

Lecture 3
Thur. 9.02.2014

Analysis of 1D 2-Body Collisions

(Ch. 3, Ch. 4, and Ch. 5 of Unit 1)

Review of (V_1, V_2) and (y_1, y_2) geometry and X2 launcher in box

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2)

(Lect. 2 topic)

→ Example of (V_1, V_2) and (y_1, y_2) data for high mass ratios: $m_1/m_2=49, 100, \dots$ ←

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.

Algebra and Geometry of “ellipse-Rotation” group product: **R = C · M**

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to *Lagrangian*, *l'Etrangian*, and *Hamiltonian* mechanics in Ch. 12

Reflections in the clothing store: “It’s all done with mirrors!”

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Solutions to Exercises 1.4.1 and 1.4.2

Geometry of X2 launcher bouncing in box (Review)

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

→ Integration of (V_1, V_2) data to space-space plots (y_1, y_2) ← (Lect. 2 topic not mentioned)

Example of (V_1, V_2) and (y_1, y_2) data for high mass ratios: $m_1/m_2=49, 100, \dots$

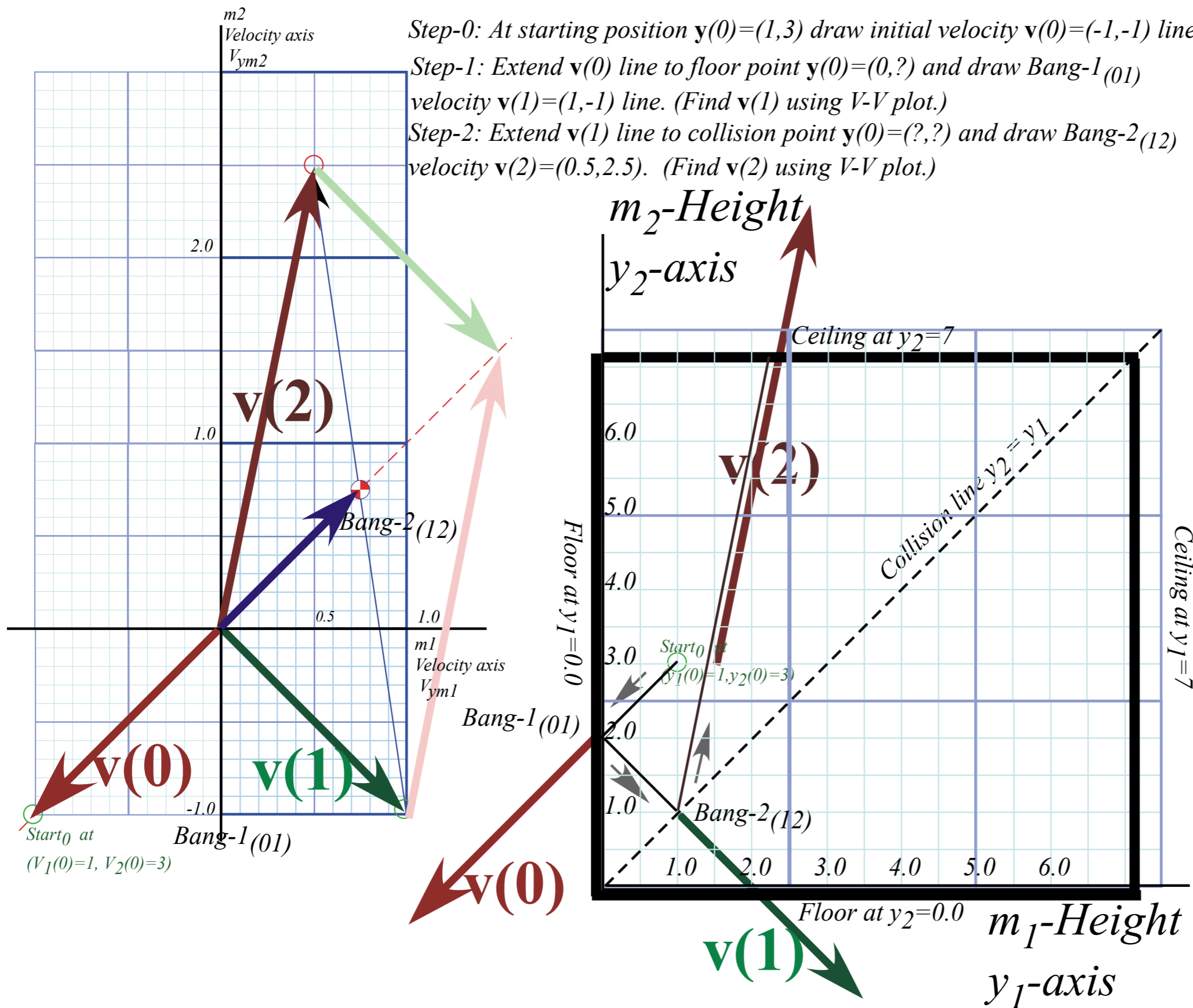
Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

Step-0: At starting position $\mathbf{y}(0)=(1,3)$ draw initial velocity $\mathbf{v}(0)=(-1,-1)$ line.

Step-1: Extend $\mathbf{v}(0)$ line to floor point $\mathbf{y}(0)=(0,?)$ and draw Bang-1(01) velocity $\mathbf{v}(1)=(1,-1)$ line. (Find $\mathbf{v}(1)$ using V-V plot.)

Step-2: Extend $\mathbf{v}(1)$ line to collision point $\mathbf{y}(0)=(?,?)$ and draw Bang-2(12) velocity $\mathbf{v}(2)=(0.5,2.5)$. (Find $\mathbf{v}(2)$ using V-V plot.)



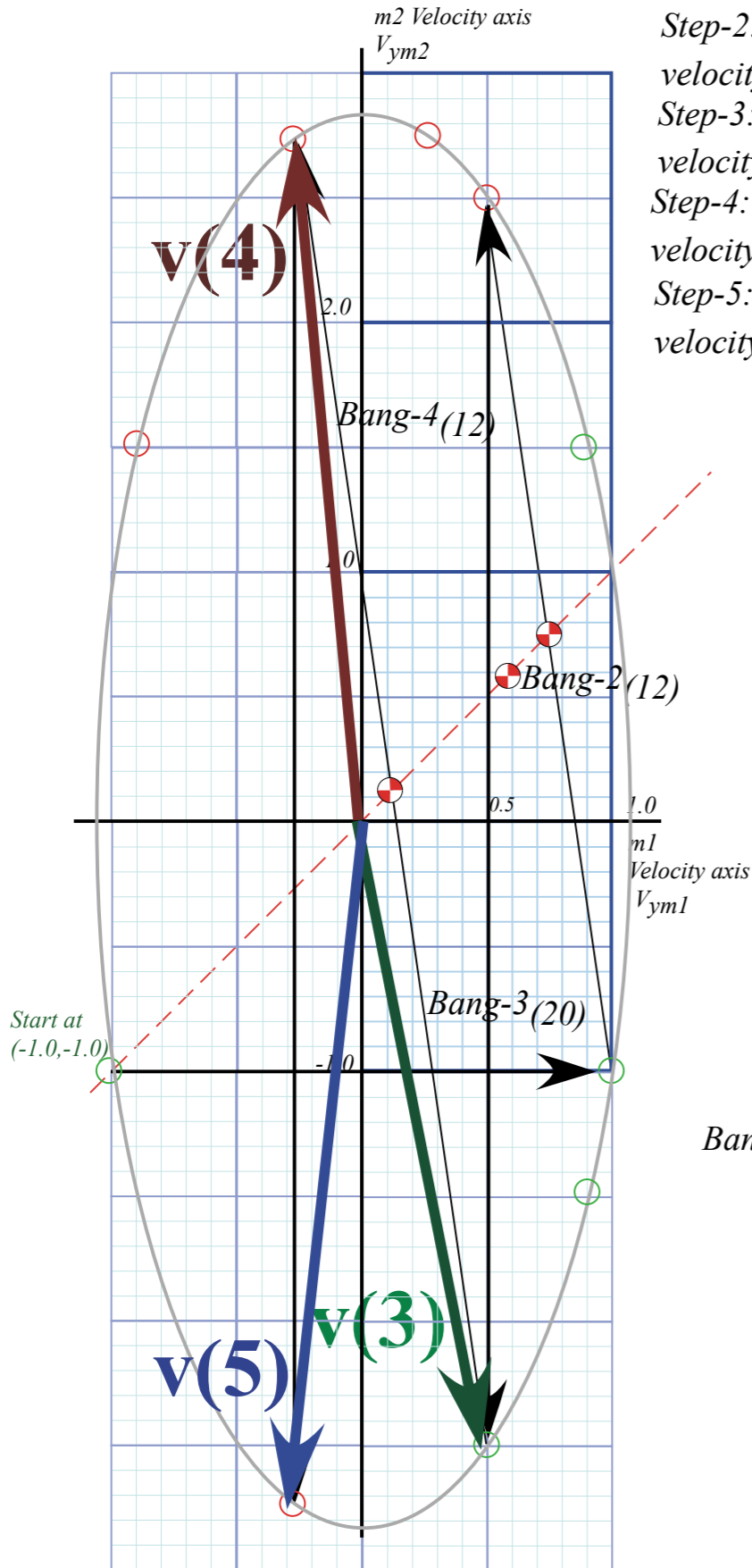
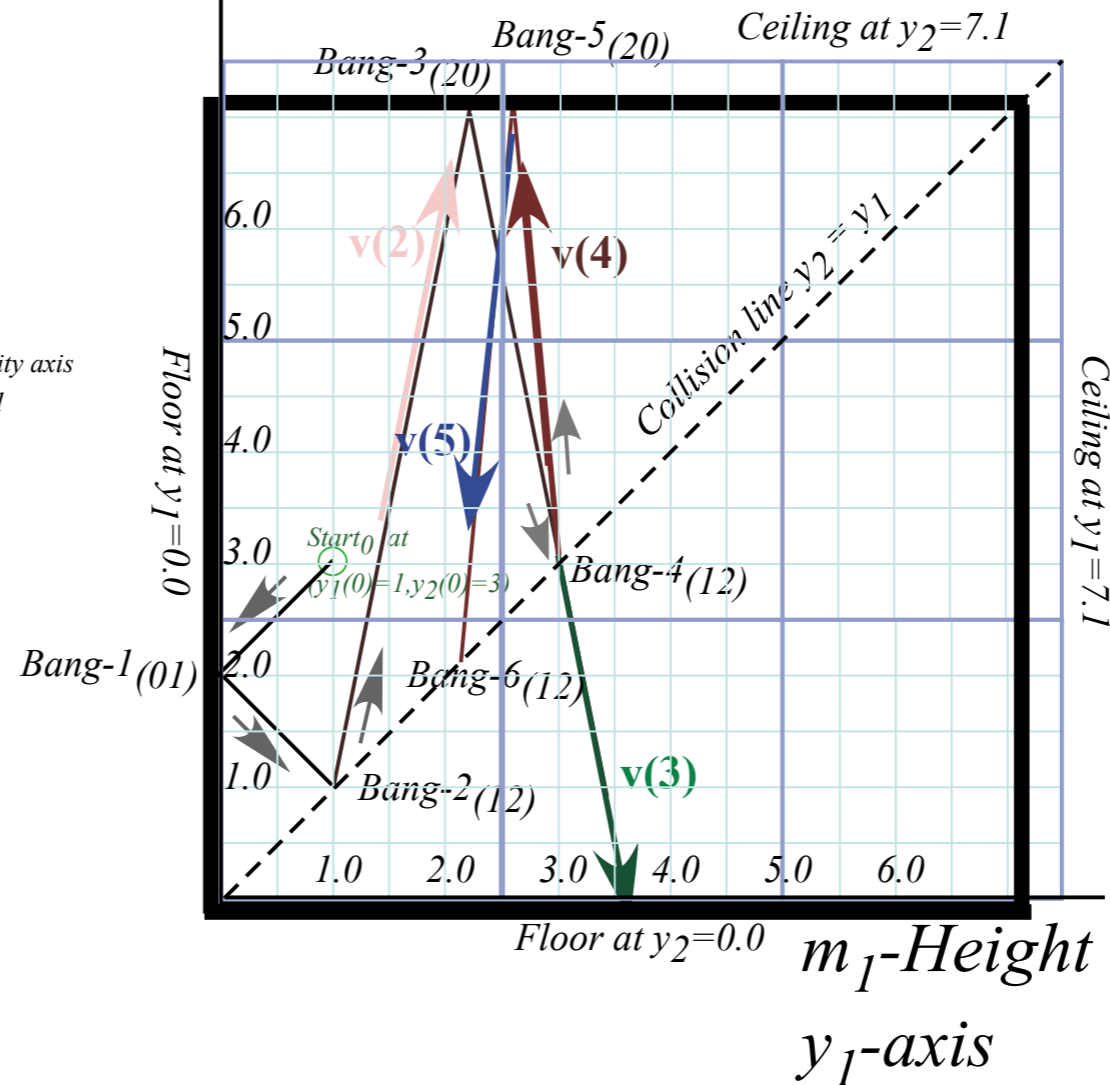
Geometric "Integration" (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

- Step-2: Extend $\mathbf{v}(2)$ line to ceiling point $\mathbf{y}(3)=(?, 7.1)$ and draw Bang-3(20) velocity $\mathbf{v}(3)=(1, -1)$ line. (Find $\mathbf{v}(3)$ using V-V plot.)
- Step-3: Extend $\mathbf{v}(3)$ line to collision point $\mathbf{y}(4)=(?, ?)$ and draw Bang-4(12) velocity $\mathbf{v}(4)=(0.5, 2.5)$. (Find $\mathbf{v}(4)$ using V-V plot.)
- Step-4: Extend $\mathbf{v}(4)$ line to ceiling point $\mathbf{y}(4)=(?, 7.1)$ and draw Bang-5(20) velocity $\mathbf{v}(5)=(1, -1)$ line. (Find $\mathbf{v}(5)$ using V-V plot.)
- Step-5: Extend $\mathbf{v}(5)$ line to collision point $\mathbf{y}(6)=(?, ?)$ and draw Bang-6(12) velocity $\mathbf{v}(6)=(0.5, 2.5)$. (Find $\mathbf{v}(6)$ using V-V plot.)

m_2 -Height

y_2 -axis



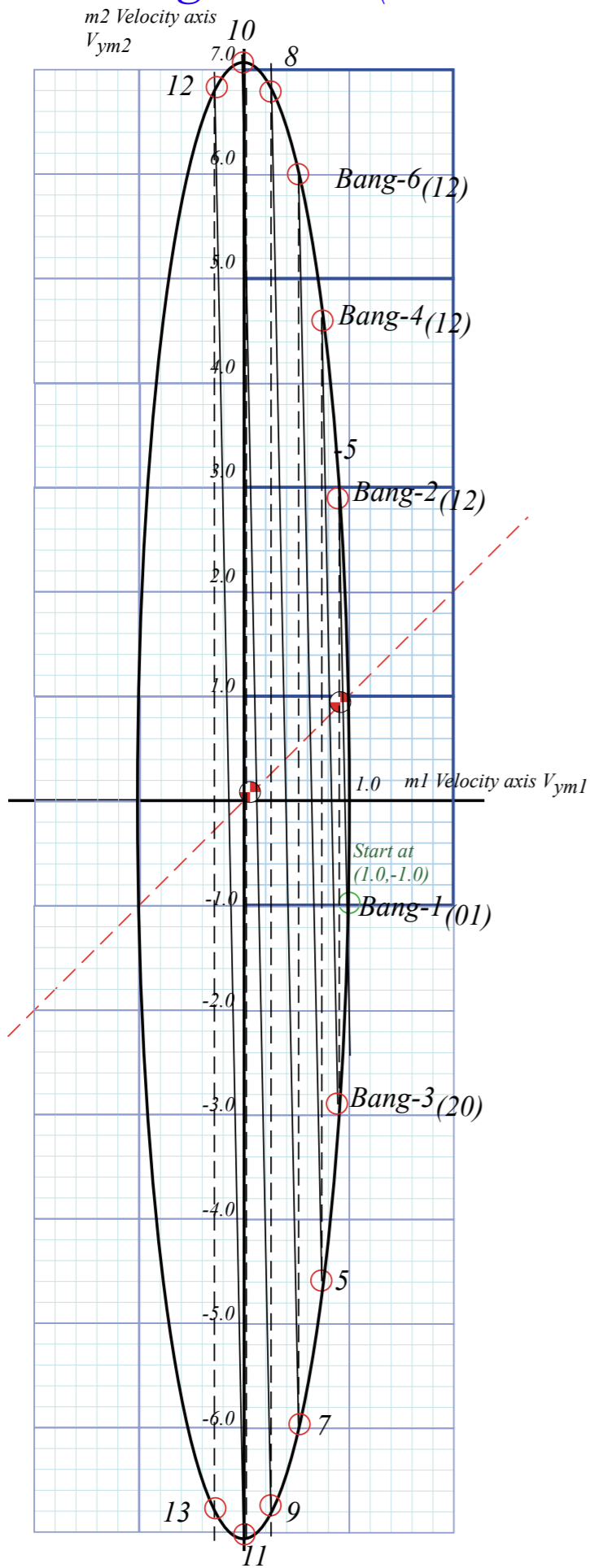
Geometry of X2 launcher bouncing in box (Review)

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2) (Lect. 2 topic not mentioned)

→ Example of (V_1, V_2) and (y_1, y_2) data for high mass ratios: $m_1/m_2=49, 100, \dots$ ←

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: $m_1=49$ and $m_2=1$

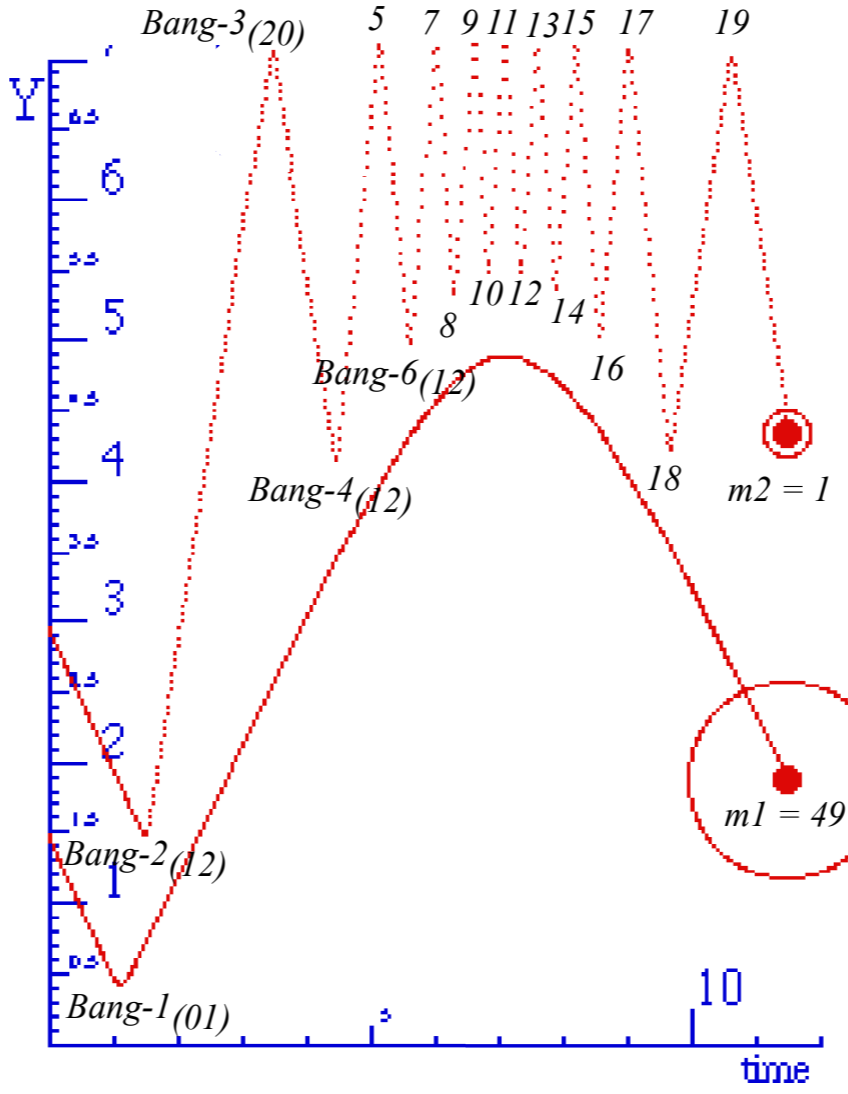
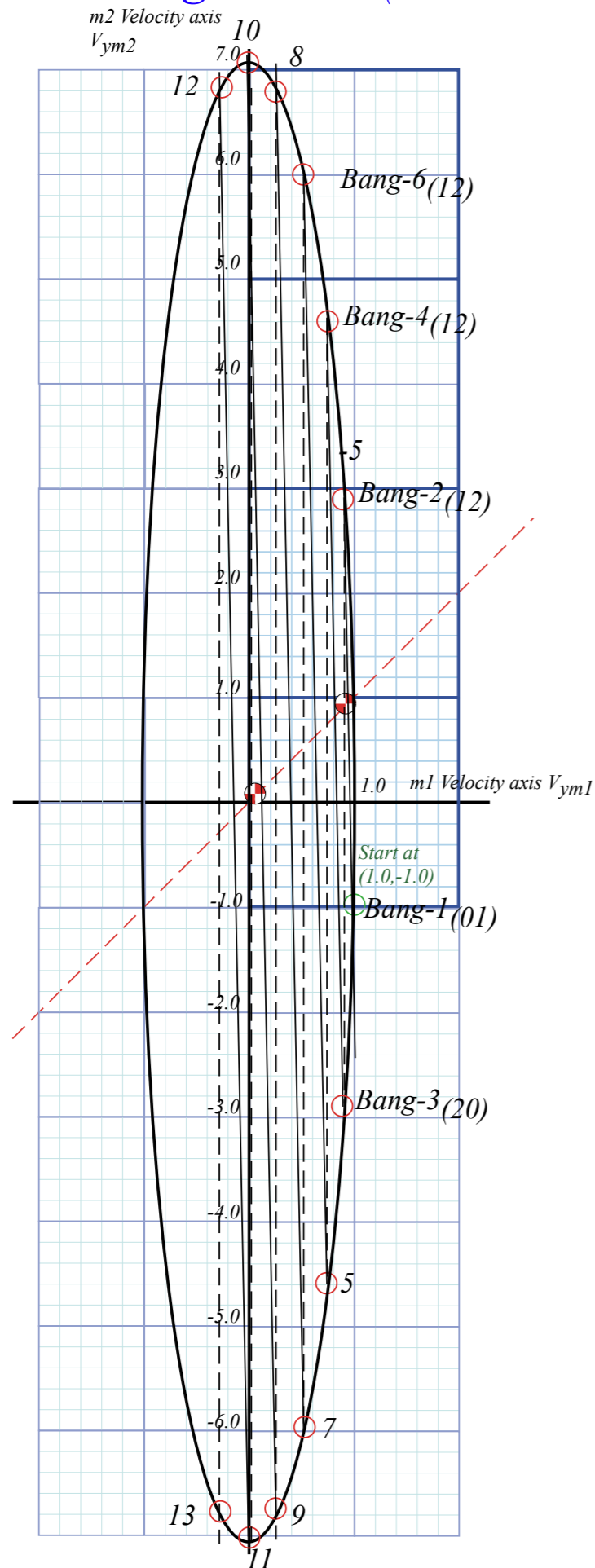


Fig. 5.1
in Unit 1

Geometric "Integration" (Converting Velocity data to Space-time trajectory)



Example with masses: $m_1=49$ and $m_2=1$

Kinetic Energy Ellipse

$$KE = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$\begin{aligned} a_1 &= \sqrt{2KE/M_1} \\ &= \sqrt{2KE/49} \\ &= \sqrt{50/49} \\ &= 1.01 \end{aligned}$$

Ellipse radius 2

$$\begin{aligned} a_2 &= \sqrt{2KE/m_2} \\ &= \sqrt{2KE/1} \\ &= \sqrt{50/1} \\ &= 7.07 \end{aligned}$$

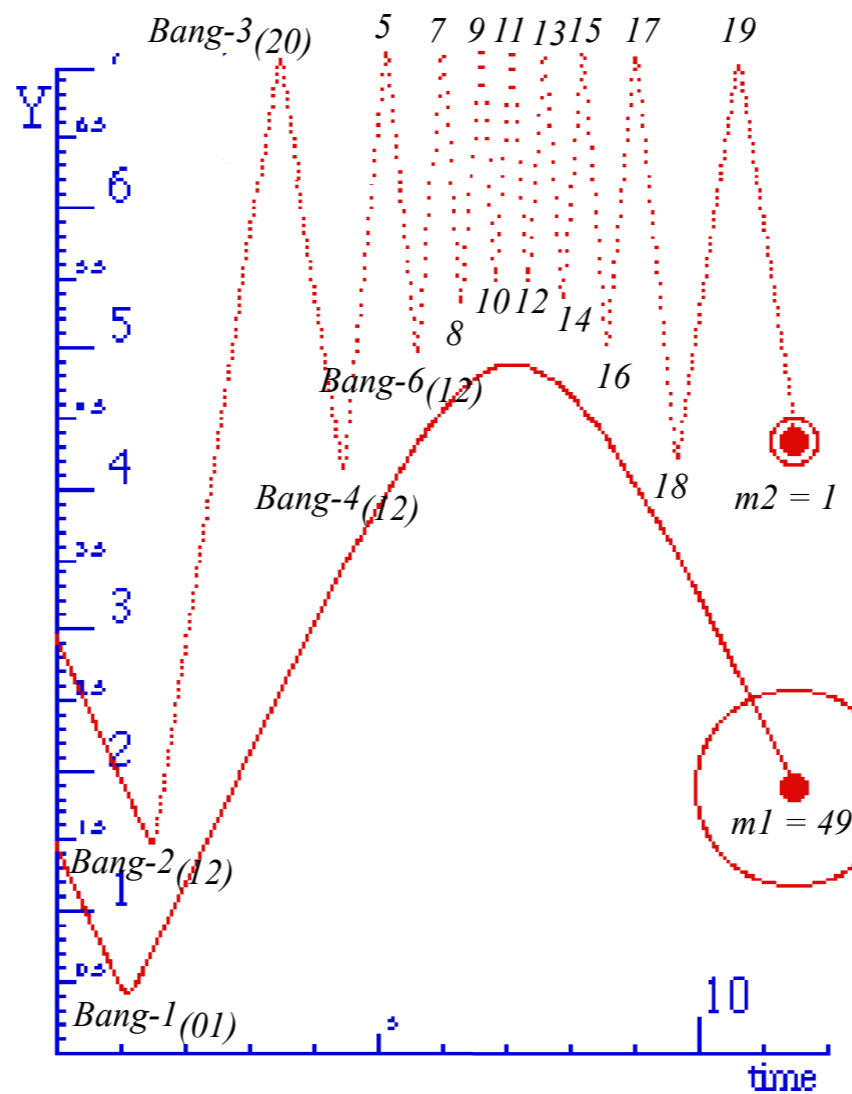


Fig. 5.1
in Unit 1

Multiple collisions calculated by matrix operator products

 *Matrix or tensor algebra of 1-D 2-body collisions*

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:
$$V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} =$$

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Multiple Collisions by Matrix Operator Products

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Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

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Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

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Multiple Collisions by Matrix Operator Products

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$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***



Multiple Collisions by Matrix Operator Products

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Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Define "ellipse-Rotation" \mathbf{R} as group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL } (0))}
 \end{aligned}$$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL } (0))} \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{(\text{after Bang-1})}
 \end{aligned}$$

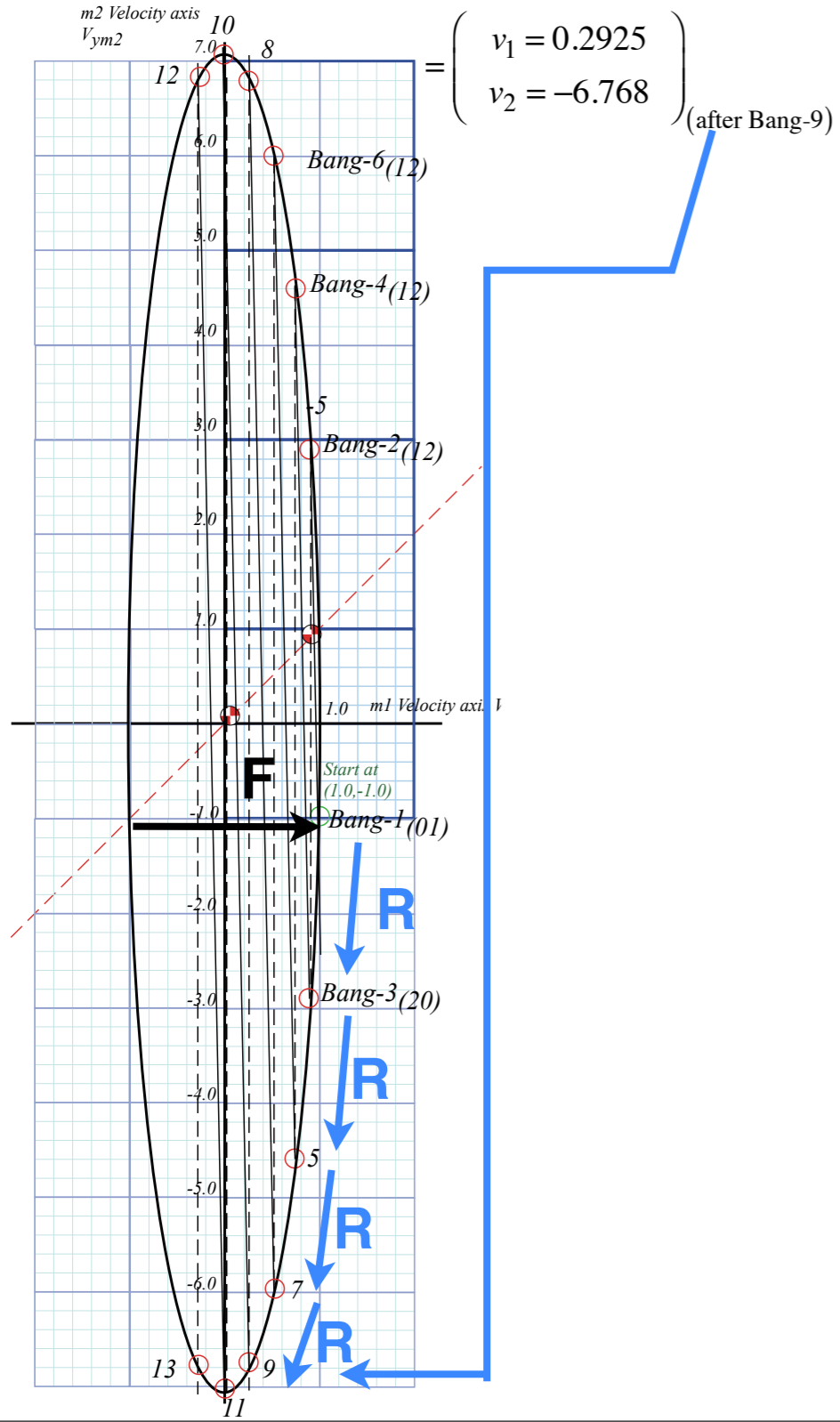
“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 \left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \mathbf{F} \left. \begin{array}{l} |IN^0\rangle \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{array} \right\}_{(INITIAL (0))} \\
 \left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{c} \mathbf{F} |IN^0\rangle \\ v_1 = 1 \\ v_2 = -1 \end{array} \right)_{(after Bang-1)}
 \end{aligned}$$

$$= \left(\begin{array}{c} v_1 = 0.2925 \\ v_2 = -6.768 \end{array} \right)_{(after Bang-9)}$$

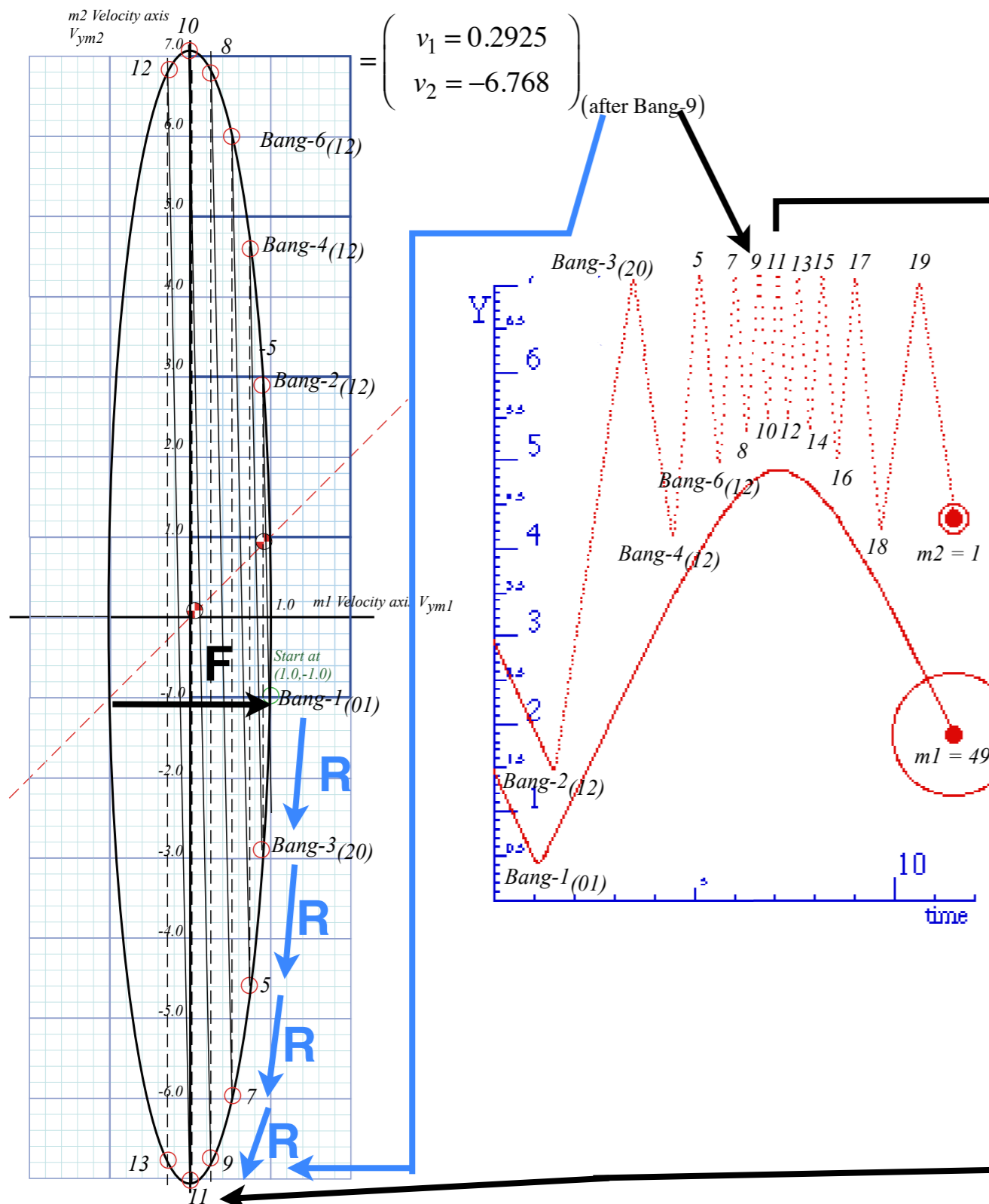
“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix} \quad (\text{INITIAL } (0)) \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix} \quad (\text{after Bang-1})
 \end{aligned}$$



“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix} \quad (\text{INITIAL } (0)) \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix} \quad (\text{after Bang-1})
 \end{aligned}$$



“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 \begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} \\
 &= \begin{pmatrix} v_1 = 0.0100 \\ v_2 = -7.071 \end{pmatrix} \quad (\text{after Bang-11})
 \end{aligned}$$

Ellipse rescaling-geometry and reflection-symmetry analysis

 *Rescaling KE ellipse to circle*

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

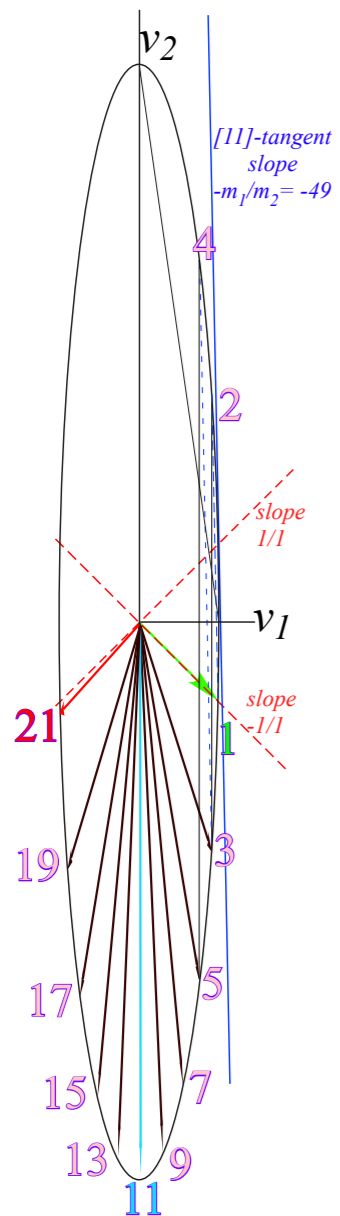
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Ellipse rescaling geometry and reflection symmetry analysis

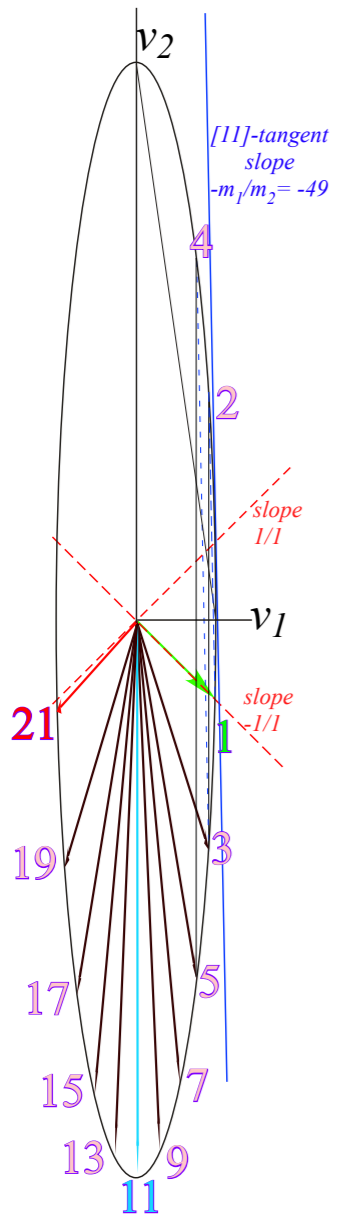
Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN_1} \\ v_2^{FIN_1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN_1} / \sqrt{m_1} \\ V_2^{FIN_1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

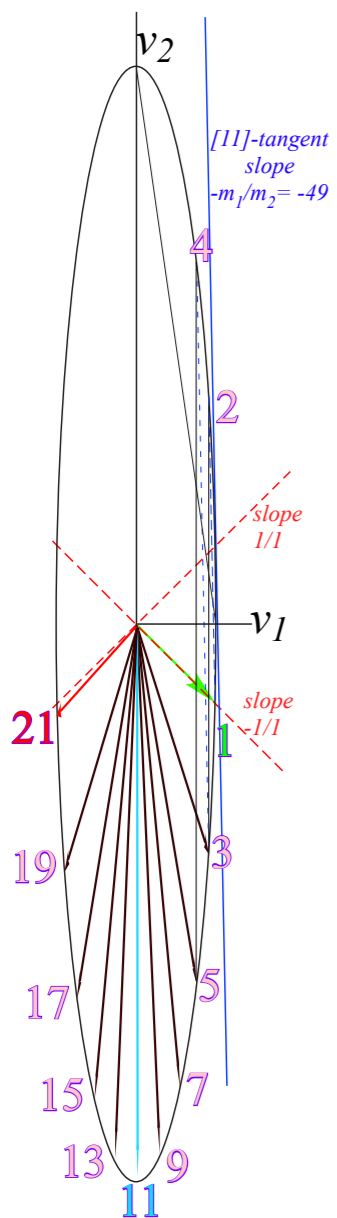


Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$



Ellipse rescaling geometry and reflection symmetry analysis

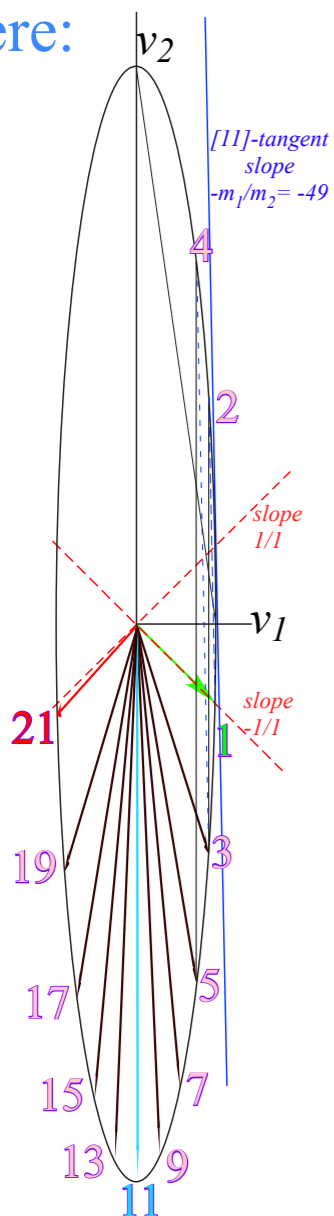
Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{I}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{I}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{I}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{I}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$



Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

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where: $\cos\theta \equiv \frac{m_1 - m_2}{m_1 + m_2}$ and: $\sin\theta \equiv \frac{2\sqrt{m_1 m_2}}{m_1 + m_2}$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$

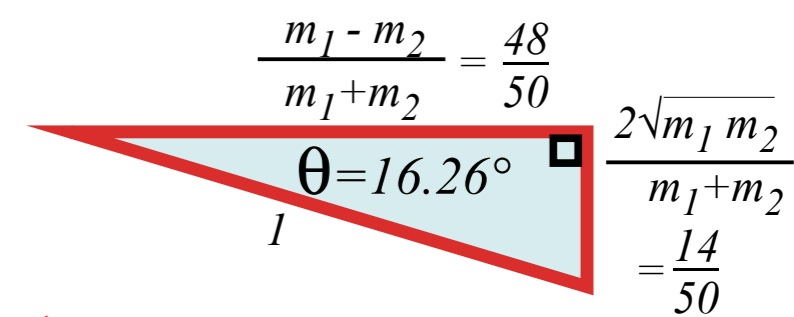
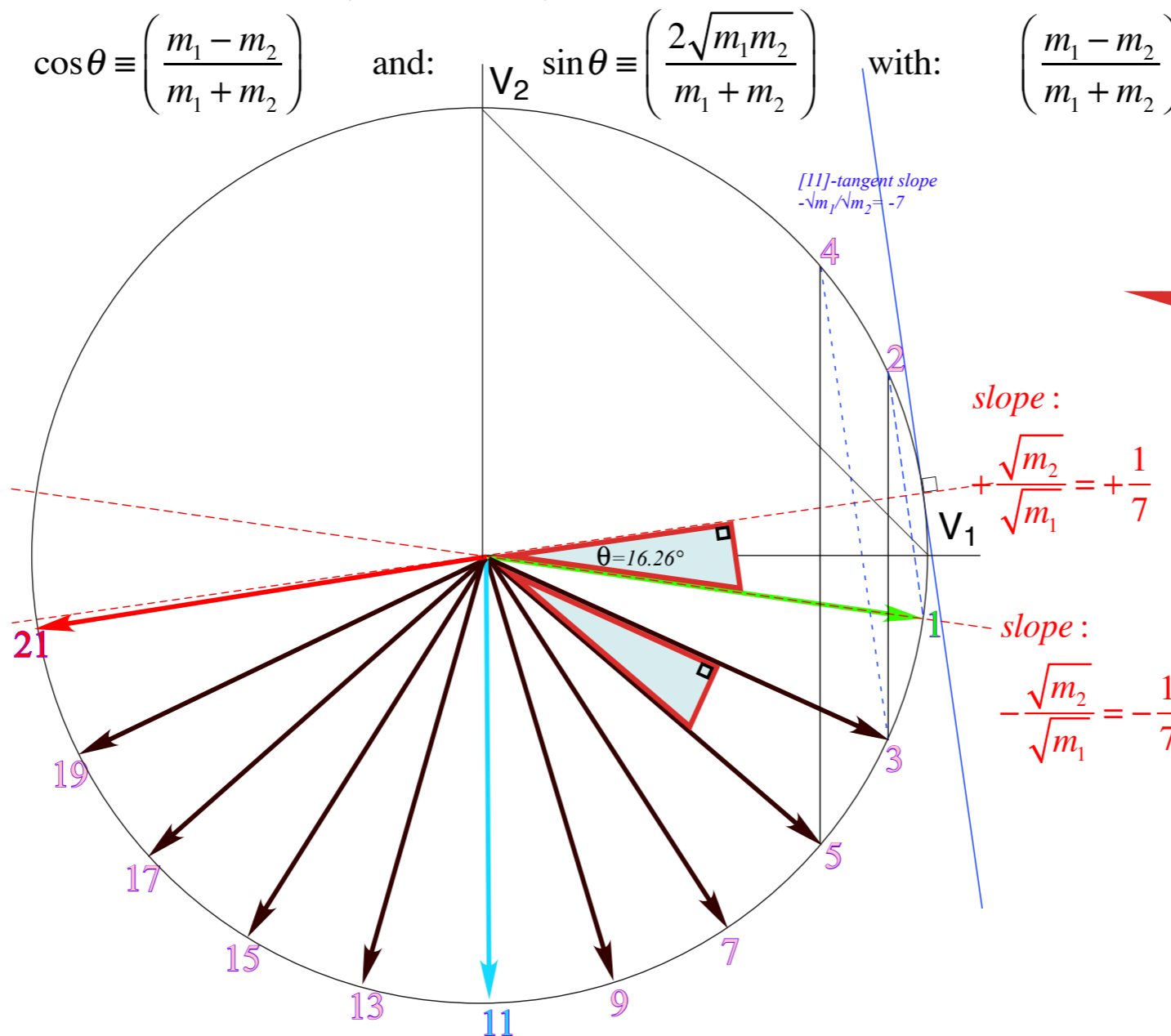
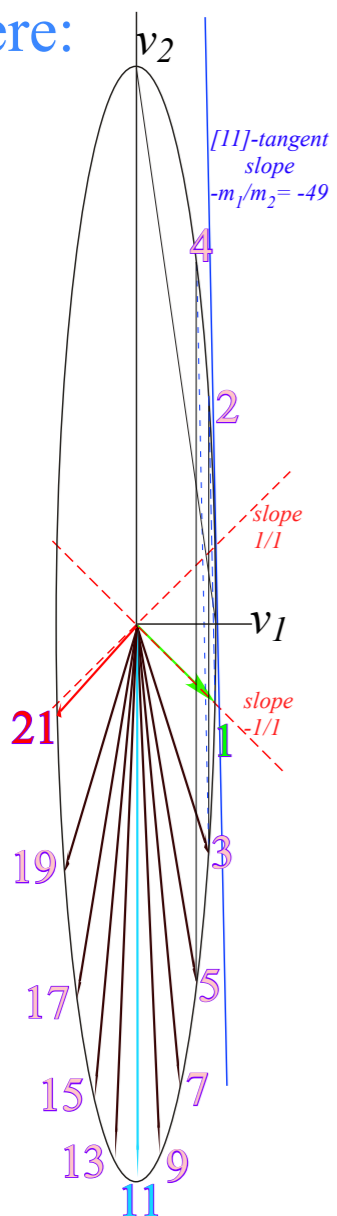


Fig. 5.2a-c
(revised)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \frac{m_1 - m_2}{m_1 + m_2}$ and: $\sin\theta \equiv \frac{2\sqrt{m_1 m_2}}{m_1 + m_2}$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2}\right)^2 = 1$

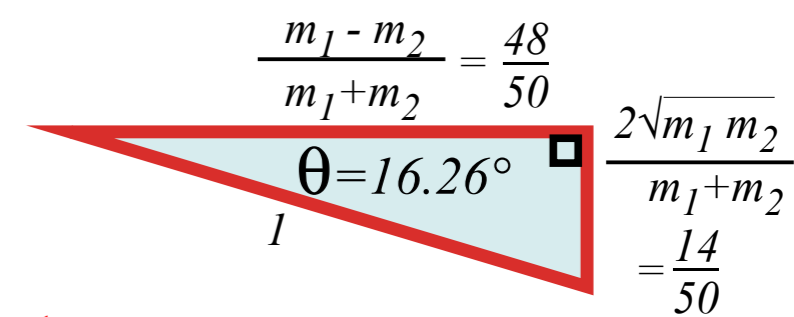
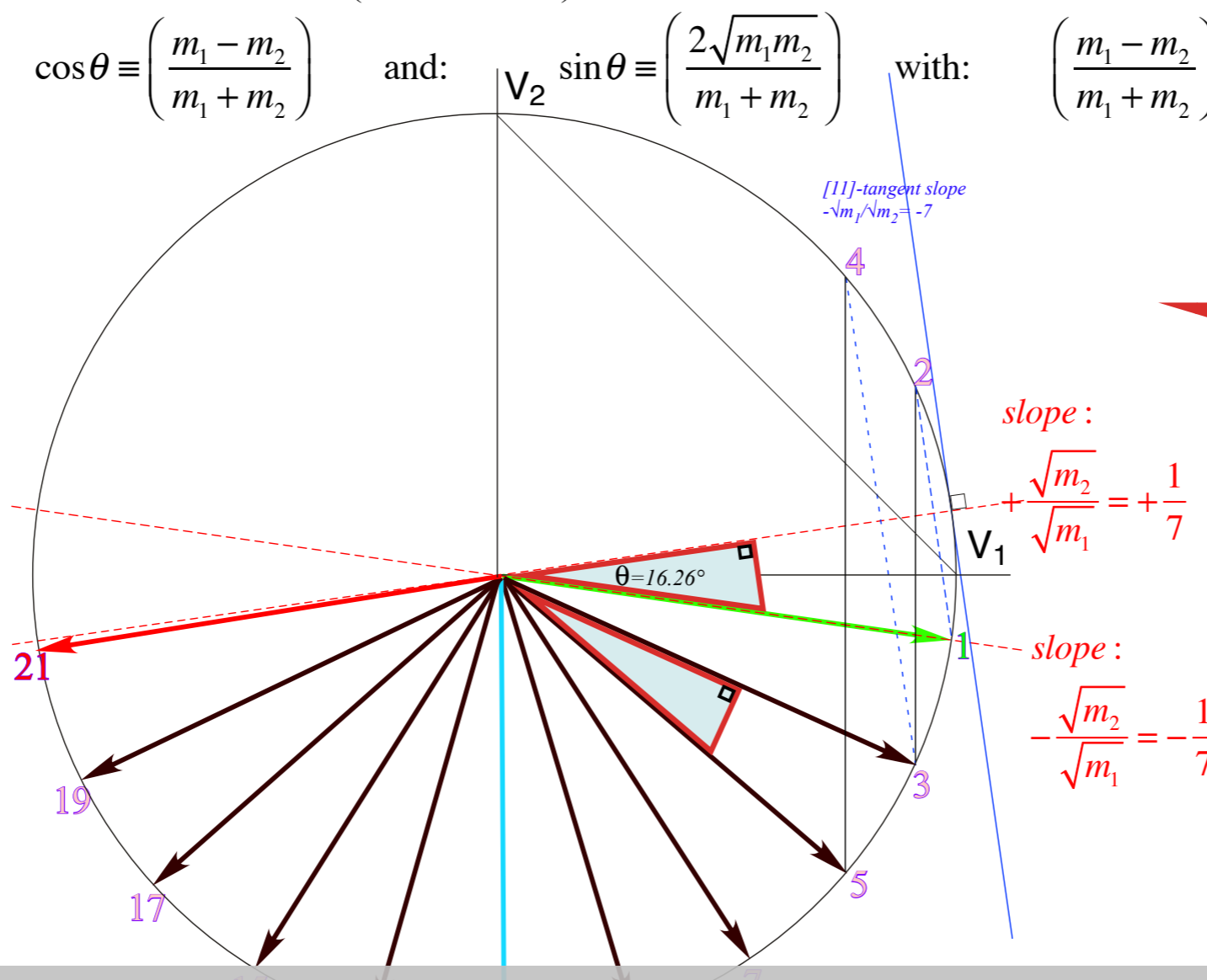
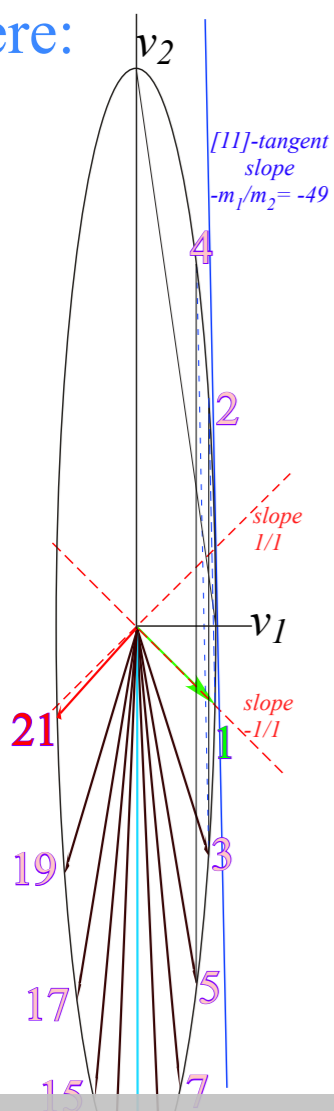


Fig. 5.2a-c
(revised)

Note: If $m_1 \cdot m_2$ is perfect-square, then θ -triangle is rational ($3^2 + 4^2 = 5^2$, etc.)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations*

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$

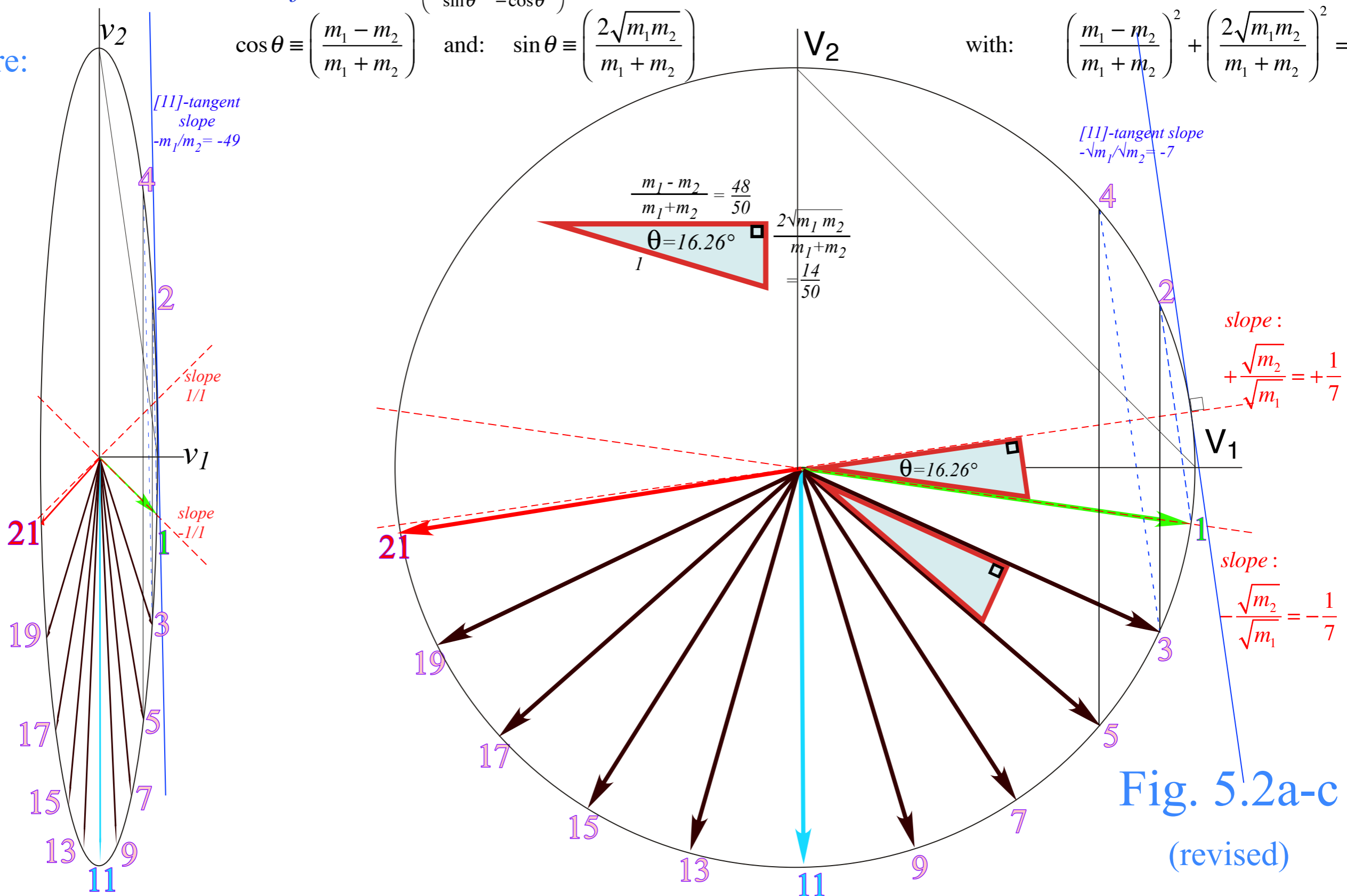


Fig. 5.2a-c
(revised)

