

Lecture 2
Thur. 8.28.2014

Analysis of 1D 2-Body Collisions
(Ch. 3 to Ch. 5 of Unit 1)

Review of elastic Kinetic Energy ellipse geometry

The X2 Superball pen launcher

Perfectly elastic “ka-bong” velocity amplification effects (Faux-Flubber)

Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

Geometric optimization and range-of-motion calculation(s)

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Review of elastic Kinetic Energy ellipse geometry

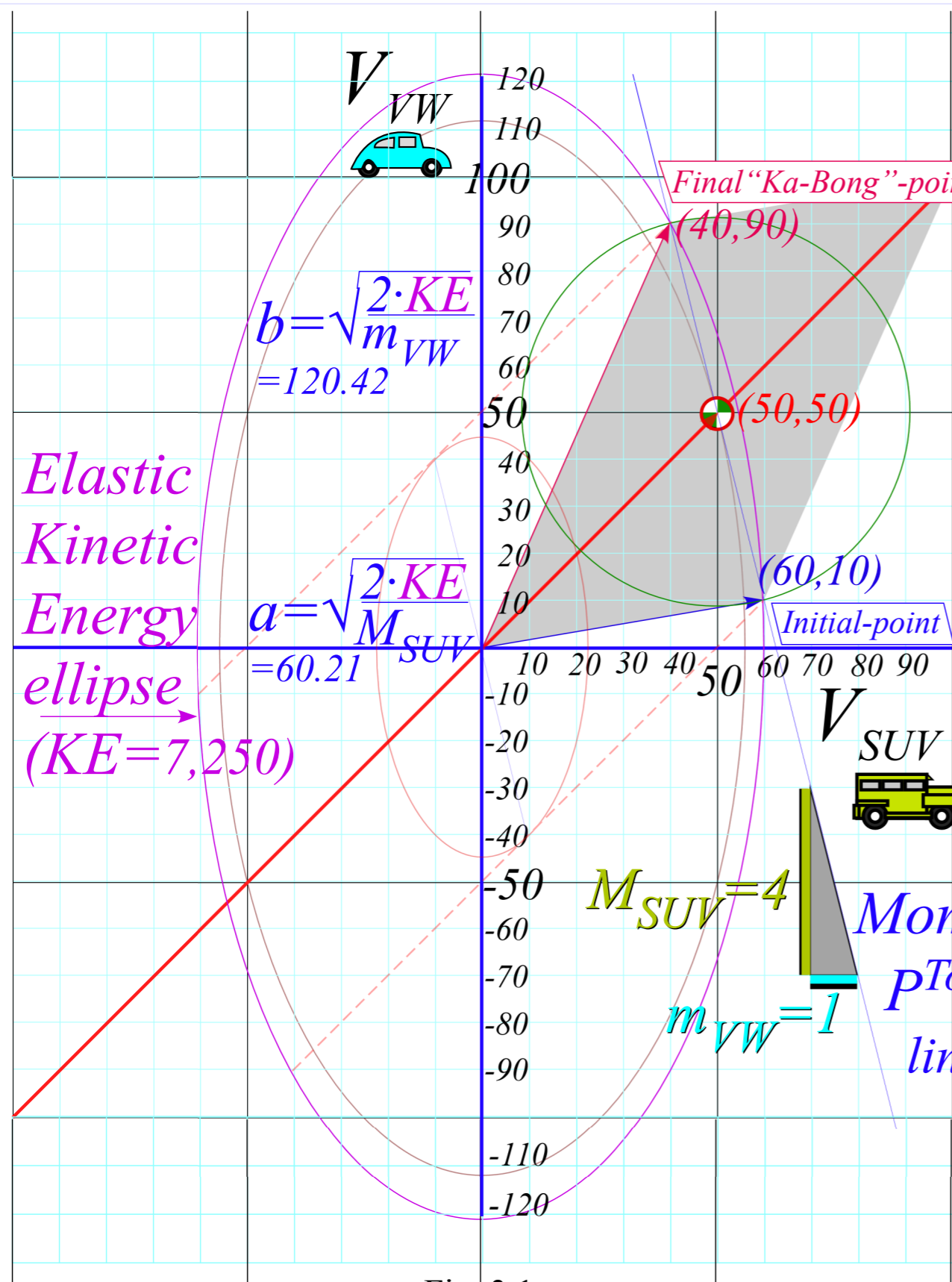


Fig. 3.1 a
in Unit 1

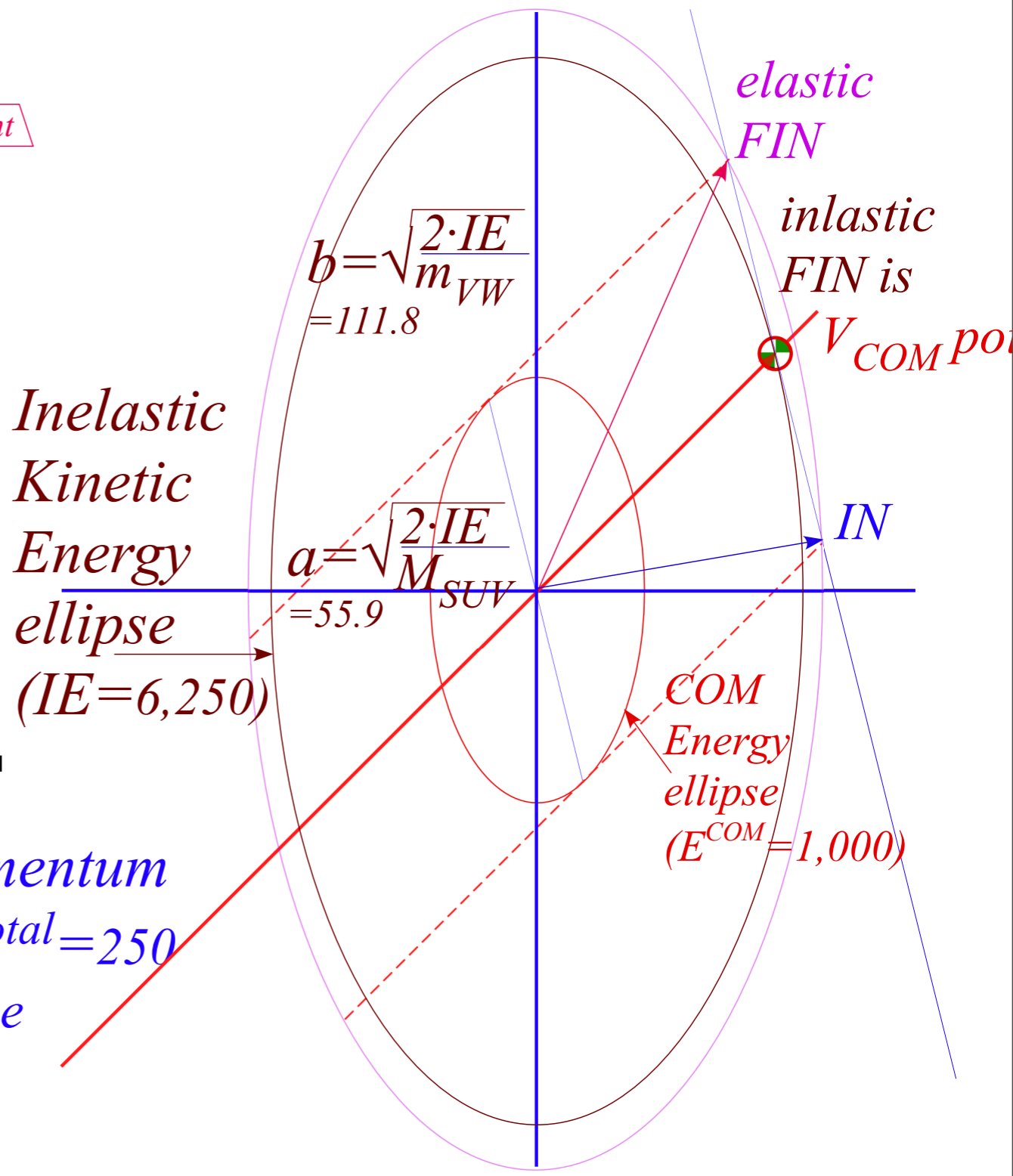
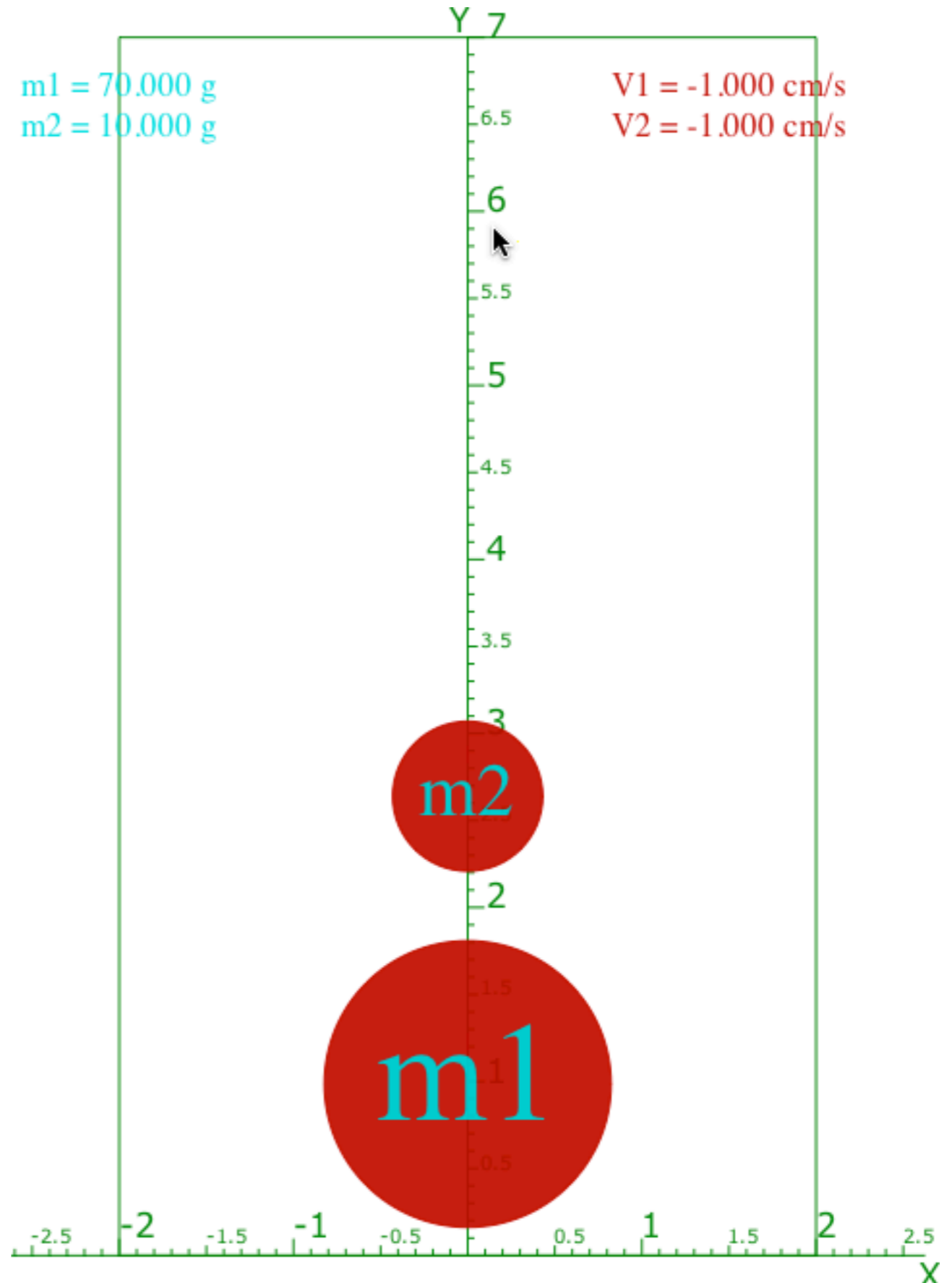
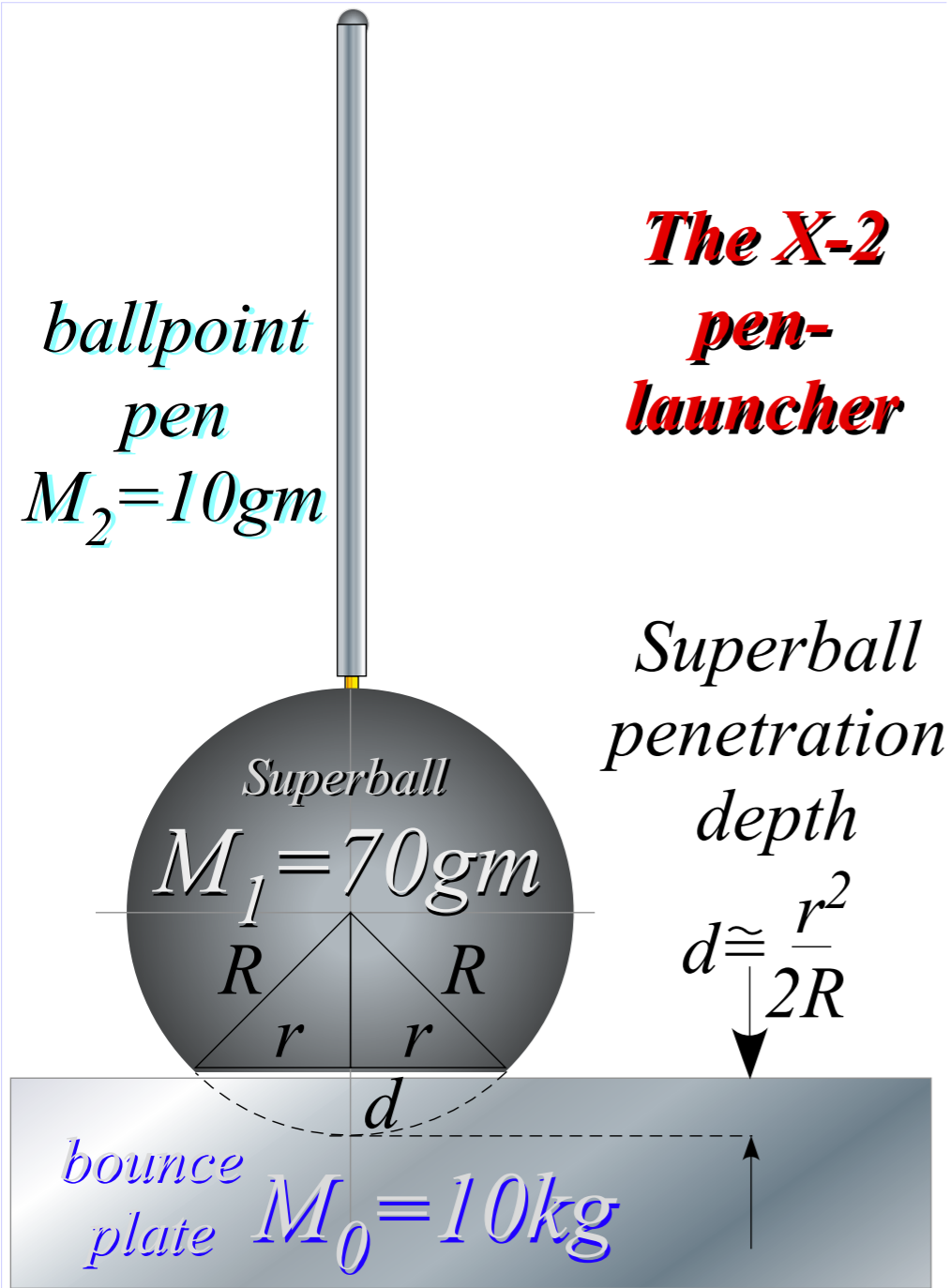


Fig. 3.1 b
in Unit 1

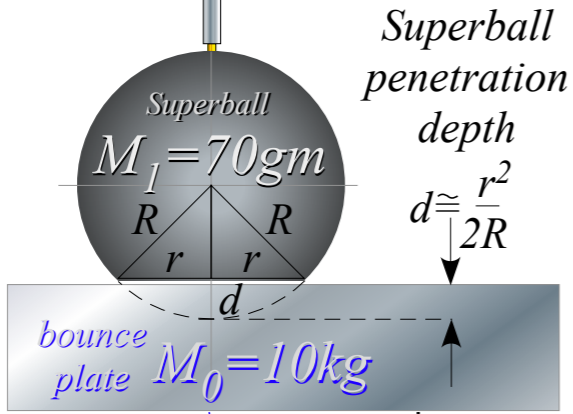
The X-2 Pen launcher and Superball Collision Simulator*



**Simulator Website:* <http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html>

ballpoint pen
 $M_2=10gm$

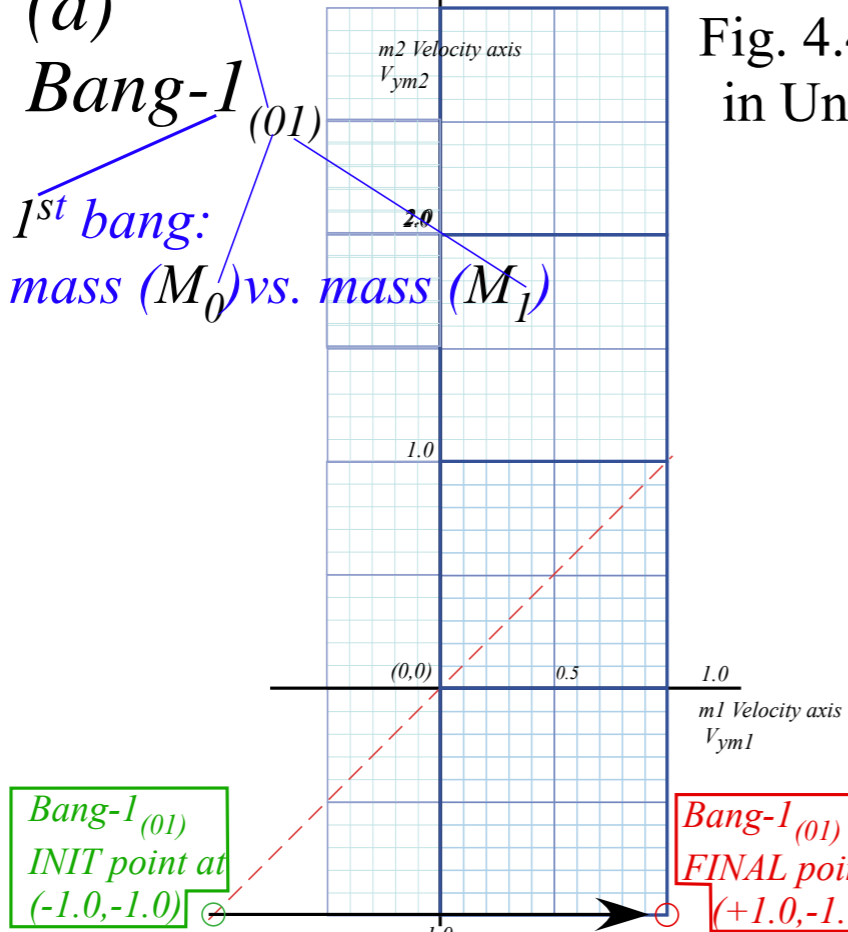
The X-2 pen-launcher



(a)

Bang-1
 (01)

1st bang:
 mass (M_0) vs. mass (M_1)



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 4.1 and Fig. 4.3 in Unit 1

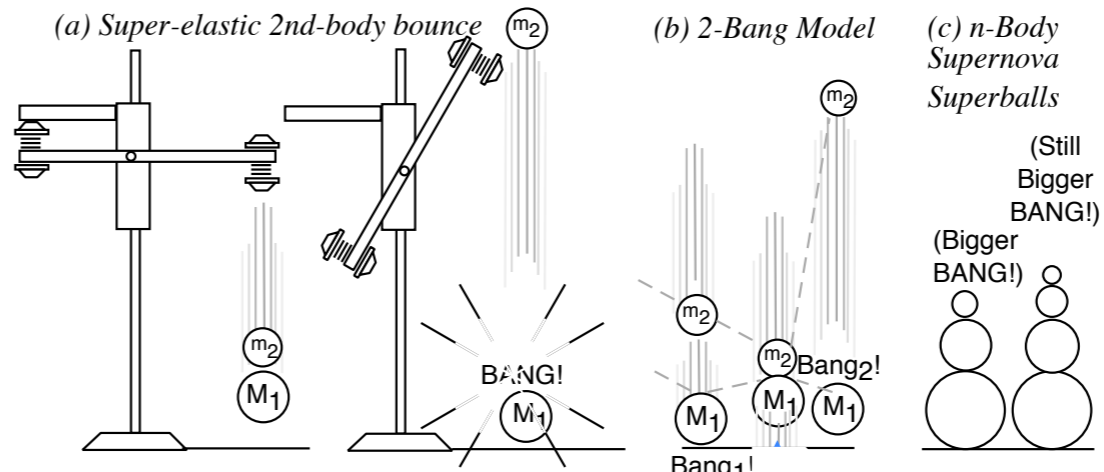
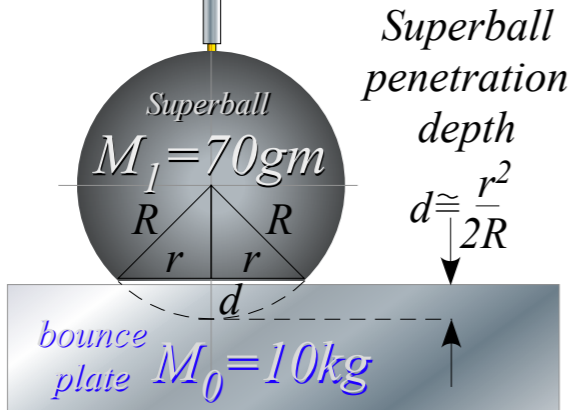


Fig. 4.4a-b in Unit 1

1st bang:
 M_1 off floor

ballpoint pen
 $M_2 = 10\text{gm}$

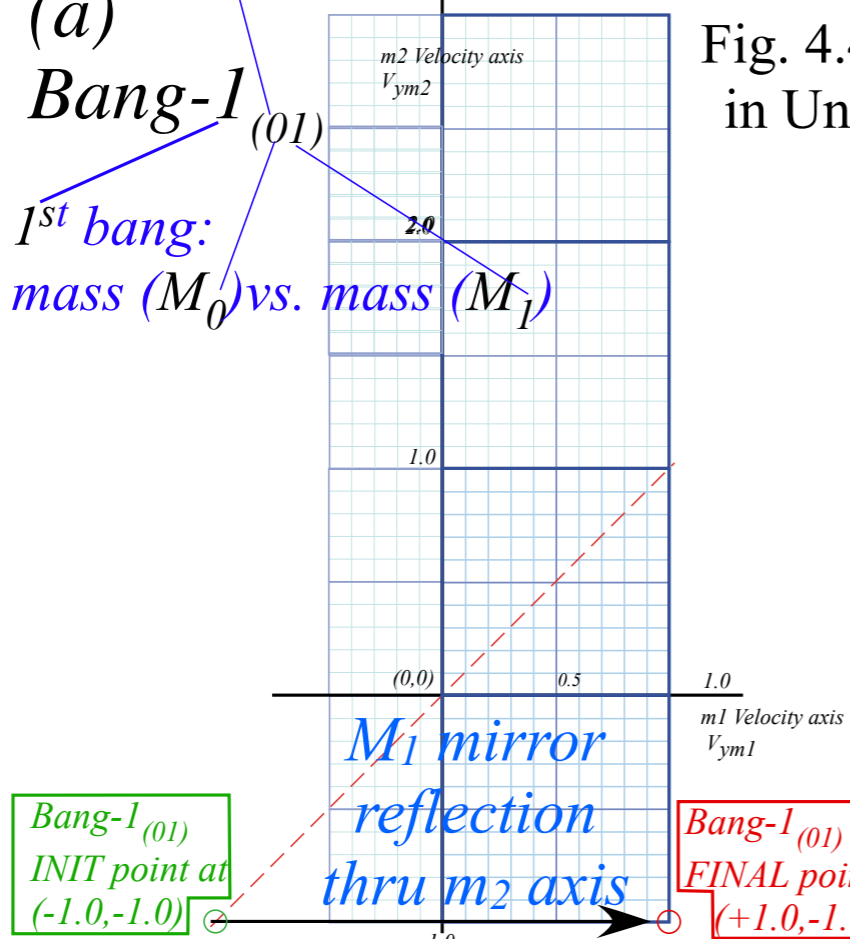
The X-2 pen-launcher



(a)

Bang-1 (01)

1st bang:
 mass (M_0) vs. mass (M_1)



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 4.1 and Fig. 4.3 in Unit 1

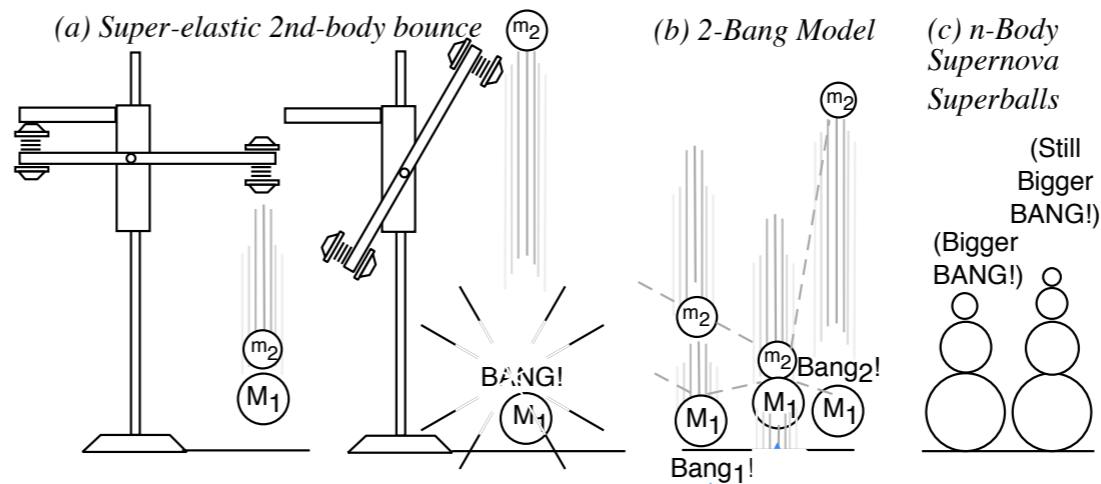
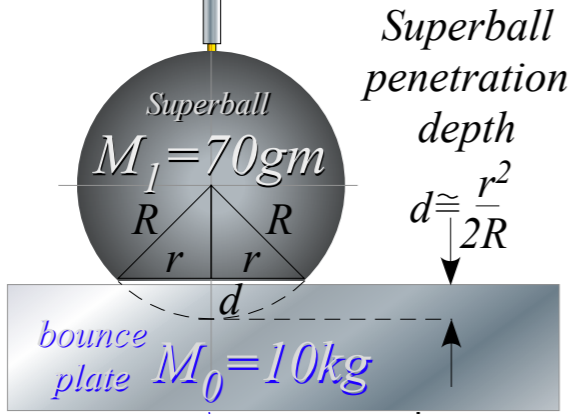


Fig. 4.4a-b in Unit 1

1st bang:
 M_1 off floor

ballpoint pen
 $M_2 = 10\text{gm}$

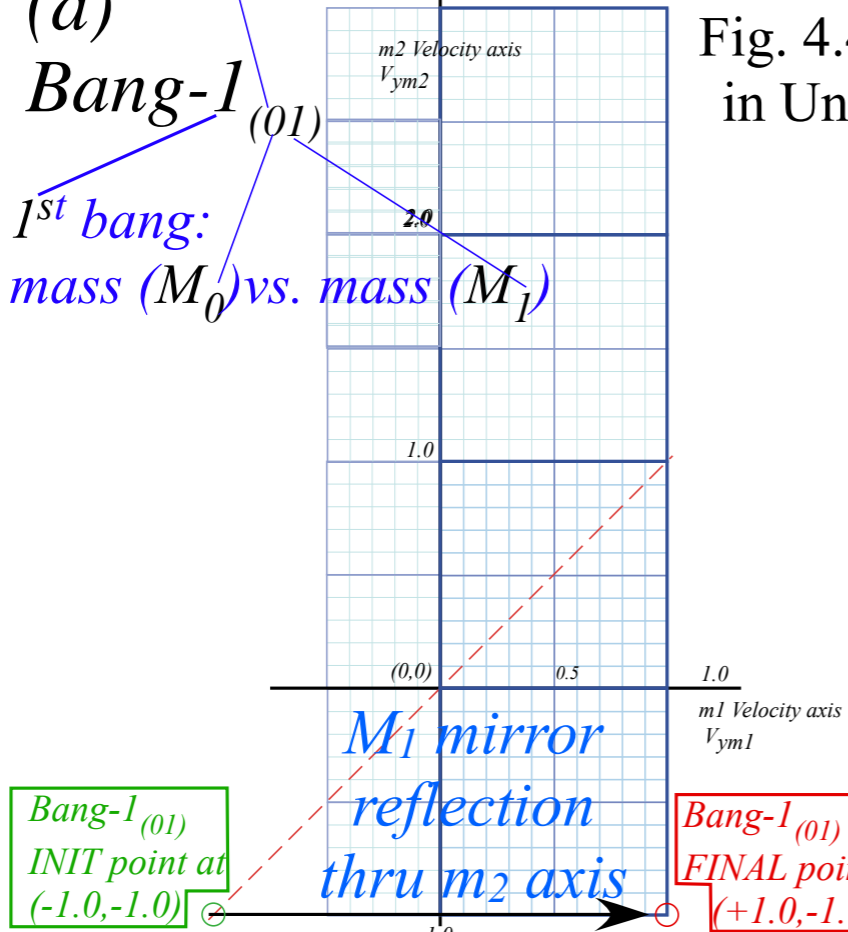
The X-2 pen-launcher



(a)

Bang-1 (01)

1st bang:
 mass (M_0) vs. mass (M_1)



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 4.1 and Fig. 4.3 in Unit 1

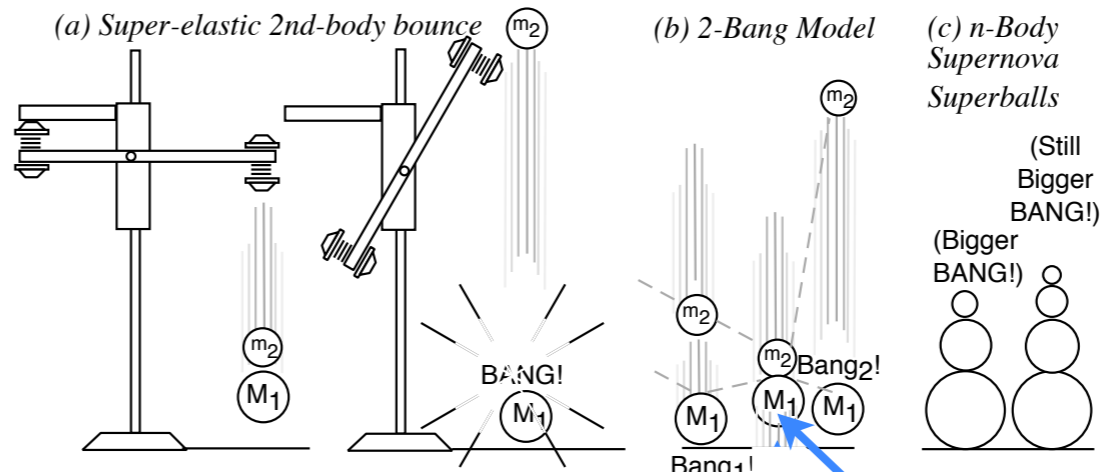


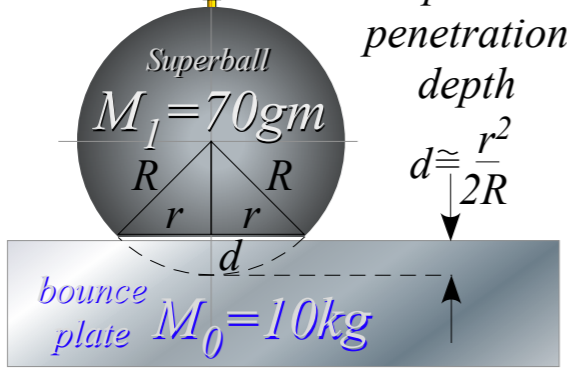
Fig. 4.4a-b in Unit 1

1st bang:
 M_1 off floor

2nd bang:
 m_2 off M_1

ballpoint pen
 $M_2 = 10\text{gm}$

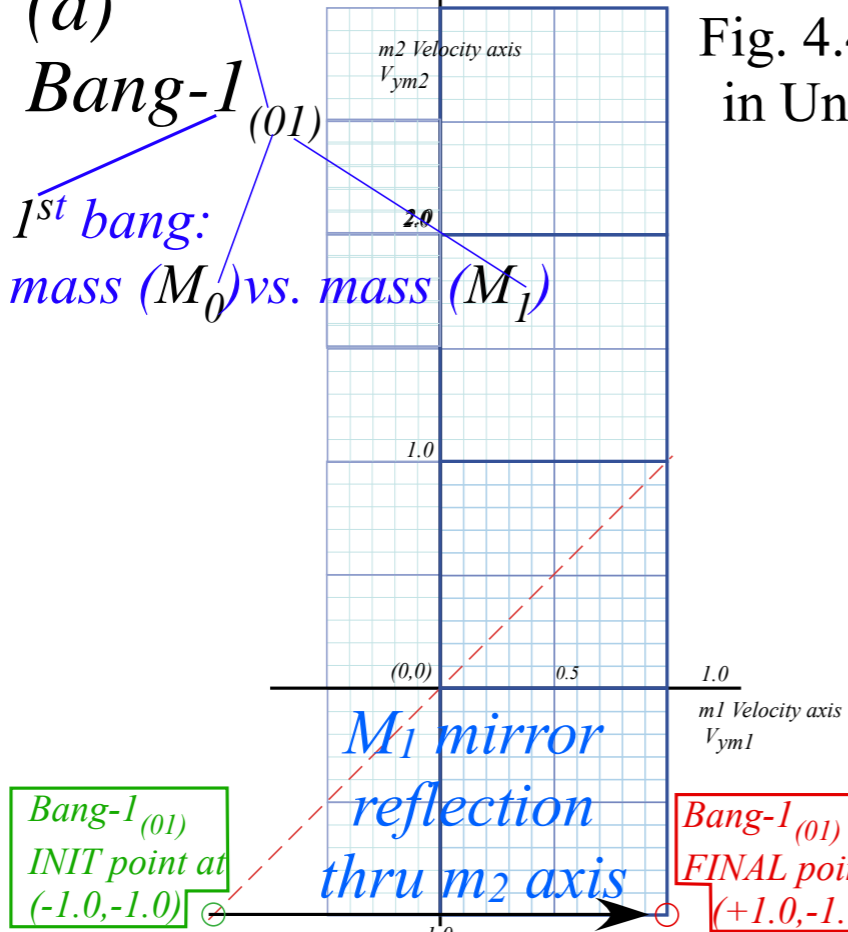
The X-2 pen-launcher



(a)

Bang-1 (01)

1st bang:
 mass (M_0) vs. mass (M_1)



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Fig. 4.1 and Fig. 4.3 in Unit 1

