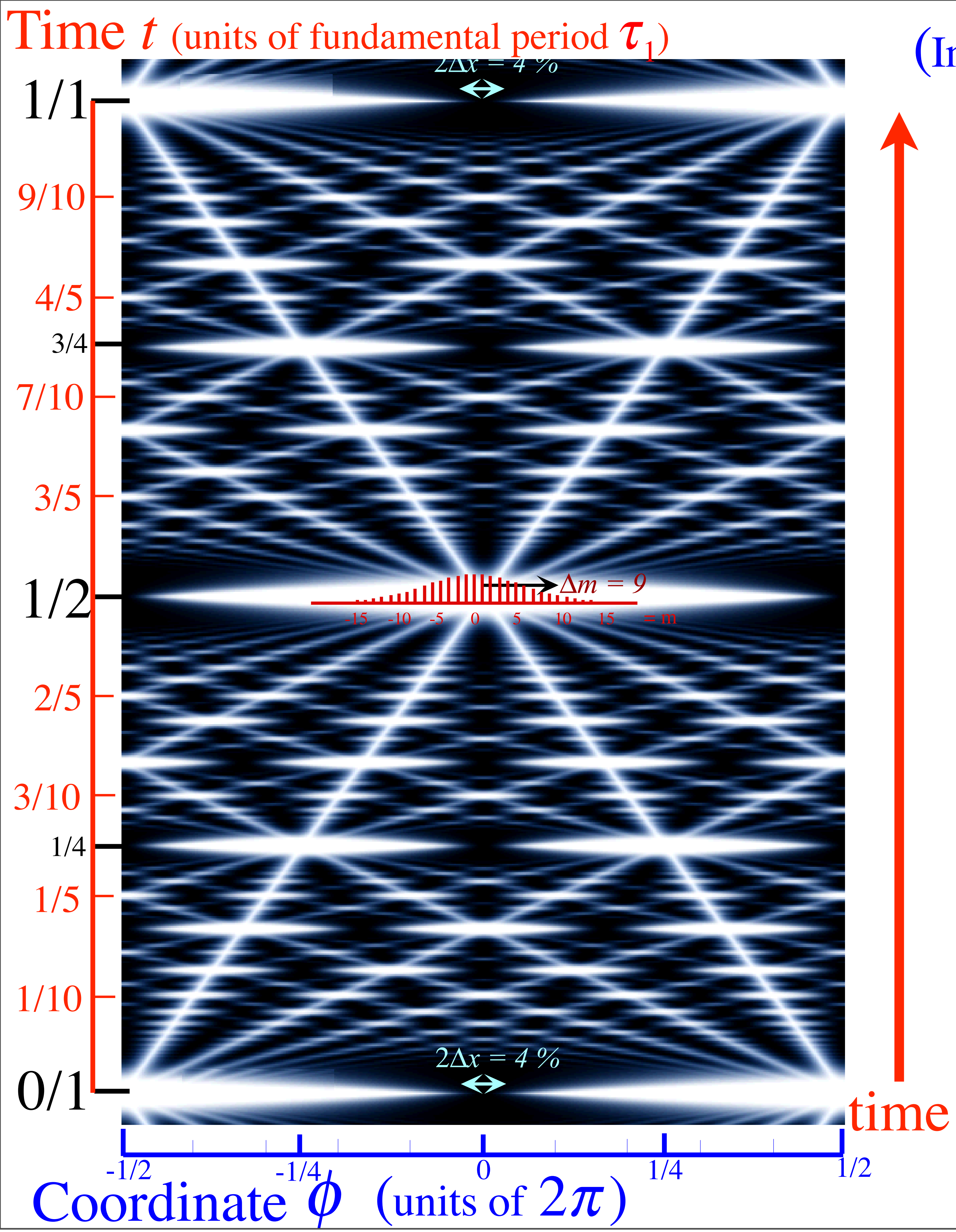
A C_3 "Three-fold Moment"

3-"cloned revival" peaks pop up at $t=\tau/3$ (Using C_3 character tables)

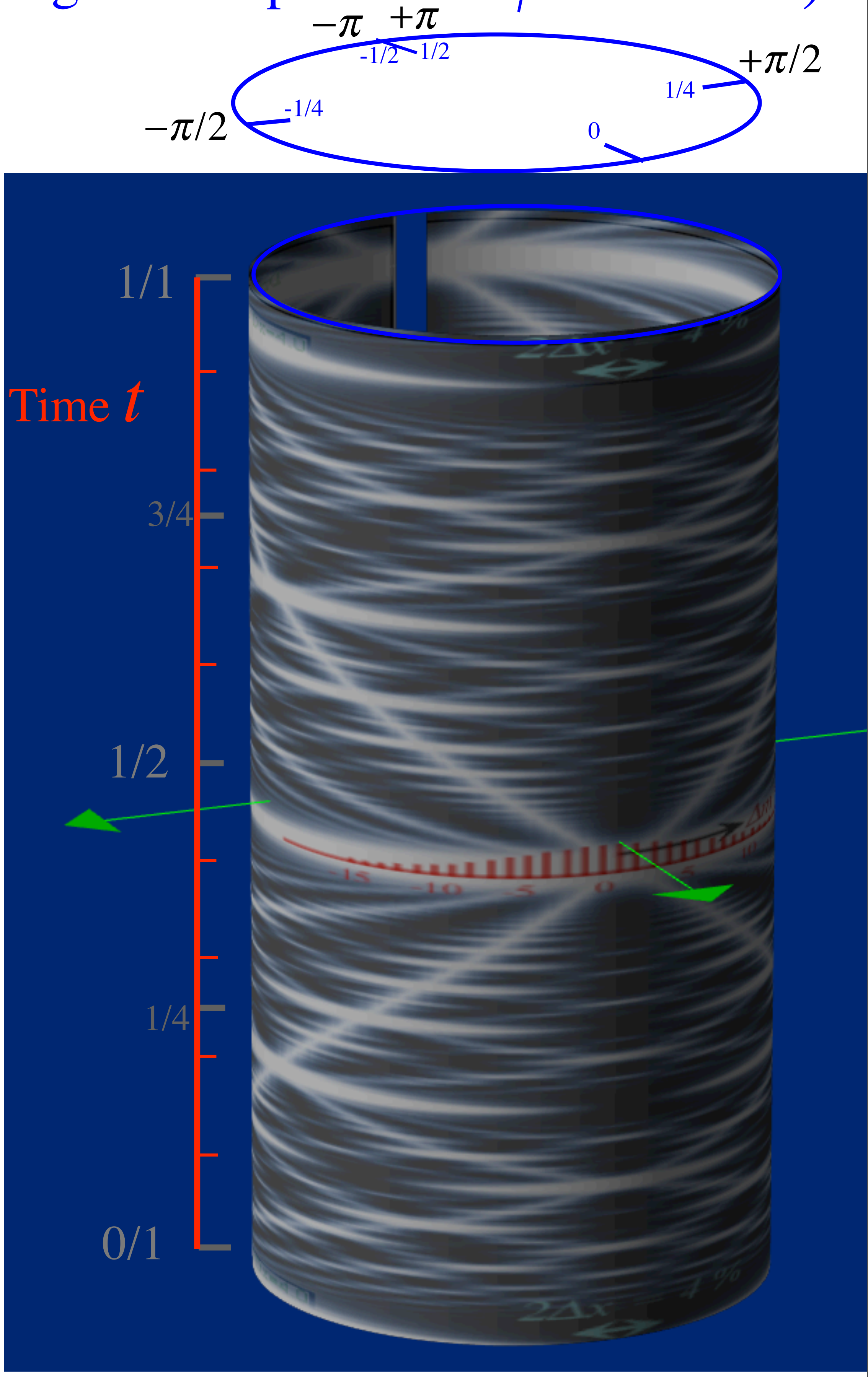


[Harter, J. Mol. Spec. 210, 166-182 (2001)]

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

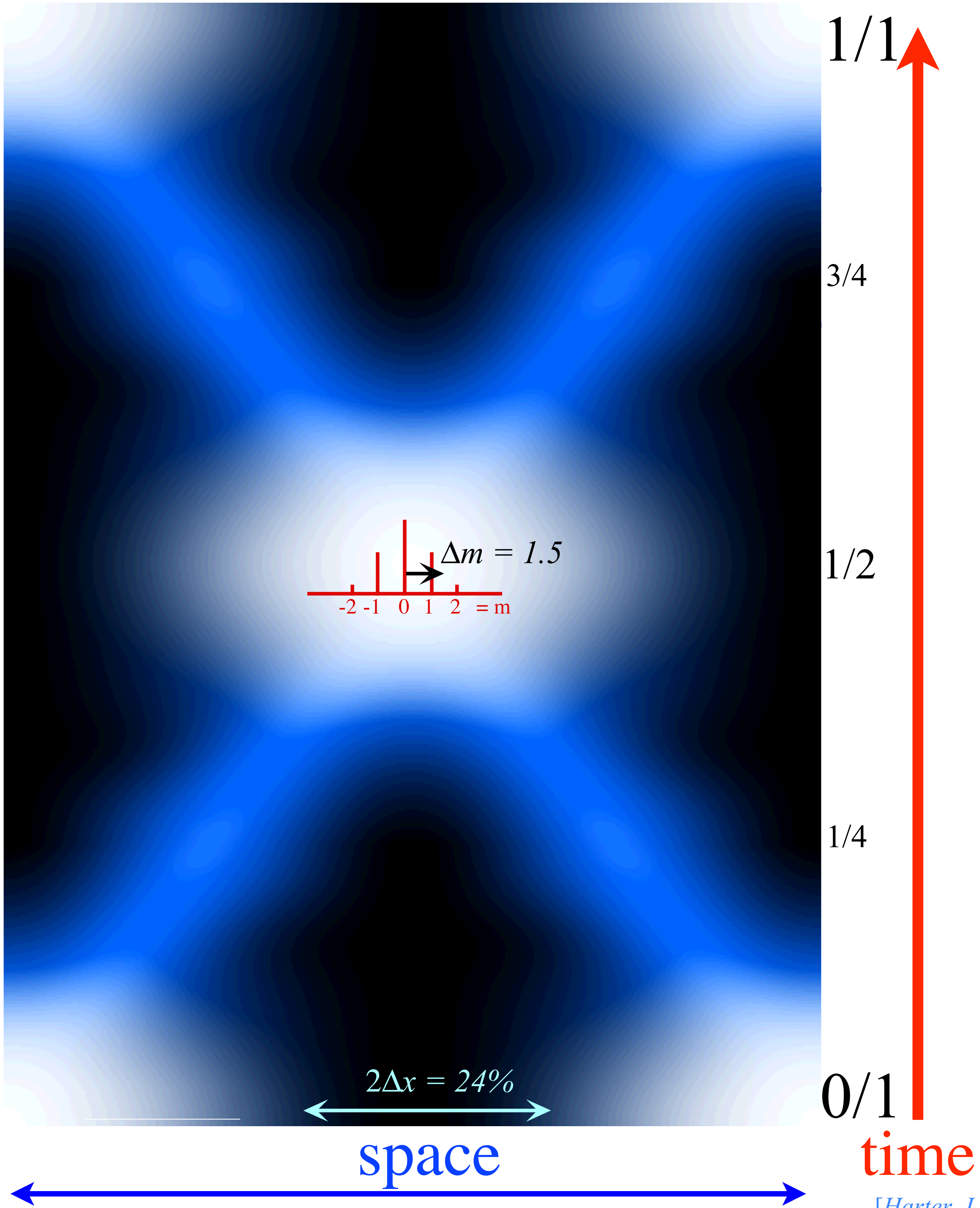


(Imagine "wrap-around" ϕ-coordinate)



N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



$|\Psi(x,t)|$ in space-time

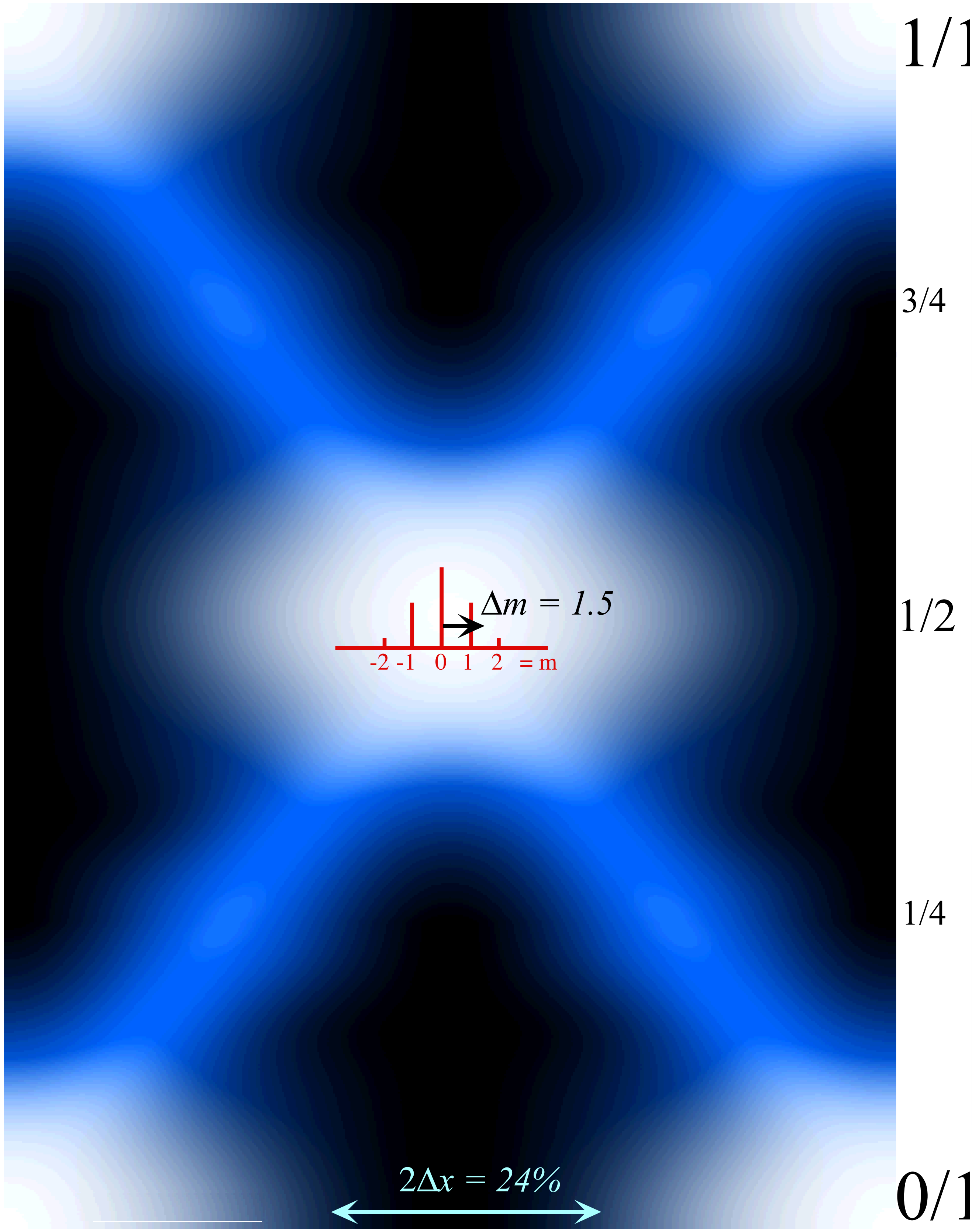
Simplest quantum revival:
Exciting first two levels
(ℓ=0 and ℓ=±1)
is like a
2-level system quantum beat
in space-time

