Lecture 10

9.23.2016

Hamiltonian vs. Lagrange mechanics in Generalized Curvilinear Coordinates (GCC)

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lectures 8-9 procedures:

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

- Deriving Hamilton's equations from Lagrange's equations
- Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m
    - Polar-coordinate example of Hamilton's equations compared to Lagrange's
        - Hamilton's equations in Runga-Kutta (computer solution) form

Examples of Hamiltonian mechanics in effective potentials

- $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ in polar coordinates and effective potential (Web Simulation: OscillatorPE - IHO)
- Coulomb orbits in polar coordinates and effective potential (Web Simulation: OscillatorPE - Coulomb)

Examples of Hamiltonian mechanics in phase plots

- 1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vertically Driven Pendulum))
- 1D-HO phase-space control (Classic Simulation of "Catcher in the Eye", Web Simulation:JerkIt)

Optional (Most likely next Lecture 11):

Parabolic and 2D-IHO orbital envelopes

Quick Review of Lagrange Relations in Lectures 9-11

→ *0th and 1st equations of Lagrange and Hamilton and their geometric relations*

Quick Review of Lagrange Relations in Lectures 8-9

0th and 1st equations of Lagrange and Hamilton

p. 25 of Lecture 8

Starts out with simple demands for explicit-dependence, "loyalty" or "fealty to the colors"

Lagrangian** and **Estrangian** have no explicit dependence on **momentum p

$$\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}$$

Hamiltonian** and **Estrangian** have no explicit dependence on **velocity v

$$\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}$$

Lagrangian** and **Hamiltonian** have no explicit dependence on **speedinum V

$$\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}$$

Such non-dependencies hold in spite of "under-the-table" matrix and partial-differential connections

$$\nabla_v L = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}}{2}$$

$$= \mathbf{M}\bullet\mathbf{v} = \mathbf{p}$$

$$\begin{pmatrix} \frac{\partial L}{\partial v_1} \\ \frac{\partial L}{\partial v_2} \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Lagrange's 1st equation(s)

$$\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$$

$$\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \frac{\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}}{2}$$

$$= \mathbf{M}^{-1}\bullet\mathbf{p} = \mathbf{v}$$

$$\begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hamilton's 1st equation(s)

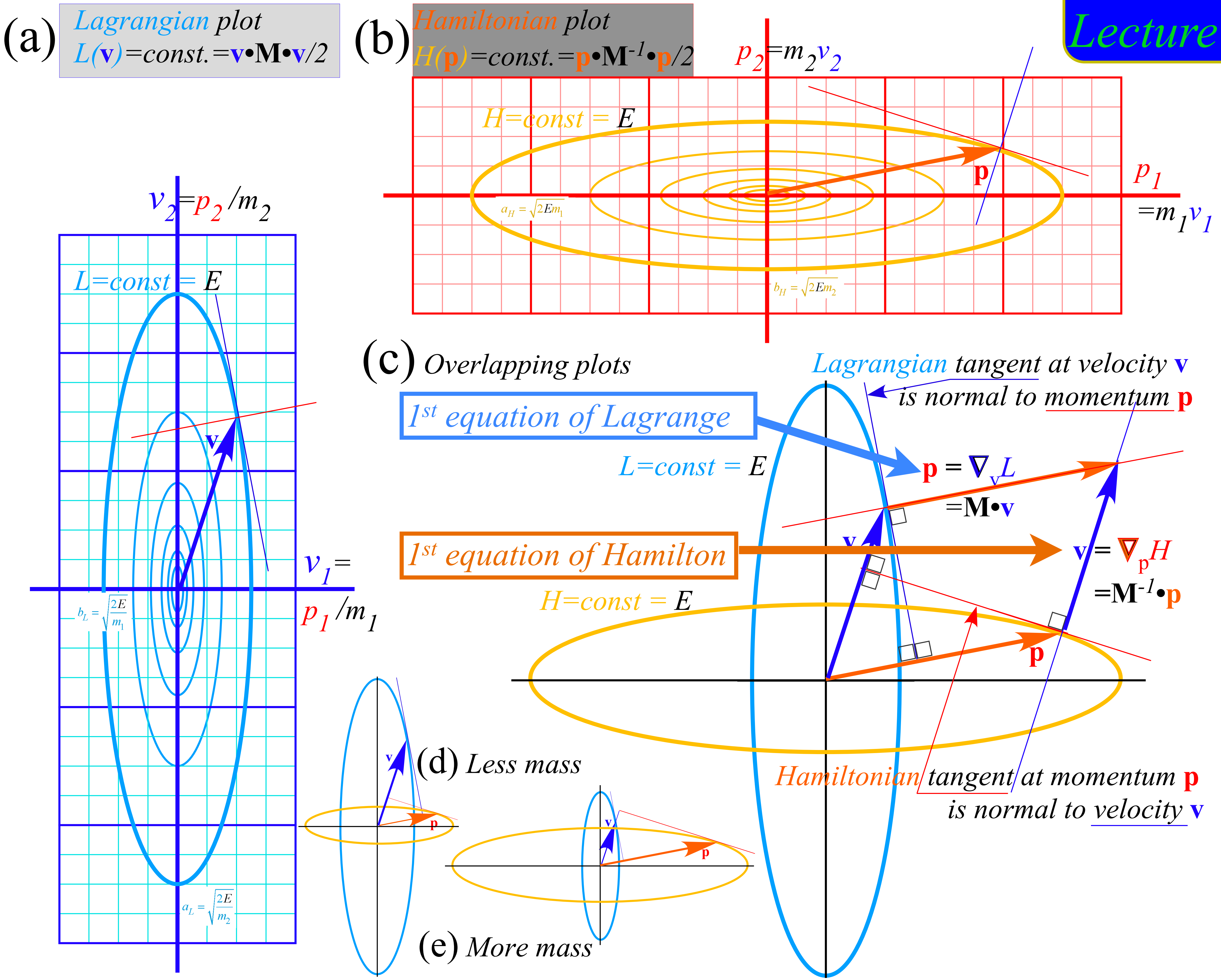
$$\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}$$

Estrangian is neglected for now.
(It is related to dual ellipse geometry
in Lecture 8 p. 71-79 and 99-101)

[†] non-dependency due to stationary-value effects as shown on p. 28-31

Unit 1
Fig. 12.2

p. 25 of Lecture 8



(a) *Lagrangian plot*
$L(\mathbf{v})=const.=\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}/2$

(b) *Hamiltonian plot*
$H(\mathbf{p})=const.=\mathbf{p}\bullet\mathbf{M}^{-1}\bullet\mathbf{p}/2$

(c) *Overlapping plots*

1st equation of Lagrange

$\mathbf{p} = \nabla_{\mathbf{v}} L$
$=\mathbf{M}\bullet\mathbf{v}$

1st equation of Hamilton

$\mathbf{v} = \nabla_{\mathbf{p}} H$
$=\mathbf{M}^{-1}\bullet\mathbf{p}$

Lagrangian tangent at velocity **v**
is normal to momentum **p**

Hamiltonian tangent at momentum **p**
is normal to velocity **v**

(d) *Less mass*

(e) *More mass*

(Review of Lecture 9)

Review of Lagrange Equations in Lecture 9

Lagrange prefers Covariant g_{mn} with Contravariant velocity $\dot{q}^m$

GCC Lagrangian definition

GCC "canonical" momentum p_m definition

→ GCC "canonical" force F_m definition

Coriolis "fictitious" forces (… and weather effects)

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

(Review of Lecture 9)

$$L(\mathbf{v})=\tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v}-U=\tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}}-U=\tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n)-U=\tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n)-U=L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)
$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix}=\begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi})=\tfrac{1}{2}M(g_{rr}\dot{r}^2+g_{\phi\phi}\dot{\phi}^2)-U(r,\phi)=\tfrac{1}{2}M(1\cdot\dot{r}^2+r^2\cdot\dot{\phi}^2)-U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r=\frac{\partial L}{\partial\dot{r}}=M\,g_{rr}\dot{r}=M\,\dot{r}$$

Nothing too surprising;
radial momentum p_r has the
usual linear $M\cdot v$ form

$$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=Mg_{\phi\phi}\dot{\phi}=Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia
factor Mr^2 automatically for the
angular momentum $p_\phi=Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r=\frac{\partial L}{\partial r}=\frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2-\frac{\partial U}{\partial r}=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal
force $Mr\omega^2$

$$\dot{p}_\phi=\frac{\partial L}{\partial\phi}=0-\frac{\partial U}{\partial\phi}$$