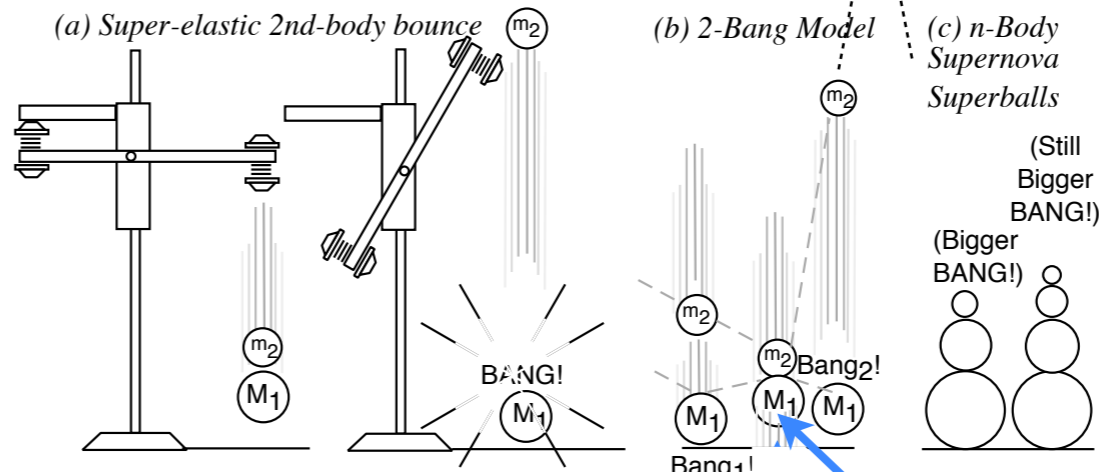


Fig. 4.4a-b in Unit 1

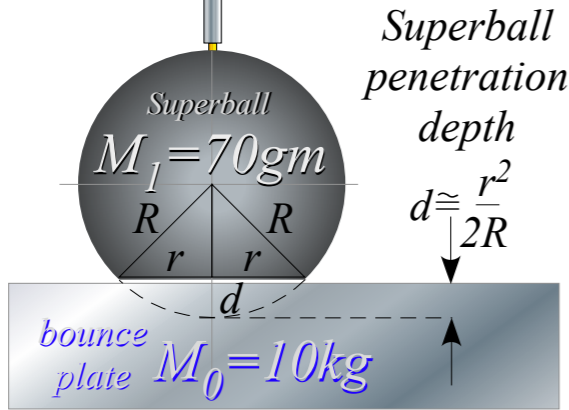
1st bang:
 M_1 off floor

2nd bang:
 m_2 off M_1

3rd bang:
 m_2 off ceiling

ballpoint pen
 $M_2 = 10\text{gm}$

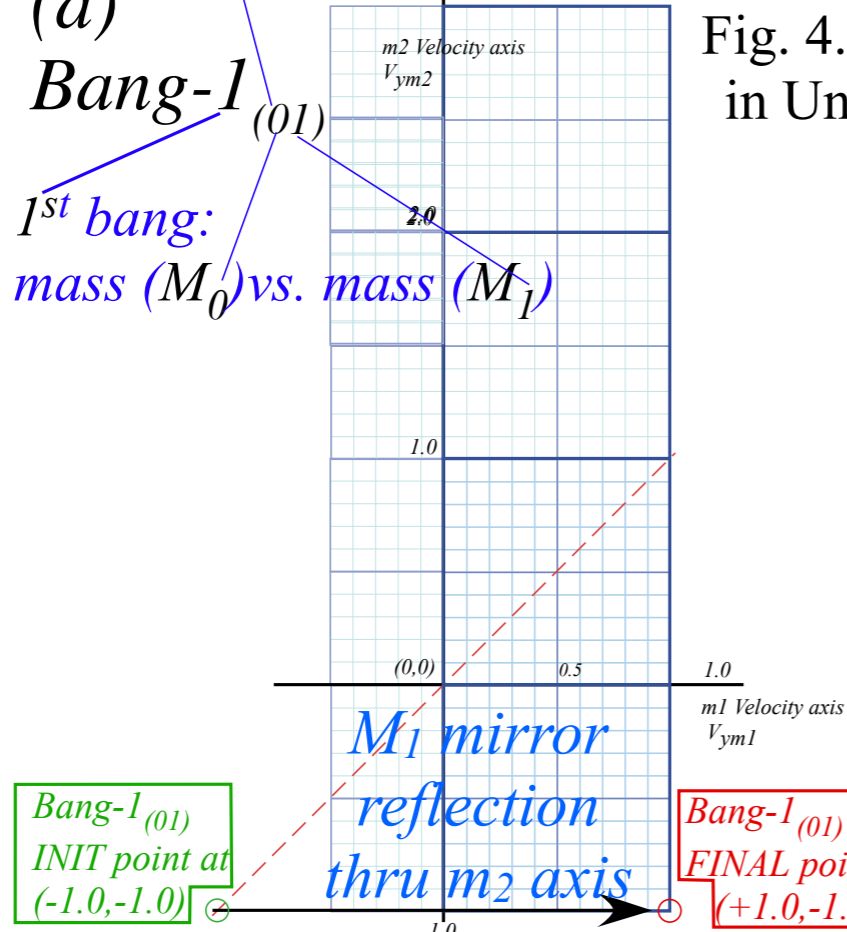
The X-2 pen-launcher



(a)

Bang-1

1st bang:
 mass (M_0) vs. mass (M_1)



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

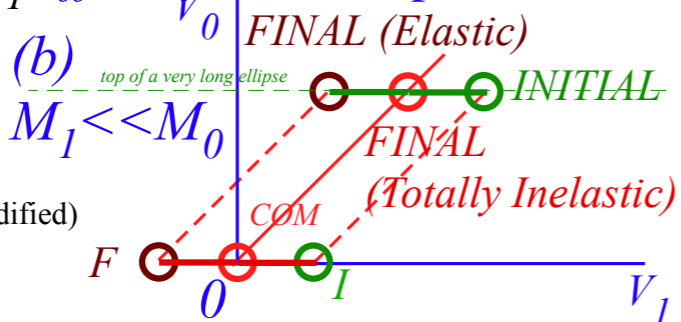
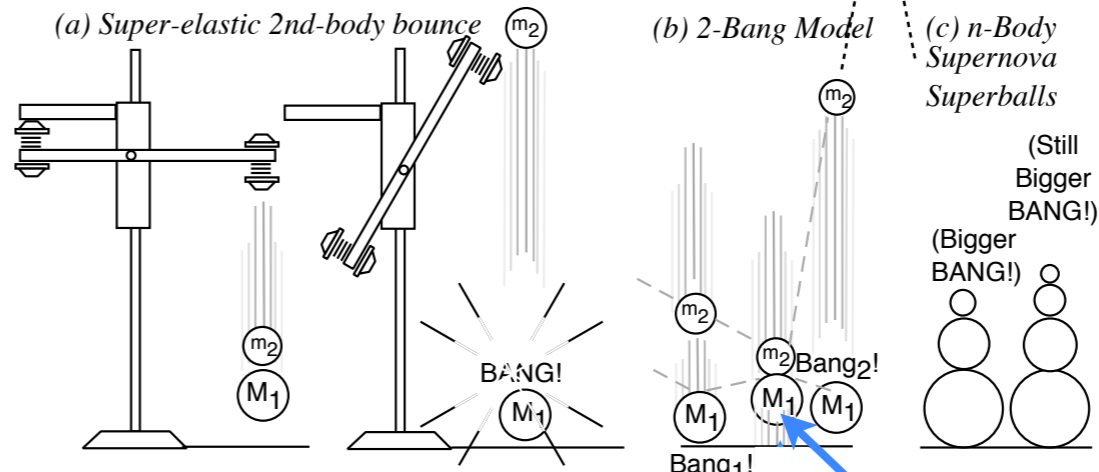


Fig. 4.2b in Unit 1 (slightly modified)

Fig. 4.1 and Fig. 4.3 in Unit 1



1st bang:
 M_1 off floor

2nd bang:
 m_2 off M_1

3rd bang:
 m_2 off ceiling

1st bang M_1 off floor "skinny-ellipse"

ballpoint pen
 $M_2 = 10\text{gm}$

The X-2 pen-launcher

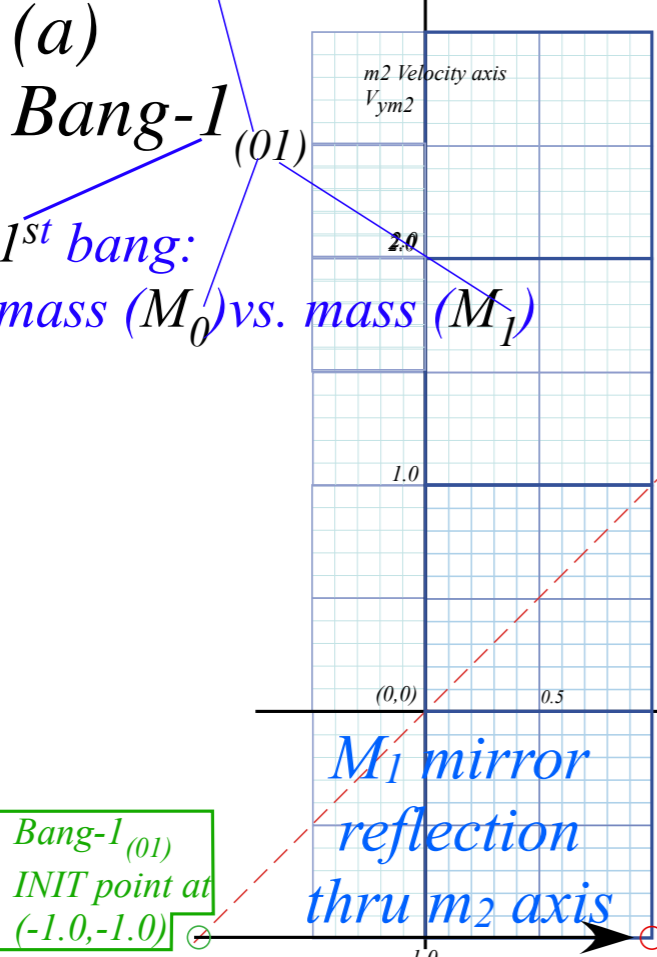
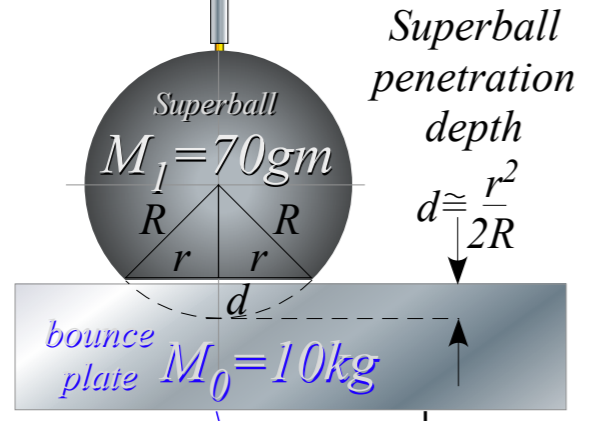


Fig. 4.4a-b in Unit 1

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

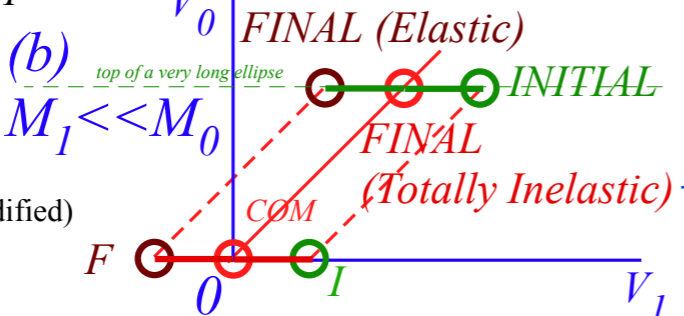
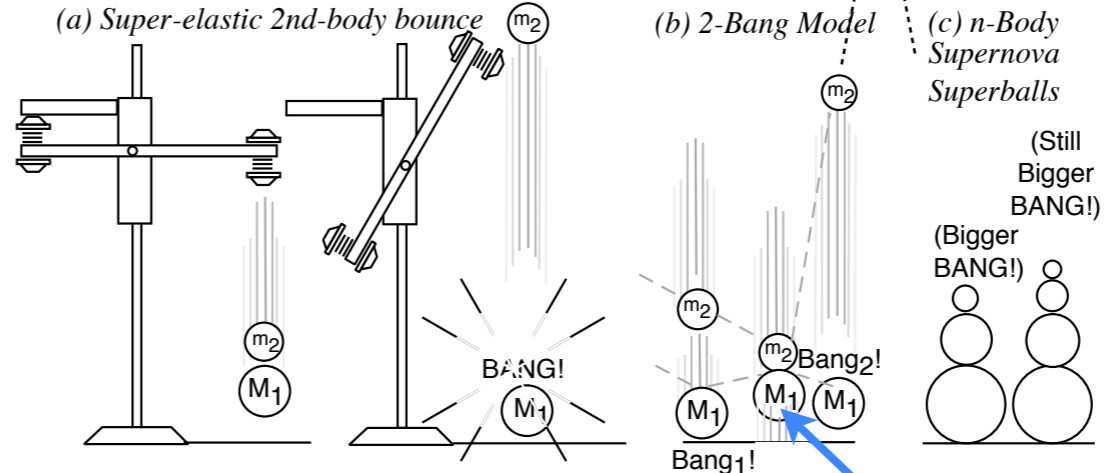


Fig. 4.2b in Unit 1 (slightly modified)

Fig. 4.1 and Fig. 4.3 in Unit 1



1st bang: M_1 off floor
 2nd bang: m_2 off M_1

3rd bang: m_2 off ceiling

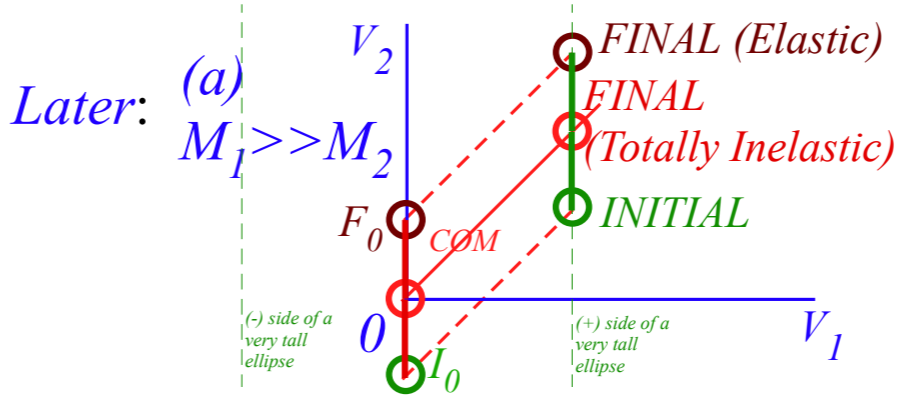
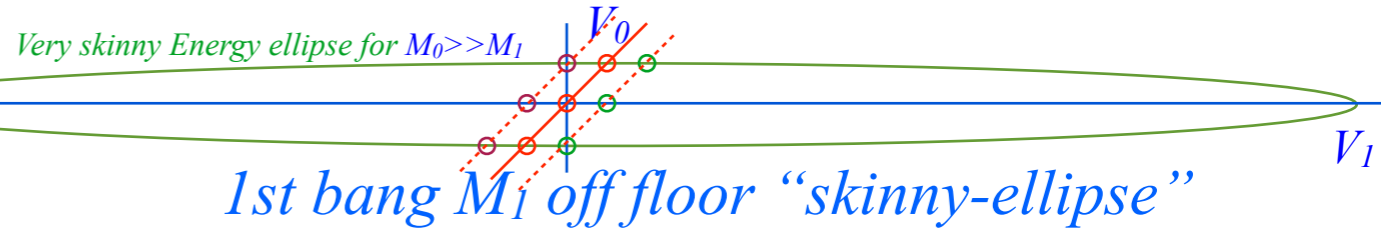
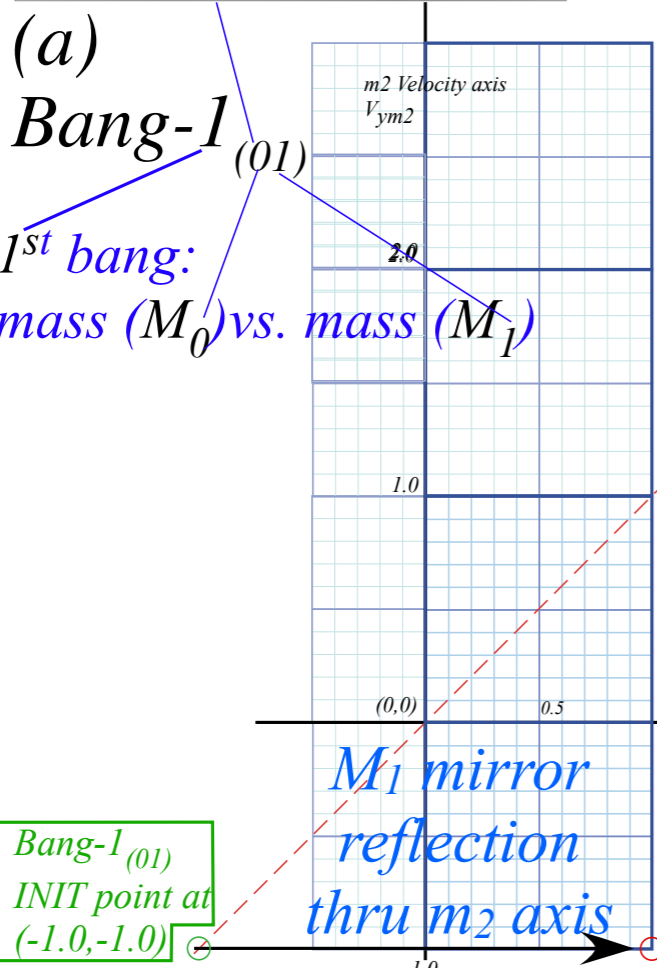
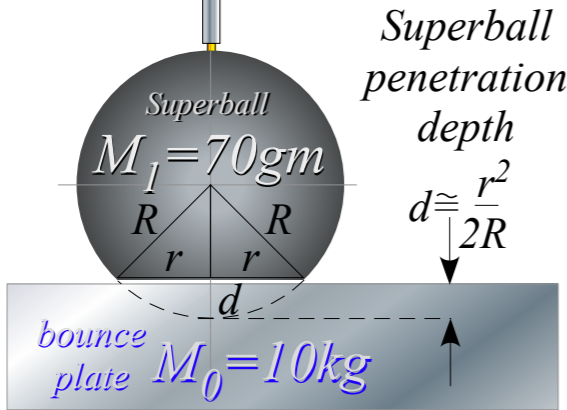


Fig. 4.2a in Unit 1 (slightly modified)



ballpoint pen
 $M_2 = 10\text{gm}$

The X-2 pen-launcher



Bang-1
 1st bang:
 mass (M_0) vs. mass (M_1)

Bang-1₍₀₁₎
 INIT point at (-1.0, -1.0)

Bang-1₍₀₁₎
 FINAL point (+1.0, -1.0)

This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

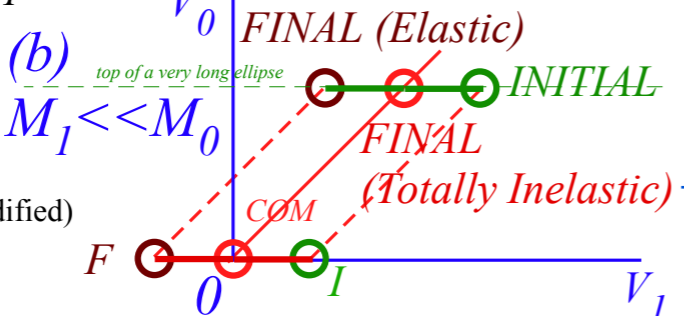
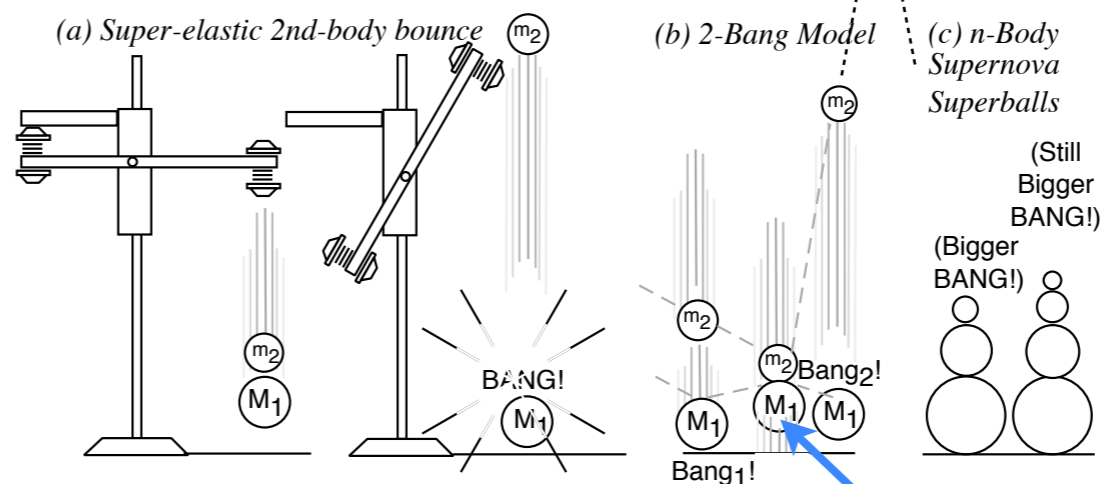


Fig. 4.2b
 in Unit 1 (slightly modified)

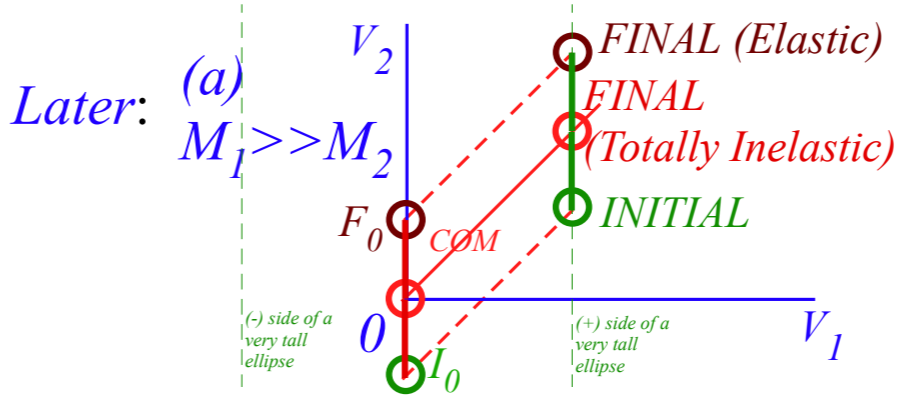
Fig. 4.1 and Fig. 4.3
 in Unit 1



1st bang:
 M_1 off floor

2nd bang:
 m_2 off M_1

3rd bang:
 m_2 off ceiling



Later:

(a)
 $M_1 \gg M_2$

Fig. 4.2a
 in Unit 1 (slightly modified)

Very skinny Energy ellipse for $M_0 \gg M_1$

1st bang M_1 off floor "skinny-ellipse"

Geometry of X2 launcher bouncing in box

 *Independent Bounce Model (IBM)*

Geometric optimization and range-of-motion calculation(t)

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)

ballpoint pen
 $M_2=10gm$

The X-2 pen-launcher

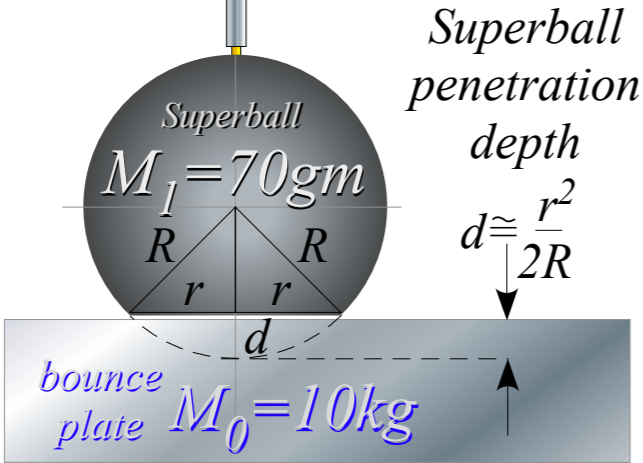


Fig. 4.1 and Fig. 4.3 in Unit 1

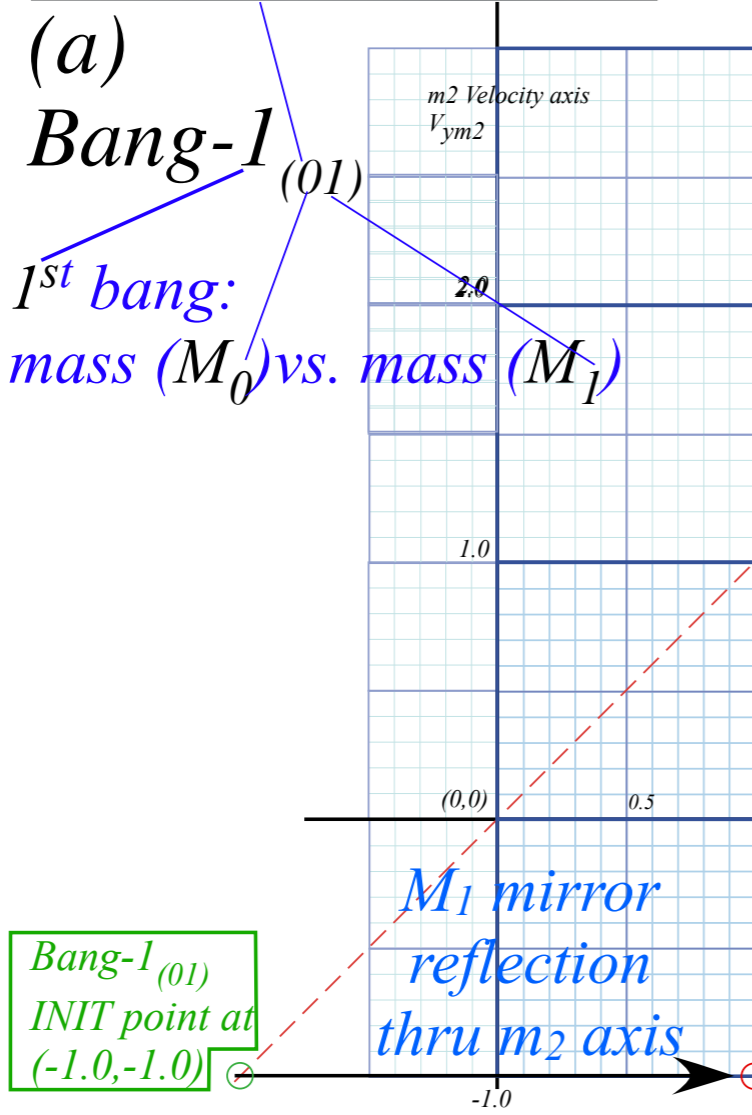
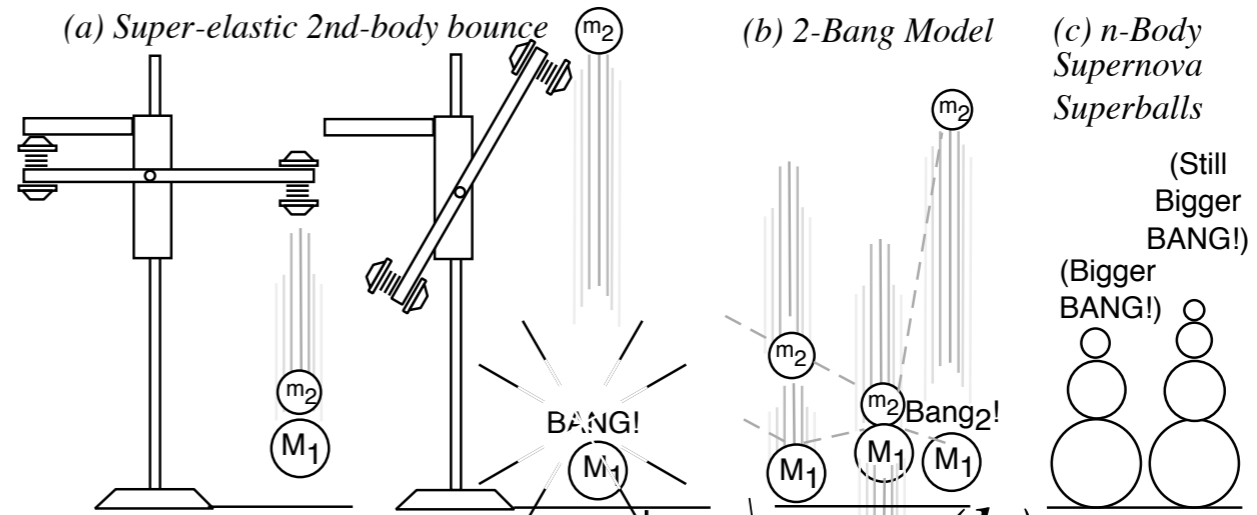
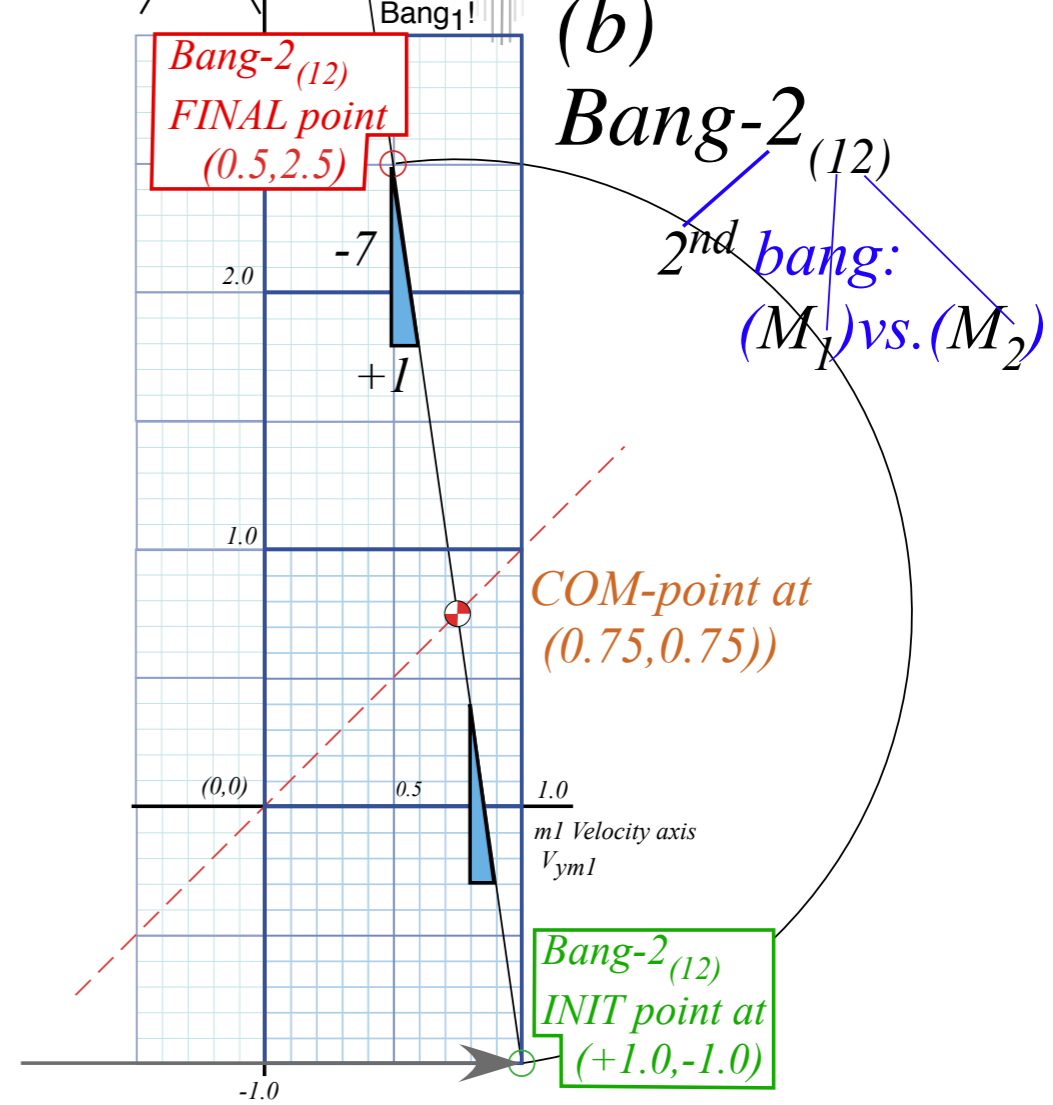