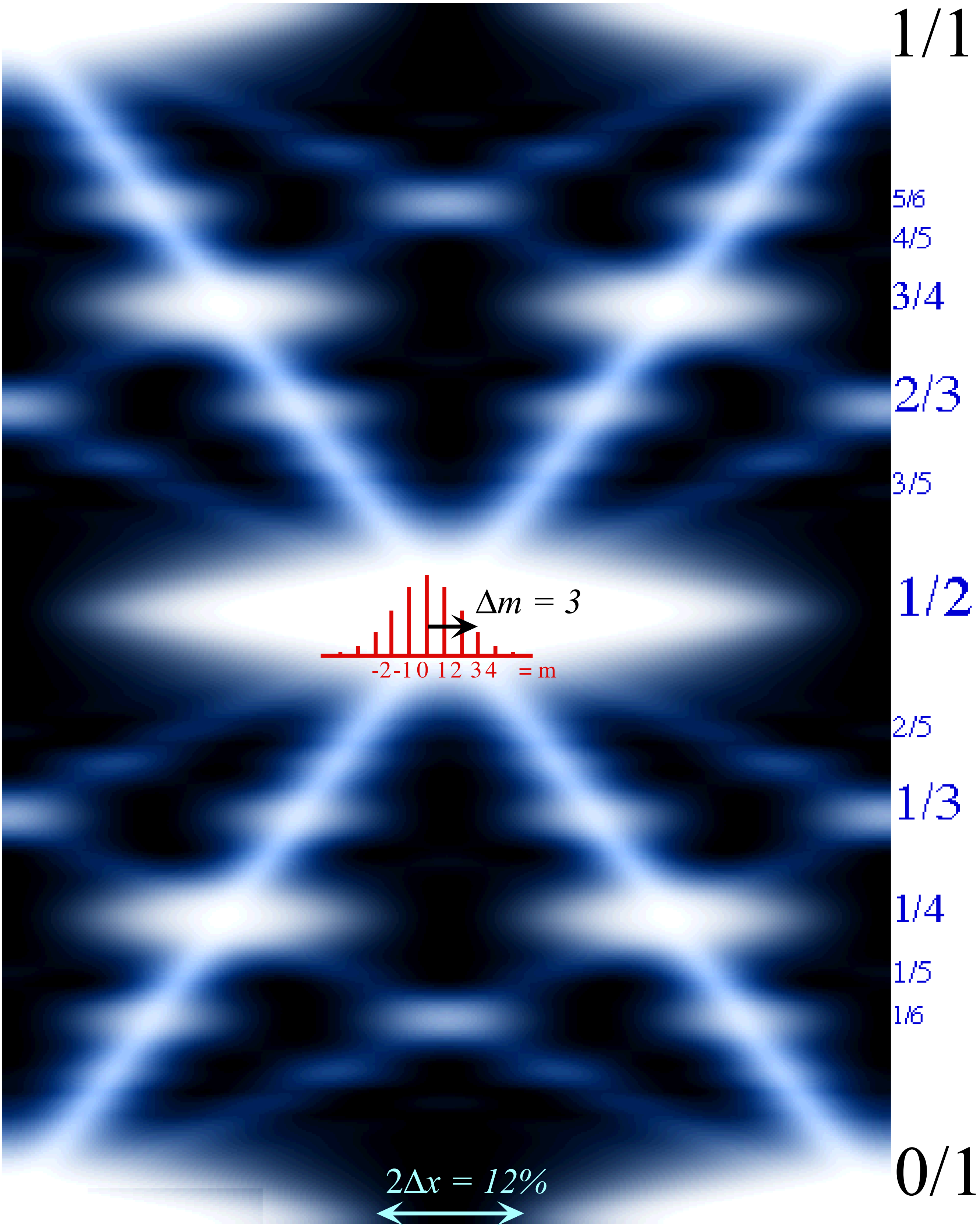
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)

(4-levels (0, ±1,±2,±3) (and some ±4) excited)

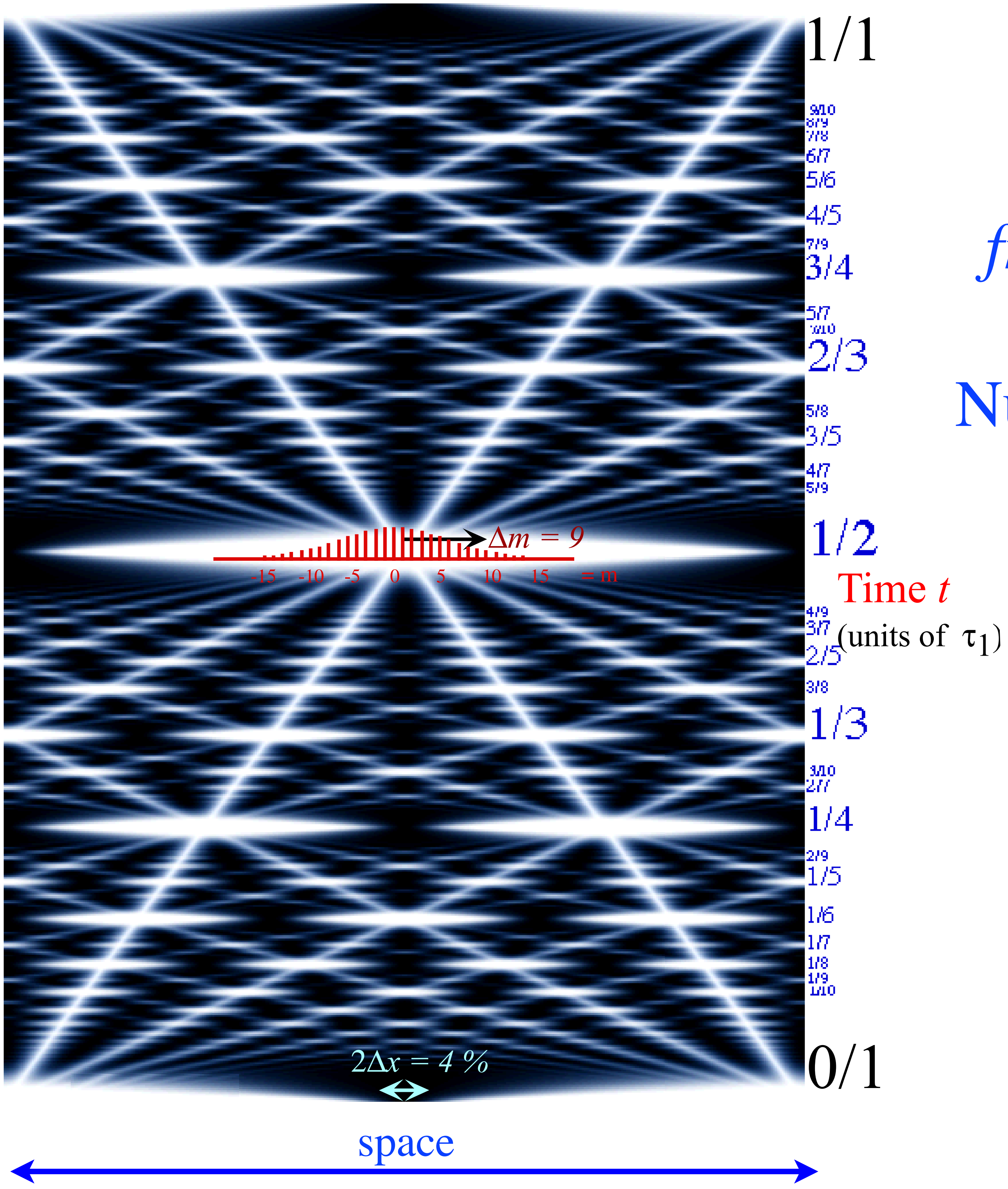




Simplest *fractional* quantum revivals: 3,4,5-level systems

N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)



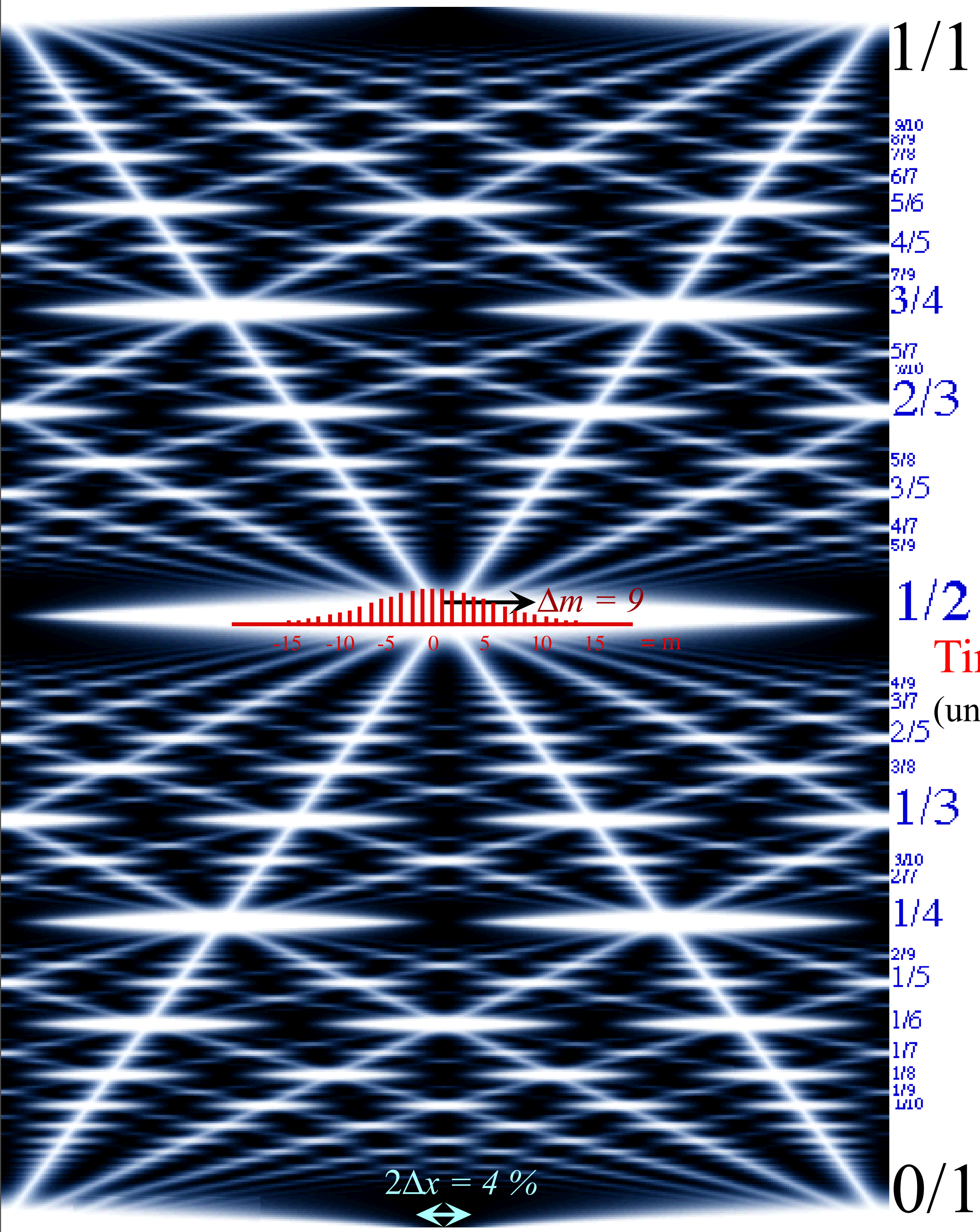
fractional quantum revivals:
in 3,4,…, N-level systems
Number increases rapidly with
number of levels
and/or bandwidth
of excitation

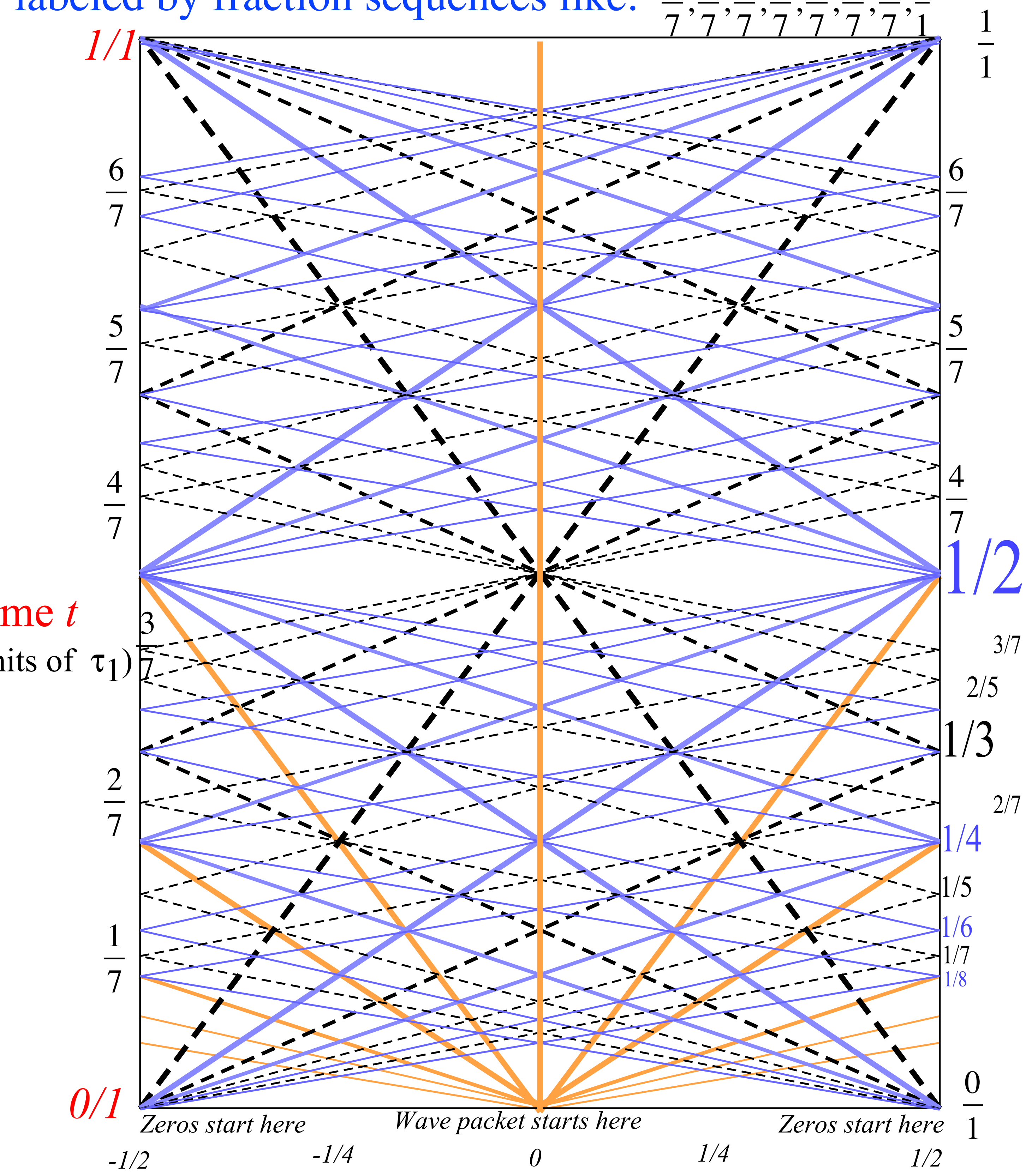
[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