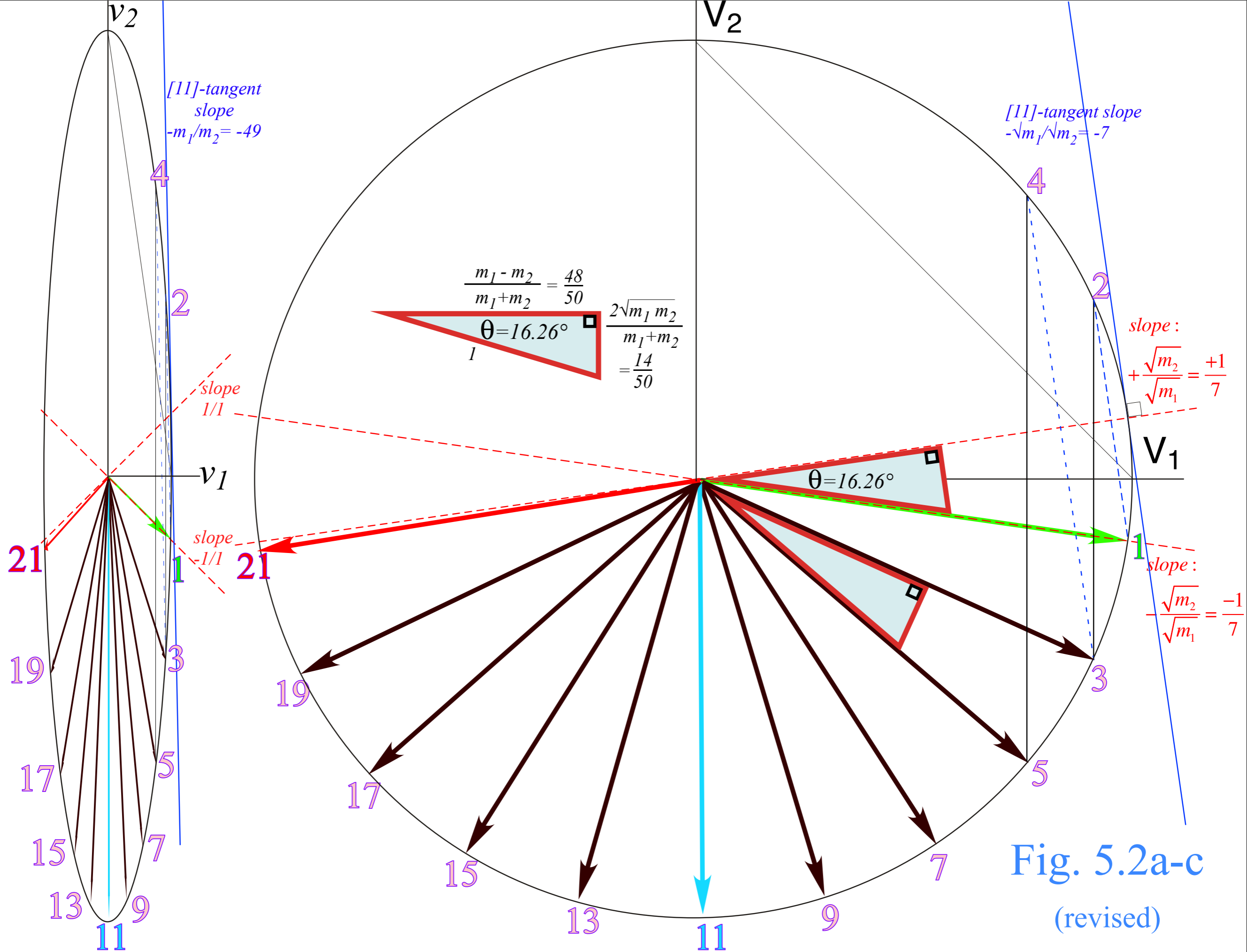


Fig. 5.2a-c
(revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

 *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12*

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Group multiplication and product table

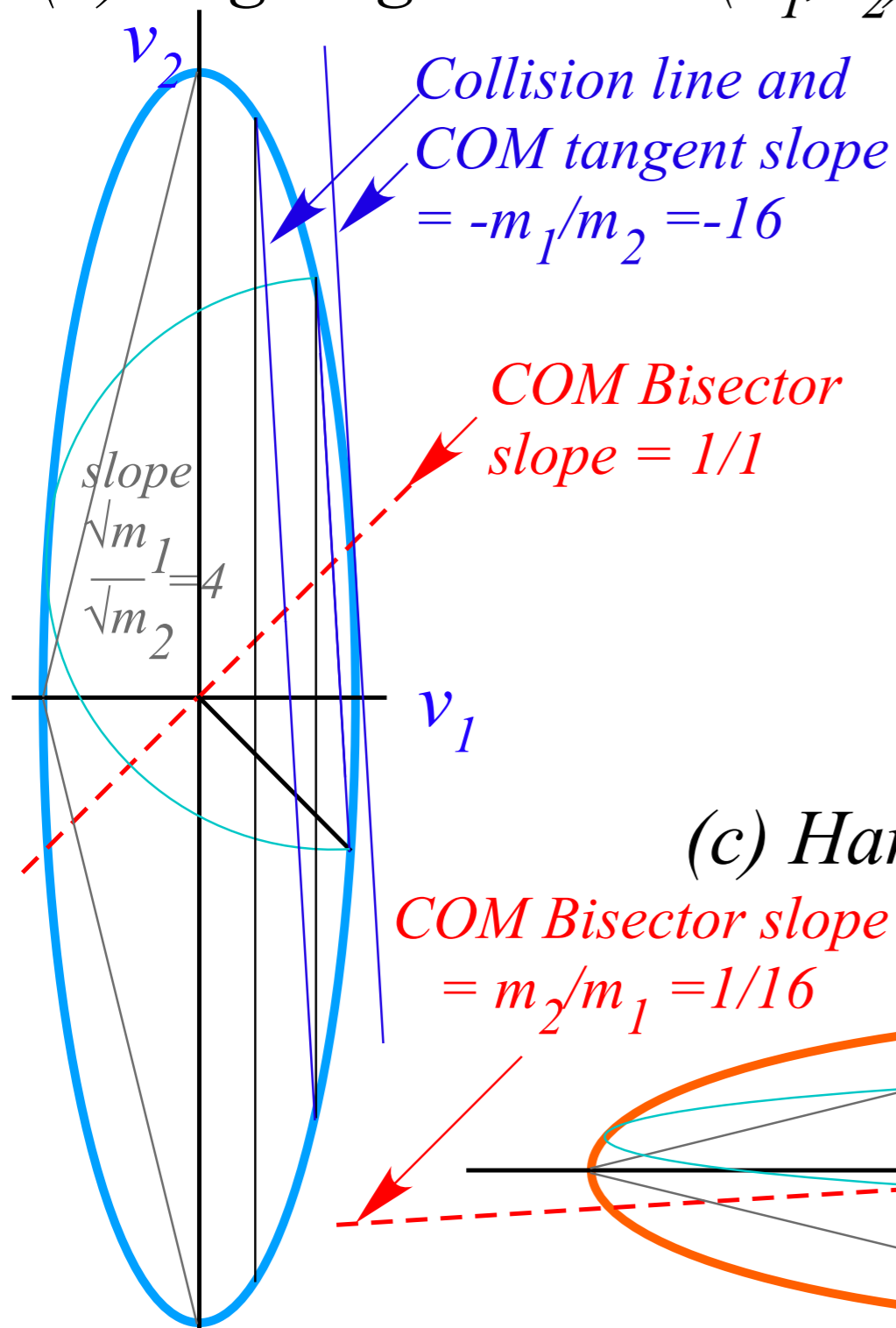
Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

What ellipse rescaling leads to... (in Ch. 9-12)

How this relates to *Lagrangian*,

and *Hamiltonian* mechanics in Ch. 12

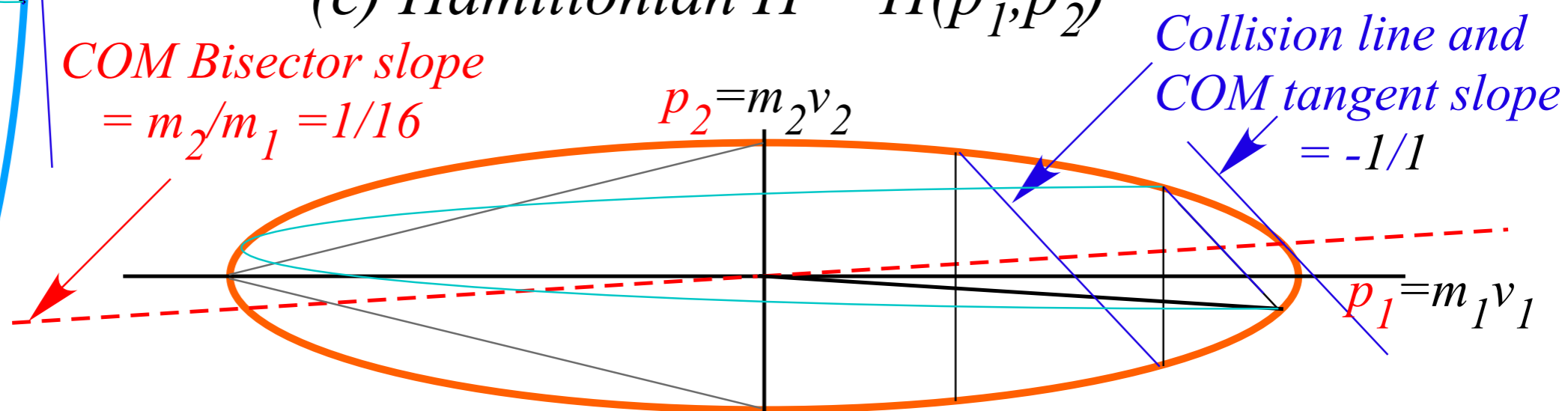
(a) Lagrangian $L = L(v_1, v_2)$



velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$
 velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

(c) Hamiltonian $H = H(p_1, p_2)$

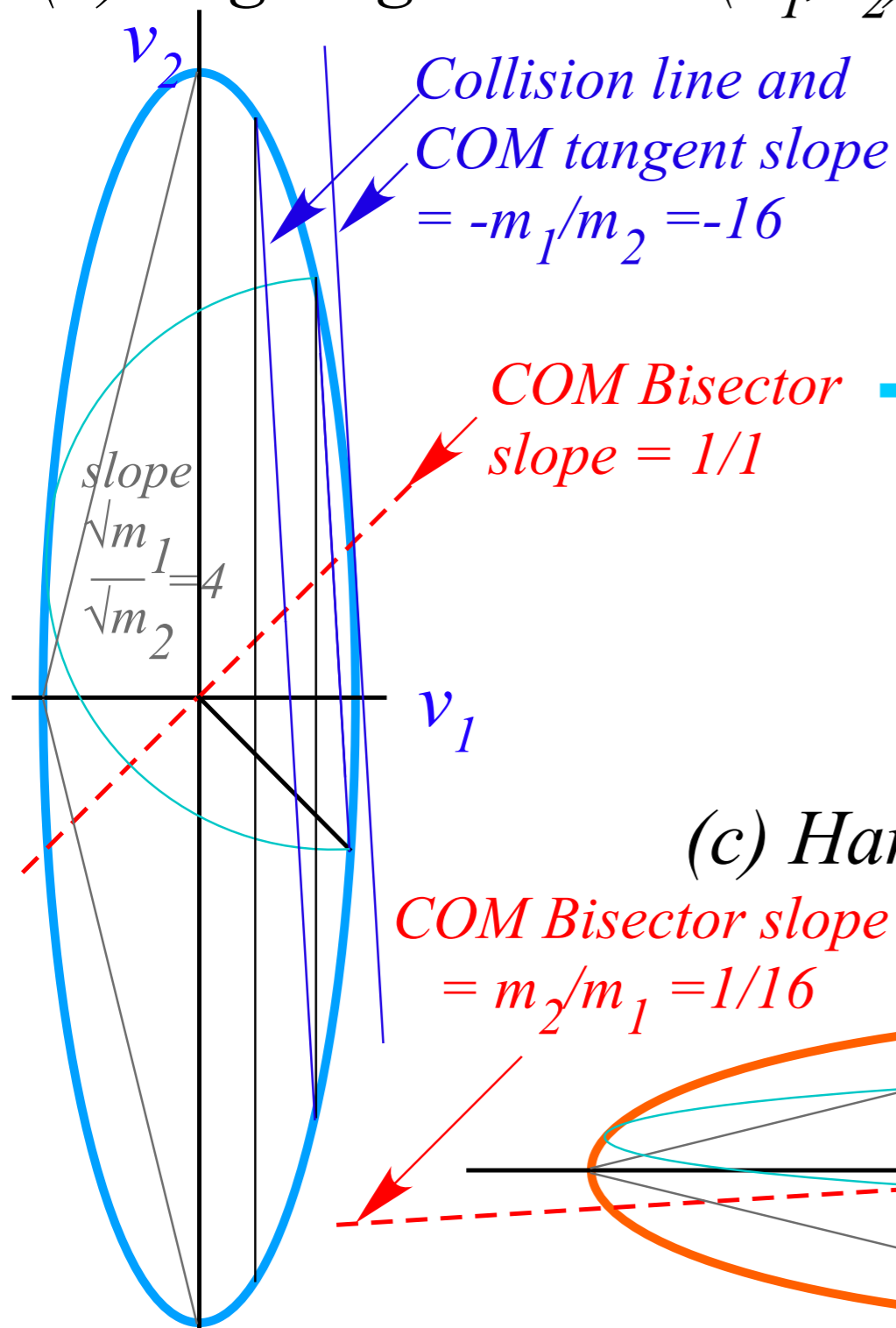
COM Bisector slope
 $= m_2/m_1 = 1/16$



What ellipse rescaling leads to... (in Ch. 9-12)

How this relates to *Lagrangian*, and *Hamiltonian* mechanics in Ch. 12

(a) Lagrangian $L = L(v_1, v_2)$

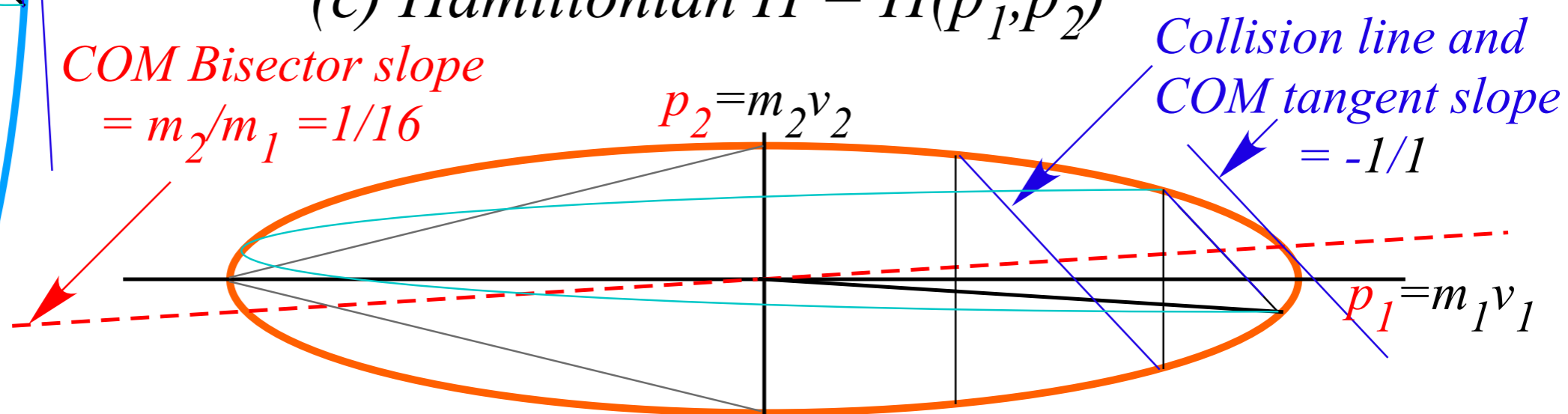


velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$
 velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

Lagrangian $L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$
 rescaled to

Hamiltonian $H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

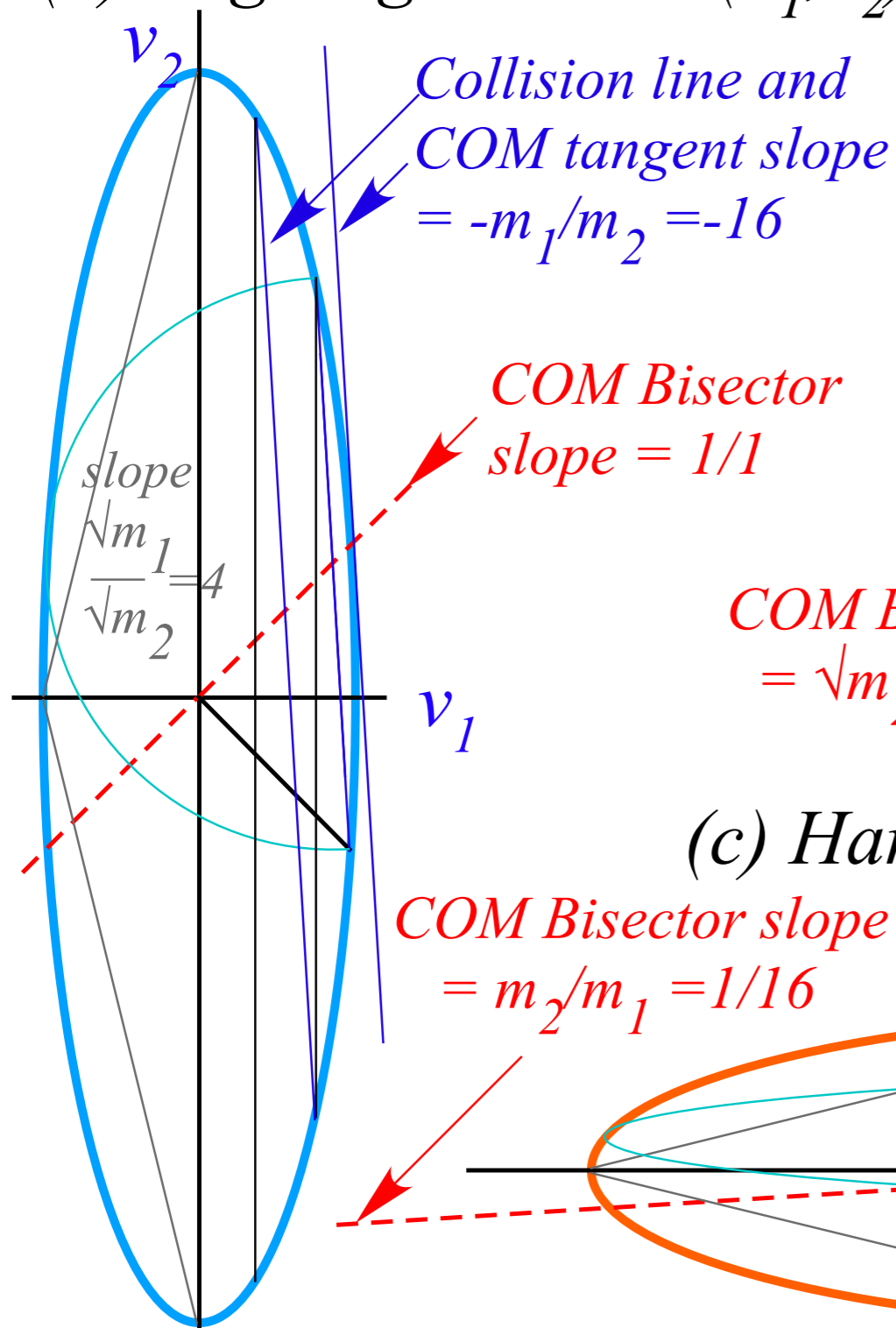
(c) Hamiltonian $H = H(p_1, p_2)$



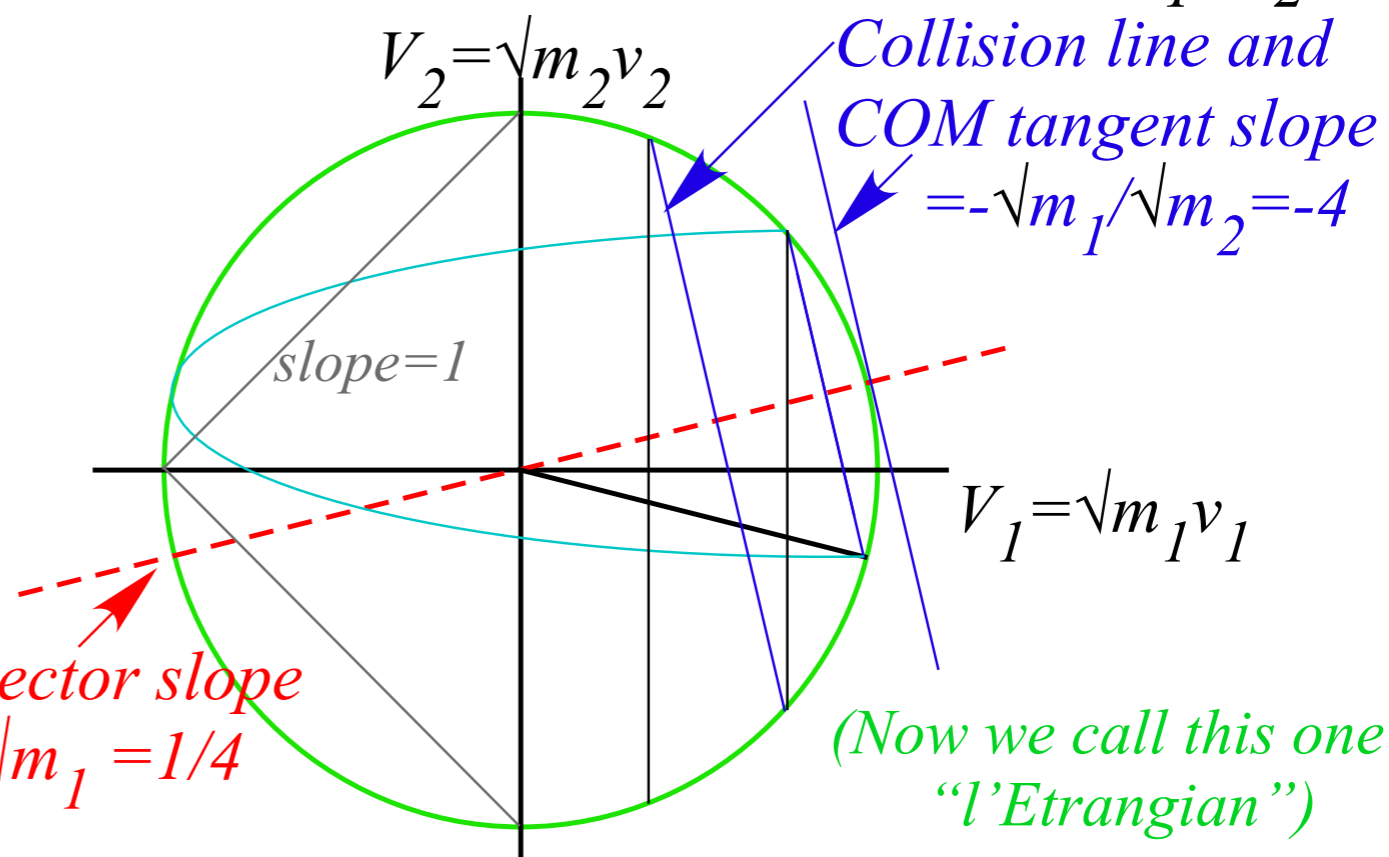
What ellipse rescaling leads to... (in Ch. 9-12)

How this relates to *Lagrangian*, *l'Etranganian*, and *Hamiltonian* mechanics in Ch. 12

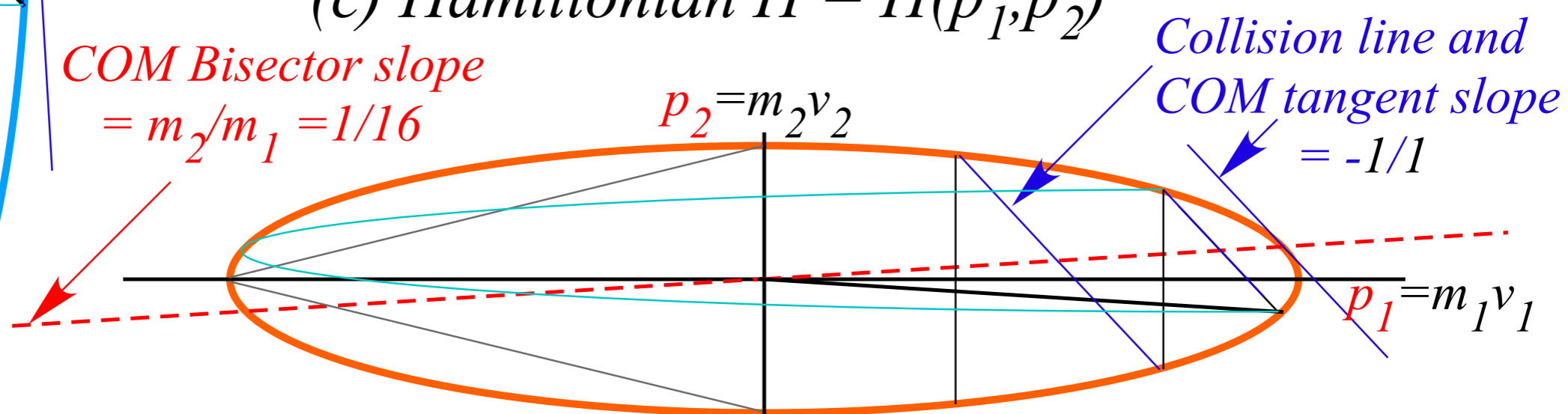
(a) Lagrangian $L = L(v_1, v_2)$



(b) Estranganian $E = E(V_1, V_2)$



(c) Hamiltonian $H = H(p_1, p_2)$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

 *Reflections in the clothing store: "It's all done with mirrors!"*

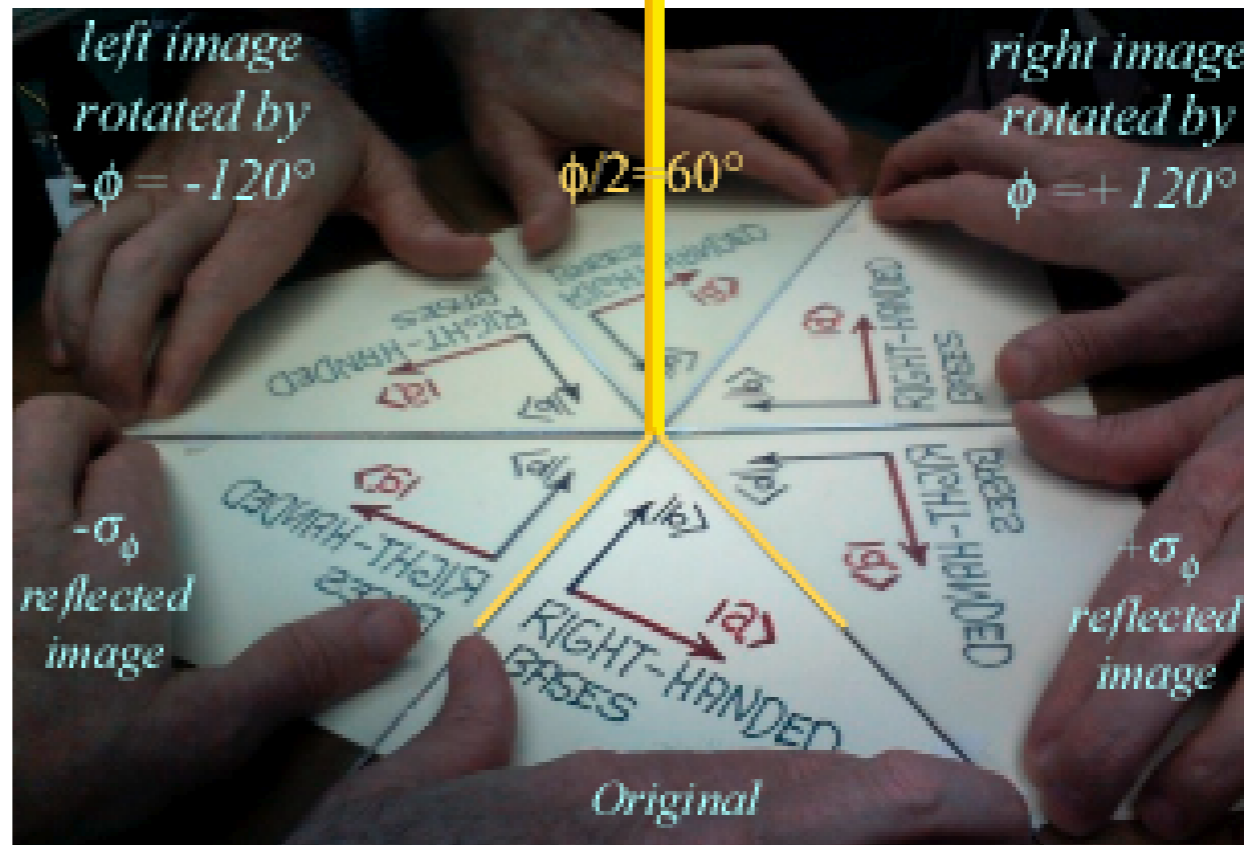
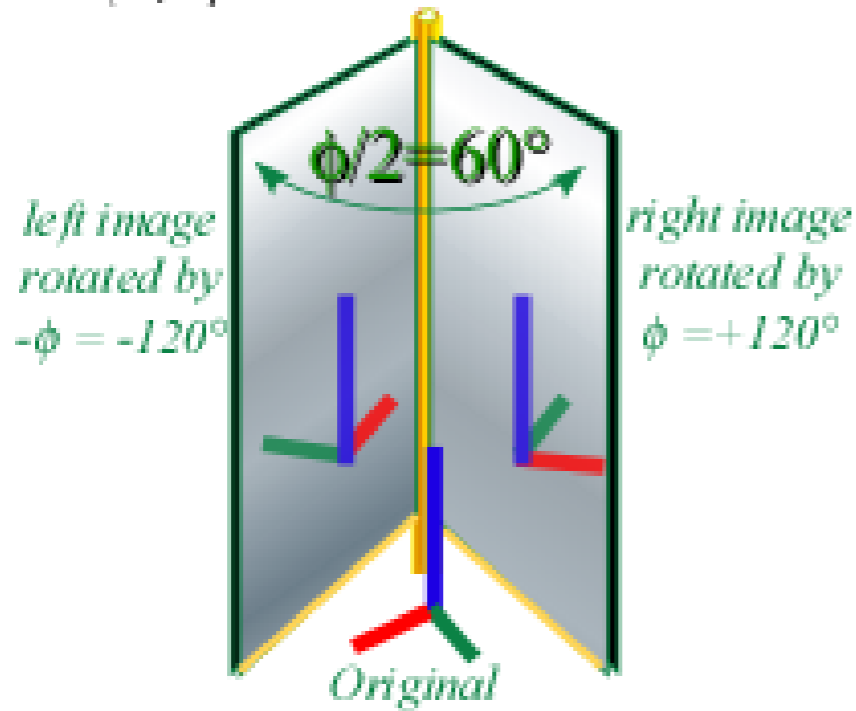
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Reflections in clothing store mirrors

(a) $\phi = \pm 120^\circ$ rotations



(b) $\phi = \pm 180^\circ$ rotations

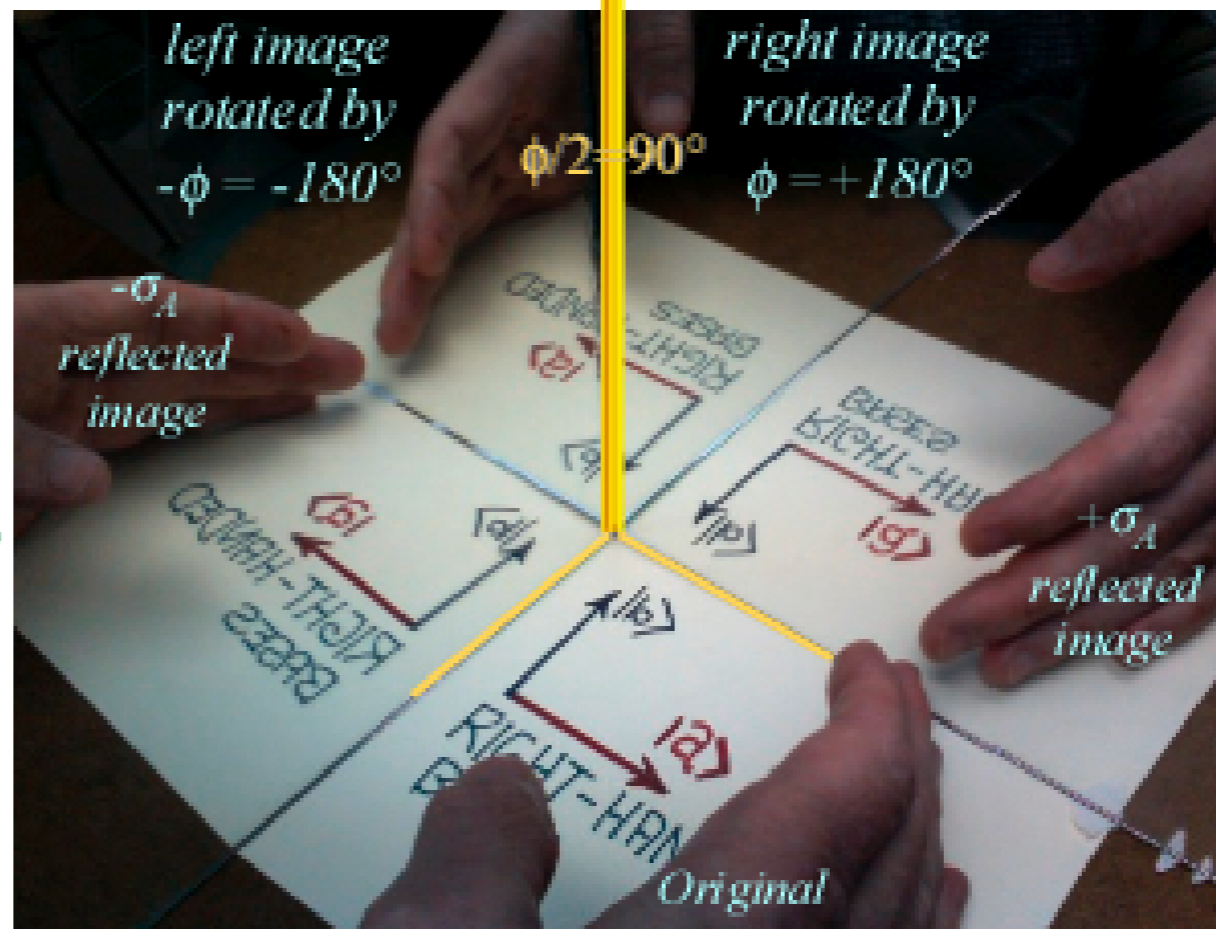
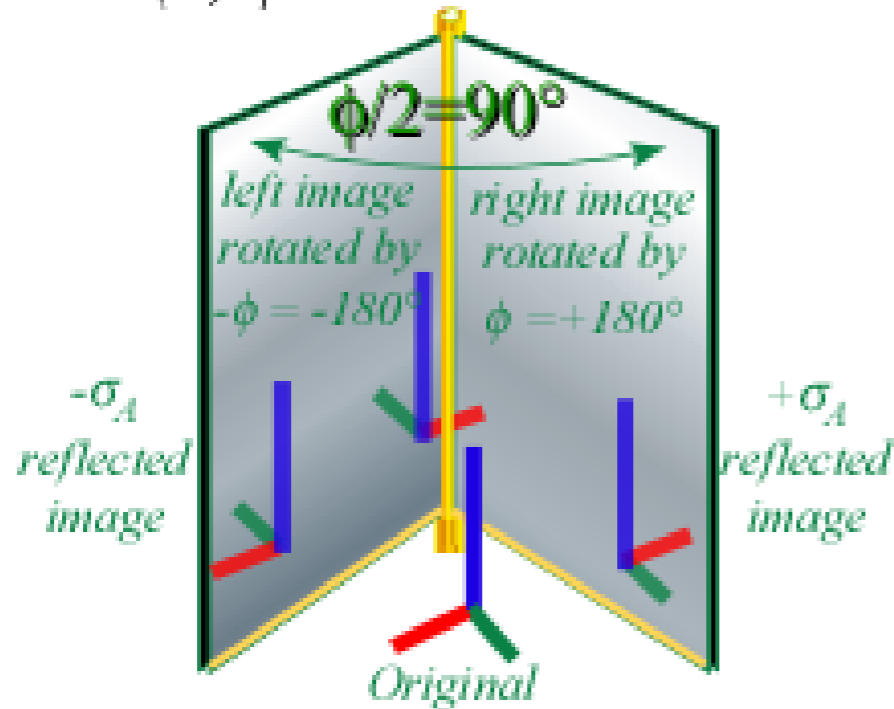
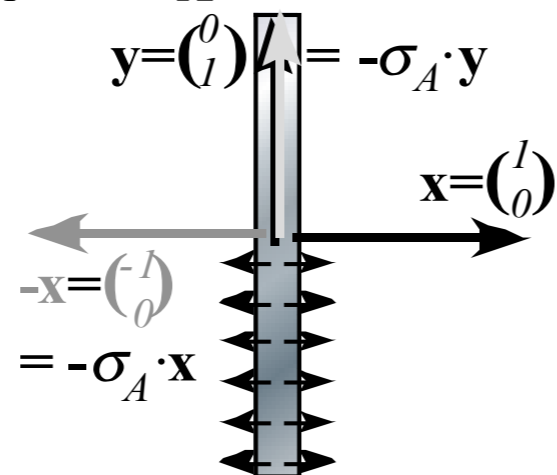
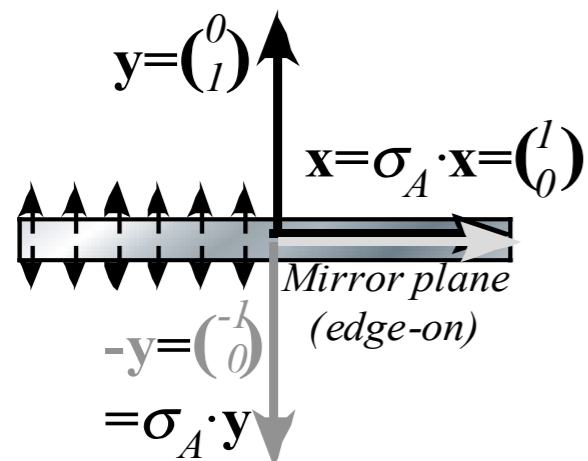


Fig. 5.4a-b

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

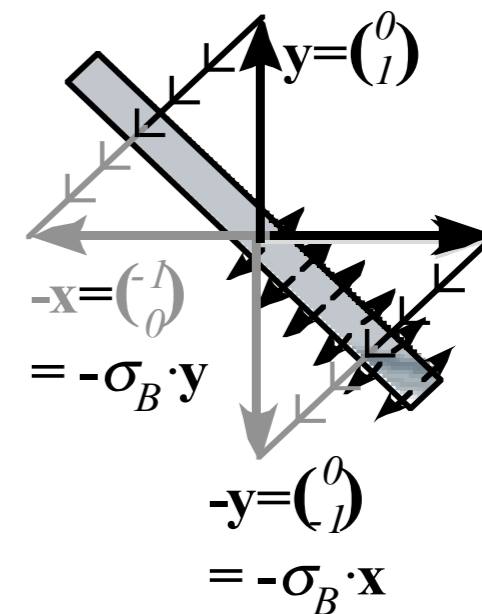
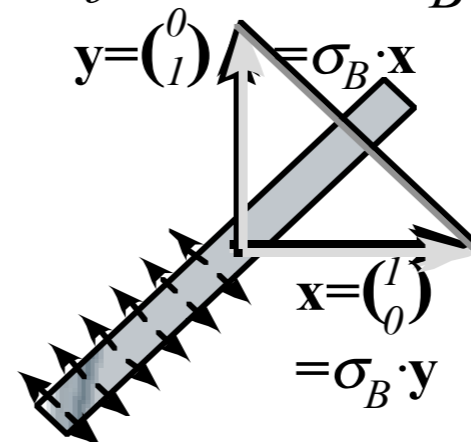
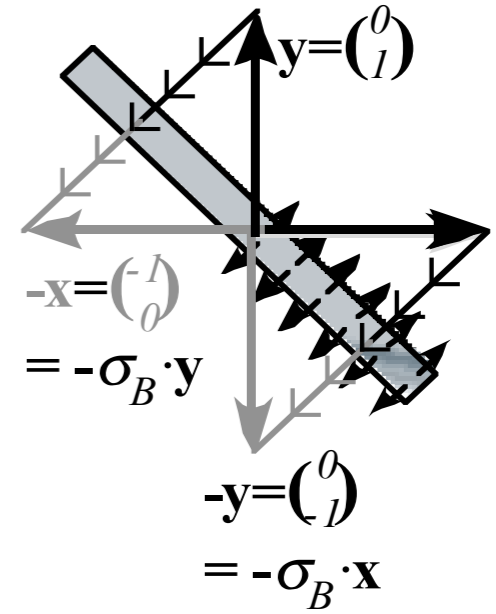
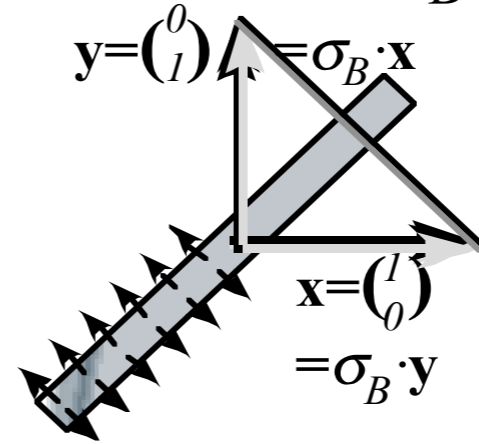
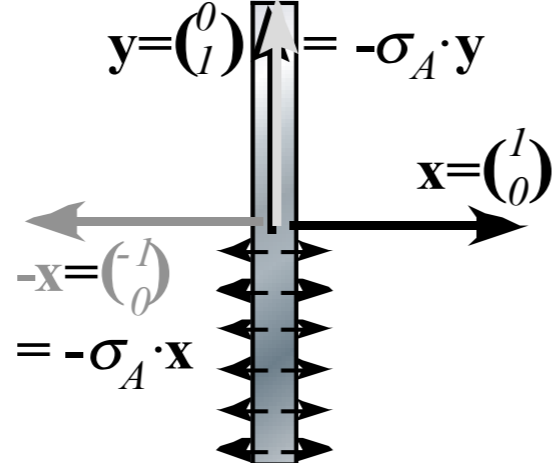
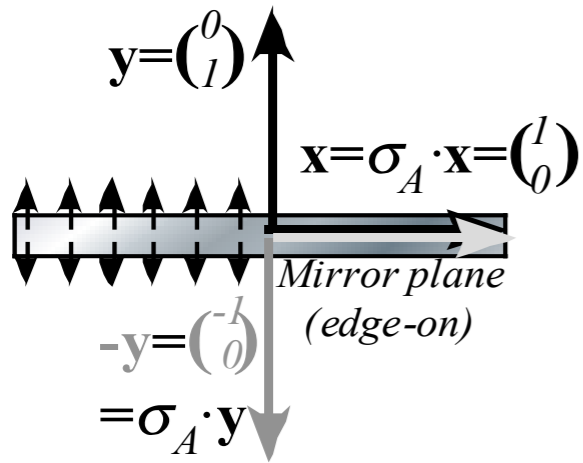


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

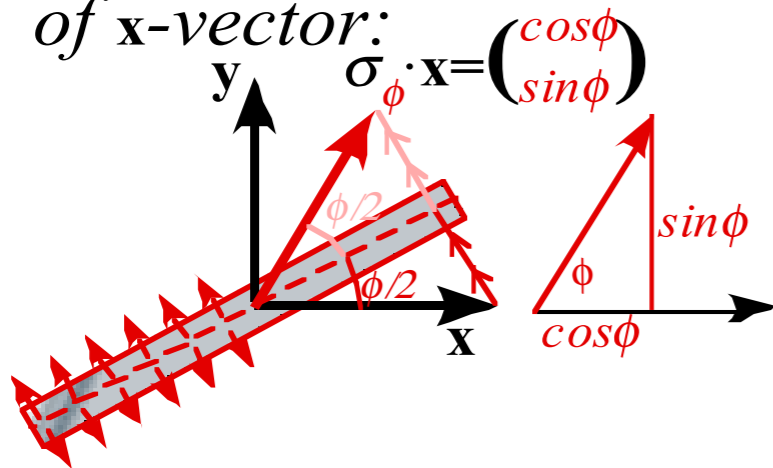
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of x -vector:



...of y -vector:

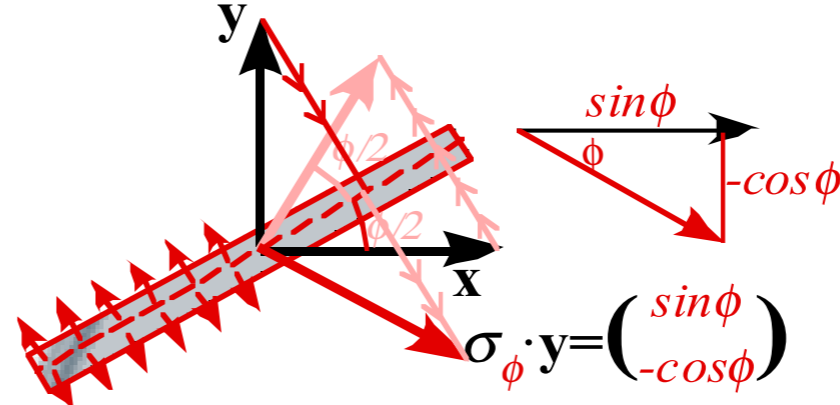
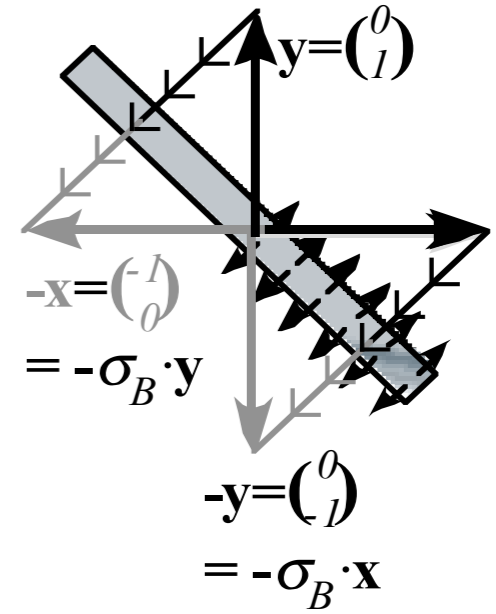
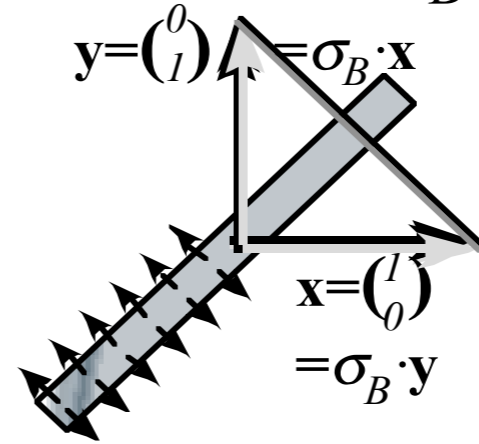
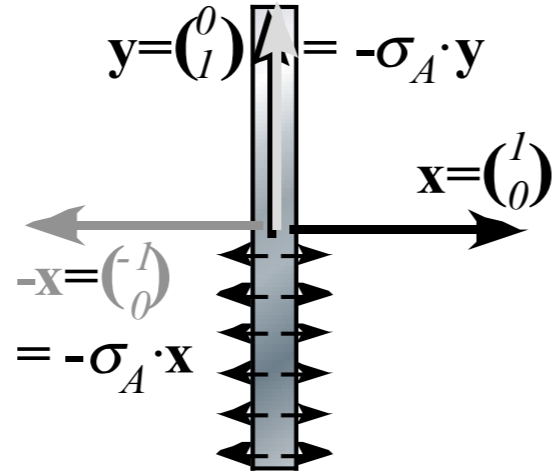
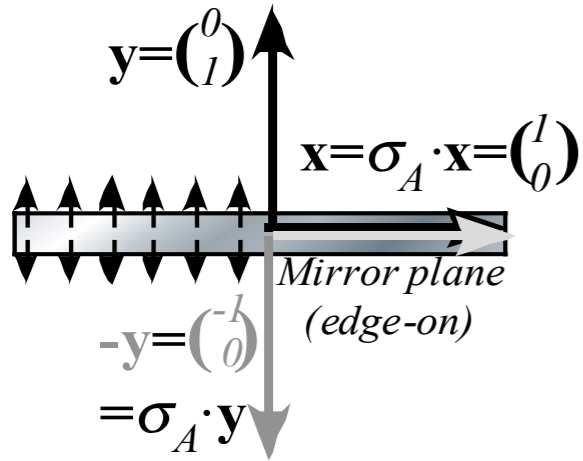


Fig. 5.3a-e

Symmetry: It's all done with mirrors!

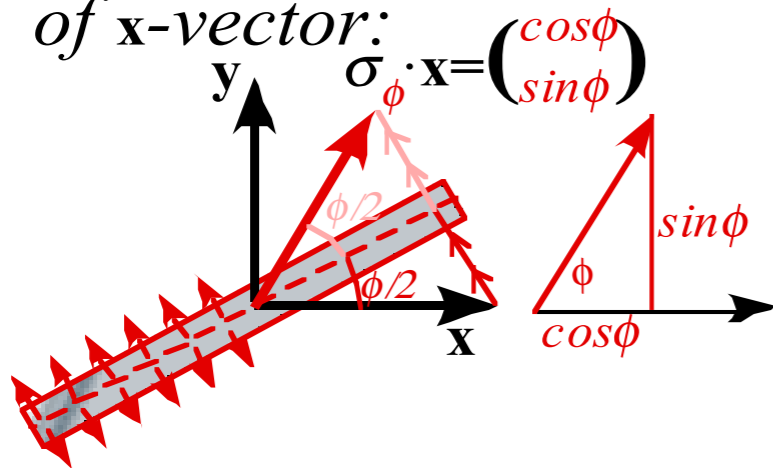
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

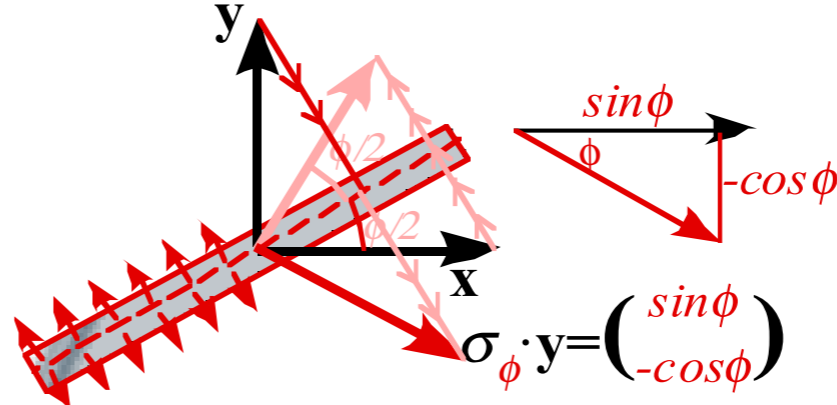


(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

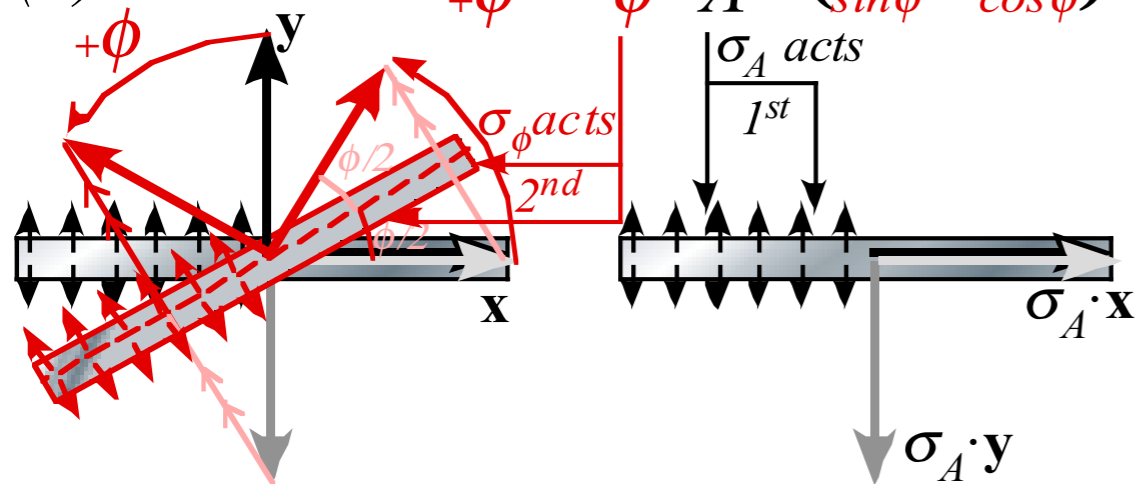
of \mathbf{x} -vector:



...of \mathbf{y} -vector:



(d) Rotation: $R_{+\phi} = \sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



(e) Rotation: $R_{-\phi} = \sigma_A \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

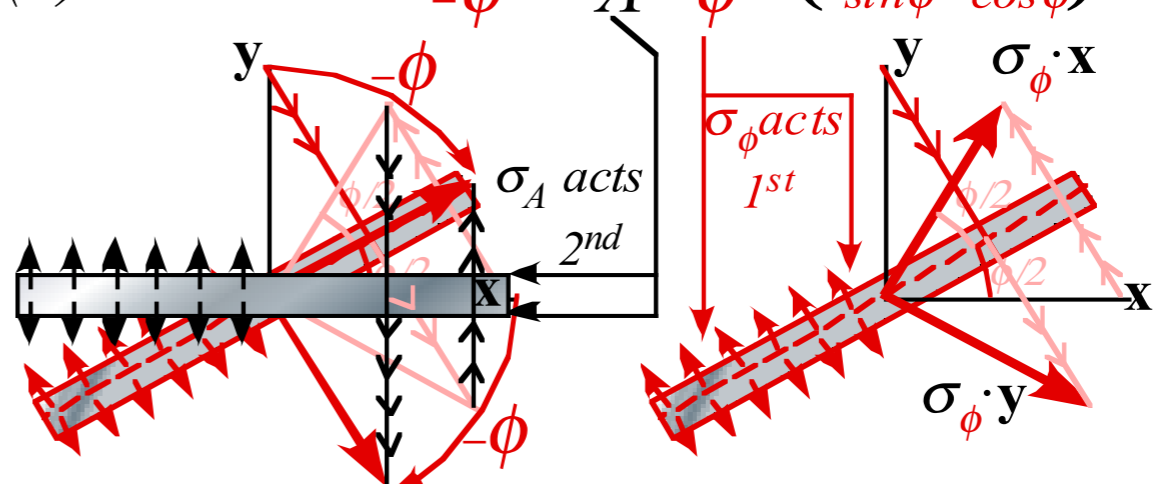


Fig. 5.3a-e

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

Product of odd number of reflections is a reflection

*... even number of reflections is a rotation (or unit-op **1**)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily

Waves patterns are non-rigid and reflect easily

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

Product of odd number of reflections is a reflection

*... even number of reflections is a rotation (or unit-op **1**)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily

Waves patterns are non-rigid and reflect easily

∴ ...wave reflections underlie modern physics

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

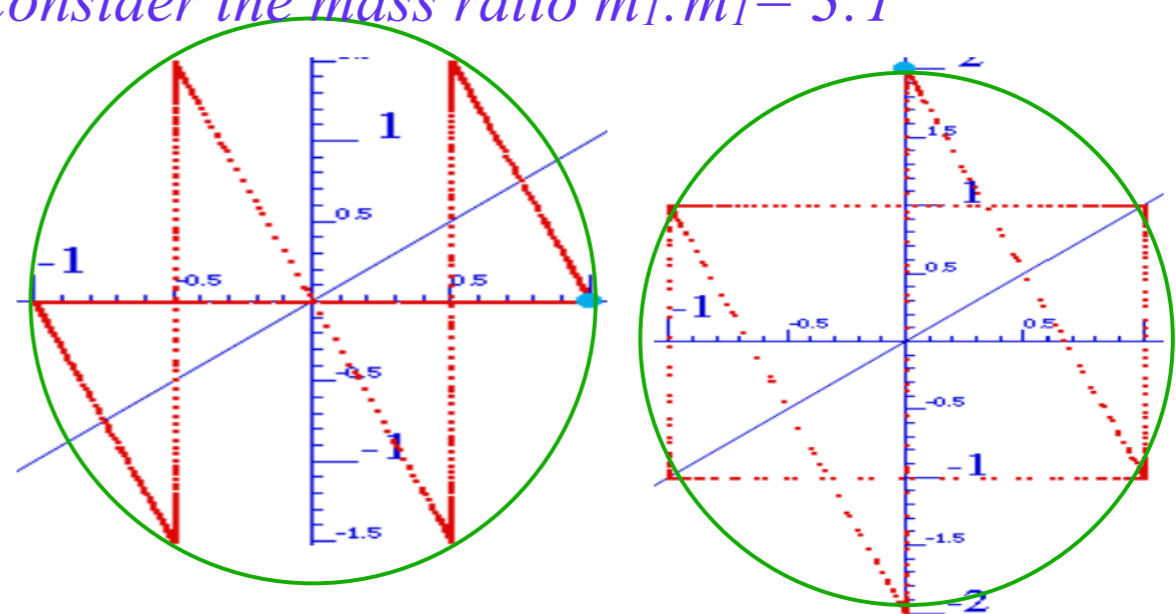
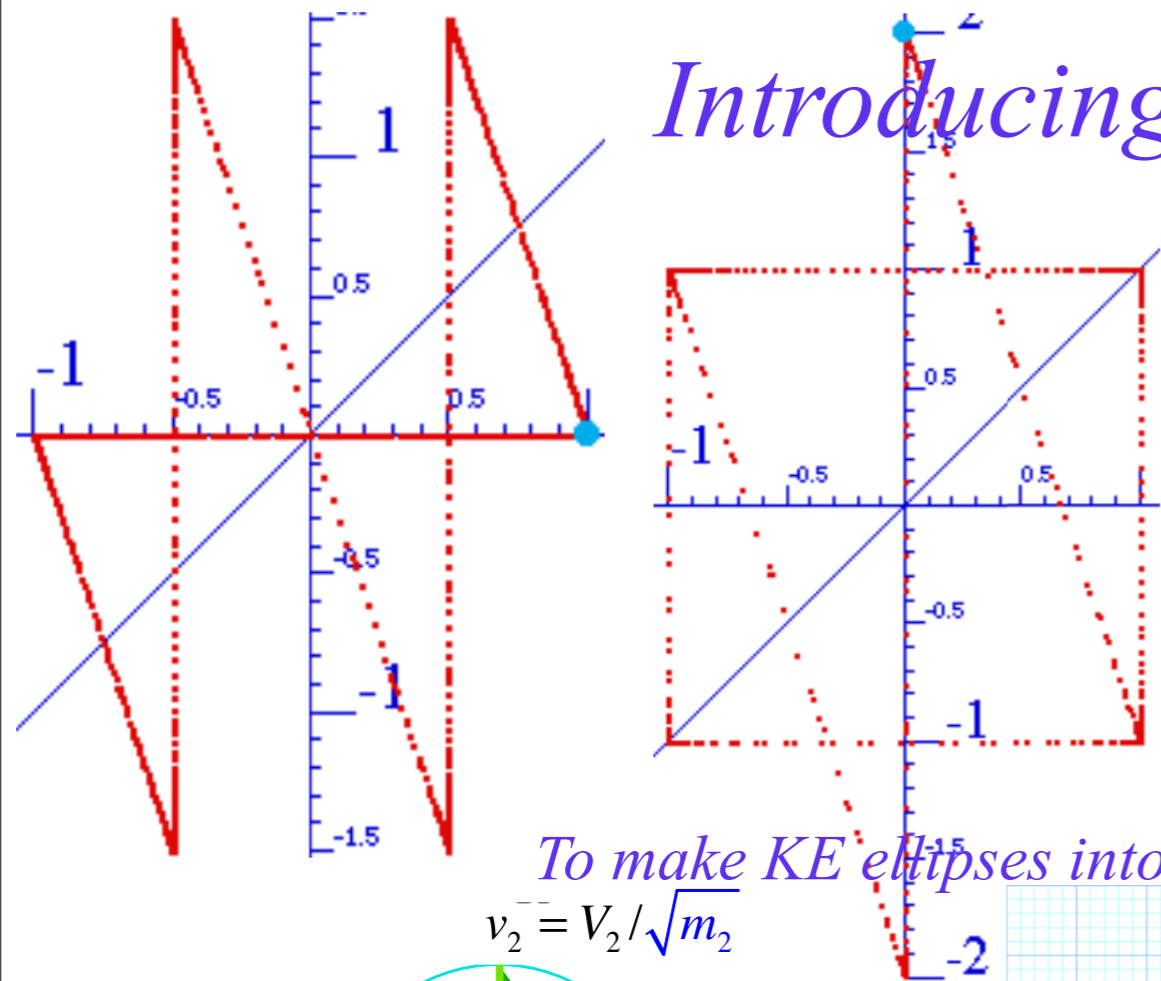
 *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*

Group multiplication and product table

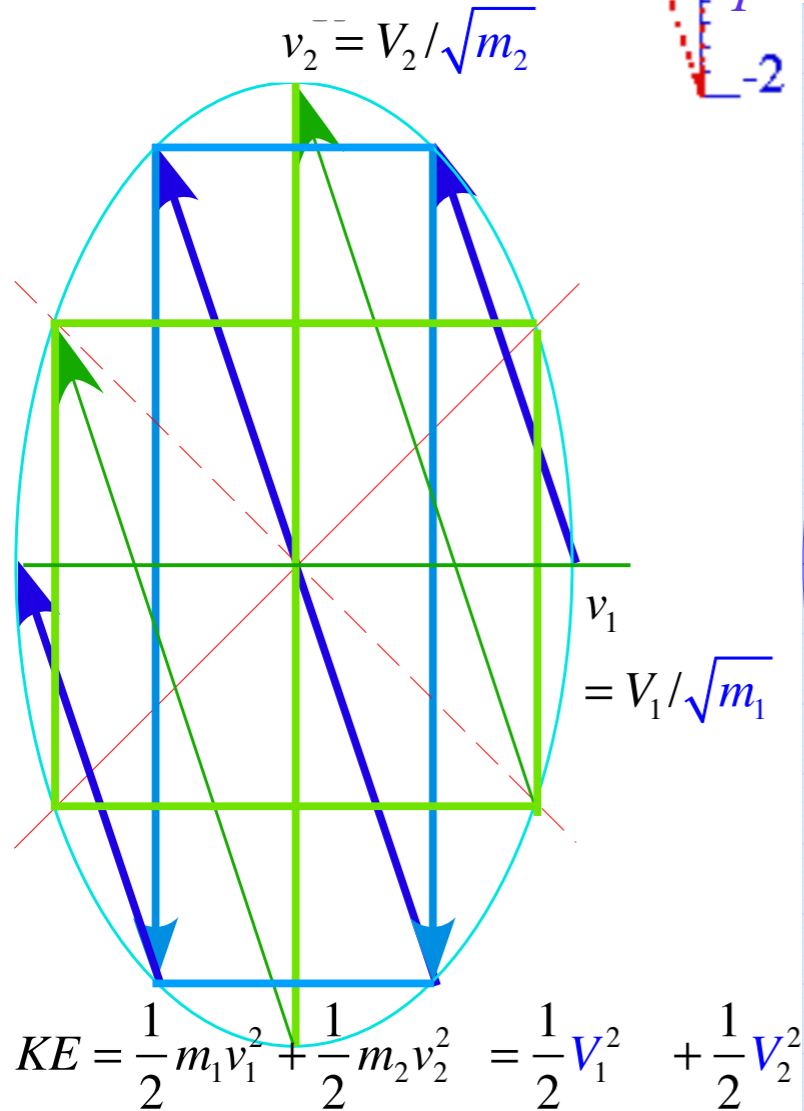
Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Introducing Symmetry Operators

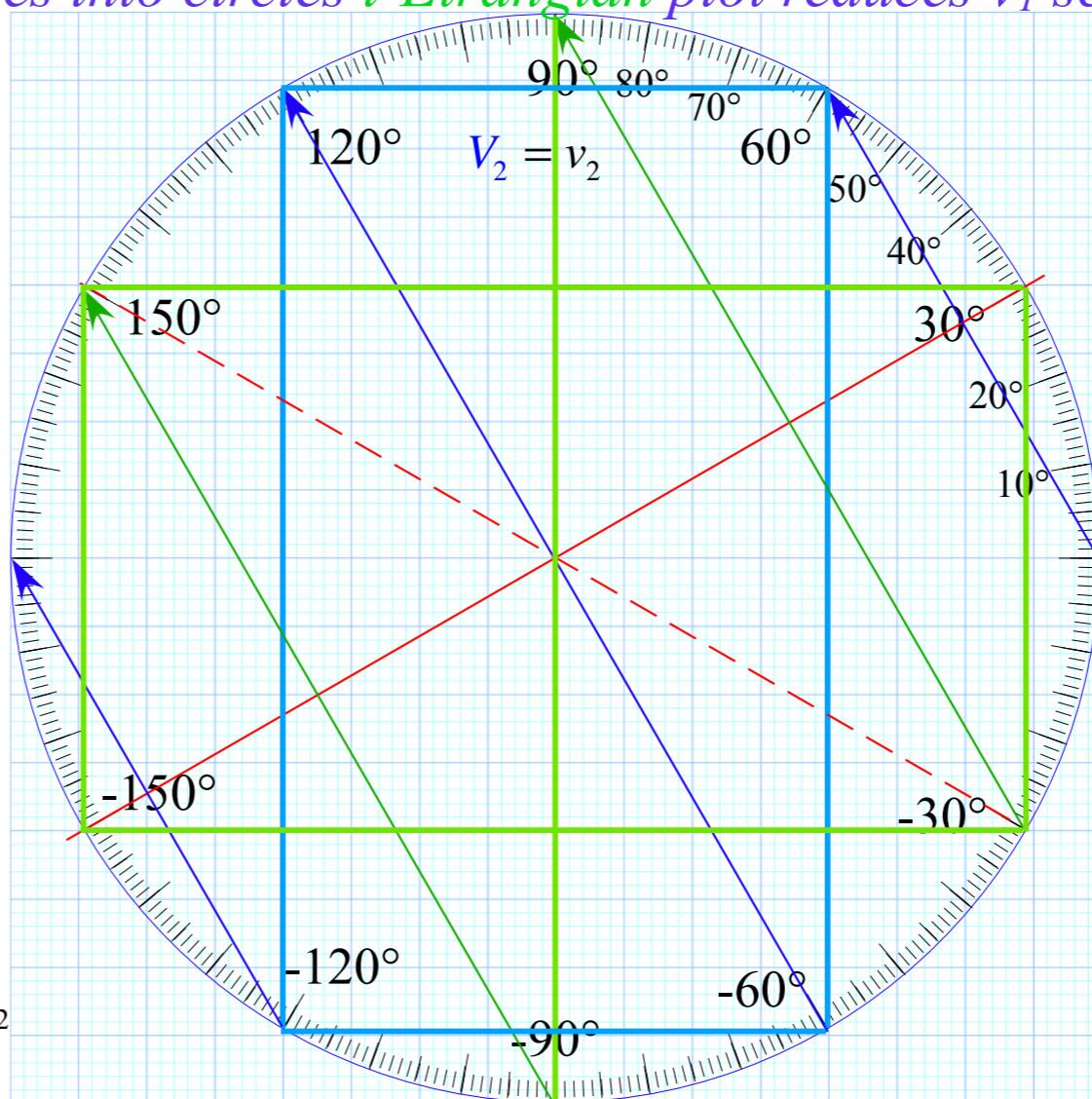
Consider the mass ratio $m_1:m_2=3:1$



To make KE ellipses into circles *l'Etranguian* plot reduces v_1 scale by $1/\sqrt{m_1}$, etc.