Angular momentum p_ϕ is conserved if
potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m\equiv\dfrac{dp_m}{dt}=\dfrac{d}{dt}M(g_{mn}\dot{q}^n)=M(\dot{g}_{mn}\dot{q}^n+g_{mn}\ddot{q}^n)$ *Equate it to $\dot{p}_m$ in 2nd L-equation:*

$$\dot{p}_r\equiv\frac{dp_r}{dt}=M\,\ddot{r}$$
$$=M\,r\,\dot{\phi}^2-\frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$\dot{p}_\phi\equiv\frac{dp_\phi}{dt}=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}$$
$$=0-\frac{\partial U}{\partial\phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if
potential U has no explicit ϕ-dependence

Rewriting GCC Lagrange equations :

(Review of Lecture 9)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$
$$= M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force
equals total
Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$
$$= 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts:
Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if
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Conventional forms

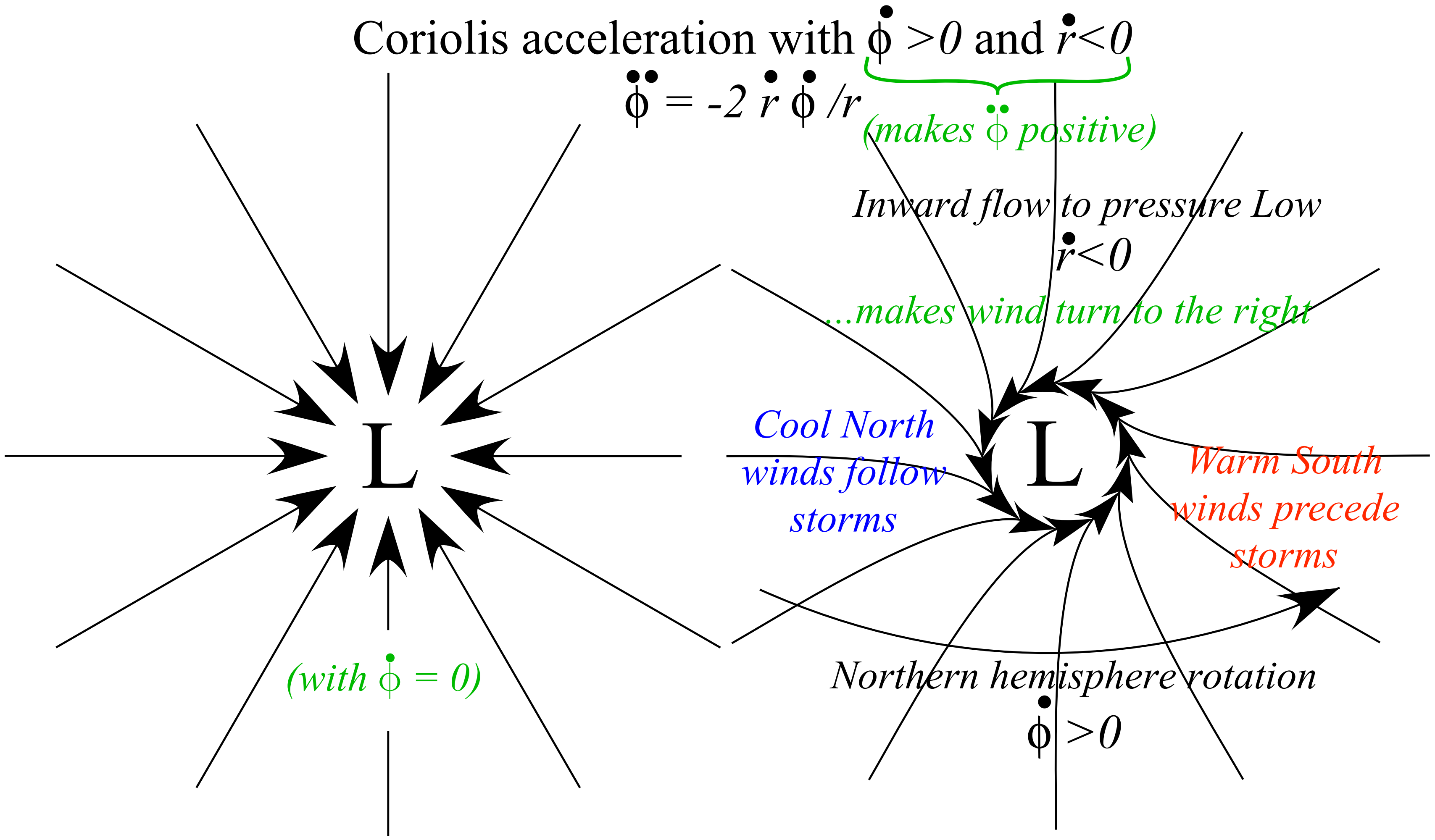
radial force: $M\,\ddot{r} = M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\,\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$



Coriolis acceleration with $\underbrace{\dot{\phi} > 0 \text{ and } \dot{r} < 0}$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}\,/r$

(makes $\ddot{\phi}$ positive)

Inward flow to pressure Low

$\dot{r} < 0$

...makes wind turn to the right

Cool North winds follow storms

Warm South winds precede storms

(with $\dot{\phi} = 0$)

Northern hemisphere rotation

$\dot{\phi} > 0$

Effect on
Northern
Hemisphere
local weather

Cyclonic flow
around lows

*Hamilton prefers Contravariant g^{mn} with Covariant **momentum** p_m*

→ *Deriving Hamilton's equations from Lagrange's equations*

Expressing Hamiltonian $H(p_m,q^n)$ using g^{mn} and covariant momentum p_m

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

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...of coordinates and velocity and time, too. (You can safely drop last chain-rule factor [1=dt/dt])

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}\frac{dt}{dt}$$

Deriving Hamilton's equations from Lagrangian theory

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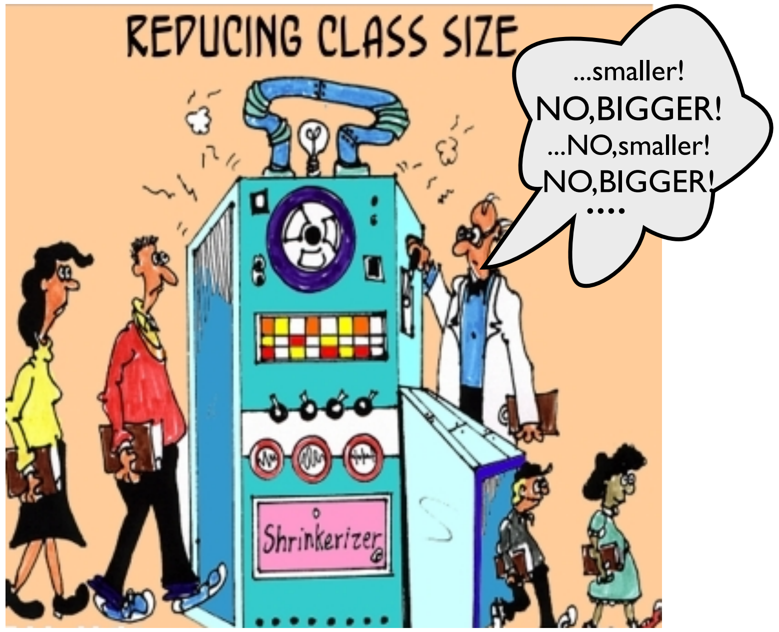
...of coordinates and velocity and <u>time</u>, too. (Imagine Mad Scientist turning U(t)-dial.)

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



Cartoonish way to imagine
explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

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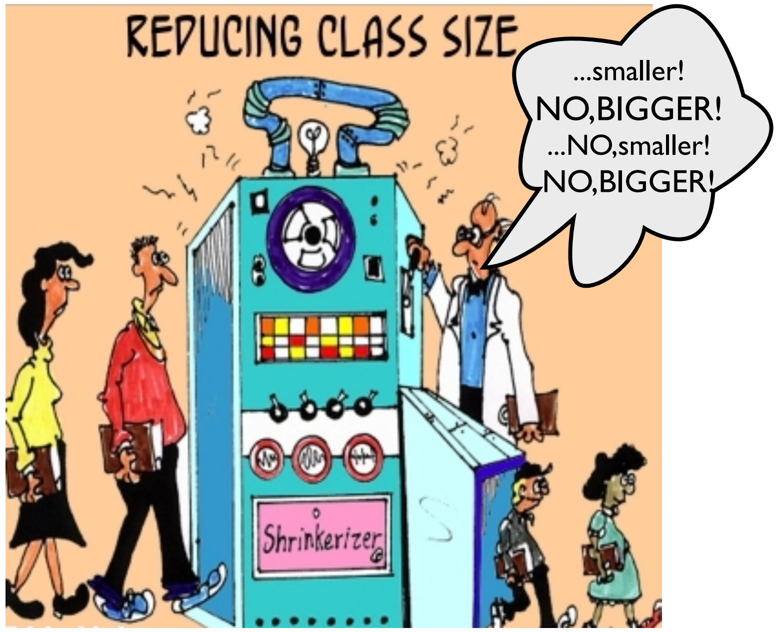
Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