Zeros (clearly) and “particle-packets” (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$

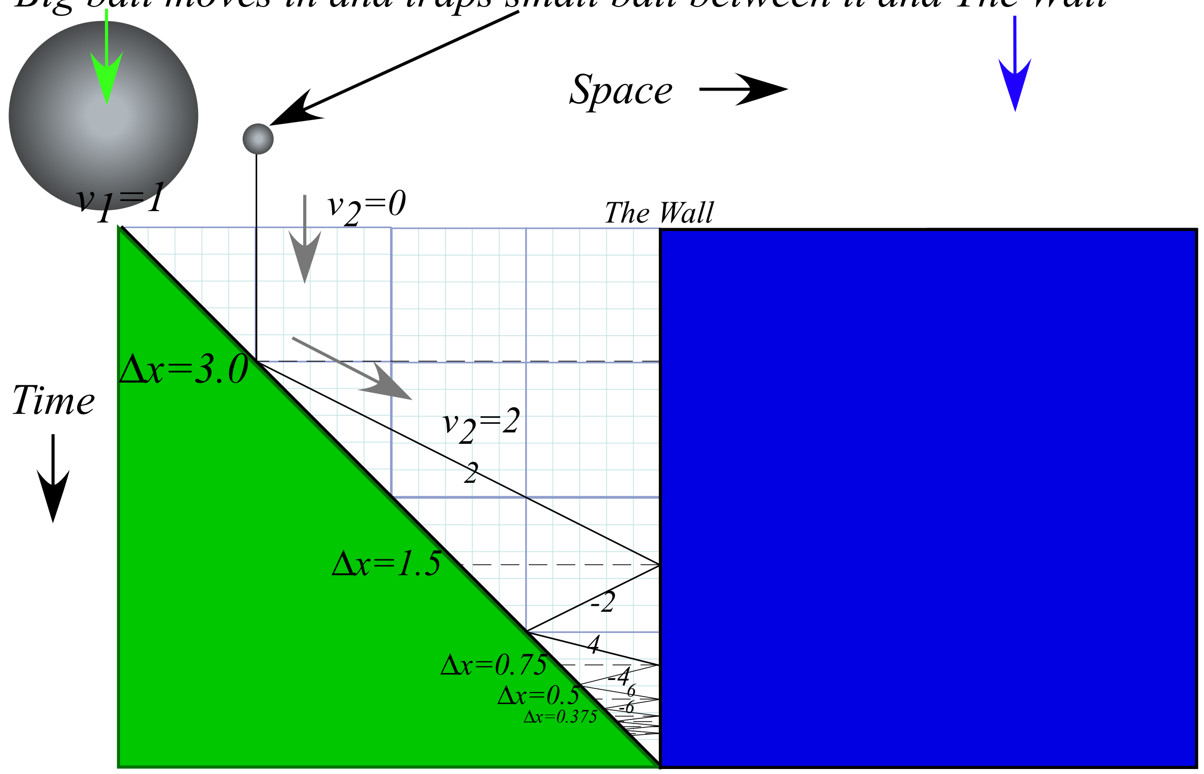




Coordinate ϕ

(units of 2π)

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



(a) Big ball moves in and traps small ball between it and The Wall

Lect. 5 (9.11.14)

The Classical “Monster Mash”

Classical introduction to

Heisenberg “Uncertainty” Relations

$$v_2 = \frac{const.}{Y} \quad \text{or:} \quad Y \cdot v_2 = const.$$

is analogous to: $\Delta x \cdot \Delta p = N \cdot \hbar$

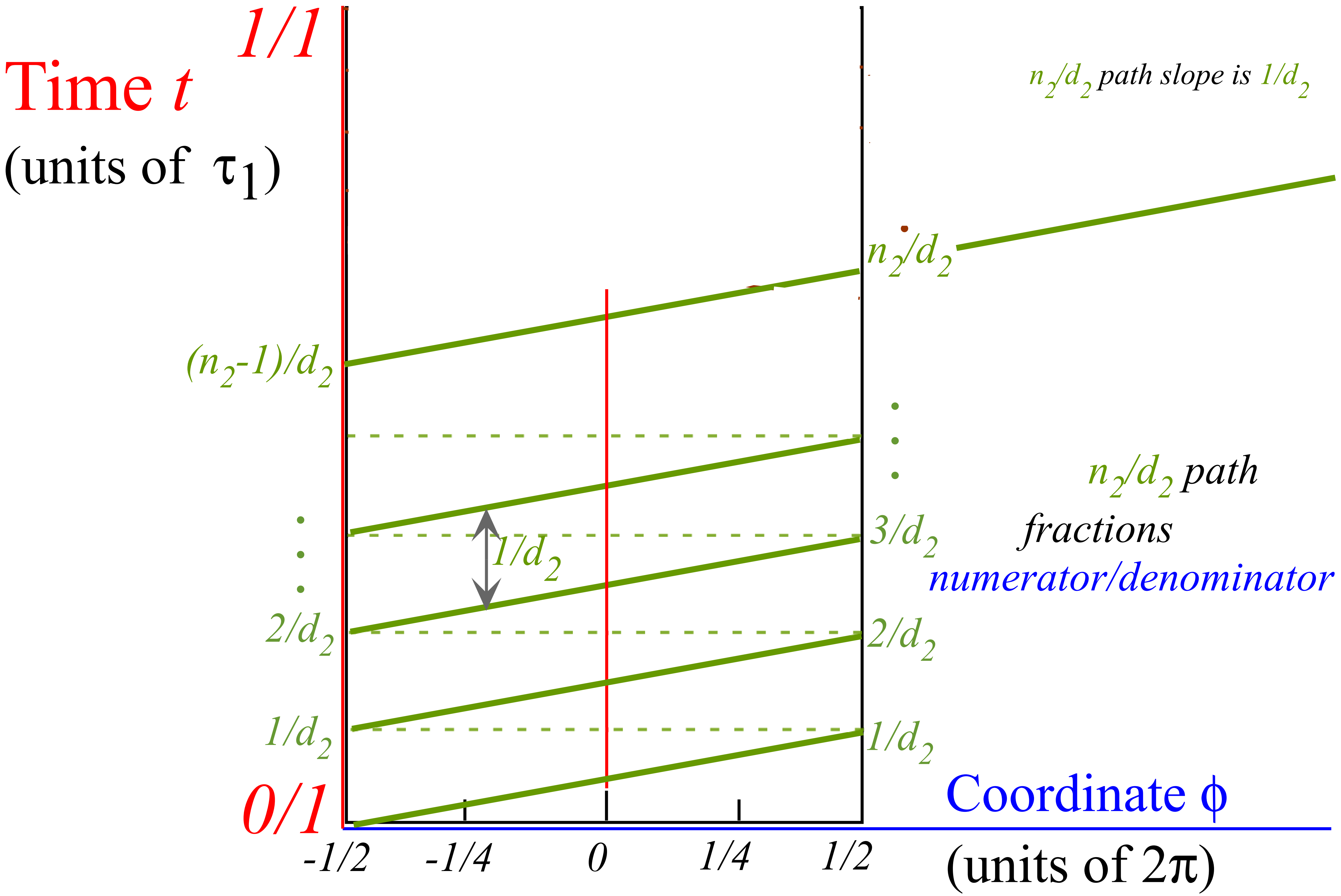
Recall classical “Monster Mash” in Lecture 5

with small-ball trajectory paths having same geometry
as revival beat wave-zero paths

Farey-Sum arithmetic of revival wave-zero paths
(How *Rational Fractions N/D* occupy real space-time)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

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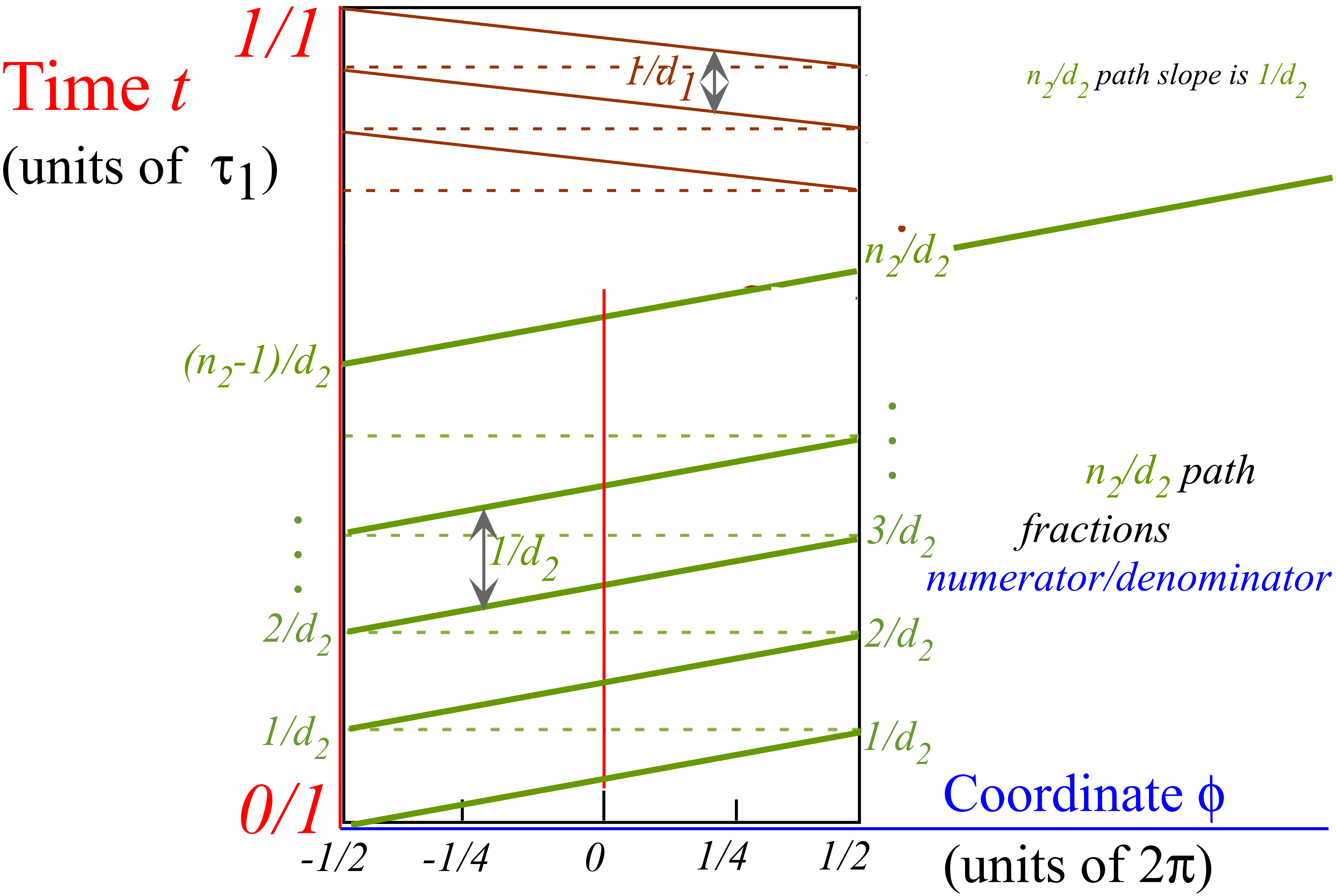

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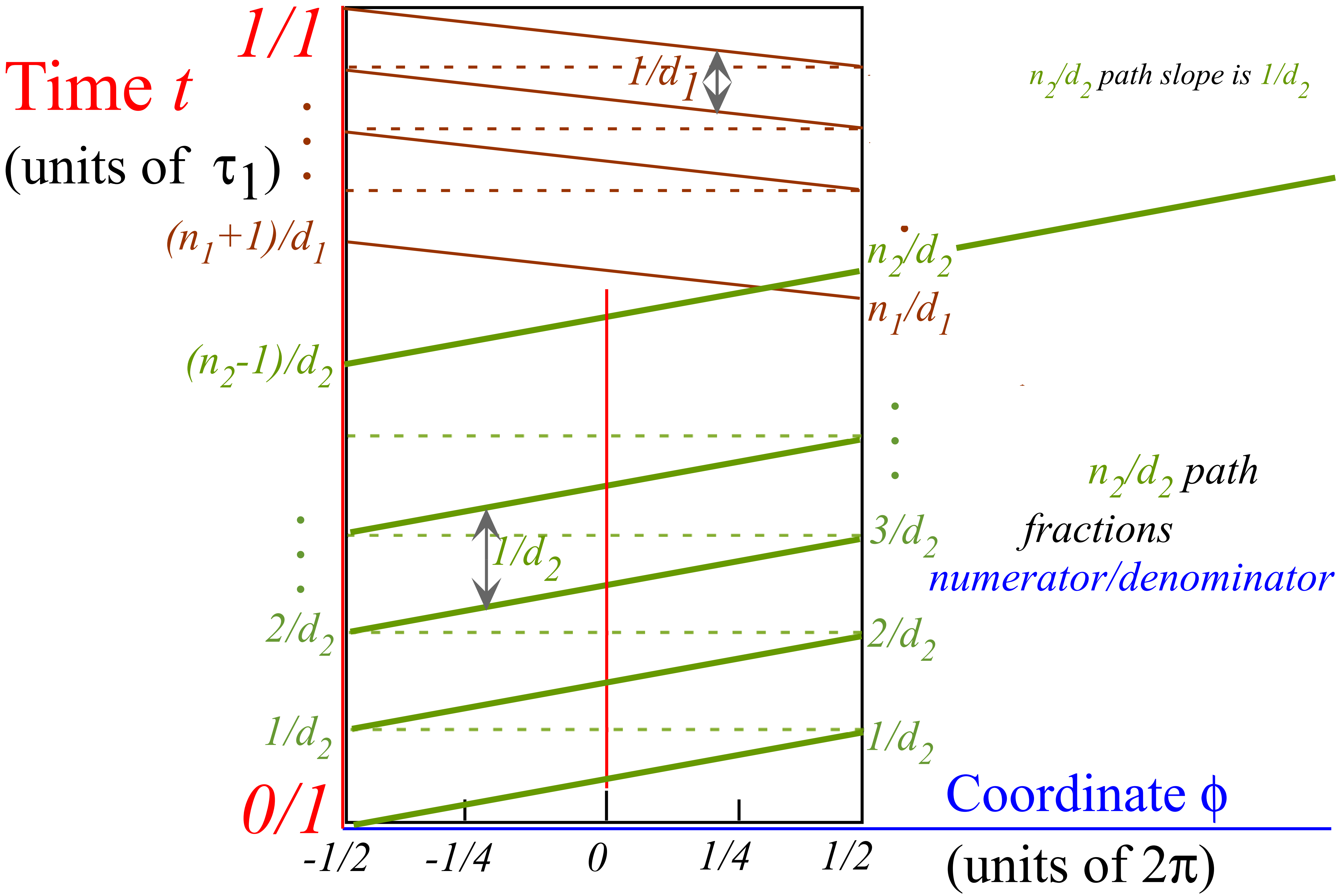

