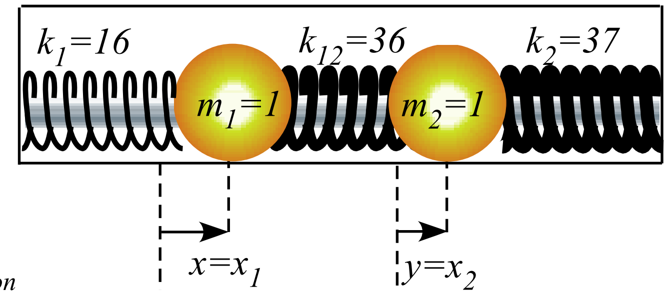
Assignment 12 - **Classical Mechanics 5103** 11/17/16

Main Reading: Classical Mechanics with a BANG! *Unit 4 Ch 4.3 thru Ch.4.4. and Ch. 4.8*

Due Wed. Nov. 23



Exercise 4.3.1 Coupled oscillation

Two identical mass $M=1kg$ blocks slide friction-free on a rod and are connected by springs $k_1=16N{\cdot}m^{-1}$ and $k_2=37N{\cdot}m^{-1}$ to ends of a box and coupled to each other by spring $k_{12}=36N{\cdot}m^{-1}$.

a. Write Lagrangian equations of motion and derive a **K**-matrix form of them.

b. Solve for eigenmodes and eigenfrequencies of system and plot their directions on an X,Y-graph. Use spectral decomposition methods of Appendix 4.C to derive eigensolution projectors and eigenvectors.

c. Given initial conditions $(X(0)=1, Y(0)=0)$, plot the resulting path in the XY-plane. Show algebraically that it is a parabola.

d. Use spectral decomposition (Appendix 4.C) to derive square-roots $\mathbf{H}=\sqrt{\mathbf{K}}$. (How many square-roots does **K** have?)

Exercise 4.4.1 U(2) view of coupled oscillation

a. Decompose the spring **K**-matrix for exercise 4.3.1 into an **H**-matrix where $\mathbf{K}=\mathbf{H}^2$ as in (4.4.8).

b. Give the resulting **H**-matrix as an *(A,B,C,D)* combination of **1**, σ_A, σ_B, and σ_C as in (4.4.9).

c. Sketch the resulting Ω-whirl vector or "crank" in real 3D *(A,B,C)*–space as in (4.4.10).

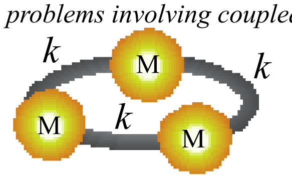
d. For $(X(0)=1, Y(0)=0)$ find initial **S**-state ("spin")vector in *(A,B,C)*–space as in (4.4.16). Show its evolution by Ω as in Fig. 4.4.2.

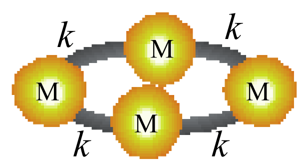
e. Plot **H**-eigenvalues $(\varepsilon_1, \varepsilon_2)$ as though they were energy levels and indicate transition rate $\Omega=\varepsilon_1-\varepsilon_2$ and mean rate $\omega=(\varepsilon_1-\varepsilon_2)/2$.

Exercise 4.4.2 U(2) B-Type coupled oscillation

Do exercises 4.3.1 and 4.4.1 for a system with two identical springs $k_1=4N{\cdot}m^{-1}=k_2$ and $M=1kg$ masses coupled to each other by spring $k_{12}=30N{\cdot}m^{-1}$. Show it is a *B*-type system. Does $(X(0)=1, Y(0)=0)$ also give a parabola? If not, what curve or function?

Possible Qualifying exam problems involving coupled oscillator





Exercise 4.8.1. Two's Company but Three's a Crowd

Three identical $M=1kg$ masses slide on a friction-free ring and are coupled by three identical $k=4N{\cdot}m^{-1}$ springs. (Left hand figure.)

a. Show that the resulting **K**-matrix can be written as a combination of three matrices that commute, satisfy $\mathbf{r}^{3m}=\mathbf{1}$, and therefore have 3rd-roots-of-unity eigenvalues $e^{im2\pi/3}=\{1, e^{i2\pi/3}, e^{-i2\pi/3}\}$.

$$\mathbf{r}^0=\mathbf{1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \mathbf{r}^1=\mathbf{r}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \mathbf{r}^2=\mathbf{r}\cdot\mathbf{r}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad \text{Note: } \begin{matrix}\mathbf{r}|1\rangle=|2\rangle\\ \mathbf{r}|2\rangle=|3\rangle\\ \mathbf{r}|3\rangle=|1\rangle\end{matrix}\ ,i.e.,\ \mathbf{r}\begin{pmatrix}1\\ \cdot\\ \cdot\end{pmatrix}=\begin{pmatrix}\cdot\\1\\ \cdot\end{pmatrix},\ \mathbf{r}^2\begin{pmatrix}1\\ \cdot\\ \cdot\end{pmatrix}=\mathbf{r}\begin{pmatrix}\cdot\\1\\ \cdot\end{pmatrix}=\begin{pmatrix}\cdot\\ \cdot\\1\end{pmatrix},\ \text{etc.}$$

b. Obtain a spectral decomposition of $\mathbf{r}^m$ and use it to get a **K**-matrix spectral decomposition, as well. Make a table of complex eigenmode phasors and their frequencies. Show how both modes and frequencies relate to the 3rd-roots $e^{im2\pi/3}$ and plot the **K**-eigenvalues versus the mode number *m*. (This is called a dispersion plot, particularly if it's done for **H**-matrix eigenvalues.)

c. Combining degenerate (equal-eigenvalue) complex eigenvector pairs to make pairs of eigenvectors with only real components.

d. Sketch the motions of each real eigenvector.

Exercise 4.8.2. ...and Four's a Mob

Do exercise 4.8.1 for four identical spring-*k*-coupled masses. (Right hand figure.)