Fig. 4.4a-b in Unit 1



This 1st bang is a floor-bounce of M_1 off very massive plate/Earth M_0

Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

 *Geometric optimization and range-of-motion calculation(s)*

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)

ballpoint pen
pen
 $M_2=10\text{gm}$

The X-2 pen-launcher

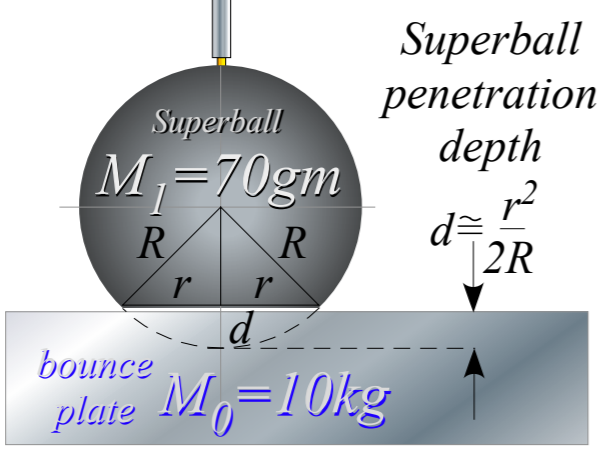


Fig. 4.1 and Fig. 4.3 in Unit 1

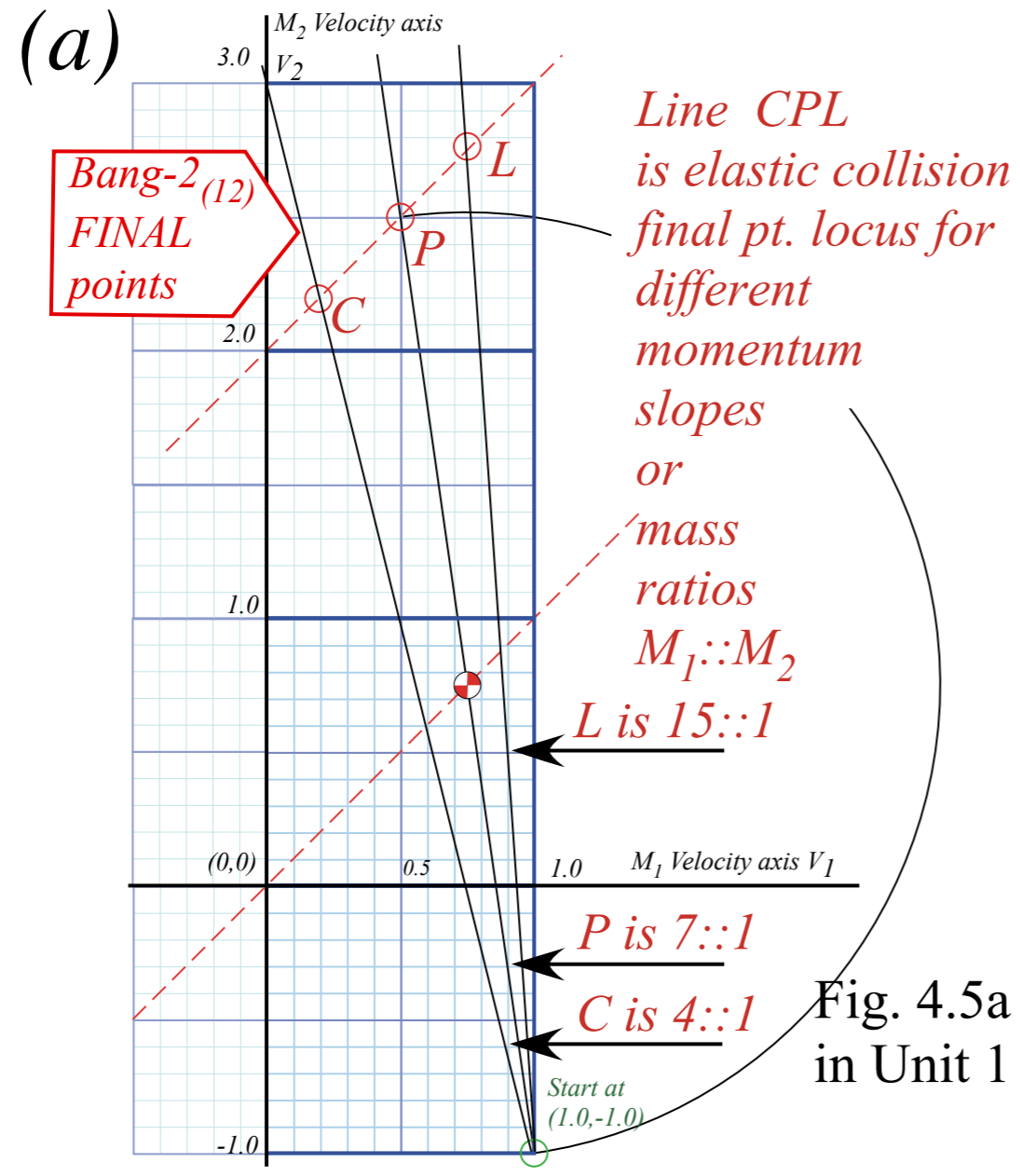
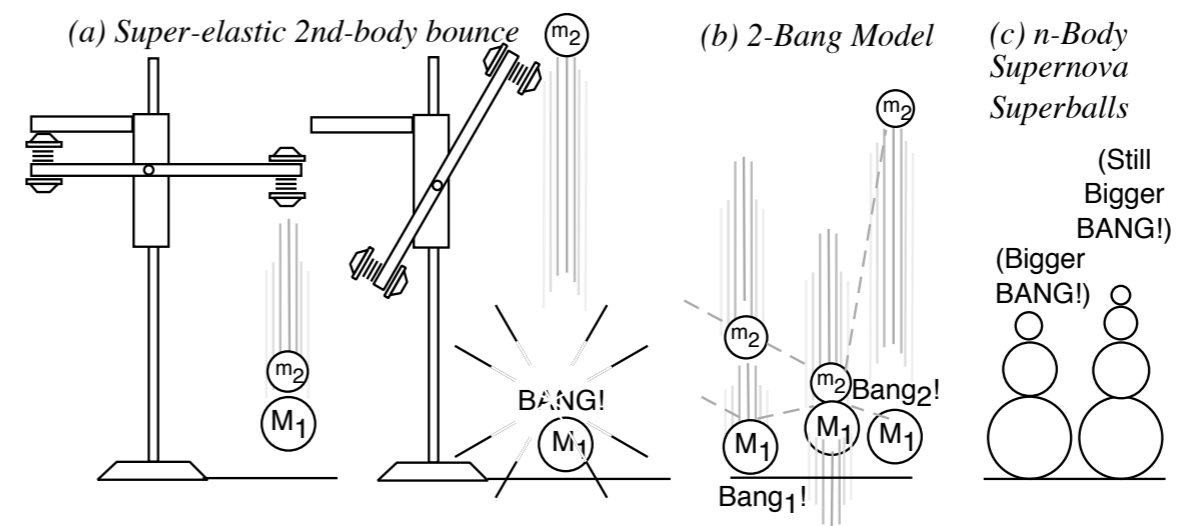
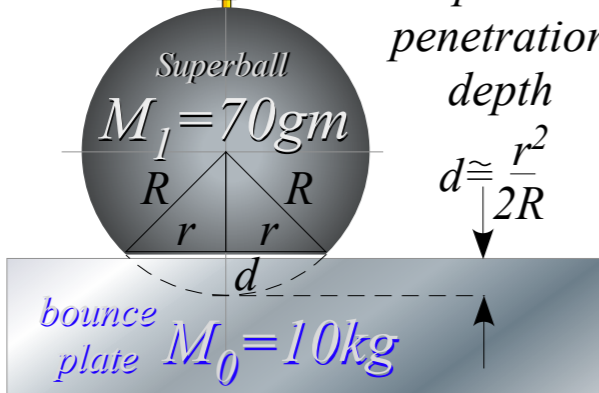


Fig. 4.5a in Unit 1

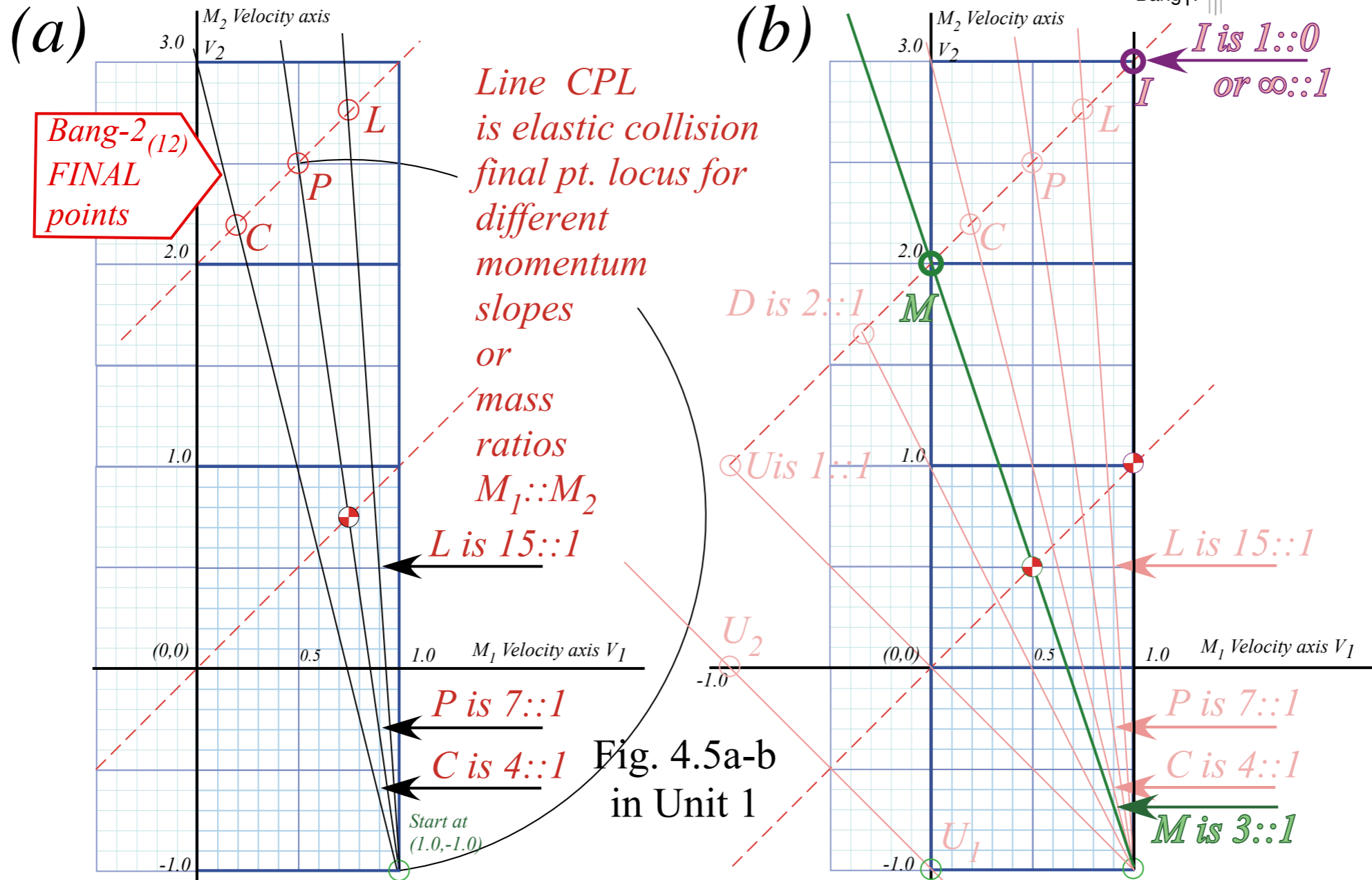
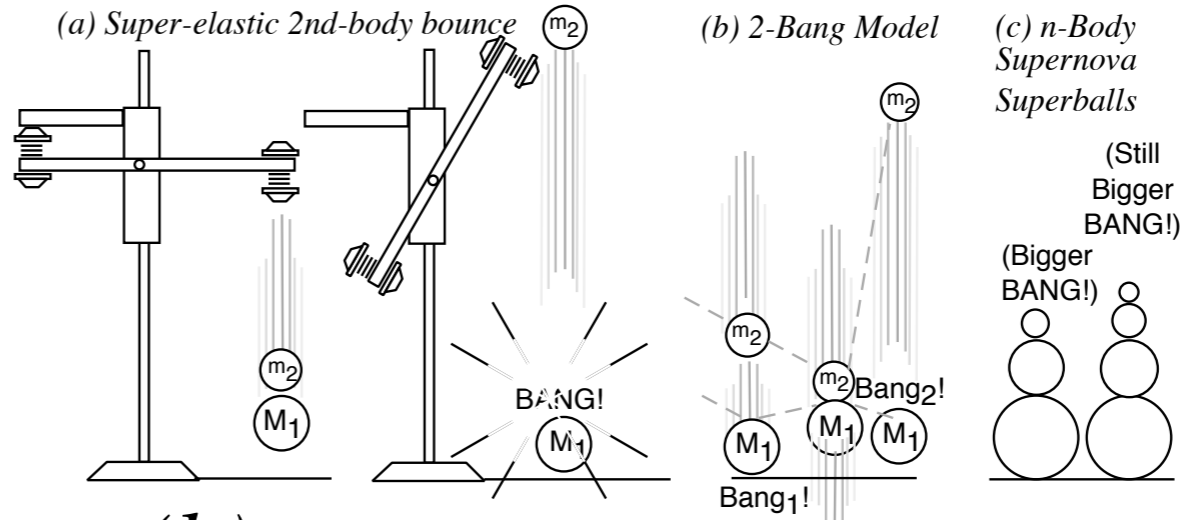
ballpoint pen
 $M_2=10\text{gm}$

The X-2 pen-launcher



Superball penetration depth

Fig. 4.1 and Fig. 4.3 in Unit 1



Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

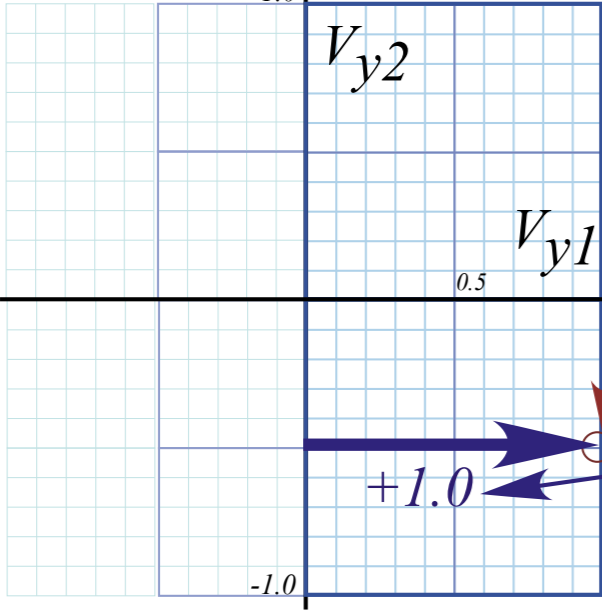
Geometric optimization and range-of-motion calculation(s)

 *Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots*

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)

Geometric "Integration" (Converting Velocity data to Spacetime)

Velocity V_{y2} vs. V_{y1} Plot

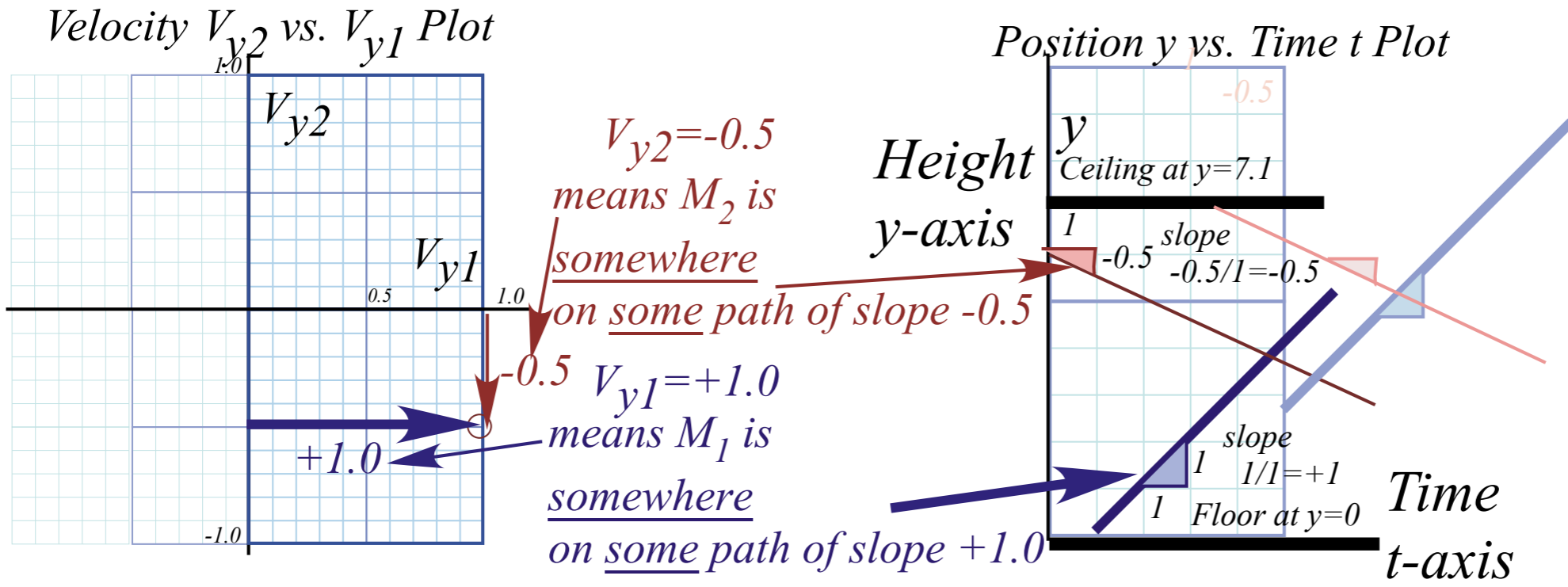


$V_{y2} = -0.5$
means M_2 is
somewhere
on some path of slope -0.5

$V_{y1} = +1.0$
means M_1 is
somewhere
on some path of slope $+1.0$

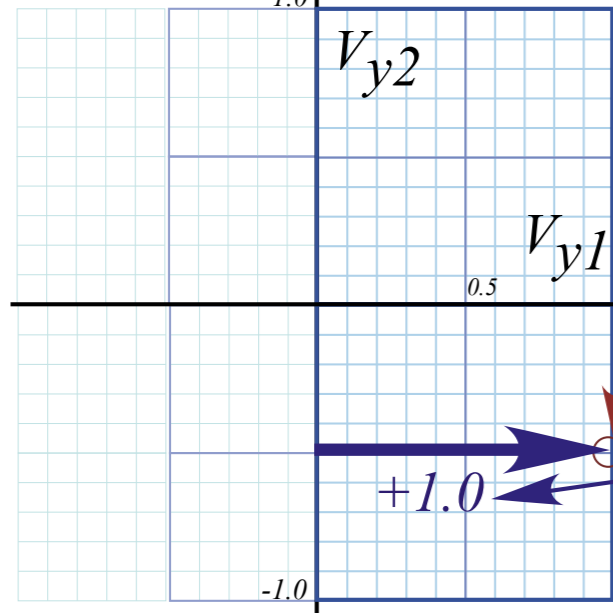
Position y vs. Time t Plot

Geometric "Integration" (Converting Velocity data to Spacetime)



Geometric "Integration" (Converting Velocity data to Spacetime)

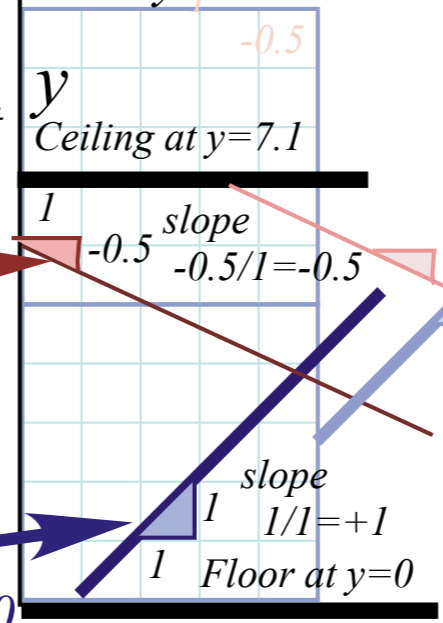
Velocity V_{y2} vs. V_{y1} Plot



$V_{y2} = -0.5$
means M_2 is
somewhere
on some path of slope -0.5

$V_{y1} = +1.0$
means M_1 is
somewhere
on some path of slope $+1.0$

Position y vs. Time t Plot



Until you specify
initial conditions $y_0(t_0)$...
...you don't know what
 v_y -line to use

Geometric "Integration" (Converting Velocity data to Spacetime)

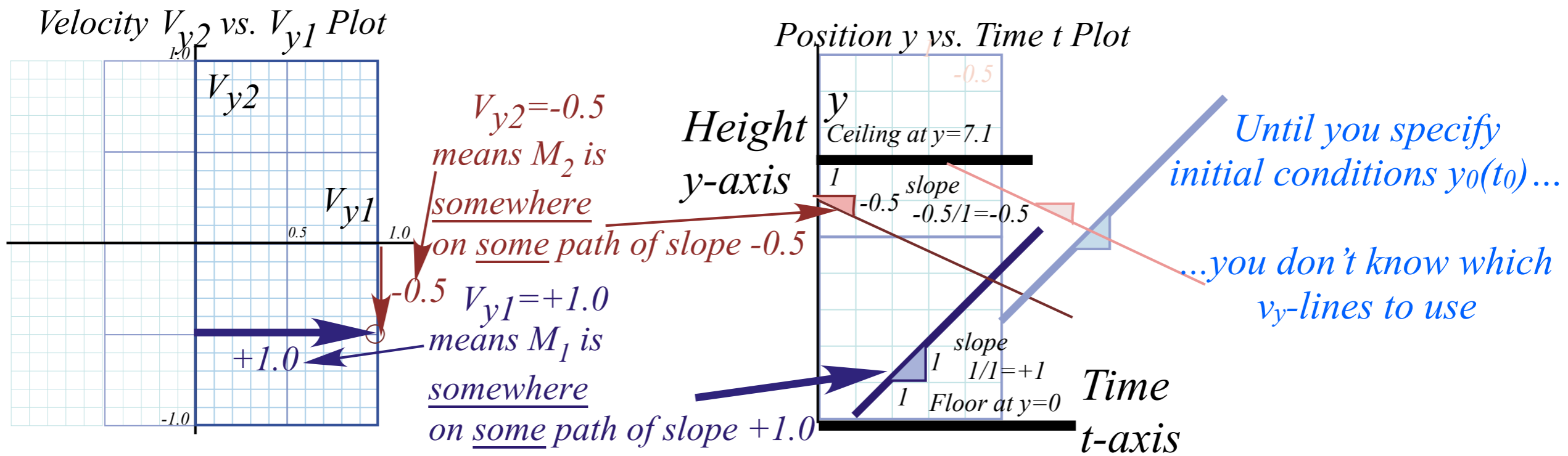
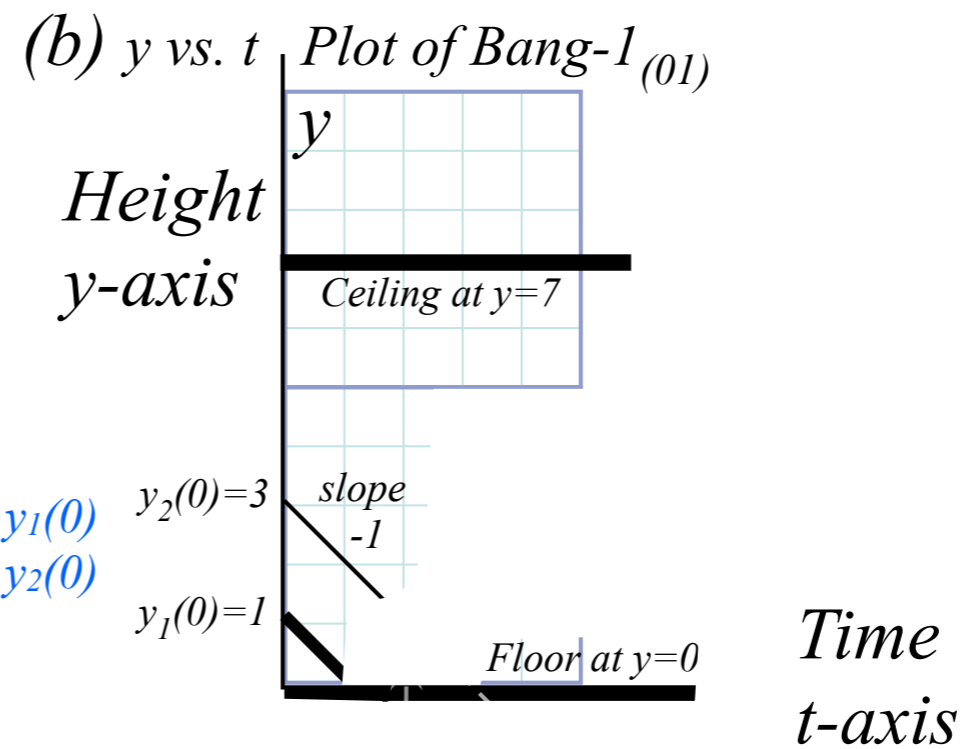
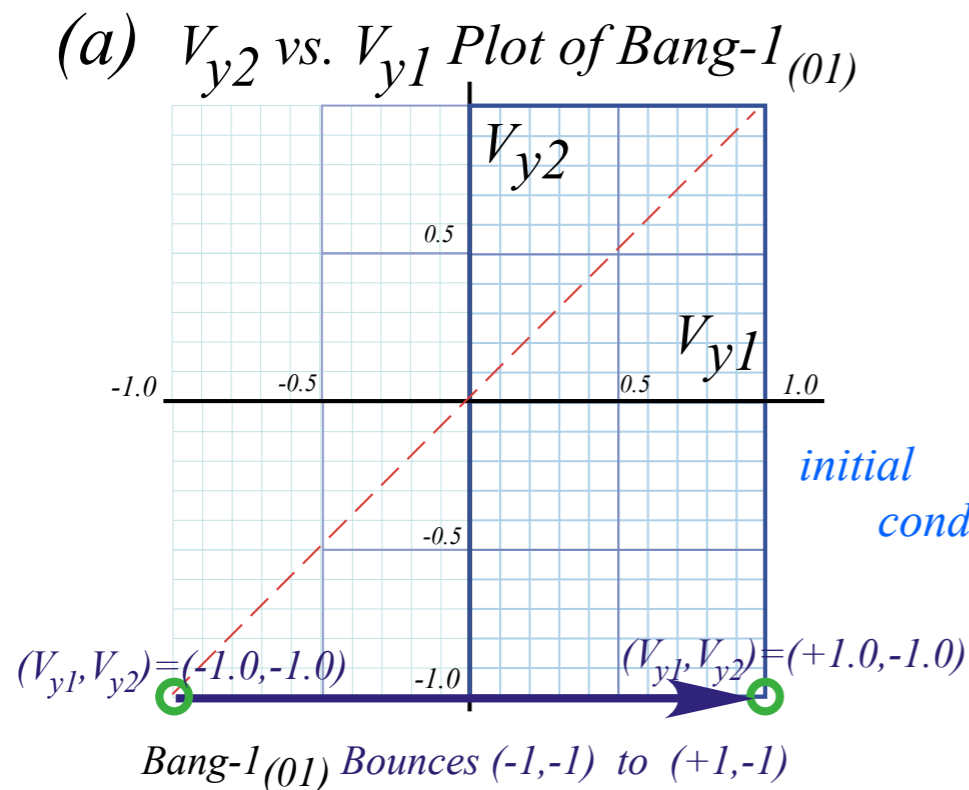


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)

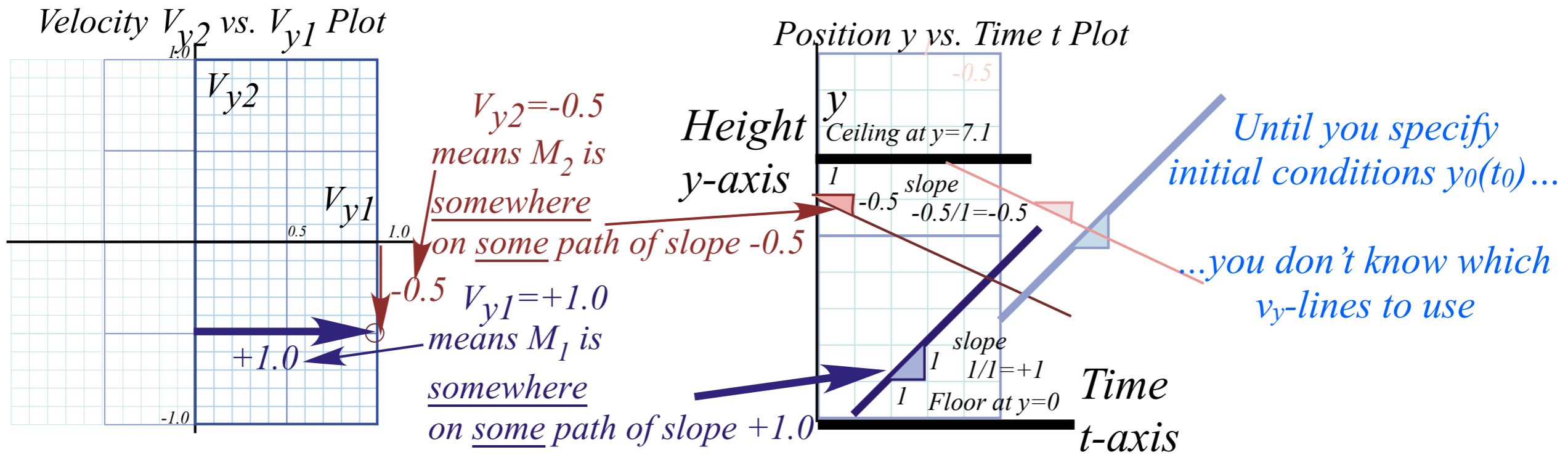
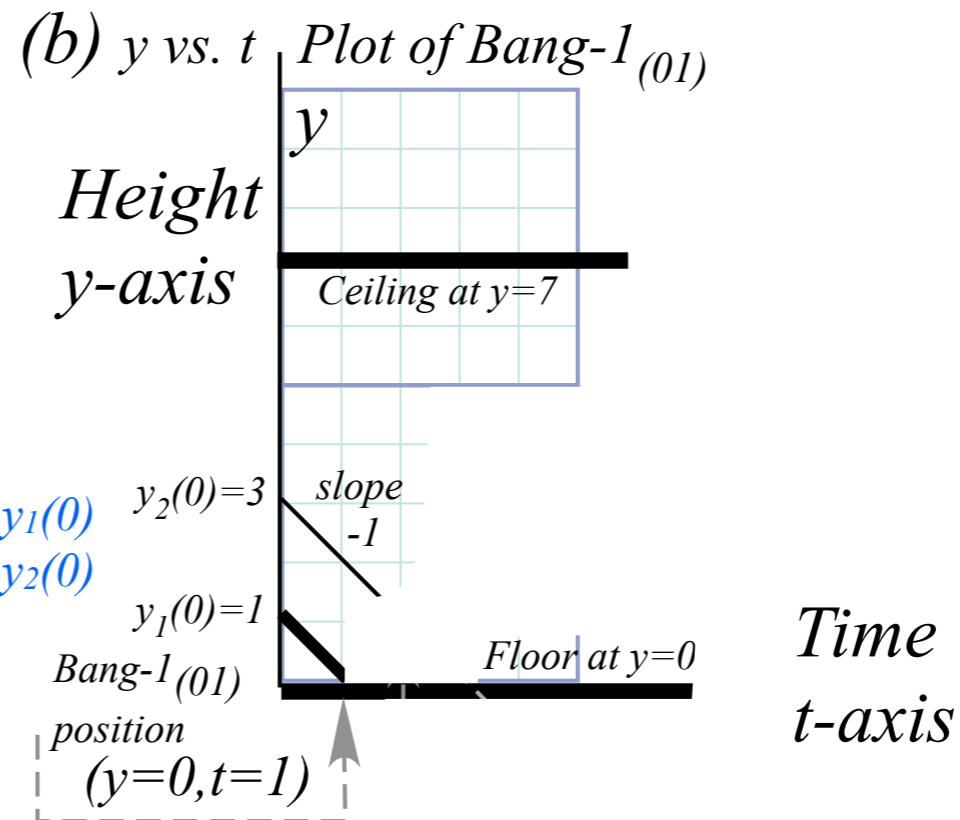
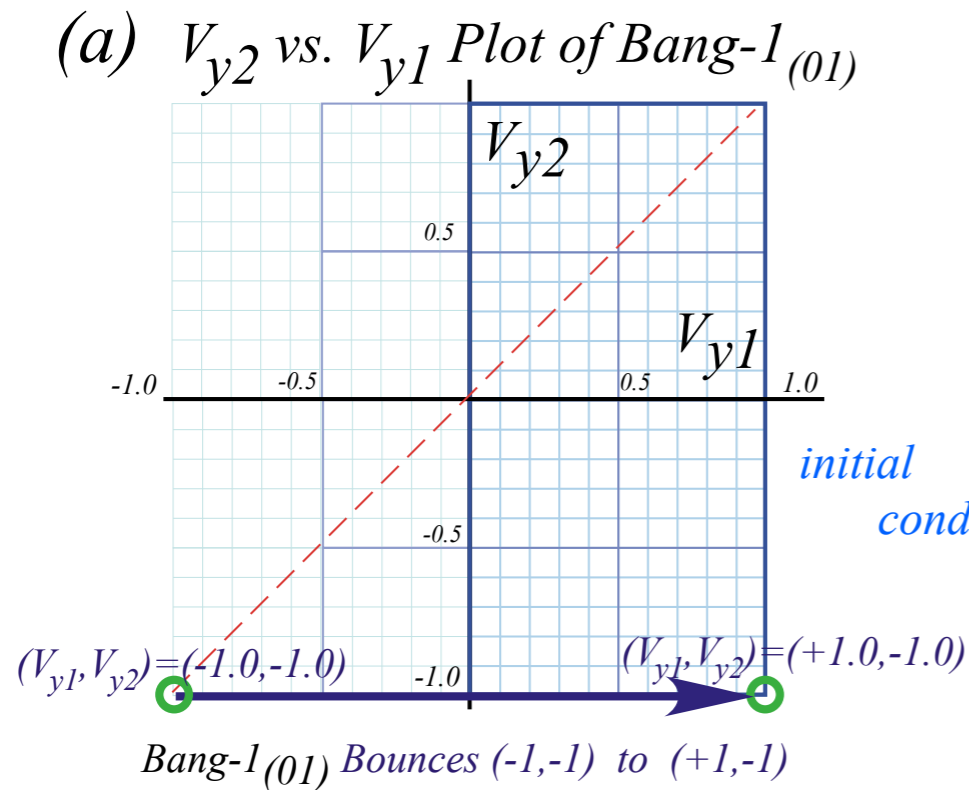


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)