Deriving Hamilton's equations from Lagrangian theory

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$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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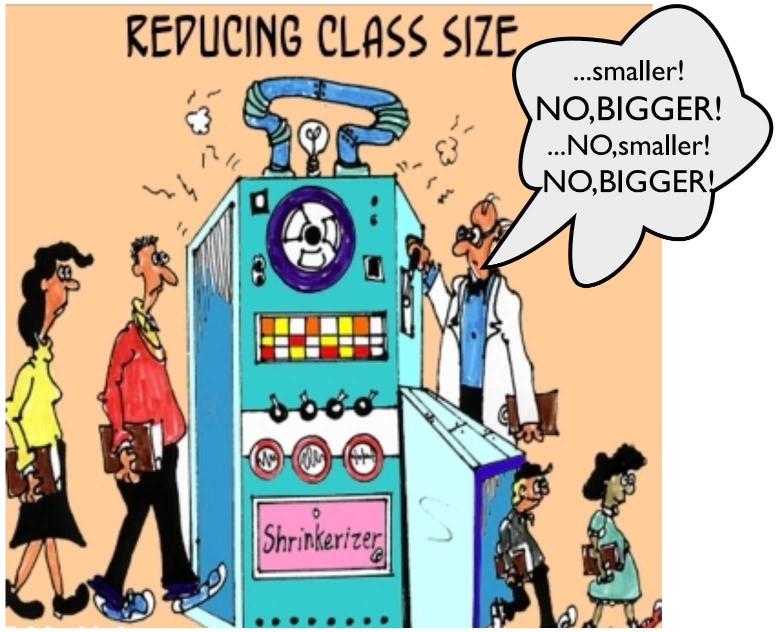
Use product rule:
$\dot{u}\frac{dv}{dt} + u\frac{d\dot{v}}{dt} = \frac{d}{dt}(u\dot{v})$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot{q}^m\right) + \frac{\partial L}{\partial t}$$



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Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

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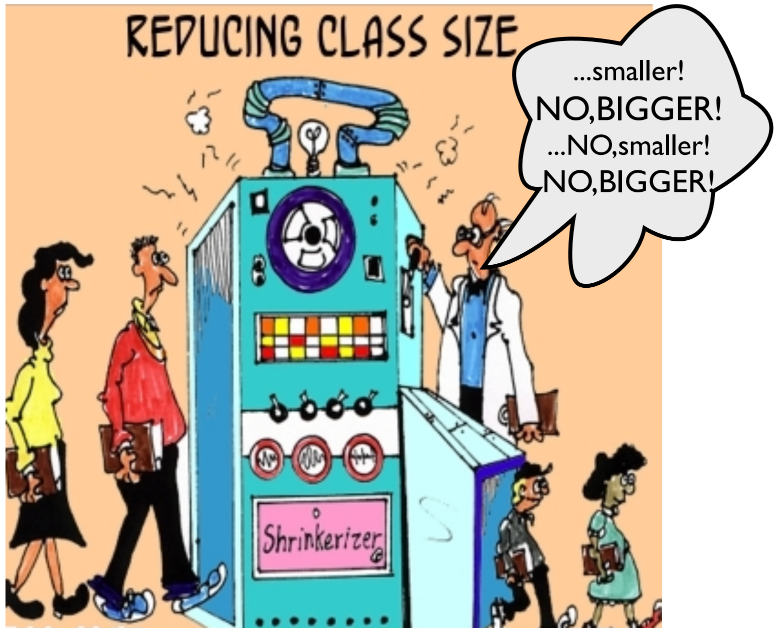

Cartoonish way to imagine explicit time dependence

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}$$

$$where: H\equiv p_m\dot{q}^m-L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$*...*

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

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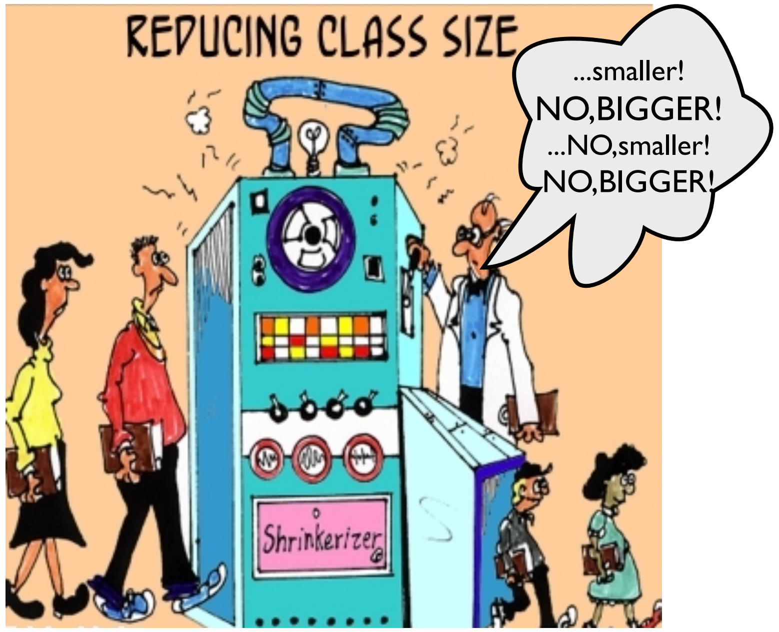
Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$



Cartoonish way to imagine explicit time dependence

and switch the dL/dt and ∂L/∂t to define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

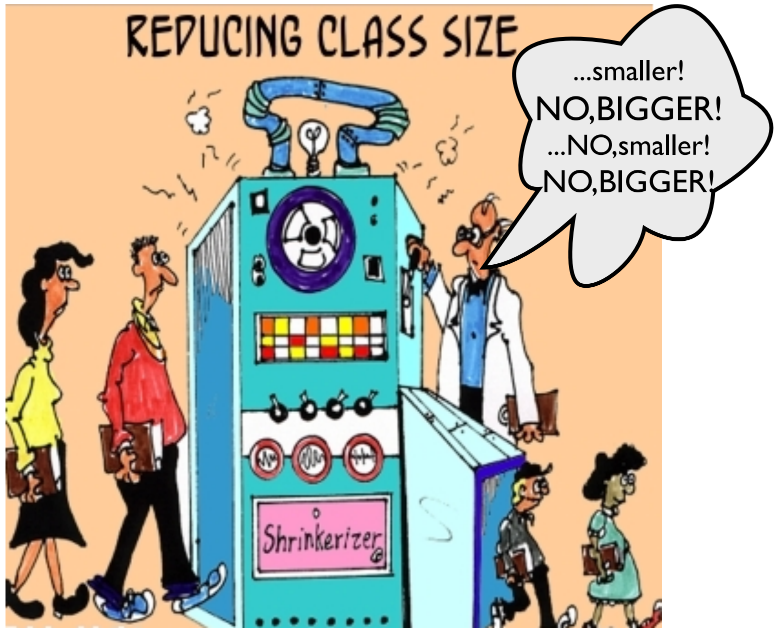
$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}\equiv\frac{dH}{dt}\qquad where: H\equiv p_m\dot{q}^m-L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot q$ ...

$$\dot L(q,\dot q, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot q^m}\frac{d\dot q^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)



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$$\dot L(q,\dot q, t) = \frac{dL}{dt} = \dot p_m \frac{dq^m}{dt} + p_m \frac{d\dot q^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule:

$$\dot u \frac{dv}{dt} + u\frac{d\dot v}{dt} = \frac{d}{dt}(u\dot v)$$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m \dot q^m\right) + \frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p} \bullet \mathbf{v} - L(\mathbf{v})$

(That's the old Legendre transform)

$$\frac{d}{dt}\left(p_m \dot q^m - L\right) = -\frac{\partial L}{\partial t} \equiv \frac{dH}{dt} \qquad where: H \equiv p_m \dot q^m - L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
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$$\dot L(q,\dot q,t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot q^m}\frac{d\dot q^m}{dt}$$

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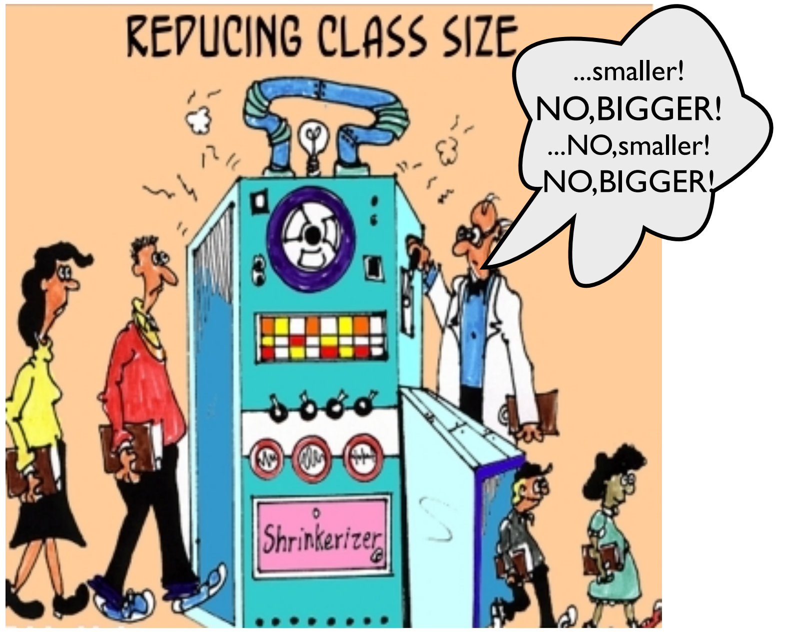