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

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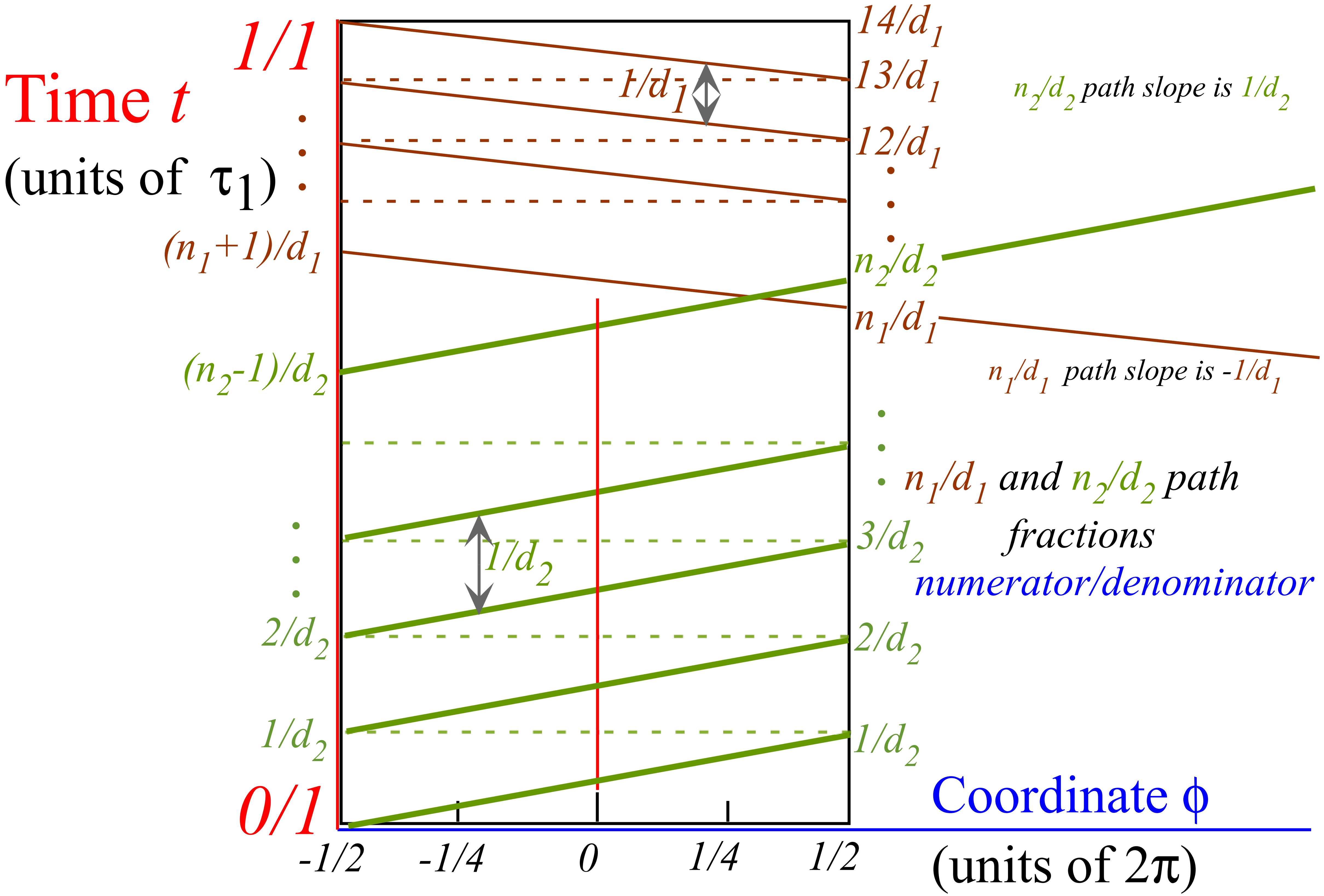

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

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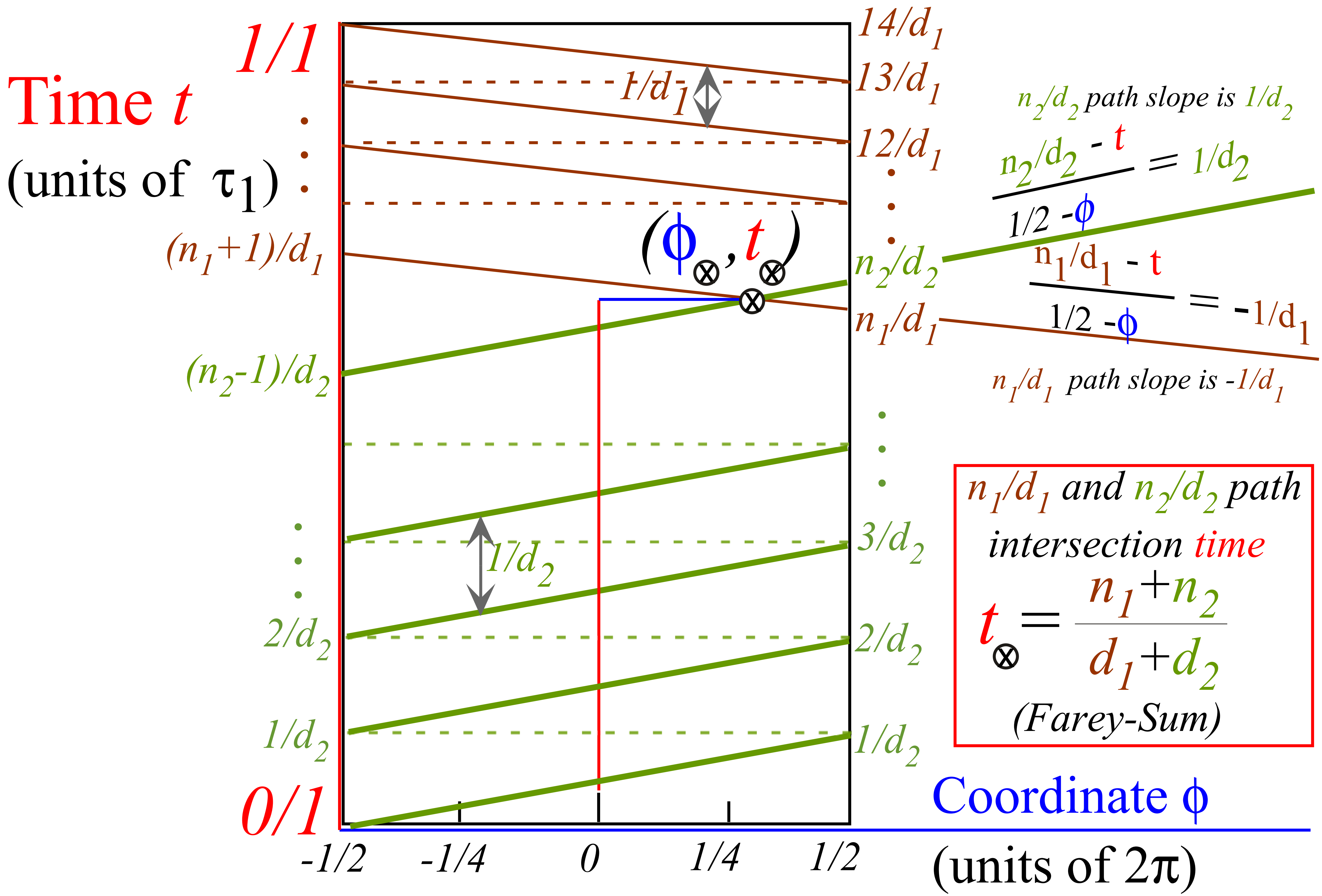


n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1+n_2}{d_1+d_2}$$

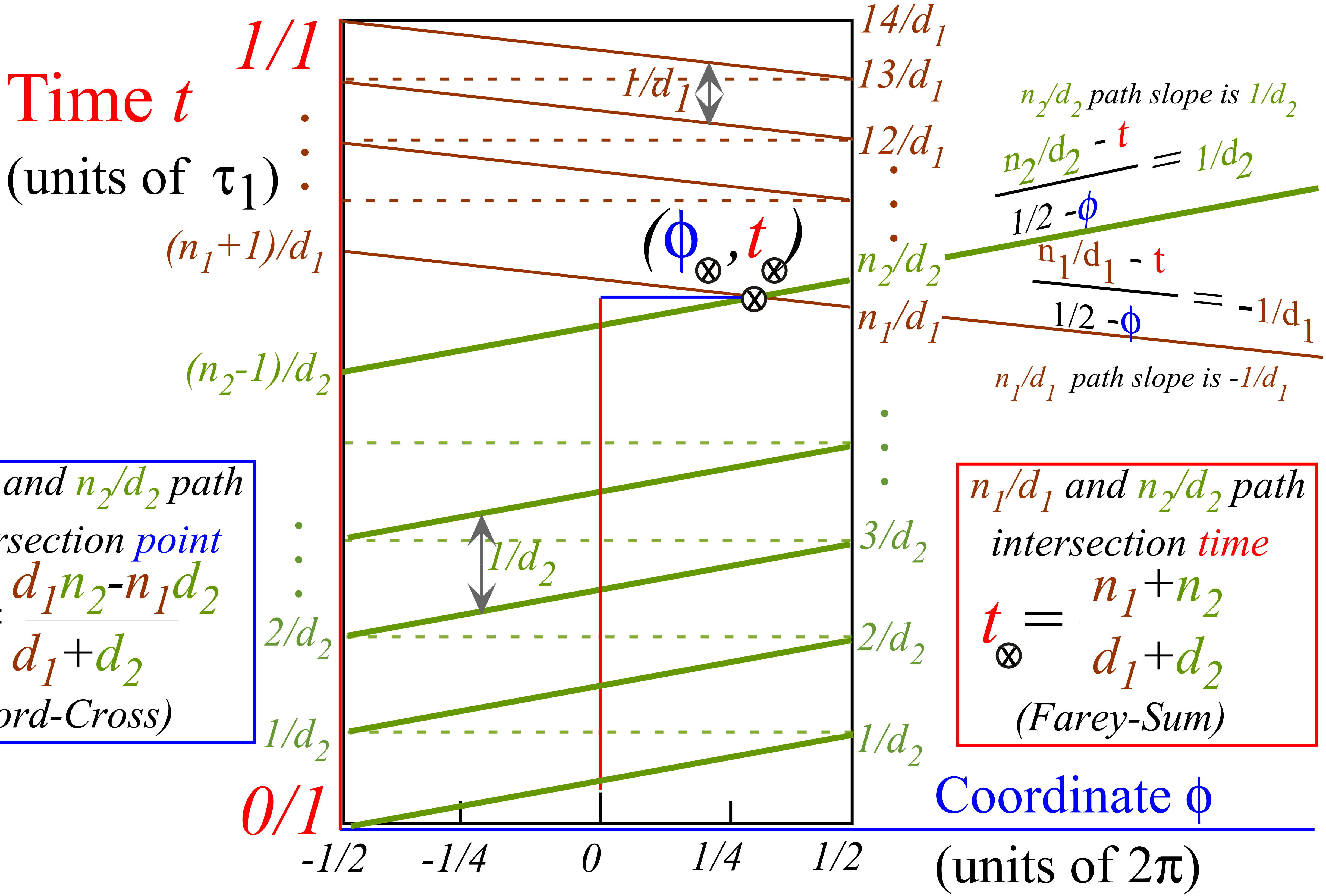
(Farey-Sum)

[John Farey, Phil. Mag.(1816)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