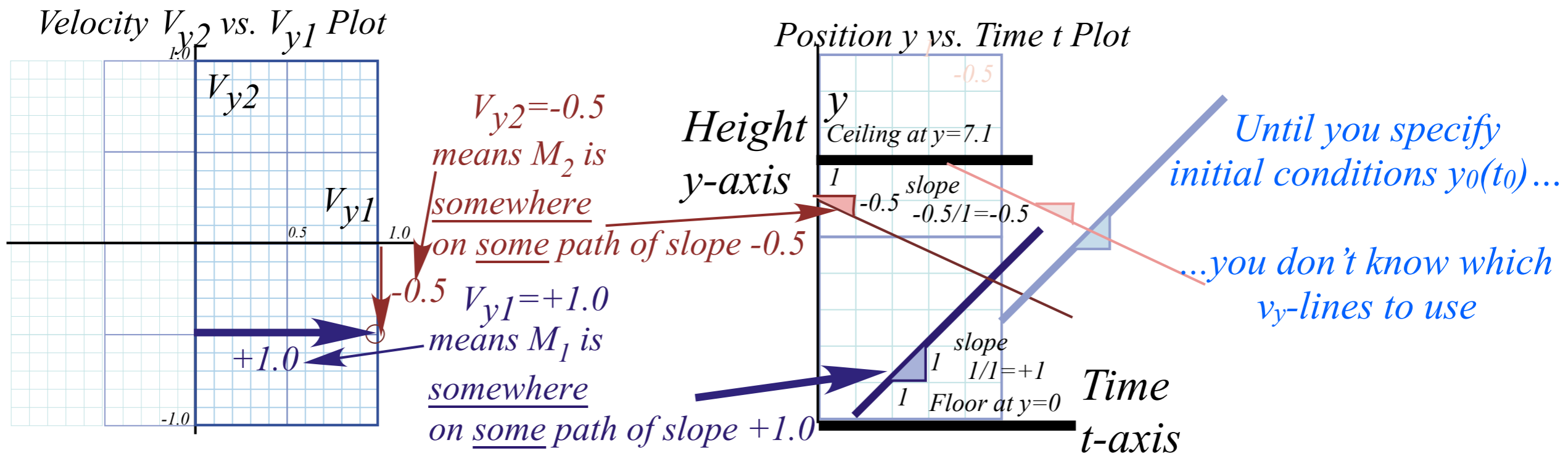
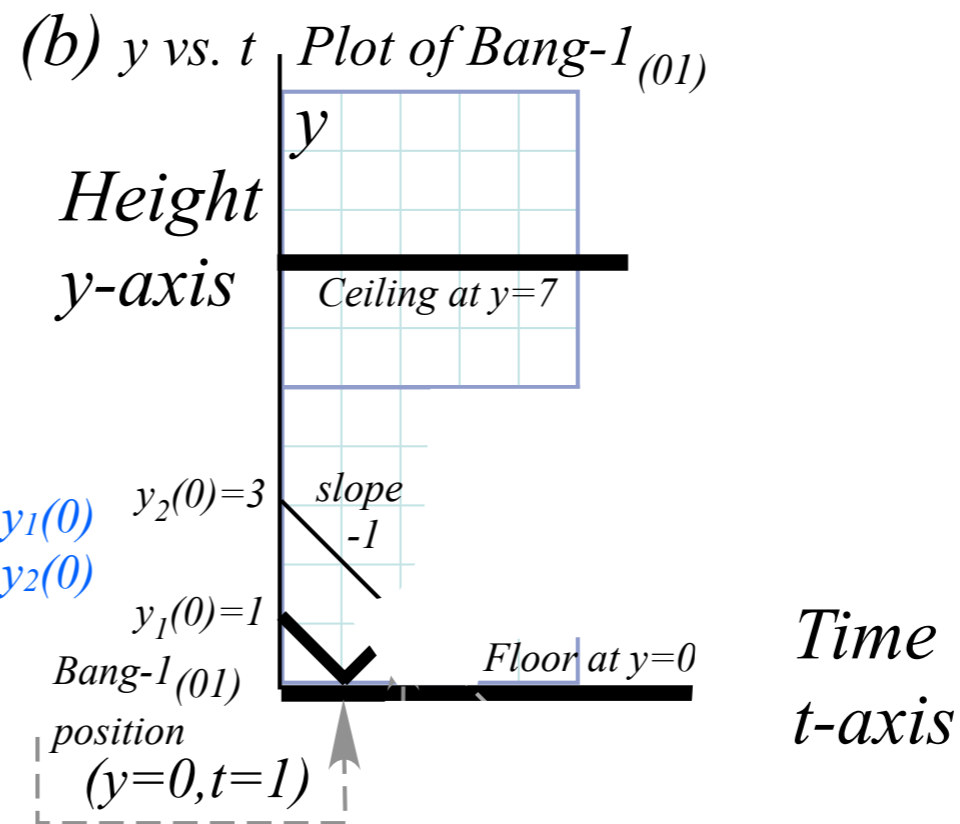
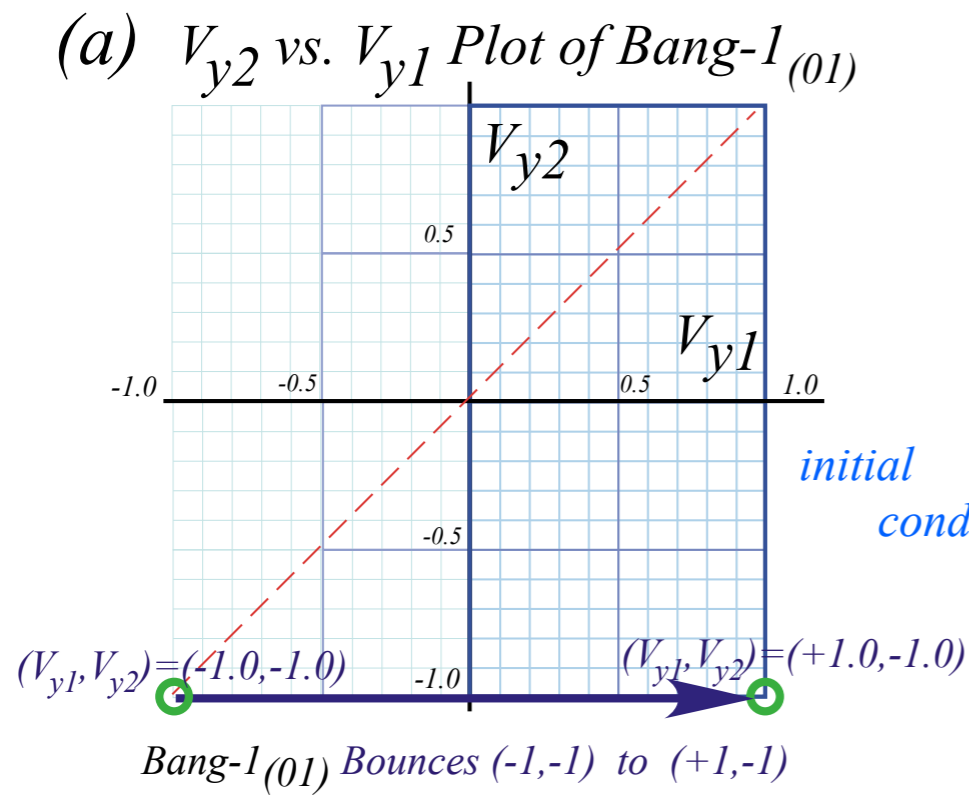


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)

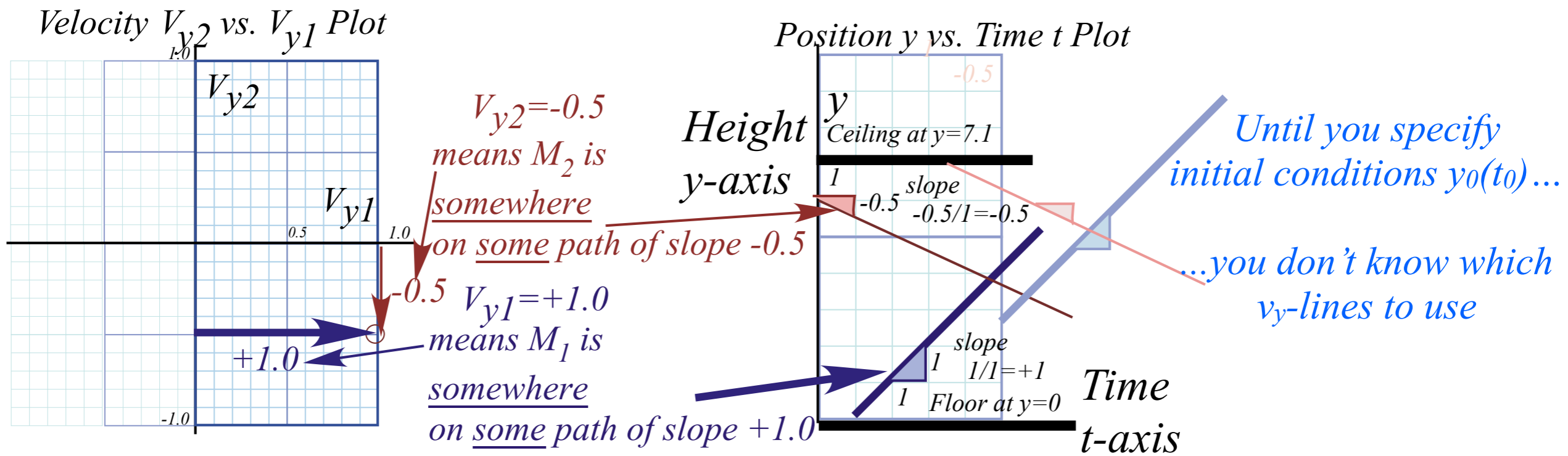
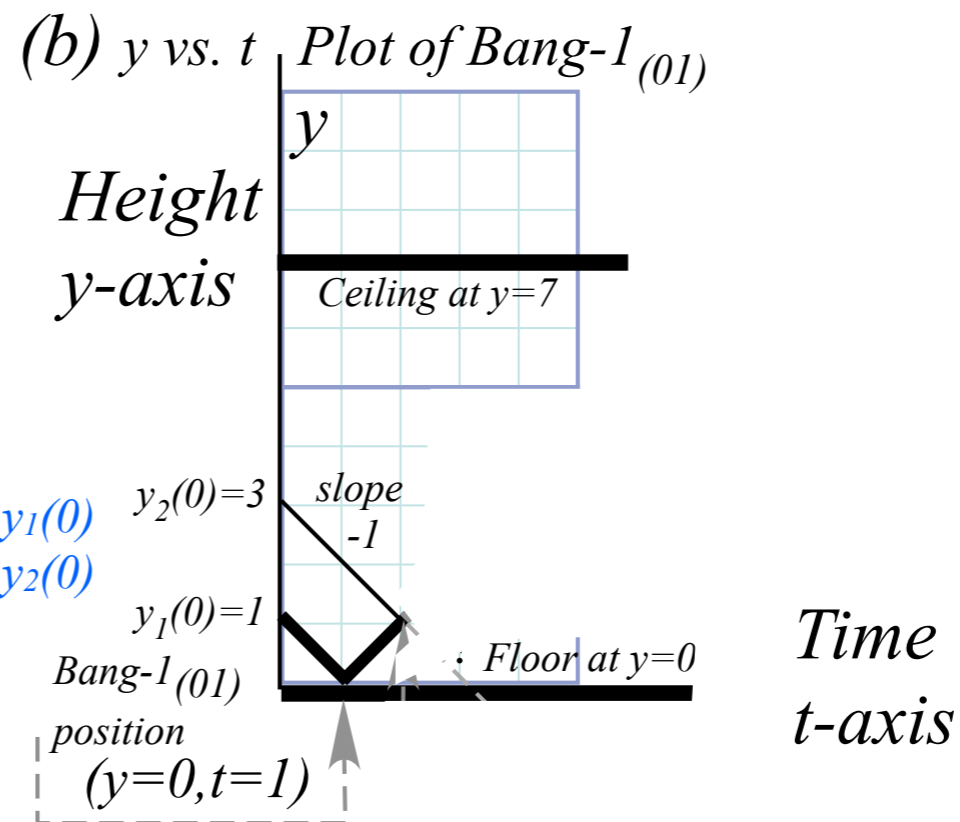
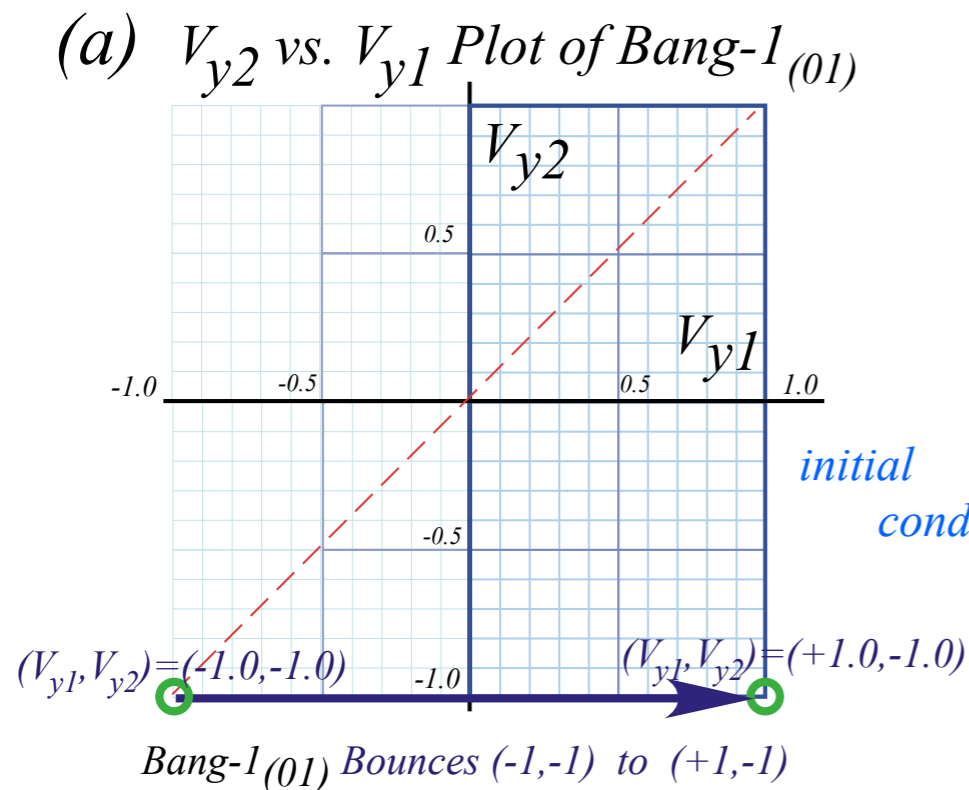


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)

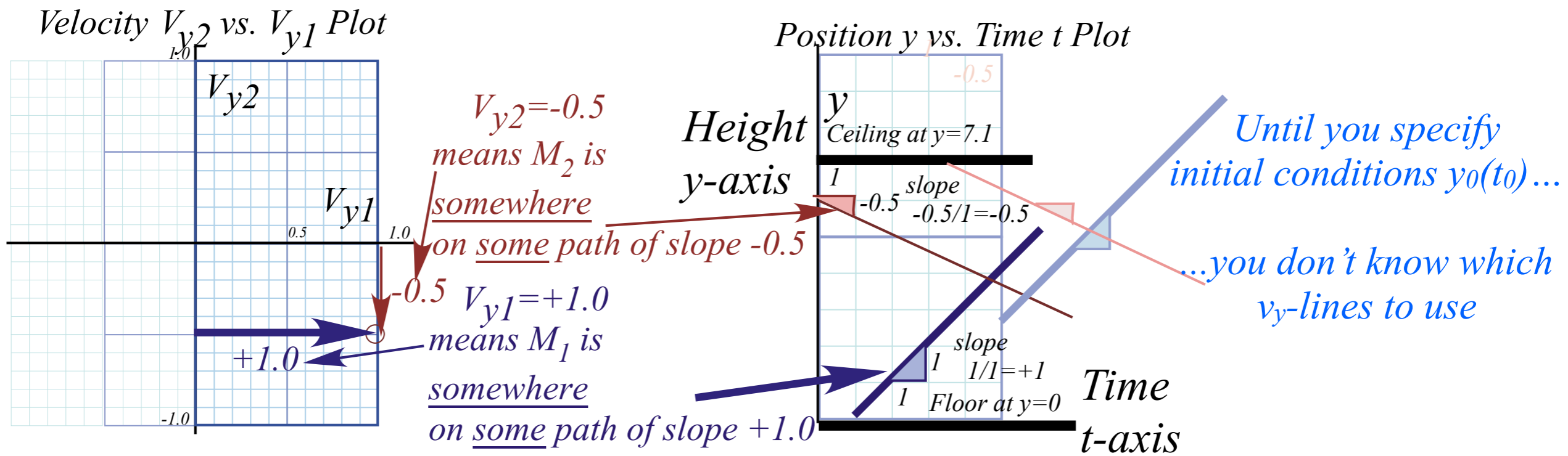
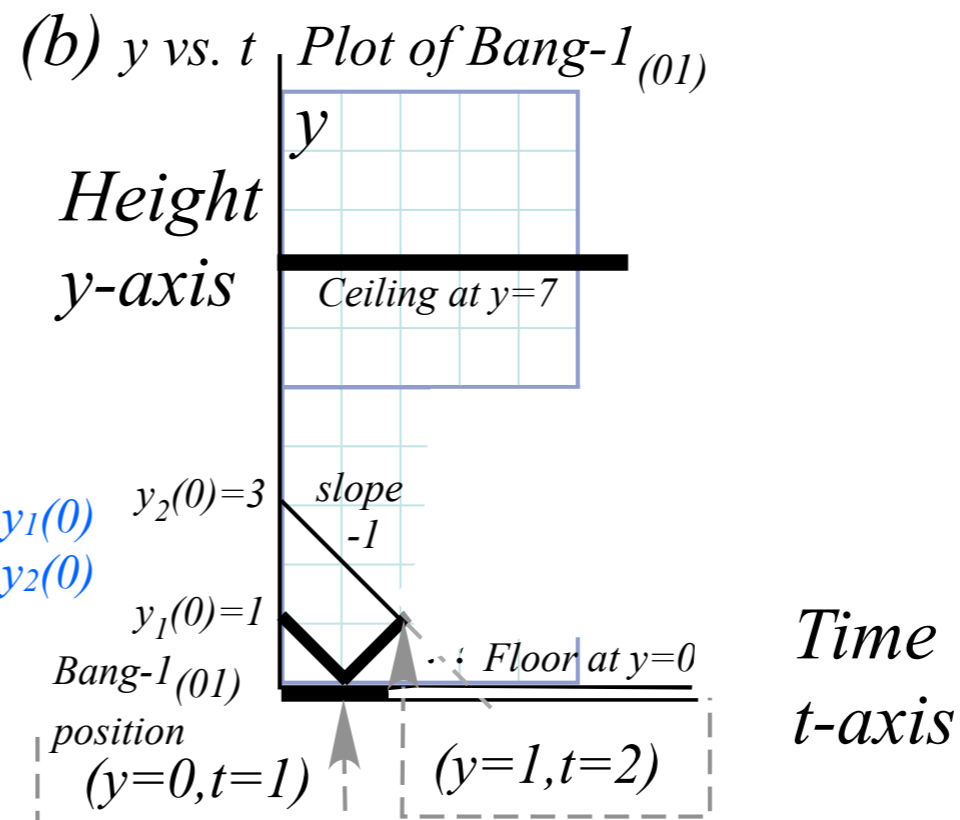
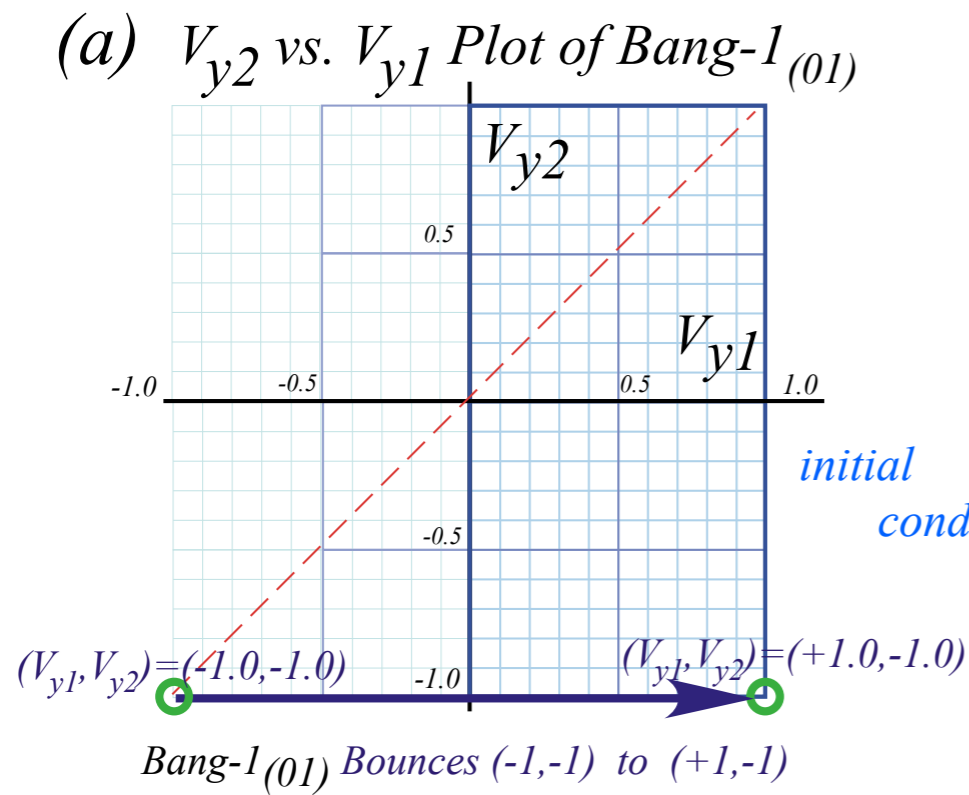


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)

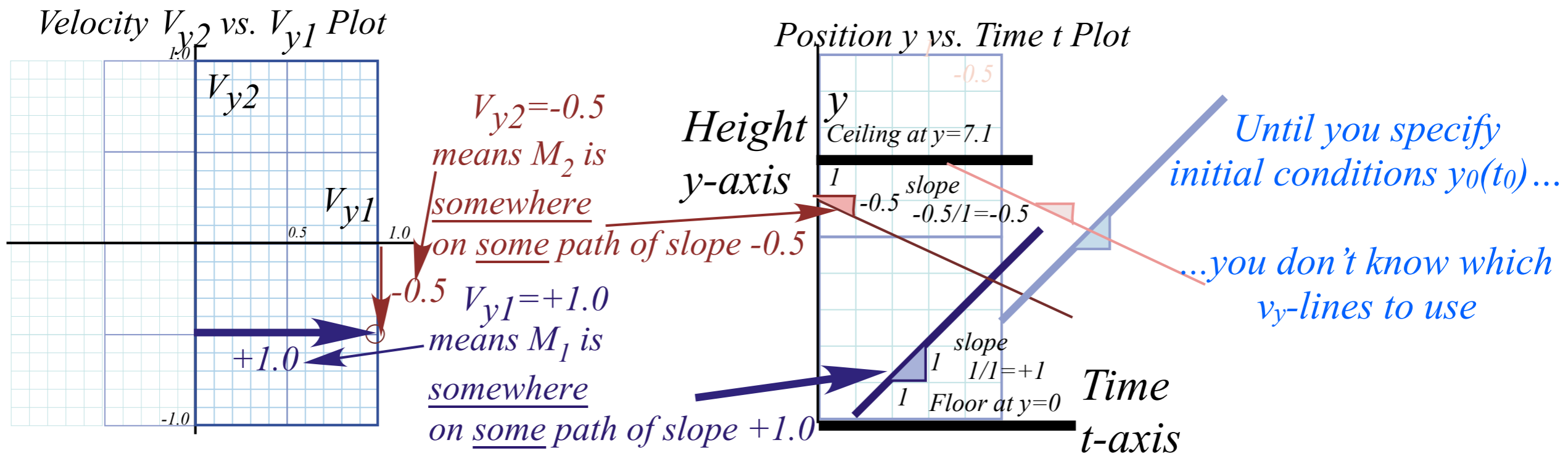
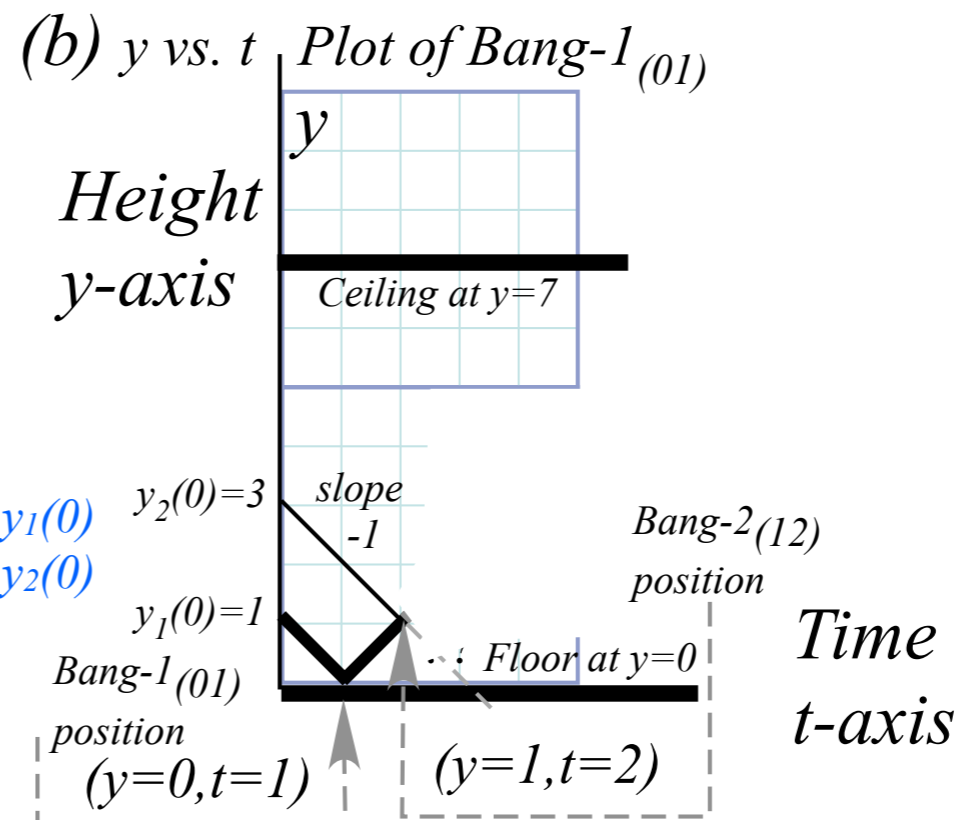
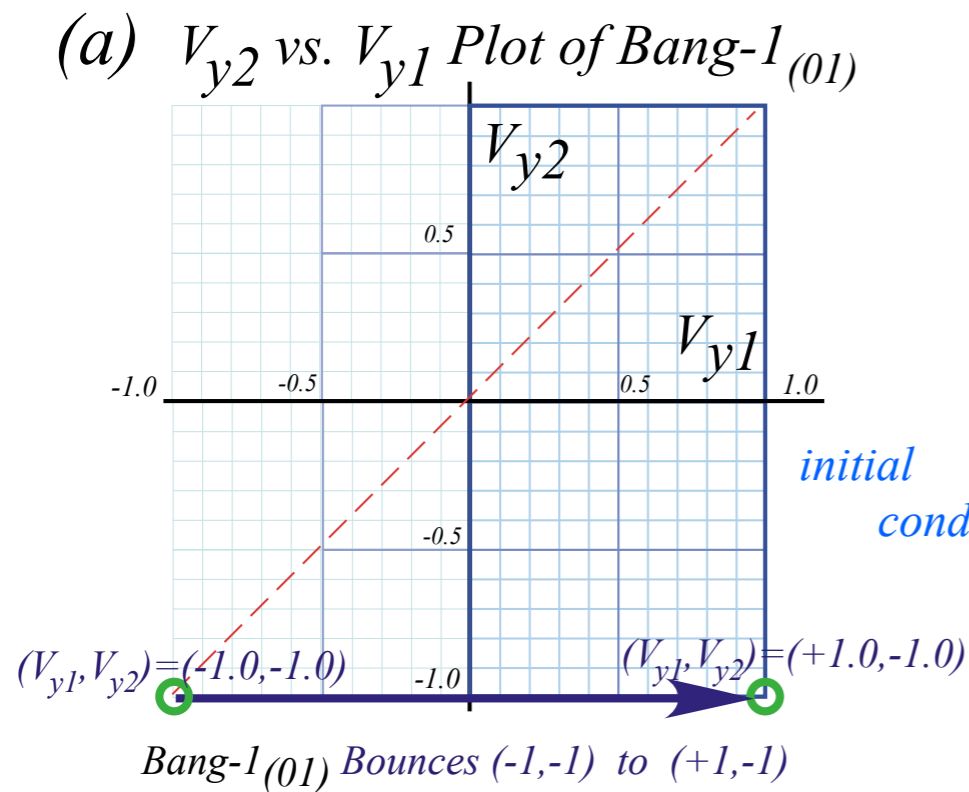
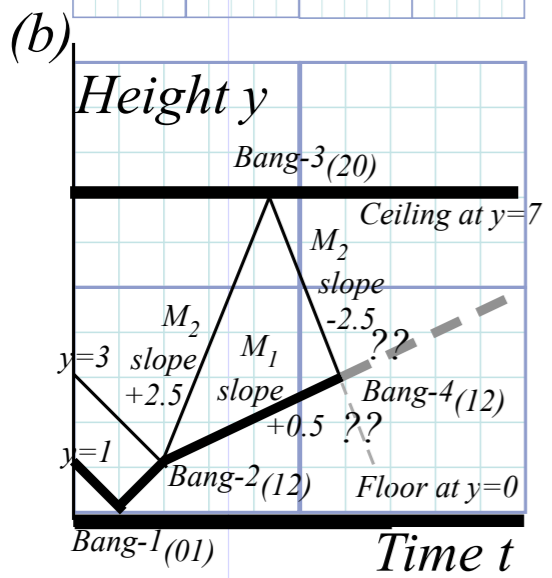
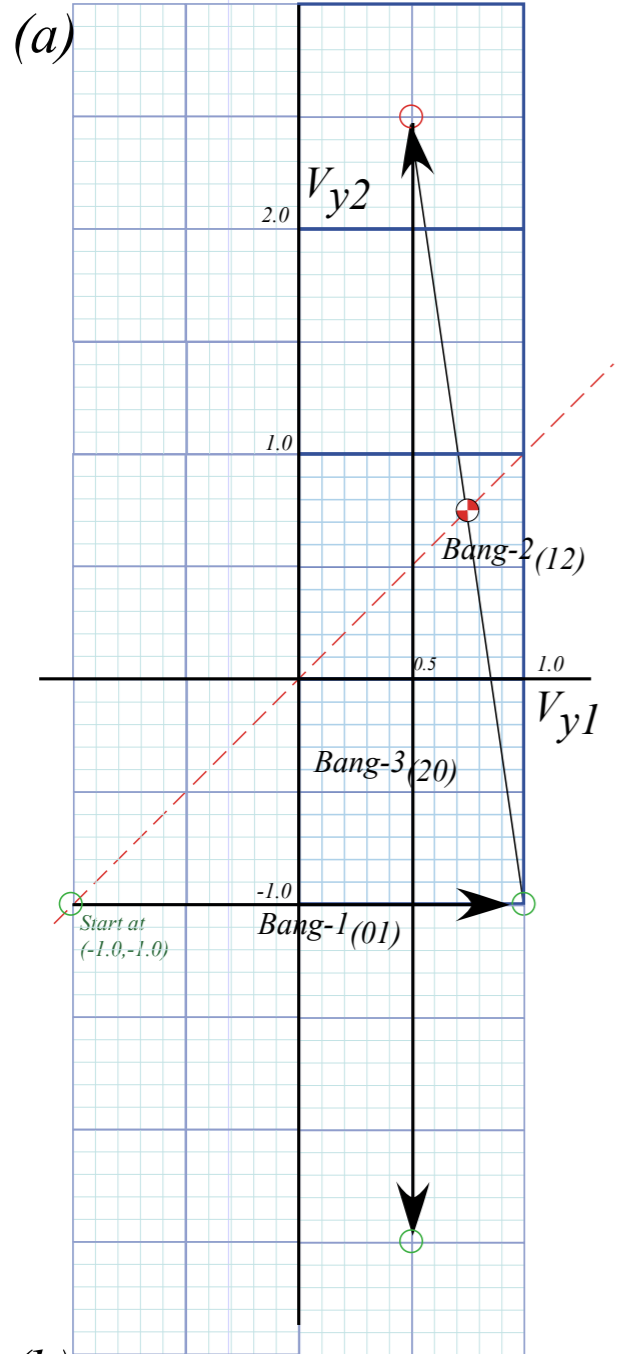


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)



Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{1}{2} + \frac{7}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

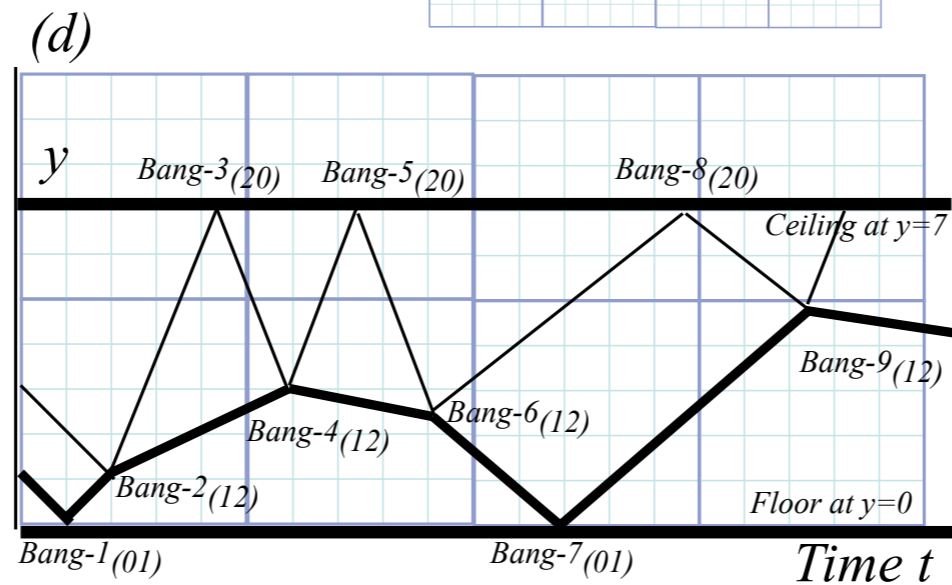
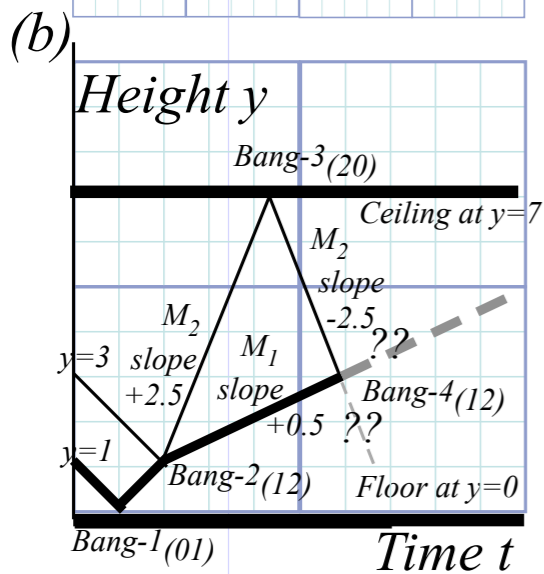
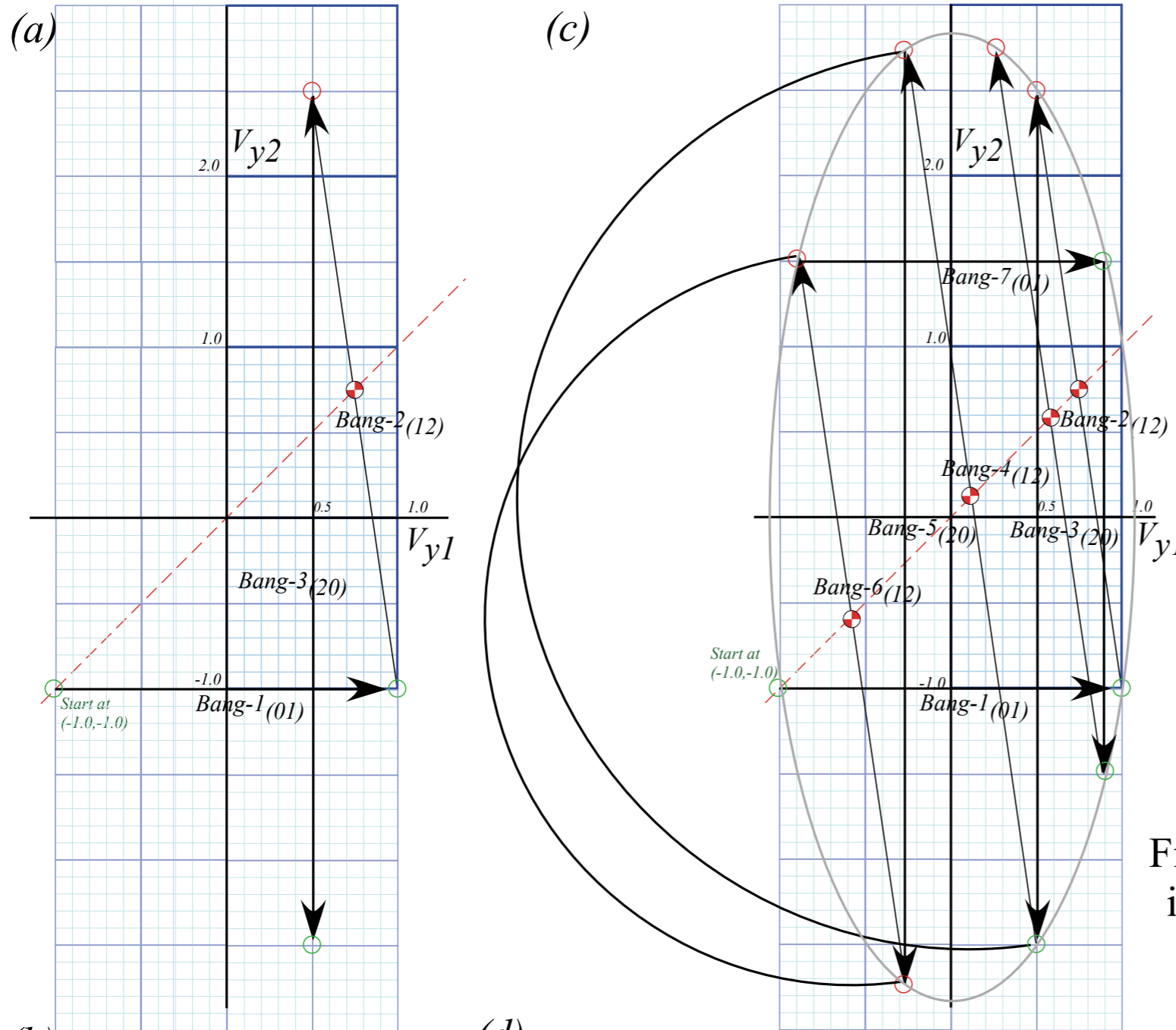