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$$\dot L(q,\dot q,t) = \frac{dL}{dt} = \dot p_m\frac{dq^m}{dt} + p_m\frac{d\dot q^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule:

$$\dot u\frac{dv}{dt} + u\frac{d\dot v}{dt} = \frac{d}{dt}(u\dot v)$$

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_m\dot q^m\right) + \frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p}) = \mathbf{p}\bullet\mathbf{v} - L(\mathbf{v})$ *(That's the old Legendre transform)*

(Recall: $\frac{\partial L}{\partial p_m} \equiv 0$ *and:* $\frac{\partial H}{\partial \dot q^m} \equiv 0$ *)*

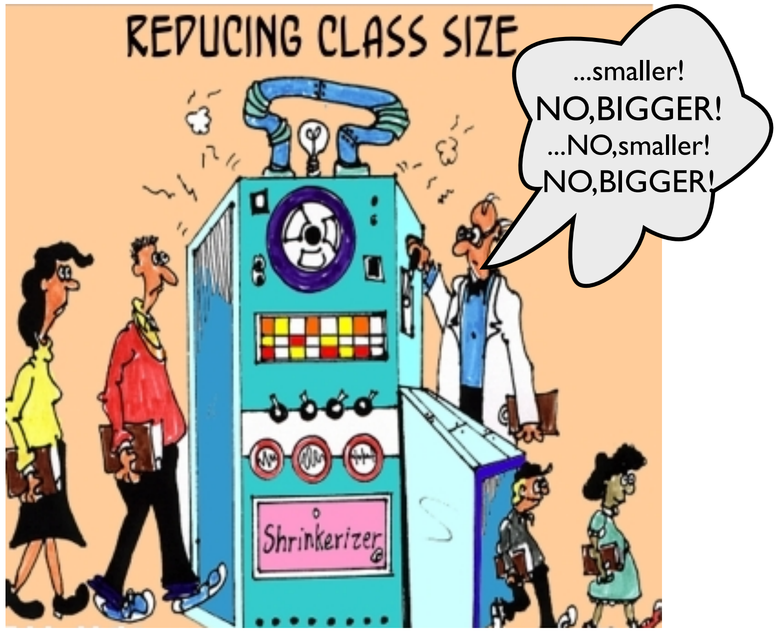
$$\frac{d}{dt}\left(p_m\dot q^m - L\right) = -\frac{\partial L}{\partial t} \equiv \frac{dH}{dt} \qquad where: H \equiv p_m\dot q^m - L$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)



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Recall Lagrange equations: $\boxed{\dot{p}_m=\frac{\partial L}{\partial q^m}}$ $\boxed{p_m=\frac{\partial L}{\partial \dot{q}^m}}$

$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\dot{p}_m\frac{dq^m}{dt}+p_m\frac{d\dot{q}^m}{dt}+\frac{\partial L}{\partial t}$$

Use product rule:

$$\dot{u}\frac{dv}{dt}+u\frac{d\dot{v}}{dt}=\frac{d}{dt}(u\dot{v})$$

$$=\frac{dL}{dt}=\frac{d}{dt}\left(p_m\dot{q}^m\right)+\frac{\partial L}{\partial t}$$

Define the Hamiltonian function $H(\mathbf{p})=\mathbf{p}\bullet\mathbf{v}-L(\mathbf{v})$

(That's the old Legendre transform)

(Recall: $\frac{\partial L}{\partial p_m}\equiv 0$ and: $\frac{\partial H}{\partial \dot{q}^m}\equiv 0$)

$$\frac{d}{dt}\left(p_m\dot{q}^m-L\right)=-\frac{\partial L}{\partial t}=\frac{dH}{dt}\quad \text{where}: H=p_m\dot{q}^m-L$$

so: $\frac{\partial H}{\partial p_m}=\frac{\partial p_m}{\partial p_m}\dot{q}^m-0$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
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...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

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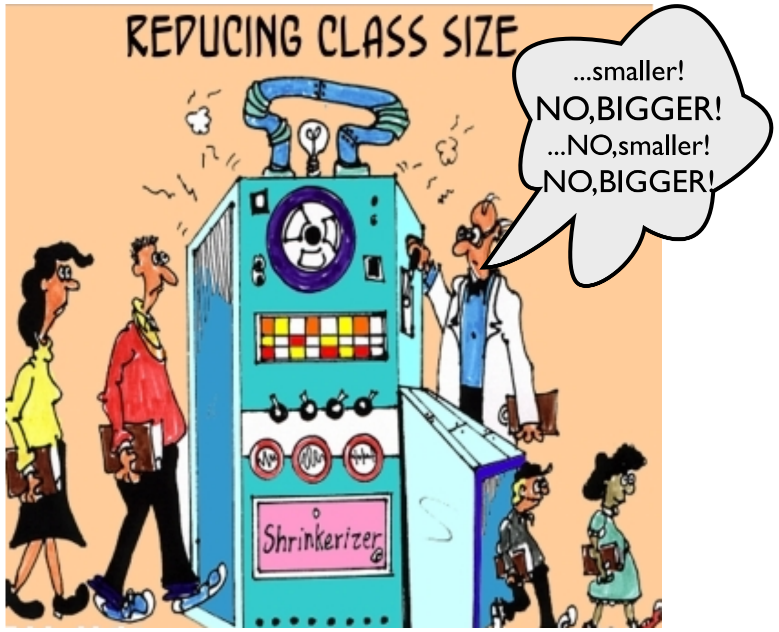

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Hamilton's 1st GCC equation

$$\boxed{\frac{\partial H}{\partial p_m}=\dot{q}^m}$$

Deriving Hamilton's equations from Lagrangian theory

Consider total time derivative of Lagrangian L=T-U
that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m}\frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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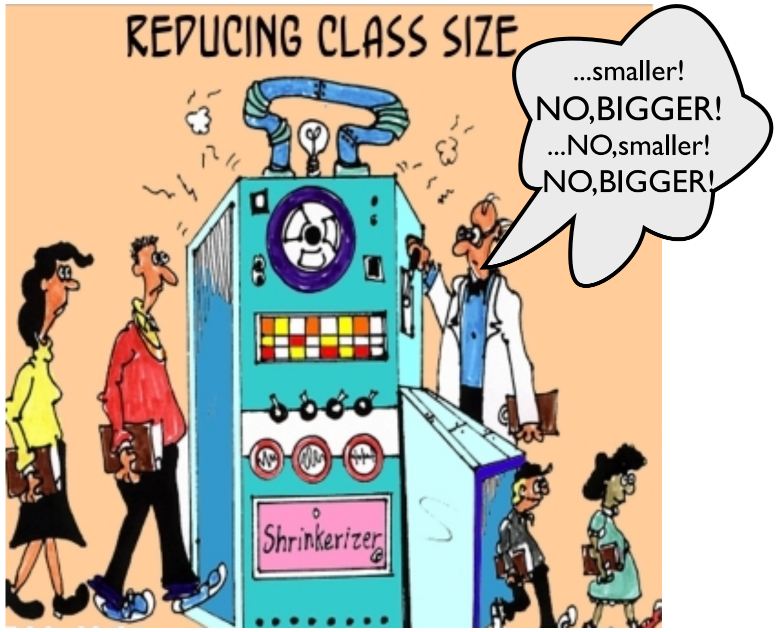