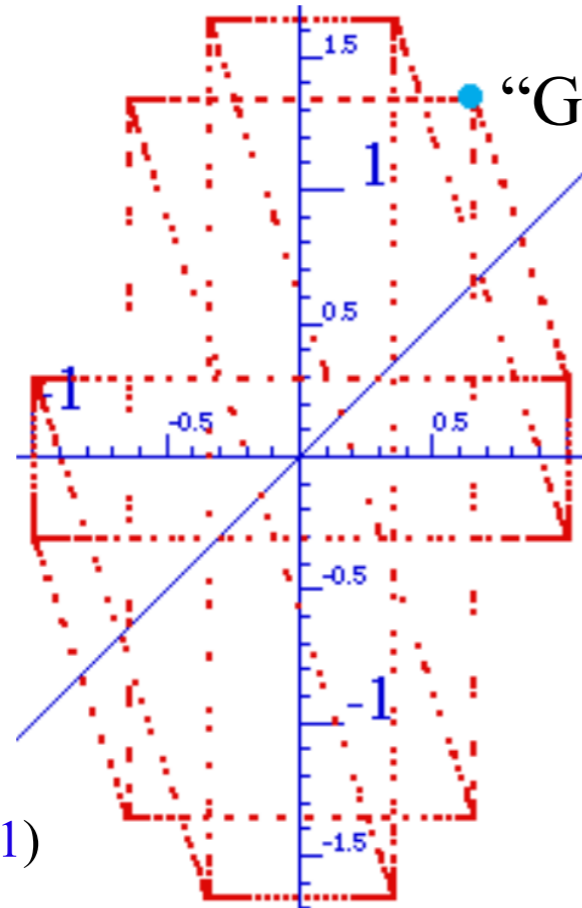
$$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$$



Here:

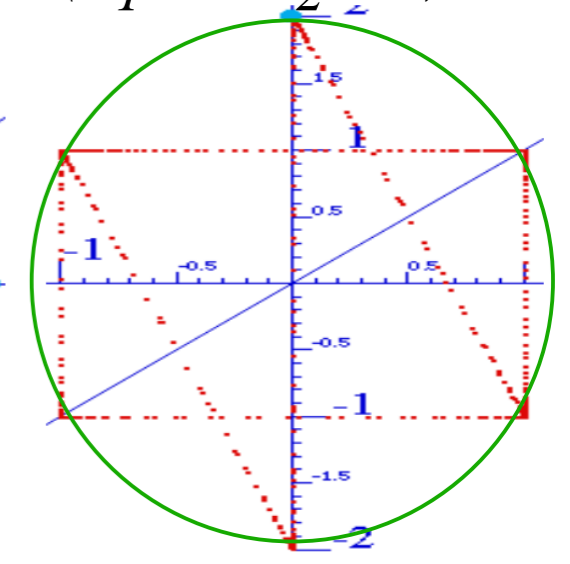
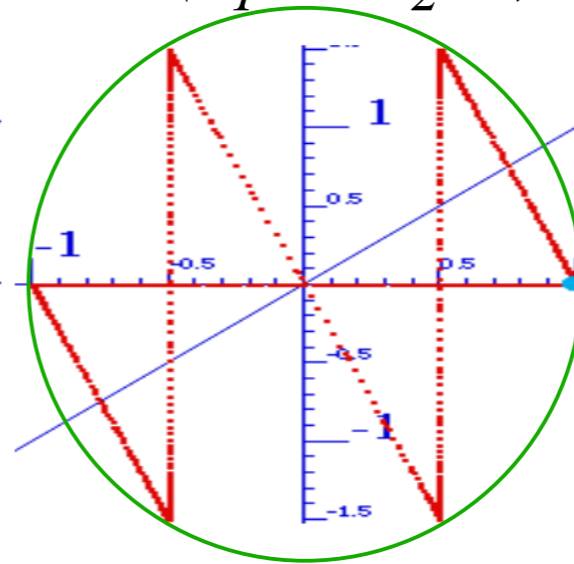
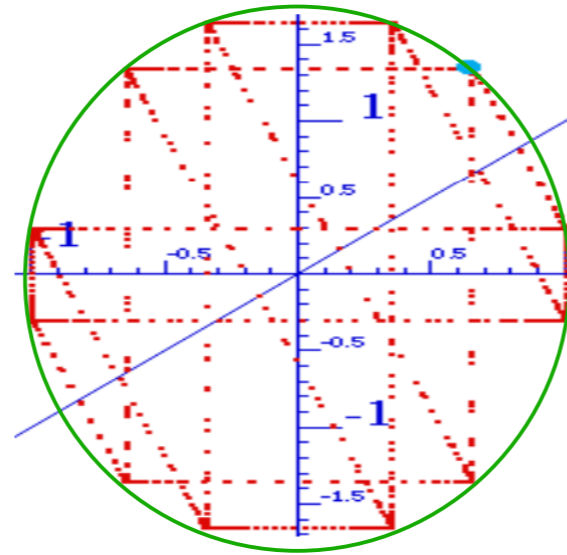
$$\frac{1}{\sqrt{m_1}} = \frac{1}{\sqrt{3}} = 0.577$$

$$\frac{1}{\sqrt{m_2}} = \frac{1}{\sqrt{1}} = 1.0$$



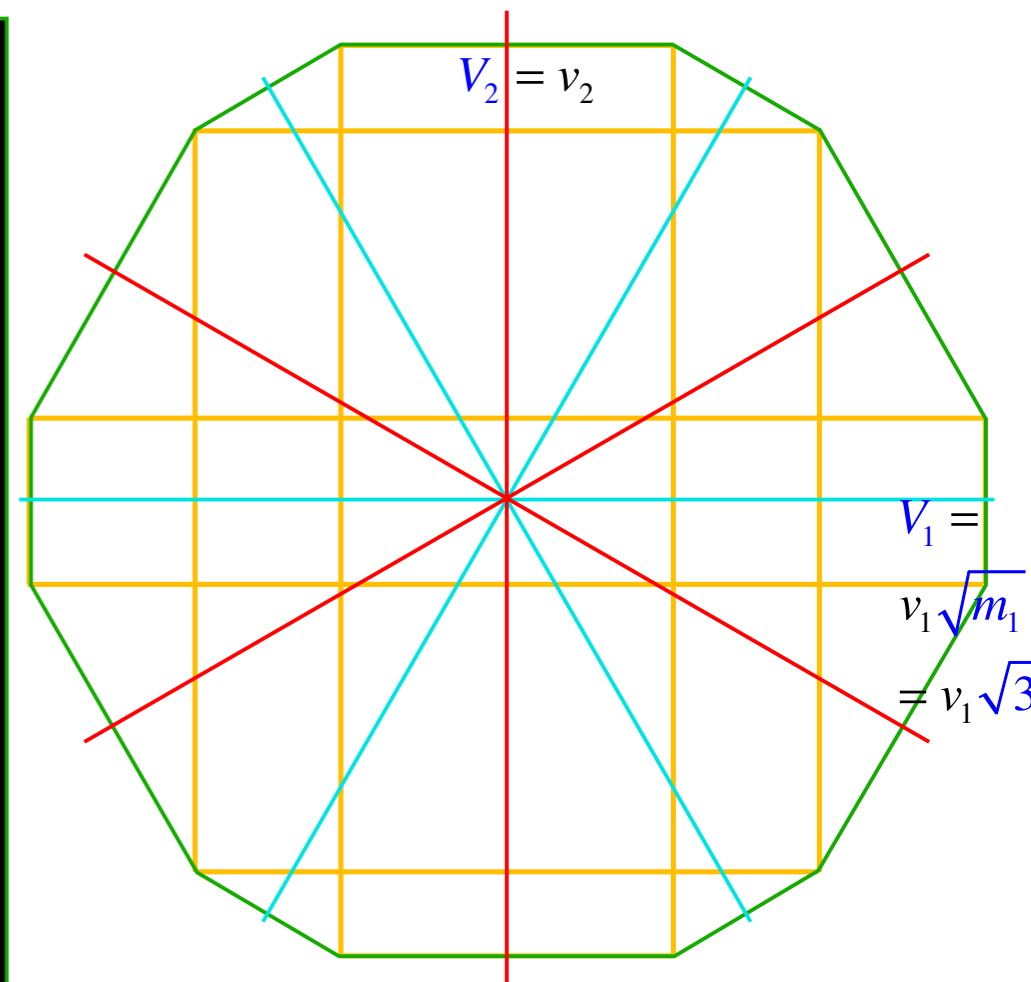
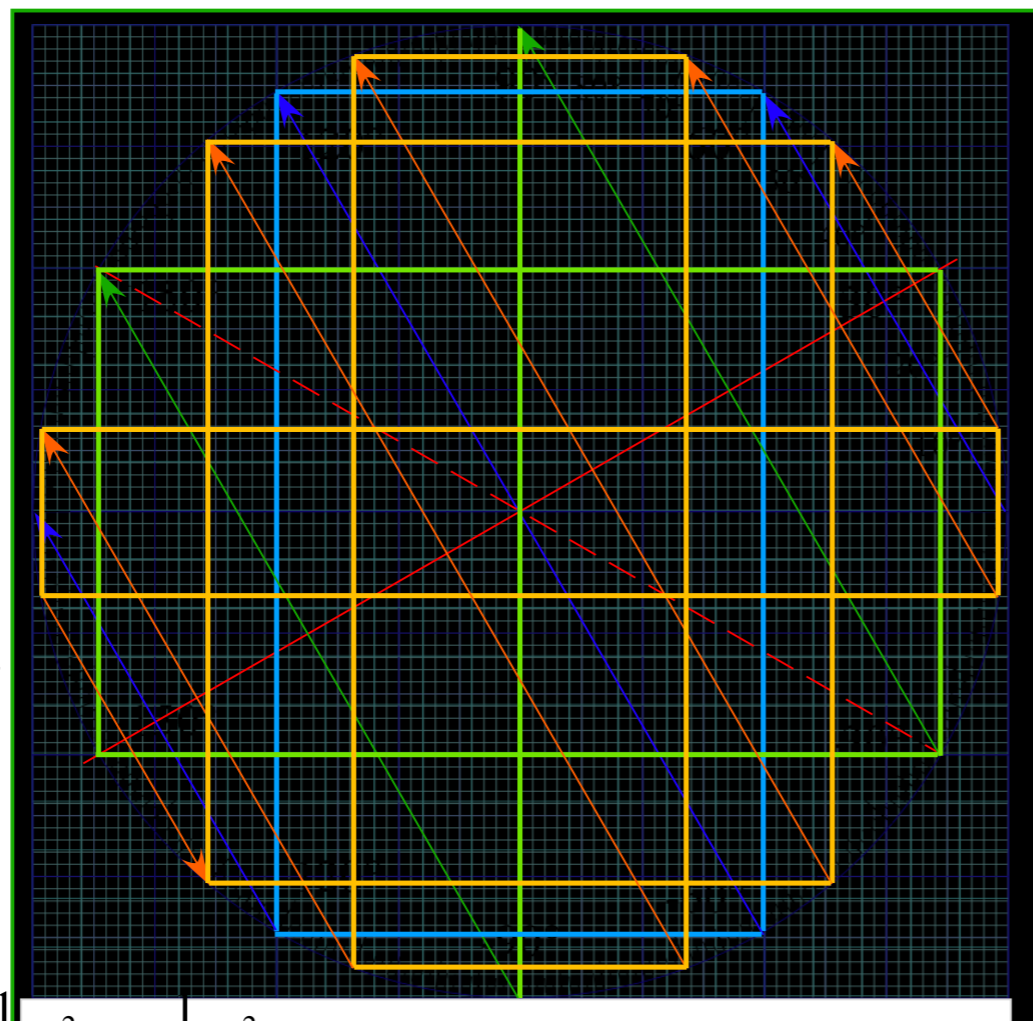
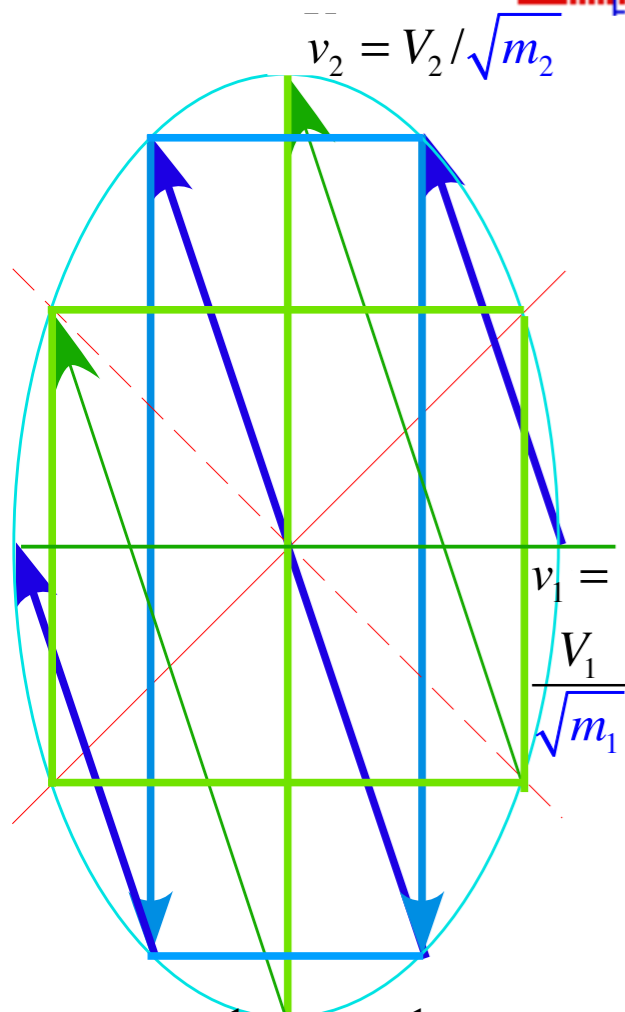
“Generic” initial velocity
 $(v_1=1.0, v_2=0.1)$

“Symmetric” initial velocity
 $(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$



$m_1/m_2=(3)/(1)$

reduce v_1 scale by $1/\sqrt{m_1} = 1/\sqrt{3}=0.577$



$$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$$

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

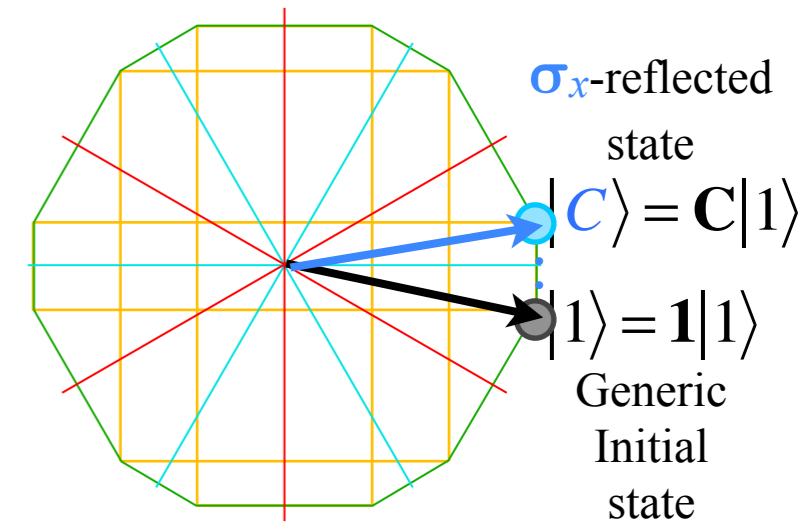
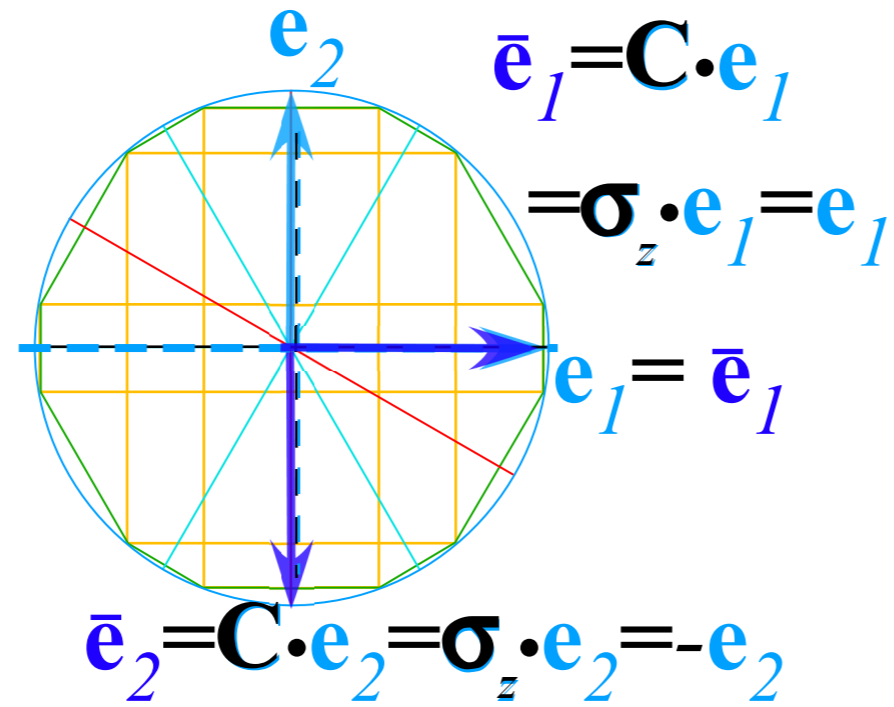
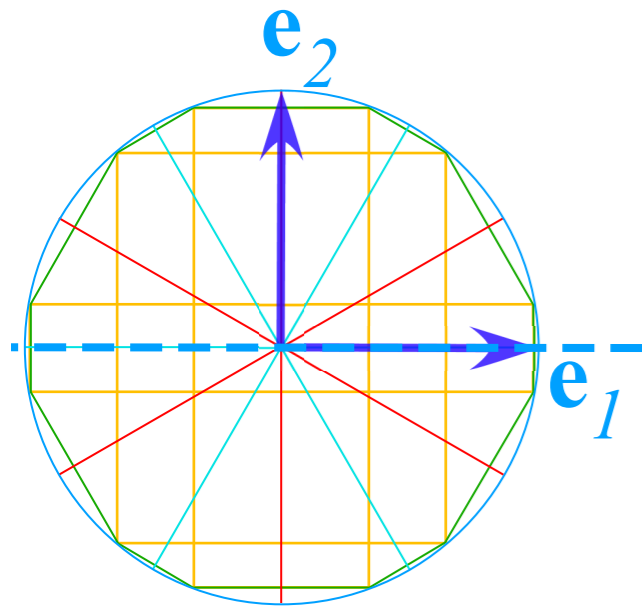
Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

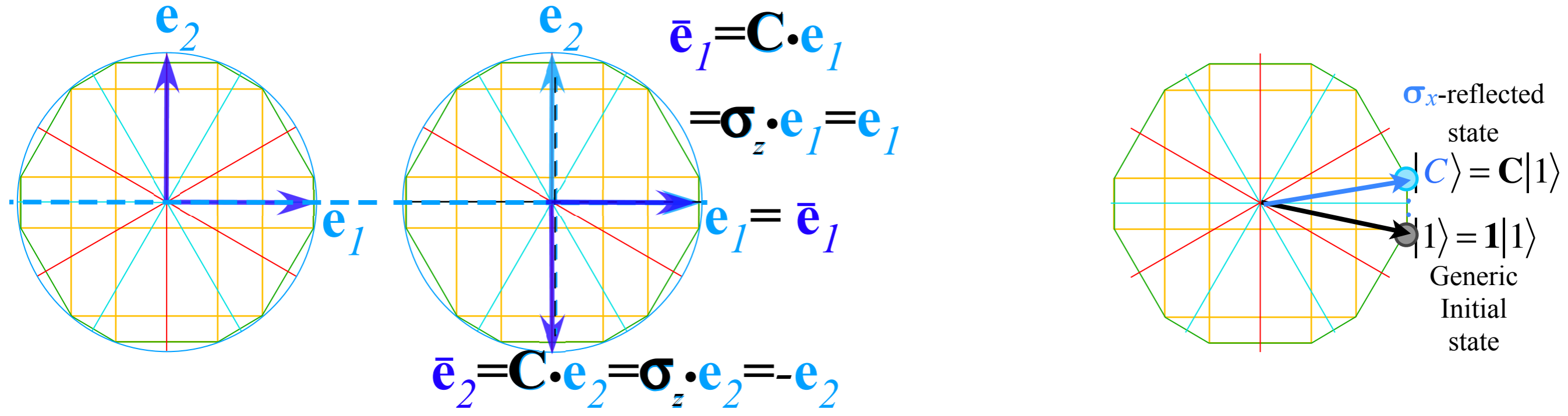
 *Group multiplication and product table*

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Effects of Ceiling Bang Matrix $\mathbf{C} = \boldsymbol{\sigma}_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



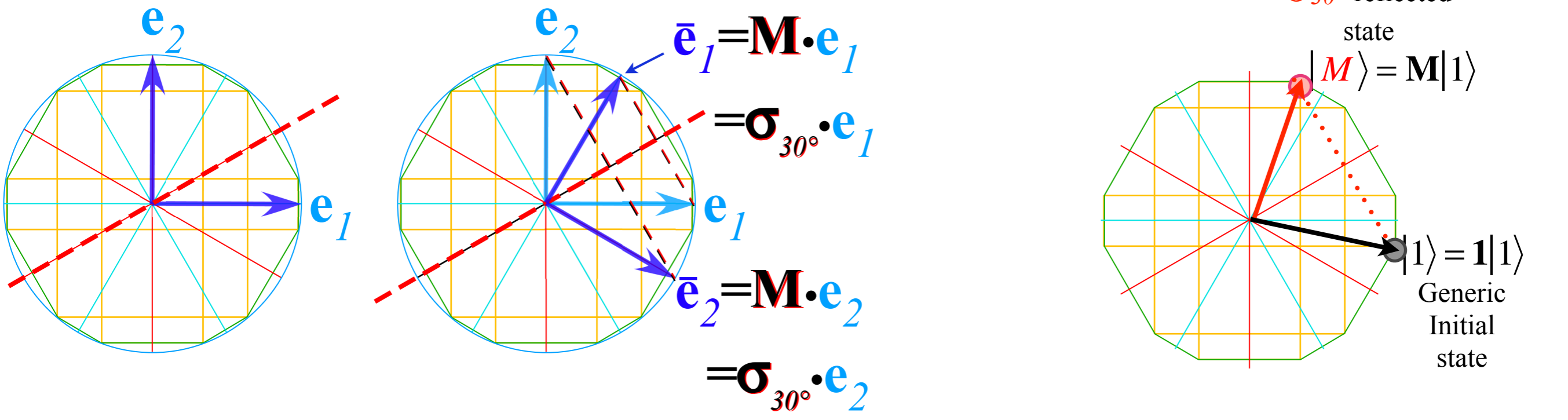
Effects of Ceiling Bang Matrix $\mathbf{C} = \boldsymbol{\sigma}_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Known as *matrix elements* or *components*

Known as *relative direction cosines*

Effects of Mass Bang Matrix $\mathbf{M} = \boldsymbol{\sigma}_{30^\circ} = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{M} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{M} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{M} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{M} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix}$

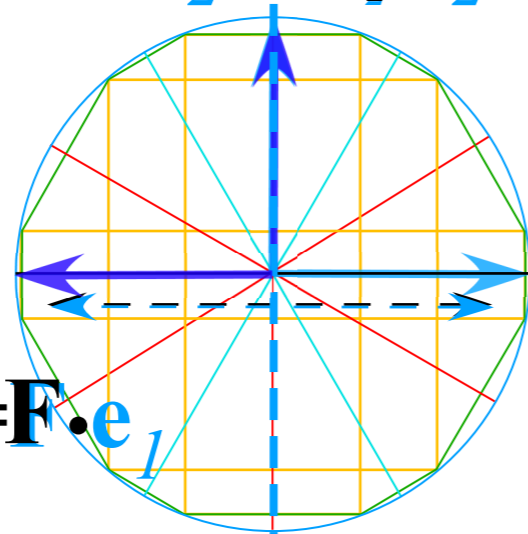
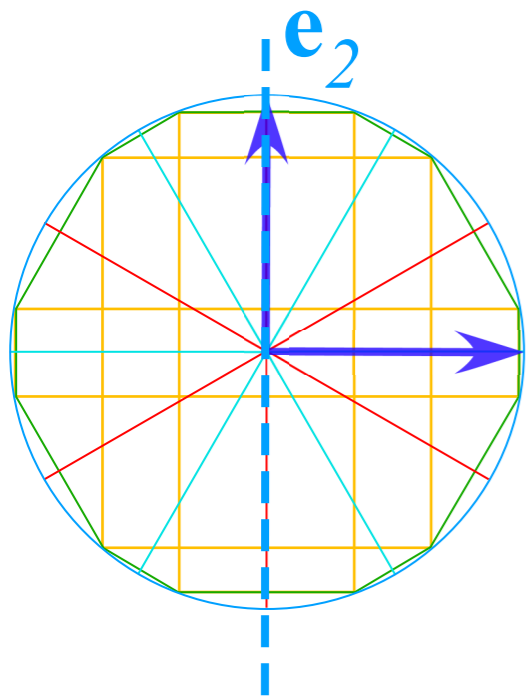


Effects of Floor Bang Matrix

$$\mathbf{F} = -\sigma_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

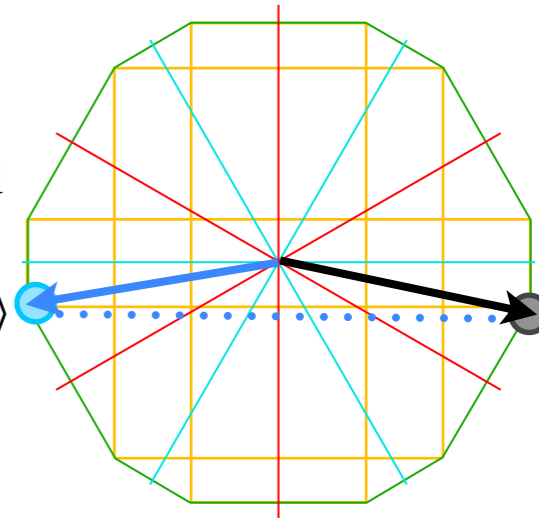
$$\bar{\mathbf{e}}_2 = \mathbf{F} \cdot \mathbf{e}_2 = -\sigma_z \cdot \mathbf{e}_2 = +\mathbf{e}_2$$

$$\bar{\mathbf{e}}_1 = \mathbf{F} \cdot \mathbf{e}_1 = -\sigma_z \cdot \mathbf{e}_1 = -\mathbf{e}_1$$



$-\sigma_z$ -reflected state

$$|F\rangle = \mathbf{F}|1\rangle$$

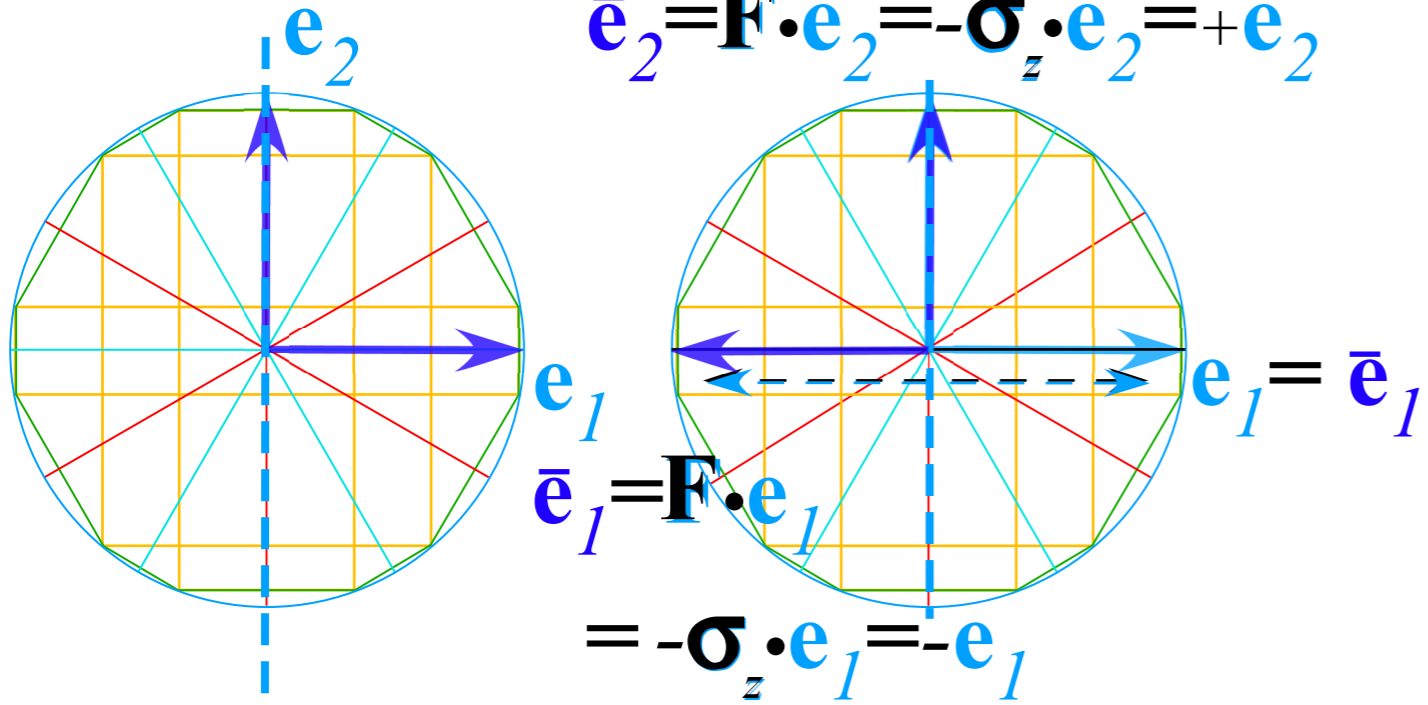


$|1\rangle = \mathbf{1}|1\rangle$
Generic Initial state

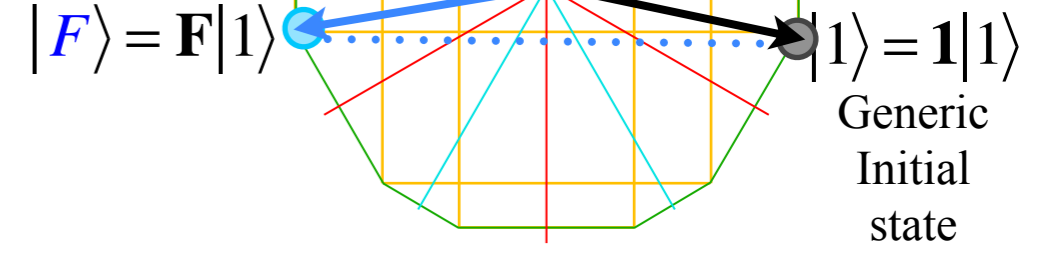
Effects of Floor Bang Matrix

$$\mathbf{F} = -\sigma_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{\mathbf{e}}_2 = \mathbf{F} \cdot \mathbf{e}_2 = -\sigma_z \cdot \mathbf{e}_2 = +\mathbf{e}_2$$

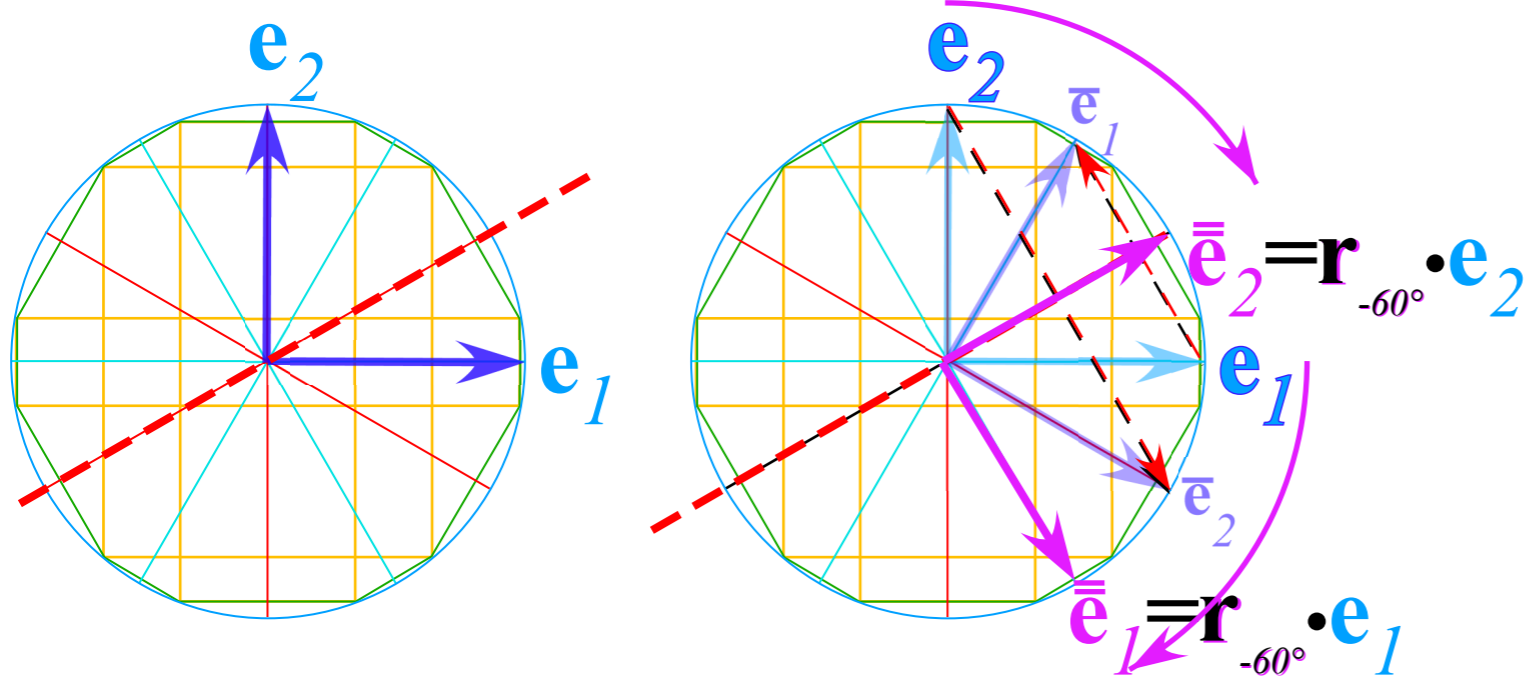


$-\sigma_z$ -reflected state

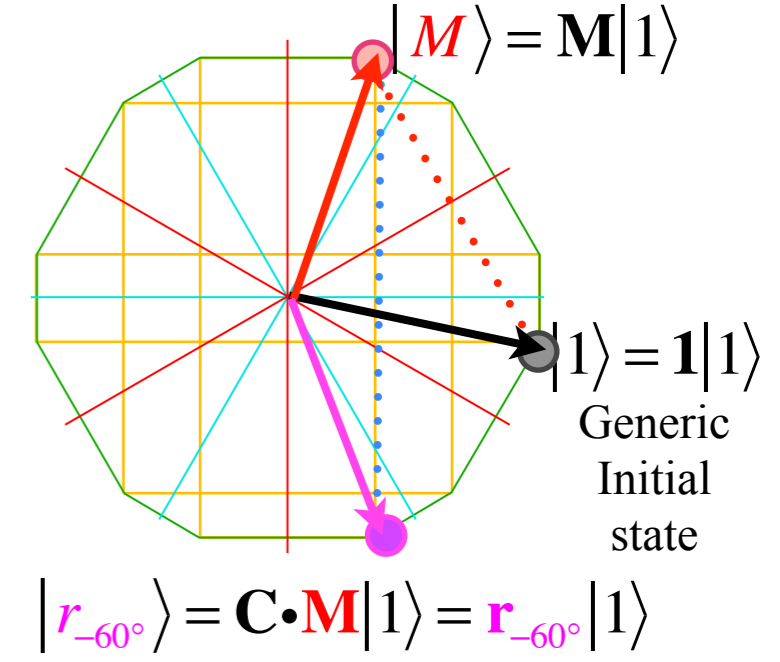


Effects of Ceiling C after Bang M:

$$\mathbf{r}_{-60^\circ} = \mathbf{C} \cdot \mathbf{M} = \sigma_z \cdot \sigma_{30^\circ}$$



σ_{30° -reflected state



σ_{30° σ_{30° -reflected state

is a \mathbf{r}_{-60° -rotated state

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)



D_6	1	r_{120}	\bar{r}_{120}	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	\bar{r}_{60}	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
1	1											
\bar{r}_{120}		1										
r_{120}			1									
σ_{60}				1								
$\bar{\sigma}_{60}$					1							
σ_z						1					\bar{r}_{60}	
I							1					
r_{60}								1				
\bar{r}_{60}									1			
$\bar{\sigma}_{30}$										1		
σ_{30}											1	
$\bar{\sigma}_z$												1

Note: $\bar{r}_{60} = I r_{120} = r_{120} I = r_{-60}$ and: $I = r_{\pm 180}$
 $\bar{r}_{120} = I r_{60} = r_{60} I = r_{-120}$ and: $I^2 = 1$
 $\sigma_{60} = I \bar{\sigma}_{30} = \bar{\sigma}_{30} I$
 $\bar{\sigma}_{60} = I \sigma_{30} = \sigma_{30} I$
 $\bar{\sigma}_z = I \sigma_z = \sigma_z I$

Easy to make hexagonal (D_6) symmetry group table:

Example 1: Find $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = \underline{\hspace{2cm}}$?

Solution: Find σ_{30° -plane and state- $|\sigma_{-60^\circ}\rangle$

Operate former on latter to get: $\sigma_{30^\circ} |\sigma_{-60^\circ}\rangle = |I\rangle$

That gives answer: $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = I$.

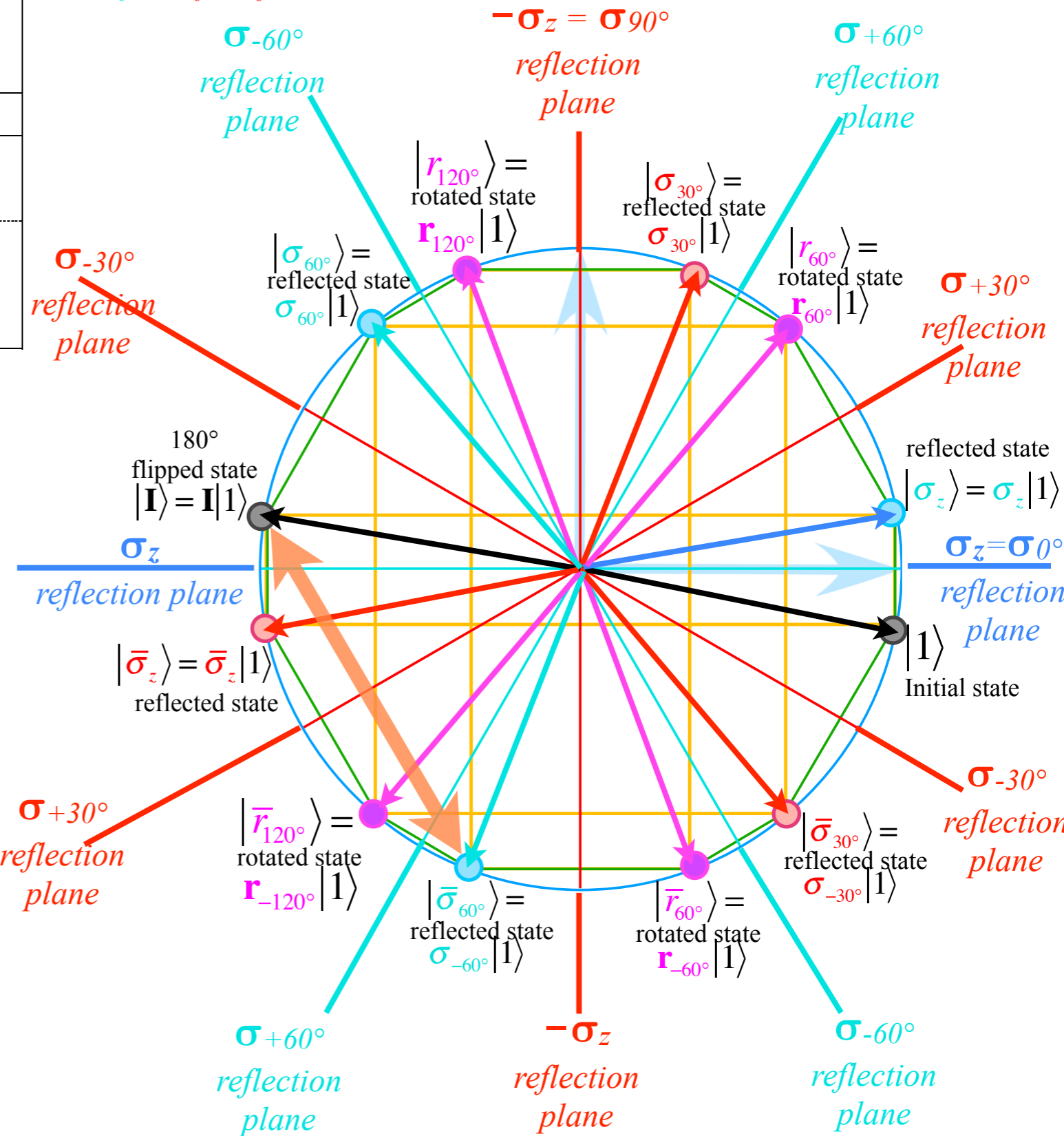
Rest of σ_{30° row follows:

σ_{30}	σ_{30}	$\bar{\sigma}_{30}$	$\bar{\sigma}_z$	\bar{r}_{60}	I	r_{60}	$\bar{\sigma}_{60}$	σ_{60}	σ_z	r_{120}	1	\bar{r}_{120}
---------------	---------------	---------------------	------------------	----------------	-----	----------	---------------------	---------------	------------	-----------	---	-----------------

Example 2: Find $r_{60^\circ} \cdot \sigma_{-60^\circ} = \underline{\hspace{2cm}}$?

Solution: Do r_{60° -rotation $r_{60^\circ} |\sigma_{-60^\circ}\rangle = |\sigma_{-30^\circ}\rangle$

That gives answer: $r_{60^\circ} \cdot \sigma_{-60^\circ} = \sigma_{-30^\circ}$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

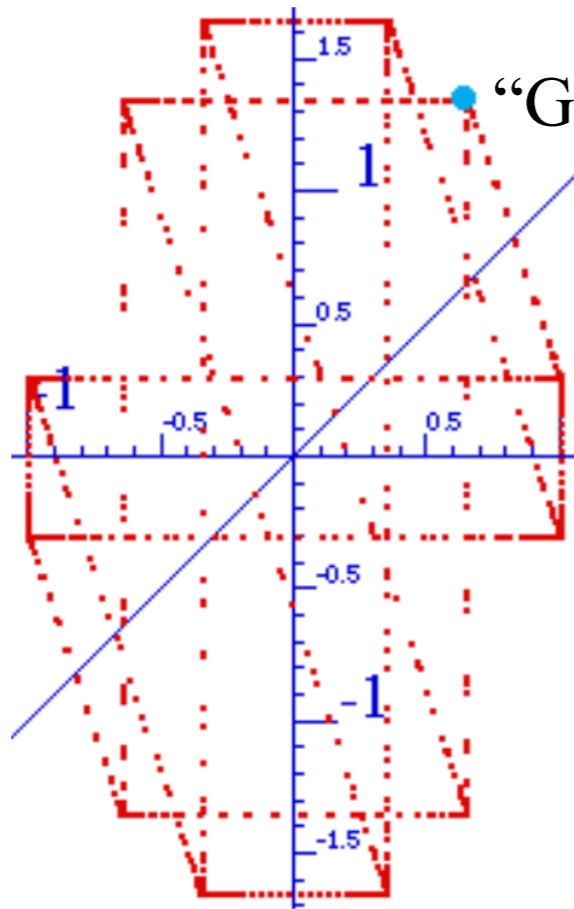
How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch. 12

Reflections in the clothing store: "It's all done with mirrors!"