Fig. 4.7a-d
in Unit 1

Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

Ellipse radius 2

$$a_1 = \sqrt{2KE / M_1}$$

$$a_2 = \sqrt{2KE / M_2}$$

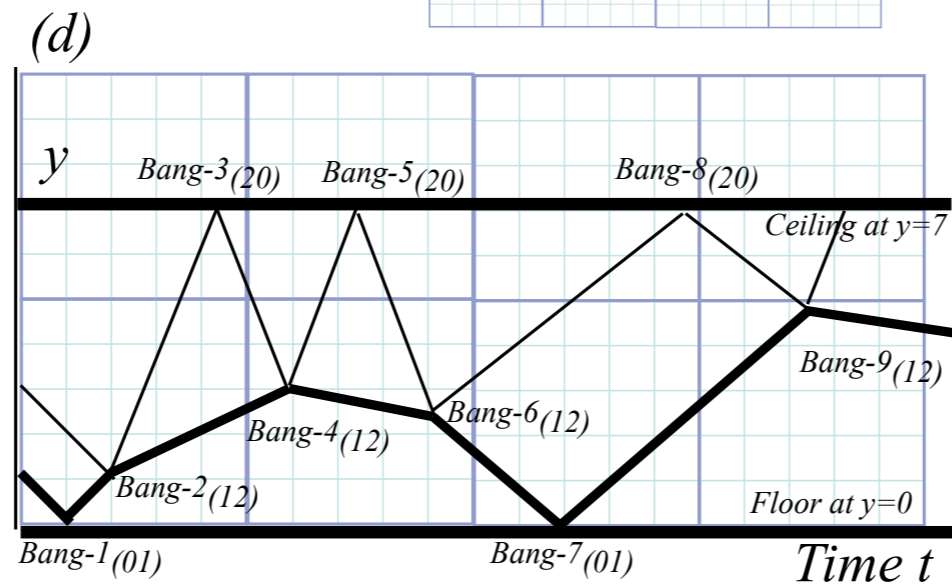
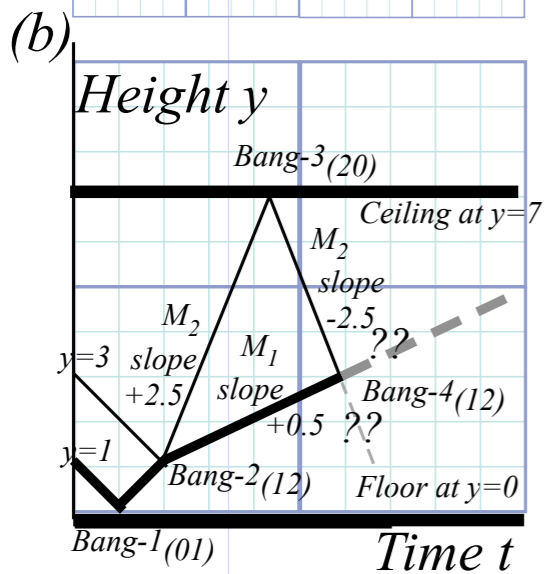
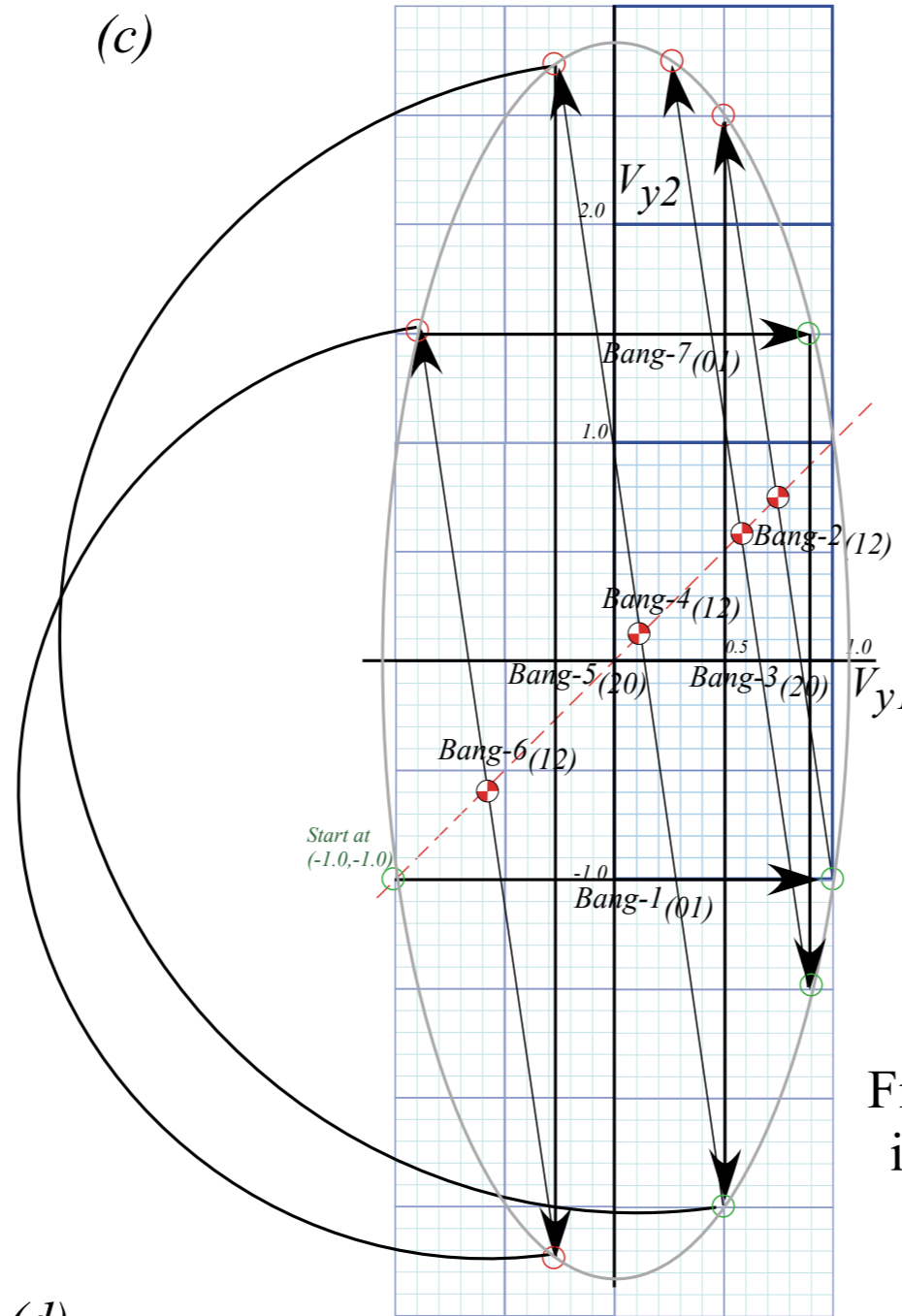
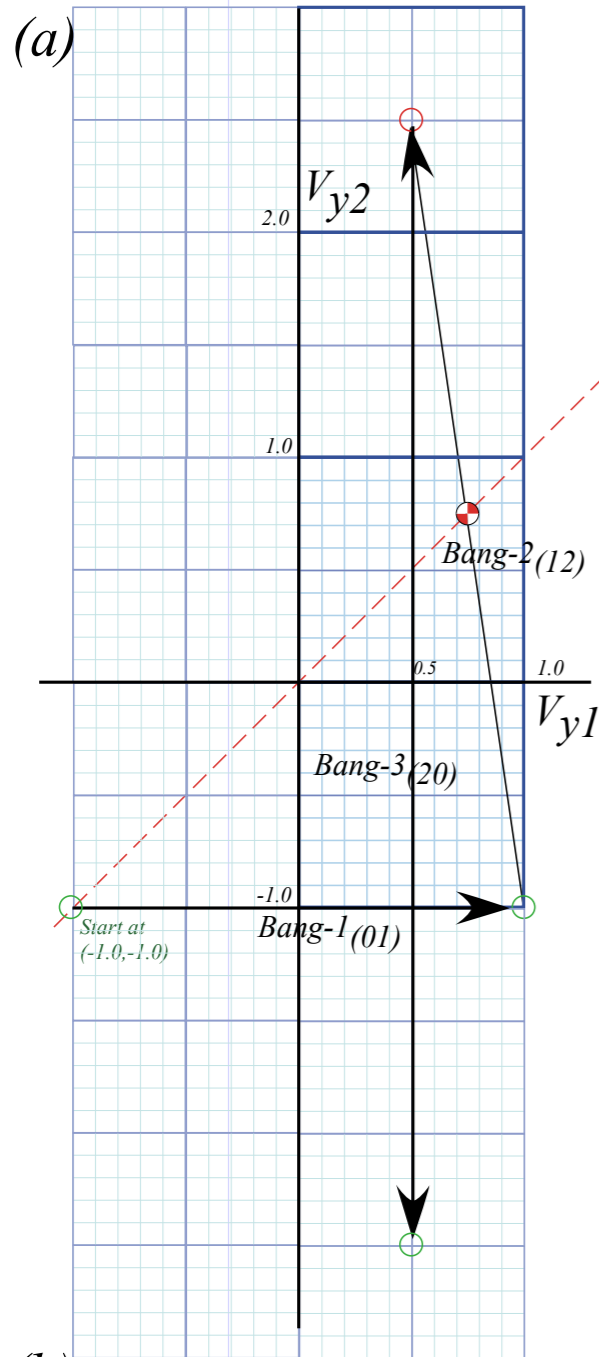


Fig. 4.7a-d
in Unit 1

Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE / M_1}$$

$$= \sqrt{2KE / 7}$$

$$= \sqrt{8/7}$$

$$= 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE / M_2}$$

$$= \sqrt{2KE / 1}$$

$$= \sqrt{8/1}$$

$$= 2.83$$

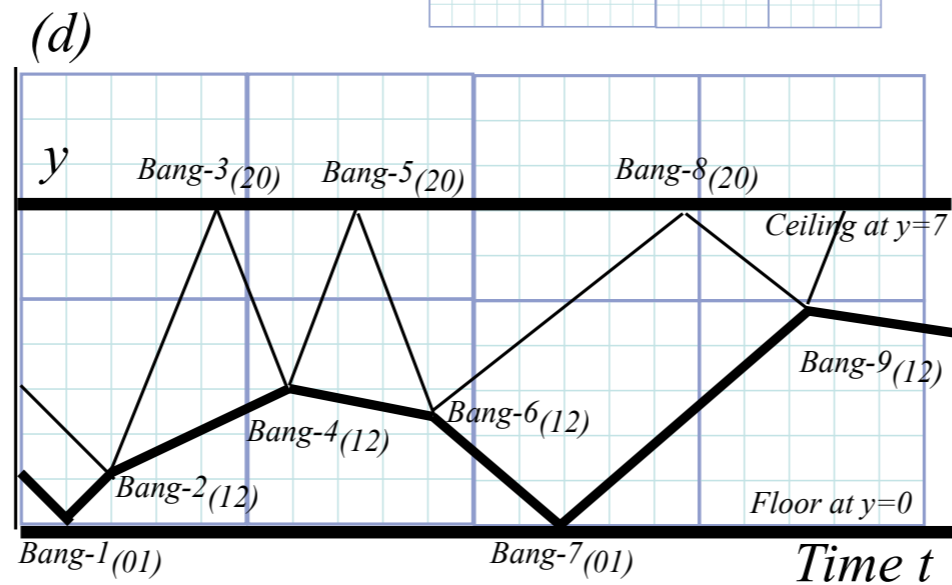
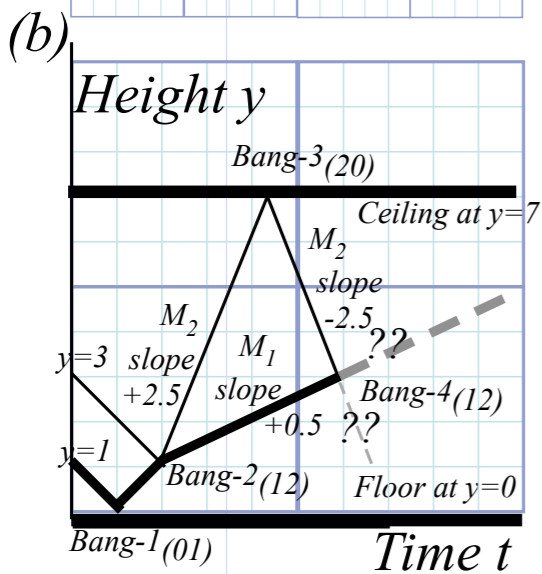
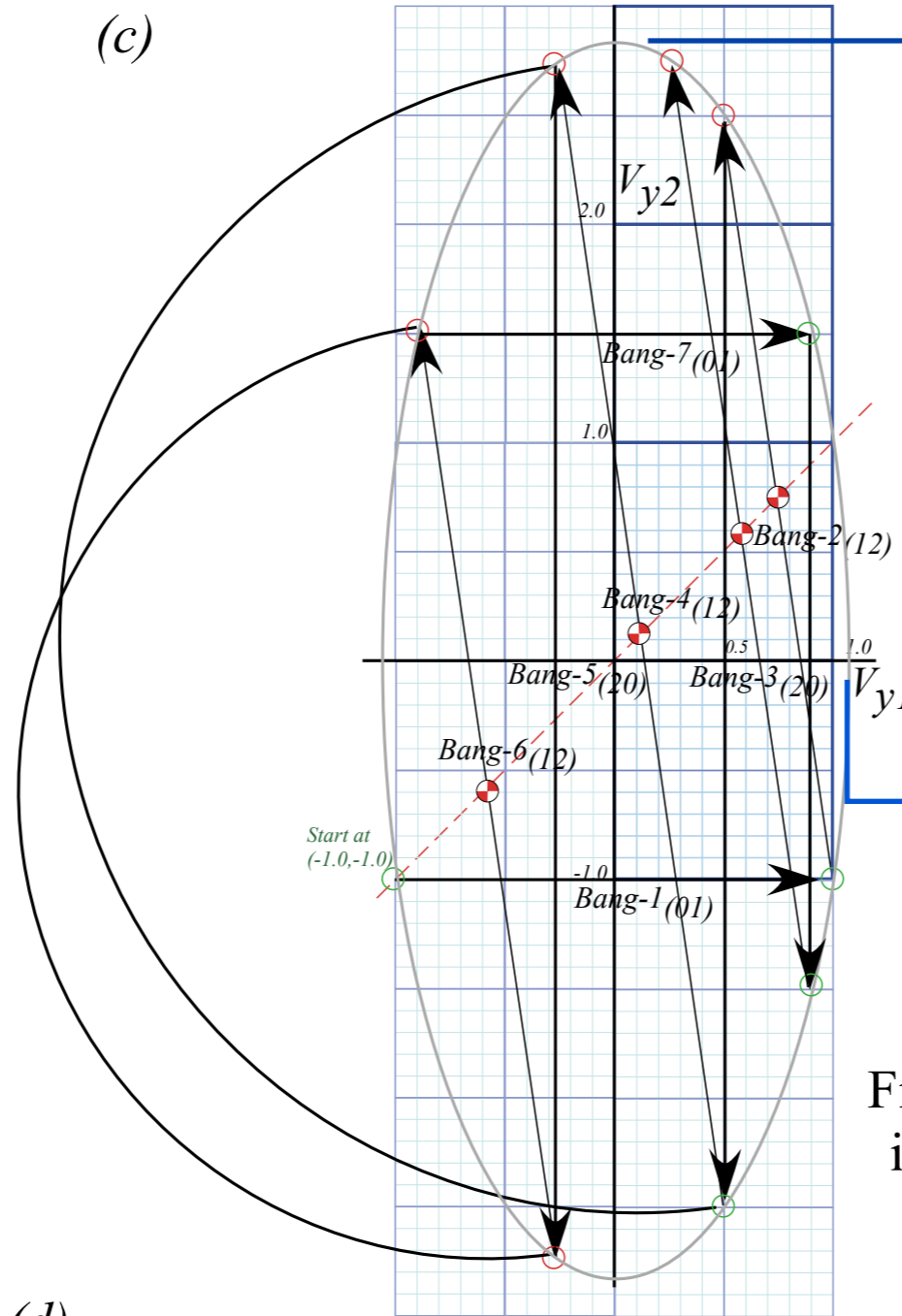
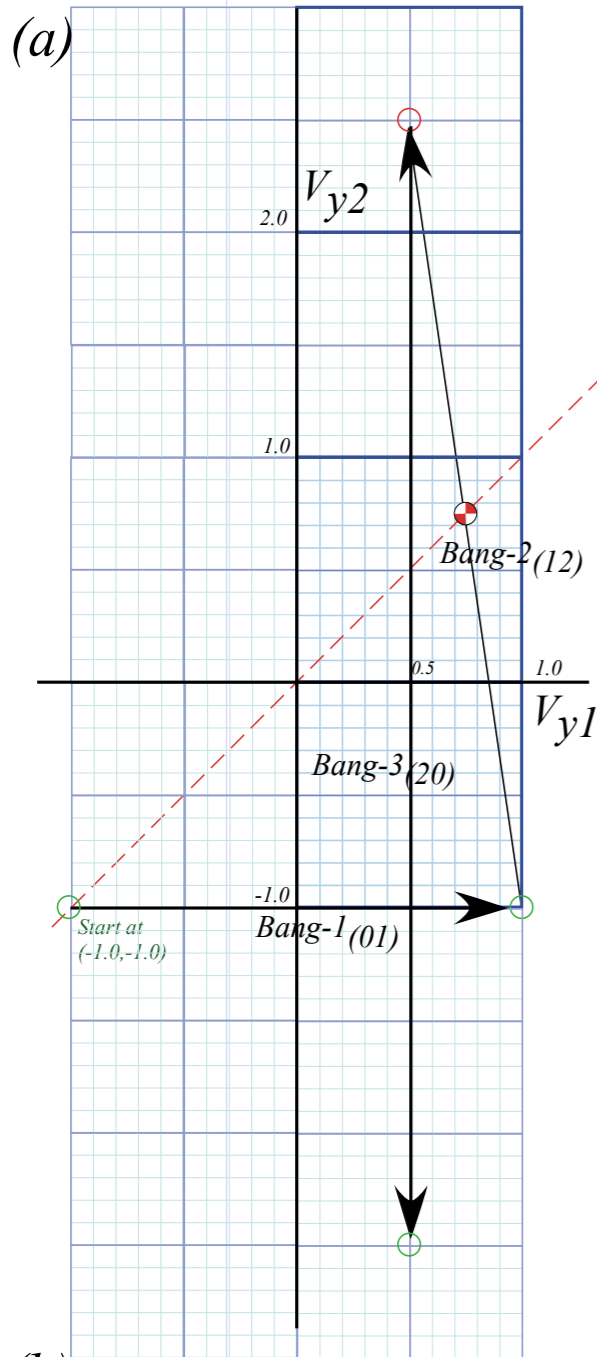


Fig. 4.7a-d
in Unit 1

Geometric "Integration" (Converting Velocity data to Spacetime)

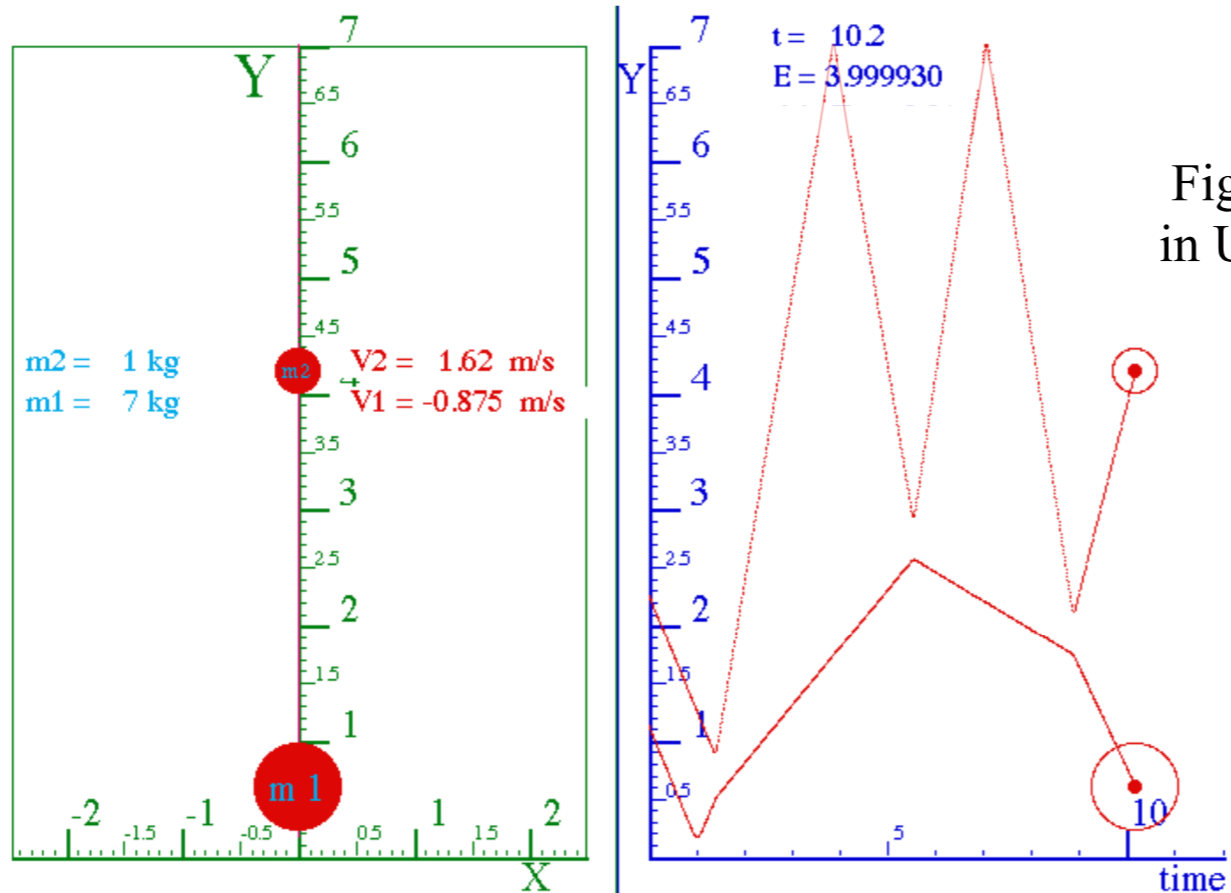


Fig. 4.8
in Unit 1

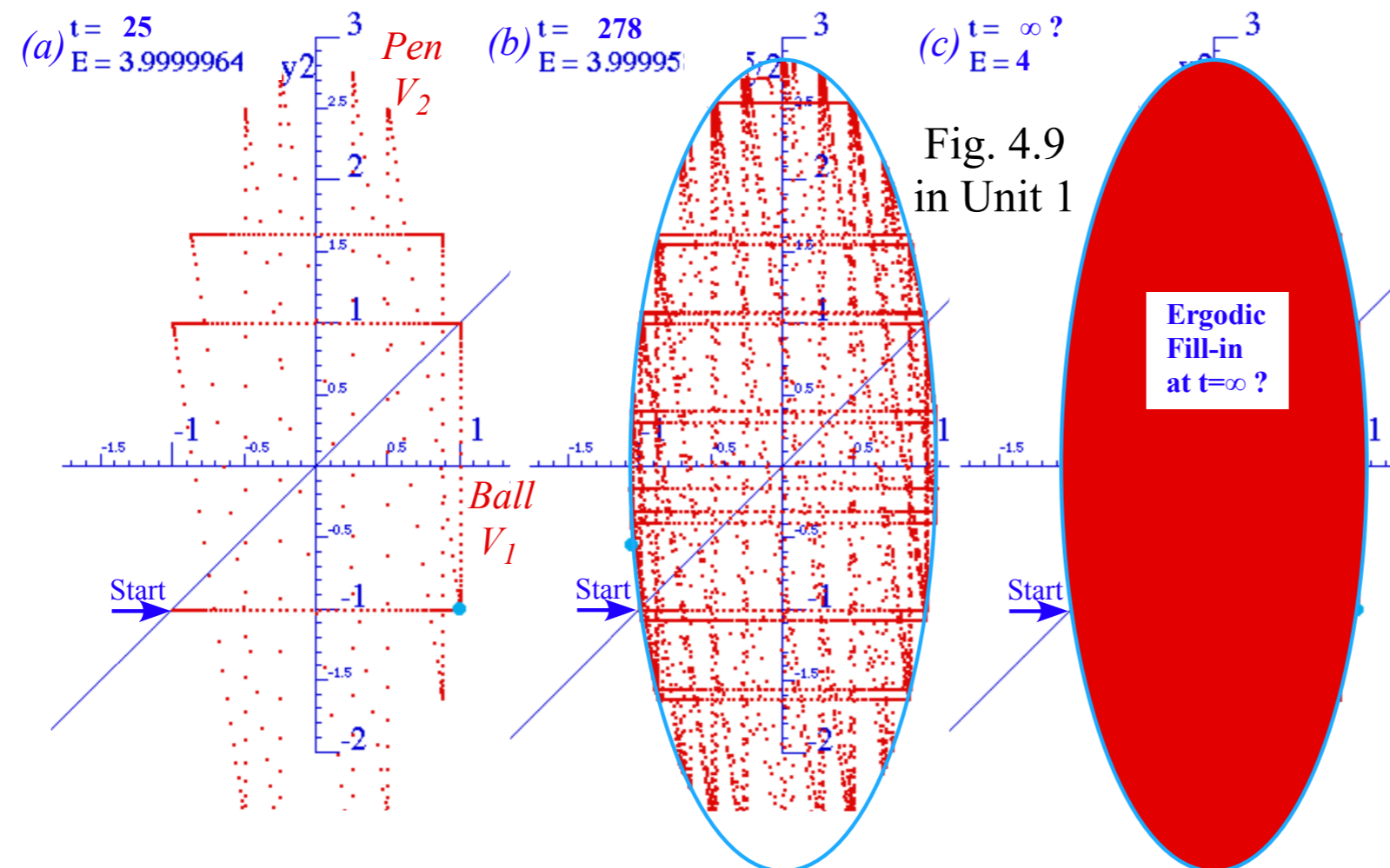


Fig. 4.9
in Unit 1

Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

Geometric optimization and range-of-motion calculation(t)

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)



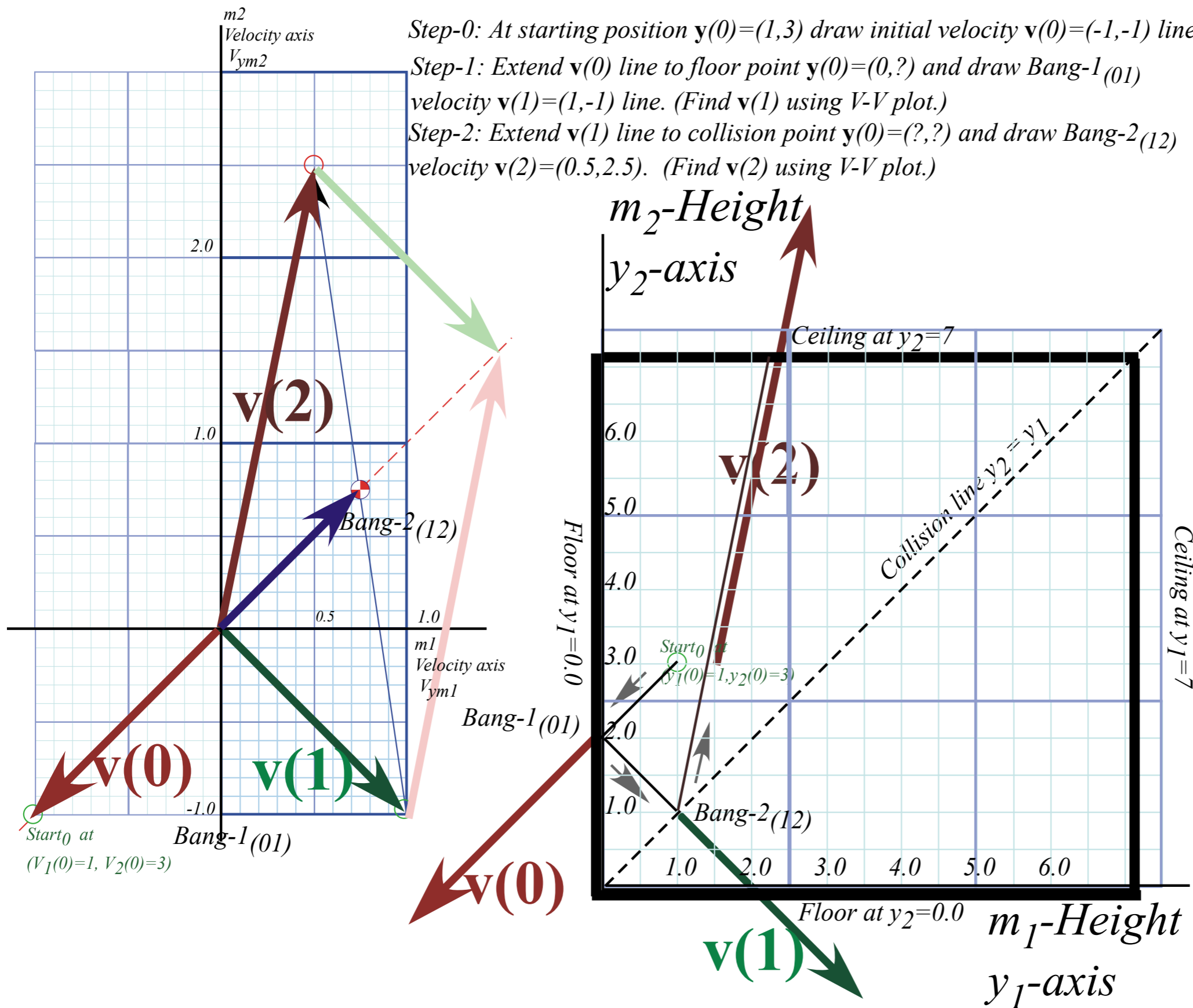
Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

Step-0: At starting position $\mathbf{y}(0)=(1,3)$ draw initial velocity $\mathbf{v}(0)=(-1,-1)$ line.

Step-1: Extend $\mathbf{v}(0)$ line to floor point $\mathbf{y}(0)=(0,?)$ and draw Bang-1(01) velocity $\mathbf{v}(1)=(1,-1)$ line. (Find $\mathbf{v}(1)$ using V-V plot.)

Step-2: Extend $\mathbf{v}(1)$ line to collision point $\mathbf{y}(0)=(?,?)$ and draw Bang-2(12) velocity $\mathbf{v}(2)=(0.5,2.5)$. (Find $\mathbf{v}(2)$ using V-V plot.)