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Hamilton's 1st GCC equation

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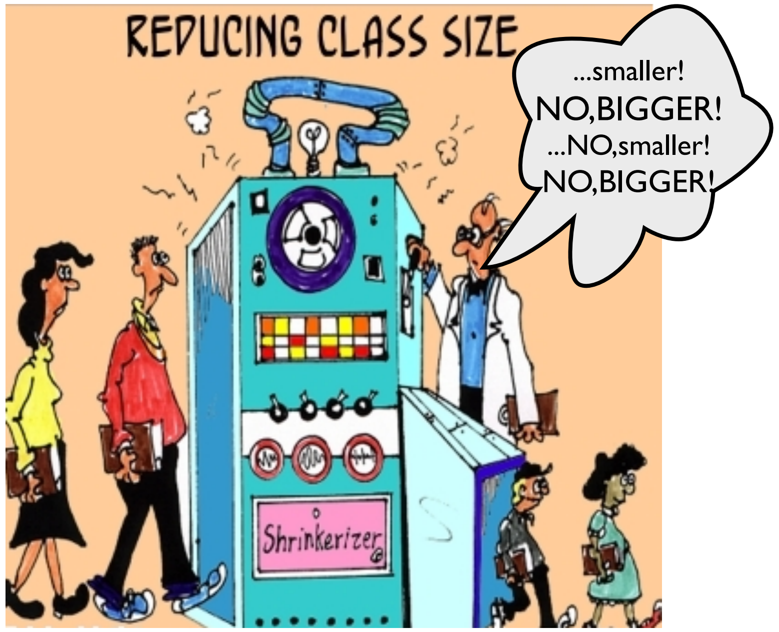

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Hamilton's 1st GCC equation

$$\boxed{\frac{\partial H}{\partial p_m} = \dot q^m}$$

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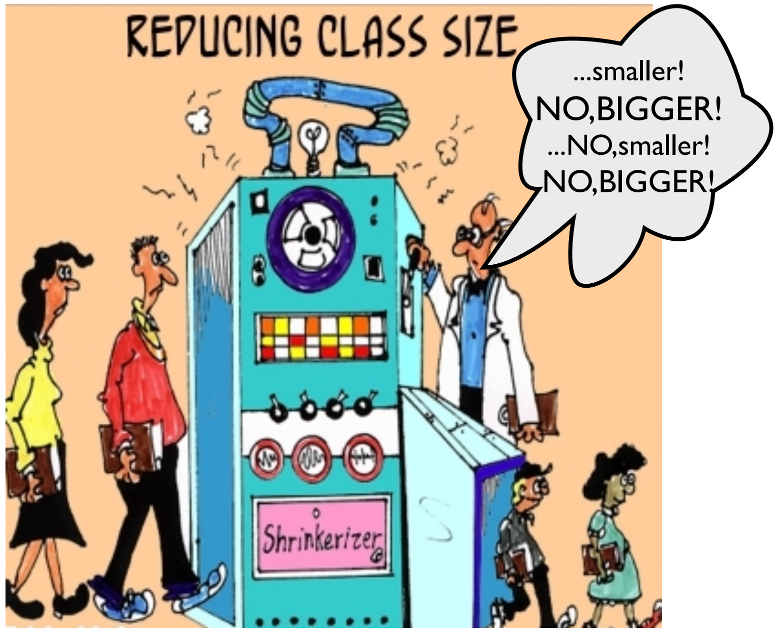

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Hamilton's 2nd GCC equation

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Deriving Hamilton's equations from Lagrangian theory

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$$\dot{L}(q,\dot{q},t)=\frac{dL}{dt}=\frac{\partial L}{\partial q^m}\frac{dq^m}{dt}+\frac{\partial L}{\partial \dot{q}^m}\frac{d\dot{q}^m}{dt}$$

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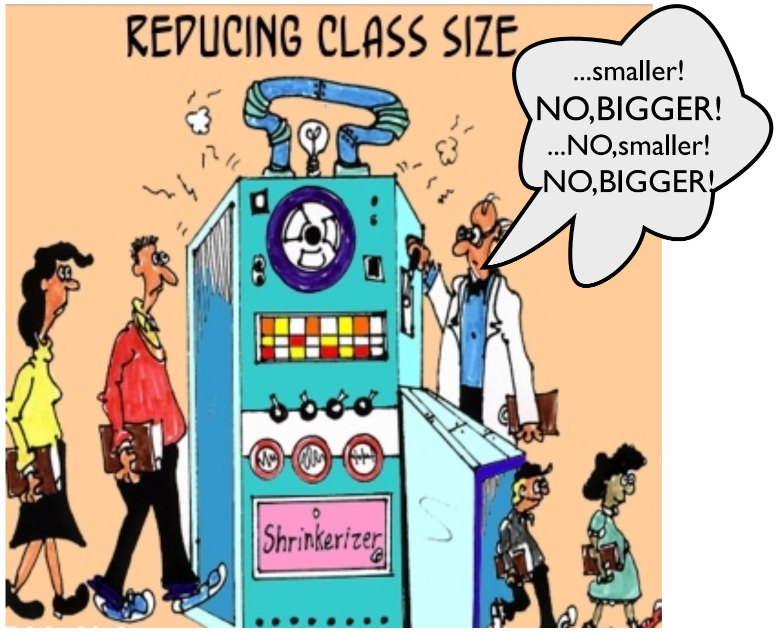

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a most **peculiar** relation involving partial vs total

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Hamilton's 1st GCC equation

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Hamilton prefers Contra*variant* g^{mn} *with* Co*variant* ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

→ *Expressing Hamiltonian* $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

Polar-coordinate example of Hamilton's equations compared to Lagrange's

Hamilton's equations in Runga-Kutta (computer solution) form

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ *with covariant metric definitions of* L *and* p_m

We already have: $H = p_m \dot{q}^m - L$ *and:* $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ *and:* $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

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details on next pages

(Formally **and** Numerically correct)

Details of metric tensor algebra:

Given: $H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$

Let: $\dot{q}^m = \frac{1}{M} g^{mn'} p_{n'}$

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Metric tensor symmetry:

$g_{mn} = g_{nm}$

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(Always applies)

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(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as:

$$L(\dot{r}, \dot{\phi}, r, \phi) = \tfrac{1}{2} M \left(g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right) - U(r, \phi)$$

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Polar coordinate Lagrangian was given as: *See covariant polar metric $g_{\mu\upsilon}$ on next page (p35)*

$$L(\dot{r}, \dot{\phi}, r, \phi) = \tfrac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \tfrac{1}{2} M (\dot{r}^2 + r^2 \cdot \dot{\phi}^2) - U(r, \phi)$$

Covariant polar metric $g_{\mu\upsilon}$ *[from p53 of Lecture 9]* *Contravariant polar metric* $g^{\mu\upsilon}$

Covariant g_{mn} *vs. Invariant* $\delta_m^{\ n}$ *vs. Contravariant* g^{mn}

$$\mathbf{E}_m \bullet \mathbf{E}_n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial \mathbf{r}}{\partial q^n} \equiv g_{mn} \qquad \mathbf{E}_m \bullet \mathbf{E}^n = \frac{\partial \mathbf{r}}{\partial q^m} \bullet \frac{\partial q^n}{\partial \mathbf{r}} = \delta_m^n \qquad \mathbf{E}^m \bullet \mathbf{E}^n = \frac{\partial q^m}{\partial \mathbf{r}} \bullet \frac{\partial q^n}{\partial \mathbf{r}} \equiv g^{mn}$$

Covariant
metric tensor
g_{mn}

Invariant
Kroneker unit tensor

$$\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Contravariant
metric tensor
g^{mn}

Polar coordinate examples (again):

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos\phi & \frac{\partial x}{\partial \phi} = -r\sin\phi \\ \frac{\partial y}{\partial r} = \sin\phi & \frac{\partial y}{\partial \phi} = r\cos\phi \end{pmatrix}$$

$$\uparrow \mathbf{E}_1 \quad \uparrow \mathbf{E}_2 \qquad \uparrow \mathbf{E}_r \qquad \uparrow \mathbf{E}_\phi$$

$$\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos\phi & \frac{\partial r}{\partial y} = \sin\phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r} \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^r = \mathbf{E}^1 \\ \leftarrow \mathbf{E}^\phi = \mathbf{E}^2 \end{matrix}$$

Covariant g_{mn}

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}_r & \mathbf{E}_r \bullet \mathbf{E}_\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}_r & \mathbf{E}_\phi \bullet \mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Invariant δ_m^n

$$\begin{pmatrix} \delta_r^r & \delta_r^\phi \\ \delta_\phi^r & \delta_\phi^\phi \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r \bullet \mathbf{E}^r & \mathbf{E}_r \bullet \mathbf{E}^\phi \\ \mathbf{E}_\phi \bullet \mathbf{E}^r & \mathbf{E}_\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Contravariant g^{mn}

$$\begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{\phi r} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}^r \bullet \mathbf{E}^r & \mathbf{E}^r \bullet \mathbf{E}^\phi \\ \mathbf{E}^\phi \bullet \mathbf{E}^r & \mathbf{E}^\phi \bullet \mathbf{E}^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of L and p_m

We already have: $H=p_m\dot{q}^m-L$ *and:* $L(\dot{q})=\frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n-U$ *and:* $p_m=\frac{\partial L}{\partial \dot{q}^m}=Mg_{mn}\dot{q}^n$

Now we combine all these:

$$H=p_m\dot{q}^m-L=\left(Mg_{mn}\dot{q}^n\right)\dot{q}^m-\left(\tfrac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n-U\right)$$

$$=Mg_{mn}\dot{q}^m\dot{q}^n-\tfrac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n+U$$

This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity$\dot{q}^m$.)

$$H=\tfrac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n+U=T+U$$

(Numerically correct ONLY!)