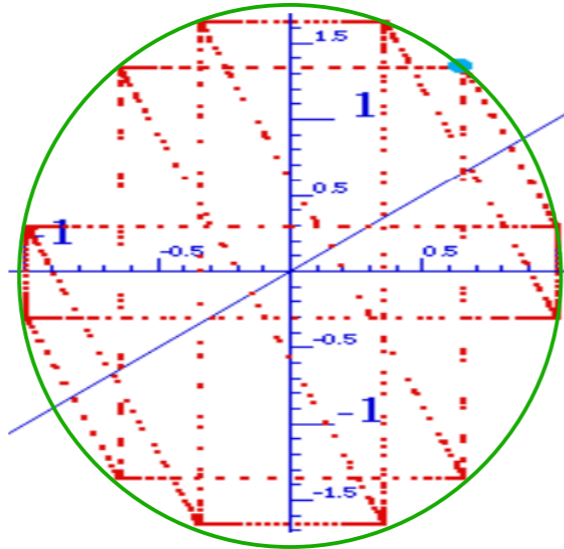
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

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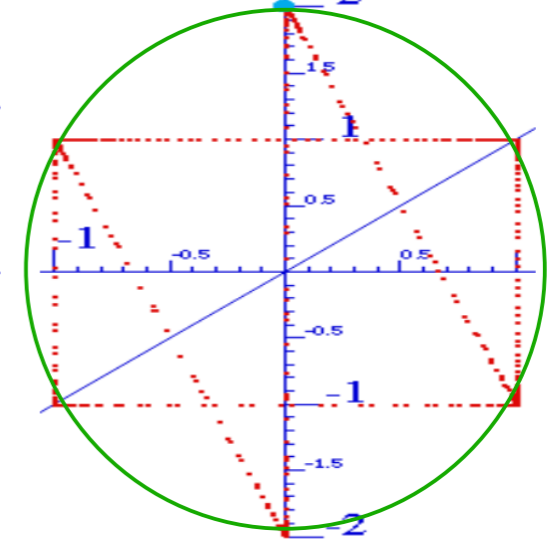
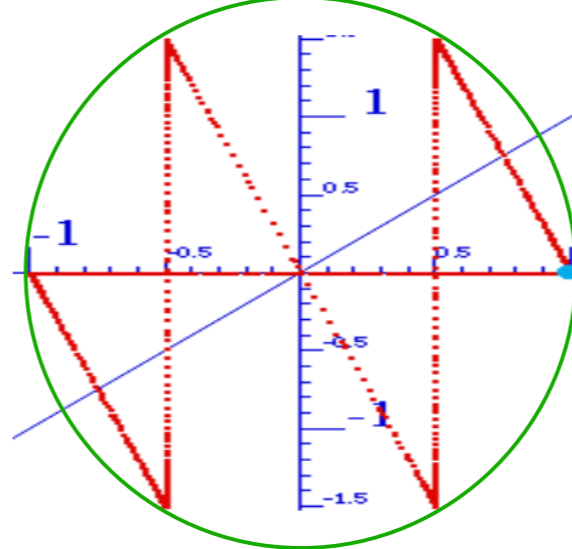
 *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*



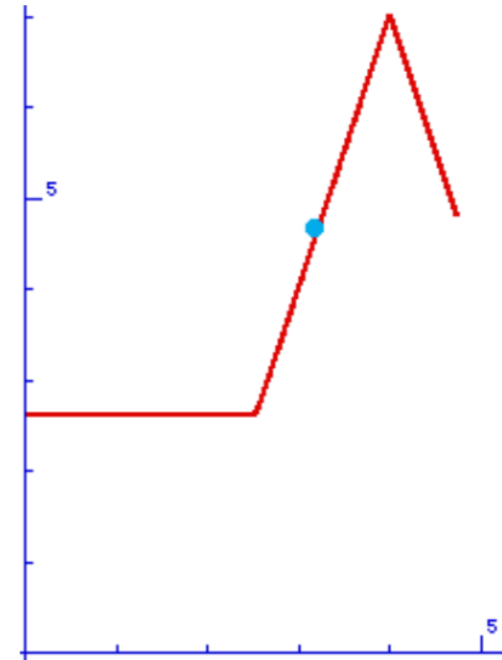
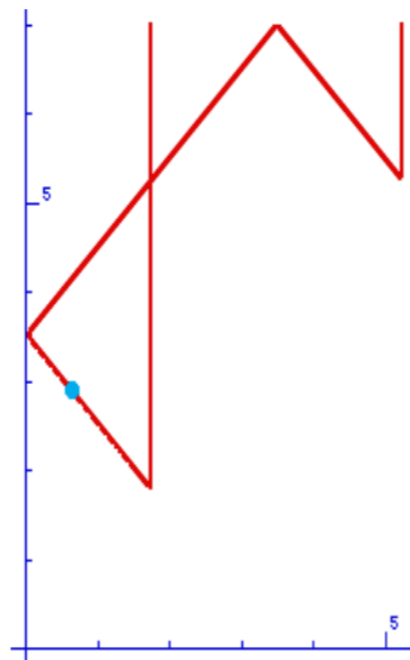
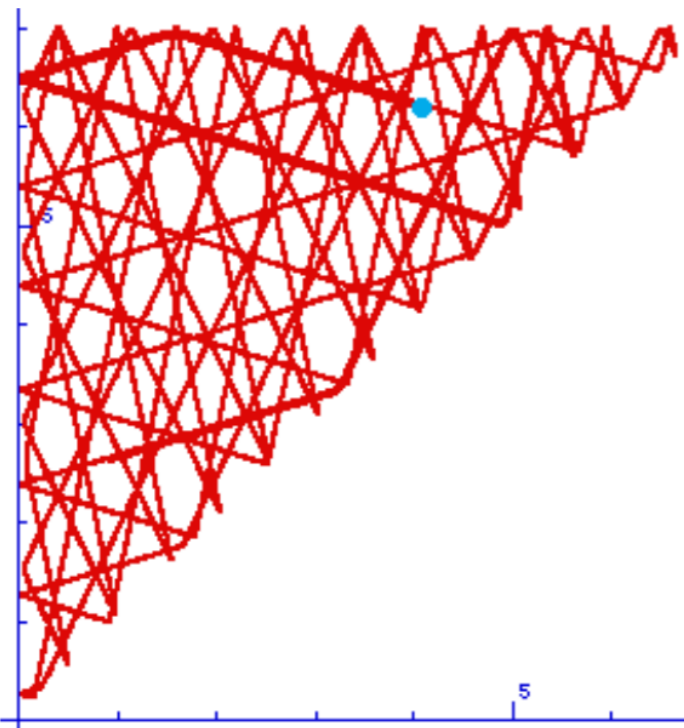
“Generic” initial velocity
 $(v_1=1.0, v_2=0.1)$

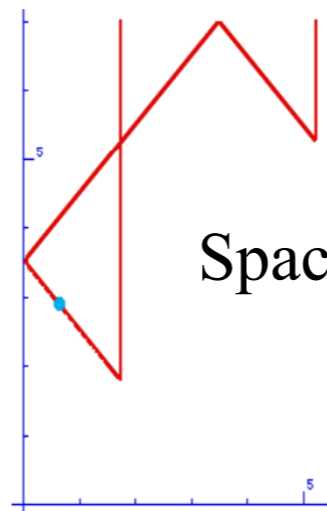
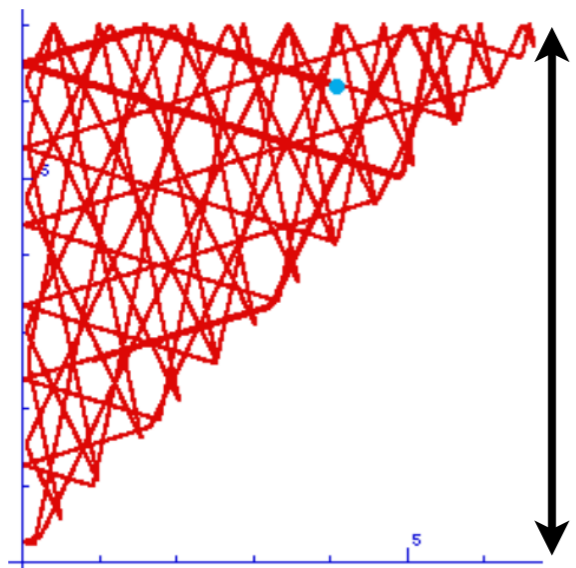


“Symmetric” initial velocity
 $(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$

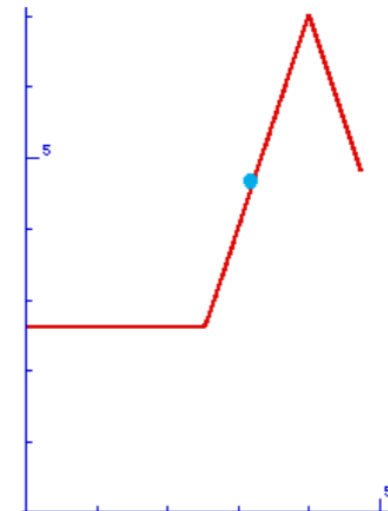


Corresponding space-space (y_1, y_2) paths



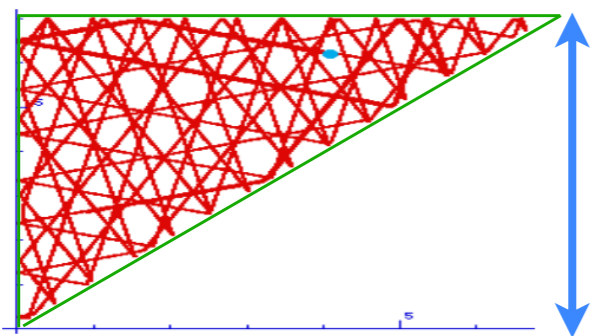


Space-space (y_1, y_2) paths

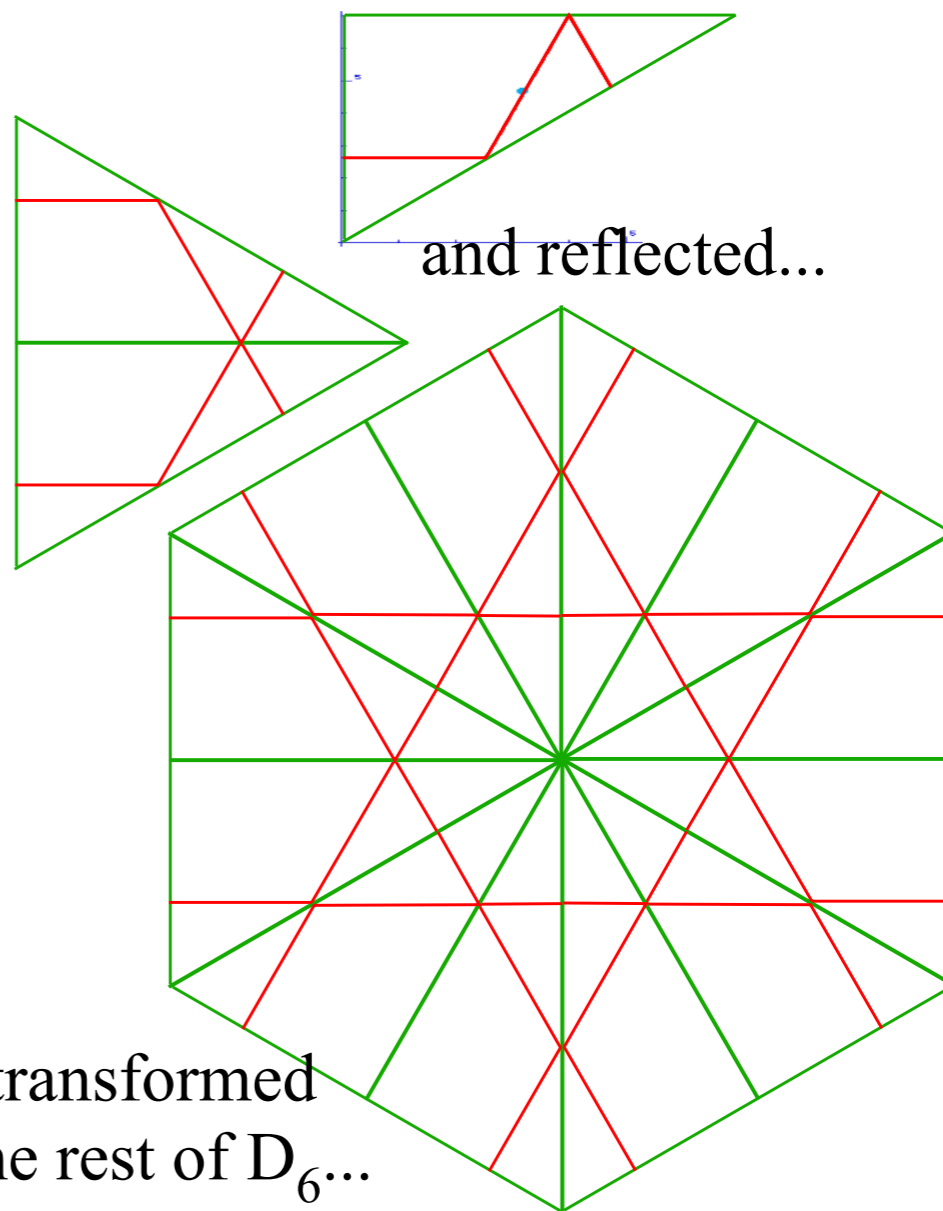
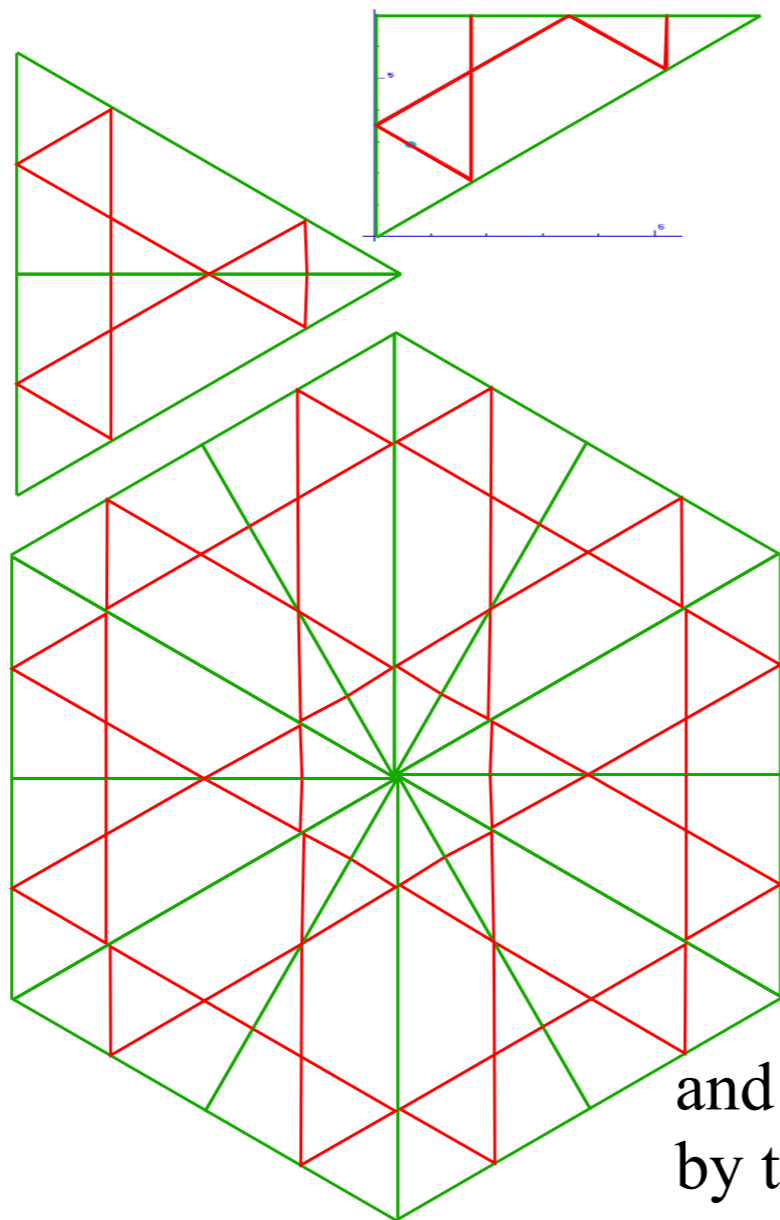


Space-space (y_1, y_2) paths scaled down by $1/\sqrt{3}$...

Scaled y down by $1/\sqrt{3}=0.577$

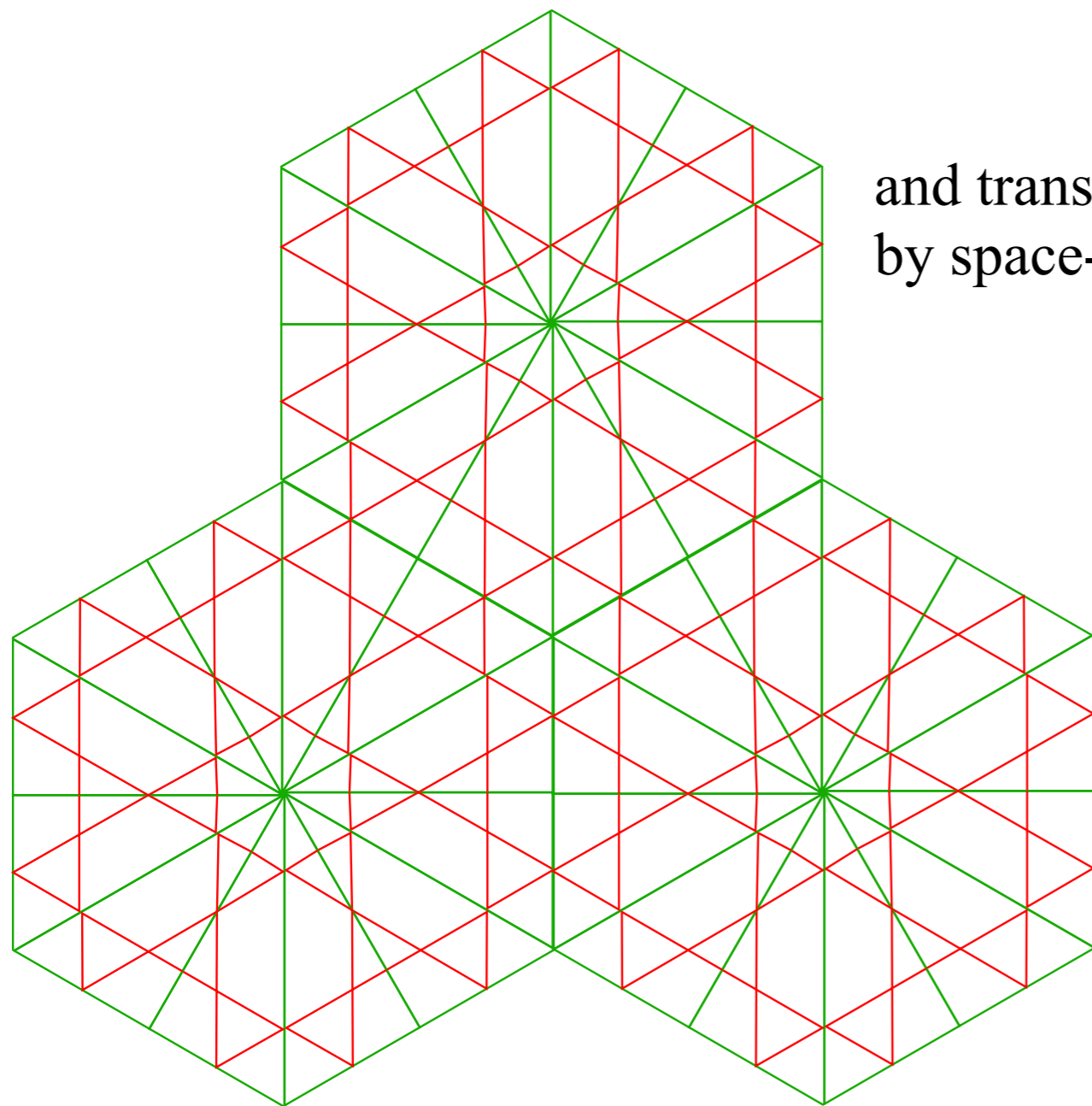


..or could have scaled x up by $\sqrt{3}=1.732$

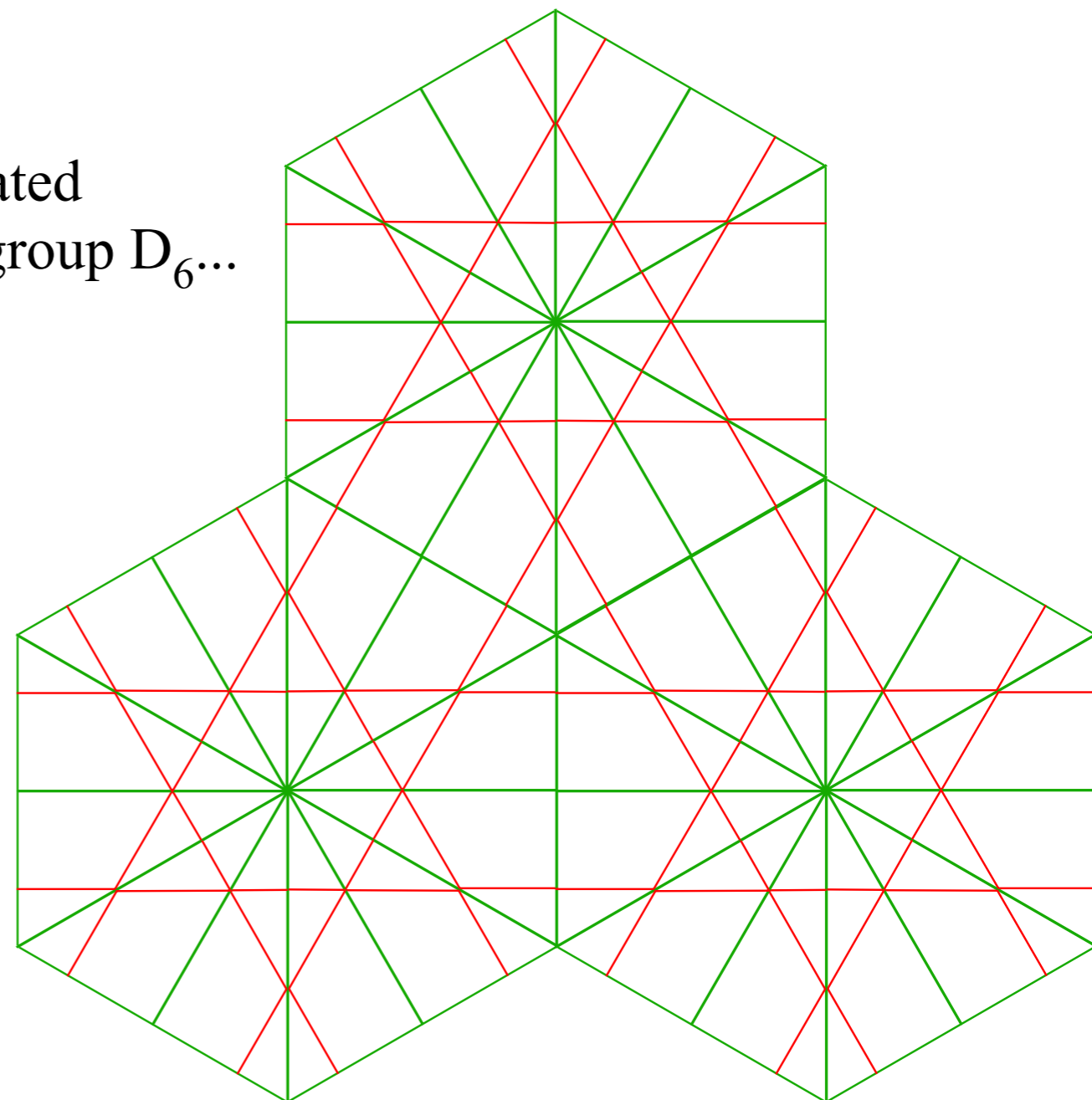


and reflected...

and transformed by the rest of D_6 ...



and translated
by space-group D_6 ...



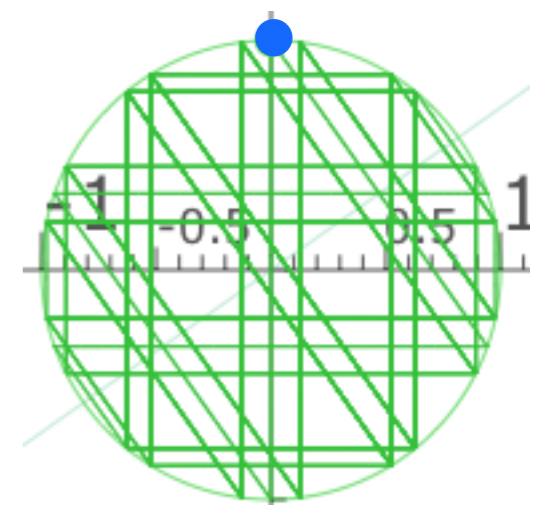
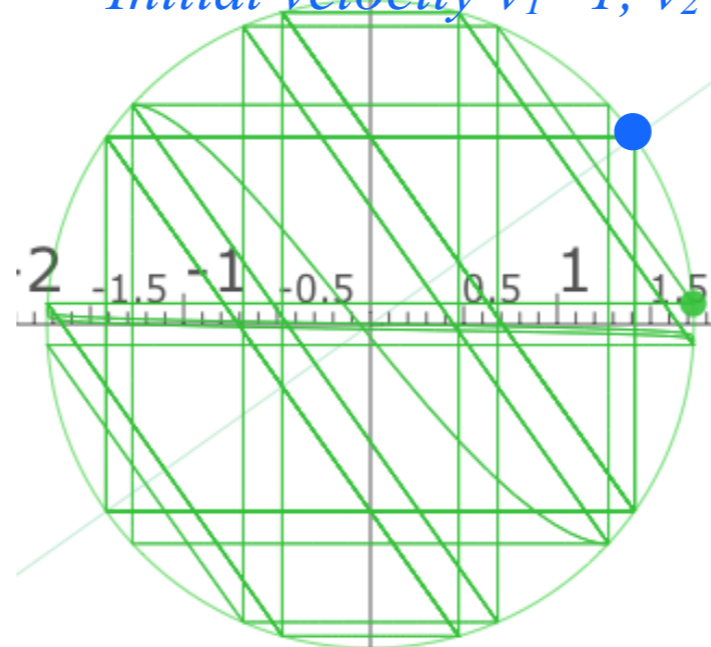
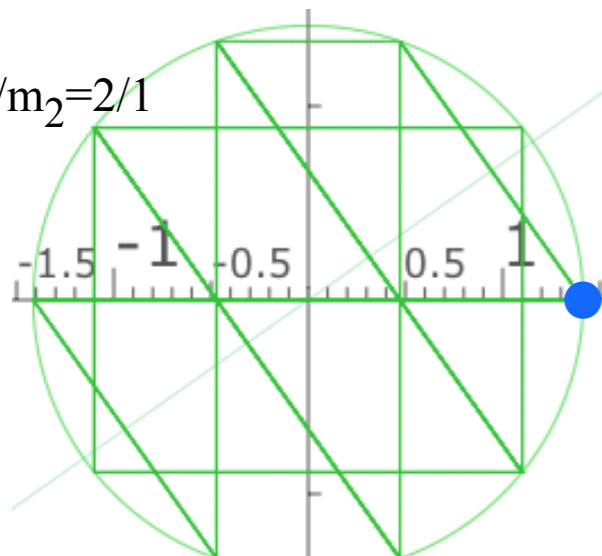
...they're just straight lines going forever.

Initial velocity $v_1=1, v_2=0$

Initial velocity $v_1=1, v_2=1$

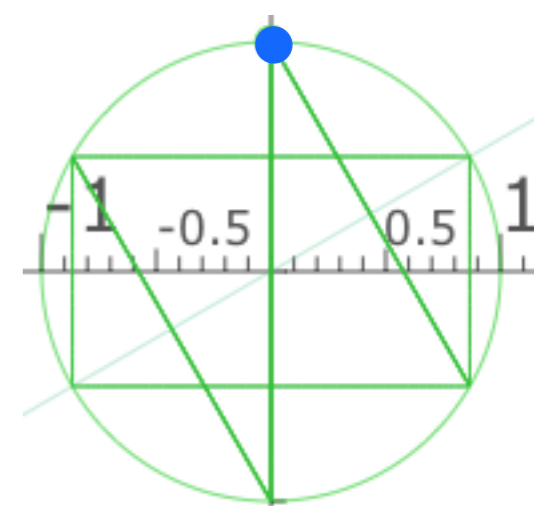
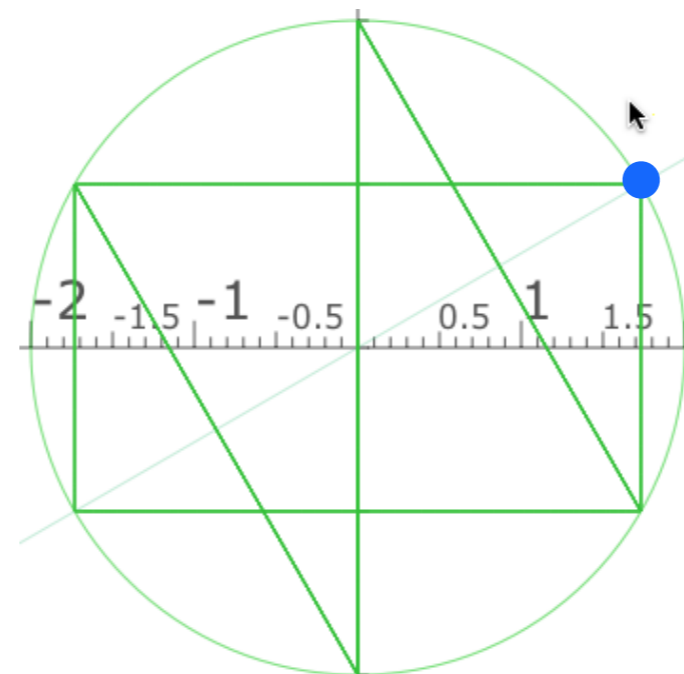
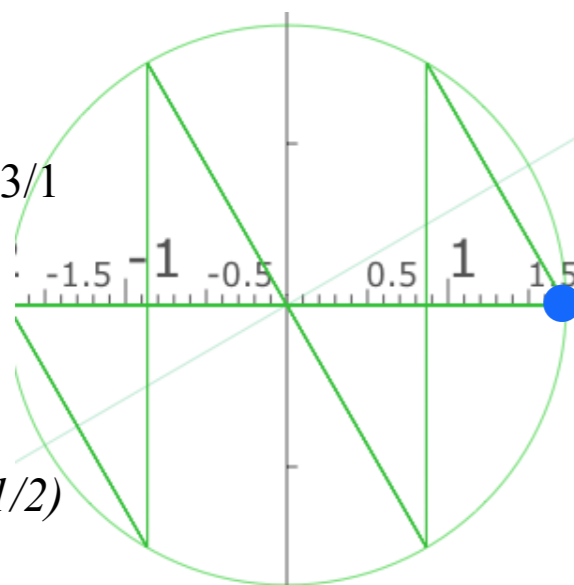
Initial velocity $v_1=0, v_2=1$

$M_1/m_2=2/1$



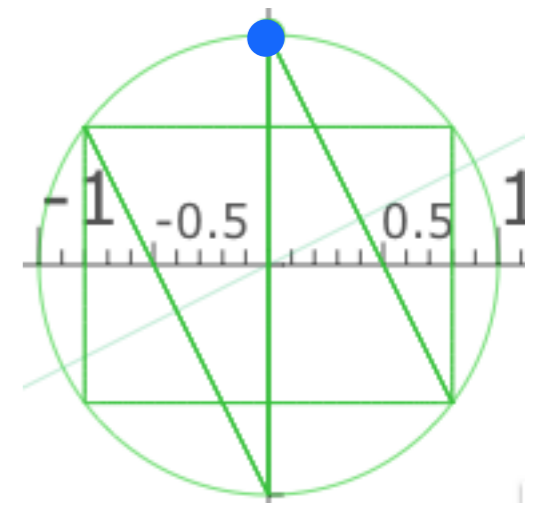
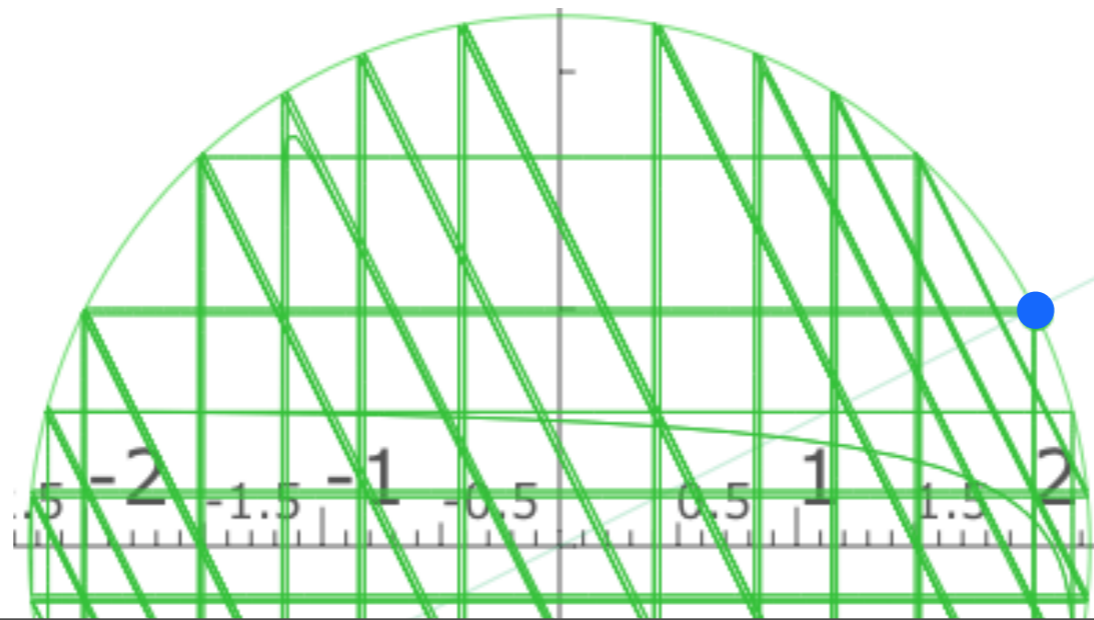
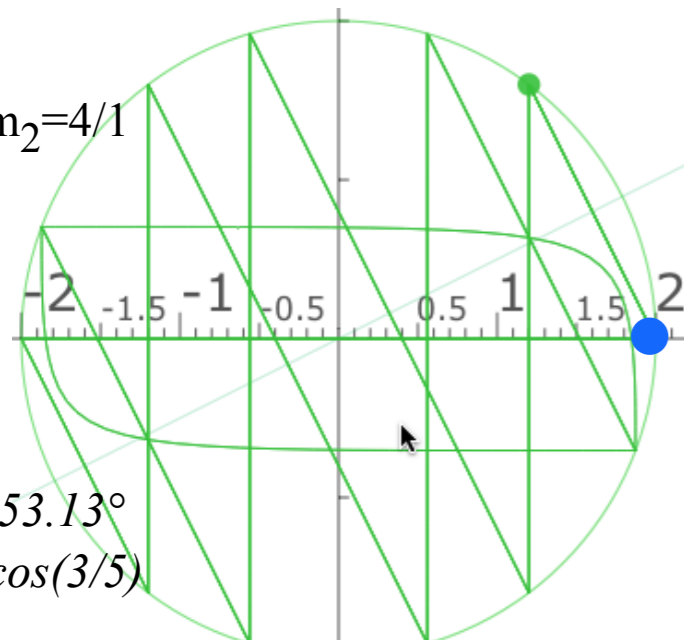
$\phi = \text{Acos}(M_1-m_2)/(M_1+m_2)$
 $= \text{Acos}(1/3) = 70.53^\circ$

$M_1/m_2=3/1$



$\phi = 60^\circ$
 $= \text{Acos}(1/2)$

$M_1/m_2=4/1$



$\phi = 53.13^\circ$
 $= \text{Acos}(3/5)$

Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE / M_1}$$

$$= \sqrt{2KE / 7}$$

$$= \sqrt{8/7}$$

$$= 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE / M_2}$$

$$= \sqrt{2KE / 1}$$

$$= \sqrt{8/1}$$

$$= 2.83$$

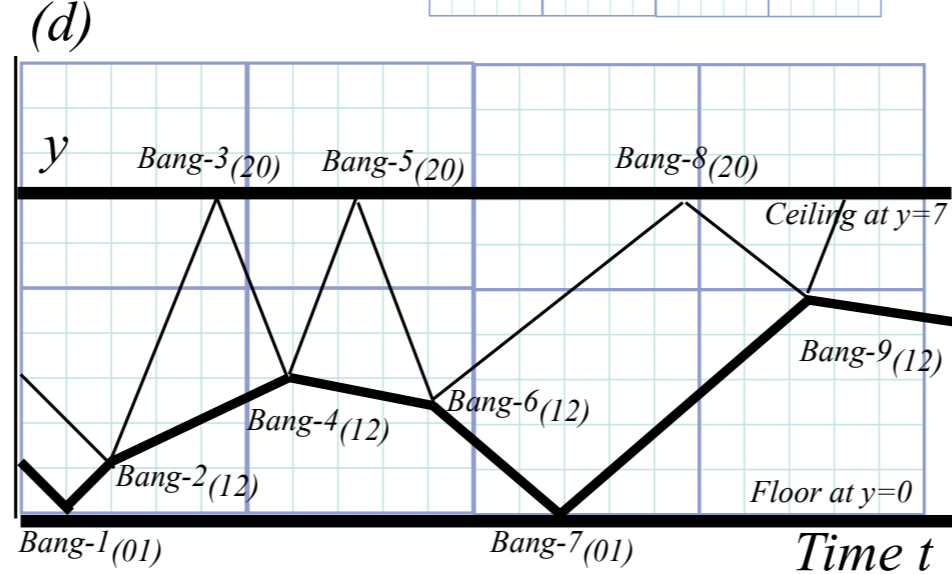
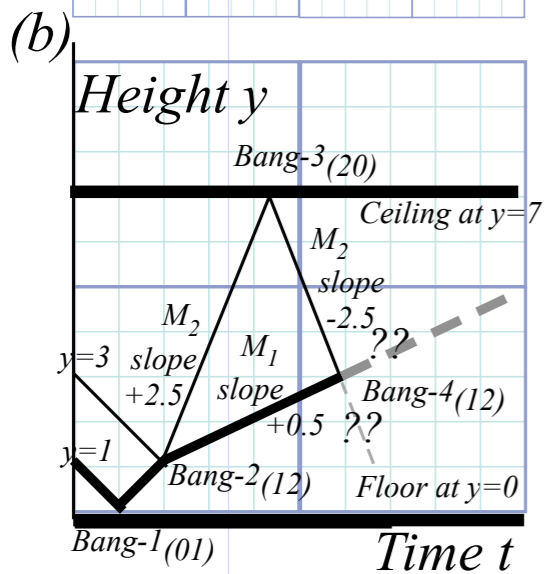
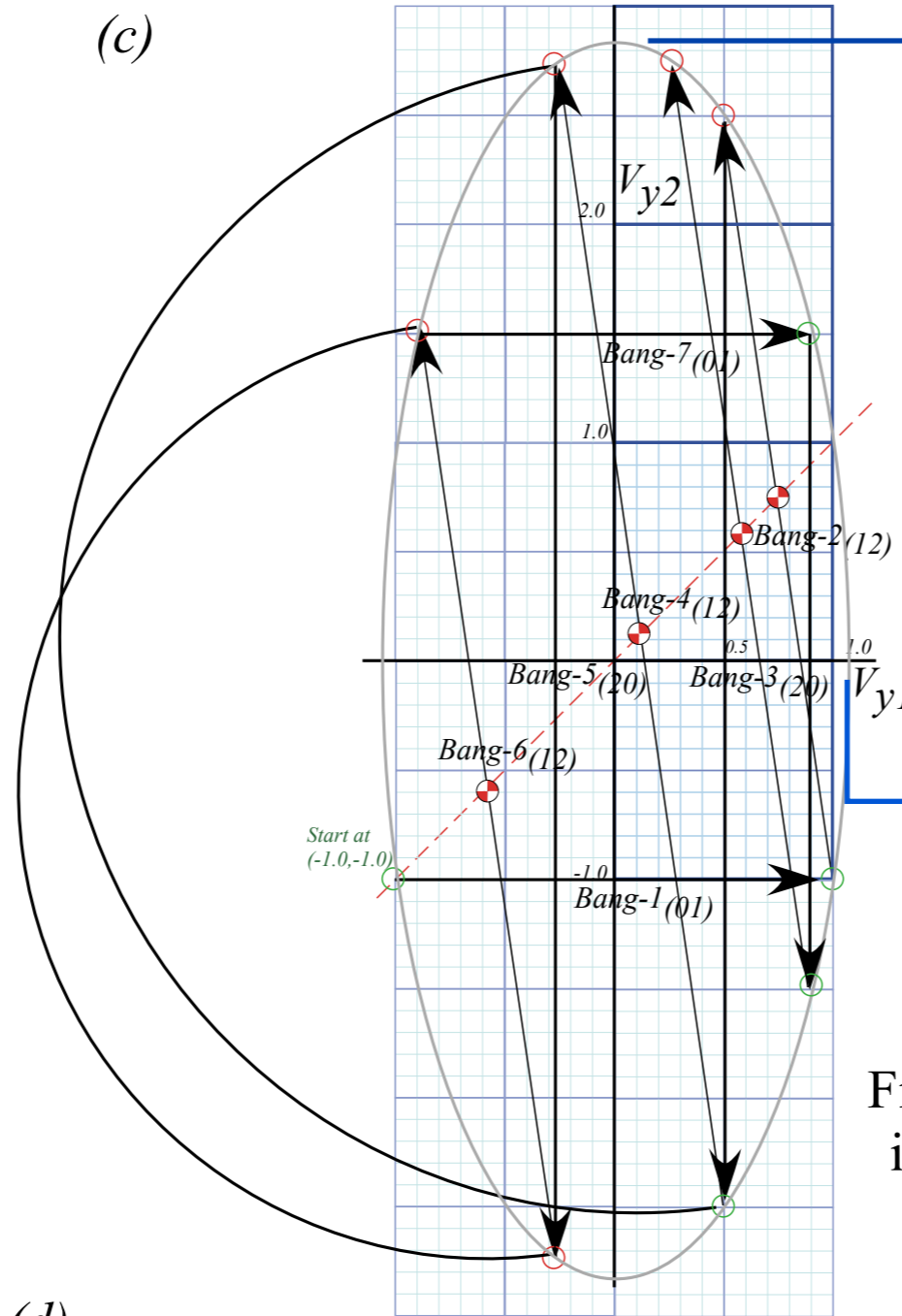
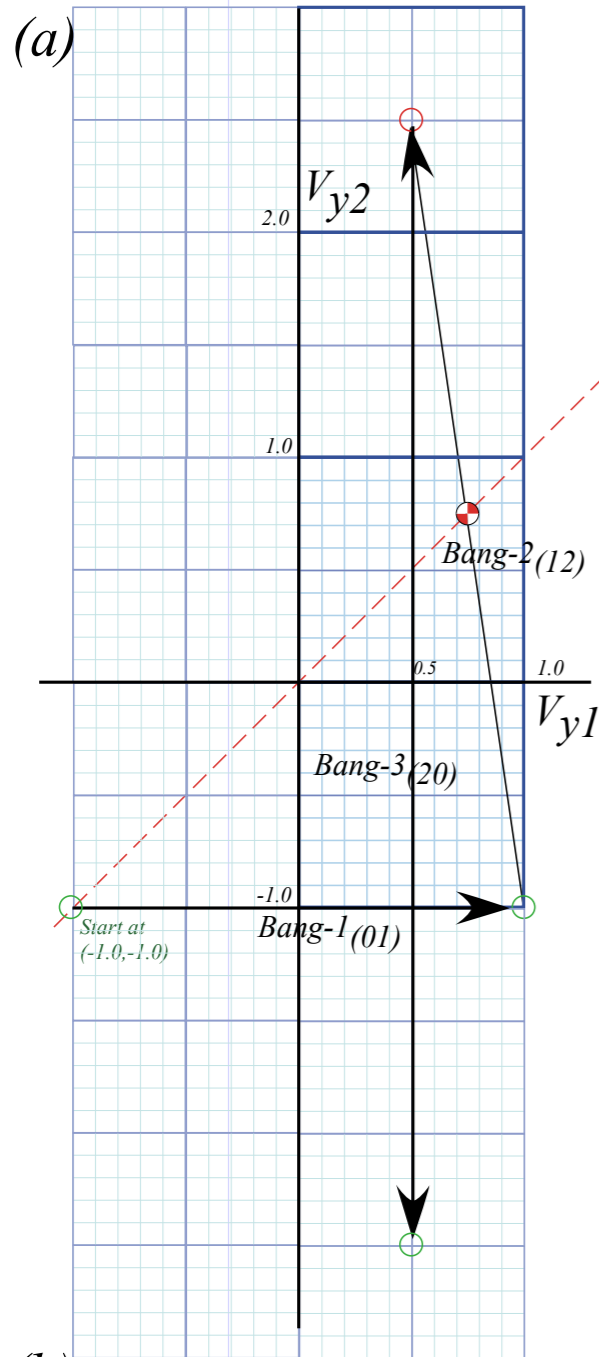


Fig. 4.7a-d
in Unit 1

Geometric "Integration" (Converting Velocity data to Spacetime)

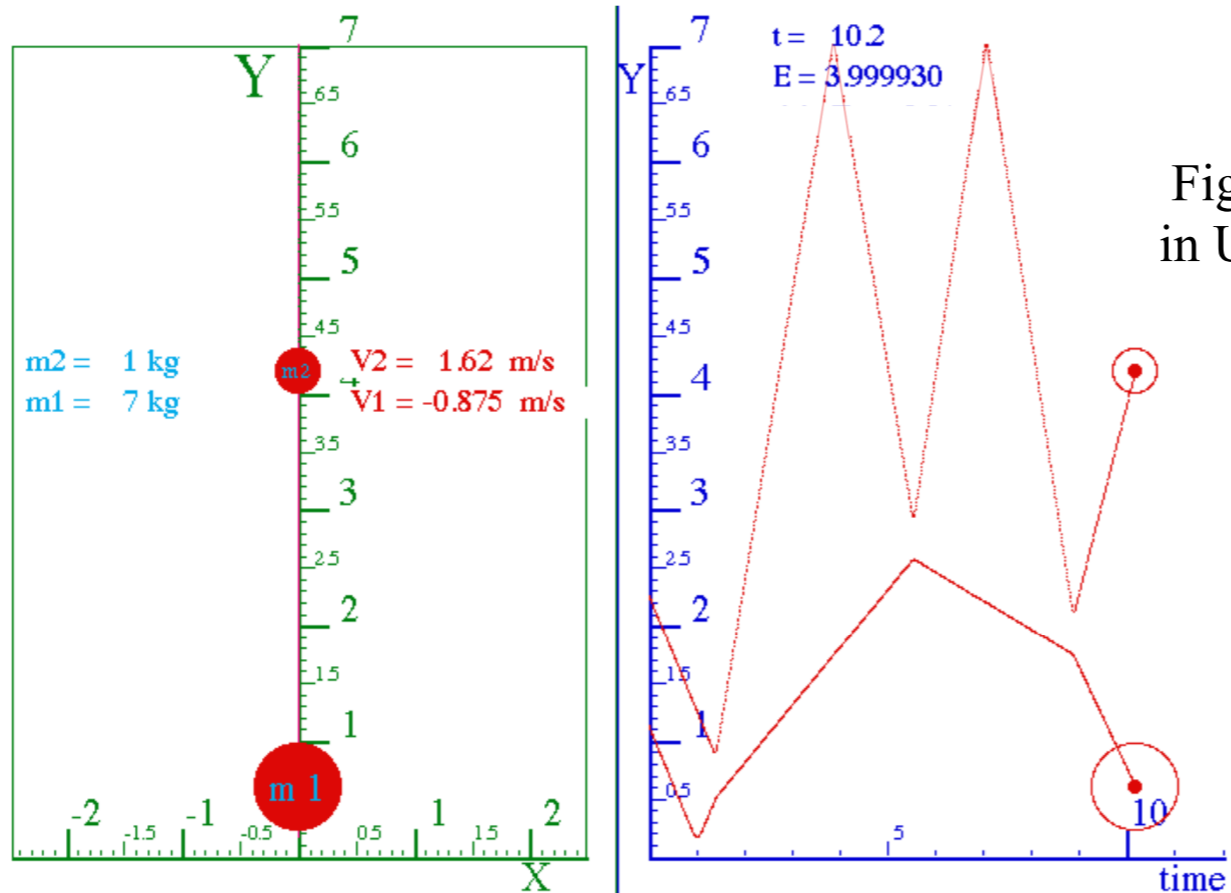


Fig. 4.8
in Unit 1

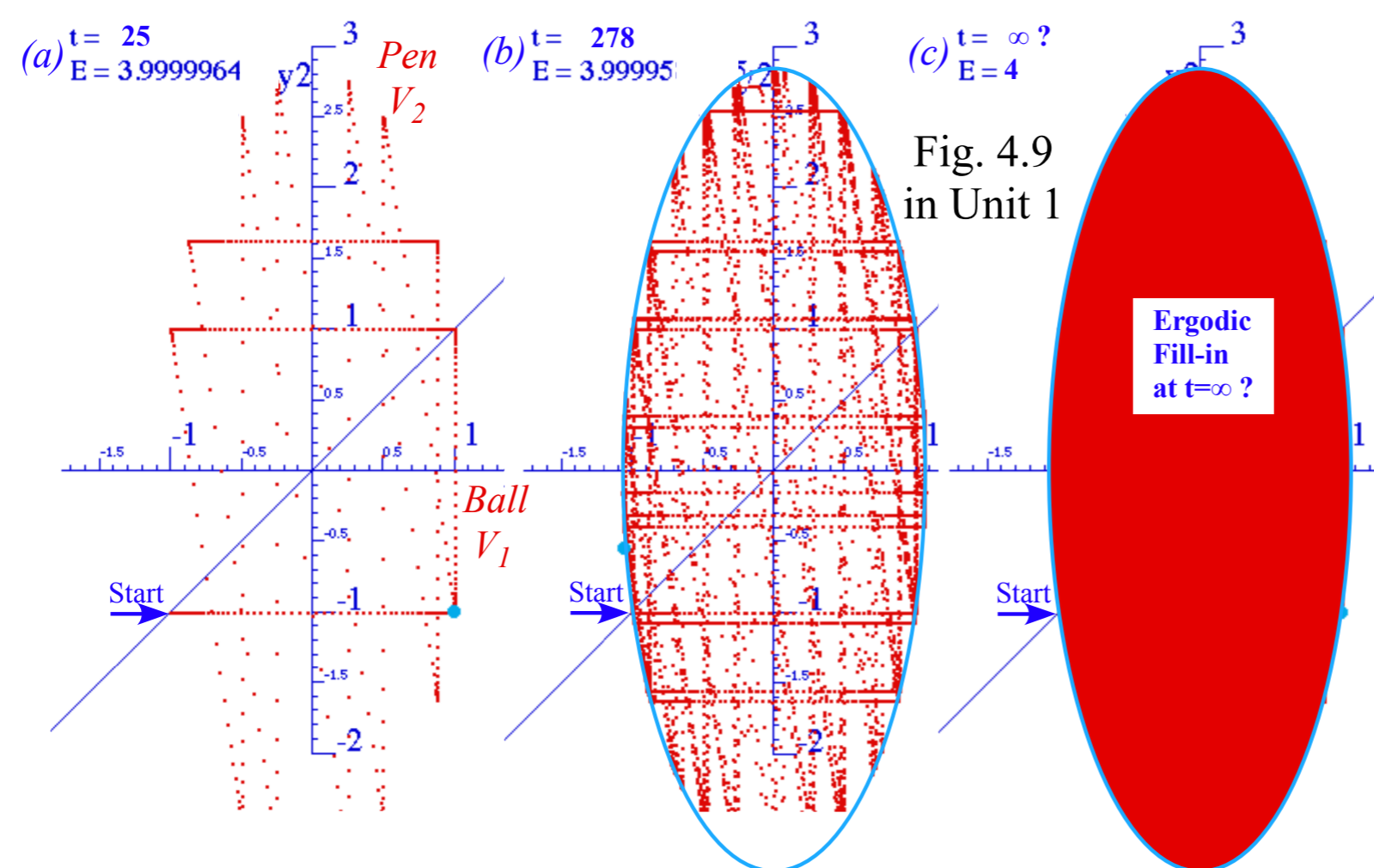


Fig. 4.9
in Unit 1

Exercise 1.4.1 and Exercise 1.4.2

Exercise 1.4.1: (a) Construct a bounce sequence plot of a mass ratio $m_1:m_2=4:1$ with the following initial values

$(x_1(0)=1.5, x_2(0)=3.0, v_1(0)=-1, v_2(0)=-1)$ and ceiling height $y_{max}=7.0$. This $4:1$ case is quasi-periodic. The collision sequence in the (v_1, v_2) plot path appears to repeat several steps then jumps to make new paths. Does the (x_1, x_2) plot also repeat those steps? Draw both plots for at least 16 collisions to analyze the sequences.

(b) Show that, with initial values $(x_1(0)=1.5, x_2(0)=3.0, v_1(0)=1, v_2(0)=0)$, the collision sequence is periodic after 12 steps in both the (v_1, v_2) plot and the (x_1, x_2) plot.

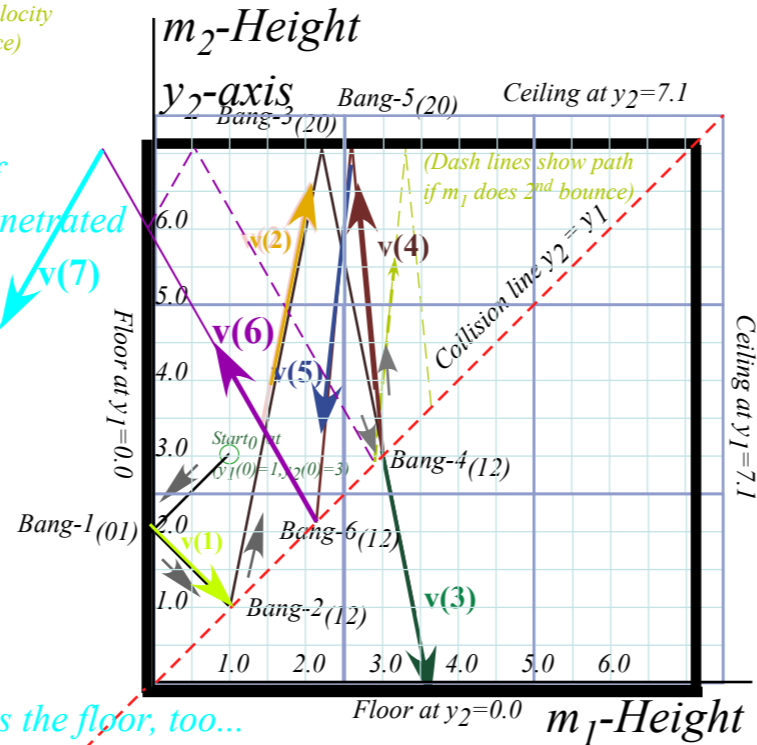
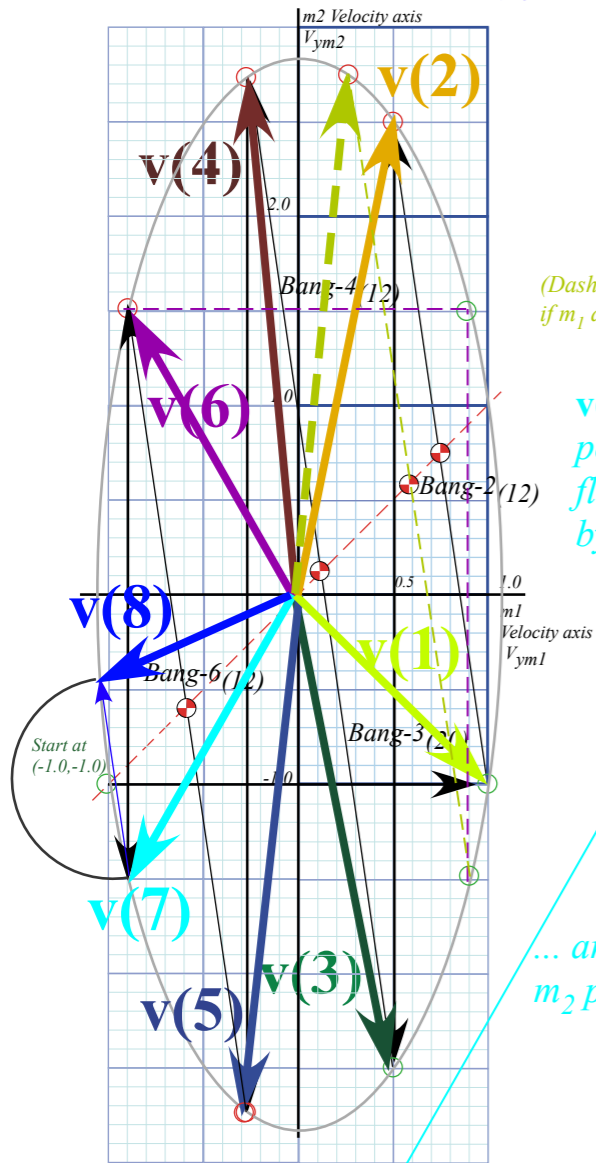
Exercise 1.4.2: Continue the (v_1, v_2) and (x_1, x_2) collision plots begun in class and shown in Fig. 4.7 and Fig. 4.11.

Continue until you reach the “gameover” point of last possible M_1 - M_2 collision assuming the floor is open after *Bang-1* so both masses can fall thru indefinitely. Show where is this last last collision.

Exercise 1.4.2 solutions from Assignment 2 are given first followed by detailed solutions of Exercise 1.4.1 from Assignment 2.

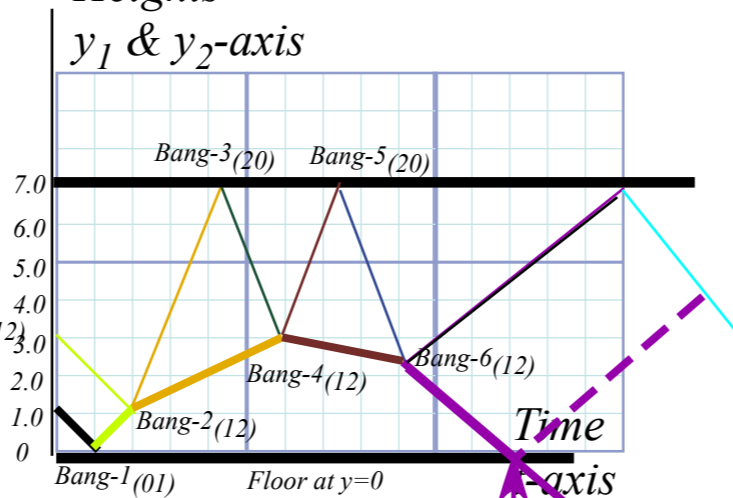
Solutions to Exercise 1.4.2 (Fig. 1.4.12 completion)

- Step-2: Extend $v(2)$ line to ceiling point $y(3)=(?, 7.1)$ and draw Bang-3(20) velocity $v(3)=(1, -1)$ line. (Find $v(3)$ using V-V plot.)
- Step-3: Extend $v(3)$ line to collision point $y(4)=(?, ?)$ and draw Bang-4(12) velocity $v(4)=(0.5, 2.5)$. (Find $v(4)$ using V-V plot.)
- Step-4: Extend $v(4)$ line to ceiling point $y(4)=(?, 7.1)$ and draw Bang-5(20) velocity $v(5)=(1, -1)$ line. (Find $v(5)$ using V-V plot.)
- Step-5: Extend $v(5)$ line to collision point $y(6)=(?, ?)$ and draw Bang-6(12) velocity $v(6)=(0.5, 2.5)$. (Find $v(6)$ using V-V plot.)

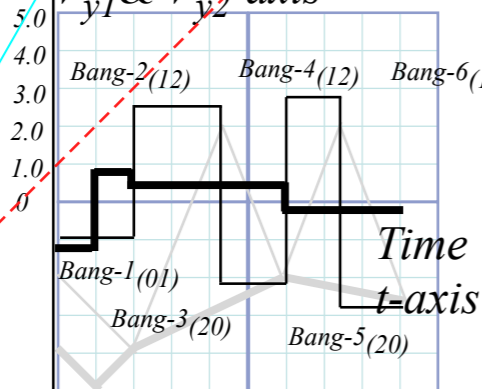


... and later m_2 penetrates the floor, too...