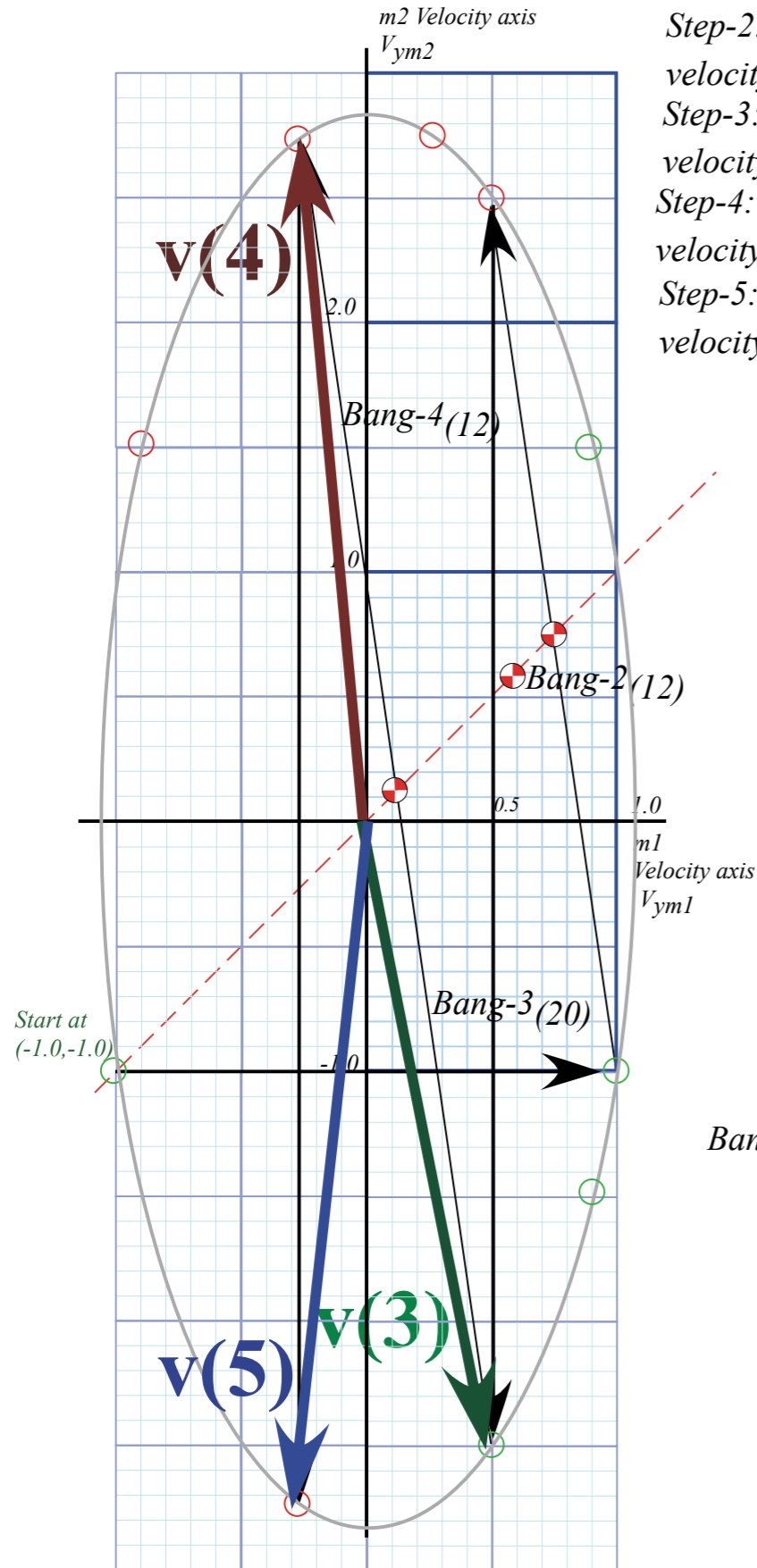
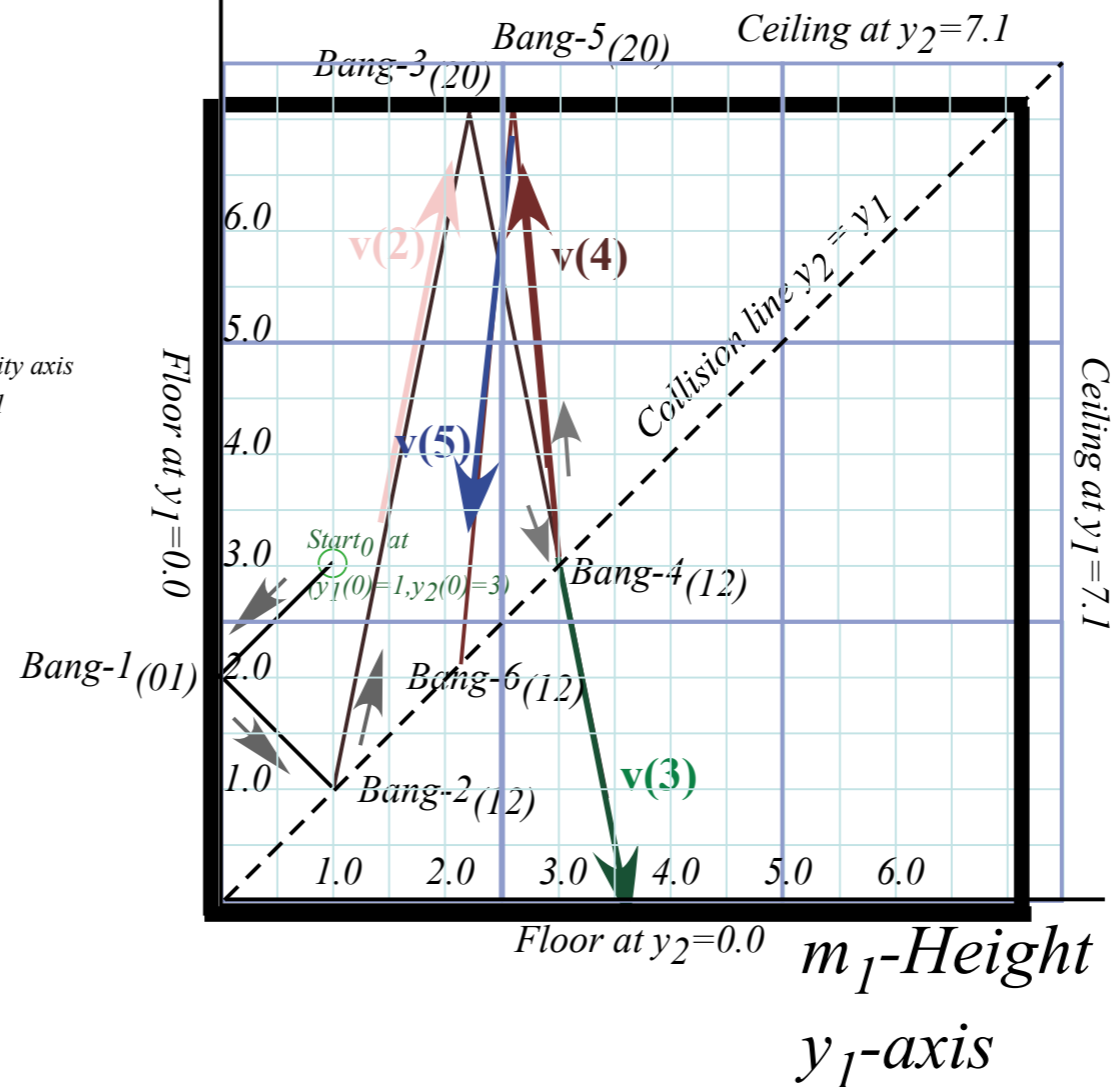
Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

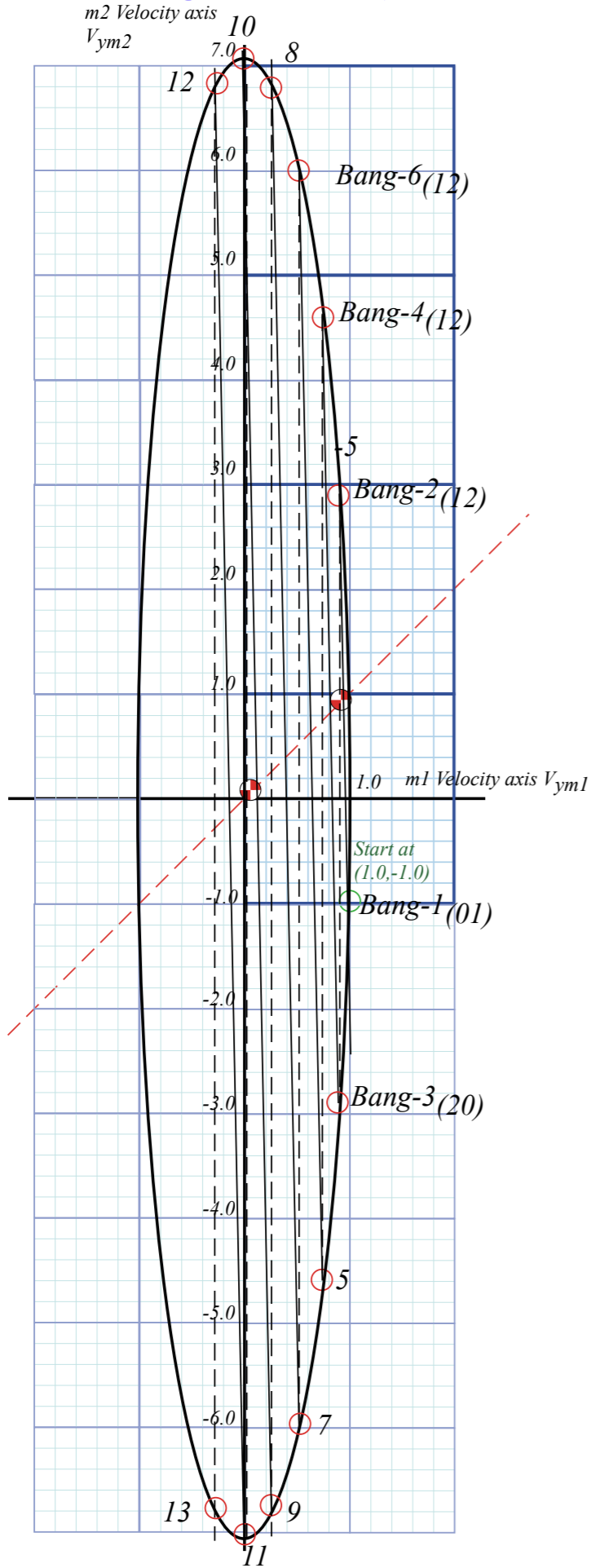
- Step-2: Extend $\mathbf{v}(2)$ line to ceiling point $\mathbf{y}(3)=(?, 7.1)$ and draw Bang-3(20) velocity $\mathbf{v}(3)=(1, -1)$ line. (Find $\mathbf{v}(3)$ using V-V plot.)
- Step-3: Extend $\mathbf{v}(3)$ line to collision point $\mathbf{y}(4)=(?, ?)$ and draw Bang-4(12) velocity $\mathbf{v}(4)=(0.5, 2.5)$. (Find $\mathbf{v}(4)$ using V-V plot.)
- Step-4: Extend $\mathbf{v}(4)$ line to ceiling point $\mathbf{y}(4)=(?, 7.1)$ and draw Bang-5(20) velocity $\mathbf{v}(5)=(1, -1)$ line. (Find $\mathbf{v}(5)$ using V-V plot.)
- Step-5: Extend $\mathbf{v}(5)$ line to collision point $\mathbf{y}(6)=(?, ?)$ and draw Bang-6(12) velocity $\mathbf{v}(6)=(0.5, 2.5)$. (Find $\mathbf{v}(6)$ using V-V plot.)

m_2 -Height

y_2 -axis



Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: $m_1=49$ and $m_2=1$

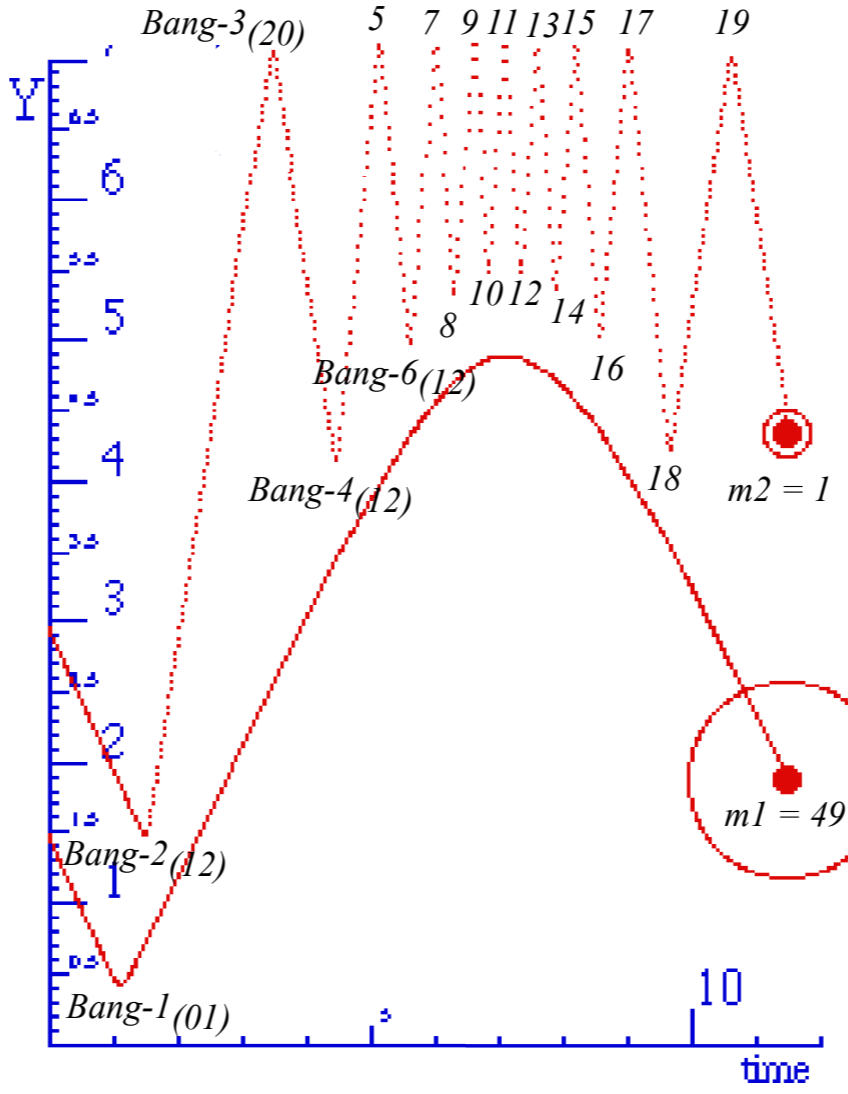
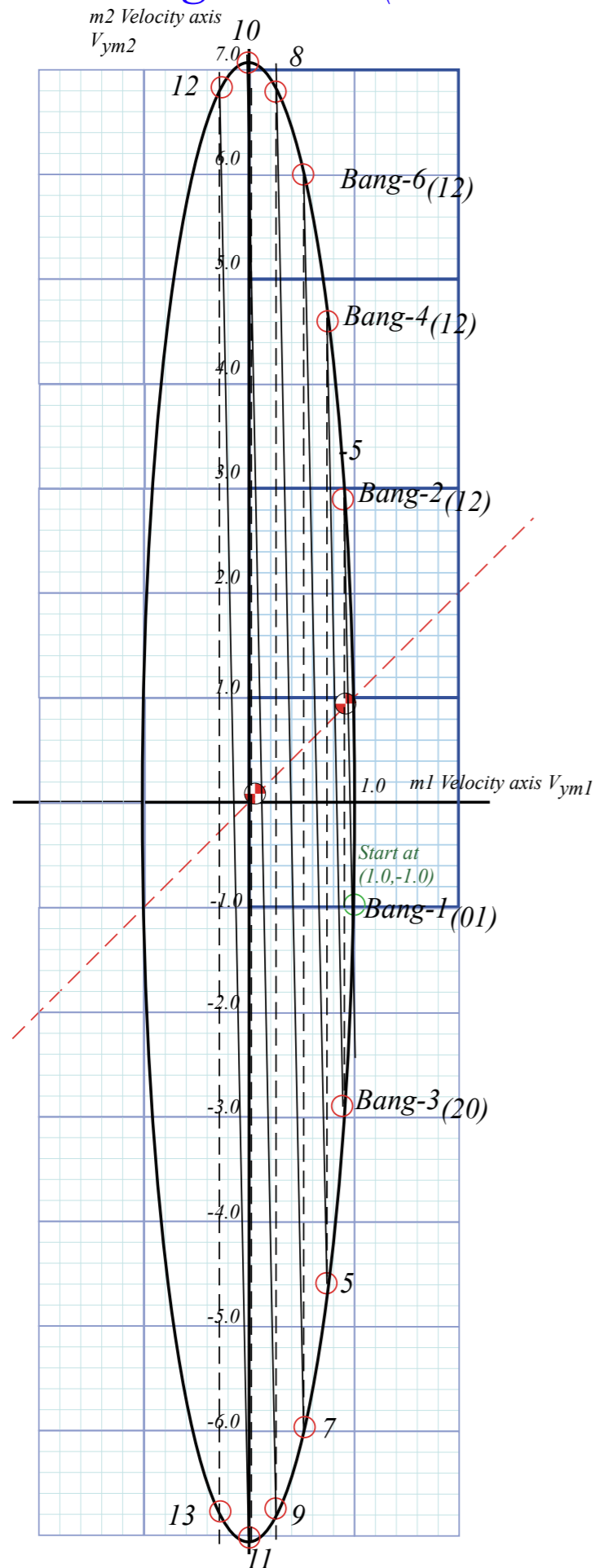


Fig. 5.1
in Unit 1

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: $m_1=49$ and $m_2=1$

Kinetic Energy Ellipse

$$KE = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$\begin{aligned} a_1 &= \sqrt{2KE/M_1} \\ &= \sqrt{2KE/49} \\ &= \sqrt{50/49} \\ &= 1.01 \end{aligned}$$

Ellipse radius 2

$$\begin{aligned} a_2 &= \sqrt{2KE/m_2} \\ &= \sqrt{2KE/1} \\ &= \sqrt{50/1} \\ &= 7.07 \end{aligned}$$

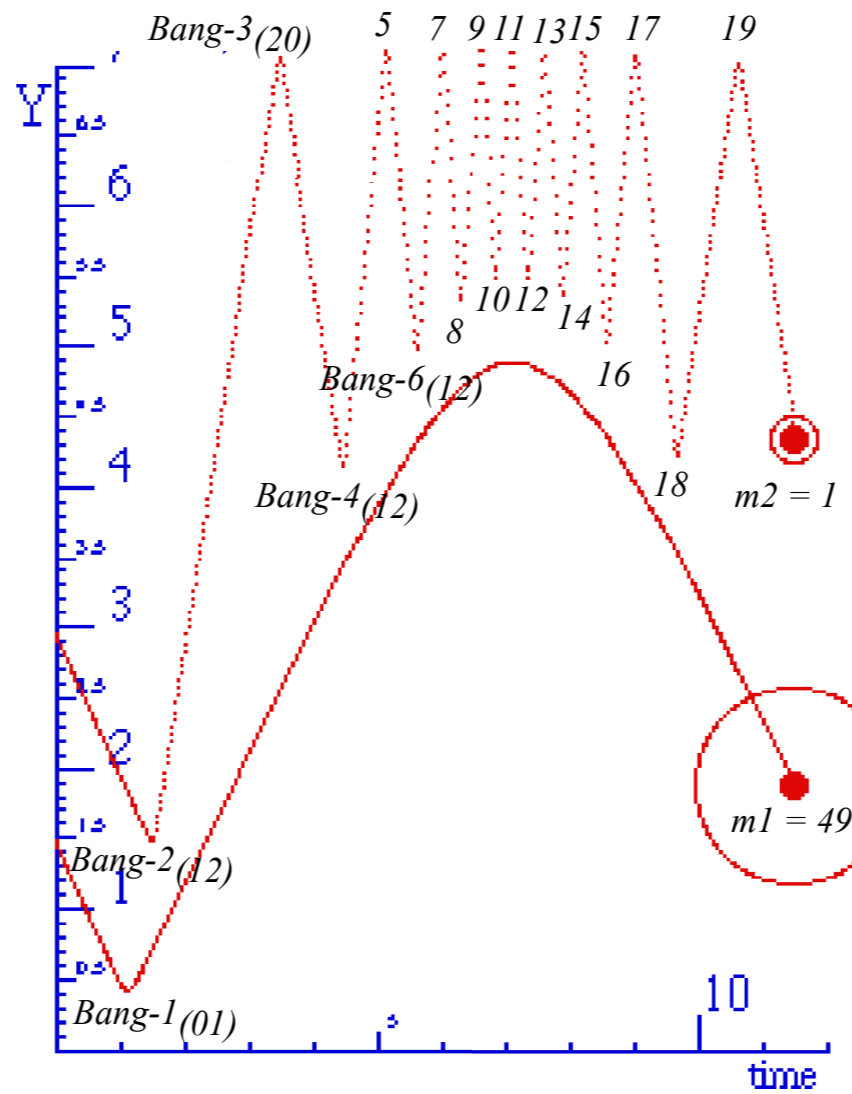


Fig. 5.1
in Unit 1

Multiple collisions calculated by matrix operator products

 *Matrix or tensor algebra of 1-D 2-body collisions*

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:
$$V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} =$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $v^{FIN} = \mathbf{M} \cdot v^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***



Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $v^{FIN} = \mathbf{M} \cdot v^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $v^{FIN} = \mathbf{M} \cdot v^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***



Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $v^{FIN} = \mathbf{M} \cdot v^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Define "ellipse-Rotation" \mathbf{R} as group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} \left(\begin{array}{c} |IN^0\rangle \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{array} \right) \\
 \left(\begin{array}{c} v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right) &= \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ 1.96 & -0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ 1.96 & -0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ 1.96 & -0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ 1.96 & -0.96 \end{array} \right) \cdot \left(\begin{array}{cc} -1 & 0 \\ 0 & +1 \end{array} \right) \left(\begin{array}{c} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{array} \right)
 \end{aligned}$$

(INITIAL (0))

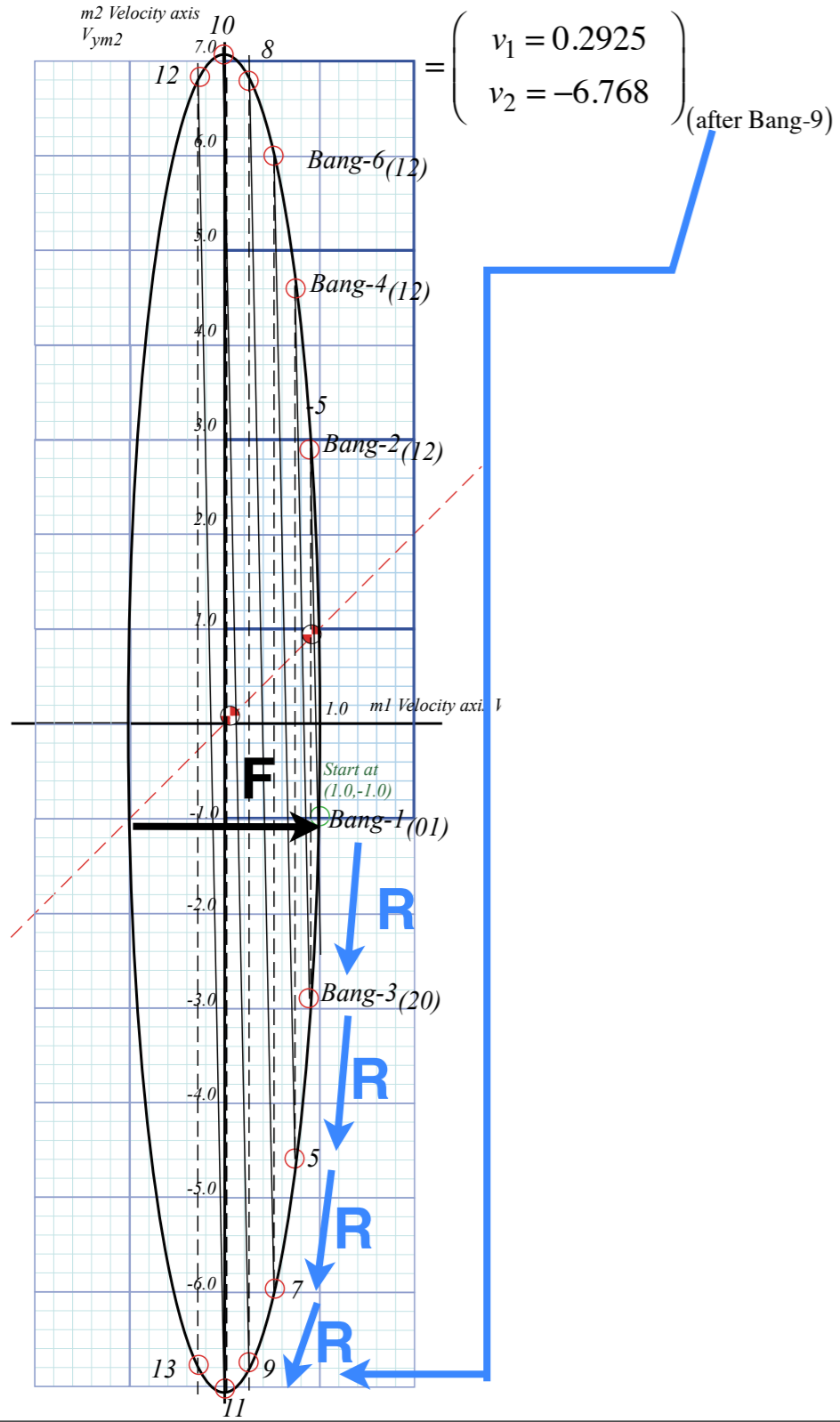
$$\begin{aligned}
 \left| \begin{matrix} FIN^9 \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{matrix} \right\rangle &= \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \mathbf{F} \left| \begin{matrix} IN^0 \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{matrix} \right\rangle_{\text{(INITIAL (0))}} \\
 \left| \begin{matrix} FIN^9 \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{matrix} \right\rangle &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{F} | IN^0 \rangle \\ v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{\text{(after Bang-1)}}
 \end{aligned}$$

“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 \begin{pmatrix} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \mathbf{F} \begin{pmatrix} |IN^0\rangle \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL } (0))} \\
 \begin{pmatrix} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} \mathbf{R} \\ \mathbf{R} \\ \mathbf{R} \\ \mathbf{R} \end{pmatrix} \cdot \mathbf{F} |IN^0\rangle \\
 &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{(\text{after Bang-1})} \\
 &= \begin{pmatrix} v_1 = 0.2925 \\ v_2 = -6.768 \end{pmatrix}_{(\text{after Bang-9})}
 \end{aligned}$$

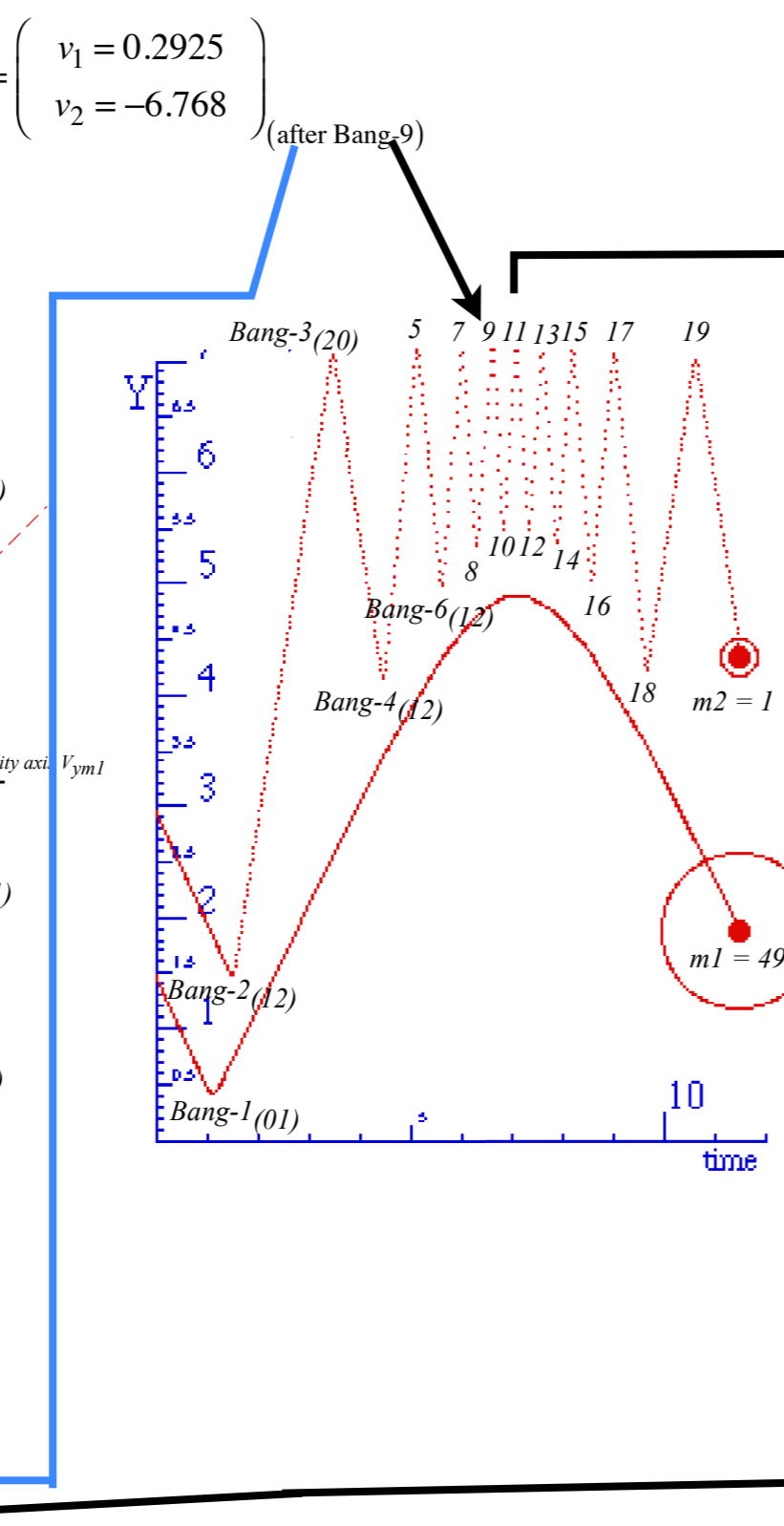
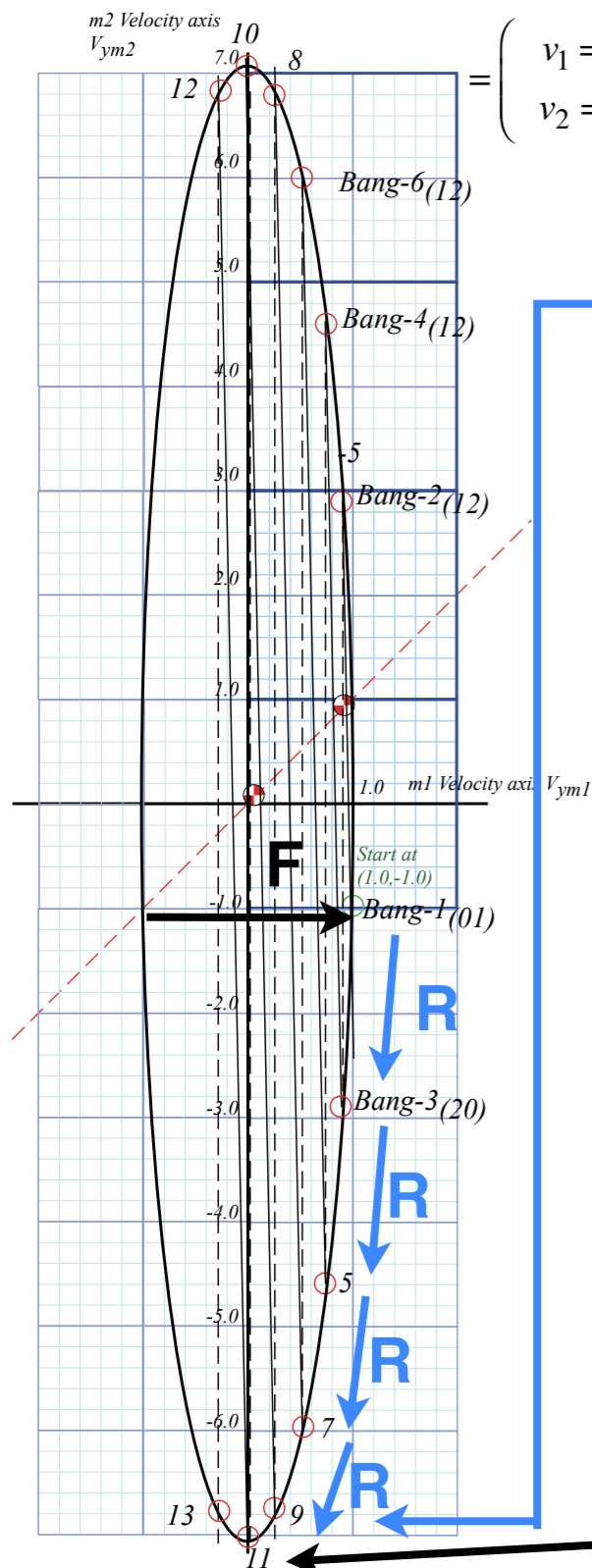
“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix} \quad (\text{INITIAL } (0)) \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix} \quad (\text{after Bang-1})
 \end{aligned}$$



“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix} \quad (\text{INITIAL } (0)) \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix} \quad (\text{after Bang-1})
 \end{aligned}$$



“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 \begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} \\
 &= \begin{pmatrix} v_1 = 0.0100 \\ v_2 = -7.071 \end{pmatrix} \quad (\text{after Bang-11})
 \end{aligned}$$

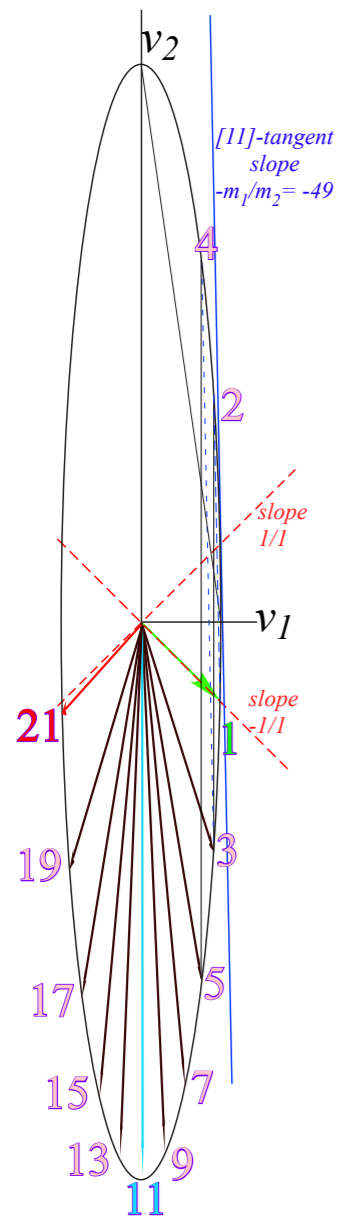
Ellipse rescaling-geometry and reflection-symmetry analysis

 *Rescaling KE ellipse to circle*

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Ellipse rescaling geometry and reflection symmetry analysis

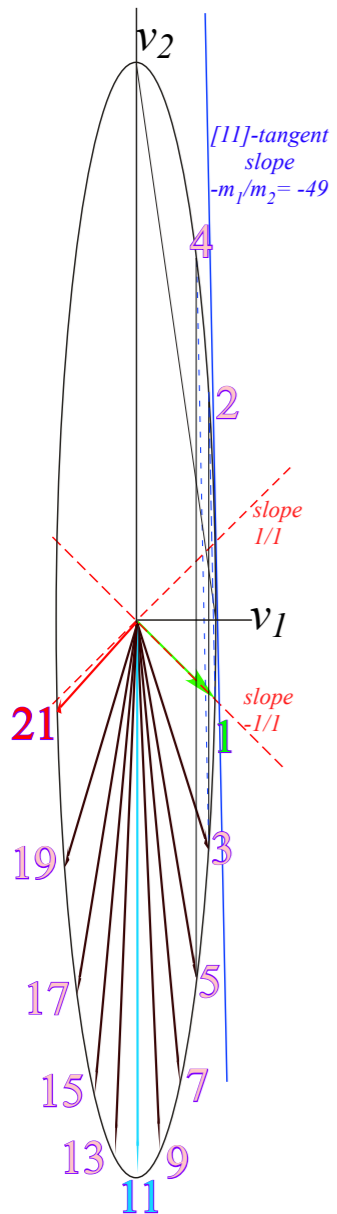
Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN_1} / \sqrt{m_1} \\ V_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

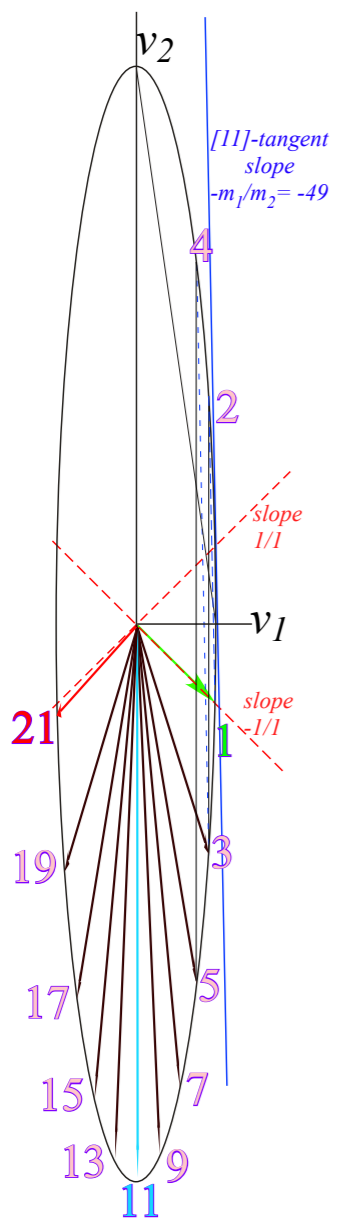


Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or:} \quad \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or:} \quad \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$



Ellipse rescaling geometry and reflection symmetry analysis

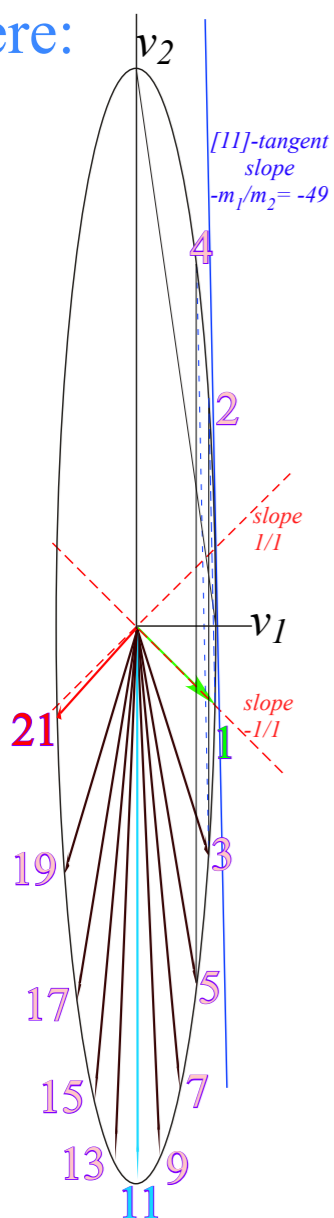
Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \frac{m_1 - m_2}{m_1 + m_2}$ and: $\sin\theta \equiv \frac{2\sqrt{m_1 m_2}}{m_1 + m_2}$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$