An inverse metric relation $\dot{q}^m=\frac{1}{M}g^{mn}p_n$ *gives correct form that depends on momentum p_m.*

$$\boxed{H=\frac{1}{2M}g^{mn}p_mp_n+U=T+U\equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as: *See covariant polar metric $g_{\mu\upsilon}$ on p39*

$$L(\dot{r},\dot{\phi},r,\phi)=\tfrac{1}{2}M(g_{rr}\dot{r}^2+g_{\phi\phi}\dot{\phi}^2)-U(r,\phi)=\tfrac{1}{2}M(\dot{r}^2+r^2\cdot\dot{\phi}^2)-U(r,\phi)$$

Polar coordinate Hamiltonian is given here:

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r{}^2+g^{\phi\phi}p_\phi{}^2)+U(r,\phi)$$

Hamilton prefers Contravariant g^{mn} with Covariant momentum p_m

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Now we combine all these:

$$H = p_m\dot{q}^m - L = \left(Mg_{mn}\dot{q}^n\right)\dot{q}^m - \left(\frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n - U\right)$$

$$= Mg_{mn}\dot{q}^m\dot{q}^n - \frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n + U$$

This gives an "illegal dependence" for the Hamiltonian (It musn't be "explicit" in velocity$\dot{q}^m$.)

$$H = \frac{1}{2}Mg_{mn}\dot{q}^m\dot{q}^n + U = T + U$$

(Numerically correct ONLY!)

An inverse metric relation $\dot{q}^m = \frac{1}{M}g^{mn}p_n$ *gives correct form that depends on momentum p_m.*

$$\boxed{H = \frac{1}{2M}g^{mn}p_m p_n + U = T + U \equiv E}$$

(Formally **and** Numerically correct)

Polar coordinate Lagrangian was given as: *See covariant polar metric $g_{\mu\upsilon}$ on p39*

$$L(\dot{r},\dot{\phi},r,\phi) = \frac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \frac{1}{2}M(\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

Polar coordinate Hamiltonian is given here: *Contravariant polar metric $g^{\mu\upsilon}$ on p35*

$$H(p_r,p_\phi,r,\phi) = \frac{1}{2M}(g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2) + U(r,\phi) = \frac{1}{2M}(p_r^2 + \frac{1}{r^2}\cdot p_\phi^2) + U(r,\phi)$$

Hamilton prefers Contravariant g^{mn} *with* Covariant ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

→ *Polar-coordinate example of Hamilton's equations compared to Lagrange's*

Hamilton's equations in Runga-Kutta (computer solution) form

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ *in 2D-polar coordinates satifies:*

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$ *Hamilton's 2nd equations:* $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

Hamiltonian $H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$ in 2D-polar coordinates satifies:

Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

$$\frac{\partial H}{\partial p_r}=\dot{r}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M}$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial r}=-\dot{p}_r$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

Polar coordinate example of Hamilton's equations

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Polar coordinate example of Hamilton's equations

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Hamilton's 1st equations: $\frac{\partial H}{\partial p_m}=\dot{q}^m$

$$\frac{\partial H}{\partial p_r}=\dot{r} \qquad \frac{\partial H}{\partial p_\phi}=\dot{\phi}$$

$$\frac{\partial H}{\partial p_r}=\frac{p_r}{M} \qquad \frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{Mr^2}$$

Hamilton's 2nd equations: $\frac{\partial H}{\partial q^m}=-\dot{p}_m$

$$\frac{\partial H}{\partial r}=-\dot{p}_r \qquad \frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial r}=-2\frac{p_\phi^2}{2Mr^3}+\frac{\partial U(r,\phi)}{\partial r}$$

Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Polar coordinate example of Hamilton's equations

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$$p_r=M\dot{r}$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

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$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

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Polar coordinate example of Hamilton's equations

$$H(p_r,p_\phi,r,\phi)=\frac{1}{2M}(g^{rr}p_r^2+g^{\phi\phi}p_\phi^2)+U(r,\phi)=\frac{1}{2M}(p_r^2+\frac{1}{r^2}\cdot p_\phi^2)+U(r,\phi)$$

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$$=Mr\dot{\phi}^2-\partial_r U(r,\phi)$$

$$\frac{\partial H}{\partial \phi}=-\dot{p}_\phi$$

$$\frac{\partial H}{\partial \phi}=\frac{\partial U(r,\phi)}{\partial \phi}$$

$$\dot{p}_\phi=-\frac{\partial U(r,\phi)}{\partial \phi}$$

$$p_\phi=Mr^2\dot{\phi} \dashrightarrow \dot{p}_\phi=2Mr\dot{r}\dot{\phi}+Mr^2\ddot{\phi}=-\partial_\phi U(r,\phi)$$

Compare these Hamilton's equations to Lagrange's on next page...

Lagrange prefers Covariant g_{mn} with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

(Review of Lecture 9)

$$L(\mathbf{v}) = \tfrac{1}{2}M\mathbf{v}\bullet\mathbf{v} - U = \tfrac{1}{2}M\dot{\mathbf{r}}\bullet\dot{\mathbf{r}} - U = \tfrac{1}{2}M(\mathbf{E}_m\dot{q}^m)\bullet(\mathbf{E}_n\dot{q}^n) - U = \tfrac{1}{2}M(g_{mn}\dot{q}^m\dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate Covariant g_{mn} metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_r\bullet\mathbf{E}_r & \mathbf{E}_r\bullet\mathbf{E}_\phi \\ \mathbf{E}_\phi\bullet\mathbf{E}_r & \mathbf{E}_\phi\bullet\mathbf{E}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \tfrac{1}{2}M(g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2) - U(r,\phi) = \tfrac{1}{2}M(1\cdot\dot{r}^2 + r^2\cdot\dot{\phi}^2) - U(r,\phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum p_m definition for each coordinate q^m:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M\, g_{rr}\dot{r} = M\,\dot{r}$$

Nothing too surprising; radial momentum p_r has the usual linear $M\cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi}\dot{\phi} = Mr^2\dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor Mr^2 automatically for the angular momentum $p_\phi = Mr^2\omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum p_m:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2}\frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 - \frac{\partial U}{\partial r} = M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt}M(g_{mn}\dot{q}^n) = M(\dot{g}_{mn}\dot{q}^n + g_{mn}\ddot{q}^n)$ Equate it to $\dot{p}_m$ in 2nd L-equation:

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\,\ddot{r}$$

$$= M\,r\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Hamilton prefers Contravariant g^{mn} *with* Covariant ***momentum*** p_m

Deriving Hamilton's equations from Lagrange's equations

Expressing Hamiltonian $H(p_m,q^n)$ *using* g^{mn} *and covariant momentum* p_m

- *Polar-coordinate example of Hamilton's equations compared to Lagrange's*

→ *Hamilton's equations in Runga-Kutta (computer solution) form*

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$p_r = M\dot{r}$

$$\dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$p_\phi = Mr^2\dot{\phi}$

$$\dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_2 = \dot{x}_2(x_1, x_2, x_3, \ldots)$$

$$\dot{x}_3 = \dot{x}_3(x_1, x_2, x_3, \ldots)$$

$$\vdots$$

Polar coordinate example: Hamilton's equations in Runga-Kutta form

$$p_r = M\dot{r} \qquad \dot{p}_r = M\ddot{r} = \frac{p_\phi^2}{Mr^3} - \frac{\partial U(r,\phi)}{\partial r}$$

$$= Mr\dot{\phi}^2 - \partial_r U(r,\phi)$$

$$p_\phi = Mr^2\dot{\phi} \qquad \dot{p}_\phi = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi} = -\partial_\phi U(r,\phi)$$

Hamiltonian eqs. in
Runga-Kutta form:

$$\dot{r} = \dot{r}(r, p_r, \phi, p_\phi) \quad = \frac{p_r}{M}$$

$$\dot{p}_r = \dot{p}_r(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi^2}{Mr^3} - \partial_r U(r,\phi)$$

$$\dot{\phi} = \dot{\phi}(r, p_r, \phi, p_\phi) \quad = \frac{p_\phi}{Mr^2}$$

$$\dot{p}_\phi = \dot{p}_\phi(r, p_r, \phi, p_\phi) \quad = -\partial_\phi U(r,\phi)$$

Runga-Kutta form:

$$\dot{x}_1 = \dot{x}_1(x_1, x_2, x_3, \ldots)$$

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Examples of Hamiltonian mechanics in effective potentials

➡ *Isotropic Harmonic Oscillator in polar coordinates and effective potential* *(Web Simulation: OscillatorPE - IHO)*

Coulomb orbits in polar coordinates and effective potential *(Web Simulation: OscillatorPE - Coulomb)*

Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and p_m-conservation

Consider polar coordinate Hamiltonian for $I_{sotropic}$ $H_{armonic}$ $O_{scillator}$ potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr} p_r{}^2 + g^{\phi\phi} p_\phi{}^2) + k \cdot r^2/2 = \frac{1}{2M}(p_r{}^2 + \frac{1}{r^2} \cdot p_\phi{}^2) + \frac{k \cdot r^2}{2} = E = const.$$

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H is not explicit function of ϕ, and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum p_ϕ is conserved constant: $p_\phi = \ell = const.$

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Same applies to any radial potential U(r)

$$E=\frac{p_r{}^2}{2M}+\overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}}+\overbrace{U(r)}^{\text{“real” PE}}}^{\text{“effective” PE}}$$

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Solving for momentum : $p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2$

$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2} = \sqrt{2M}\sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2} \cdot r^2}$$

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Radial KE is: $\frac{M\dot{r}^2}{2}=E-\frac{\ell^2}{2Mr^2}-\frac{k}{2}\cdot r^2$

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Radial velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}$$

Effective potential analysis (Reducing 2D-problem to 1D-problem)

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Radial KE is: $\frac{M\dot{r}^2}{2}=E-\frac{\ell^2}{2Mr^2}-\frac{k}{2}\cdot r^2$

Radial velocity:

$$\dot{r}=\frac{dr}{dt}=\sqrt{\frac{2E}{M}-\frac{\ell^2}{M^2r^2}-\frac{k}{M}\cdot r^2}$$

Time vs r: $t=\int_{r<}^{r>}\frac{dr}{\sqrt{\frac{2E}{M}-\frac{\ell^2}{M^2r^2}-\frac{k}{M}\cdot r^2}}$

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Same applies to any radial potential U(r)

$$E=\frac{p_r^2}{2M}+\overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}}+\overbrace{U(r)}^{\text{“real” PE}}}^{\text{“effective” PE}}$$

Solving for momentum : $p_r^2=2ME-\frac{\ell^2}{r^2}-Mk\cdot r^2$

$$p_r=M\dot{r}=\sqrt{2ME-\frac{\ell^2}{r^2}-Mk\cdot r^2}=\sqrt{2M}\sqrt{E-\frac{\ell^2}{2Mr^2}-\frac{k}{2}\cdot r^2}$$

Called the "quadrature" or 1/4-cycle solution if $r_<=0$ and $r_>=$max amplitude

Radial KE is: $\frac{M\dot{r}^2}{2}=E-\frac{\ell^2}{2Mr^2}-\frac{k}{2}\cdot r^2$

Radial velocity:

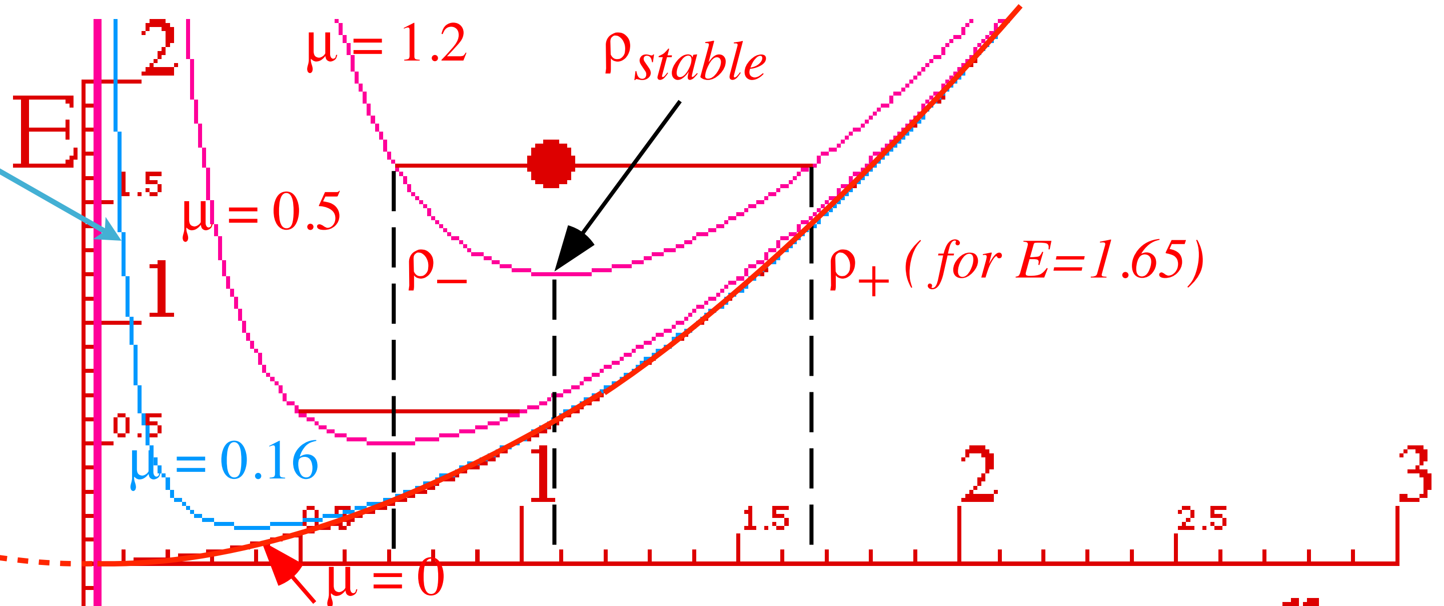
$$\dot{r}=\frac{dr}{dt}=\sqrt{\frac{2E}{M}-\frac{\ell^2}{M^2r^2}-\frac{k}{M}\cdot r^2}$$

Time vs r: $t=\int_{r<}^{r>}\frac{dr}{\sqrt{\frac{2E}{M}-\frac{\ell^2}{M^2r^2}-\frac{k}{M}\cdot r^2}}$

Time vs r for any radial U(r):

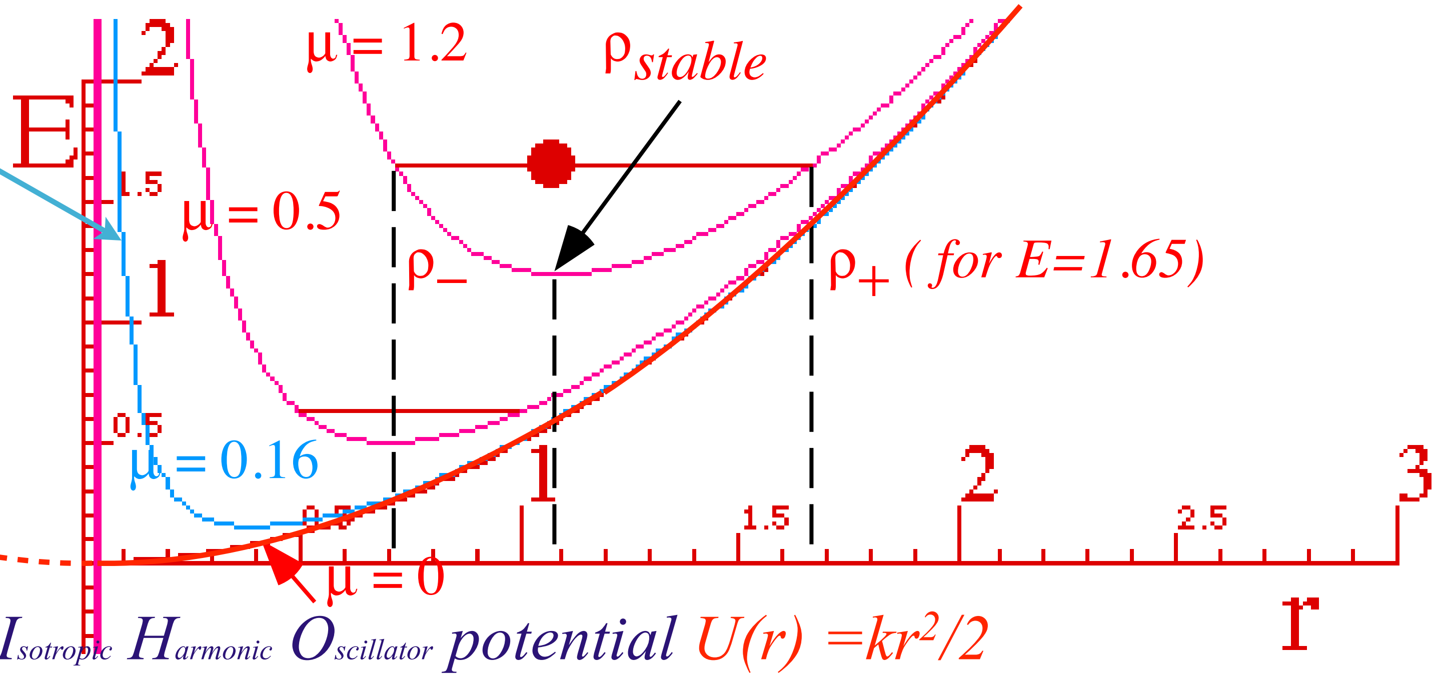
$$t=\int_{r<}^{r>}\frac{dr}{\sqrt{\frac{2E}{M}-\frac{\ell^2}{M^2r^2}-\frac{2U(r)}{M}}}$$