Heights y_1 & y_2 -axis



Velocities V_{y1} & V_{y2} -axis

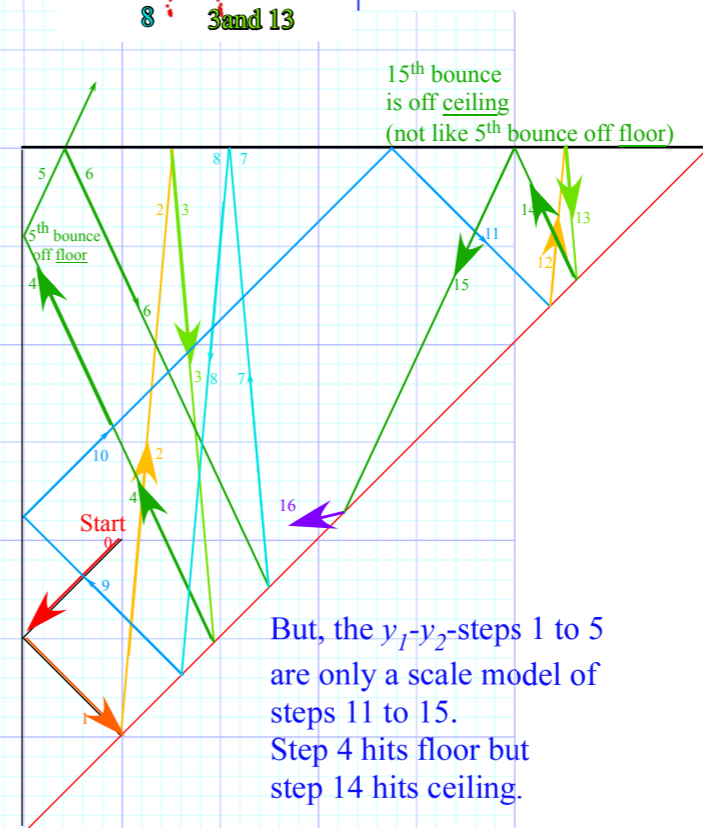
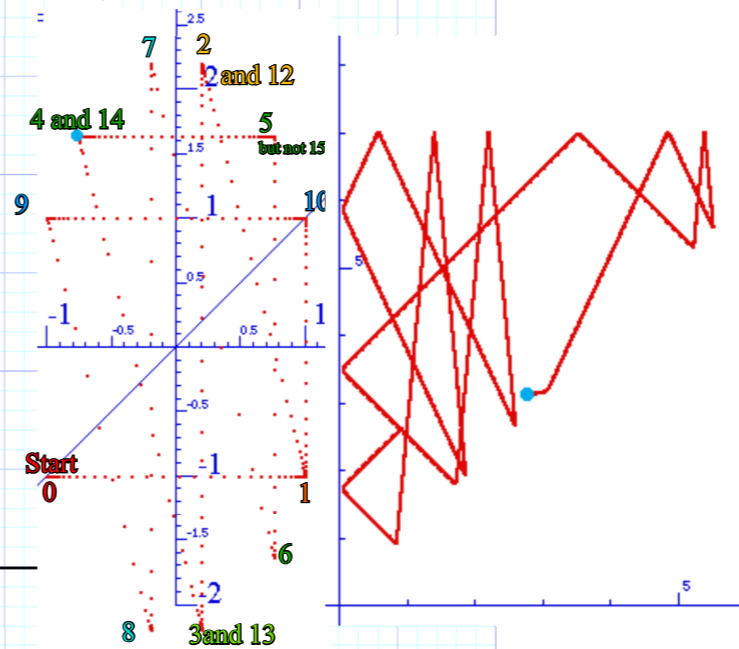
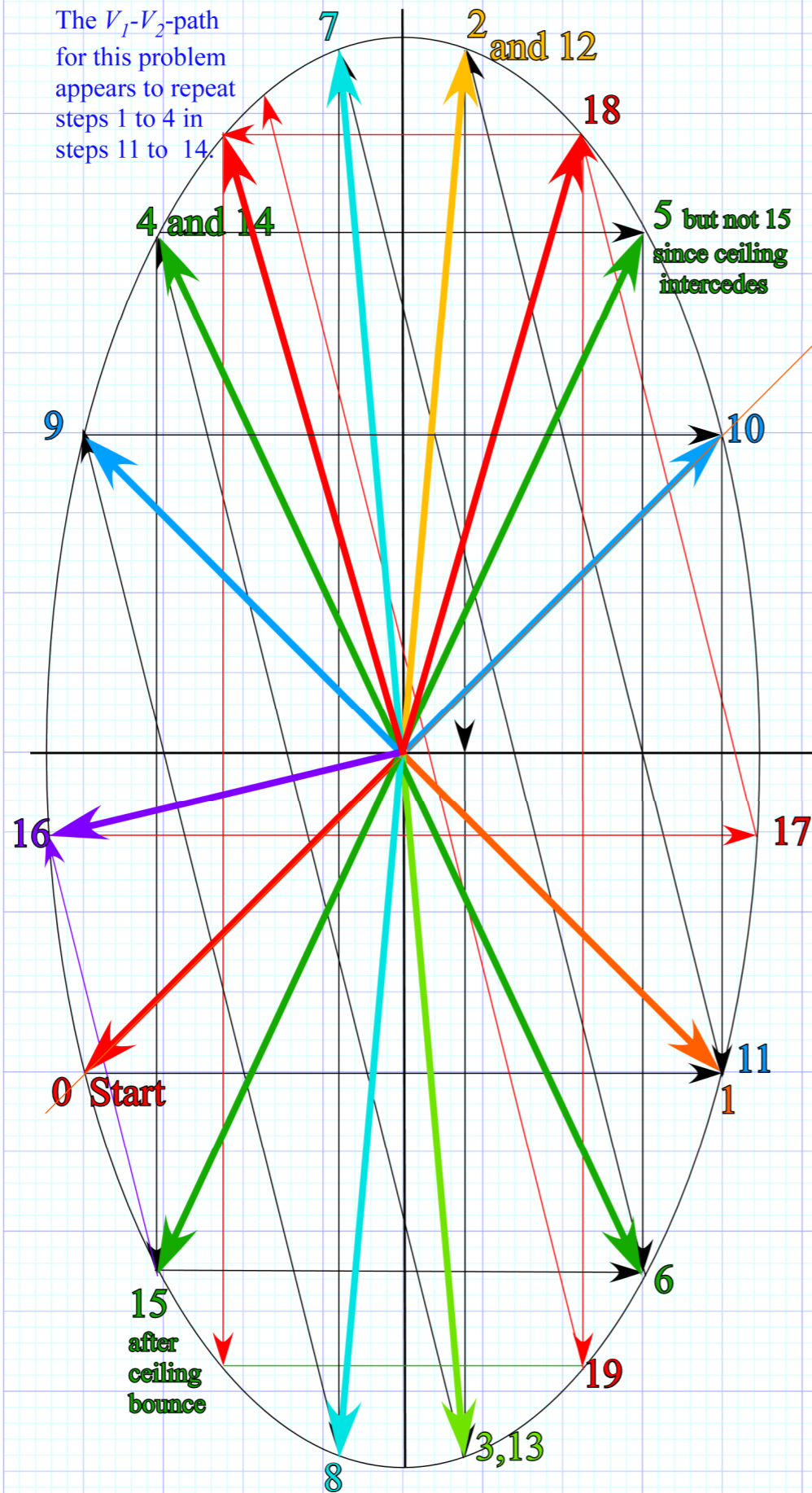


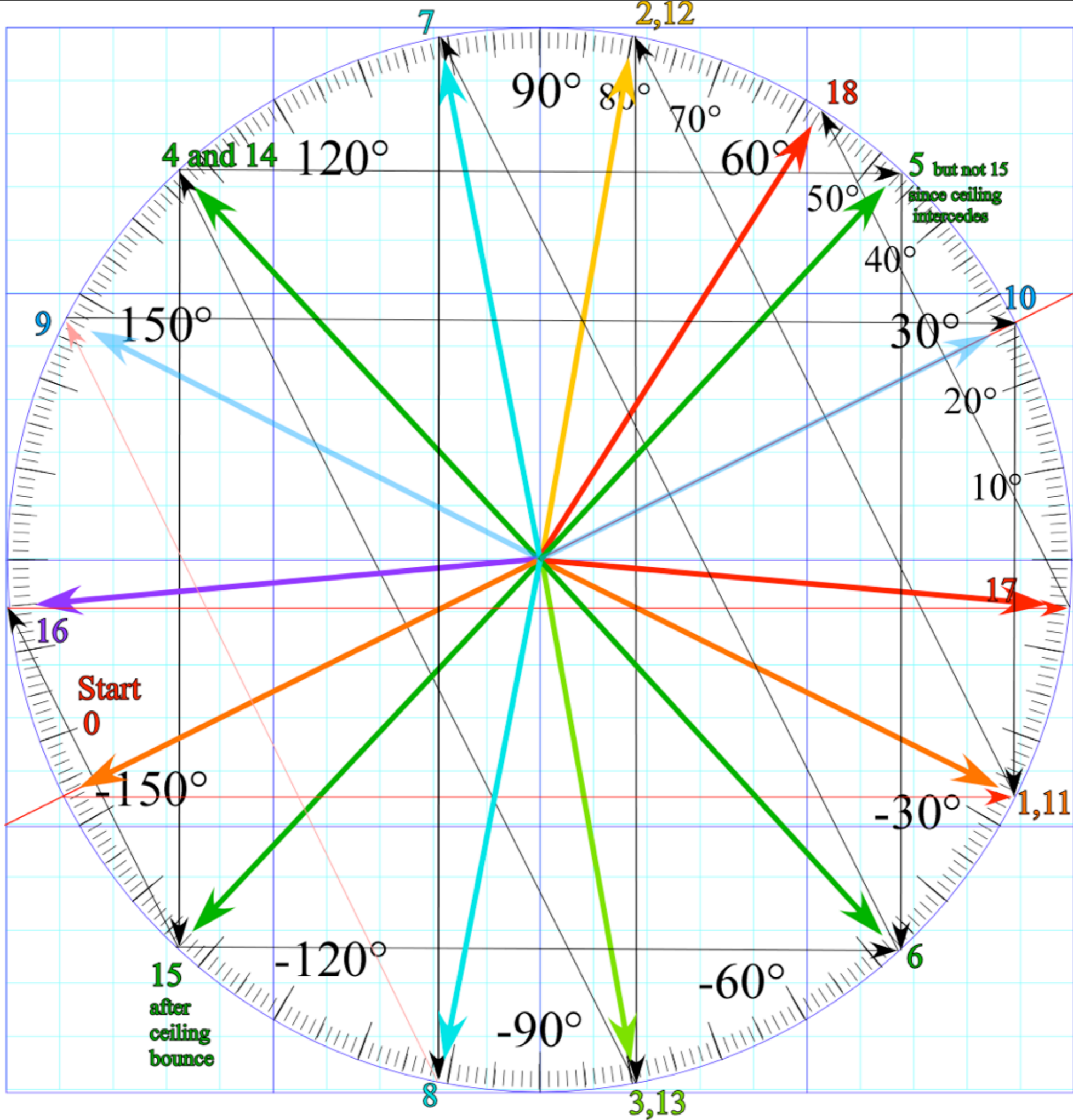
“GameOver collision” occurs way down here!

$v(8)$

First part of Exercise 1.4.1 has pen-ball initial values $v_1(0) = -1 = v_2(0)$

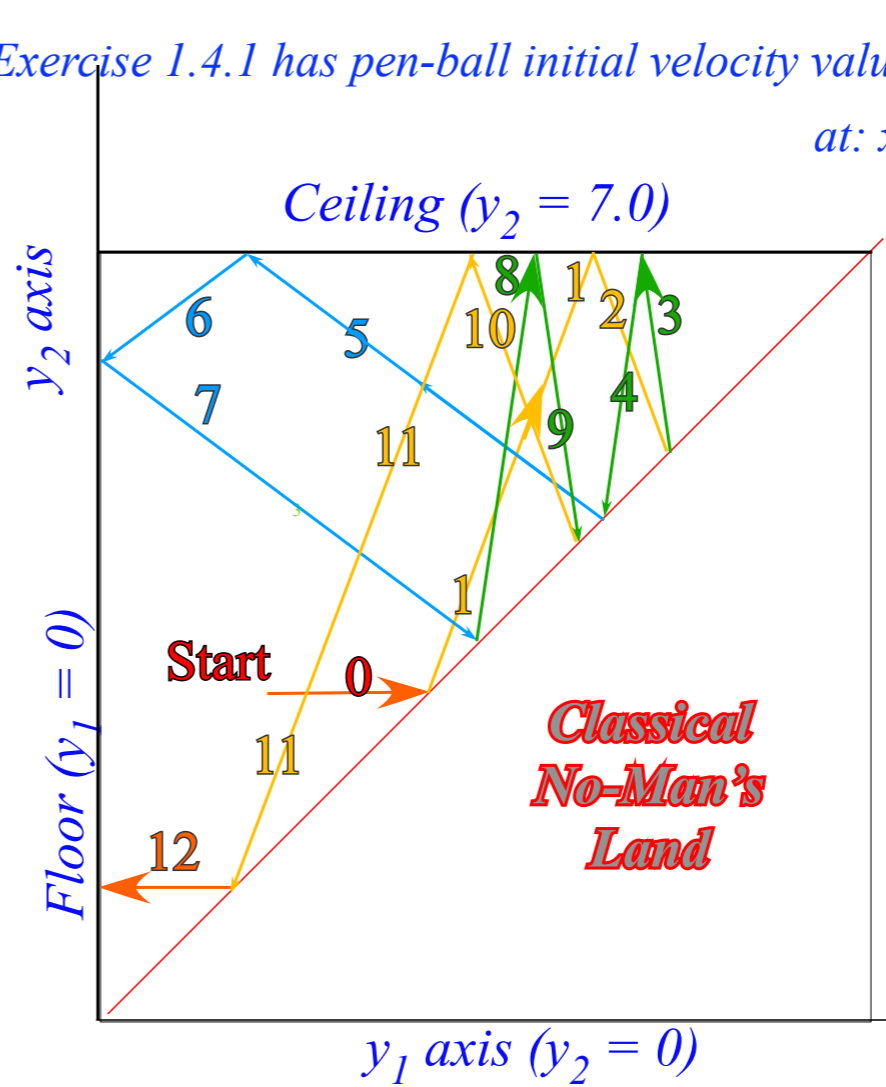
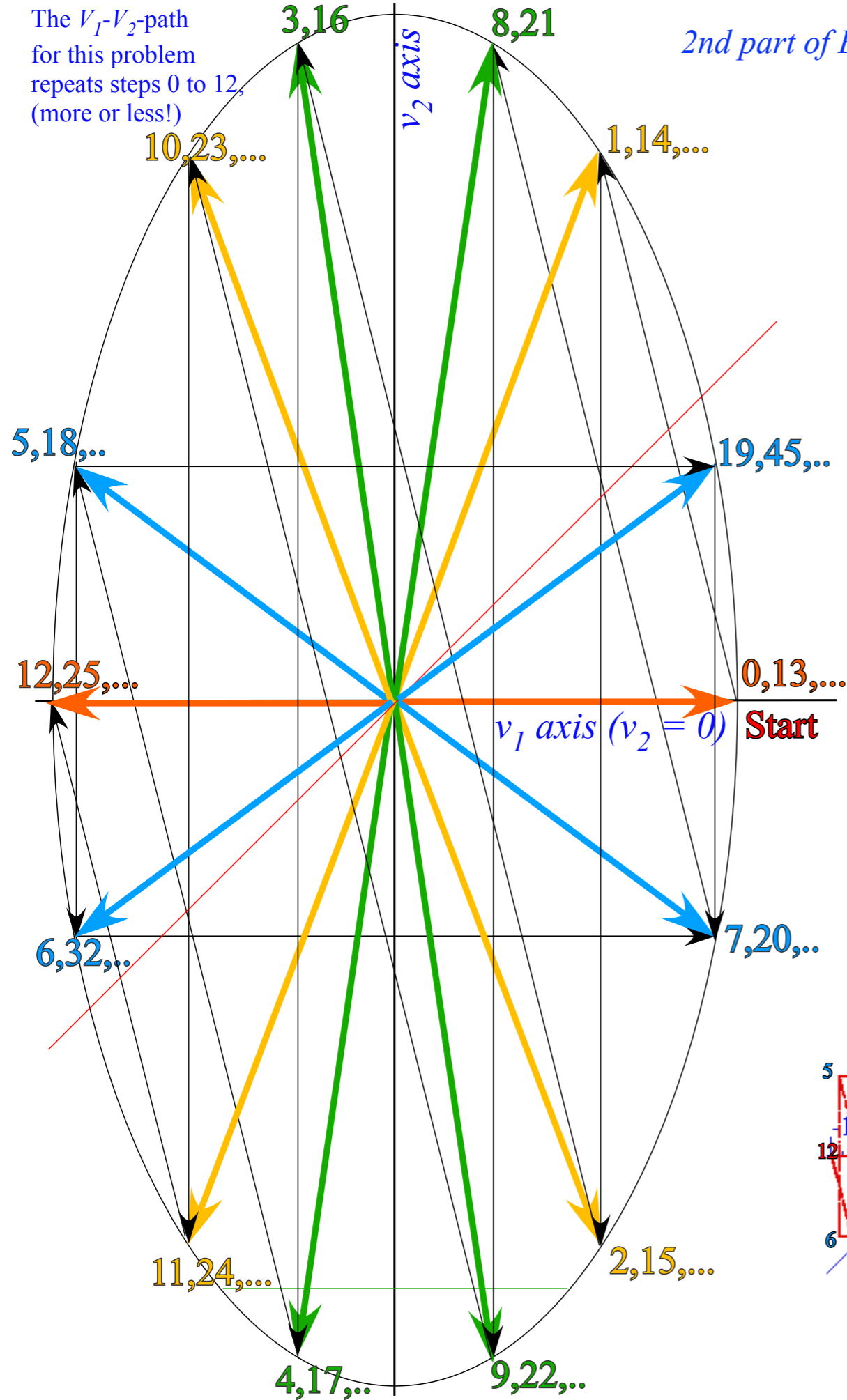
The V_1 - V_2 -path for this problem appears to repeat steps 1 to 4 in steps 11 to 14.



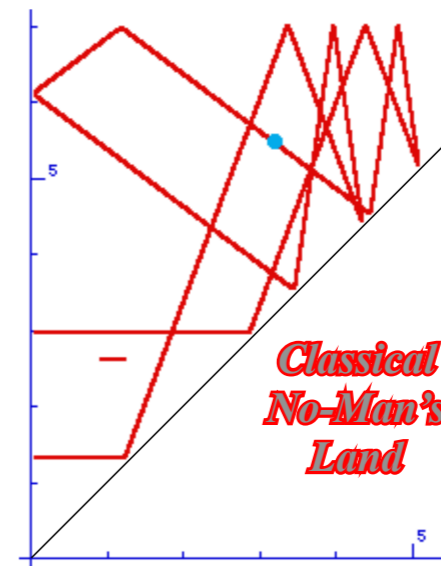
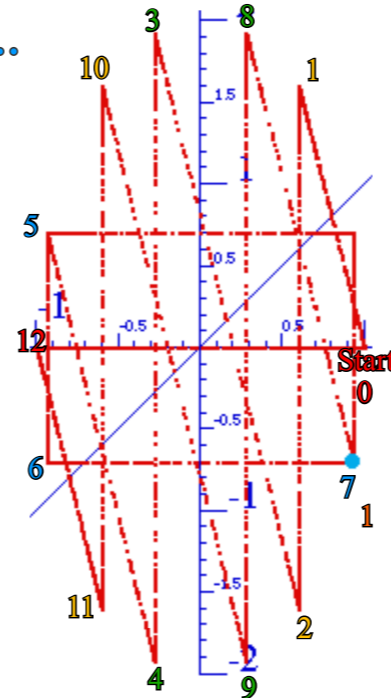


The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)

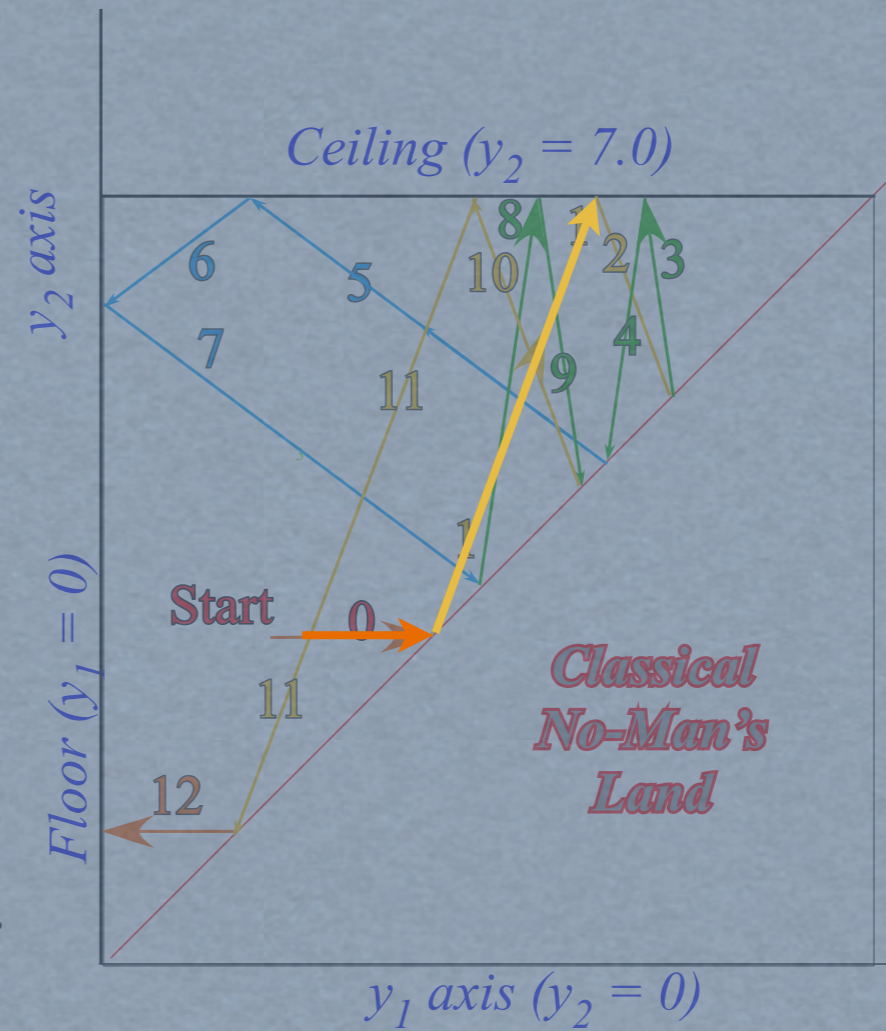
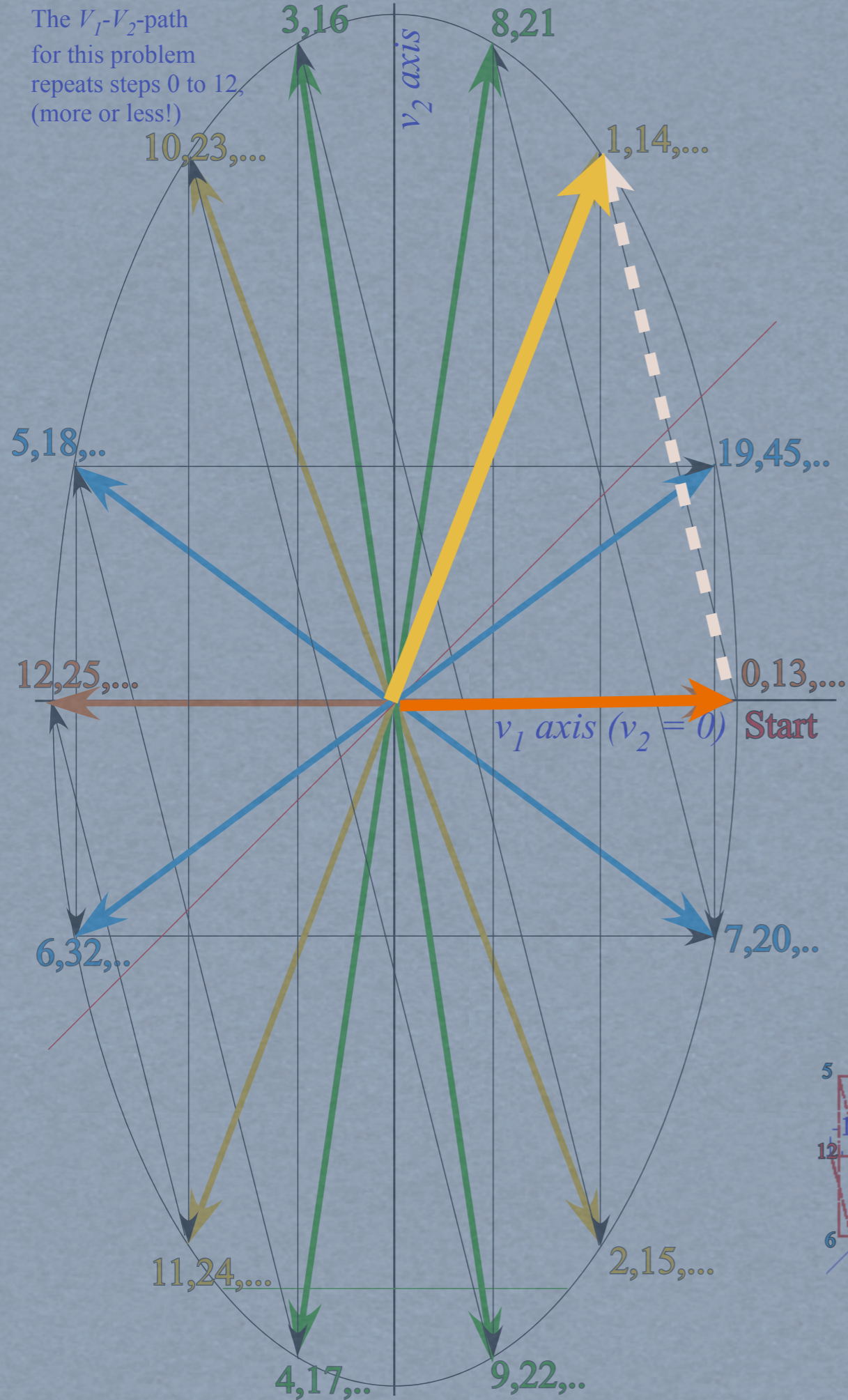
2nd part of Exercise 1.4.1 has pen-ball initial velocity values $v_1(0)=1$ and $v_2(0)=0$ at: $x_1(0)=1.5$ and $x_2(0)=3.0$



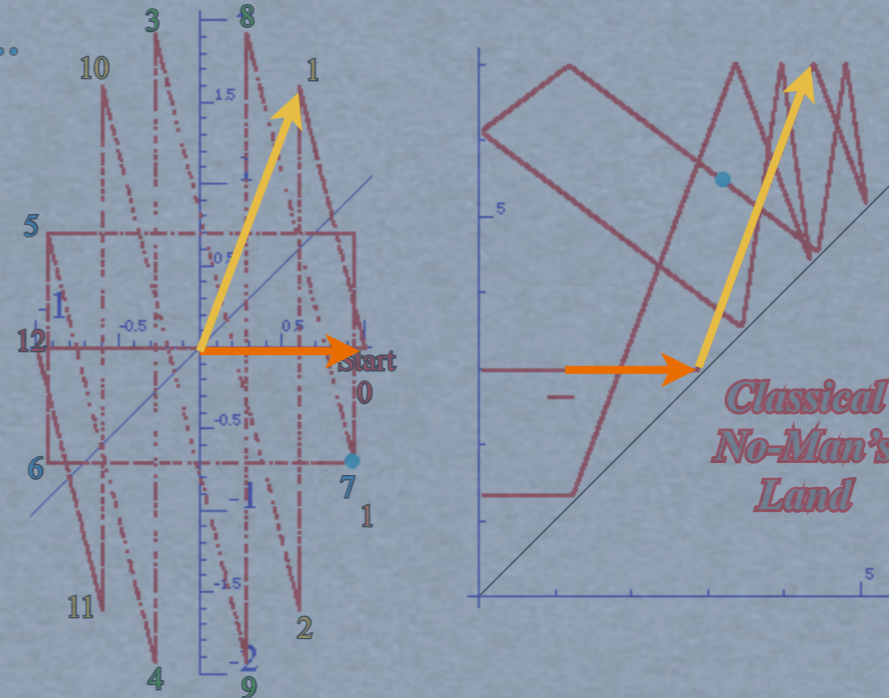
Simulations by BounceIt



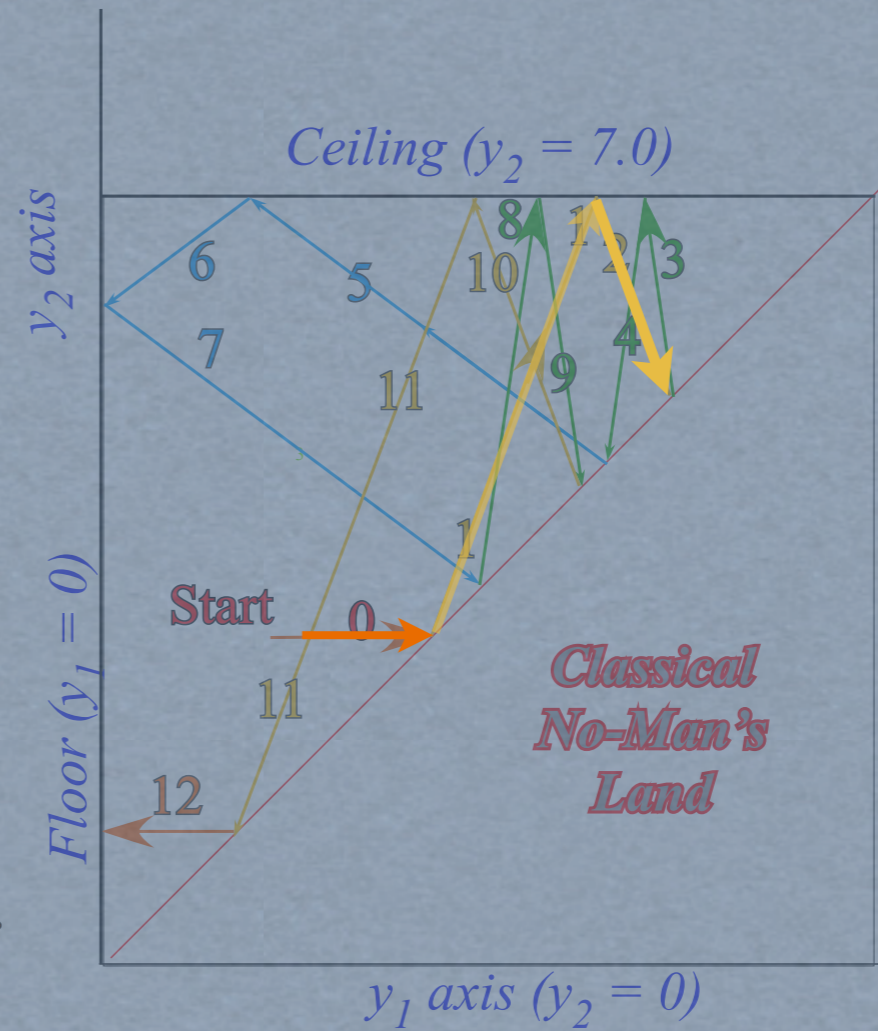
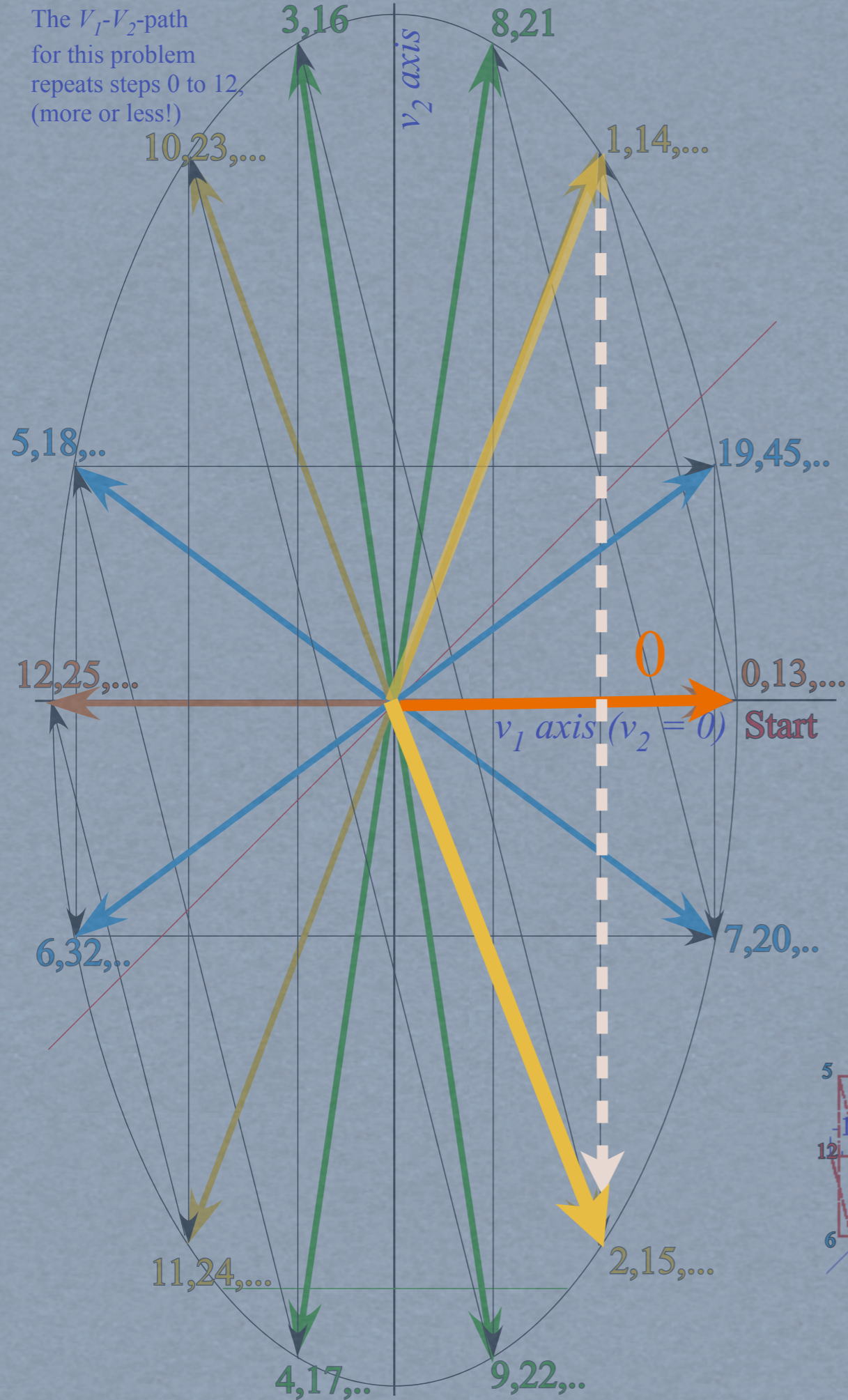
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



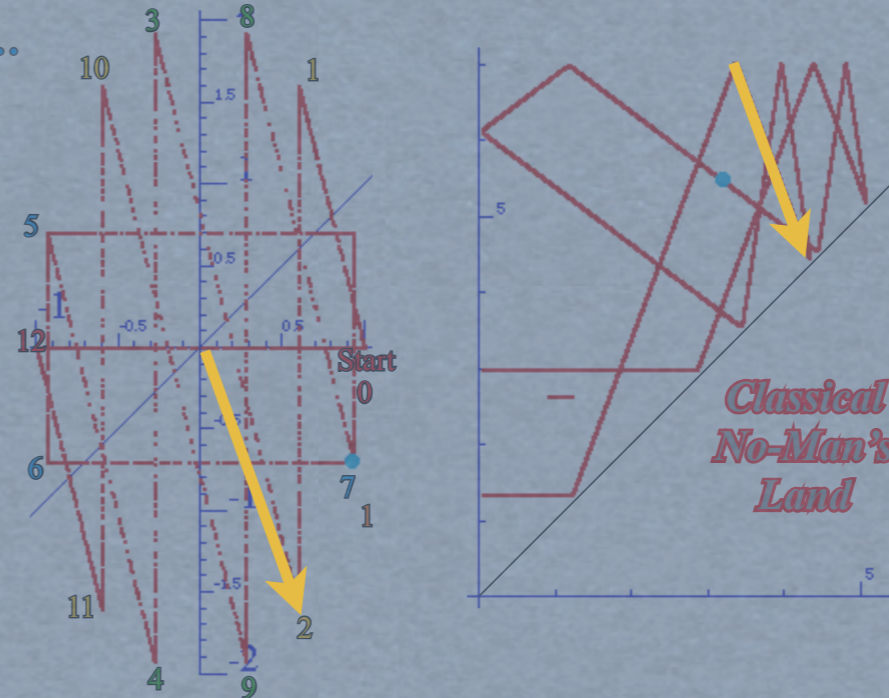
Simulations by BounceIt



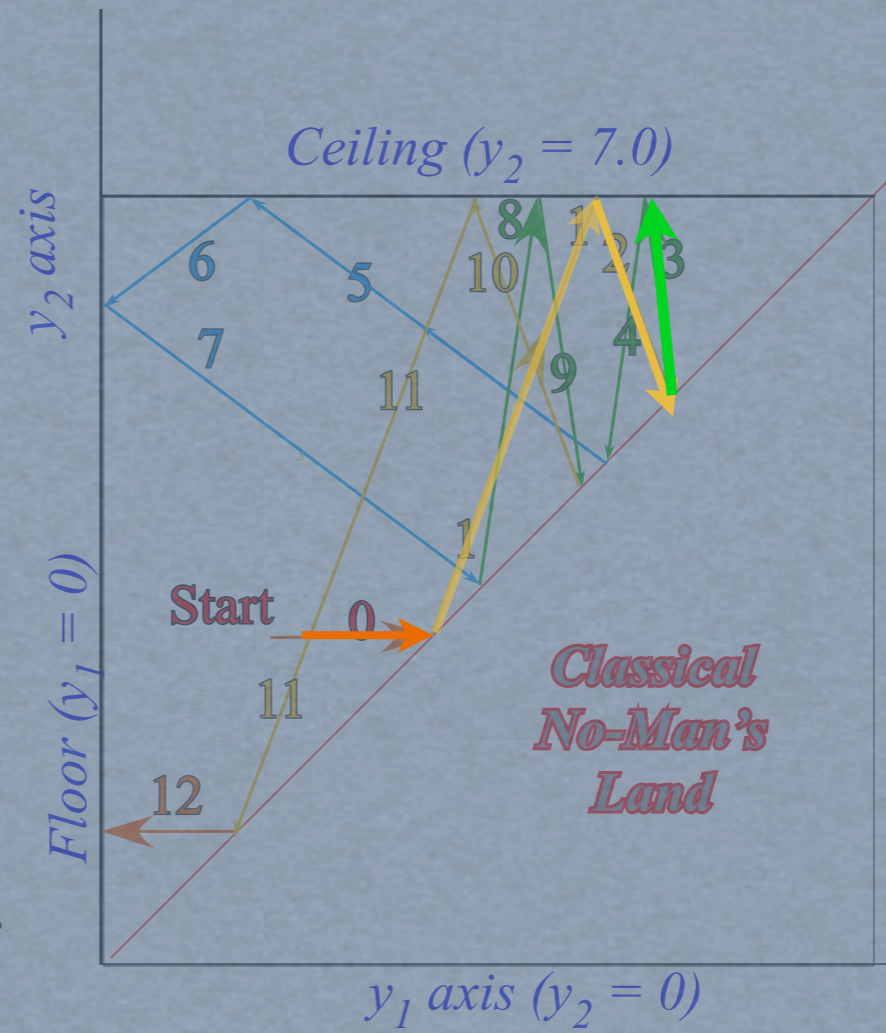
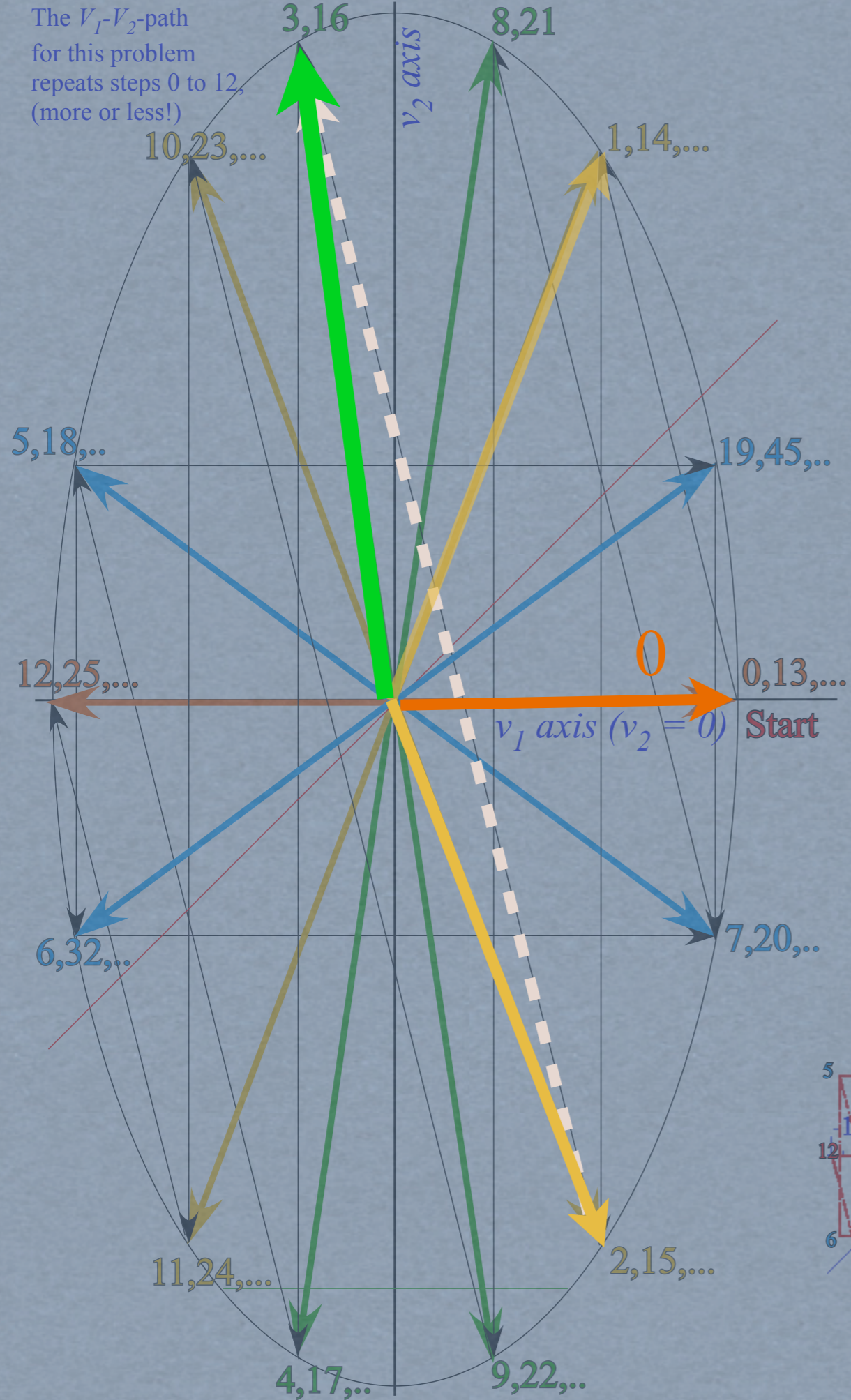
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



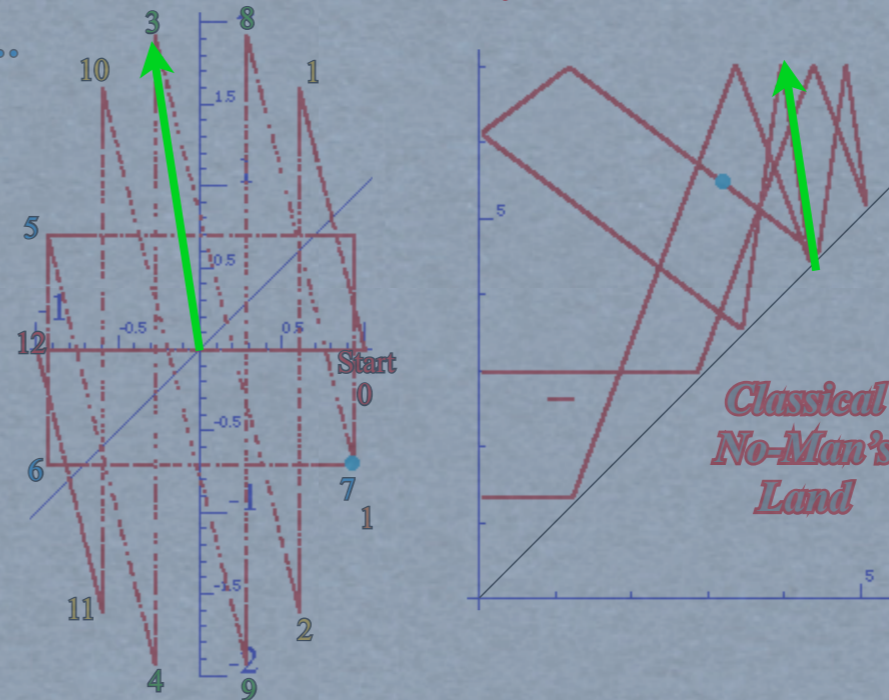
Simulations by *BounceIt*



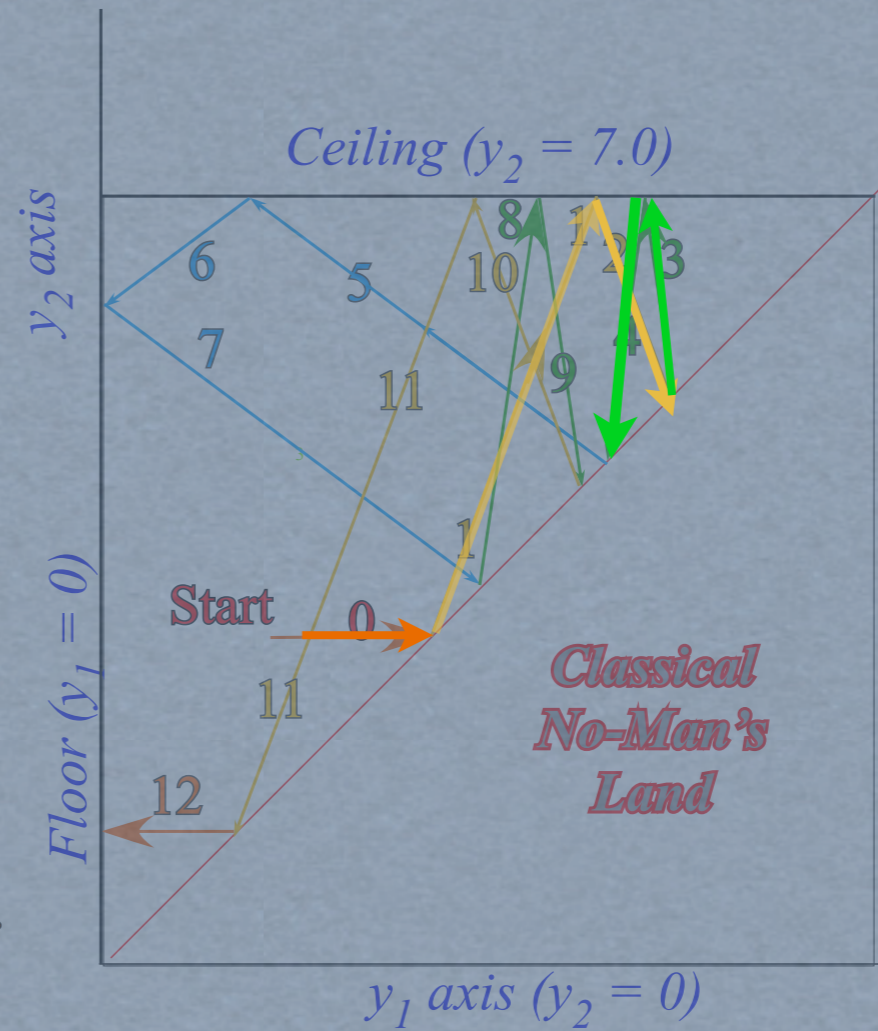
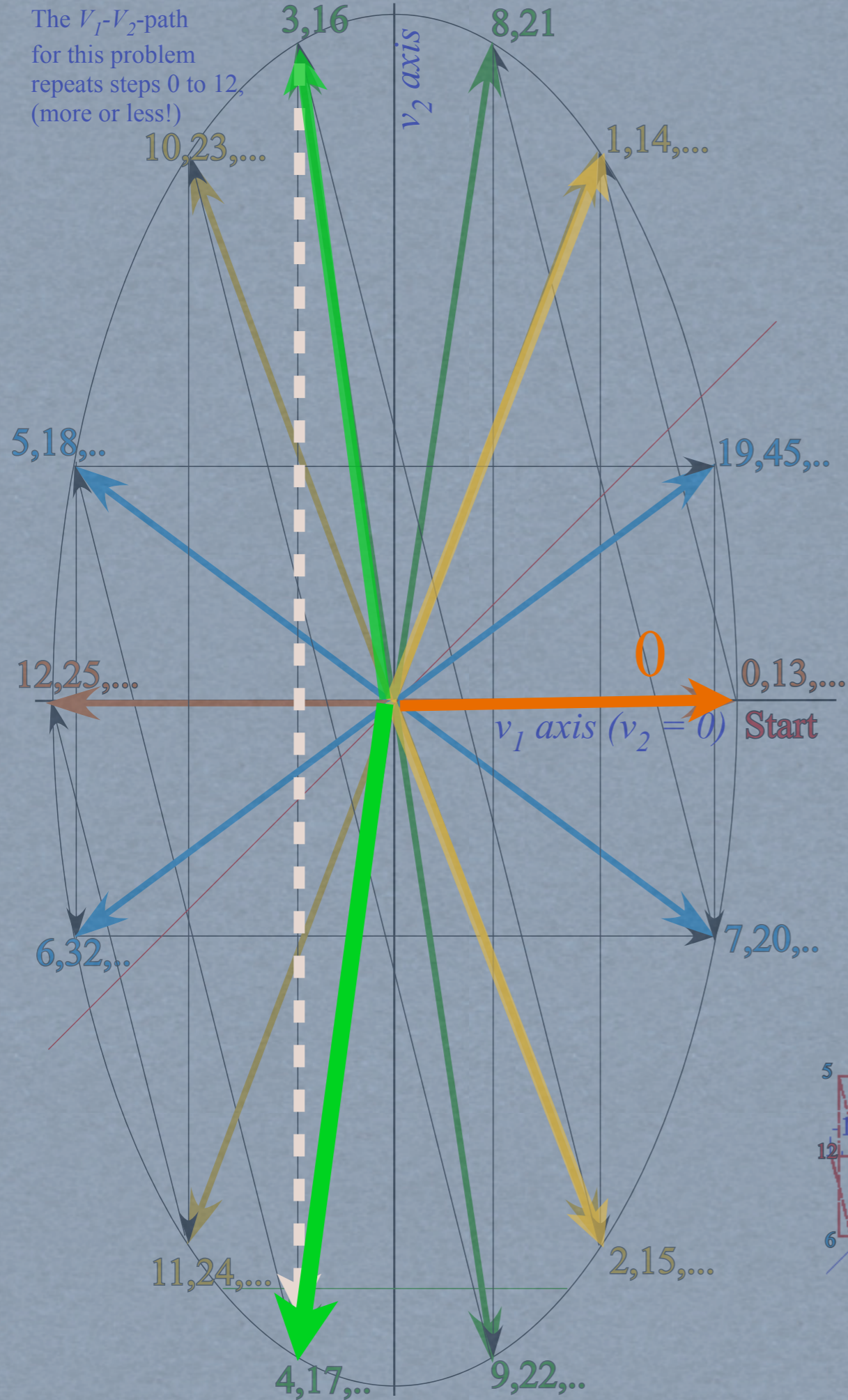
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



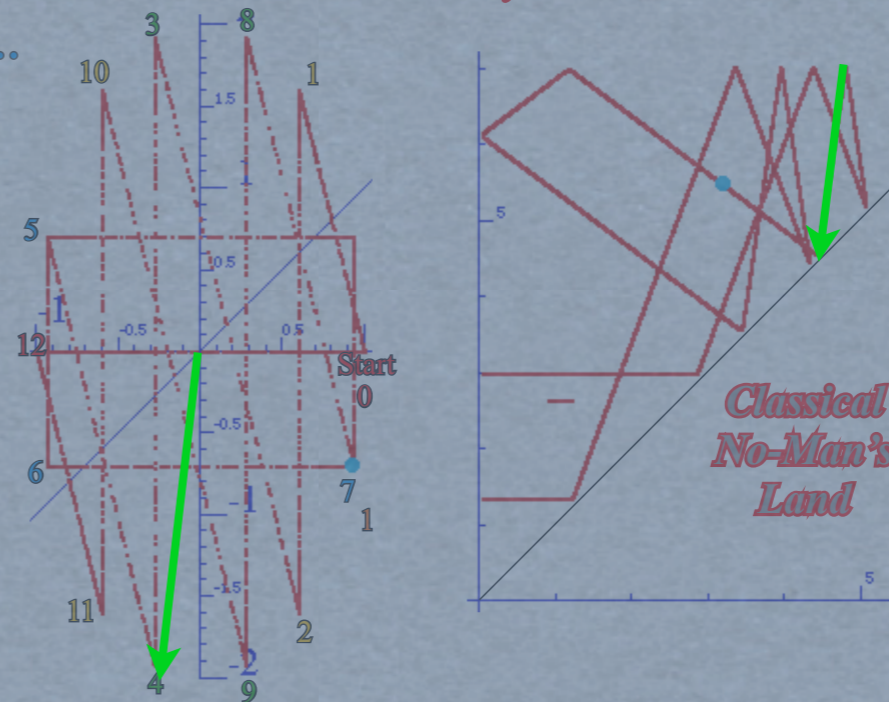
Simulations by *BouncIt*



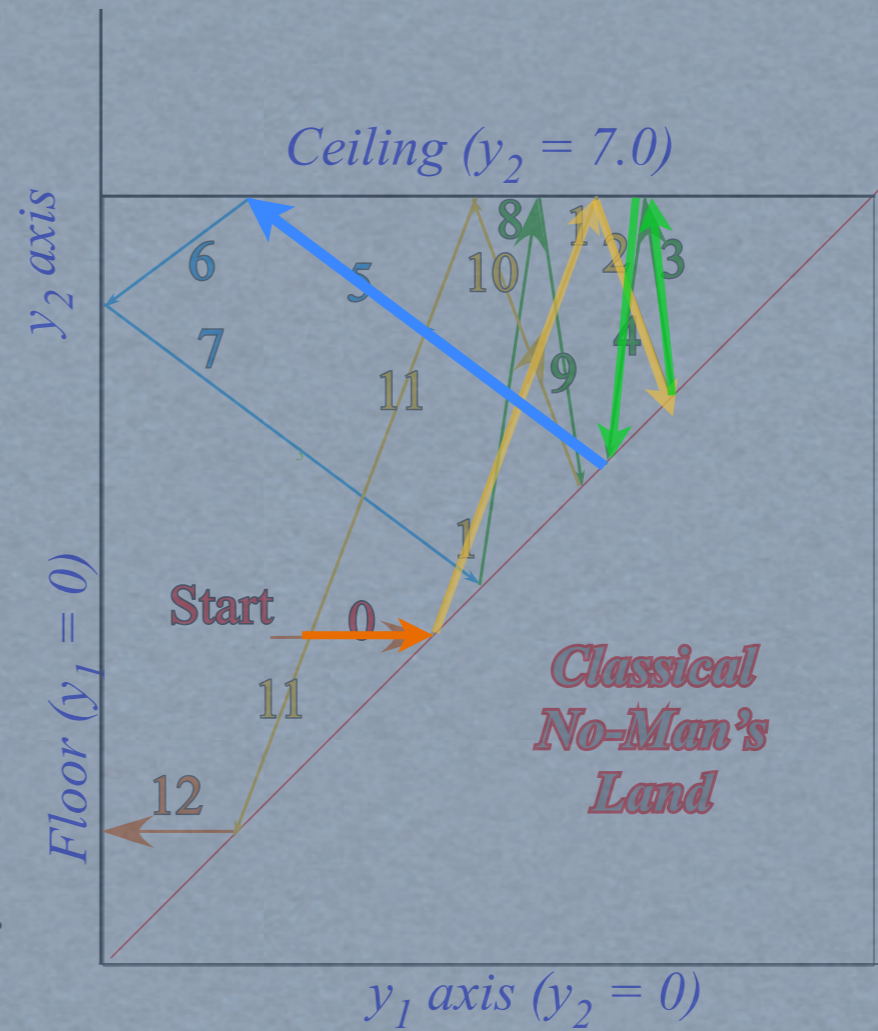
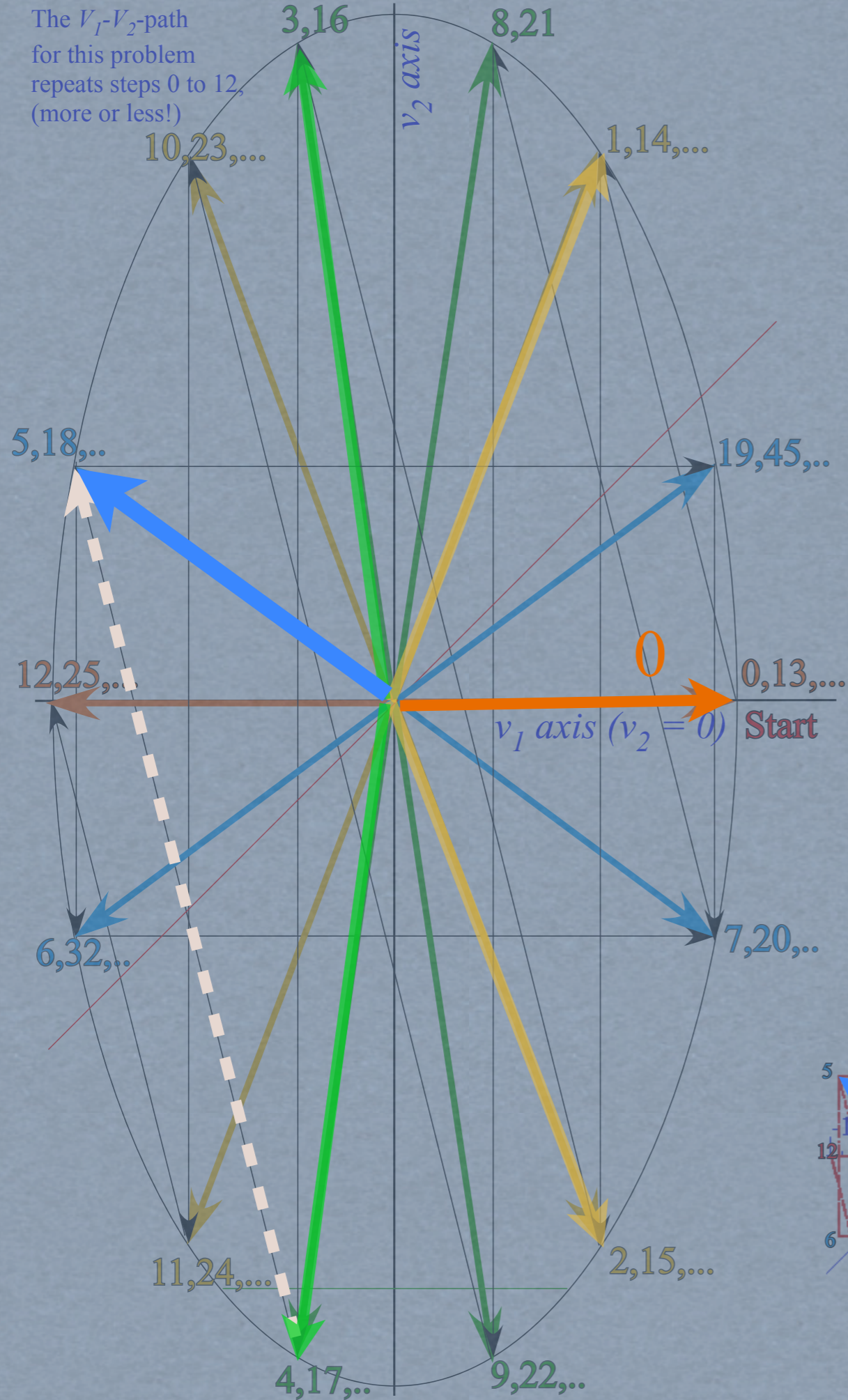
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