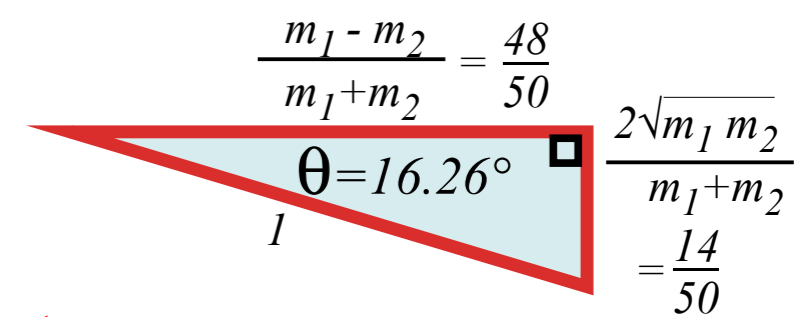
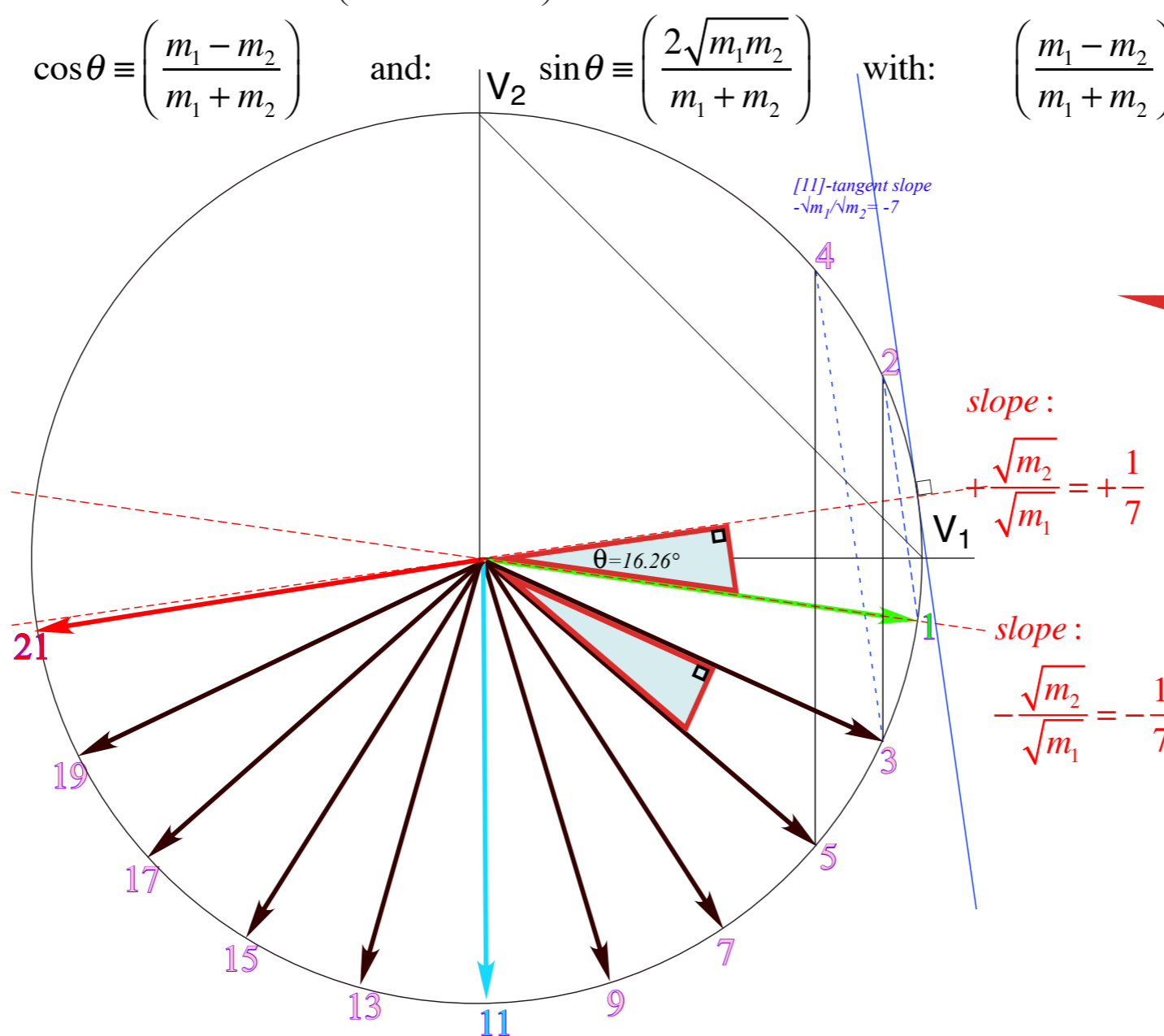
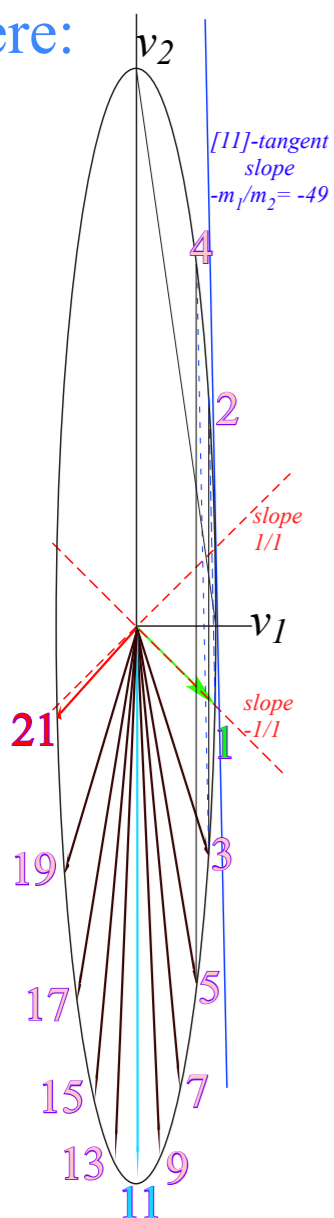


Fig. 5.2a-c
(revised)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \frac{m_1 - m_2}{m_1 + m_2}$ and: $\sin\theta \equiv \frac{2\sqrt{m_1 m_2}}{m_1 + m_2}$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$

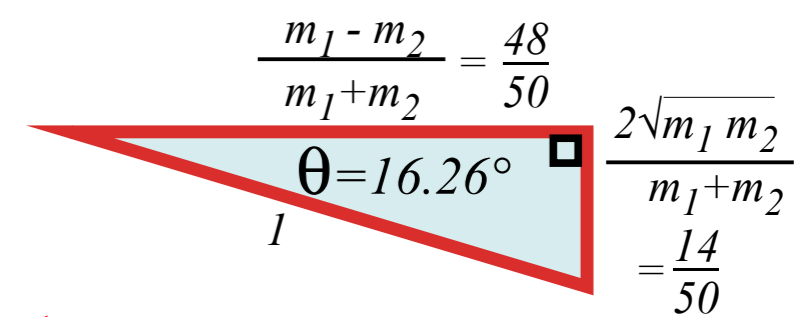
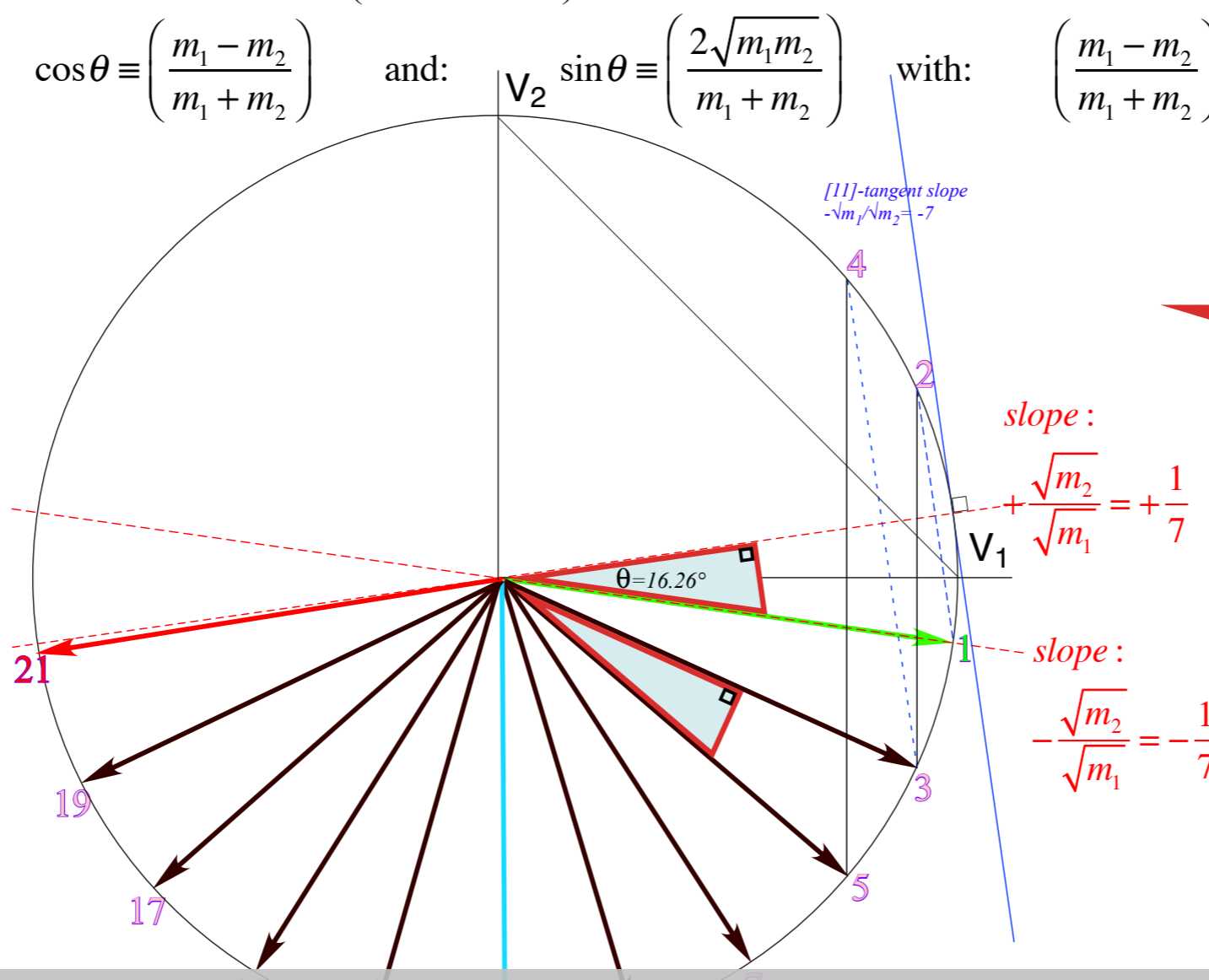
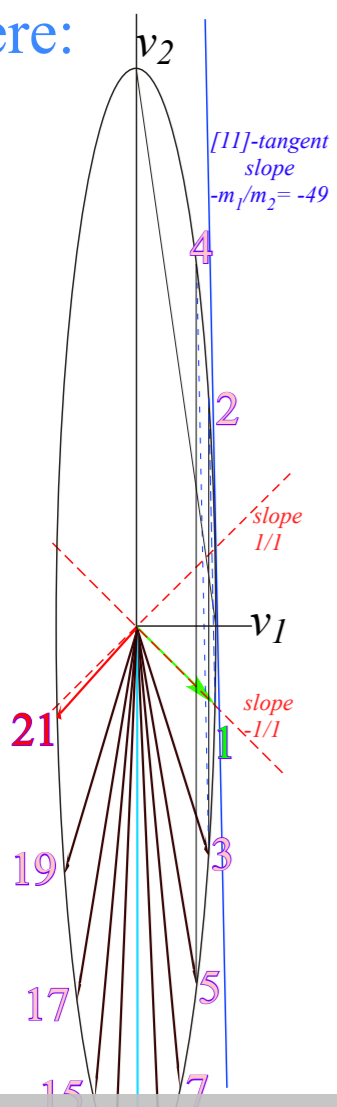


Fig. 5.2a-c
(revised)

Note: If $m_1 \cdot m_2$ is perfect-square, then θ -triangle is rational ($3^2 + 4^2 = 5^2$, etc.)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations*

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$

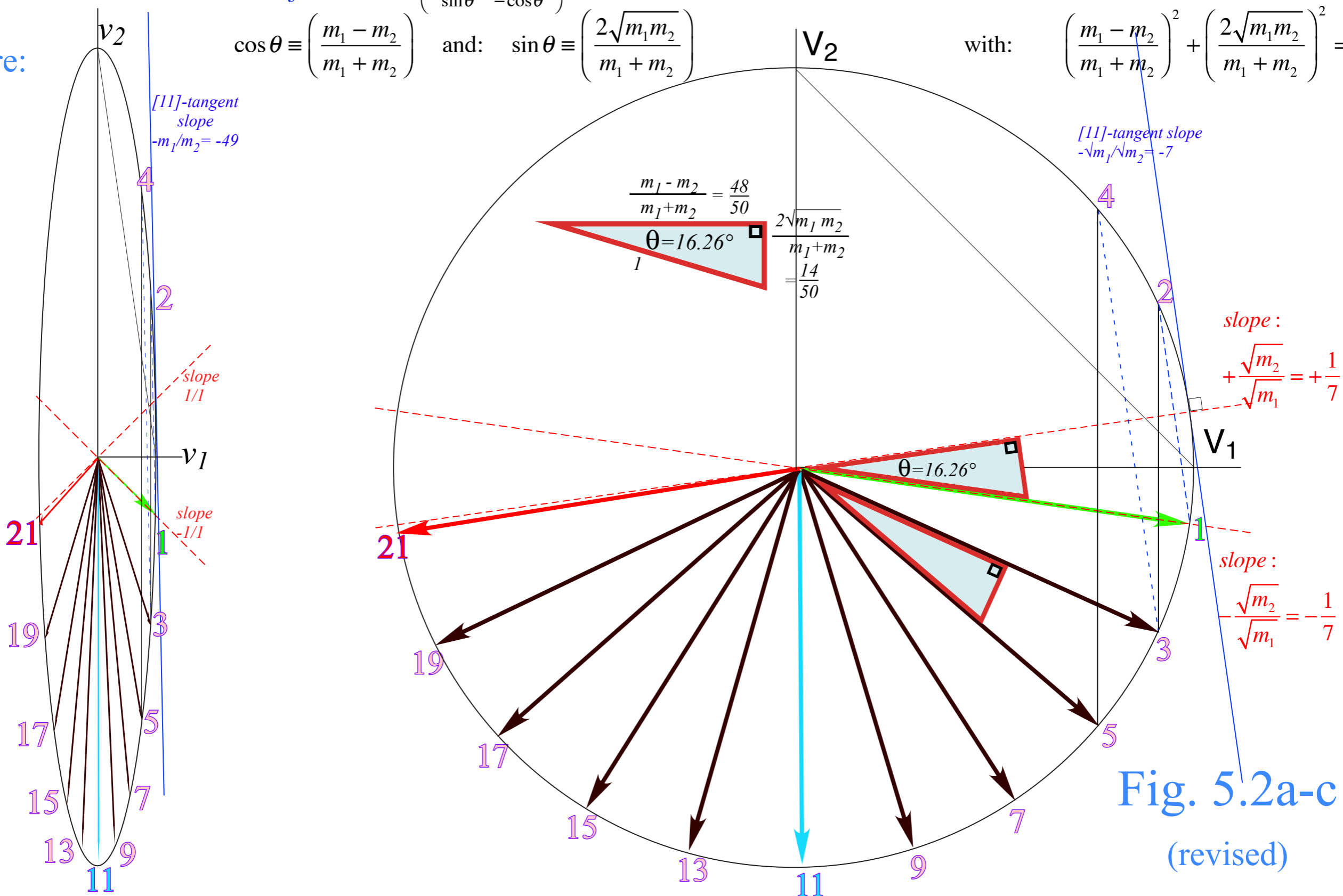


Fig. 5.2a-c
(revised)

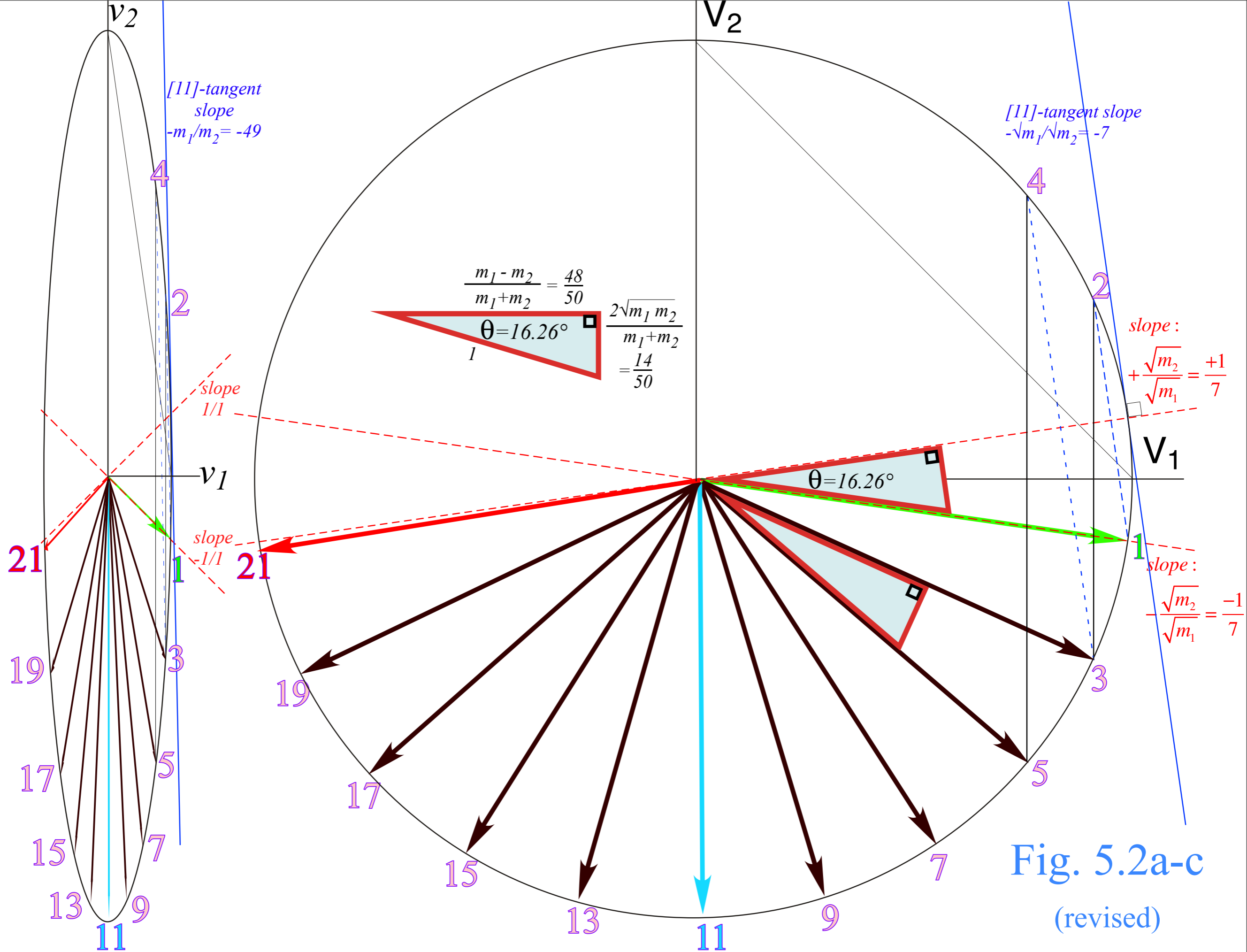


Fig. 5.2a-c
(revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

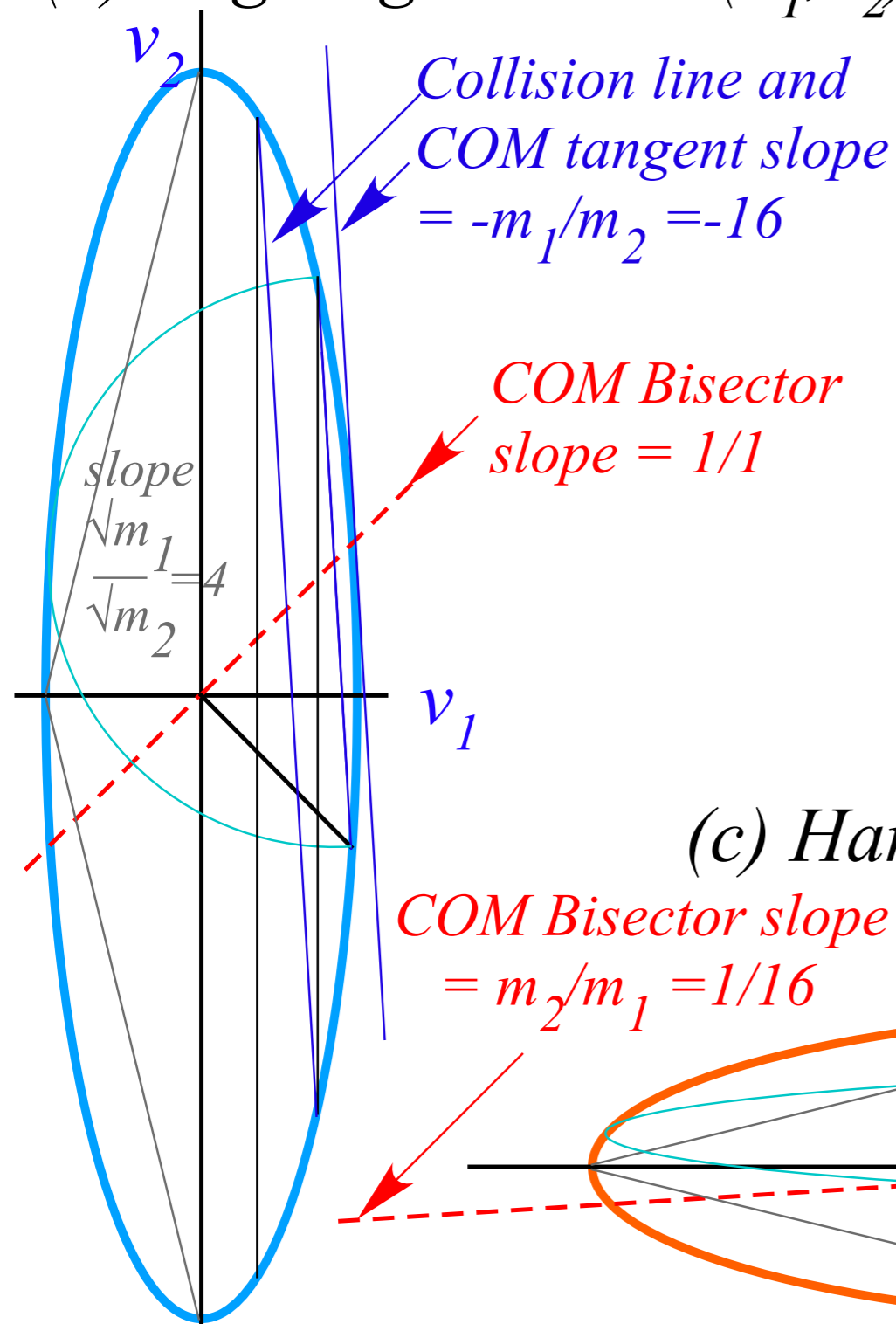
How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12



What ellipse rescaling leads to...(in Ch. 9-12)

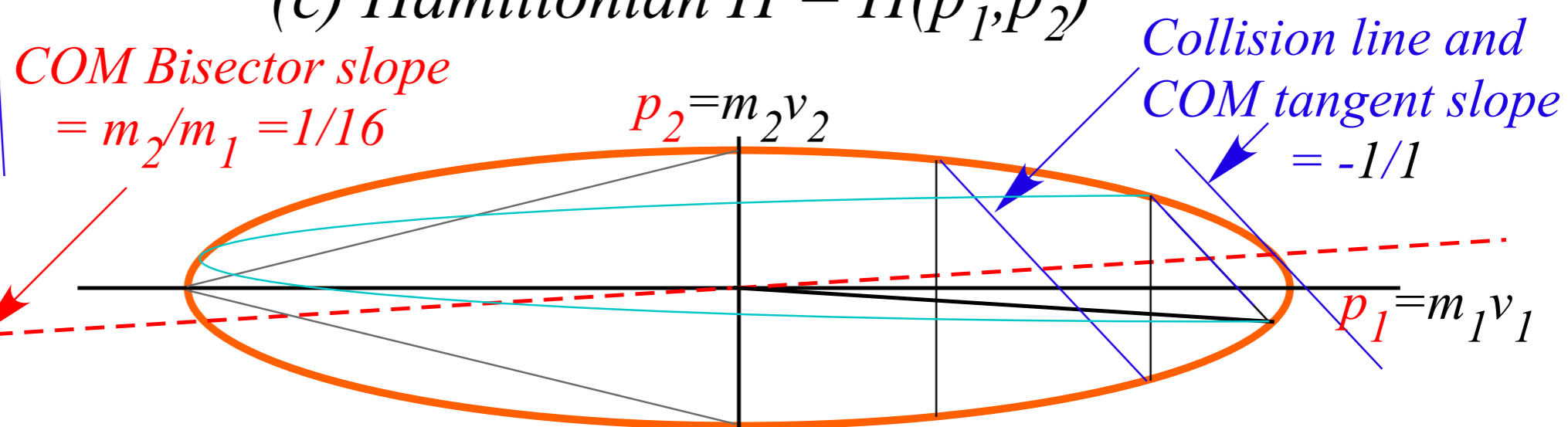
How this relates to Lagrangian, and Hamiltonian mechanics in Ch. 12

(a) Lagrangian $L = L(v_1, v_2)$



velocity v_1 rescaled to momentum: $p_1 = m_1 v_1$
 velocity v_2 rescaled to momentum: $p_2 = m_2 v_2$

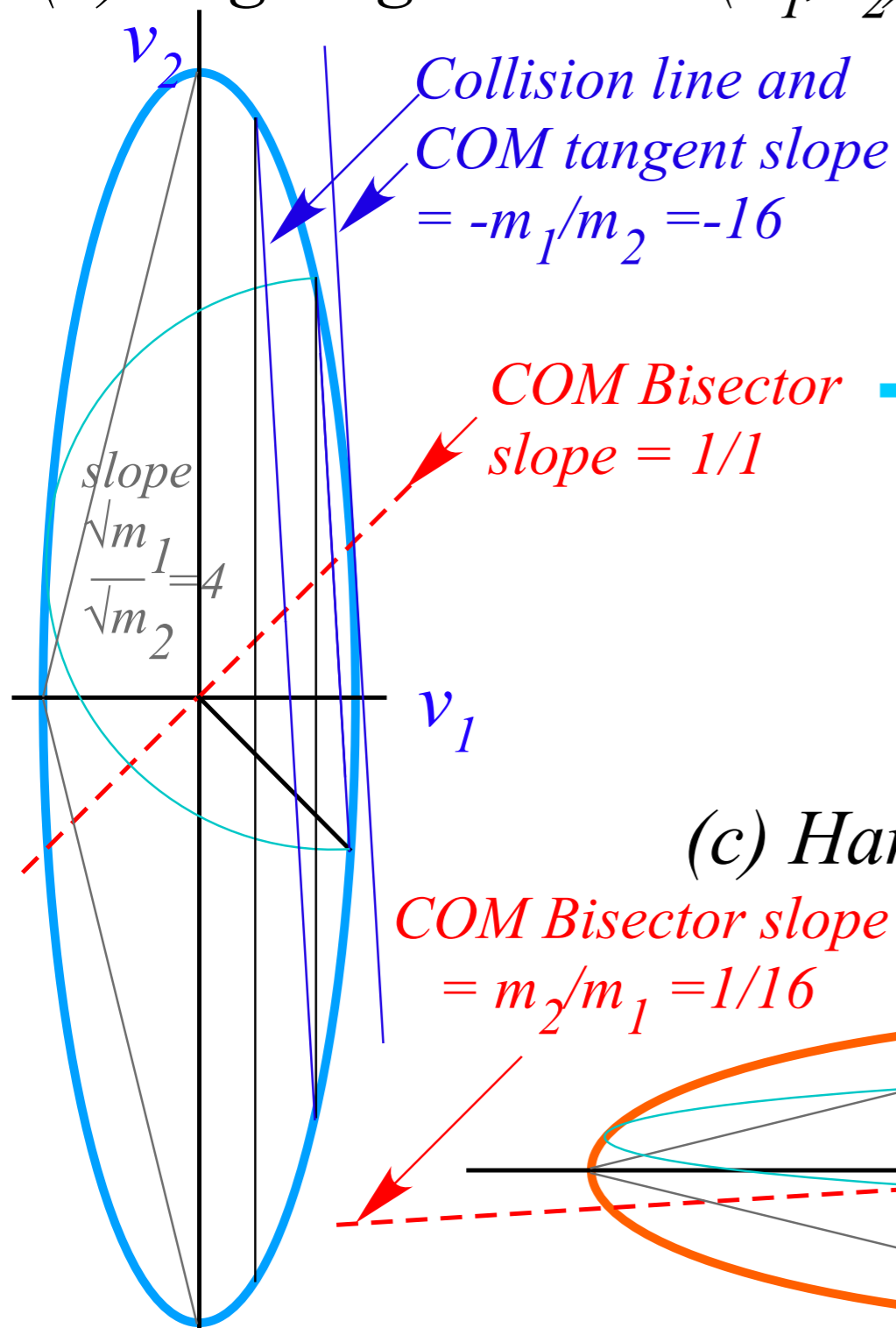
(c) Hamiltonian $H = H(p_1, p_2)$



What ellipse rescaling leads to...(in Ch. 9-12)

How this relates to Lagrangian, and Hamiltonian mechanics in Ch. 12

(a) Lagrangian $L = L(v_1, v_2)$

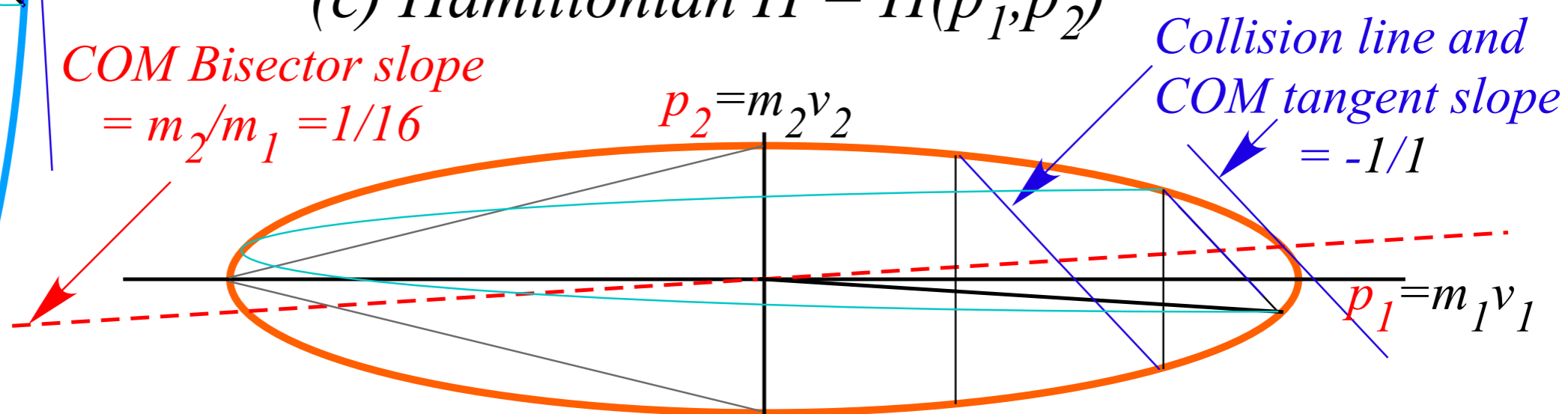


velocity v_1 rescaled to momentum: $p_1 = m_1 v_1$
 velocity v_2 rescaled to momentum: $p_2 = m_2 v_2$

Lagrangian $L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$
 rescaled to

Hamiltonian $H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

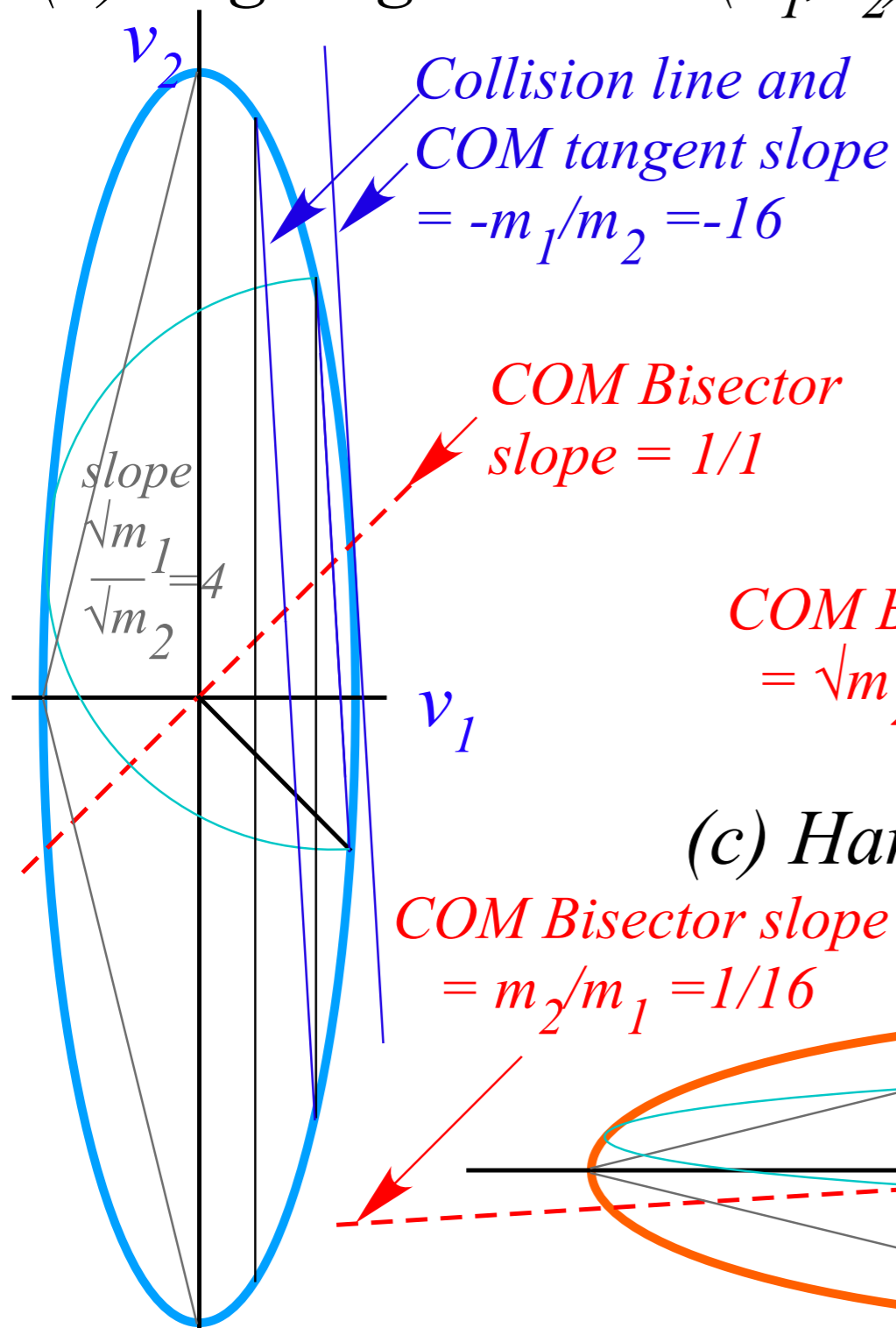
(c) Hamiltonian $H = H(p_1, p_2)$



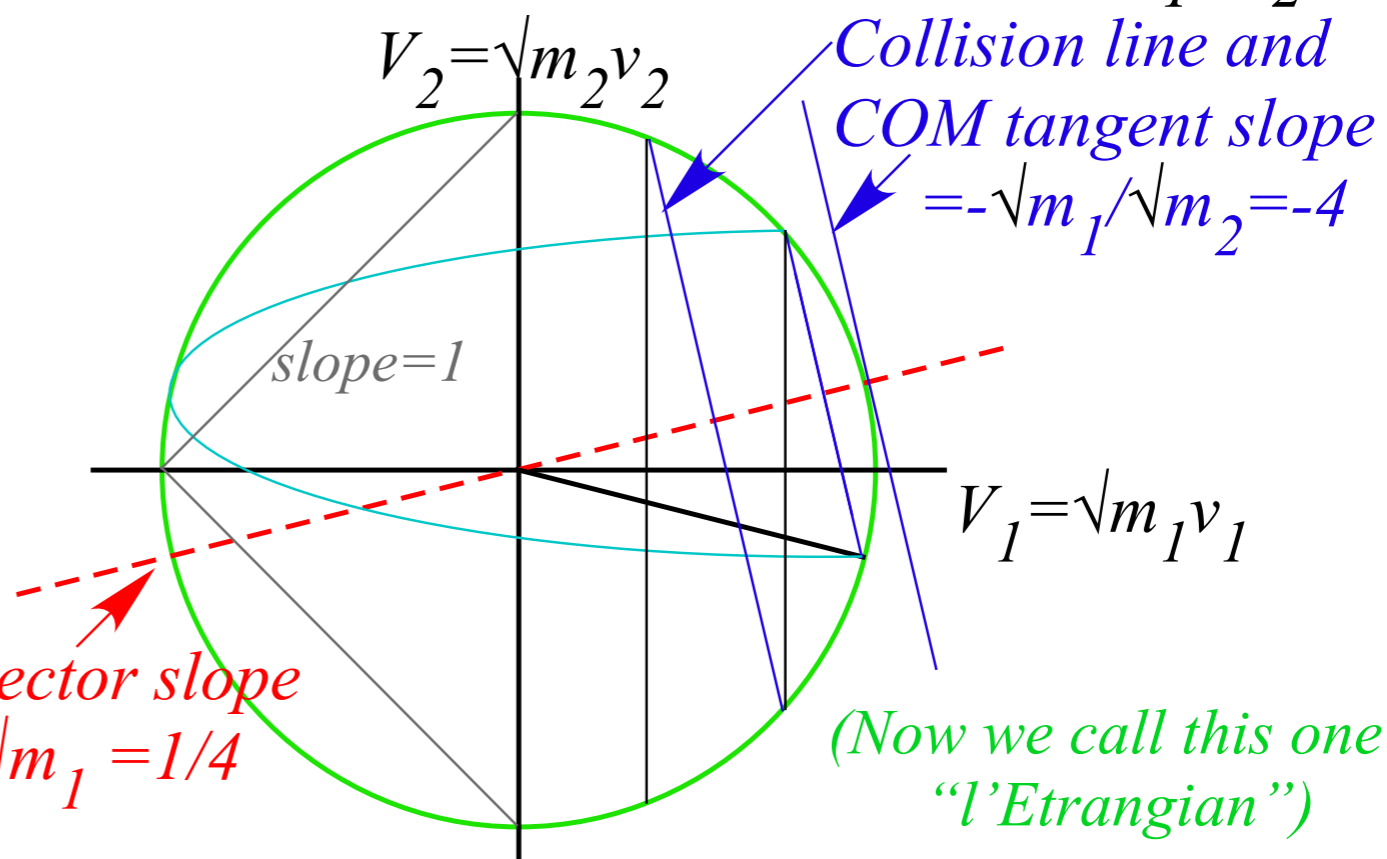
What ellipse rescaling leads to...(in Ch. 9-12)