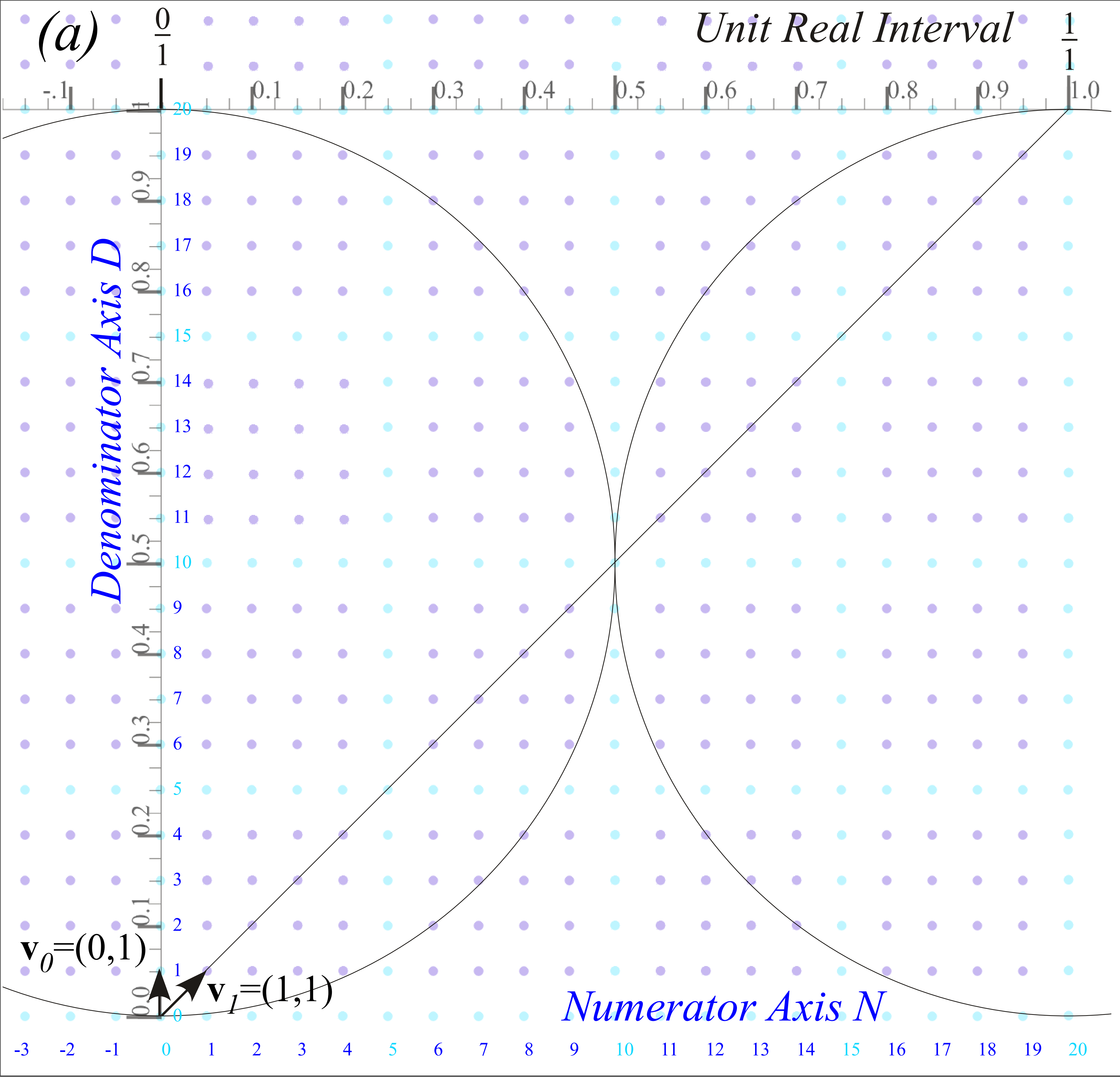
(Farey-Sum)

[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

[John Farey, Phil. Mag.(1816)]

Ford-Circle geometry of revival paths
(How *Rational Fractions N/D* occupy real space-time)



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has
diameter *1*

Harter and Alvason Li
Int. Symposium on
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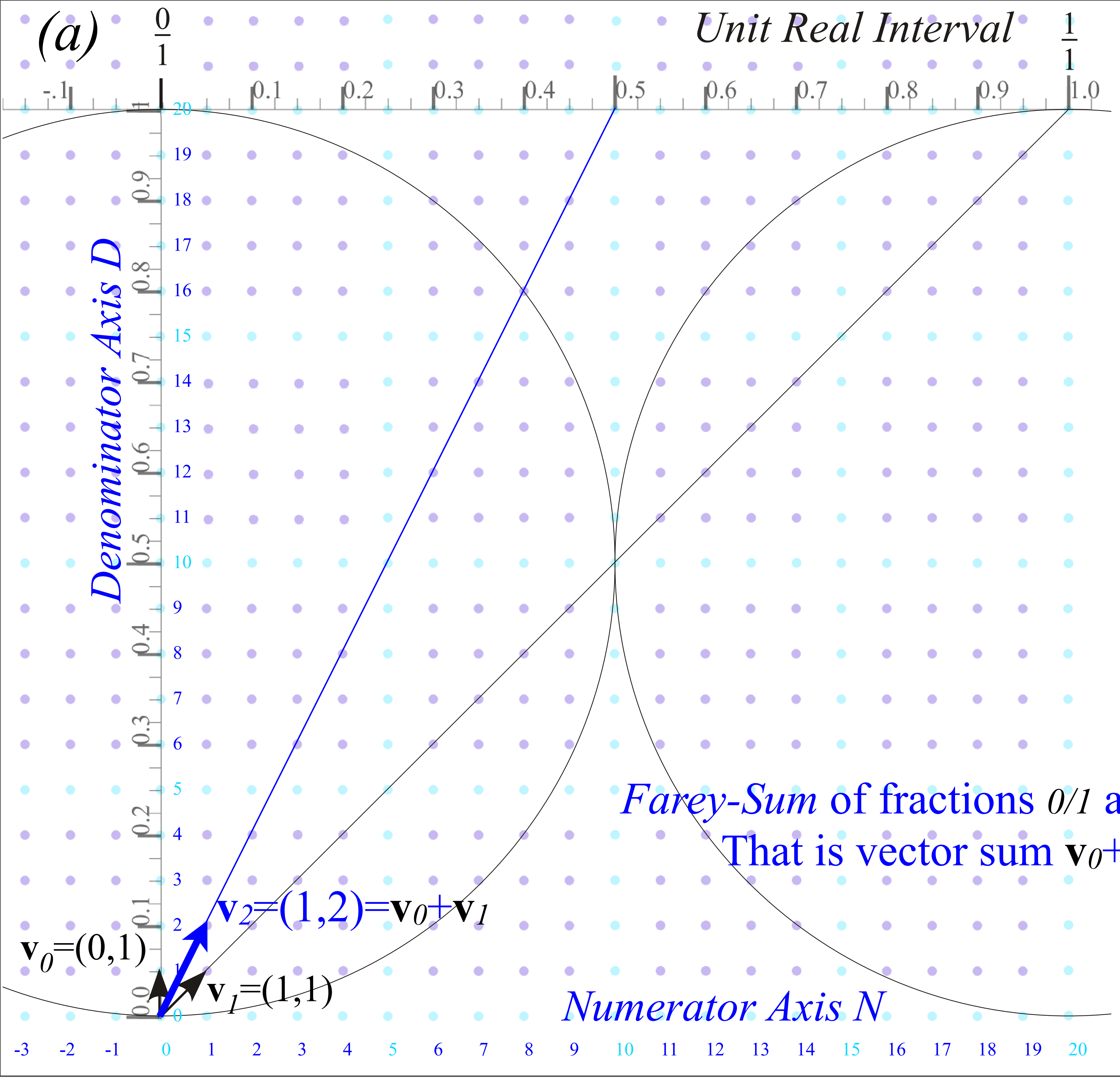
OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

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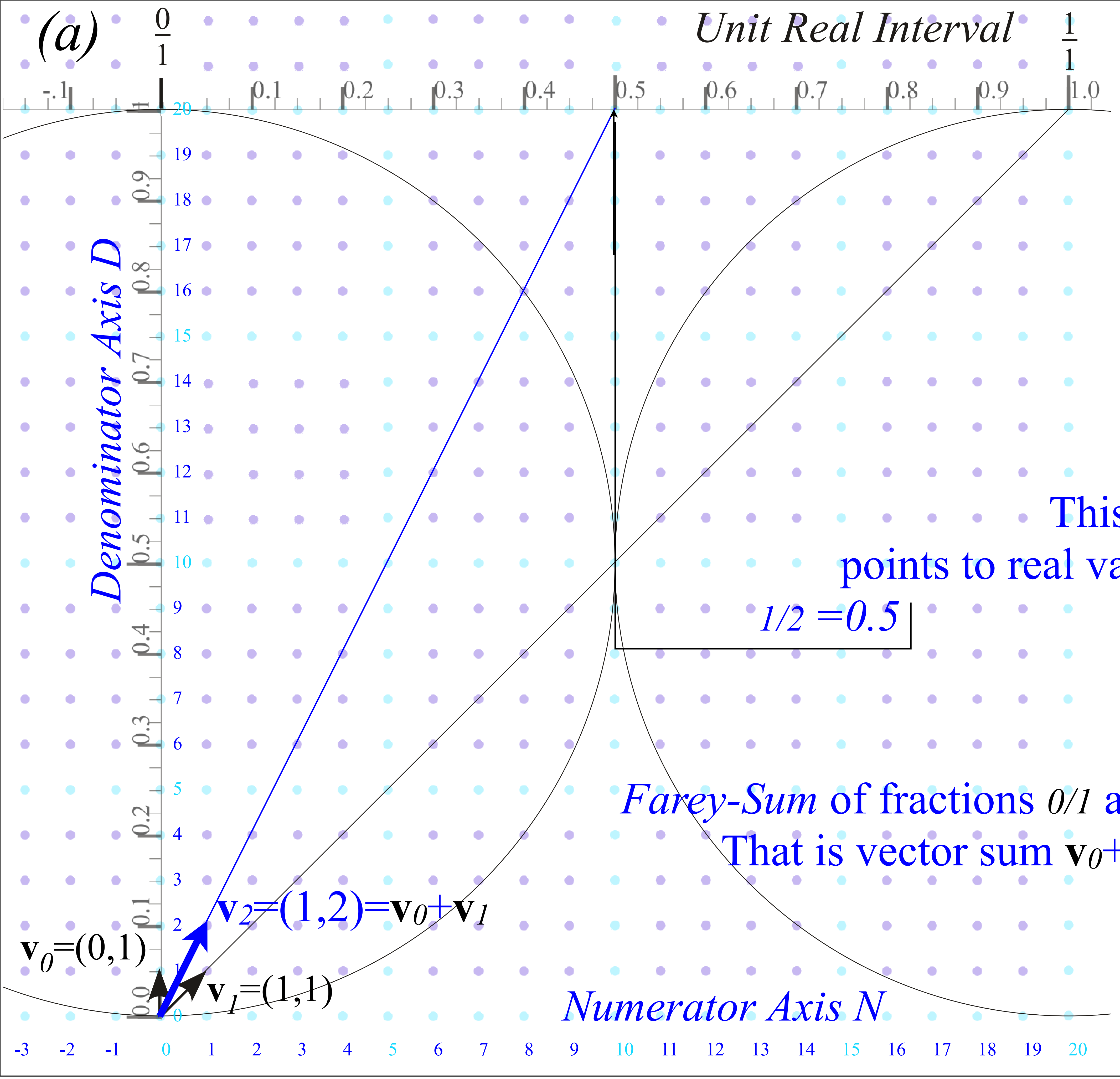

This vector $\mathbf{v}_2$ points to real value *1/2 =0.5*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

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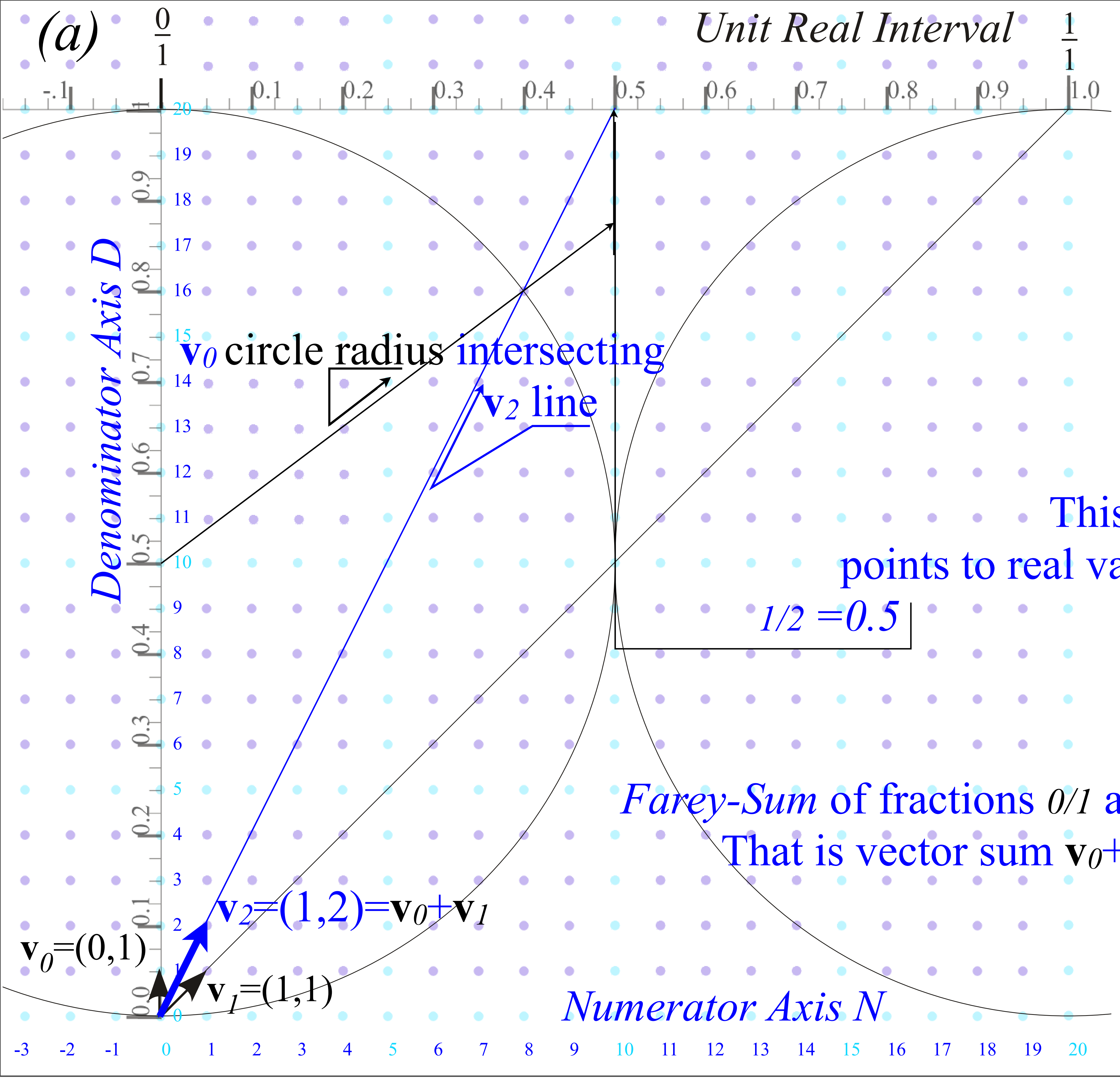

$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line

This vector $\mathbf{v}_2$ points to real value *1/2 =0.5*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$

$\mathbf{v}_0=(0,1)$

$\mathbf{v}_1=(1,1)$

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Farey Sum related to vector sum and *Ford Circles*