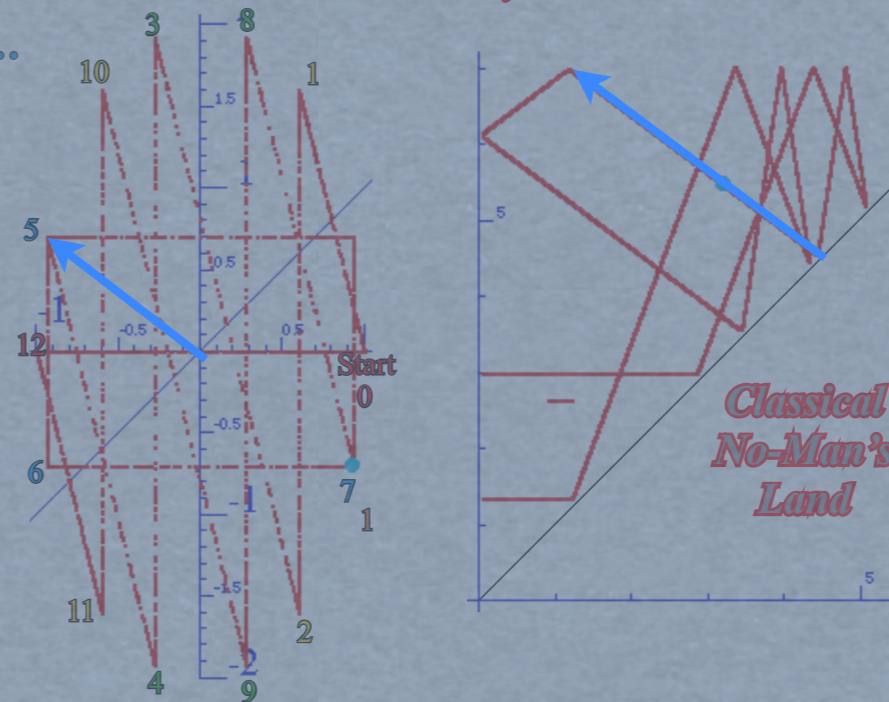
Simulations by *BounceIt*



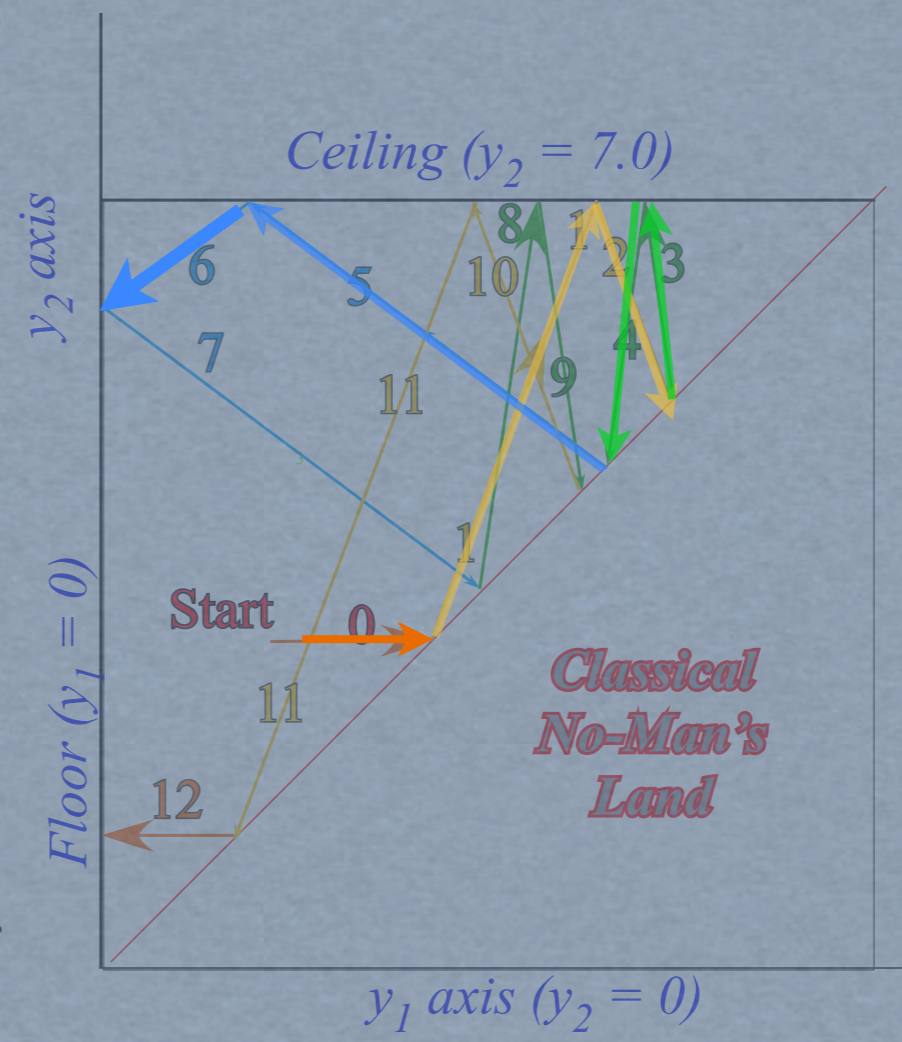
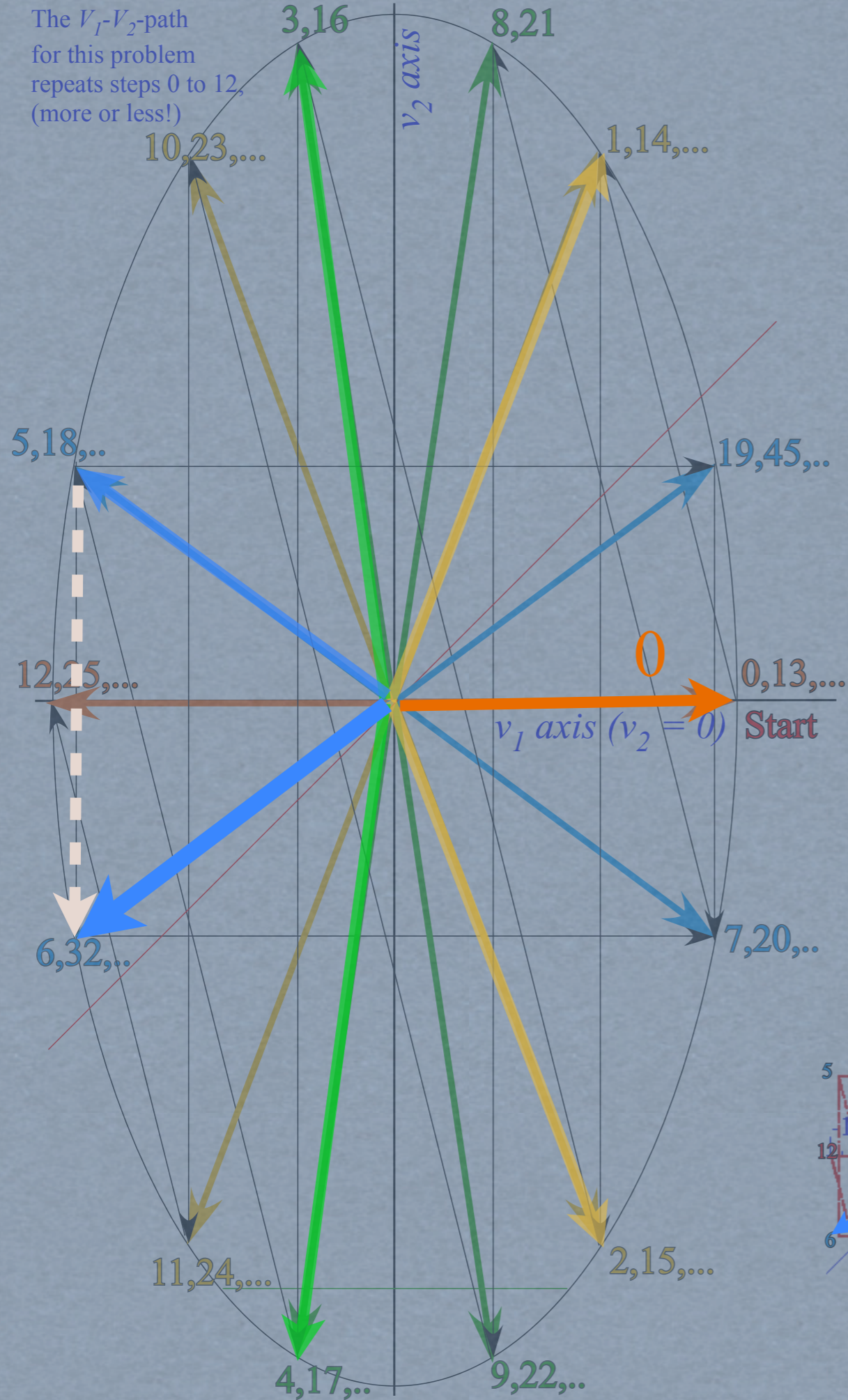
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



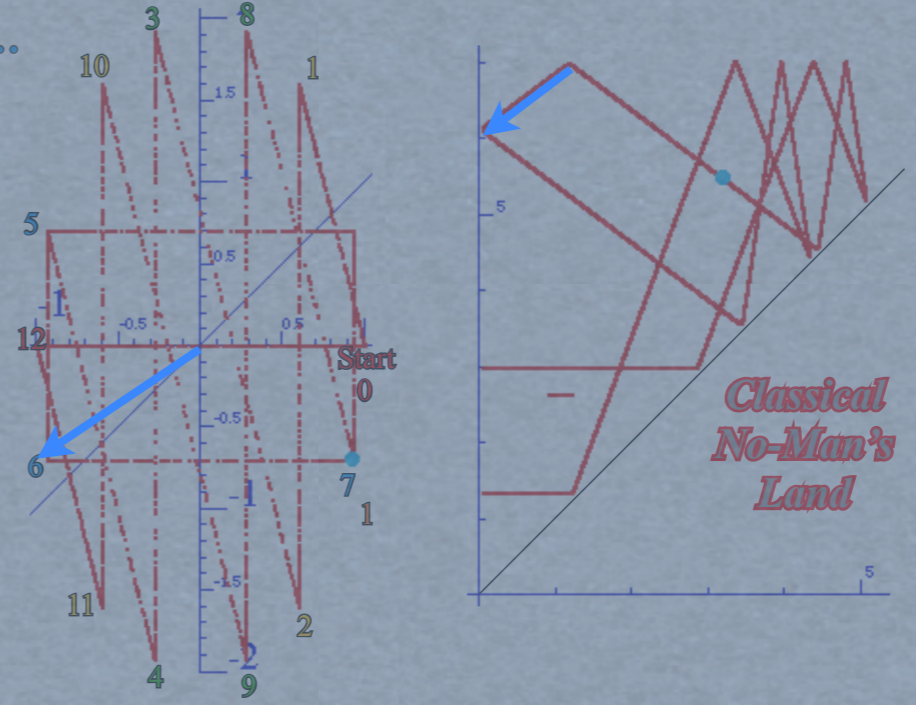
Simulations by *BounceIt*



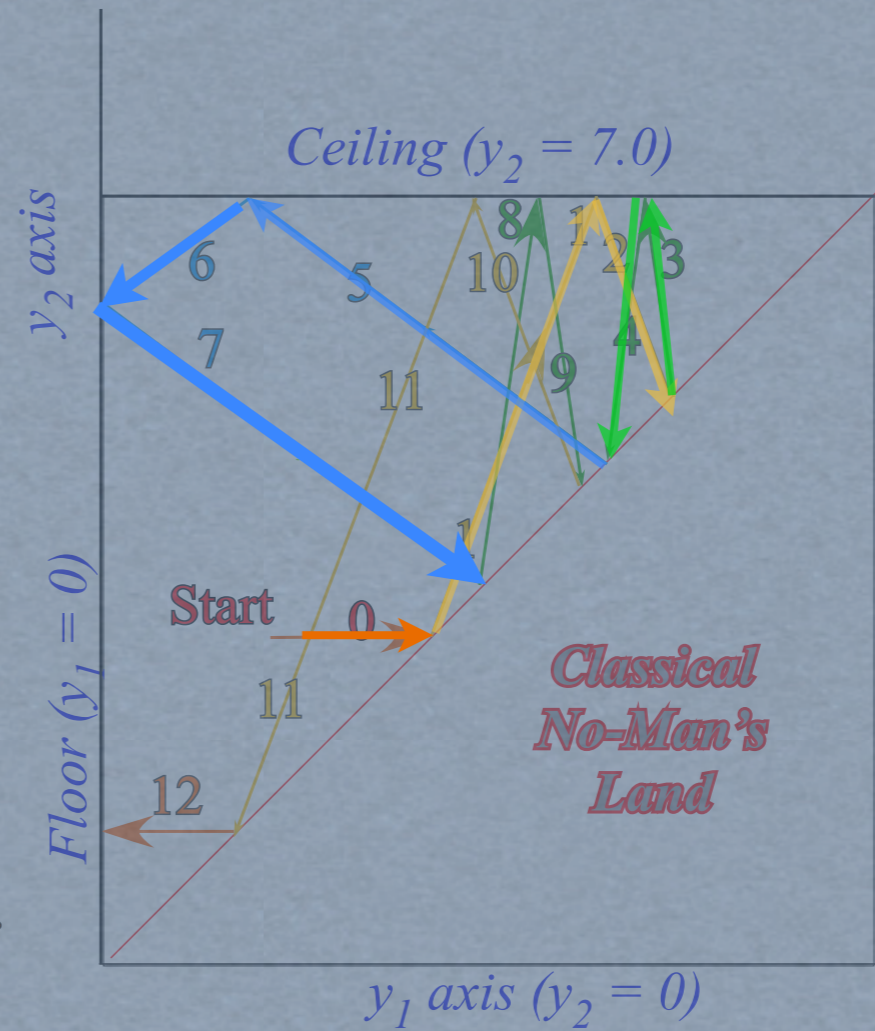
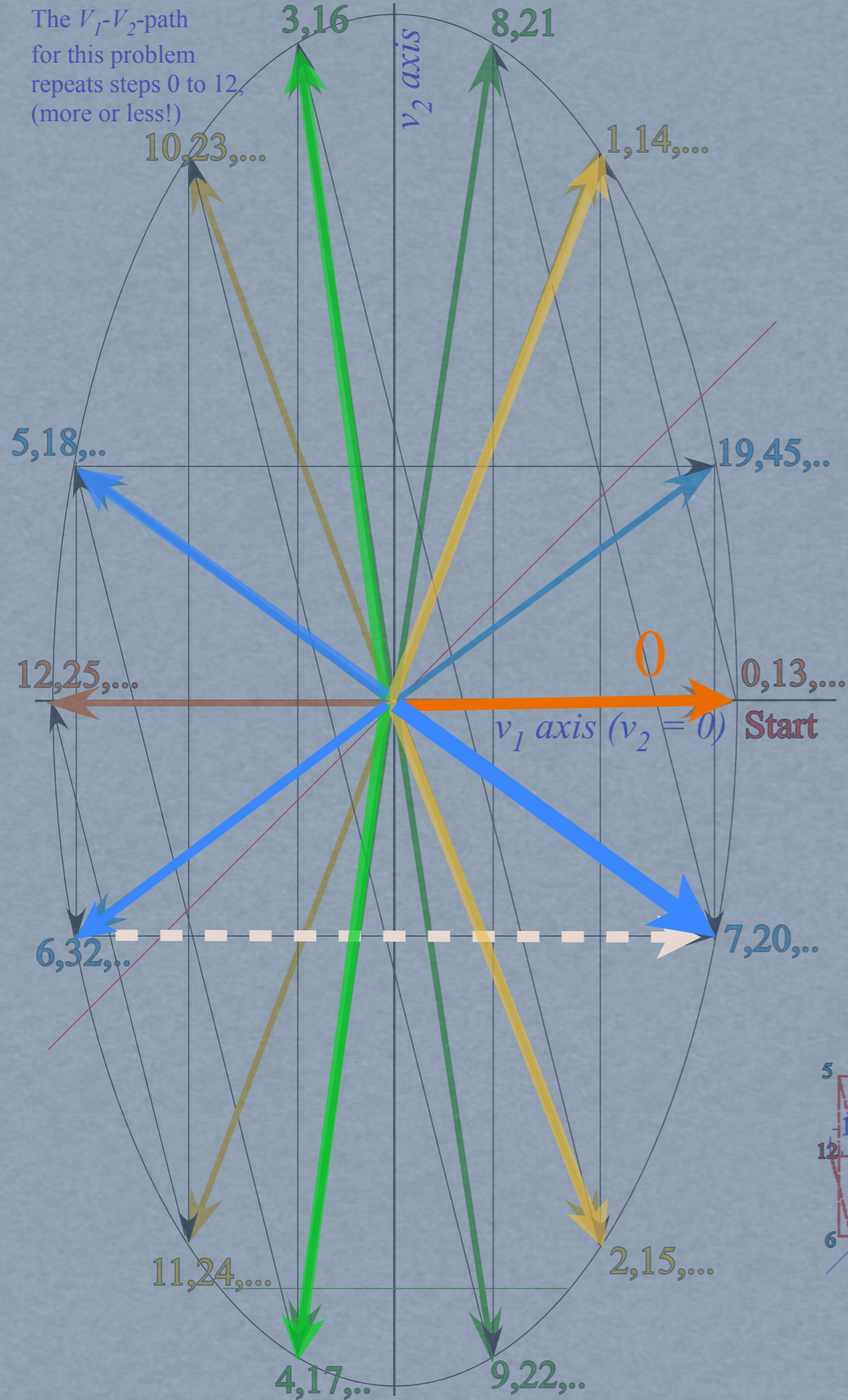
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



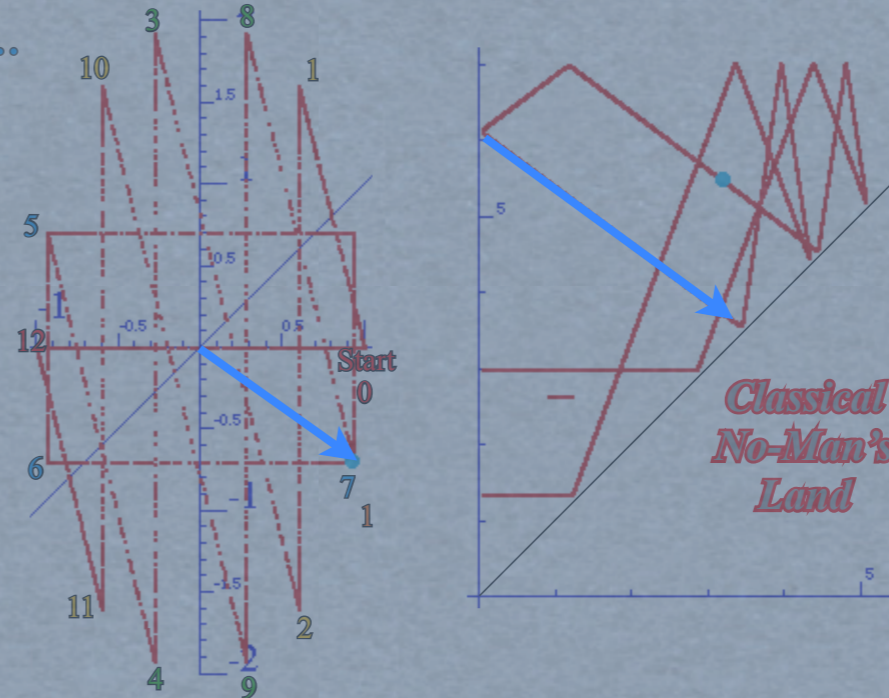
Simulations by *BounceIt*



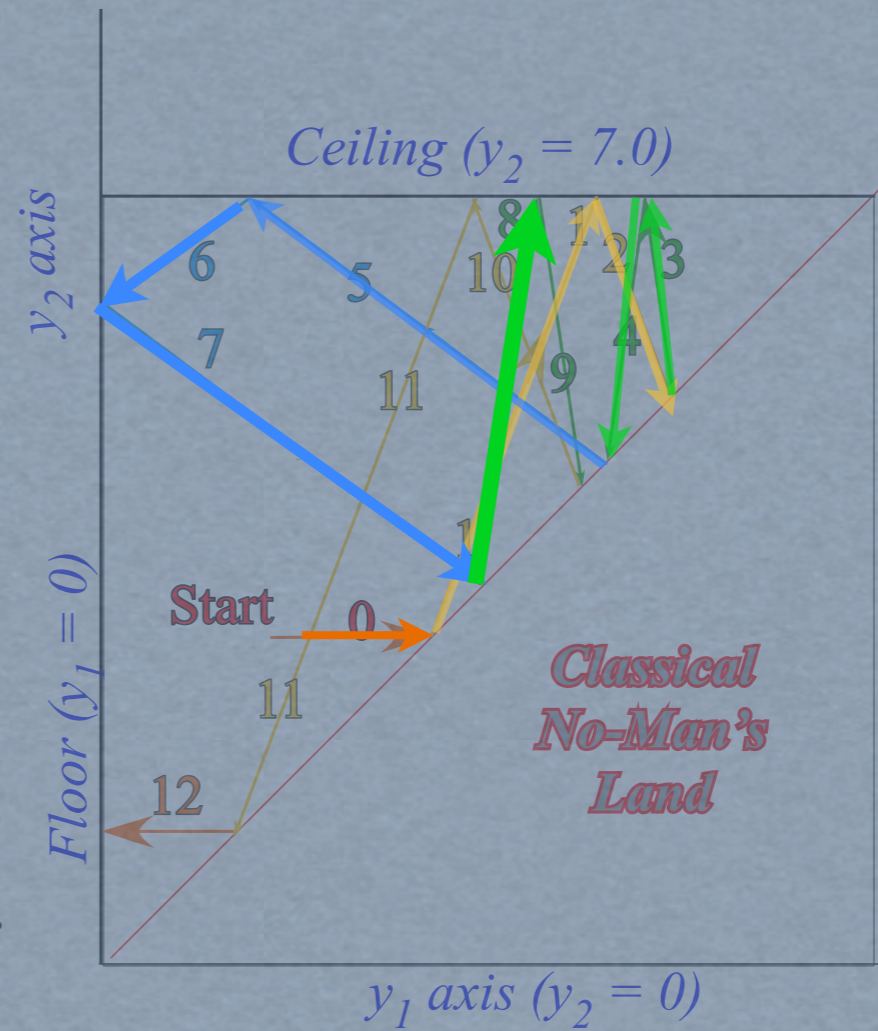
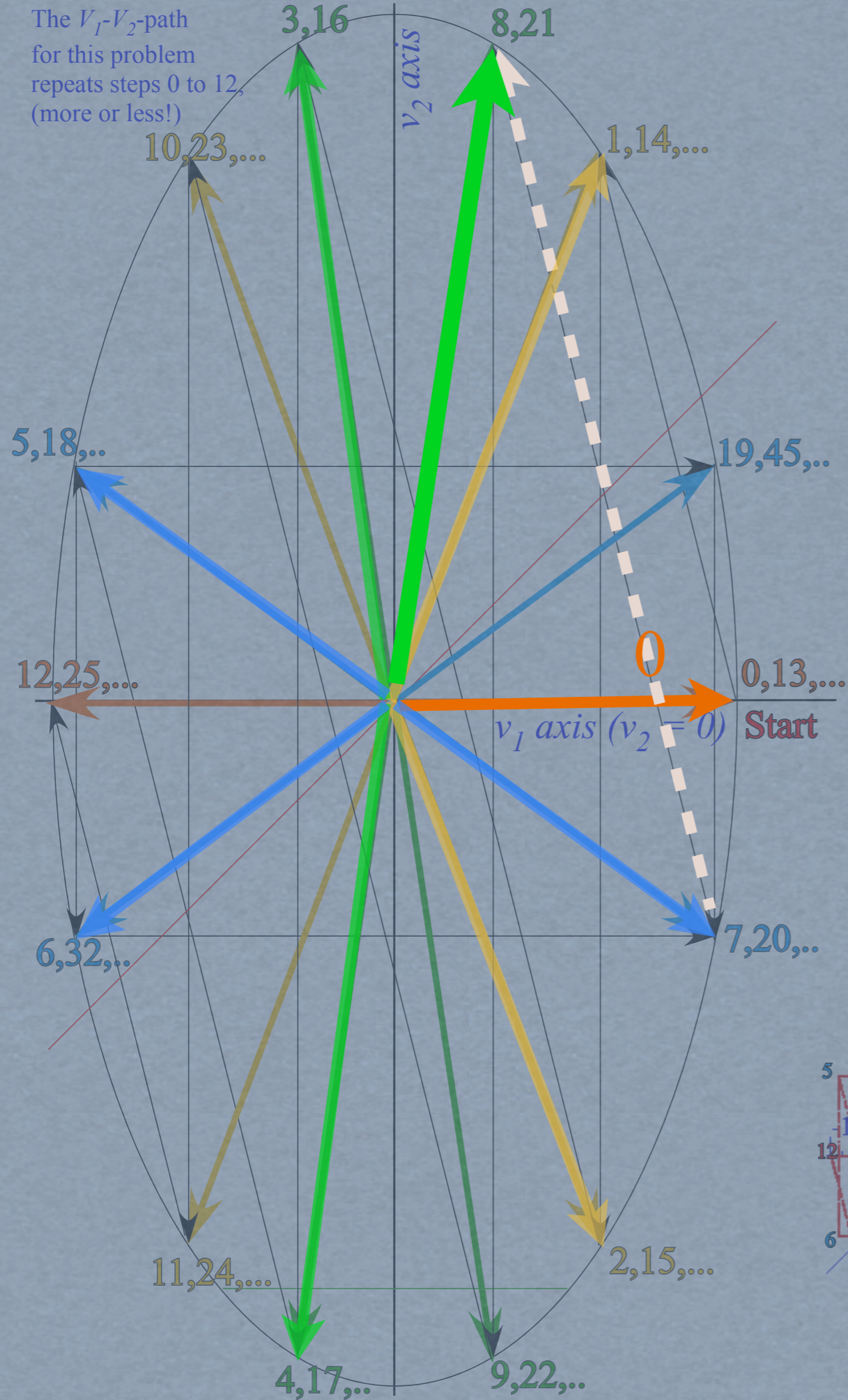
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



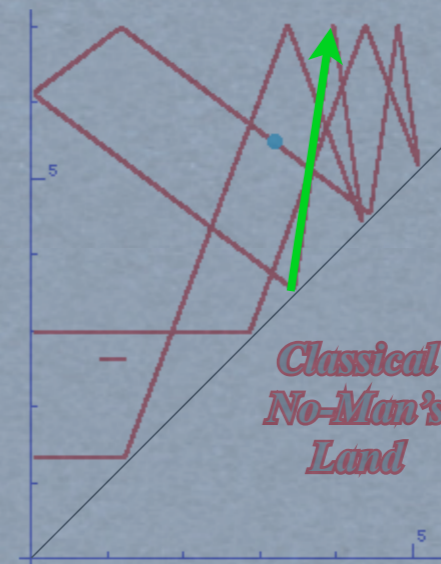
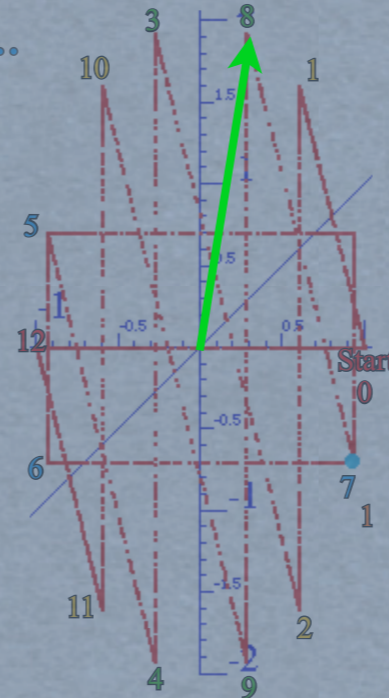
Simulations by *BounceIt*



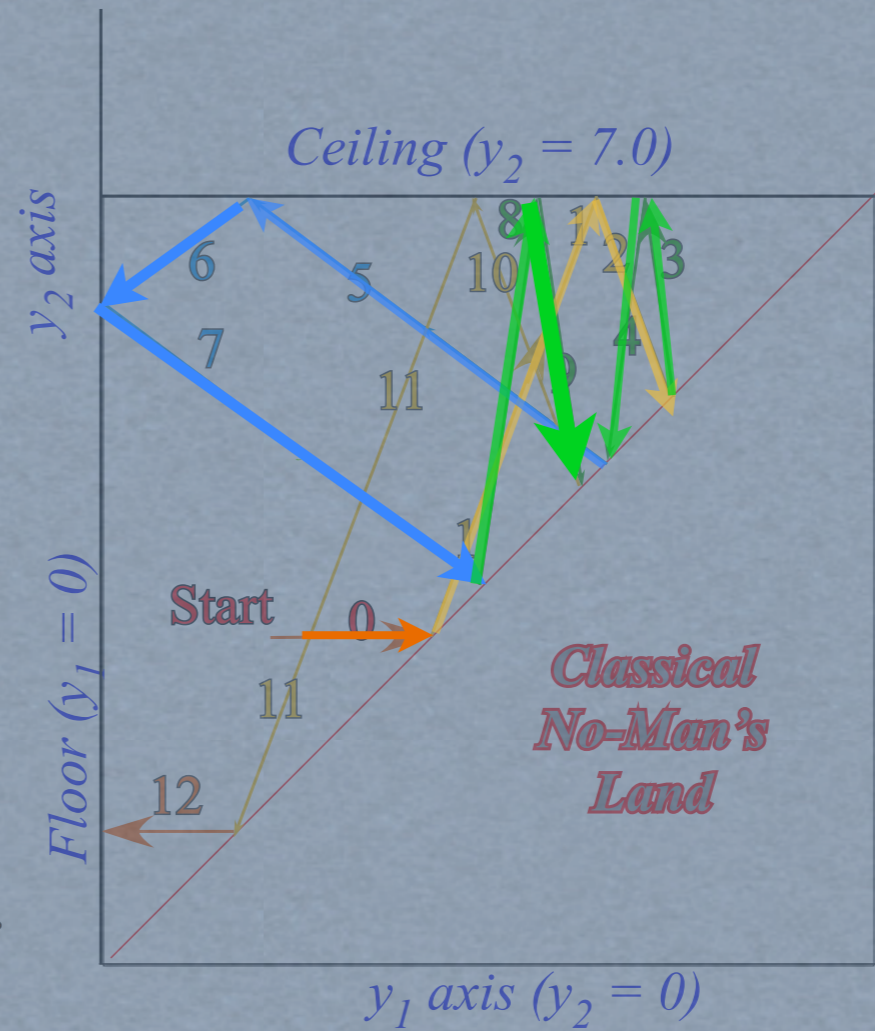
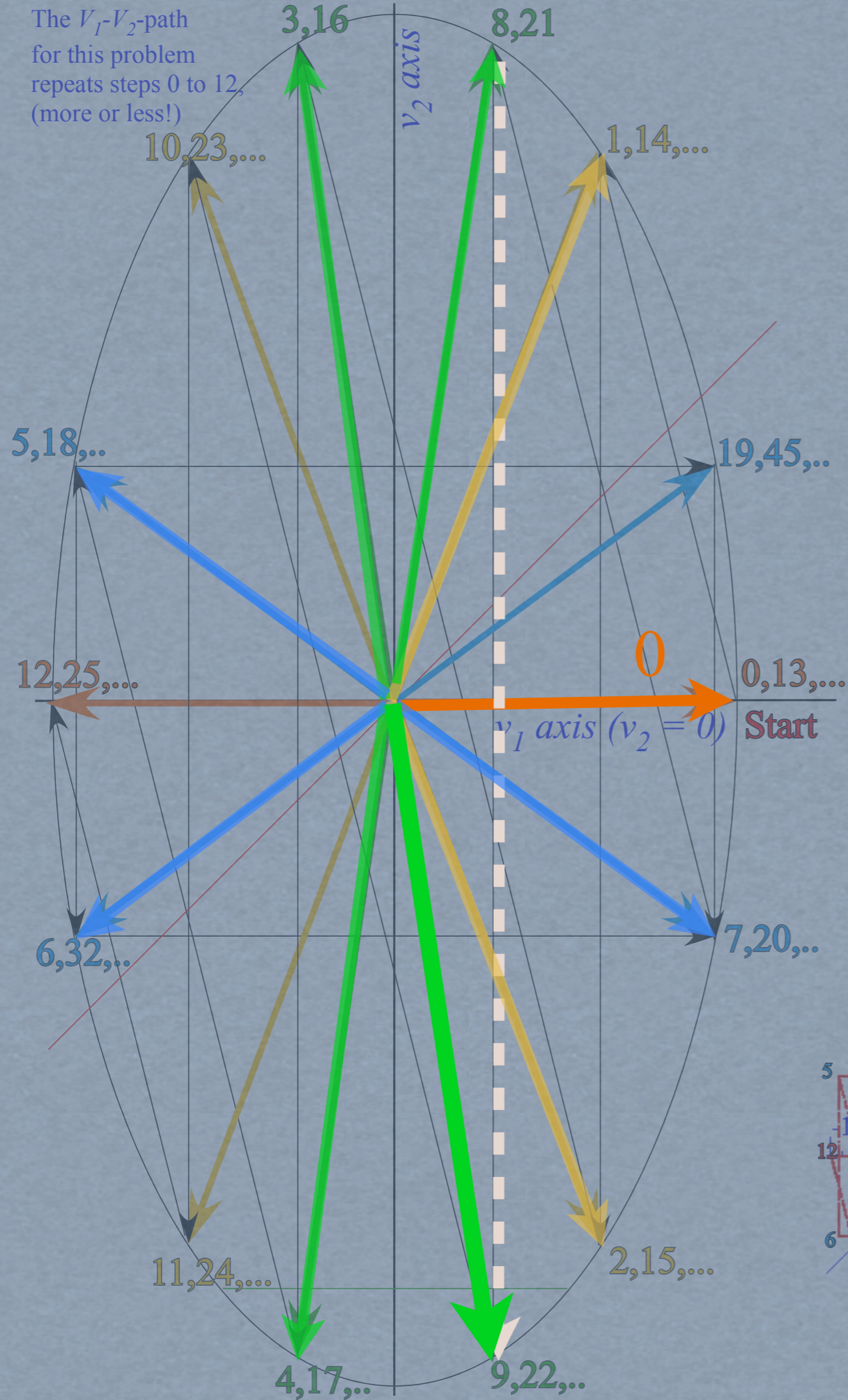
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



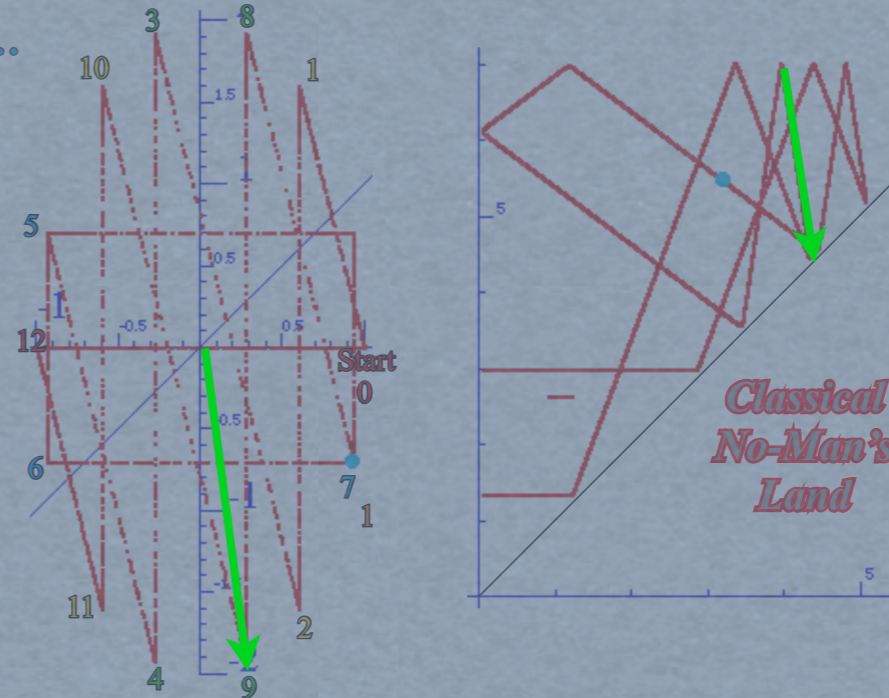
Simulations by *BounceIt*



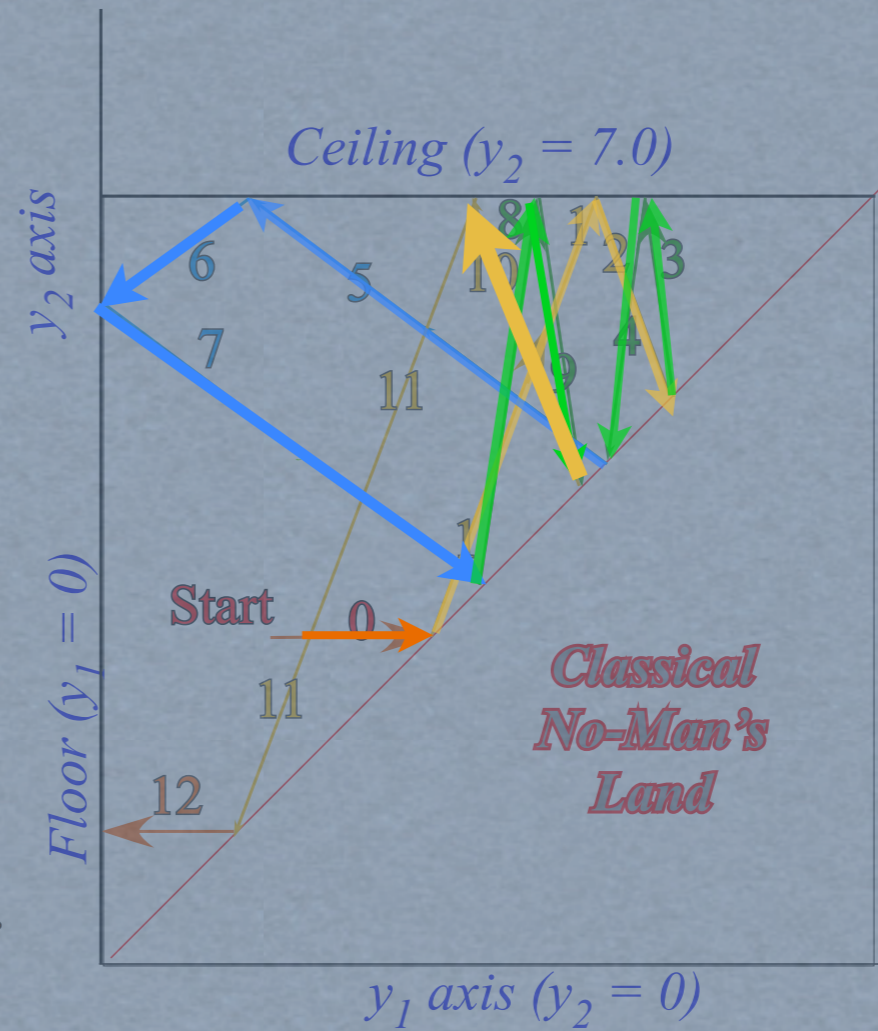
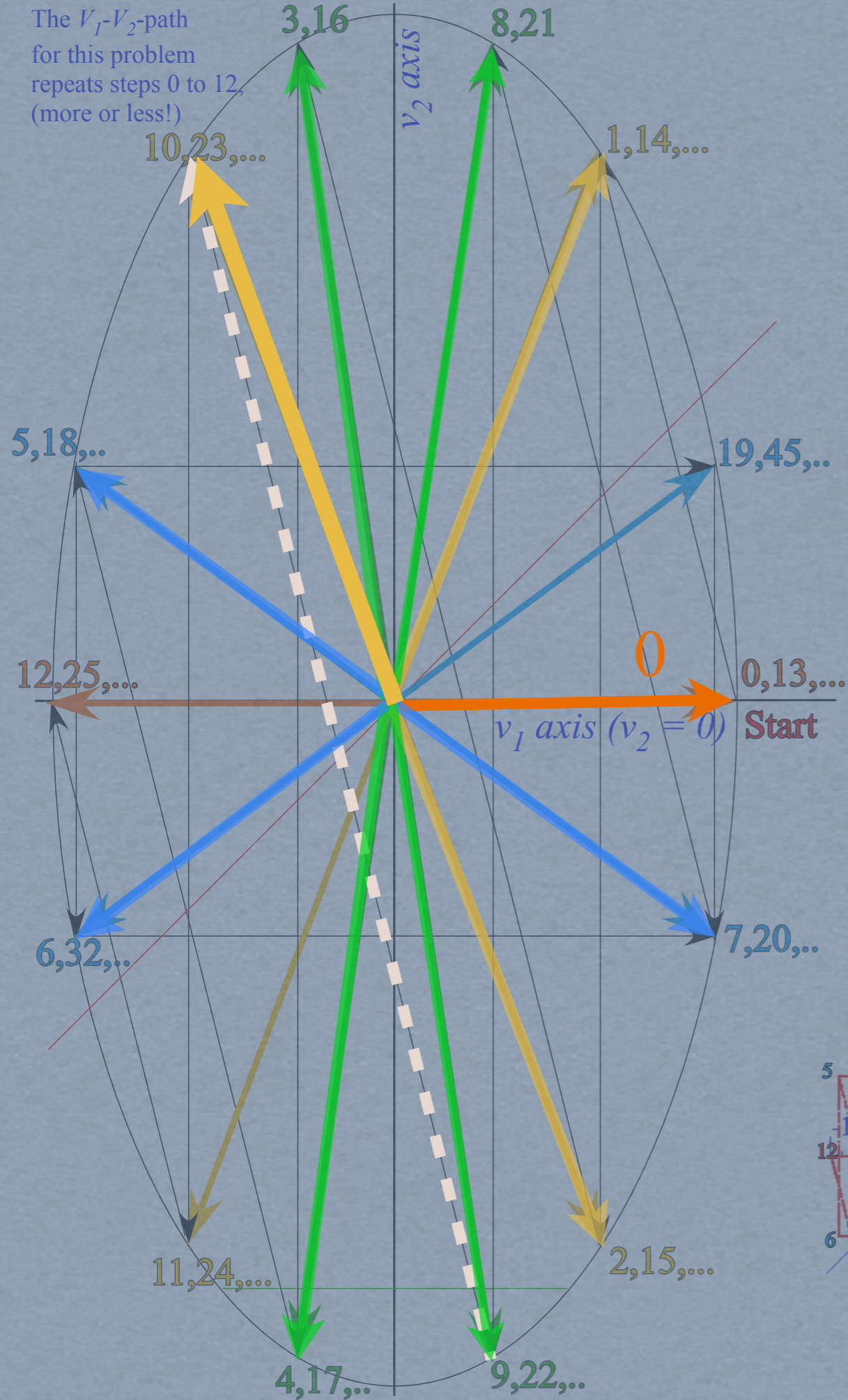
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



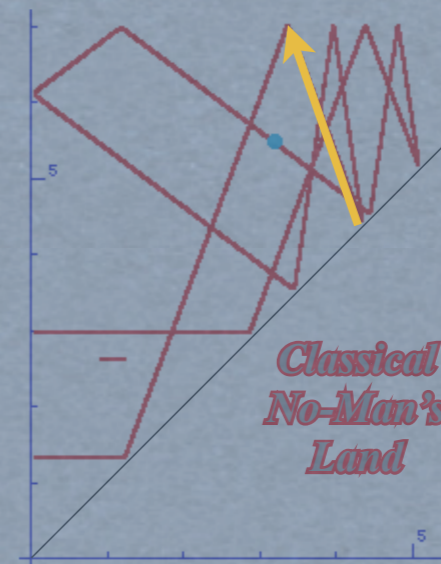
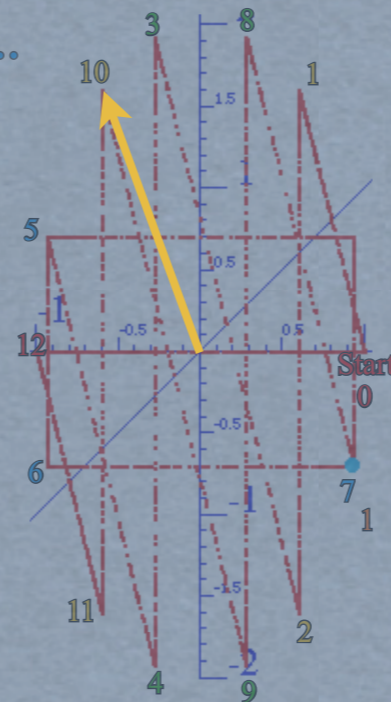
Simulations by BounceIt



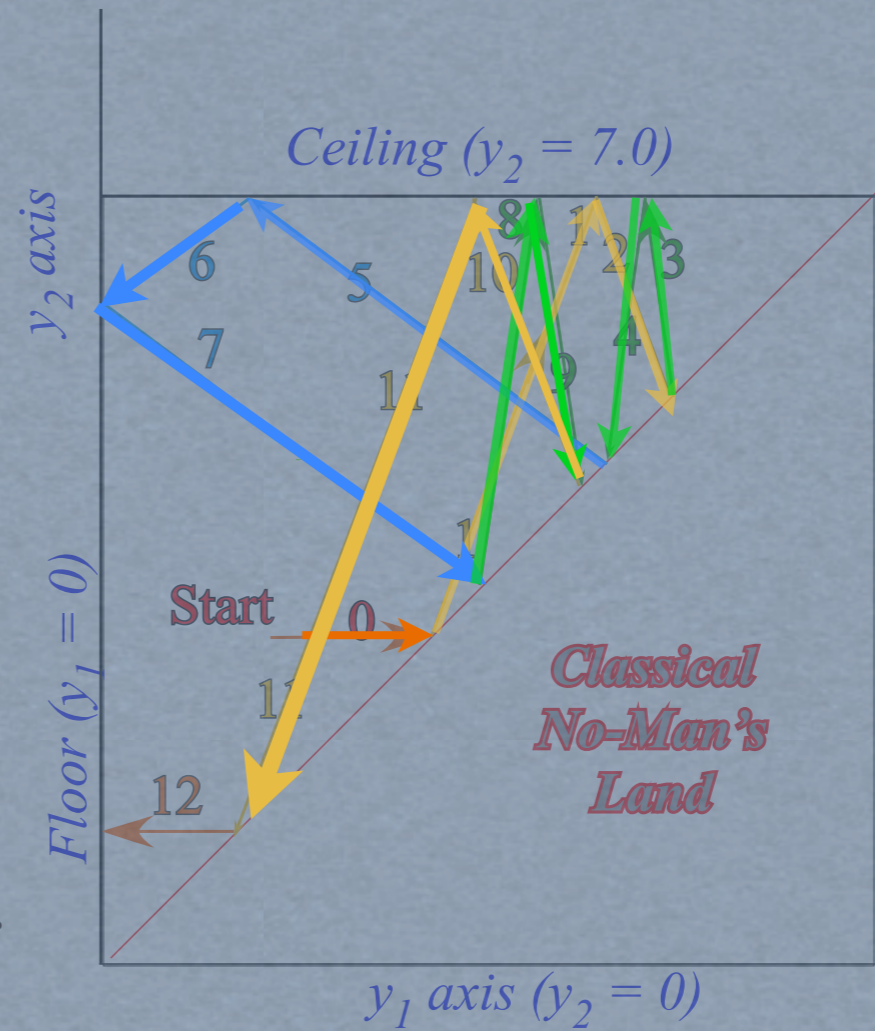
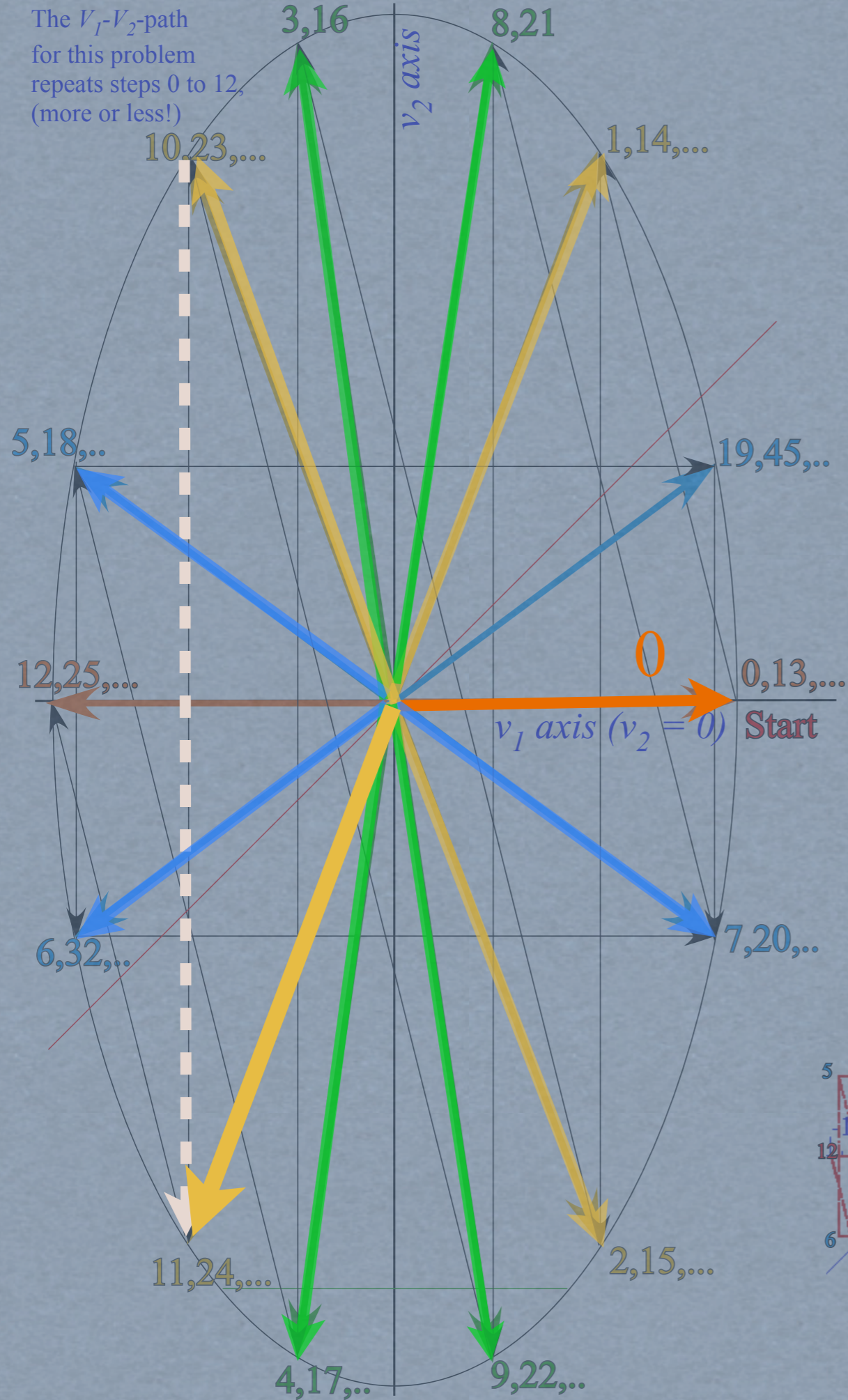
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



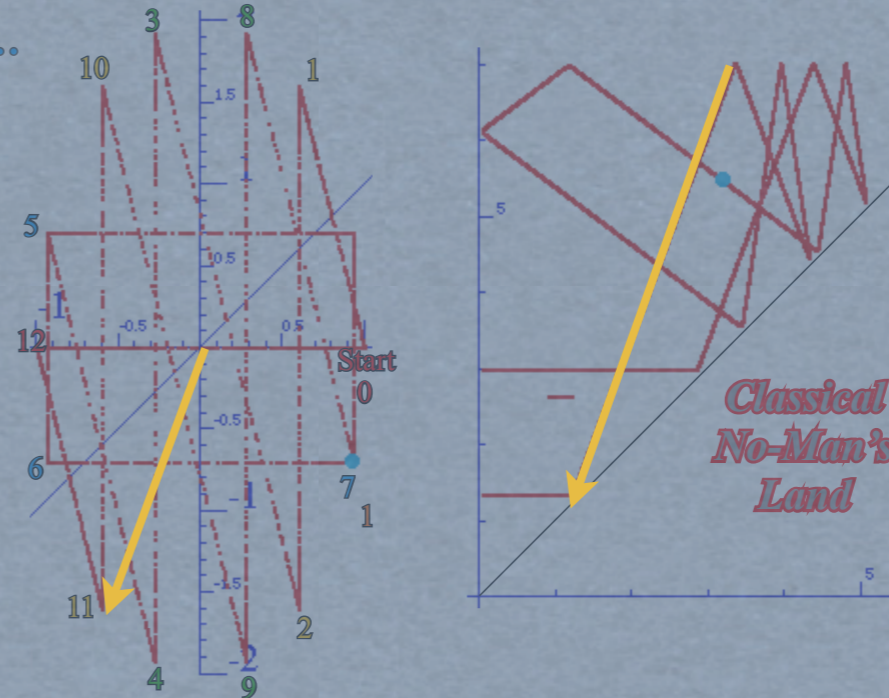
Simulations by *BounceIt*



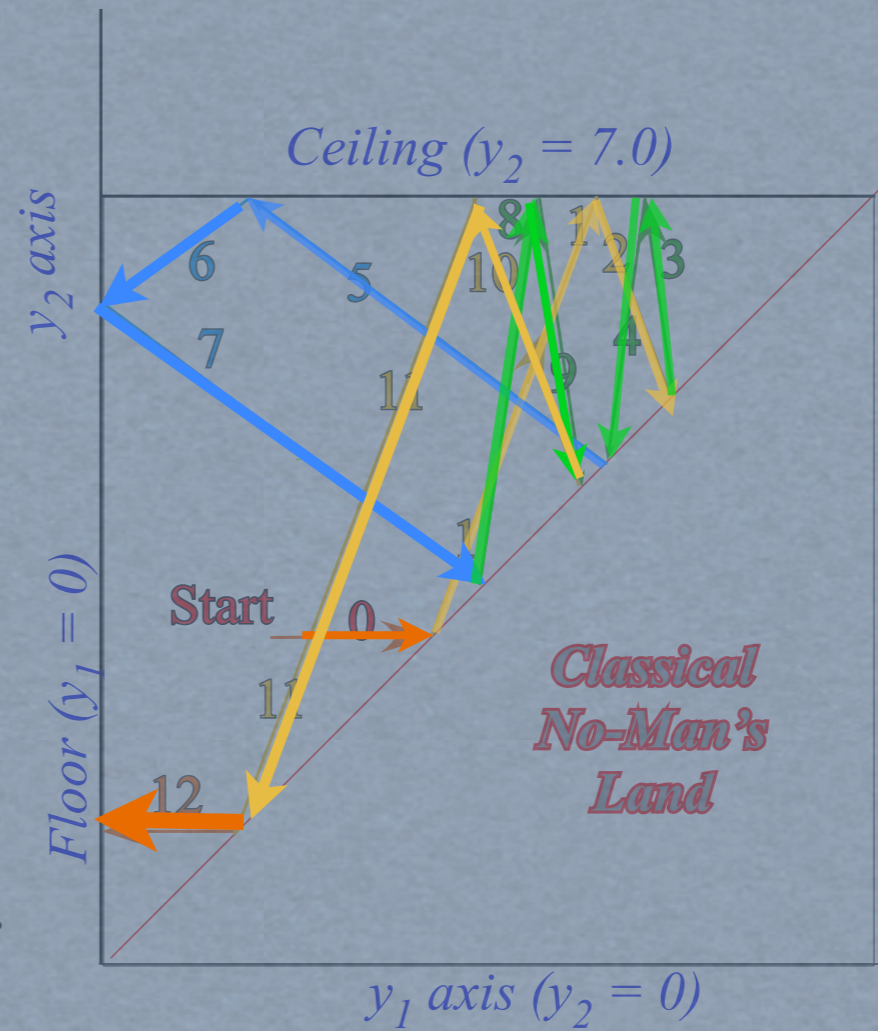
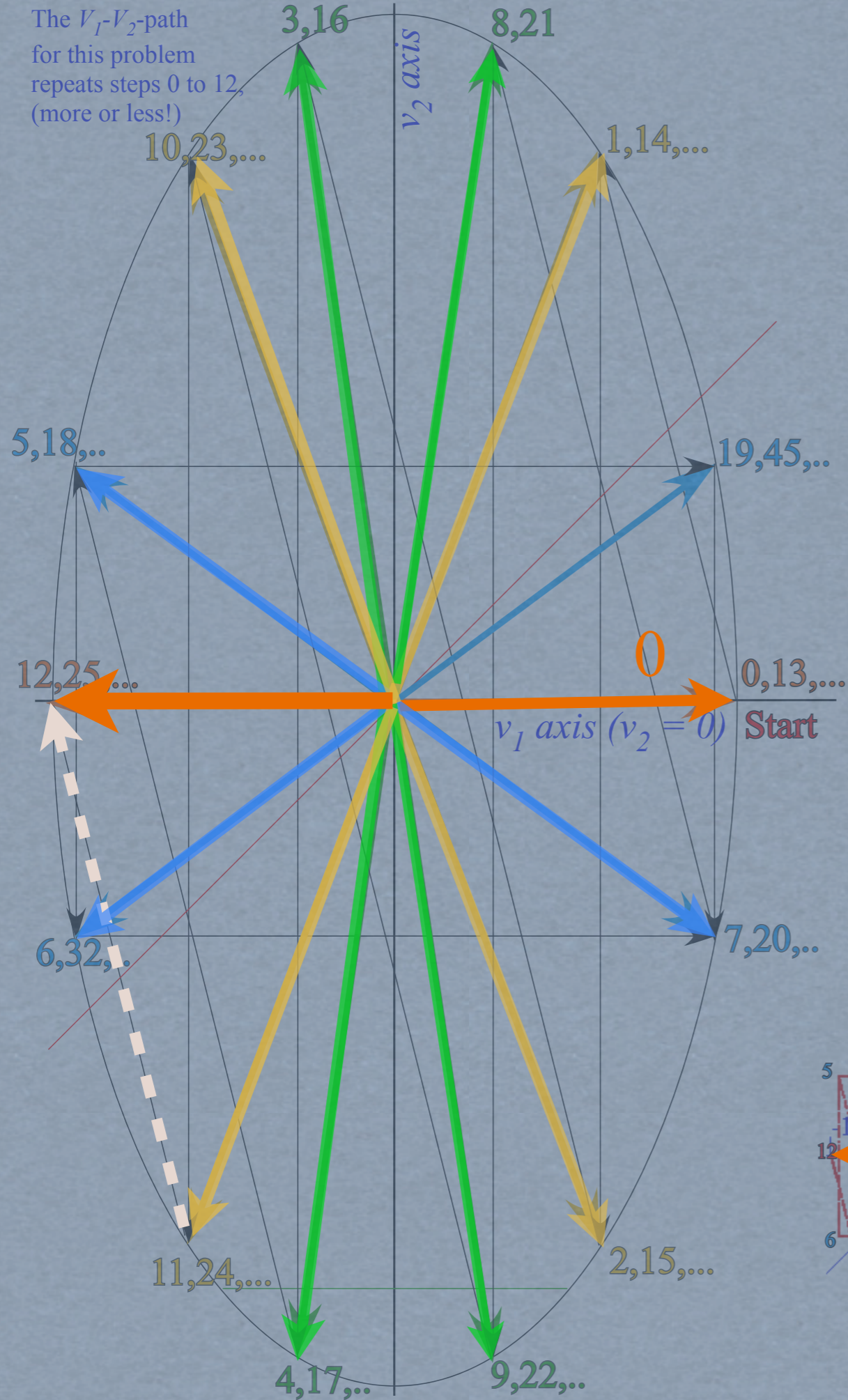
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