How this relates to *Lagrangian*, *l'Etrangian*, and *Hamiltonian* mechanics in Ch. 12

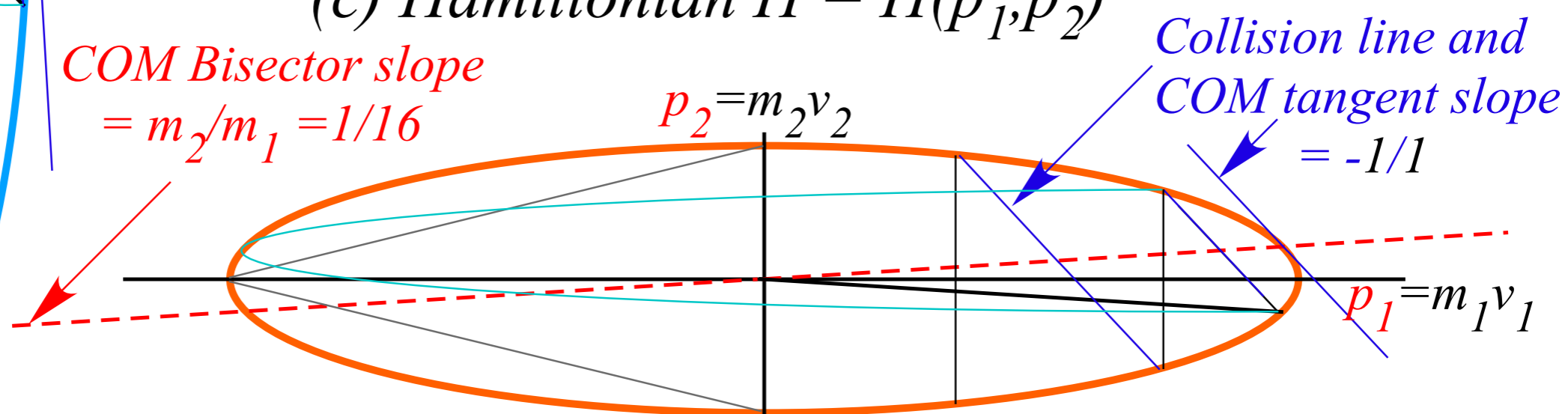
(a) Lagrangian $L = L(v_1, v_2)$



(b) Estrangian $E = E(V_1, V_2)$



(c) Hamiltonian $H = H(p_1, p_2)$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

(Preview of Lecture 3)

 *Reflections in the clothing store: "It's all done with mirrors!"*

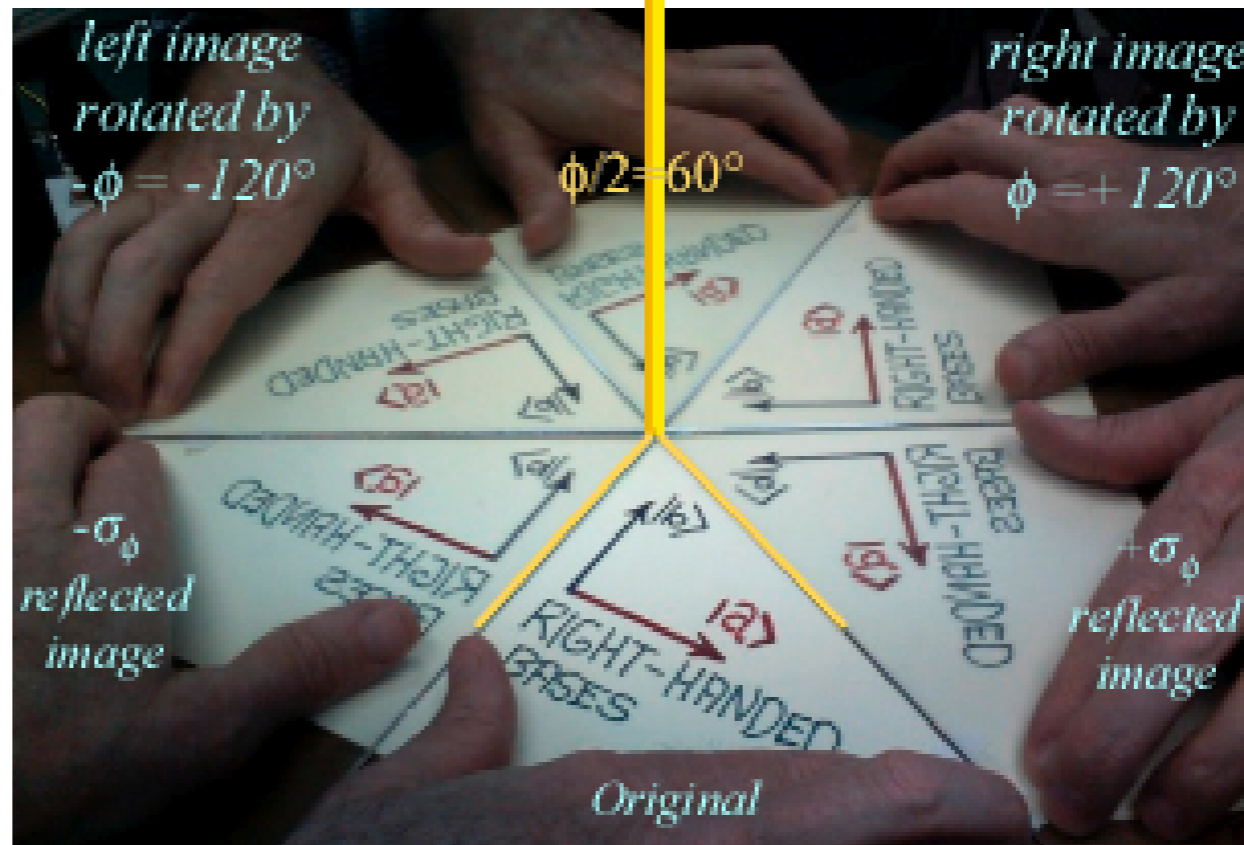
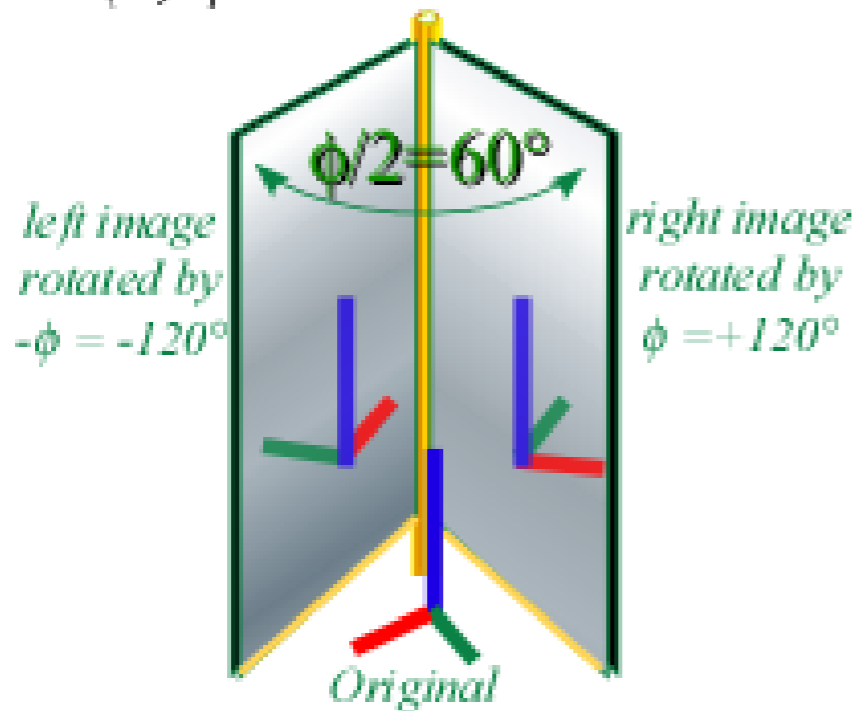
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Reflections in clothing store mirrors

(a) $\phi = \pm 120^\circ$ rotations



(b) $\phi = \pm 180^\circ$ rotations

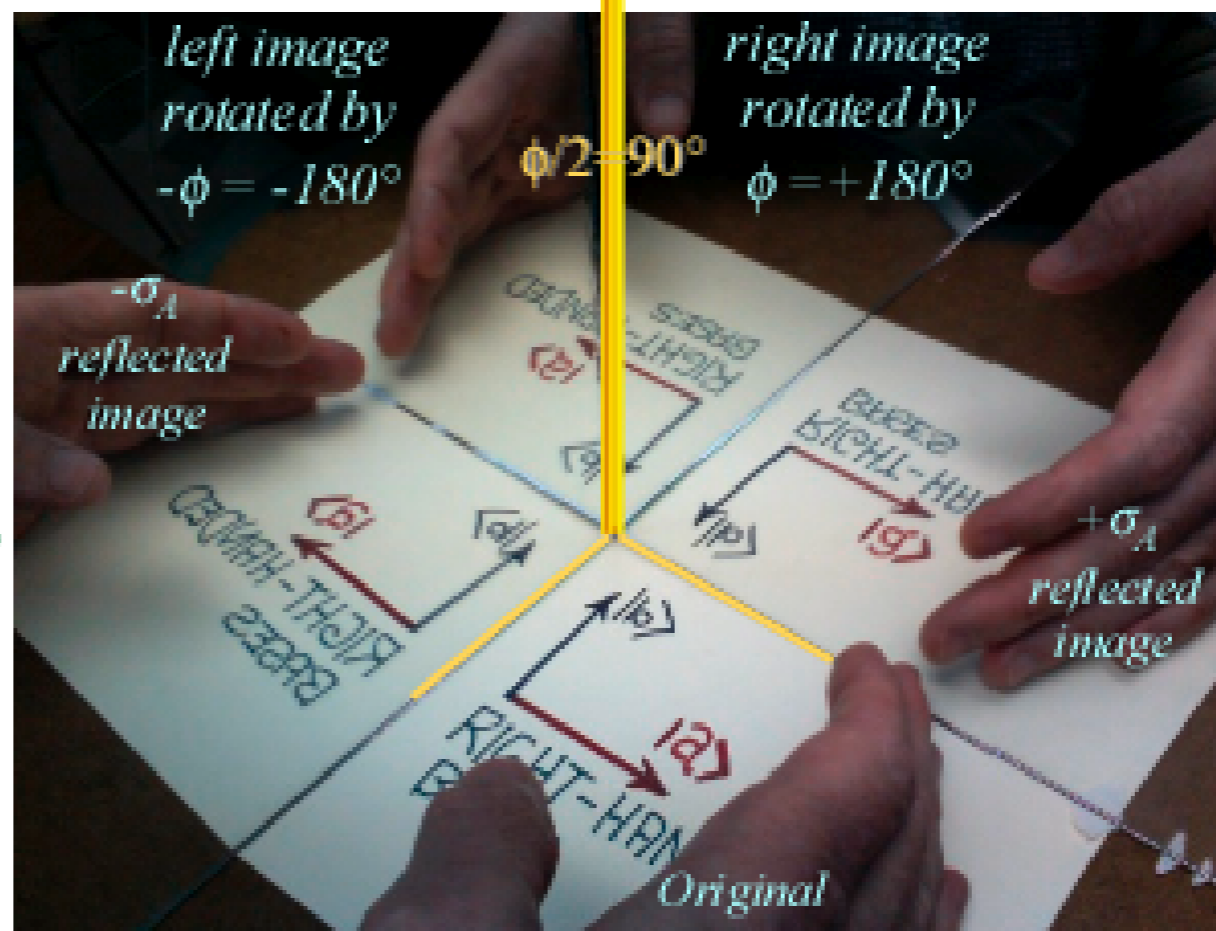
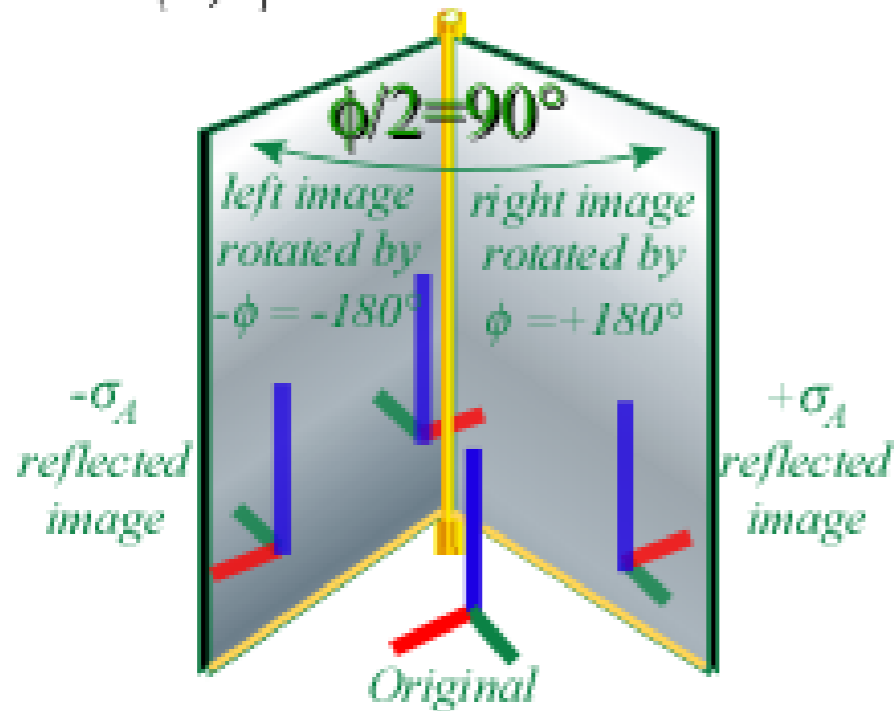
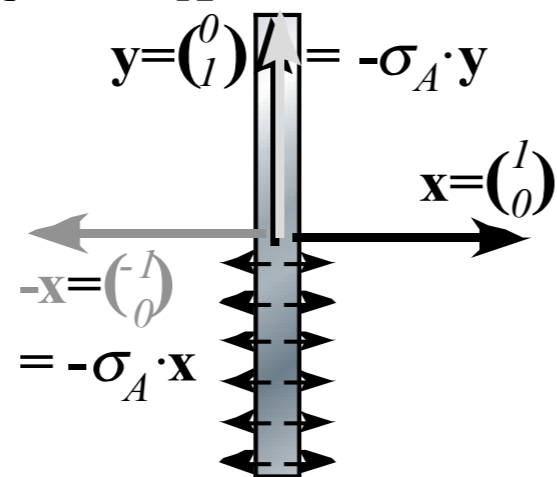
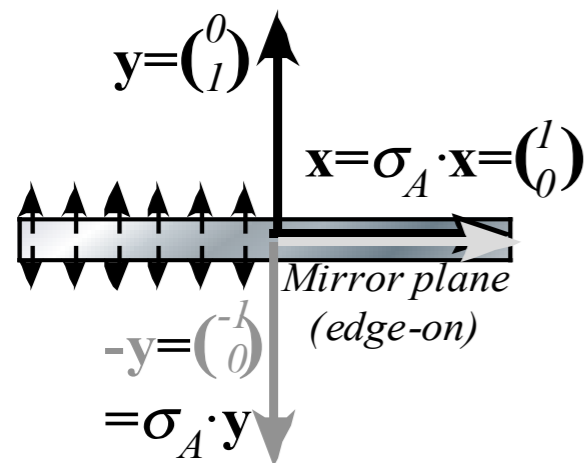


Fig. 5.4a-b

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

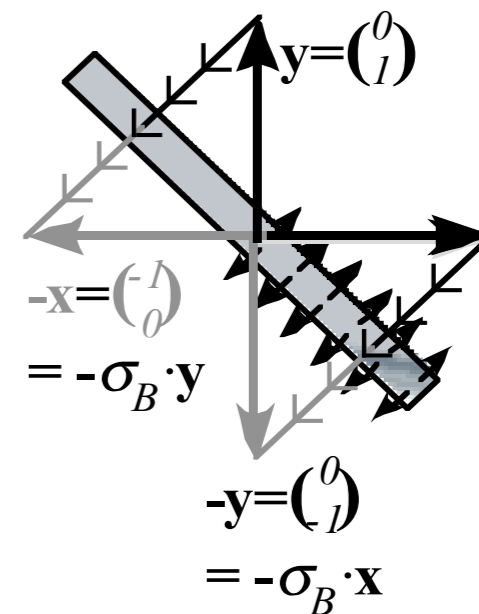
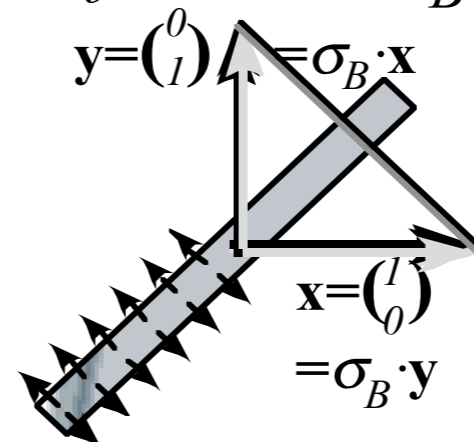
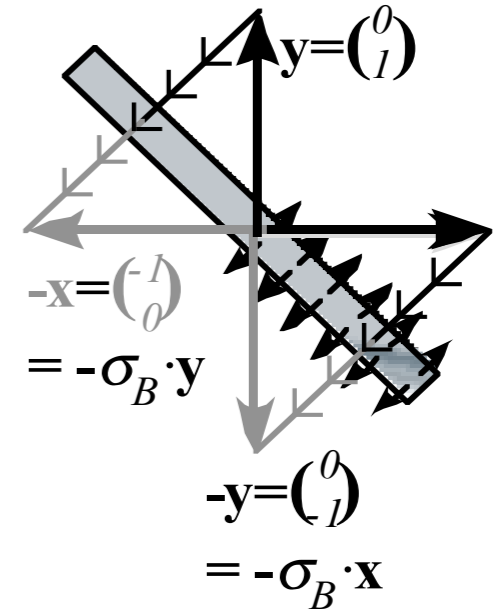
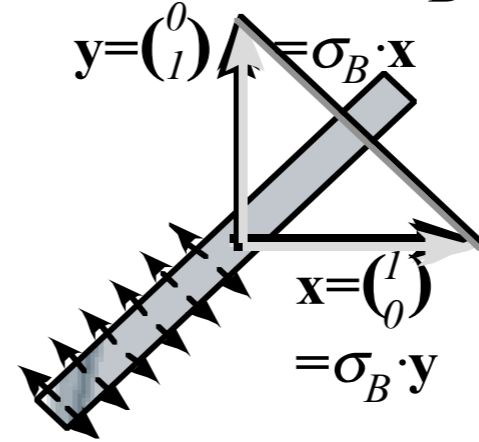
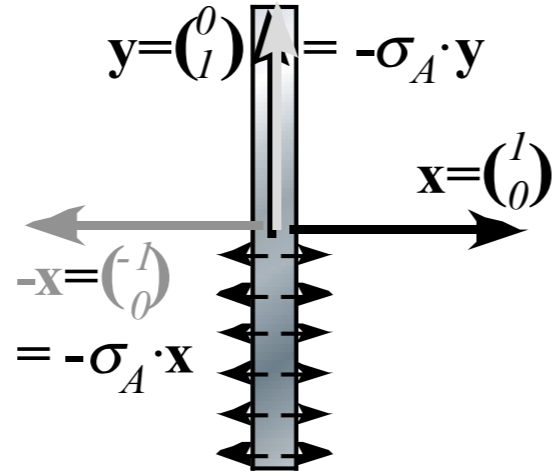
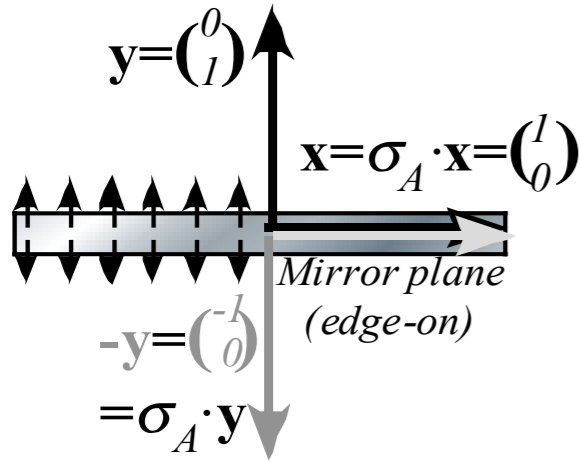


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

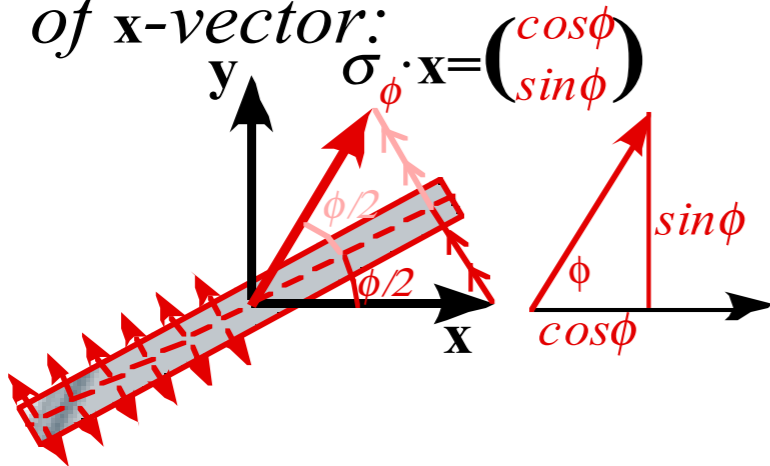
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of x -vector:



...of y -vector:

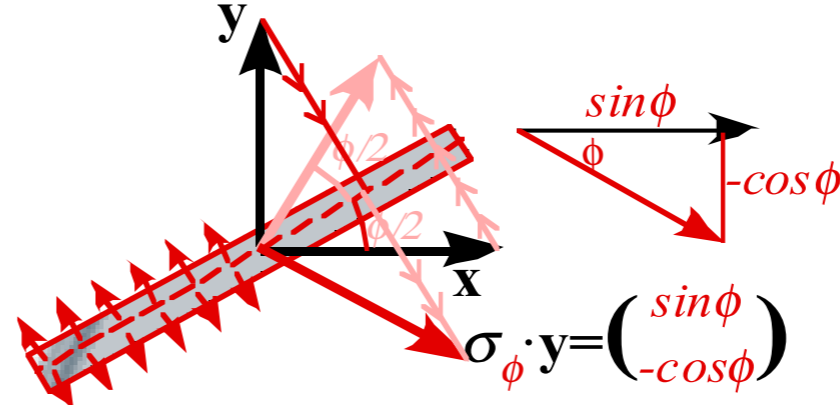
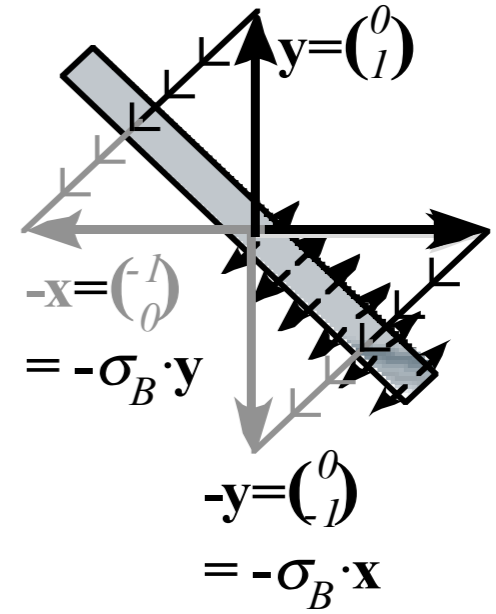
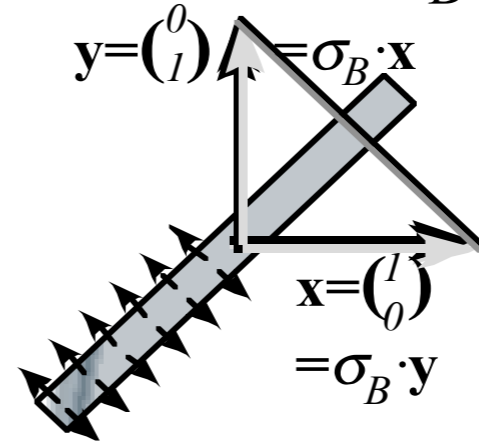
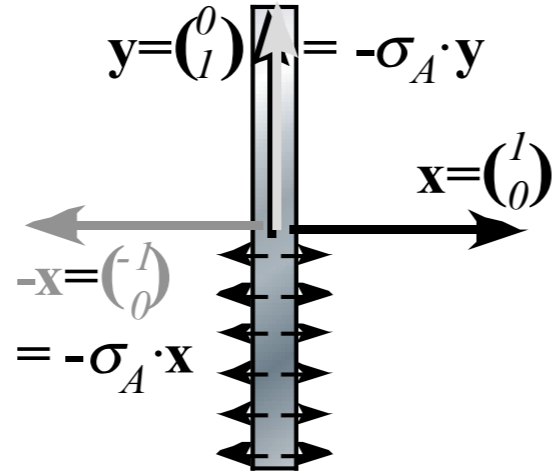
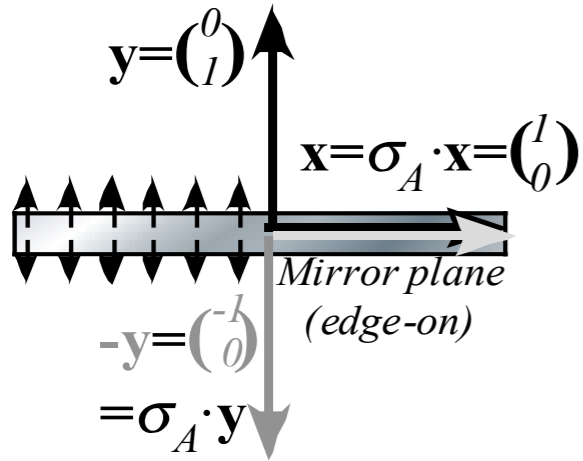


Fig. 5.3a-e

Symmetry: It's all done with mirrors!

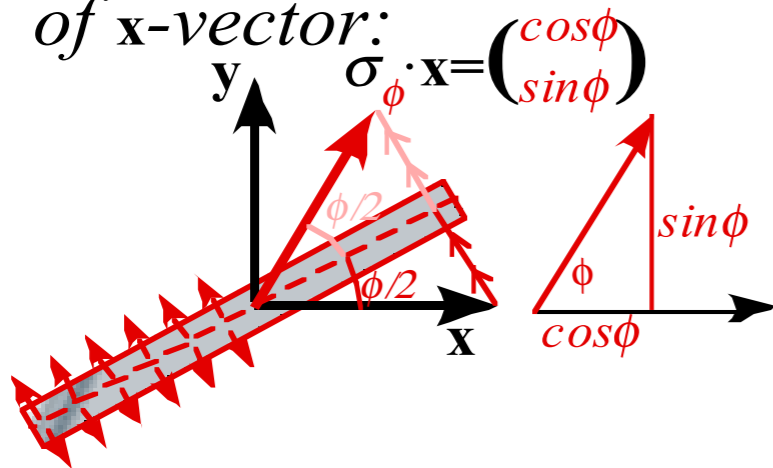
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

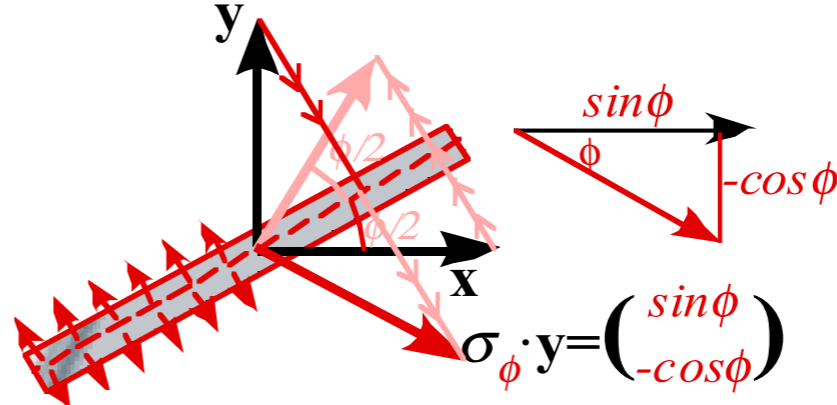


(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

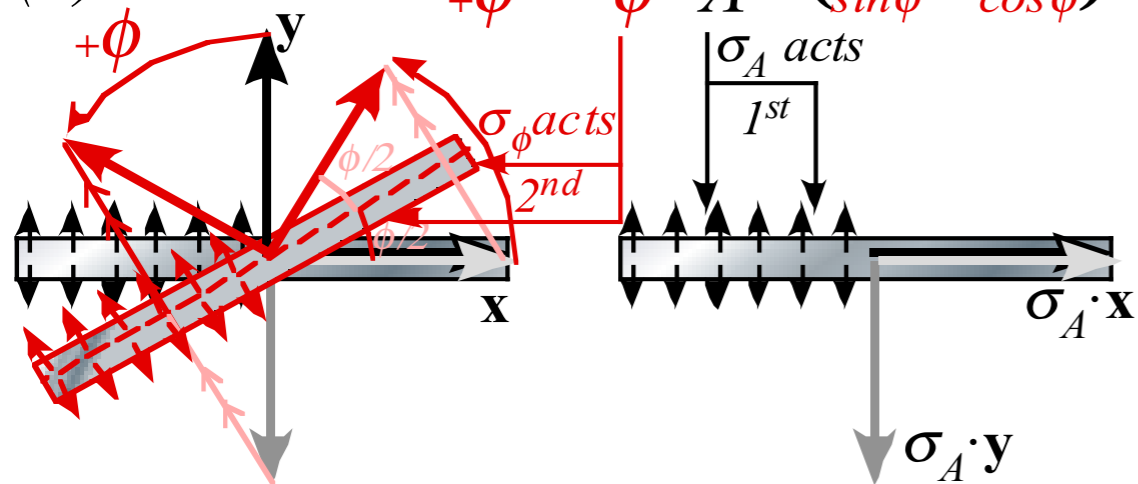
of x -vector:



...of y -vector:



(d) Rotation: $R_{+\phi} = \sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



(e) Rotation: $R_{-\phi} = \sigma_A \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

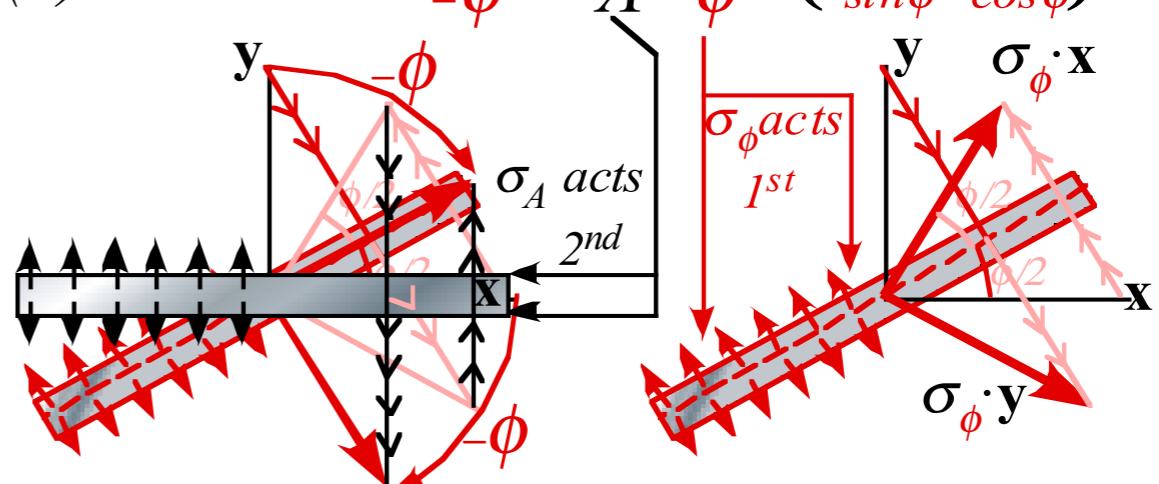


Fig. 5.3a-e