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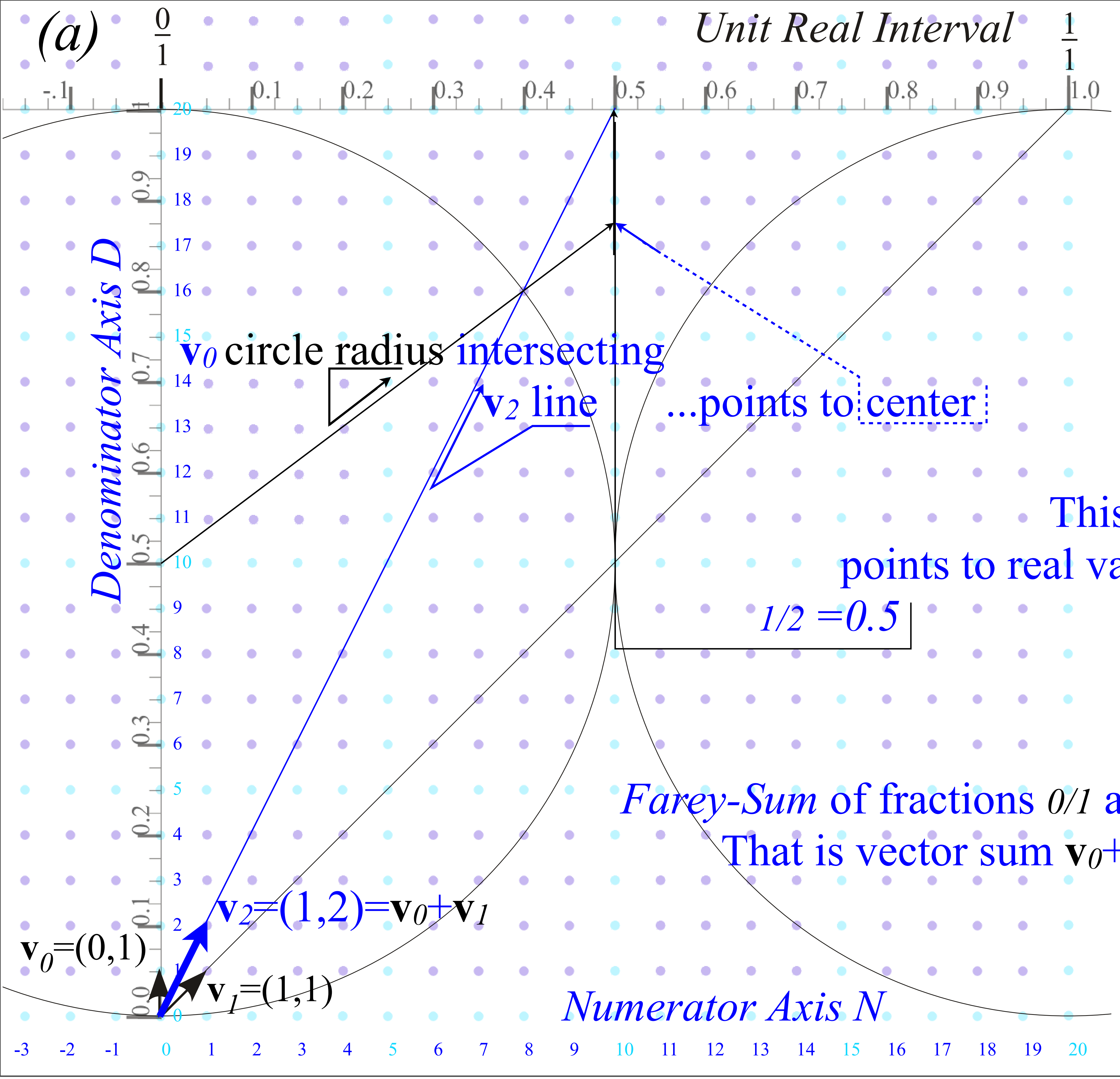

$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line

...points to center

This vector $\mathbf{v}_2$ points to real value *1/2 =0.5*

$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$

$\mathbf{v}_0=(0,1)$

$\mathbf{v}_1=(1,1)$

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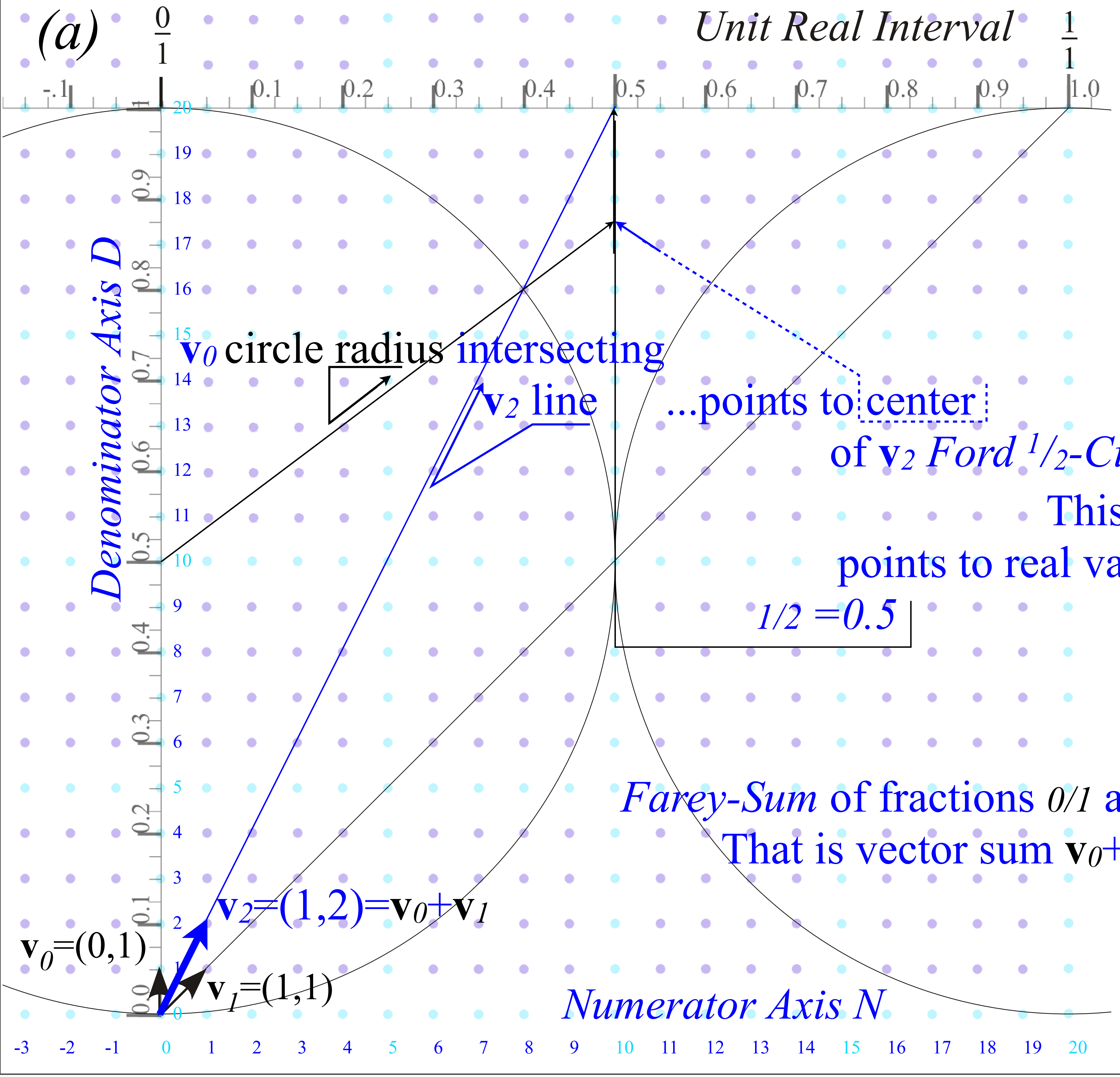
Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*



$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line

...points to center of $\mathbf{v}_2$ *Ford* $^1/_2$-*Circle*

This vector $\mathbf{v}_2$ points to real value *1/2 =0.5*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$

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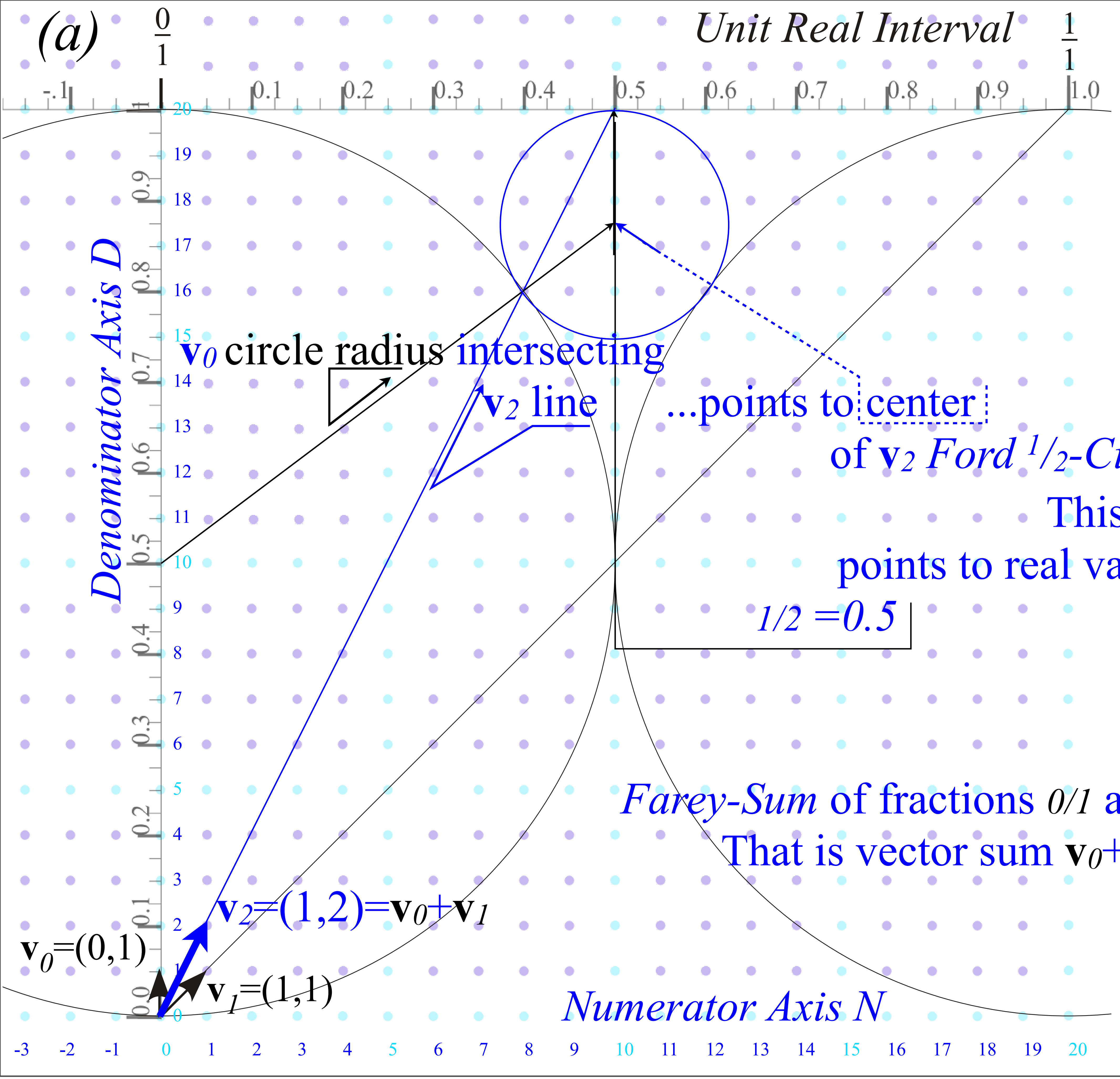
1/1-circle has diameter *1*



$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line

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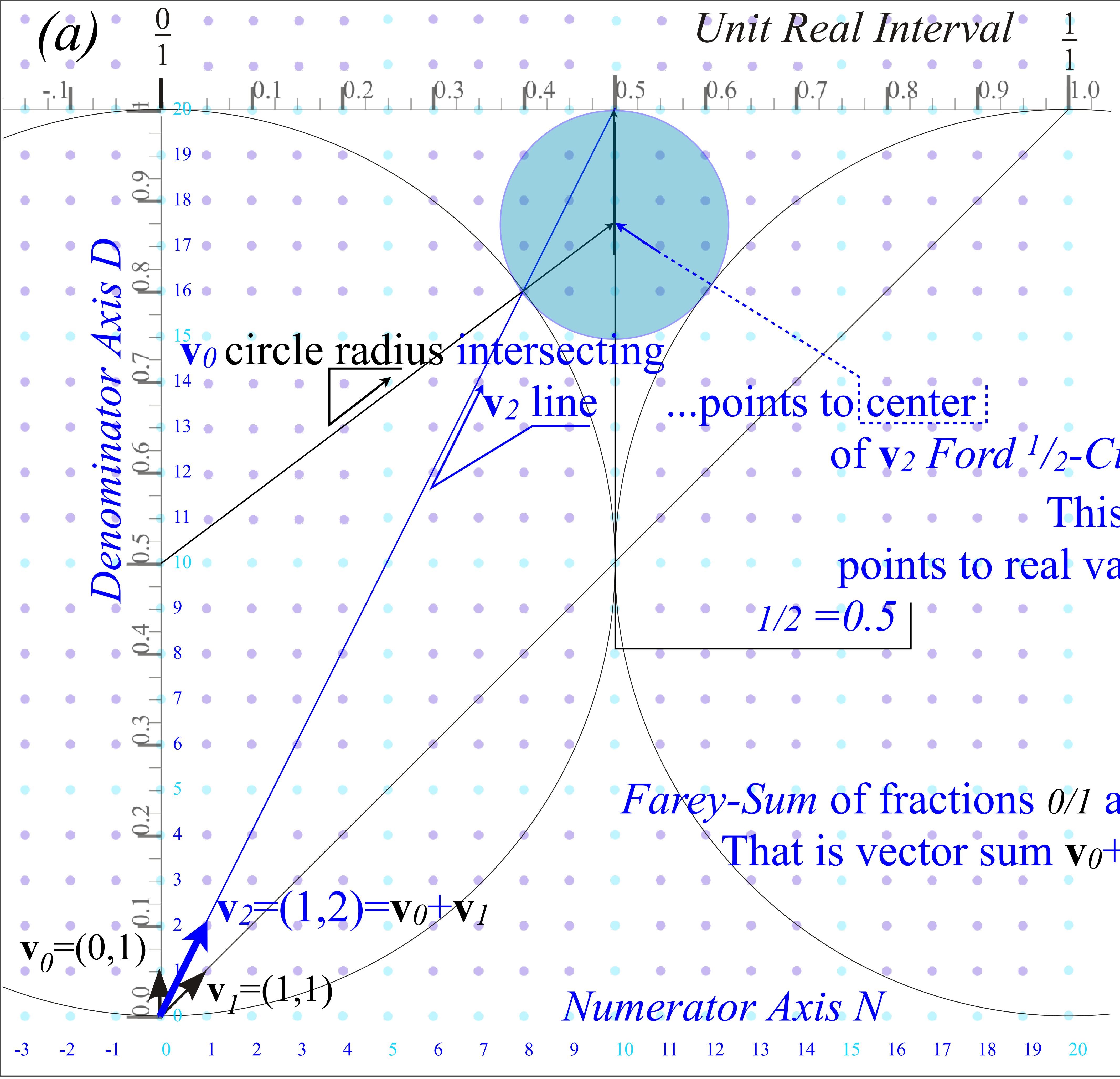