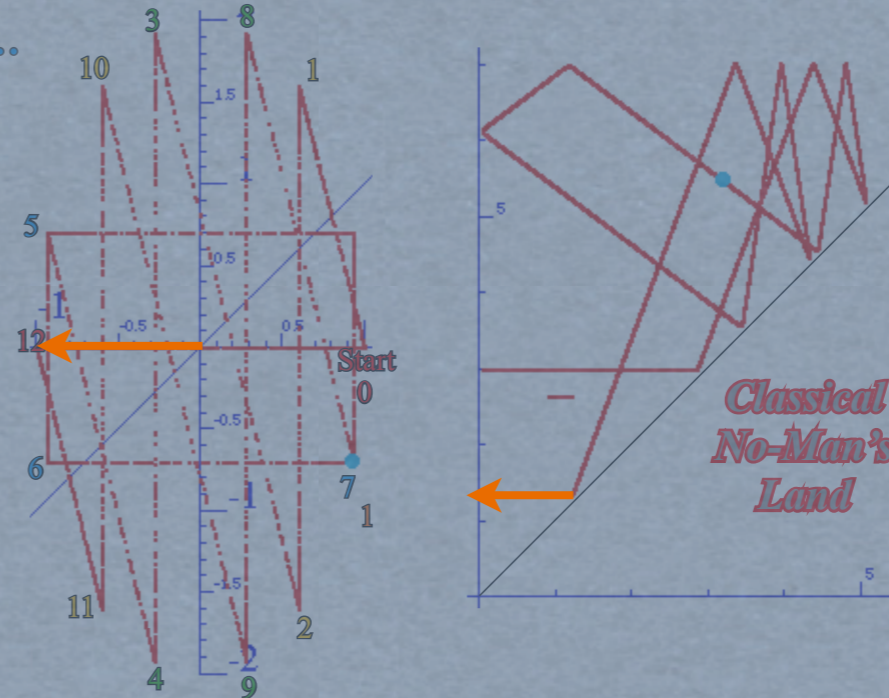
Simulations by BounceIt



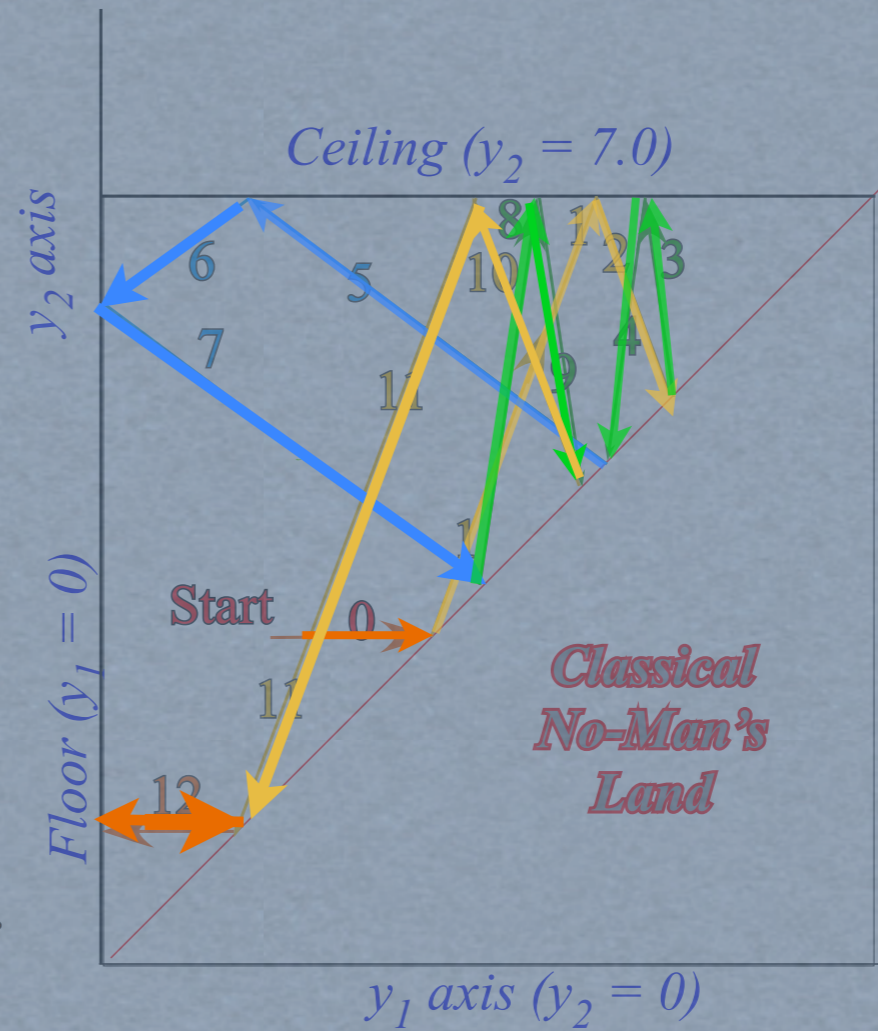
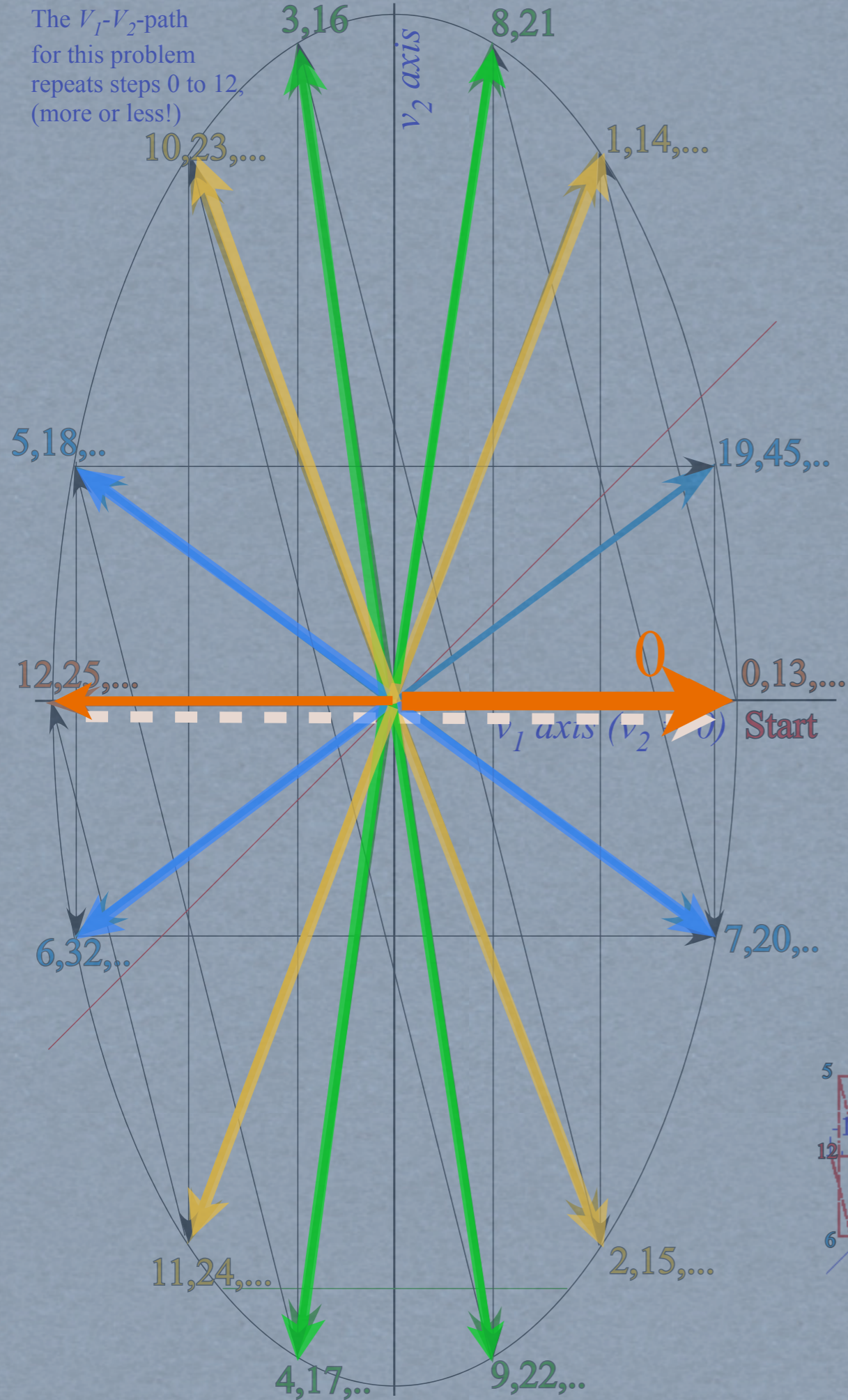
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



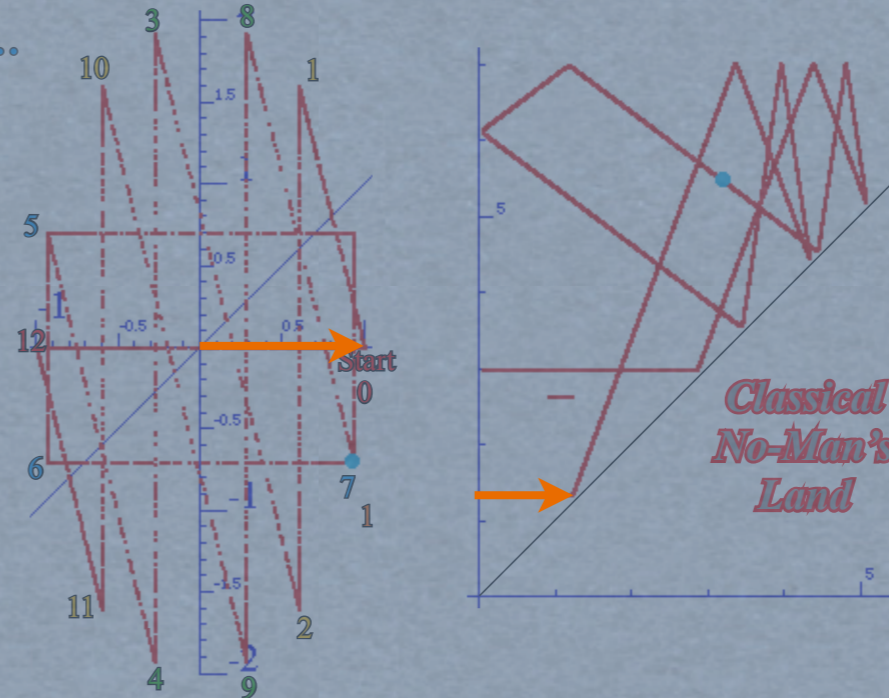
Simulations by *BounceIt*



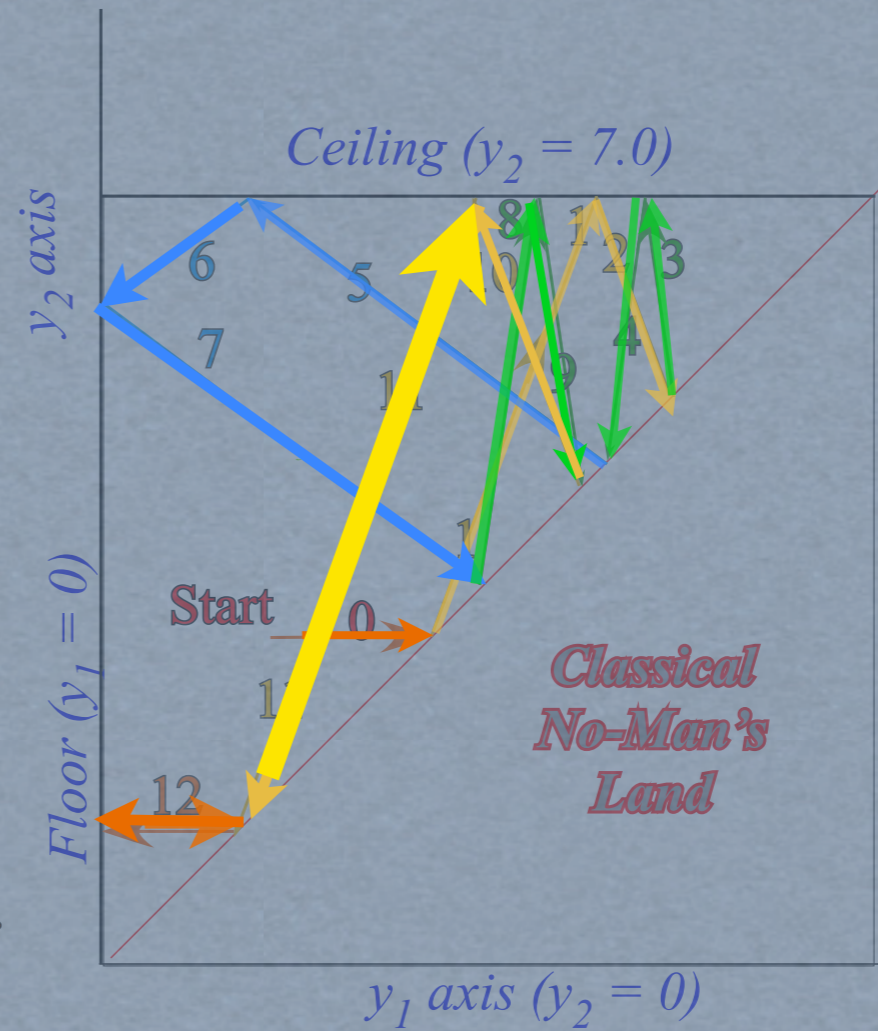
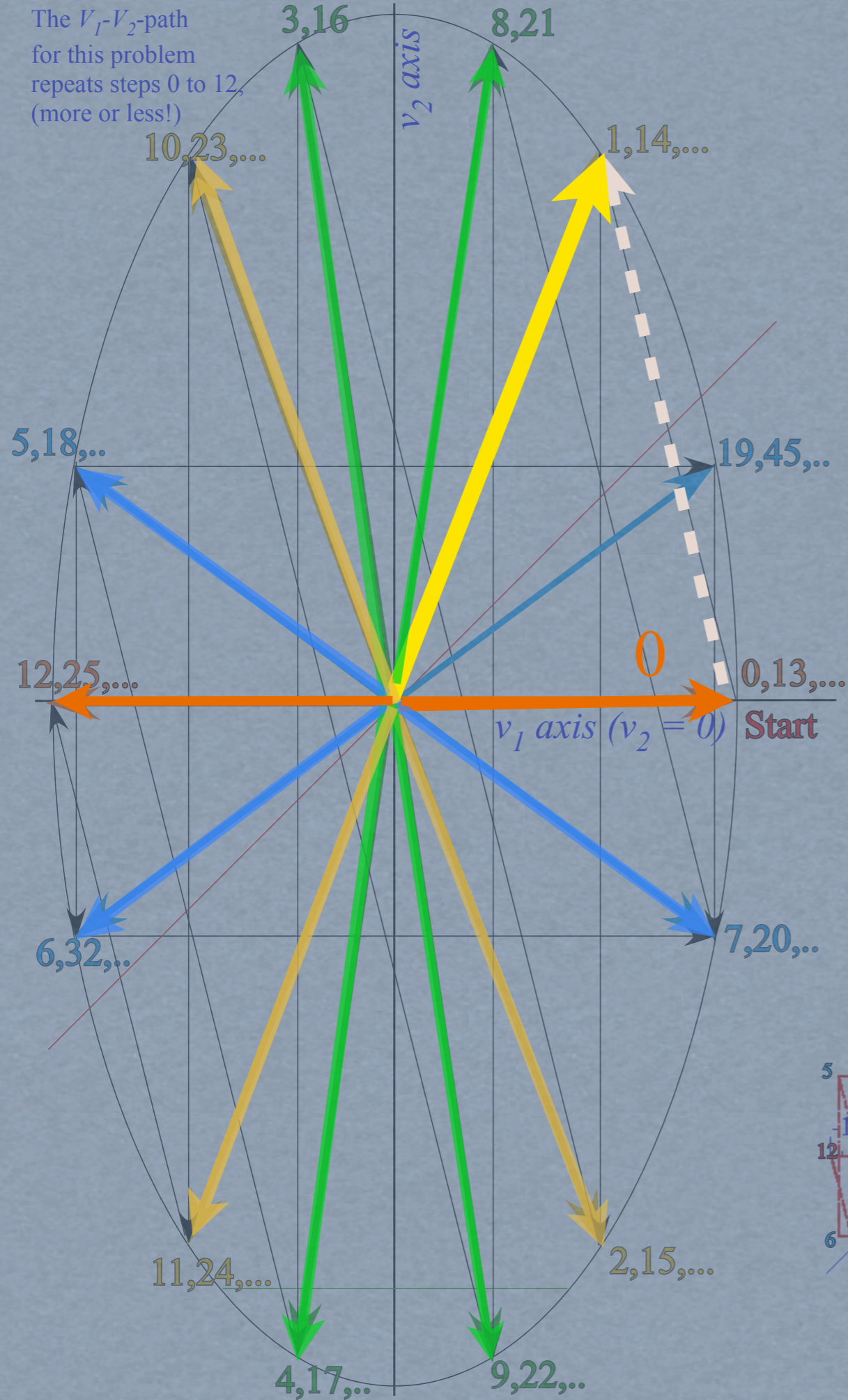
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



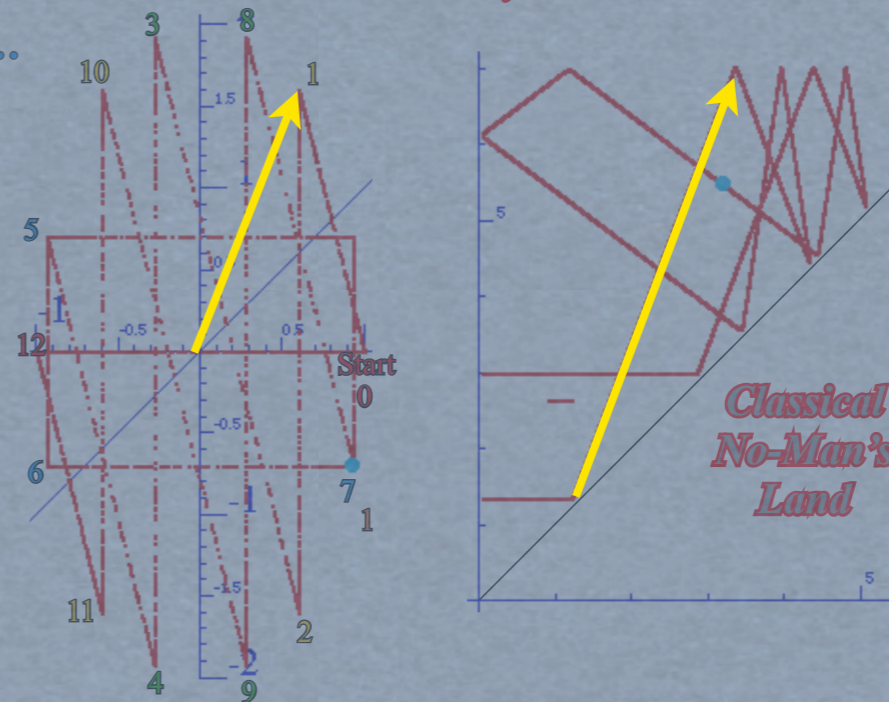
Simulations by *BounceIt*



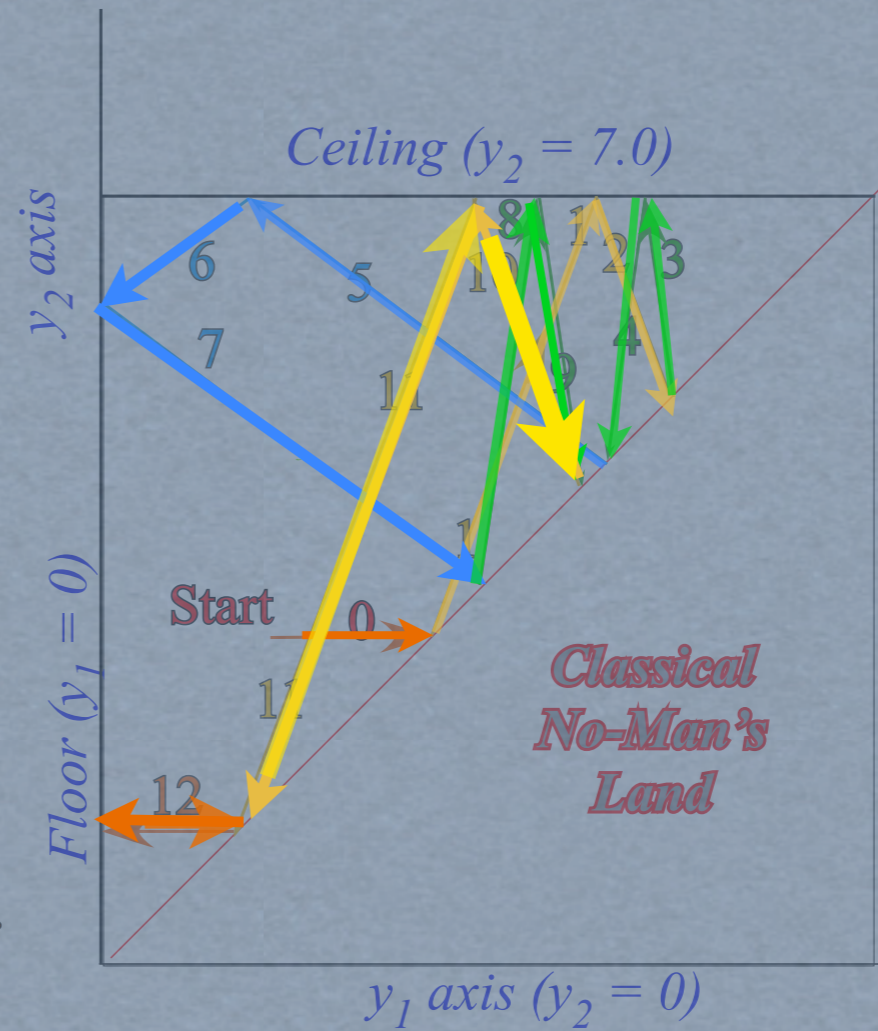
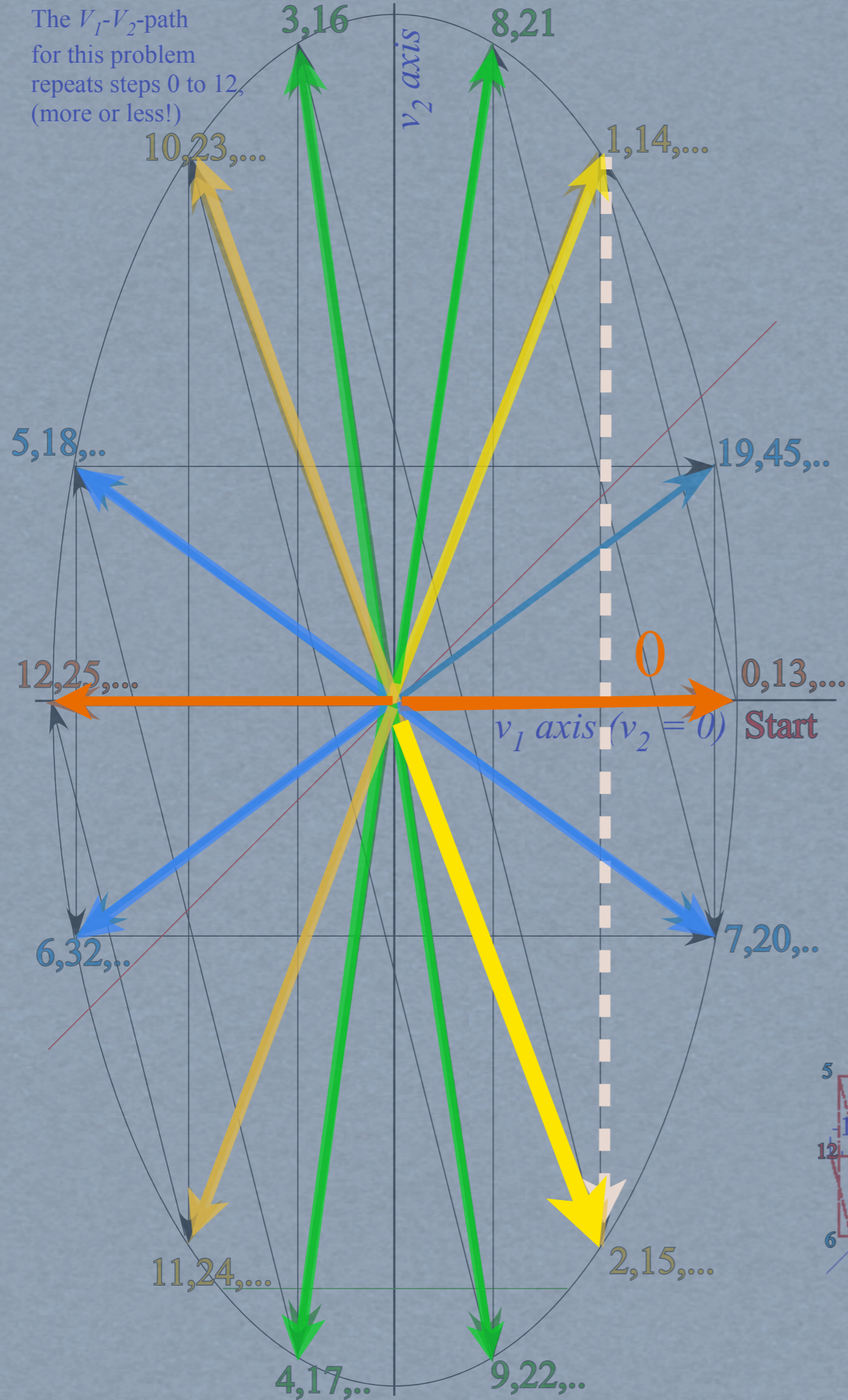
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



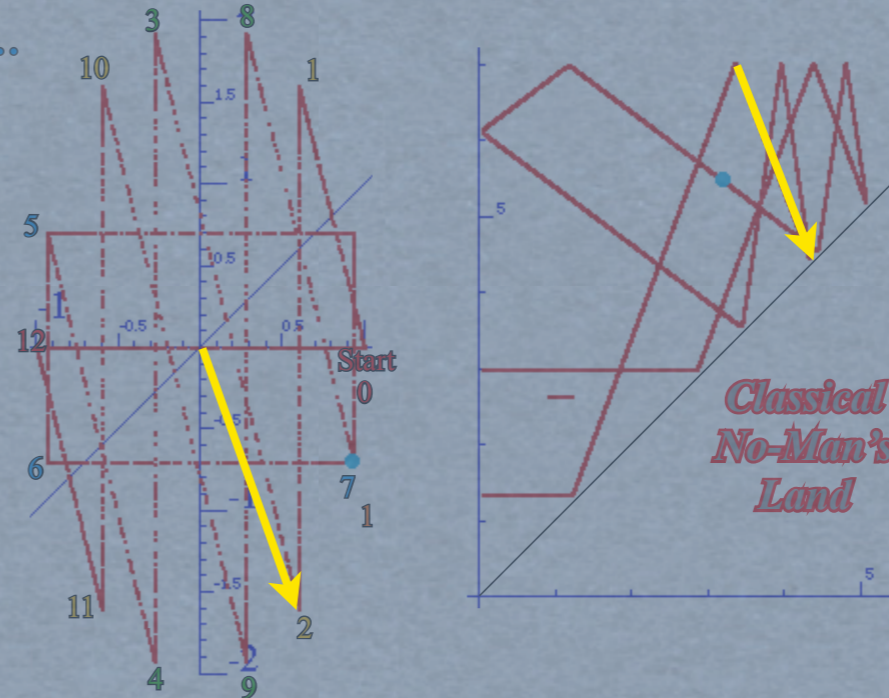
Simulations by *BounceIt*



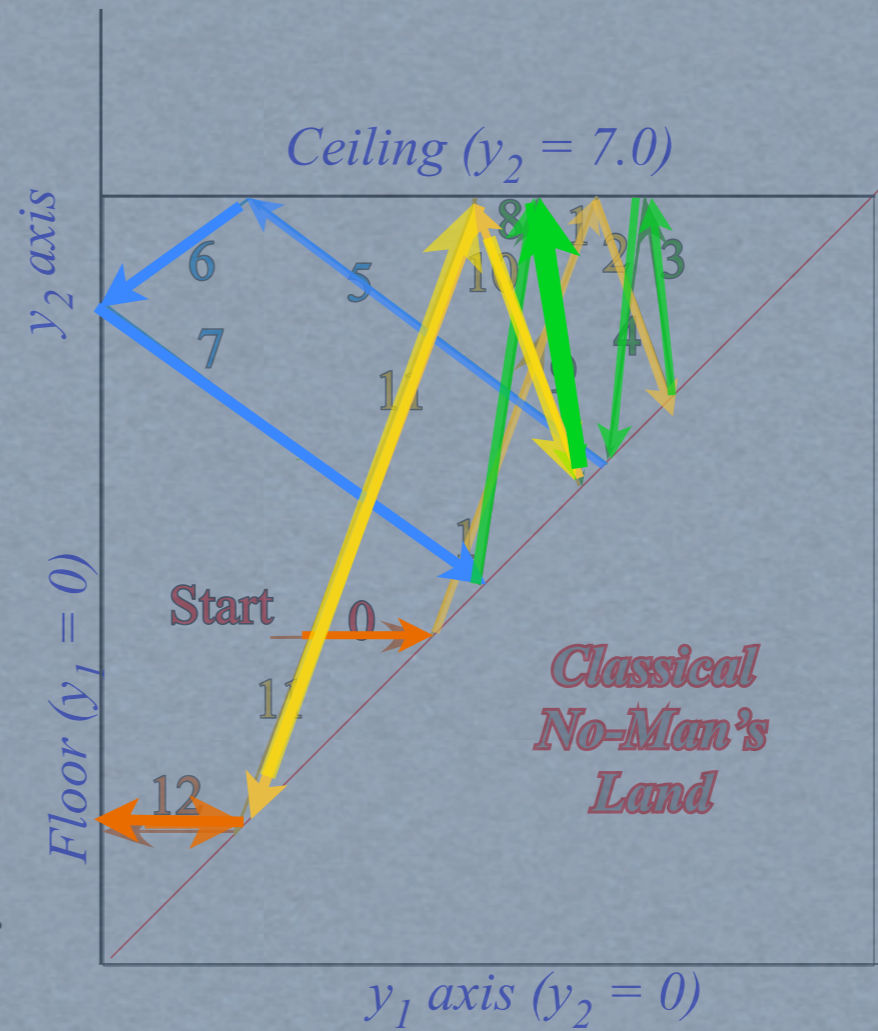
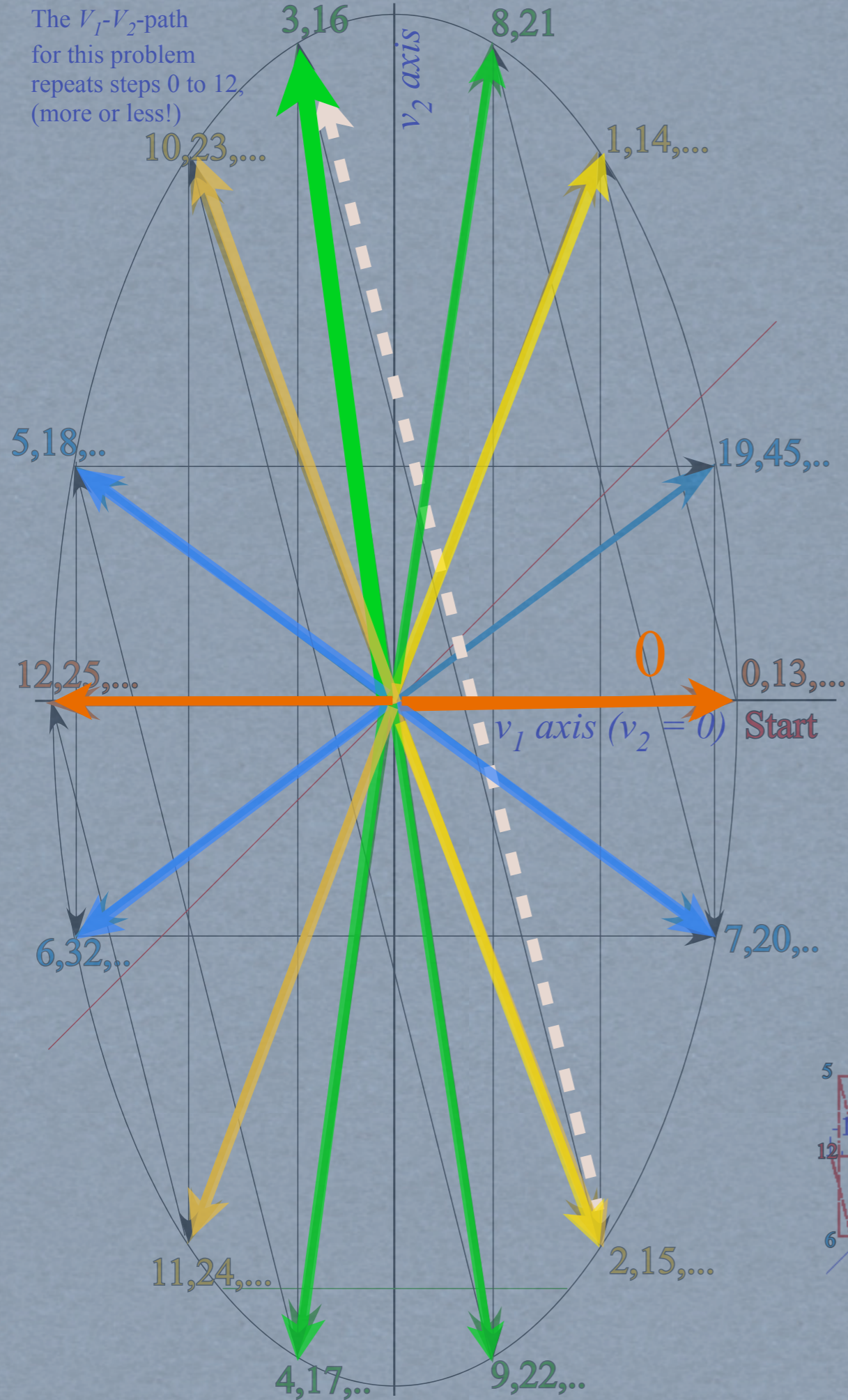
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



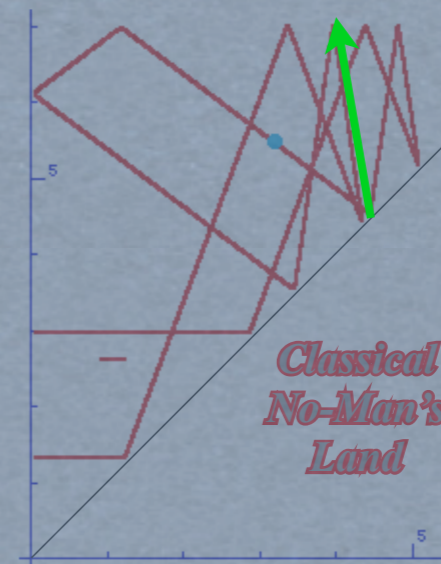
Simulations by *BounceIt*



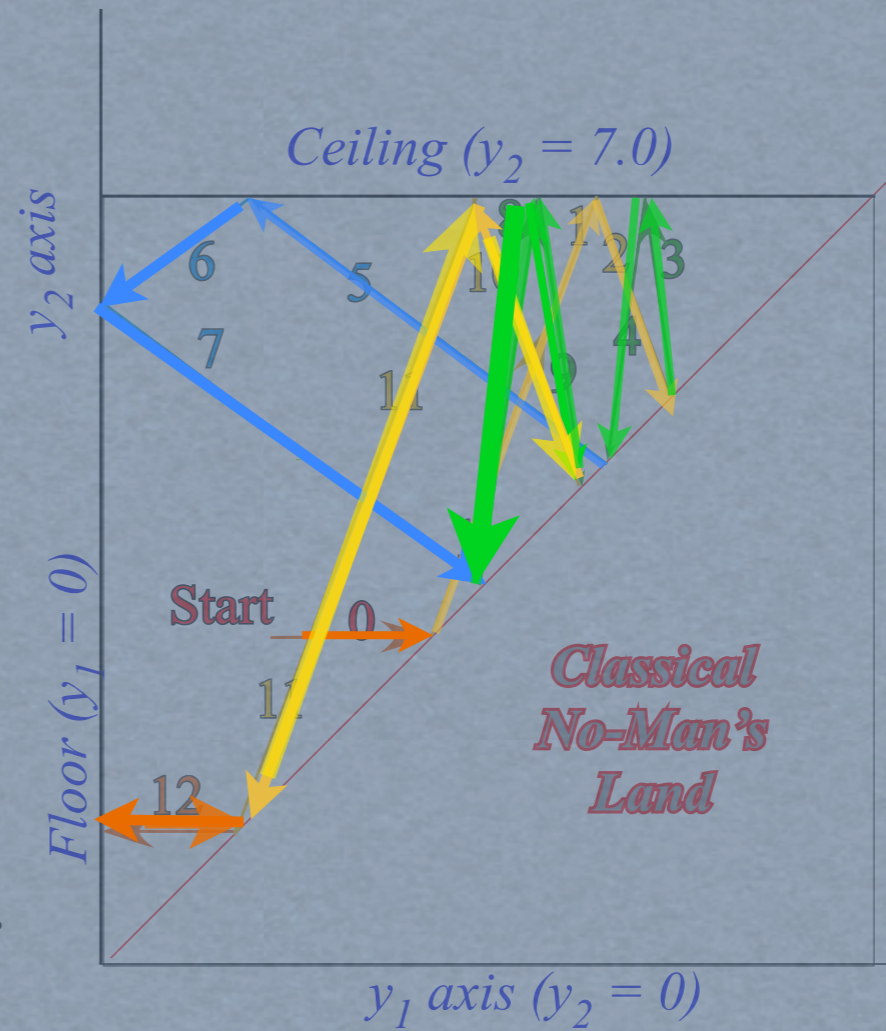
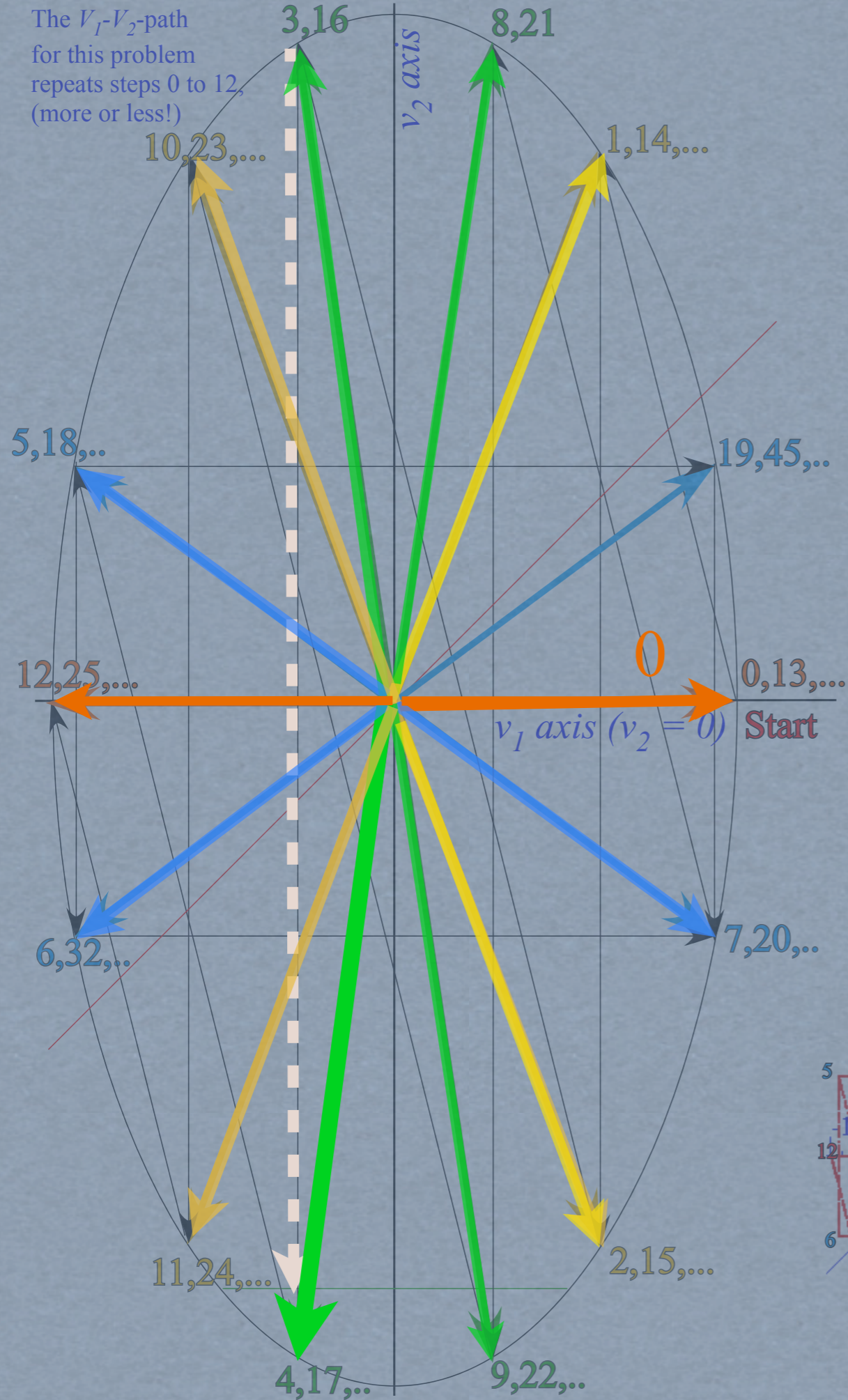
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



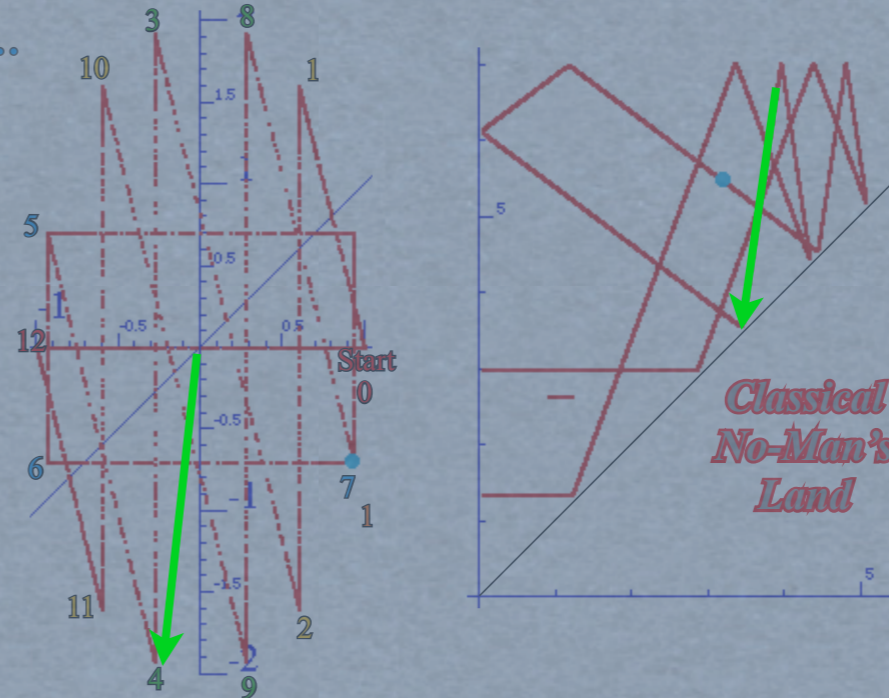
Simulations by *BounceIt*



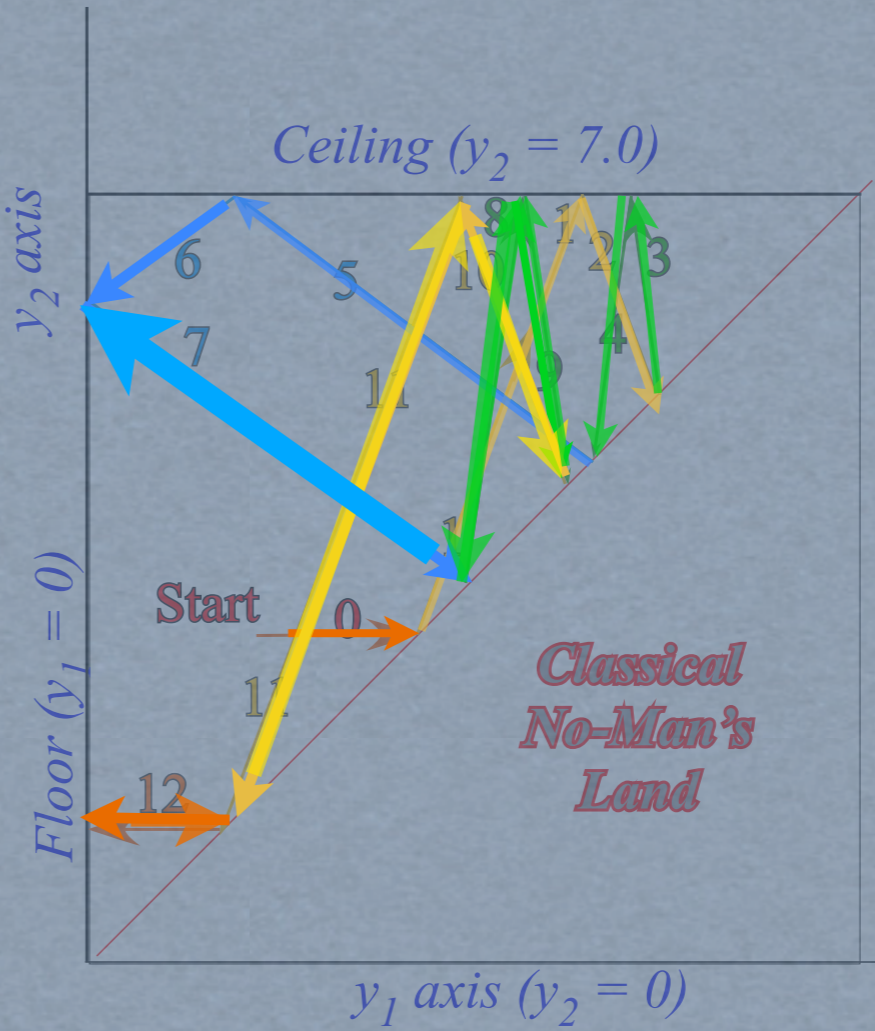
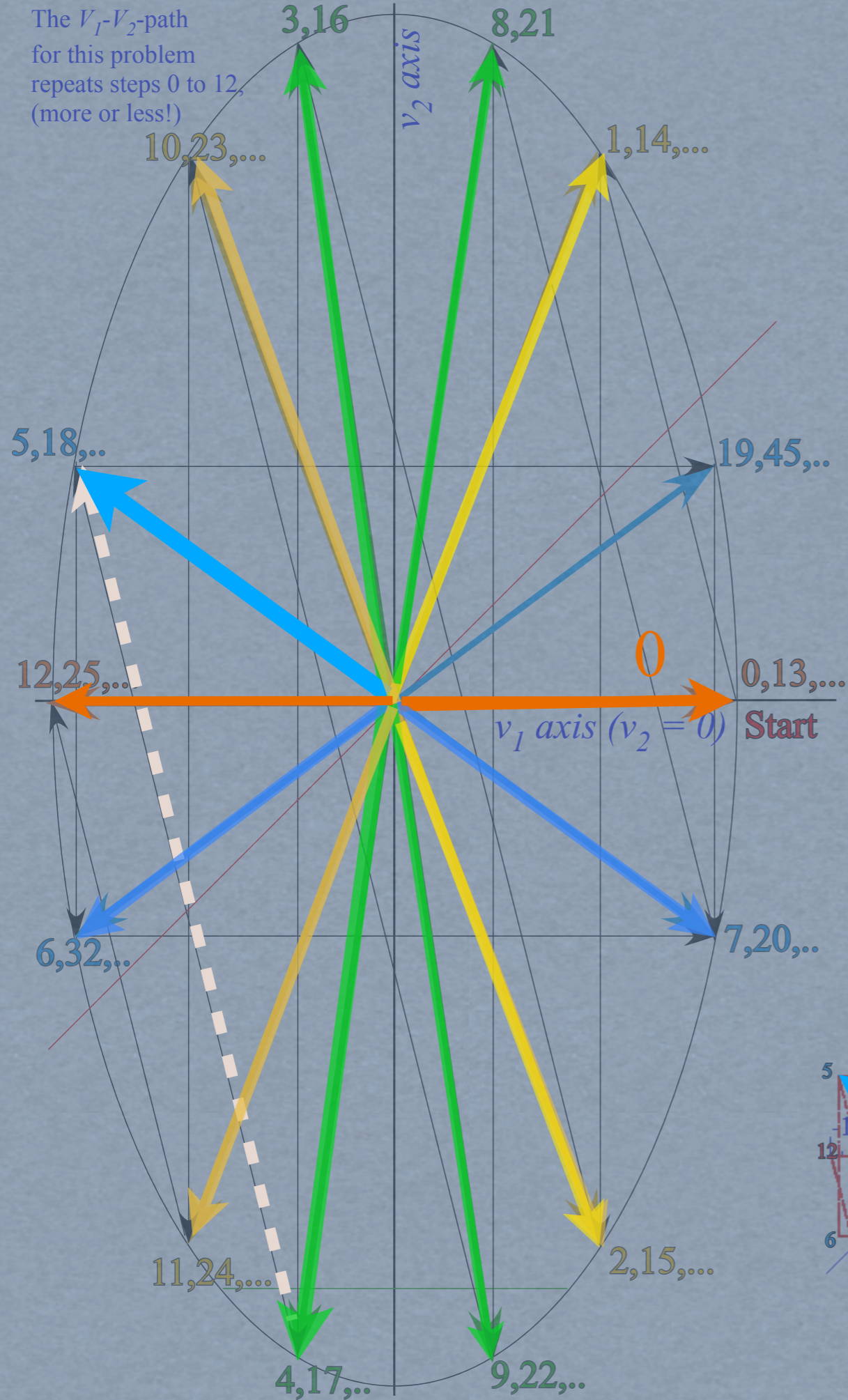
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



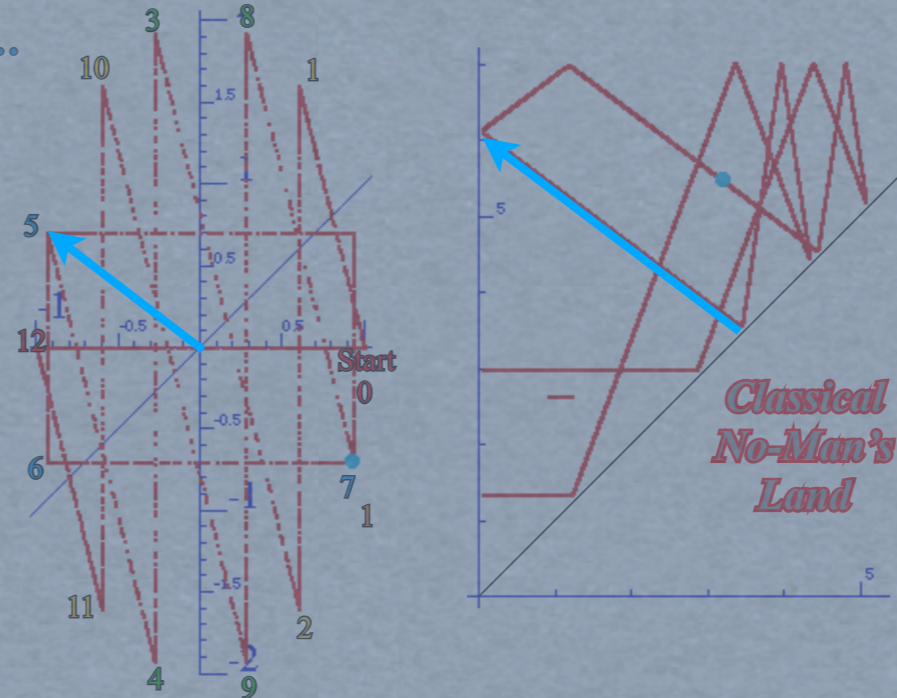
Simulations by BounceIt



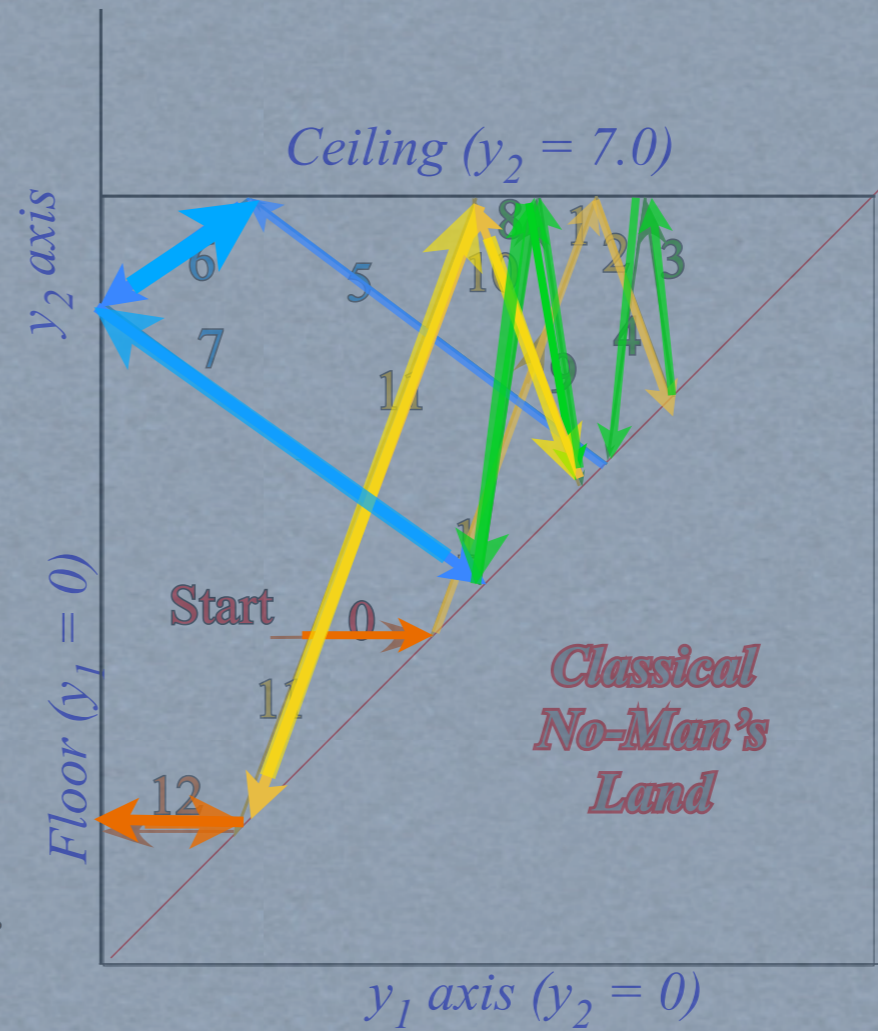
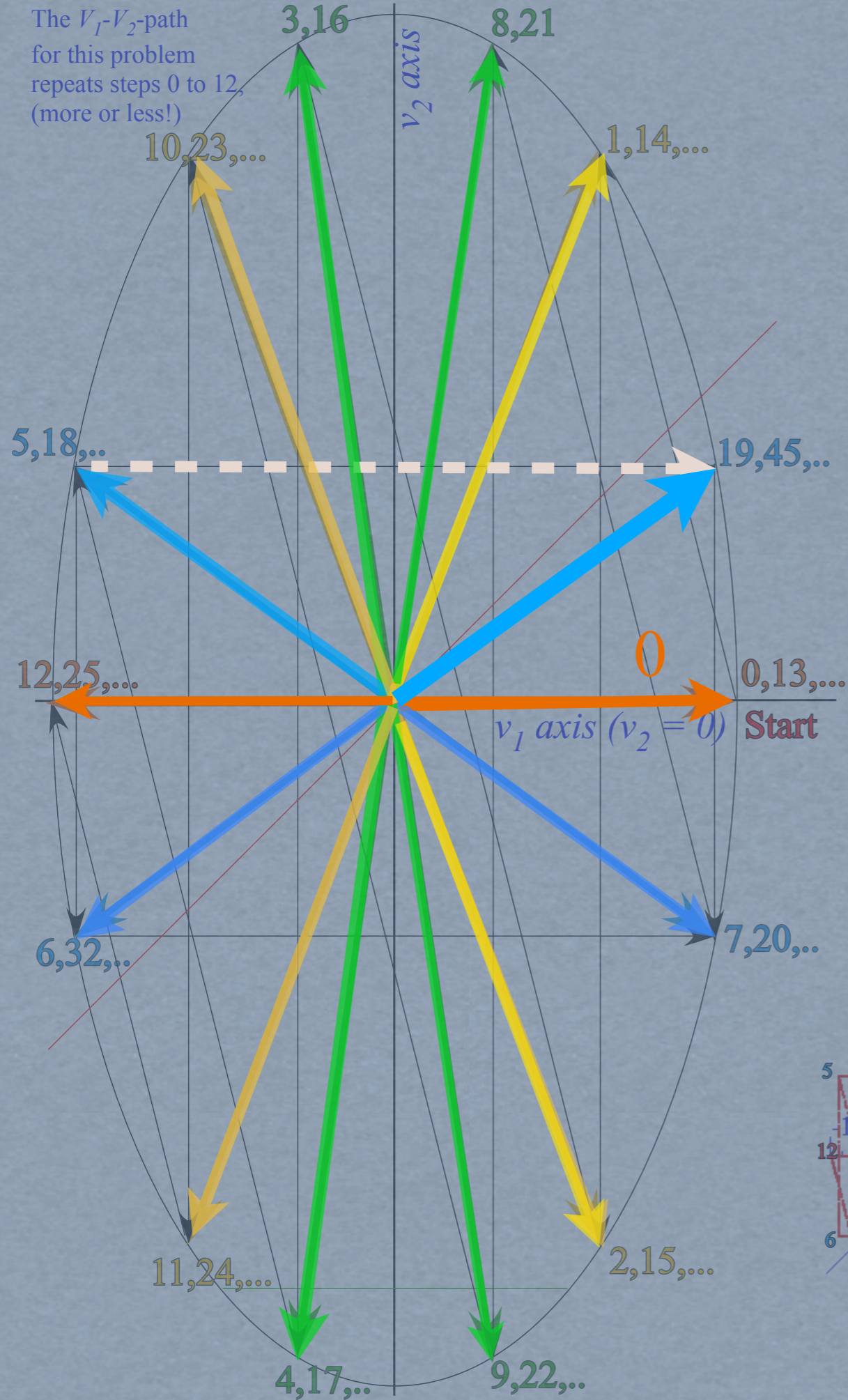
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