$\mathbf{v}_0$ circle radius intersecting

$\mathbf{v}_2$ line

...points to center of $\mathbf{v}_2$ *Ford* $^1/_2$*-Circle*

This vector $\mathbf{v}_2$ points to real value

1/2 =0.5

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$

$\mathbf{v}_0=(0,1)$

$\mathbf{v}_1=(1,1)$

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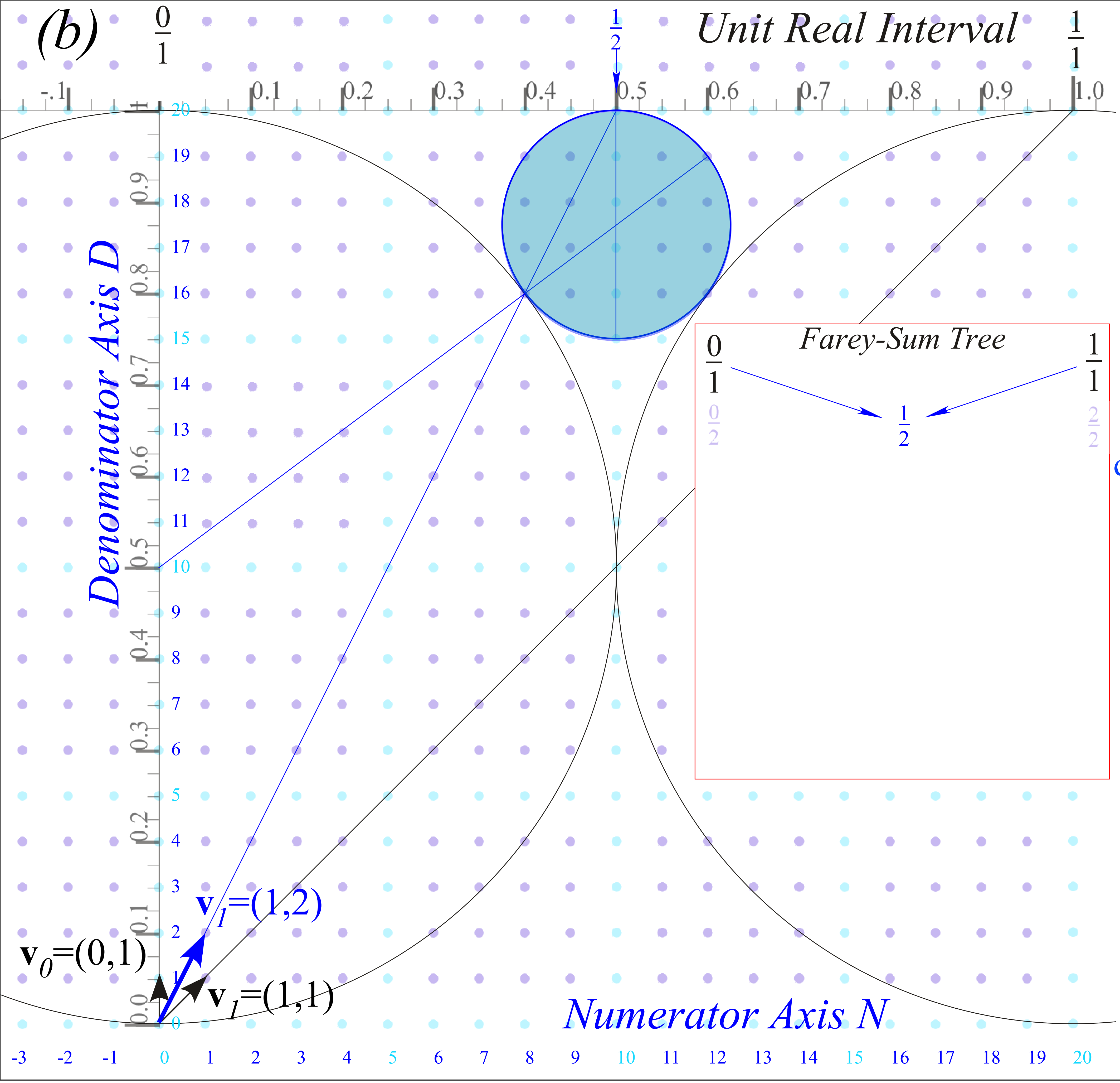
Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

1/2-circle has diameter *1/2*^{2}*=1/4*



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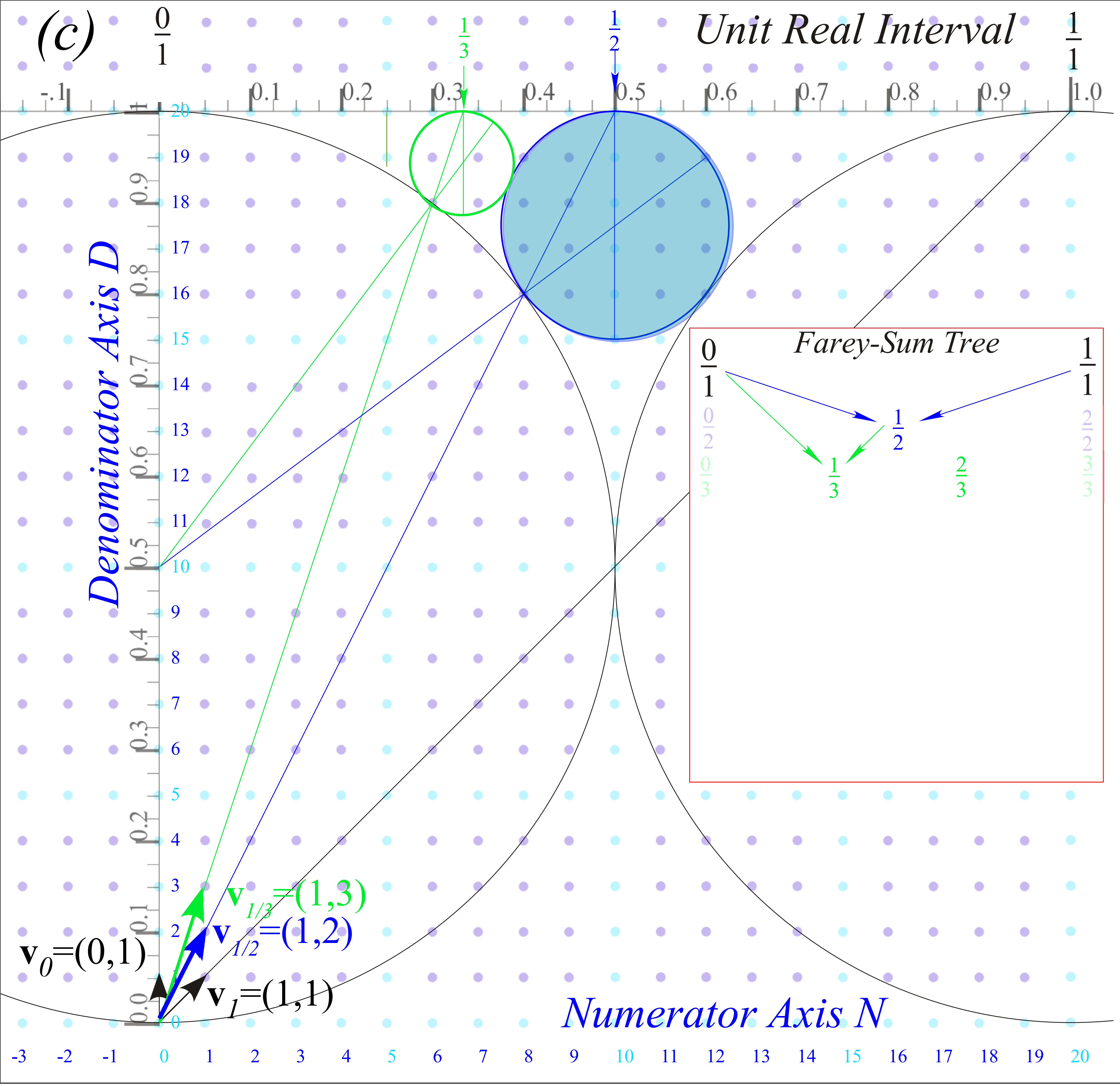

Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

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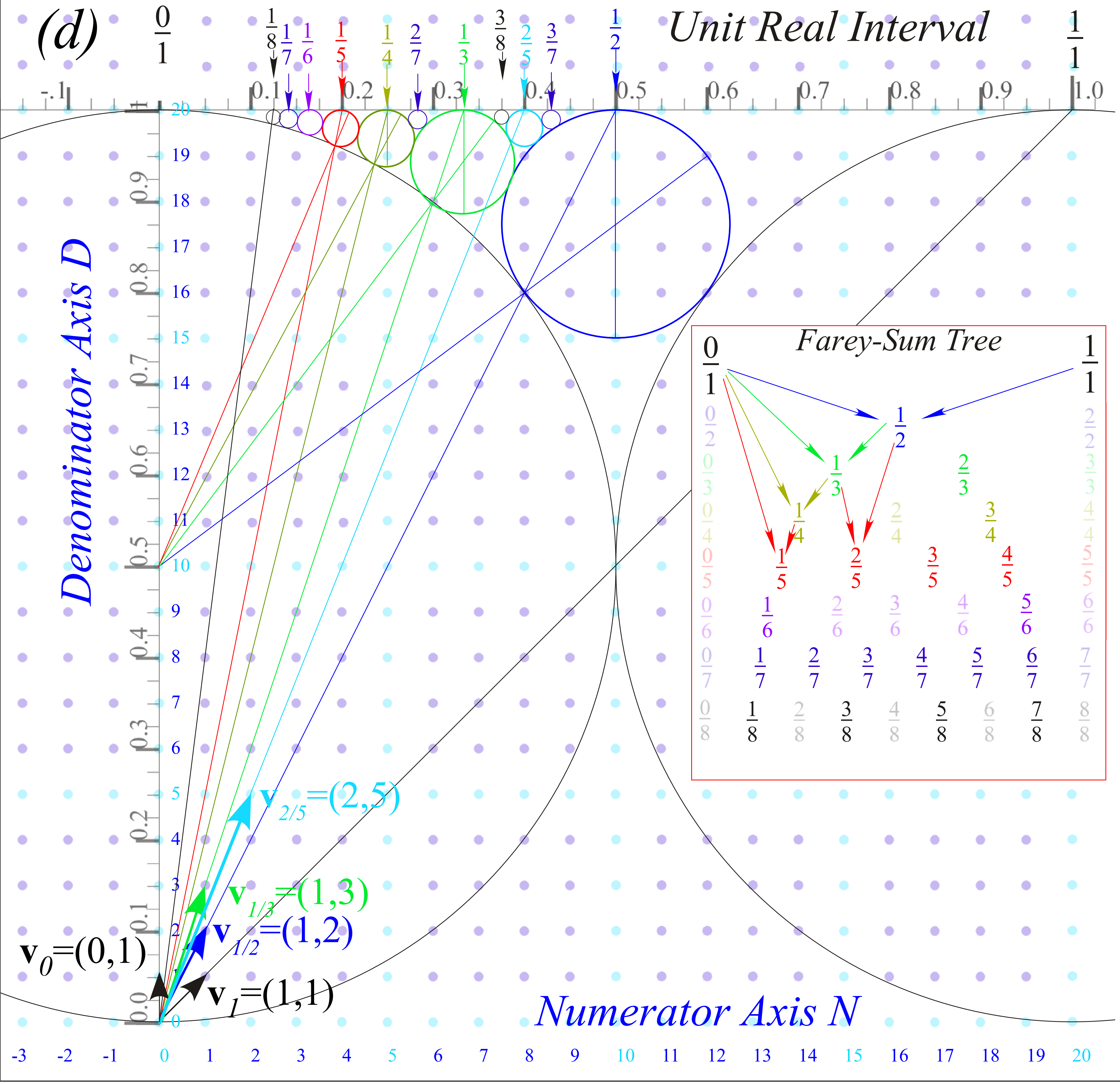
Farey Sum related to vector sum and *Ford Circles*



1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

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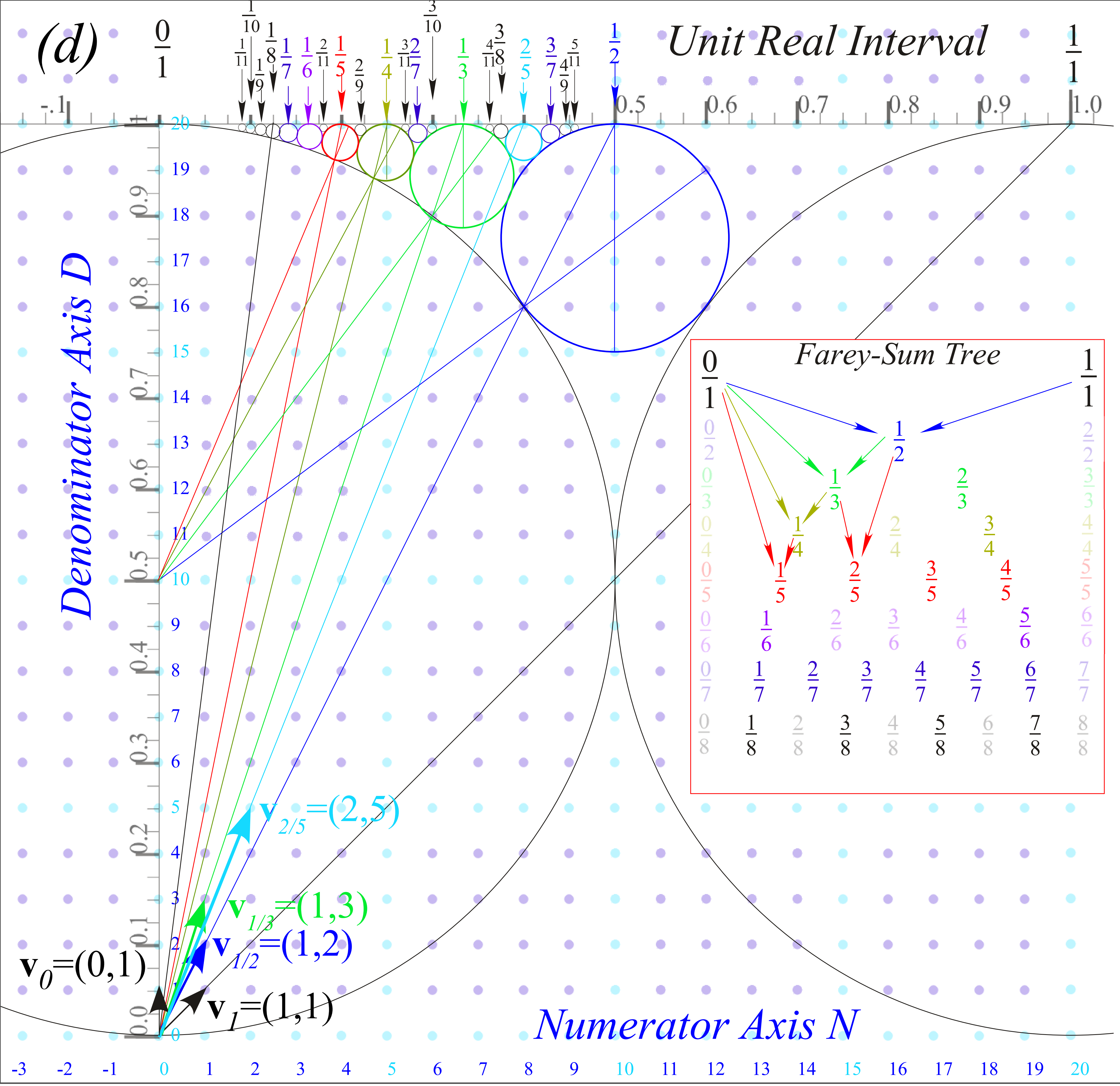
Farey Sum related to vector sum and *Ford Circles*