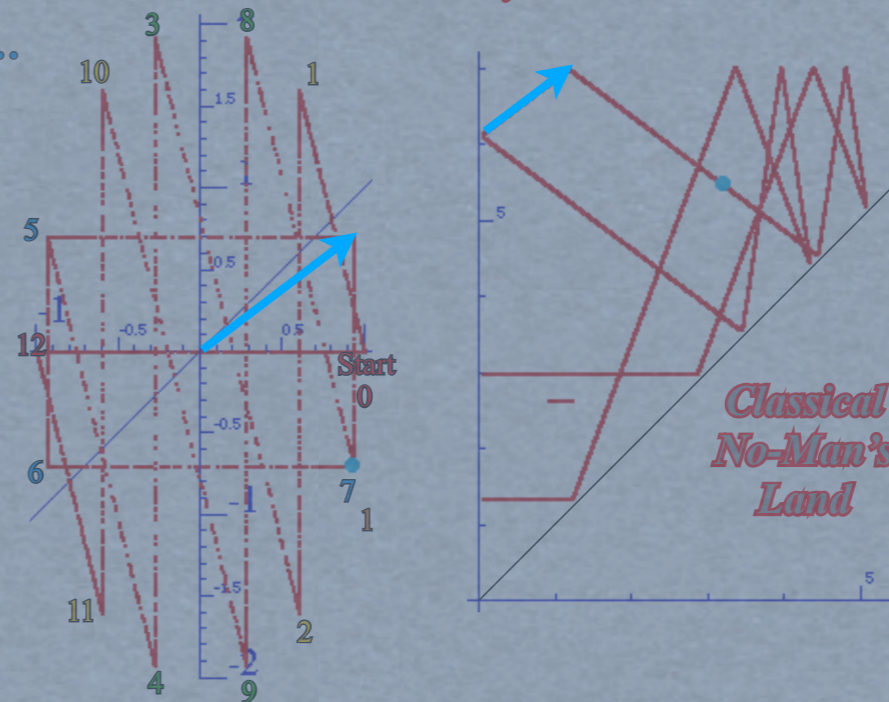
Simulations by *BounceIt*



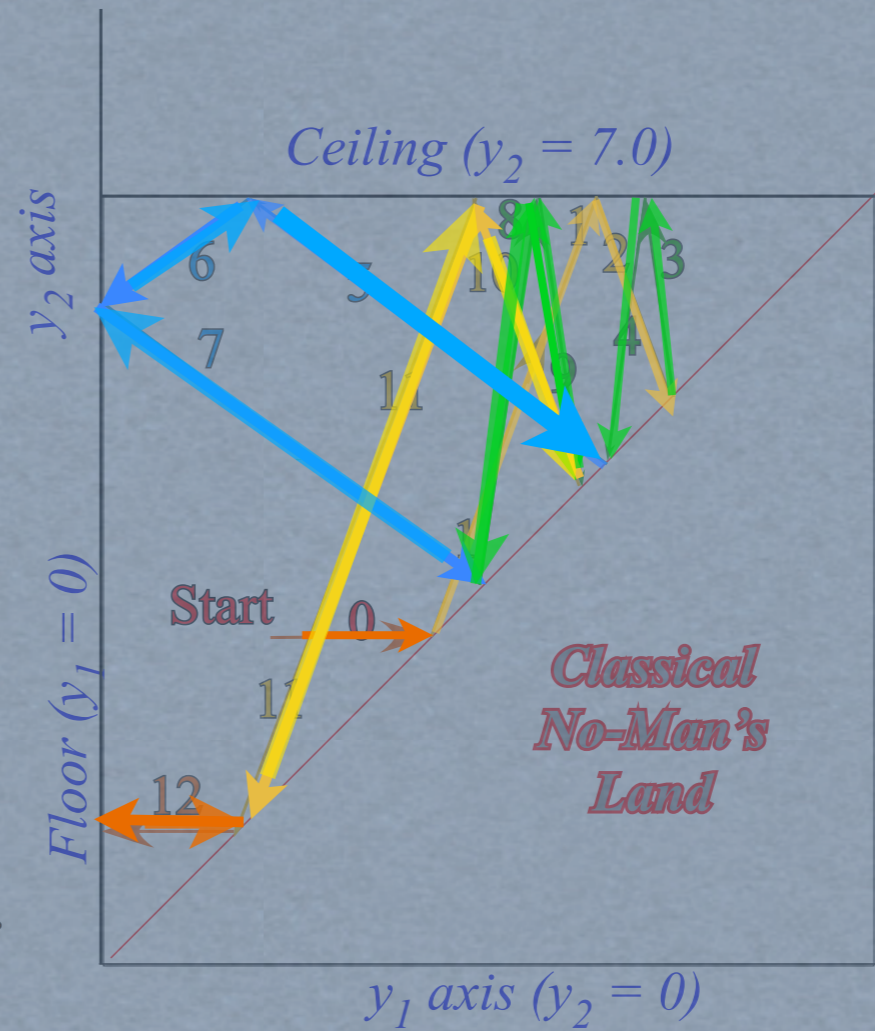
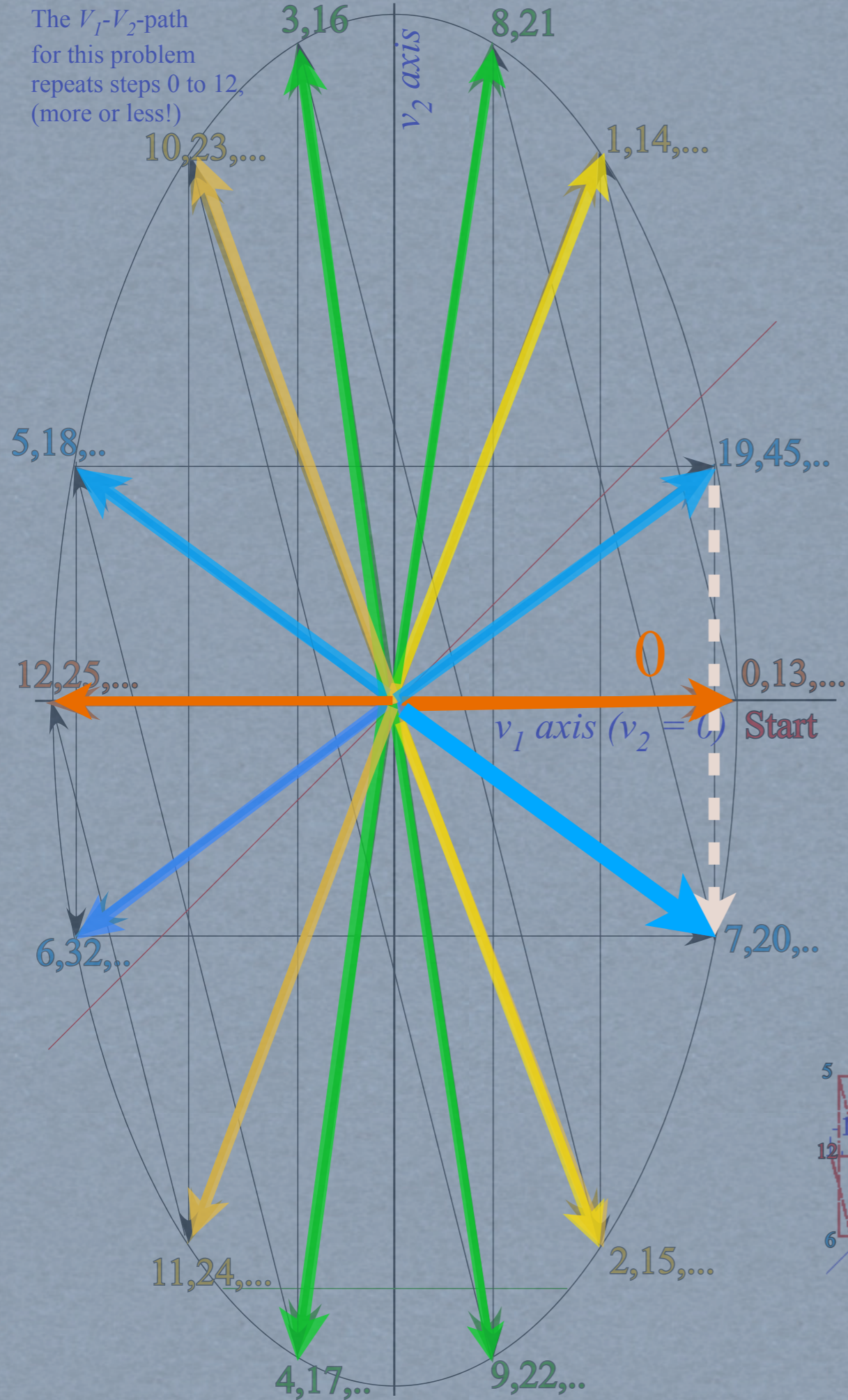
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



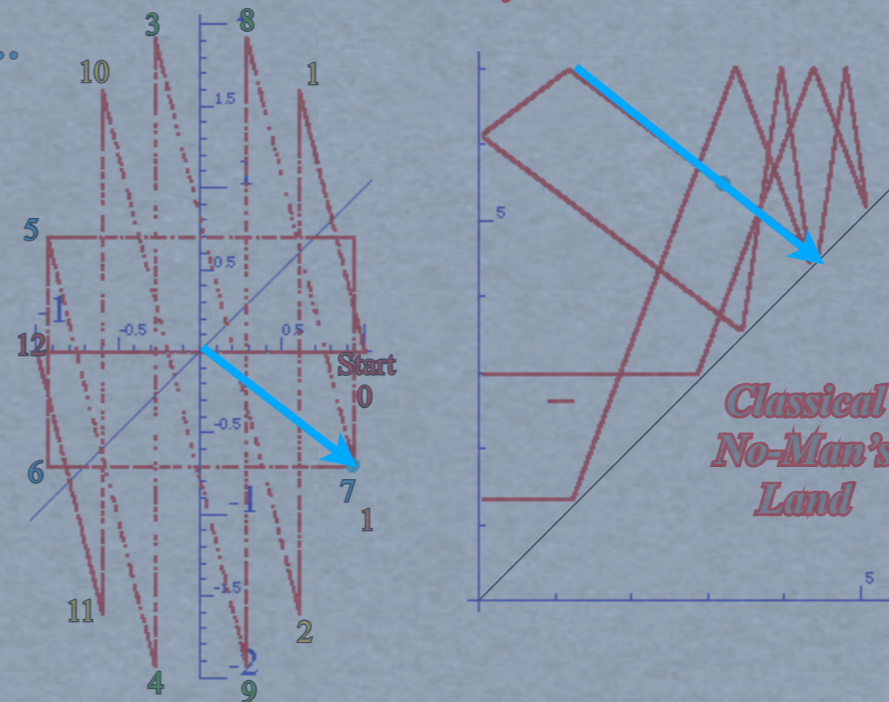
Simulations by *BounceIt*



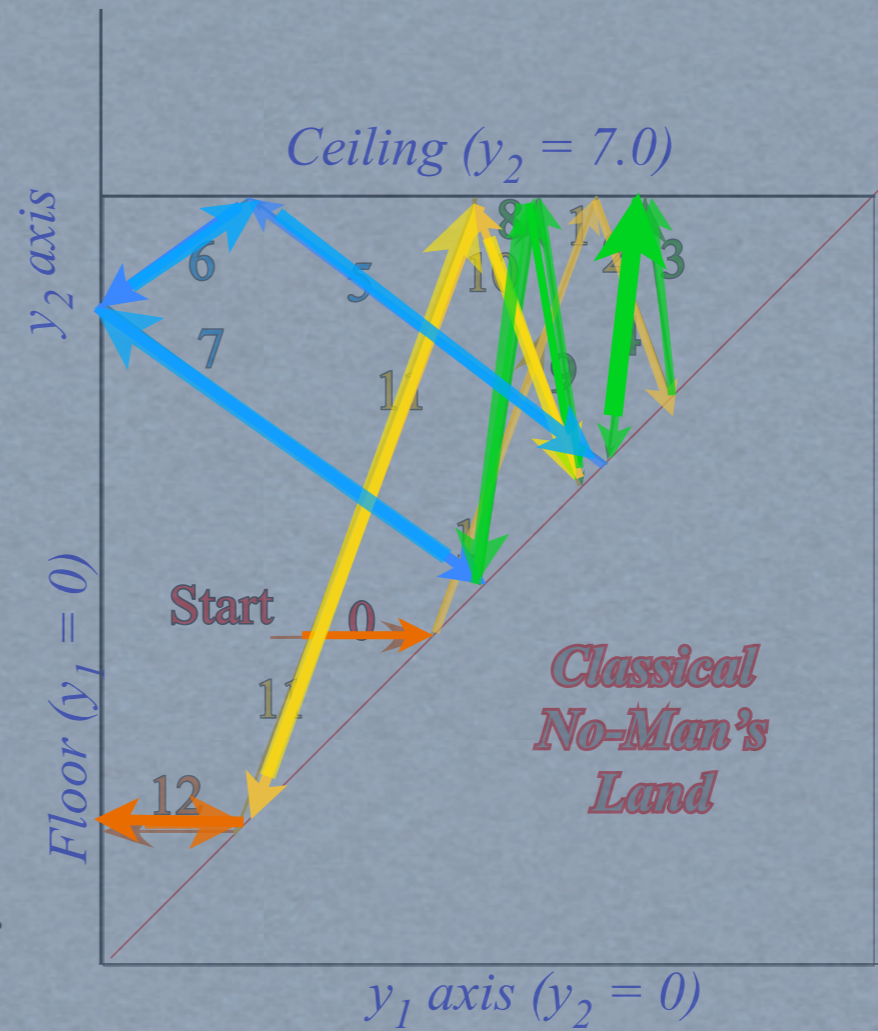
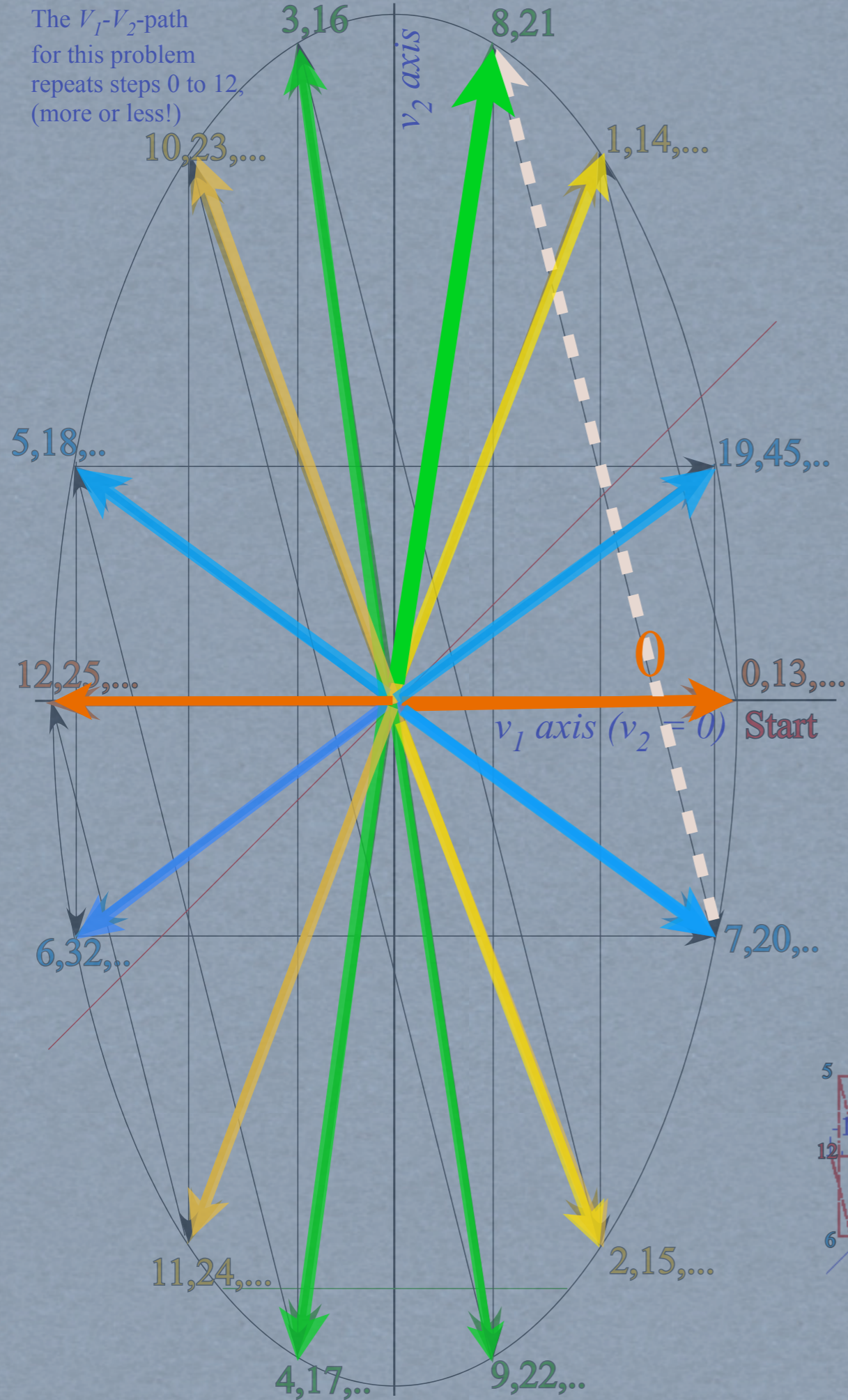
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



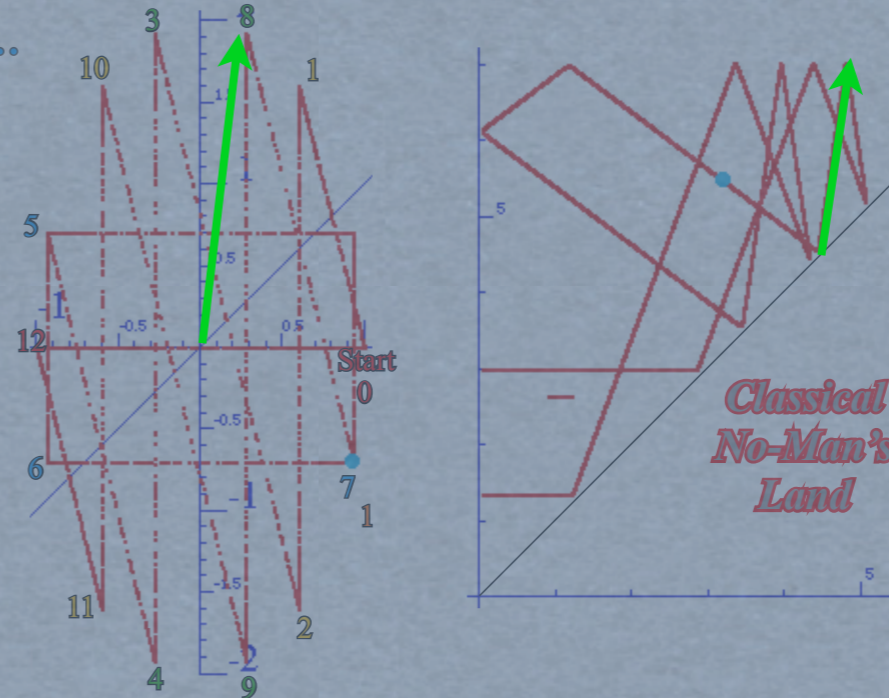
Simulations by *BounceIt*



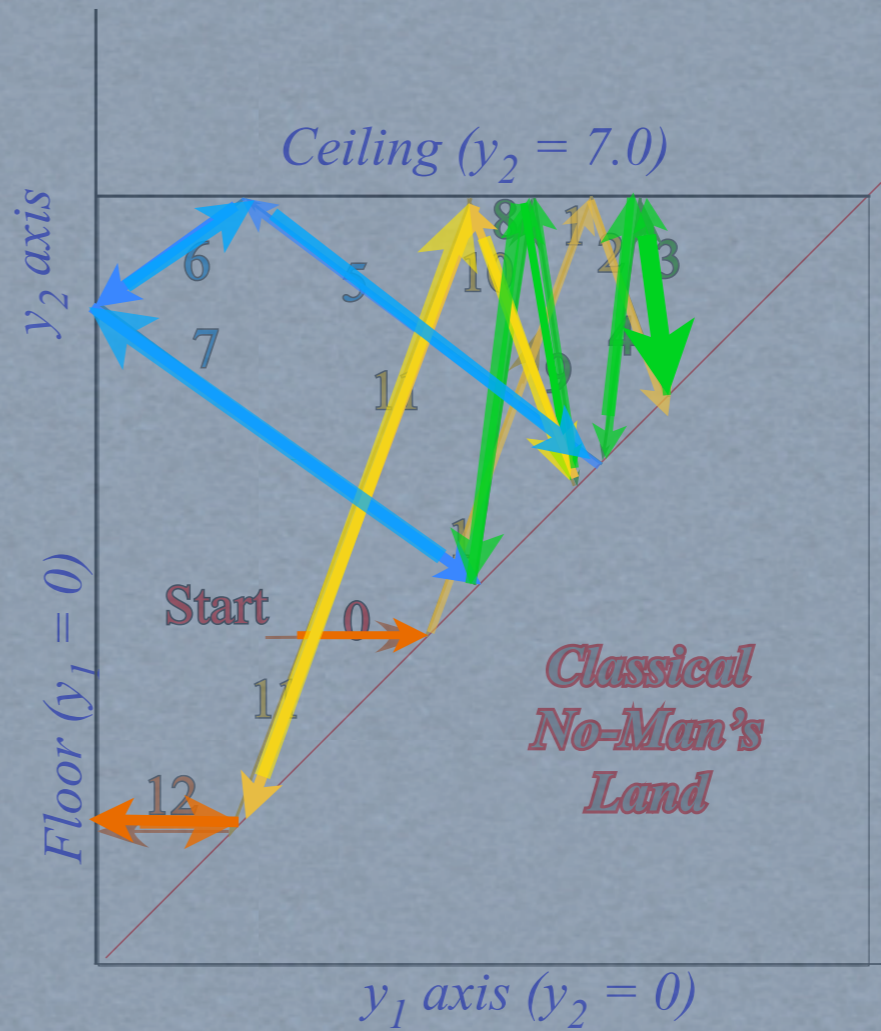
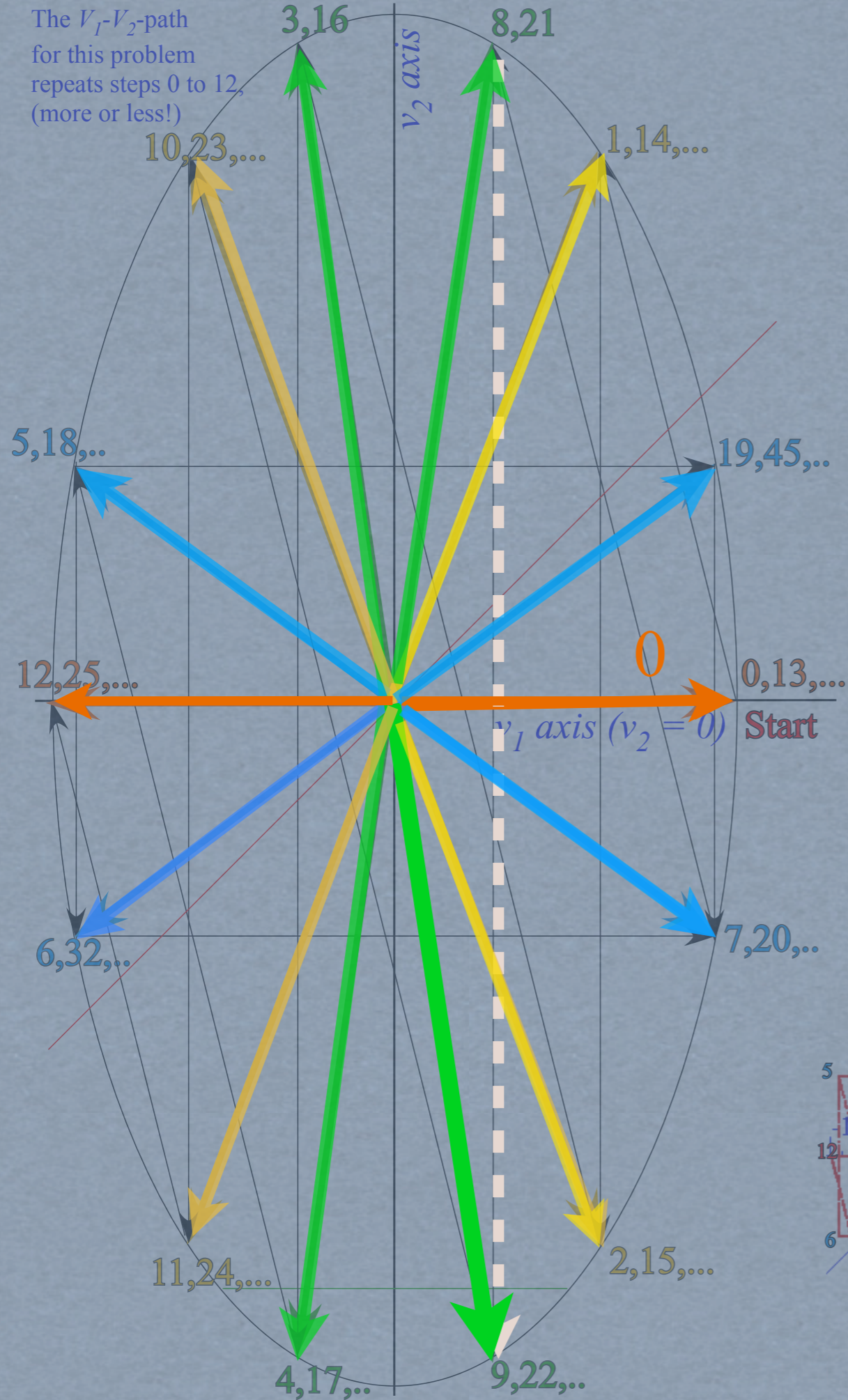
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



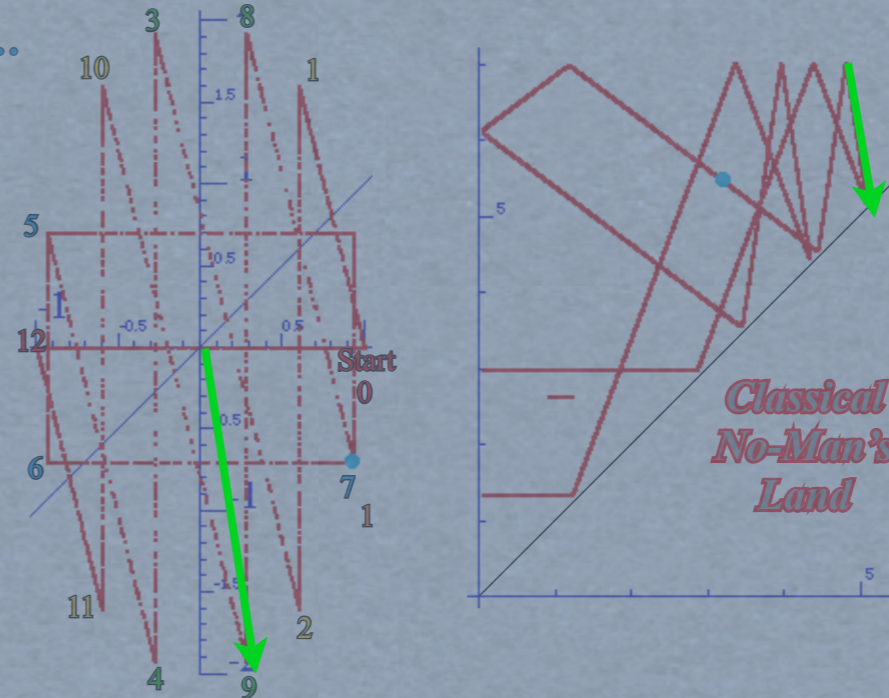
Simulations by *BounceIt*



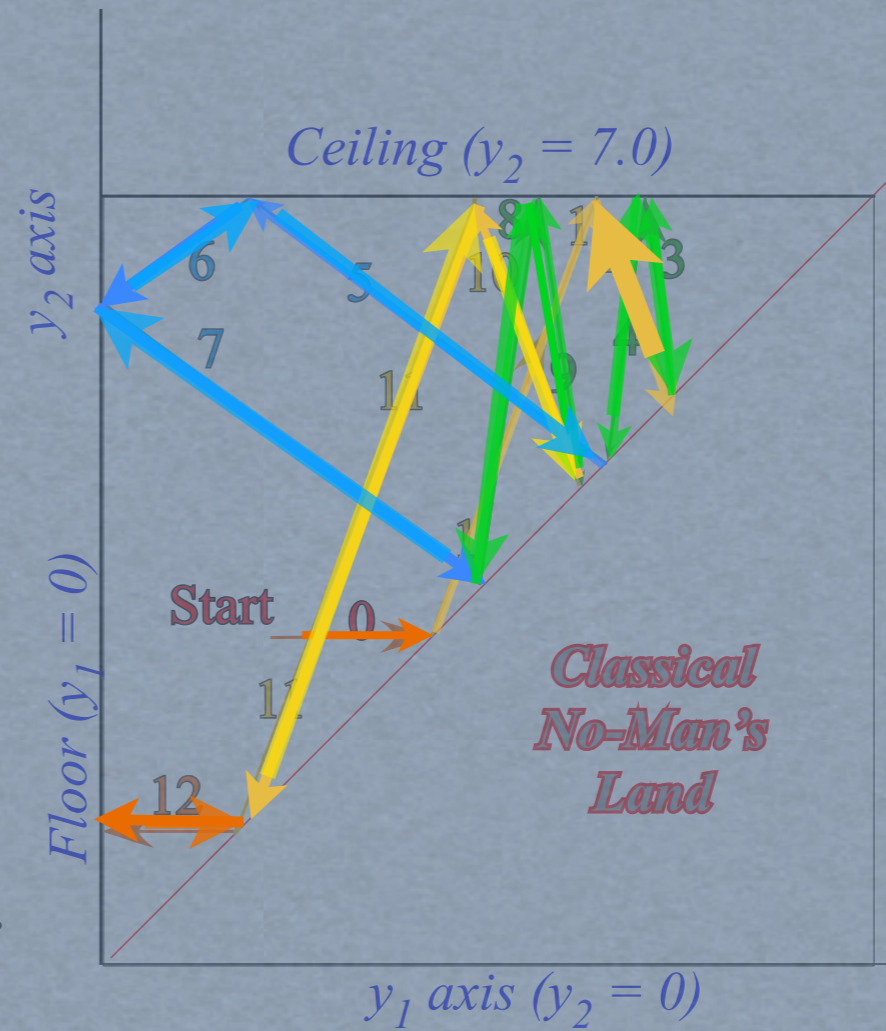
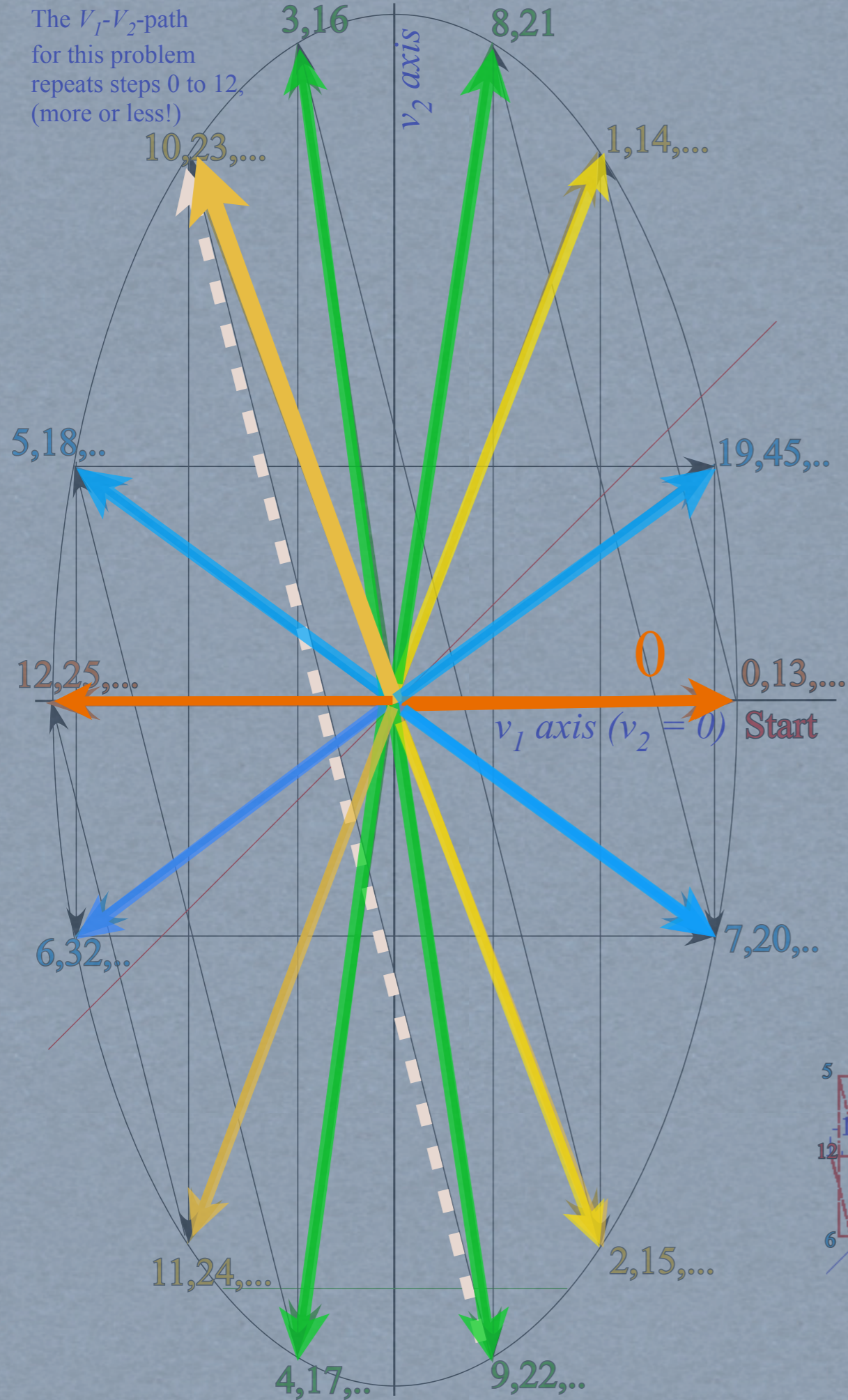
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



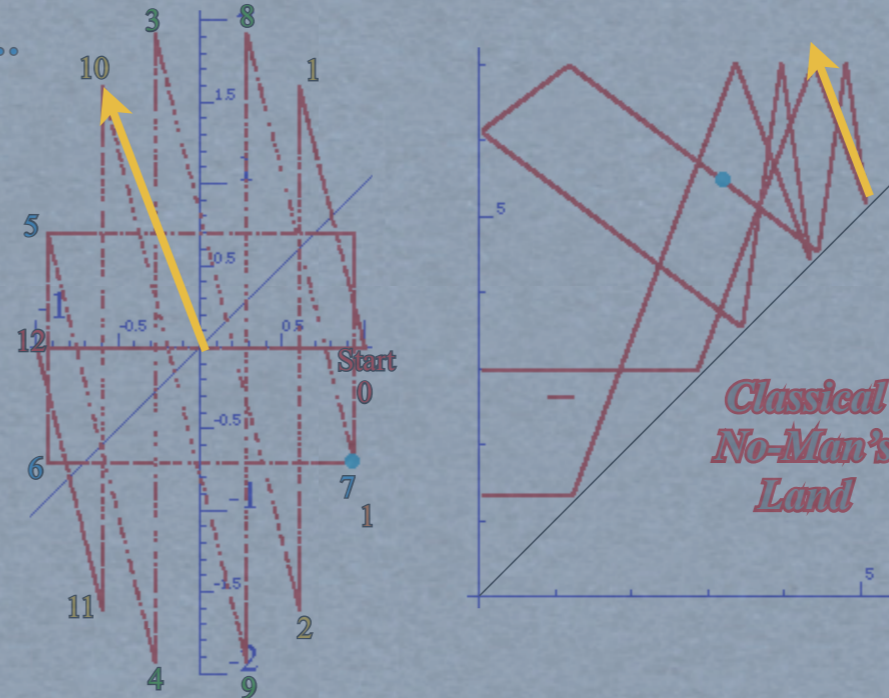
Simulations by *BounceIt*



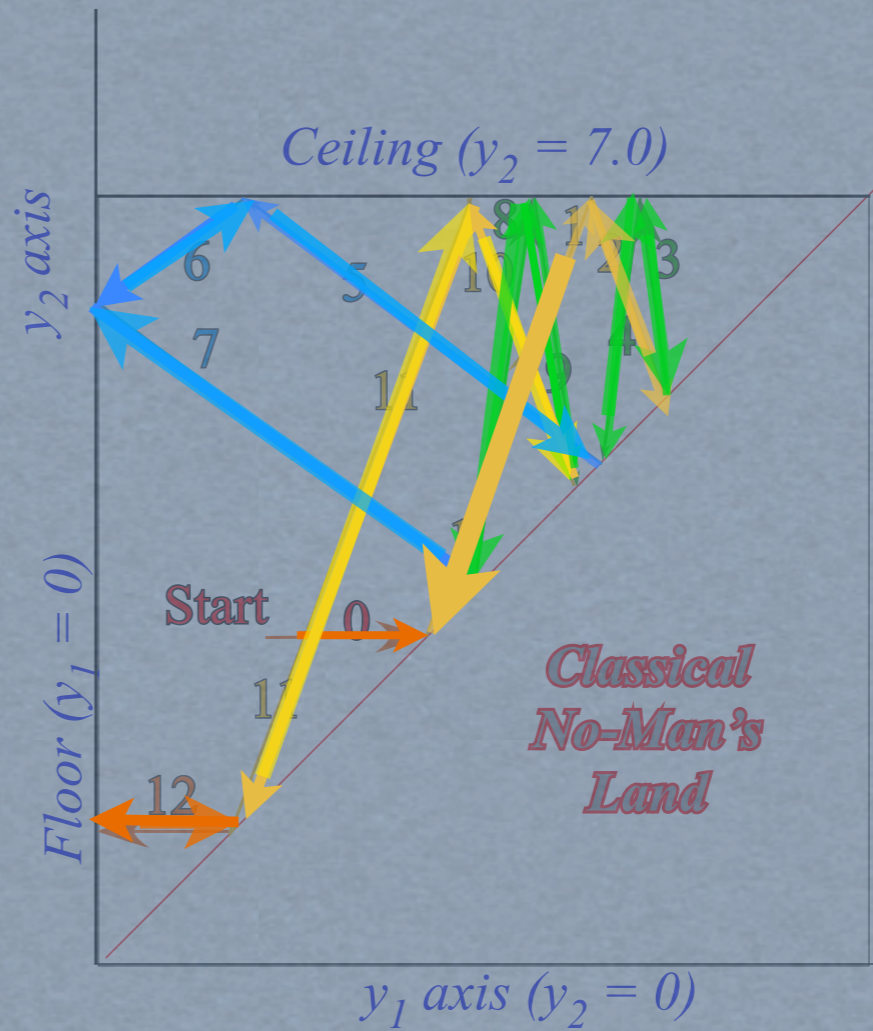
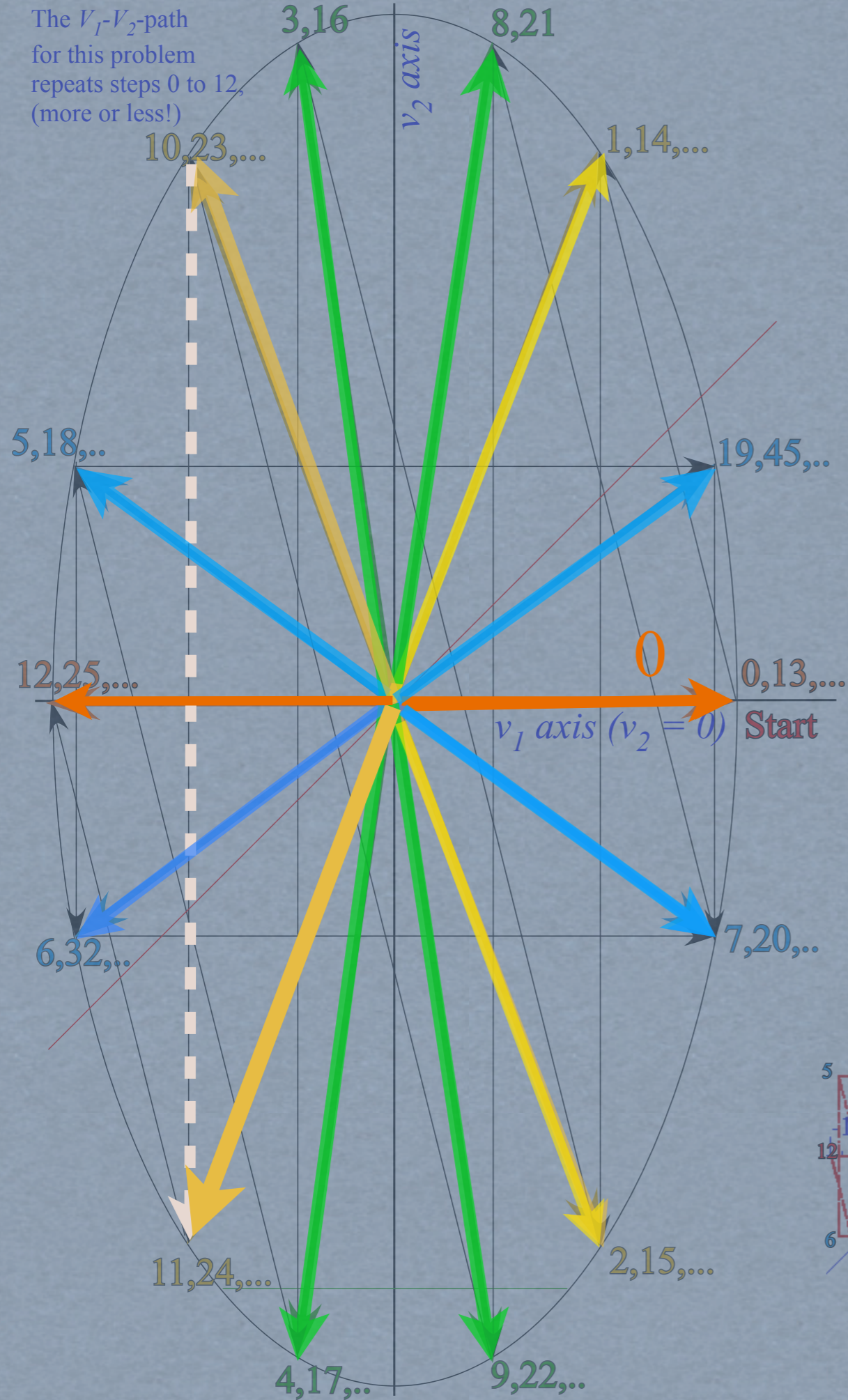
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



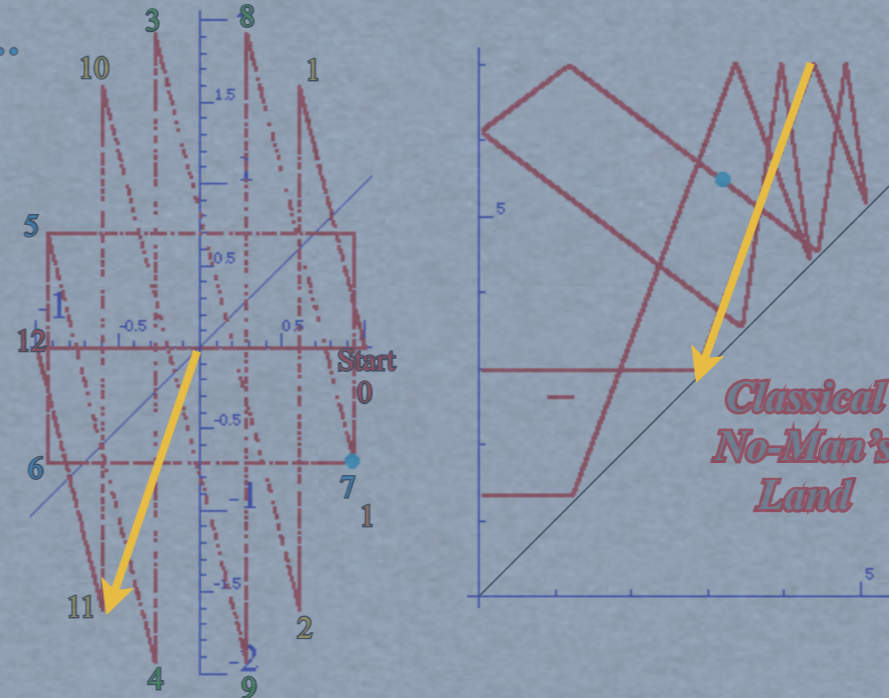
Simulations by *BounceIt*



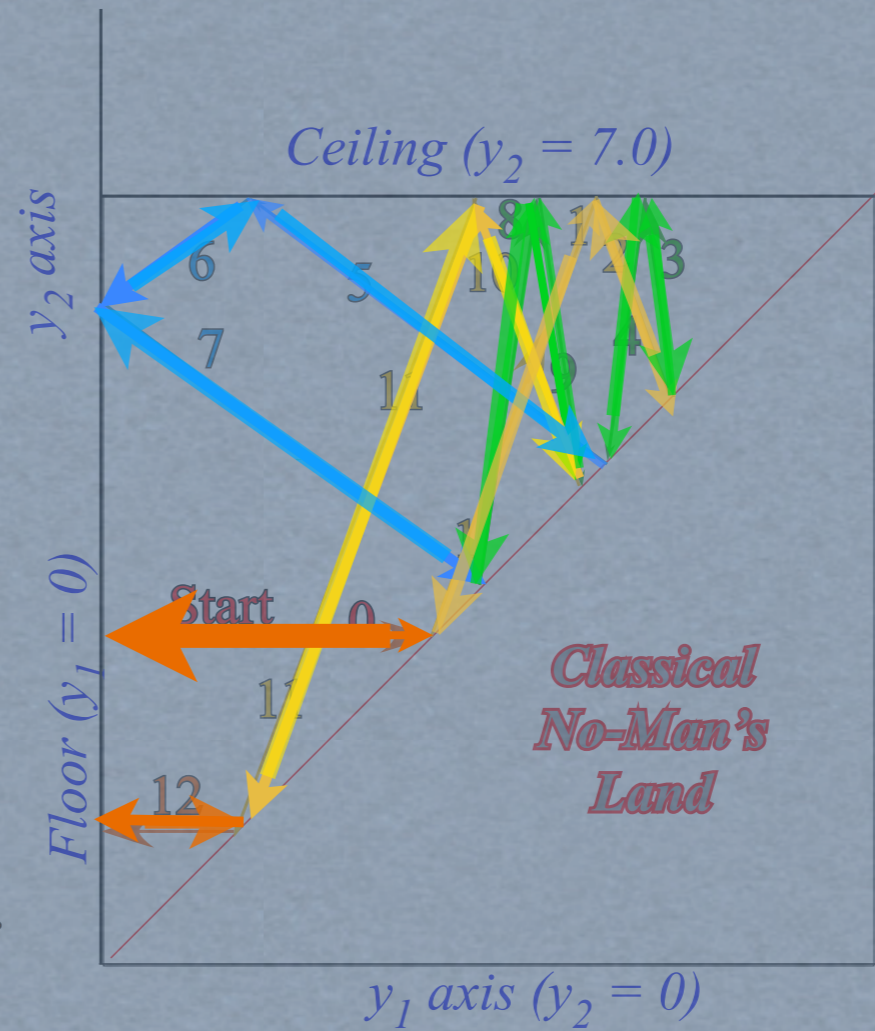
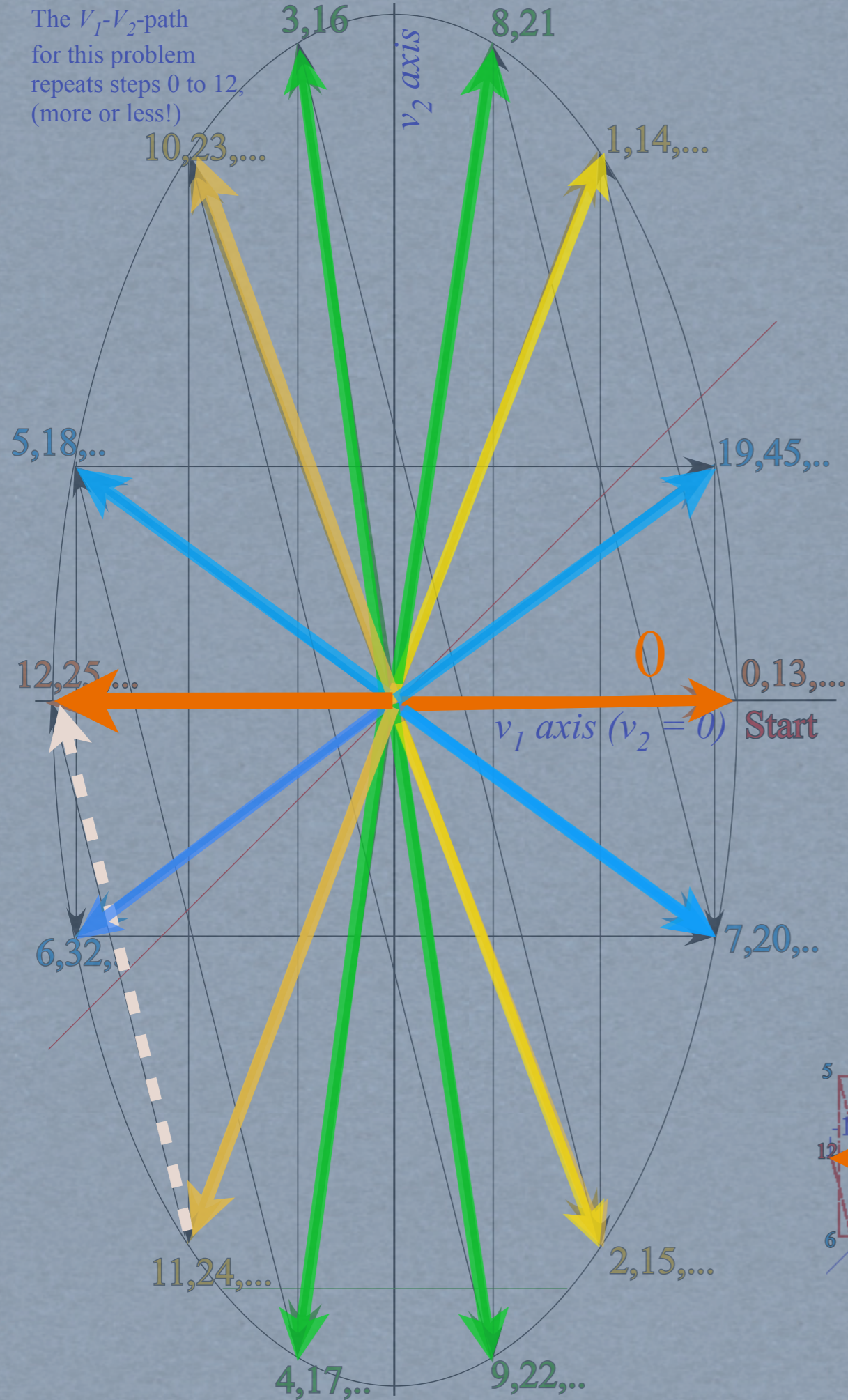
The V_1 - V_2 -path
for this problem
repeats steps 0 to 12,
(more or less!)



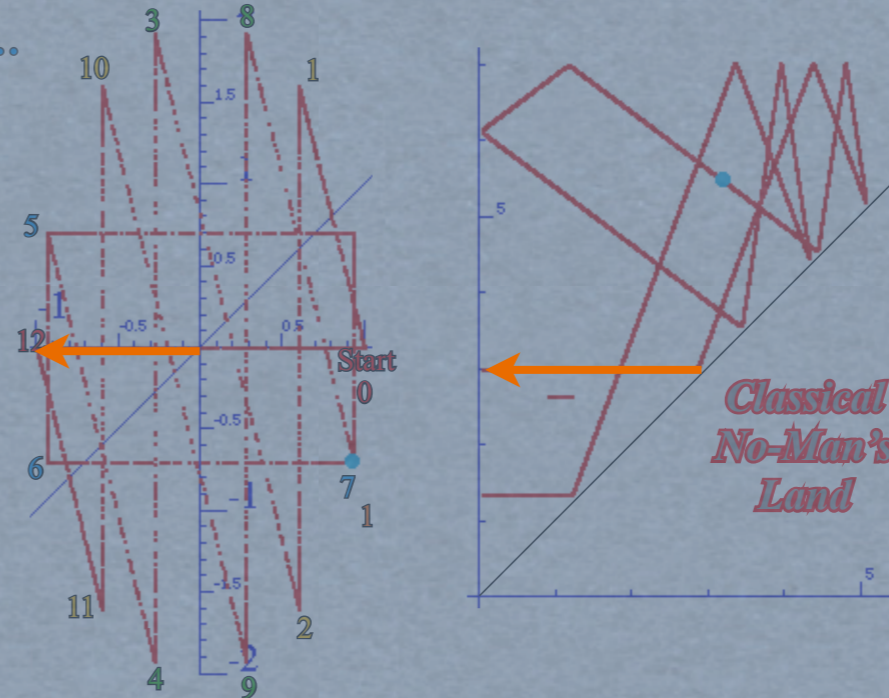
Simulations by *BounceIt*



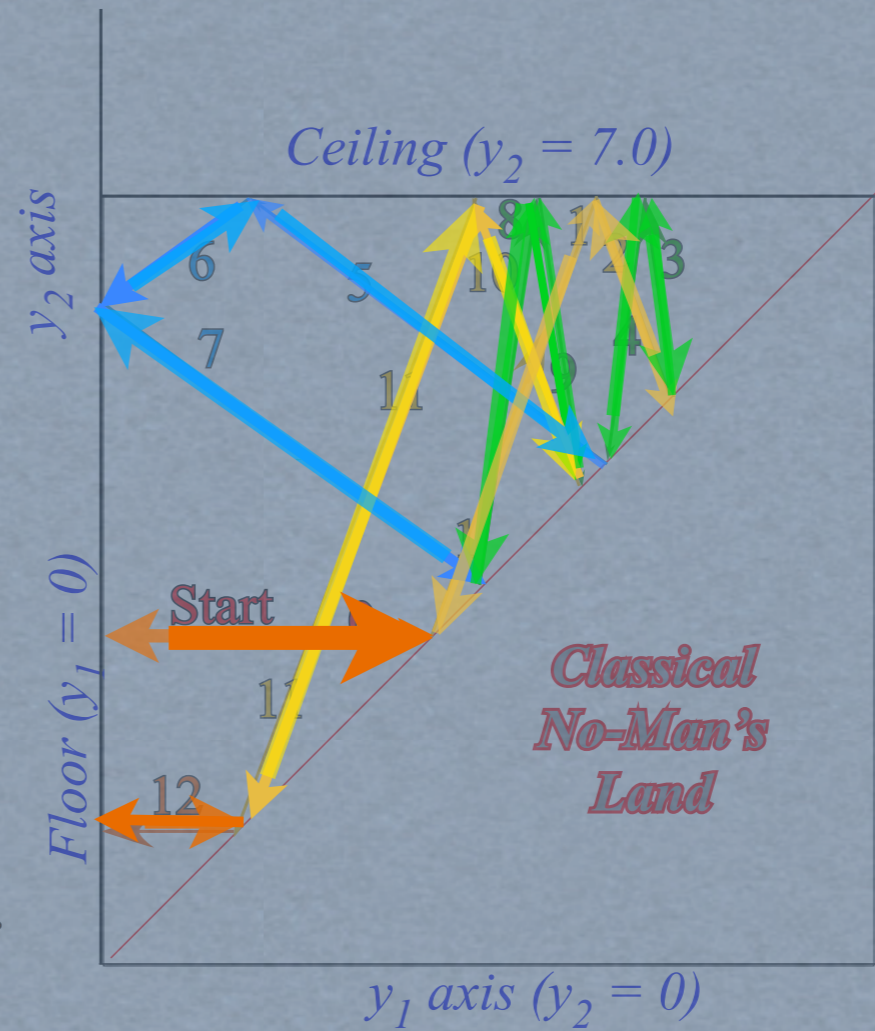
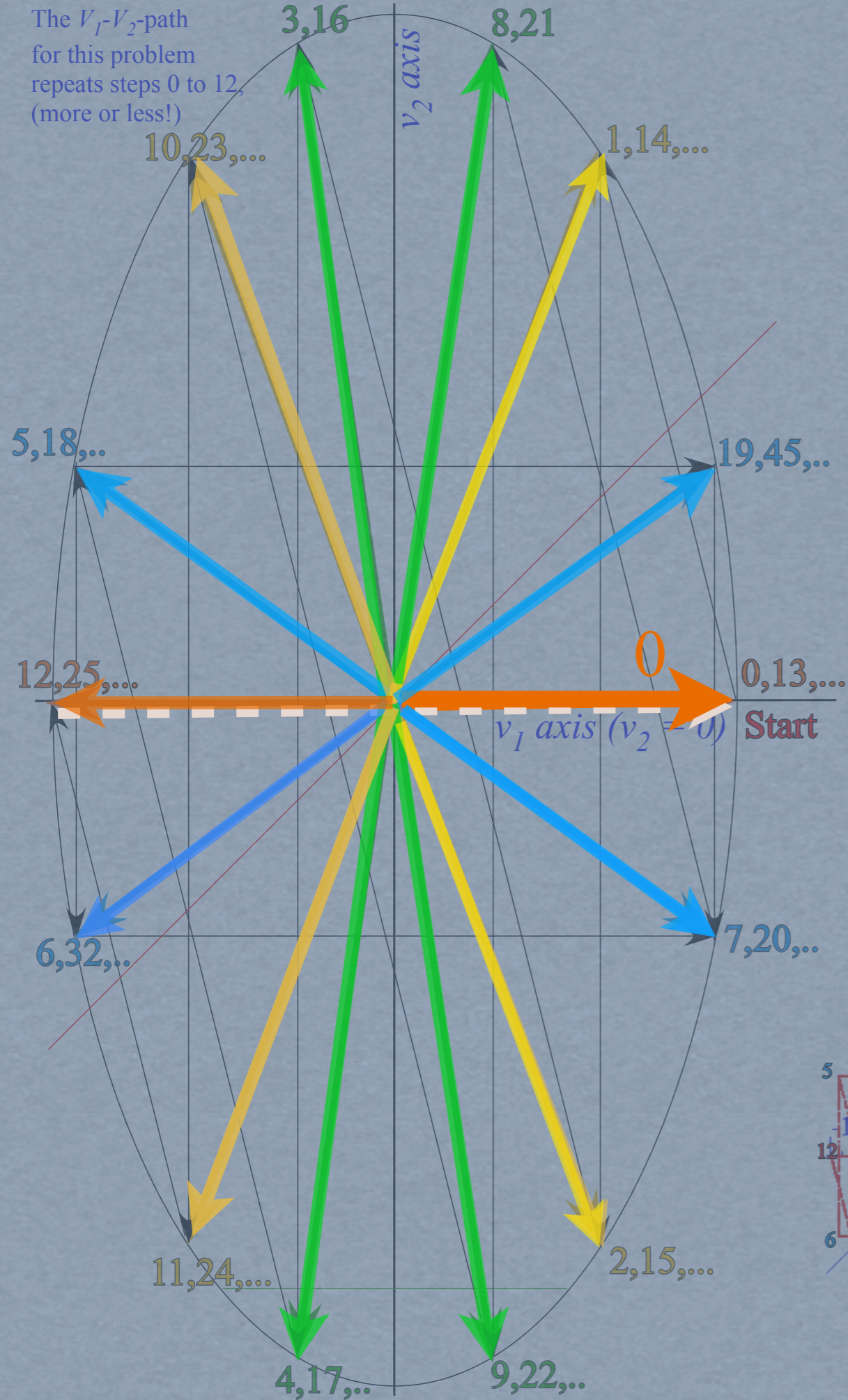
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



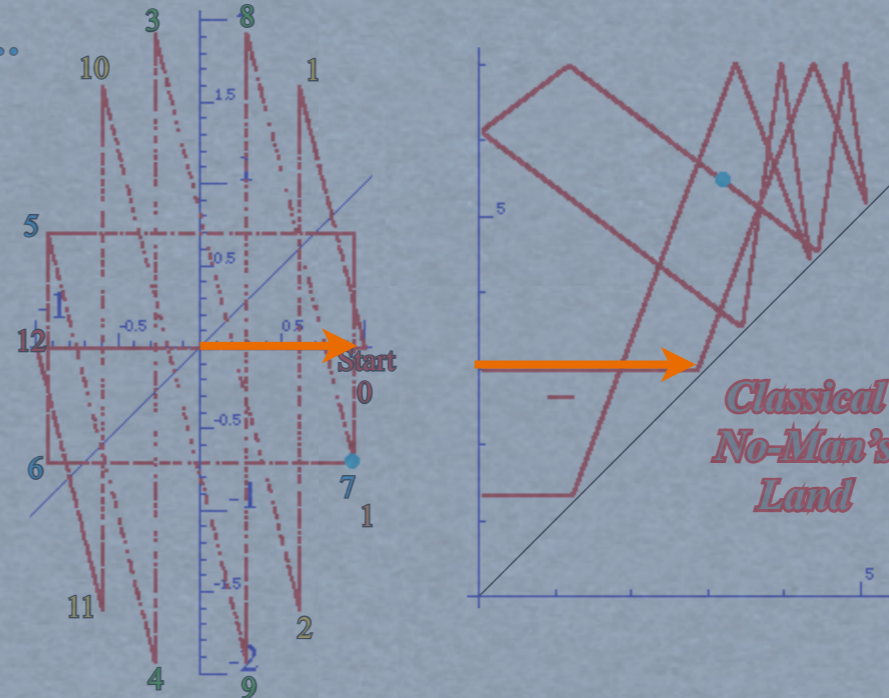
Simulations by *BounceIt*



The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by *BounceIt*



Estrangian plot of
 $m_1/m_2=4/1$
 collision
 sequence
 shows symmetry

(sort of)

c.o.m. lines
 (cons. of mom.)
 have slope
 $-\sqrt{m_2}/\sqrt{m_1}=-2/1$

COM line
 has slope
 $\sqrt{m_2}/\sqrt{m_1}=1/2$