1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

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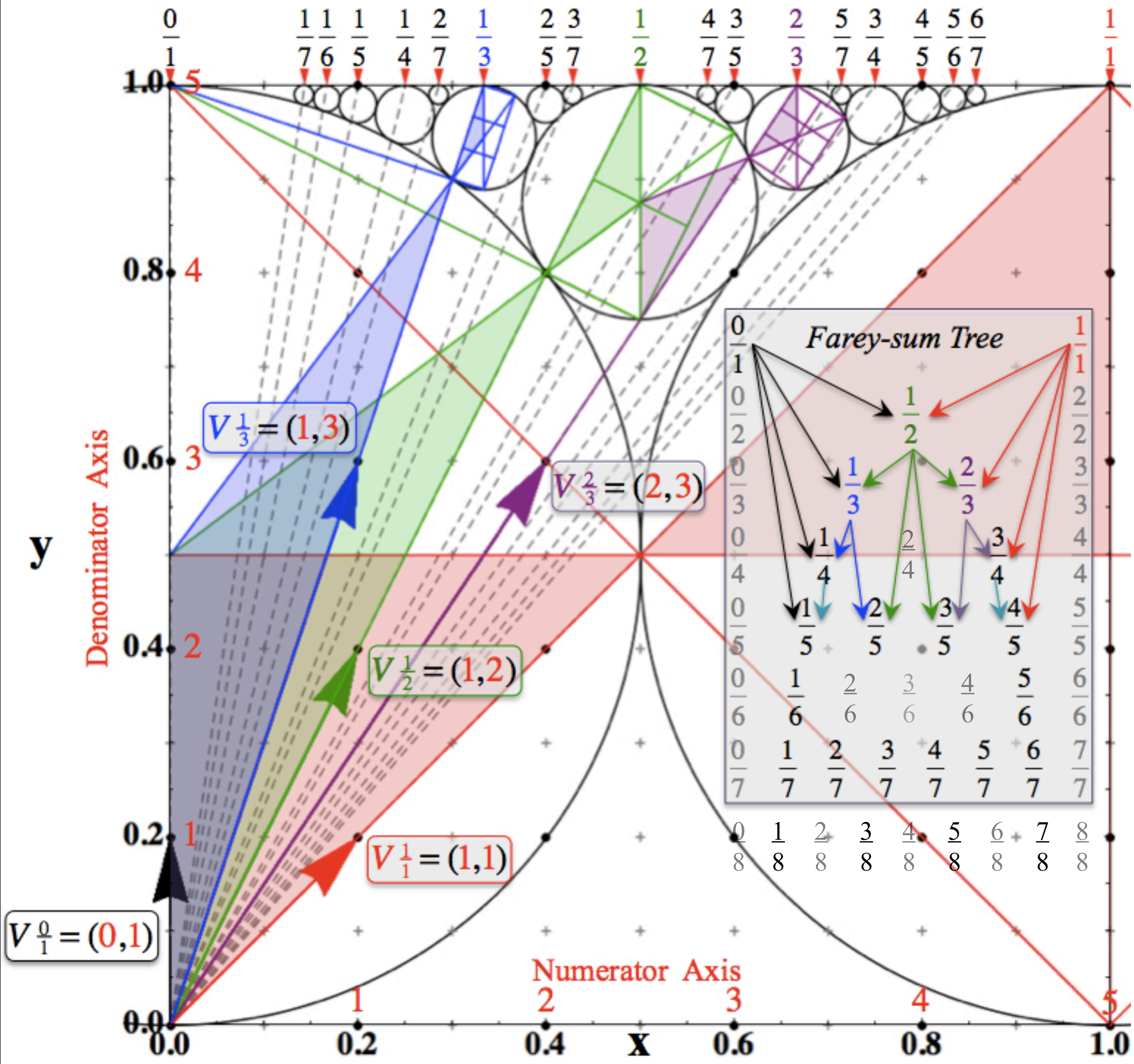

Thales Rectangles

provide
analytic geometry
of
fractal structure

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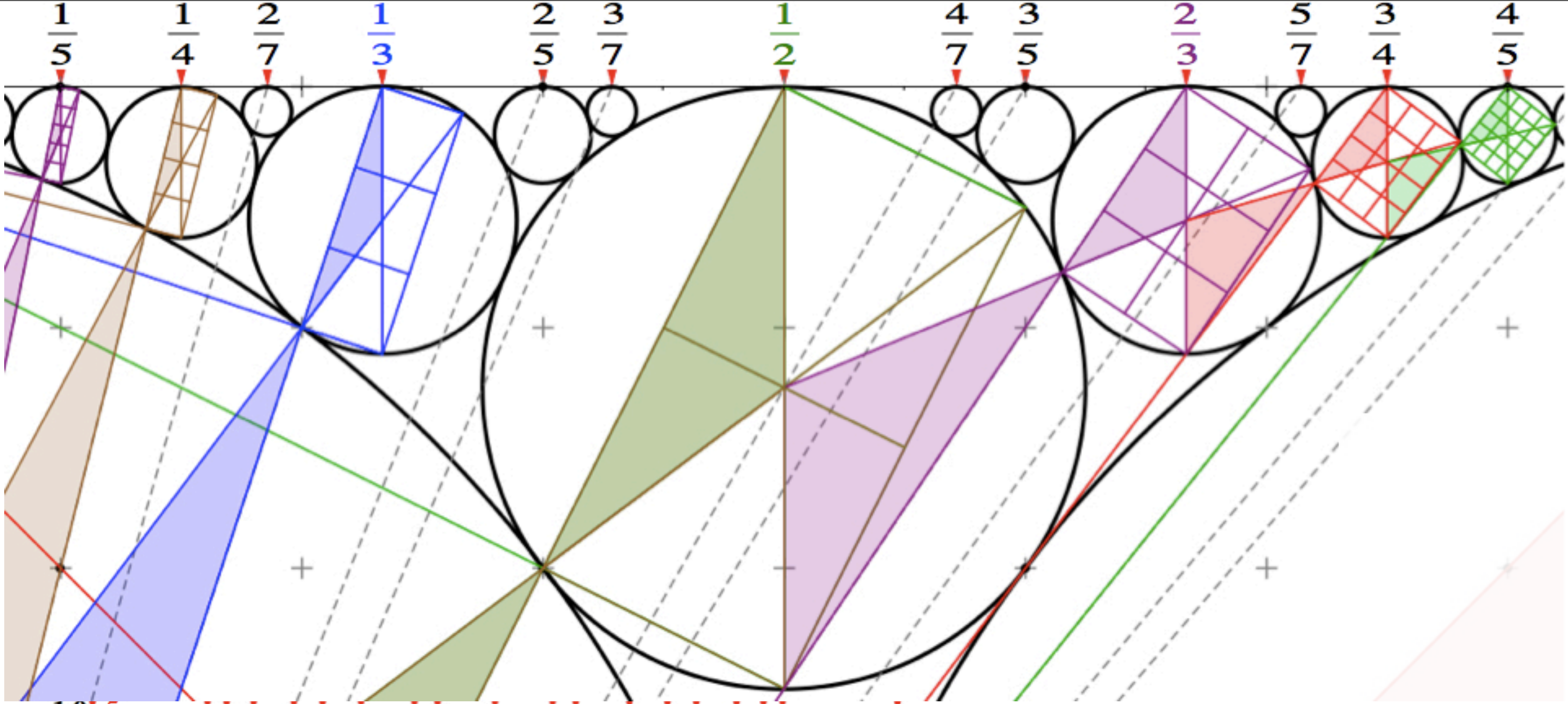
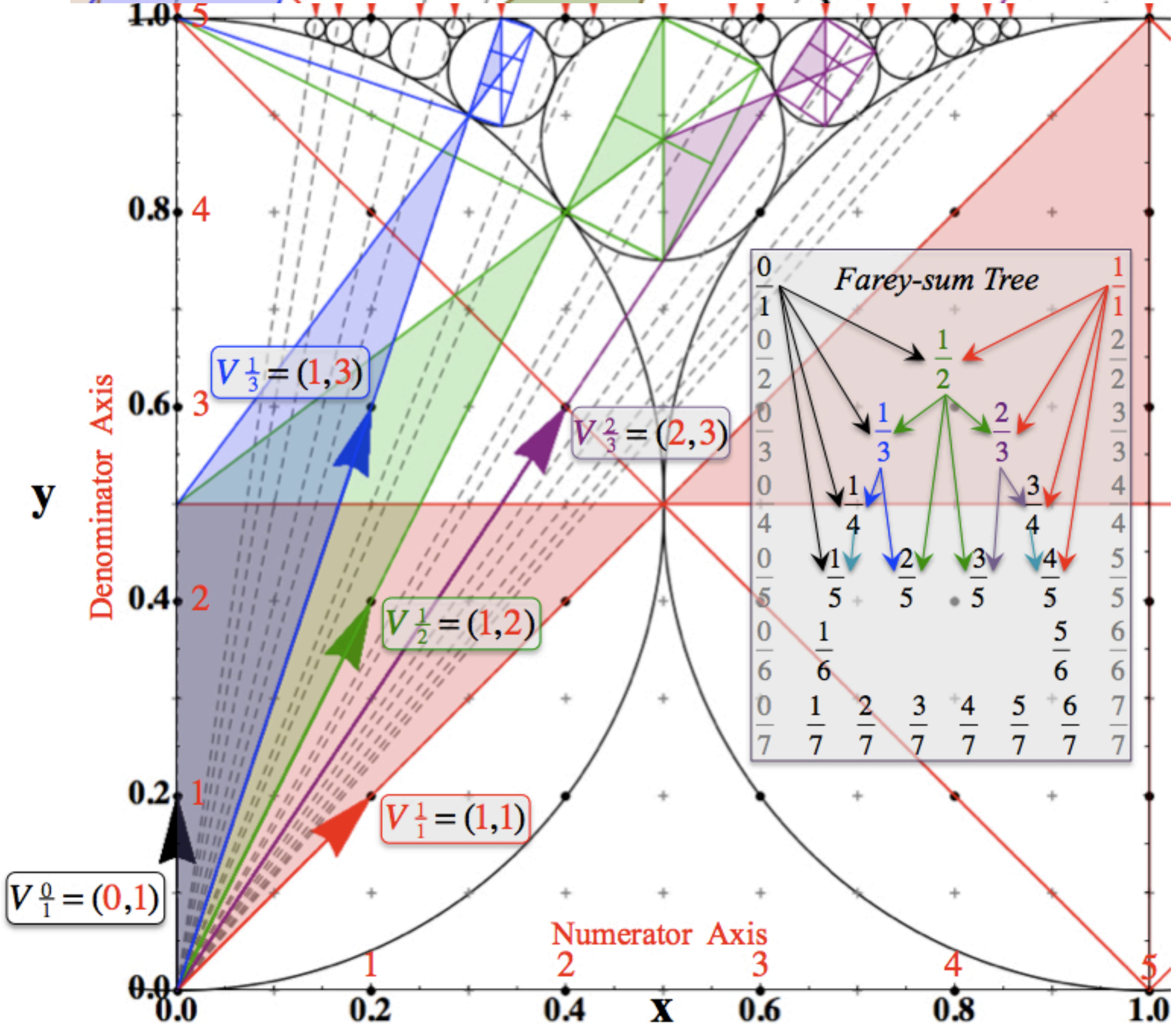
“Quantized” Thales Rectangles

provide
analytic geometry
of
fractal structure





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Relating C_N symmetric H and K matrices to differential wave operators

Relating C_N symmetric H and K matrices to wave differential operators

The 1st neighbor **K** matrix relates to a 2nd *finite-difference* matrix of 2nd x-derivative for high C_N.

$$\mathbf{K} = k(2\mathbf{1} - \mathbf{r} - \mathbf{r}^{-1}) \text{ analogous to:} -k\frac{\partial^2}{\partial x^2}$$

1st derivative momentum: $p = \frac{\hbar}{i}\frac{\partial y}{\partial x} \approx \frac{\hbar}{i}\frac{y(x+\Delta x)-y(x)}{(\Delta x)}$

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \frac{\hbar}{i}\begin{pmatrix} \cdot \\ y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ y_4 - y_3 \\ \cdot \end{pmatrix}$$

2nd derivative KE: $2mE = -\hbar^2\frac{\partial^2 y}{\partial x^2} \approx \frac{y(x+\Delta x)-2y(x)+y(x-\Delta x)}{(\Delta x)^2}$

$$-\hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \end{pmatrix}$$

H and **K** matrix equations are finite-difference versions of quantum and classical wave equations.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathbf{H}|\psi\rangle \quad \textit{(}\mathbf{H}\textit{-matrix equation)}$$

$$-\frac{\partial^2}{\partial t^2}|y\rangle = \mathbf{K}|y\rangle \quad \textit{(}\mathbf{K}\textit{-matrix equation)}$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)|\psi\rangle \quad \textit{(Scrodinger equation)}$$

$$-\frac{\partial^2}{\partial t^2}|y\rangle = -k\frac{\partial^2}{\partial x^2}|y\rangle \quad \textit{(Classical wave equation)}$$

Square p^2 gives 1st neighbor **K** matrix. Higher order p^3, p^4,.. involve 2nd, 3rd, 4th..neighbor **H**

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$p^4 \cong \begin{pmatrix} \ddots & \vdots & 1 & 0 & & \\ \cdots & 6 & -4 & 1 & 0 & \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ & 0 & 1 & -4 & 6 & -4 \\ & & 0 & 1 & -4 & 6 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\overline{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \ \overline{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\overline{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \overline{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$