$$E = \frac{p_r^{\,2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$

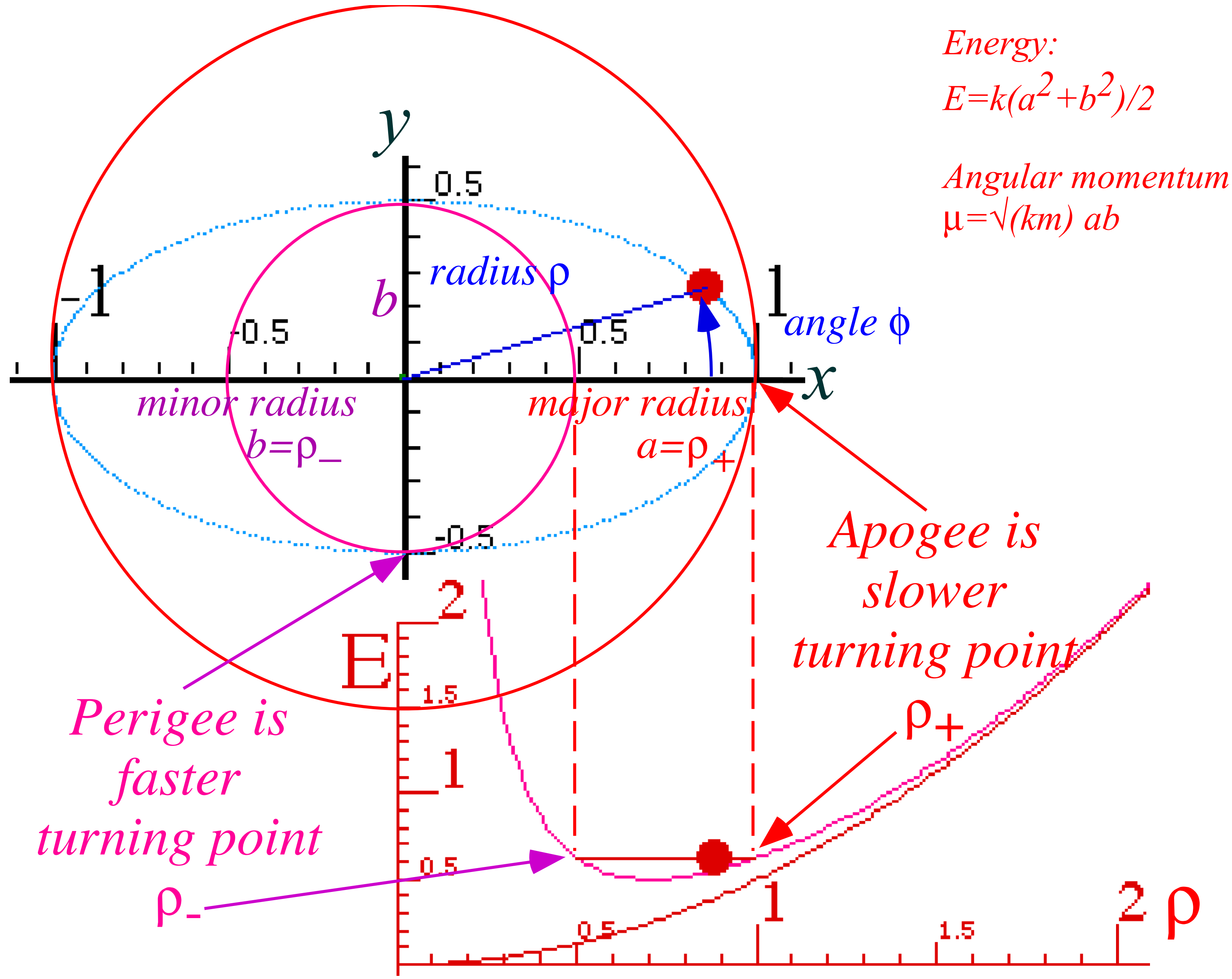
$$E = \frac{p_r^2}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} + \overbrace{\frac{k}{2}r^2}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



Hamiltonian dynamics for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$

Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$



Apogee is slower turning point

Perigee is faster turning point

Examples of Hamiltonian mechanics in effective potentials

I

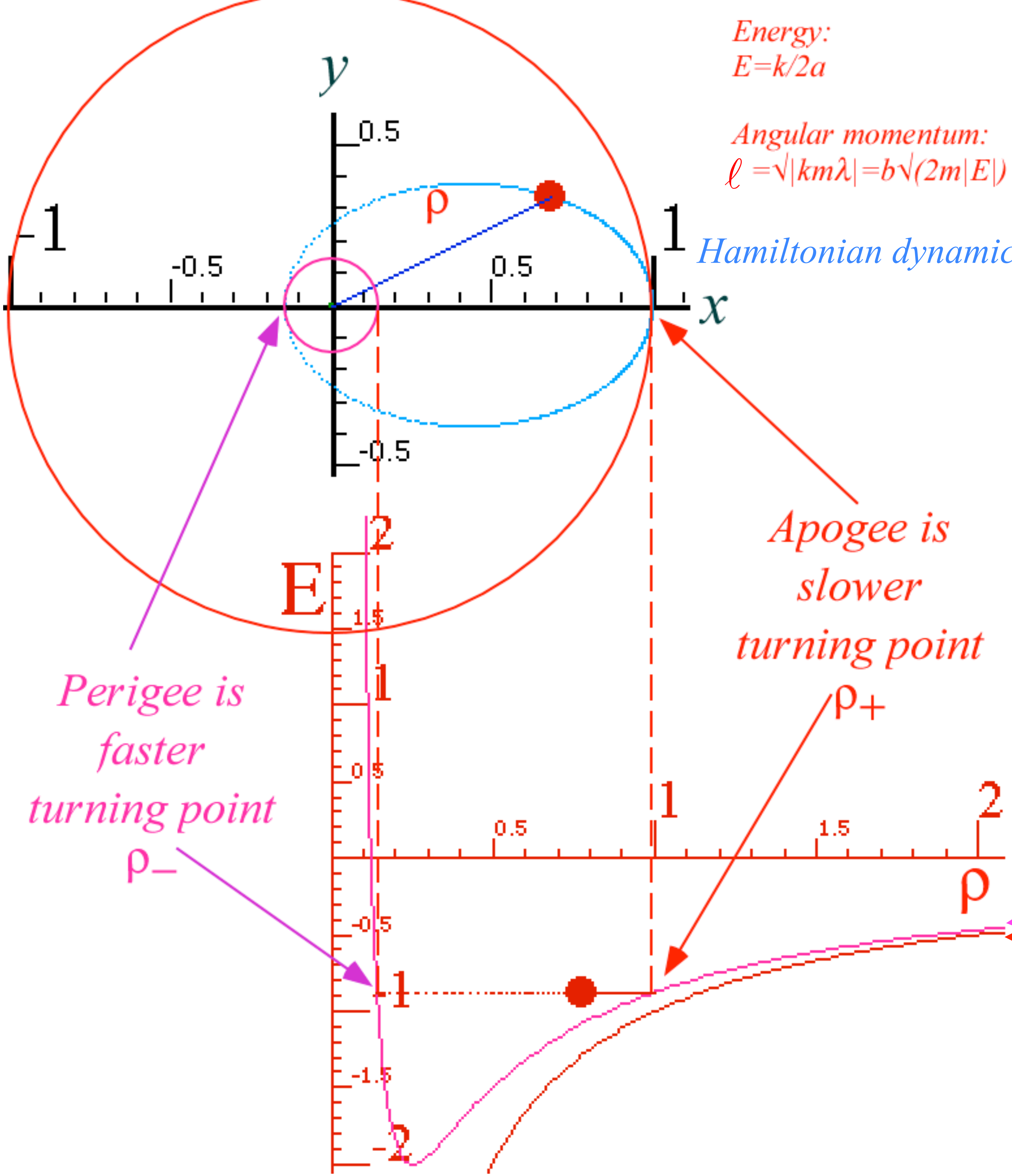
Isotropic Harmonic Oscillator in polar coordinates and effective potential *(Web Simulation: OscillatorPE - IHO)*

→ *Coulomb orbits in polar coordinates and effective potential* *(Web Simulation: OscillatorPE - Coulomb)*

Hamiltonian dynamics for Coulomb potential $U(r) = -k/r$

Energy:
$E=k/2a$

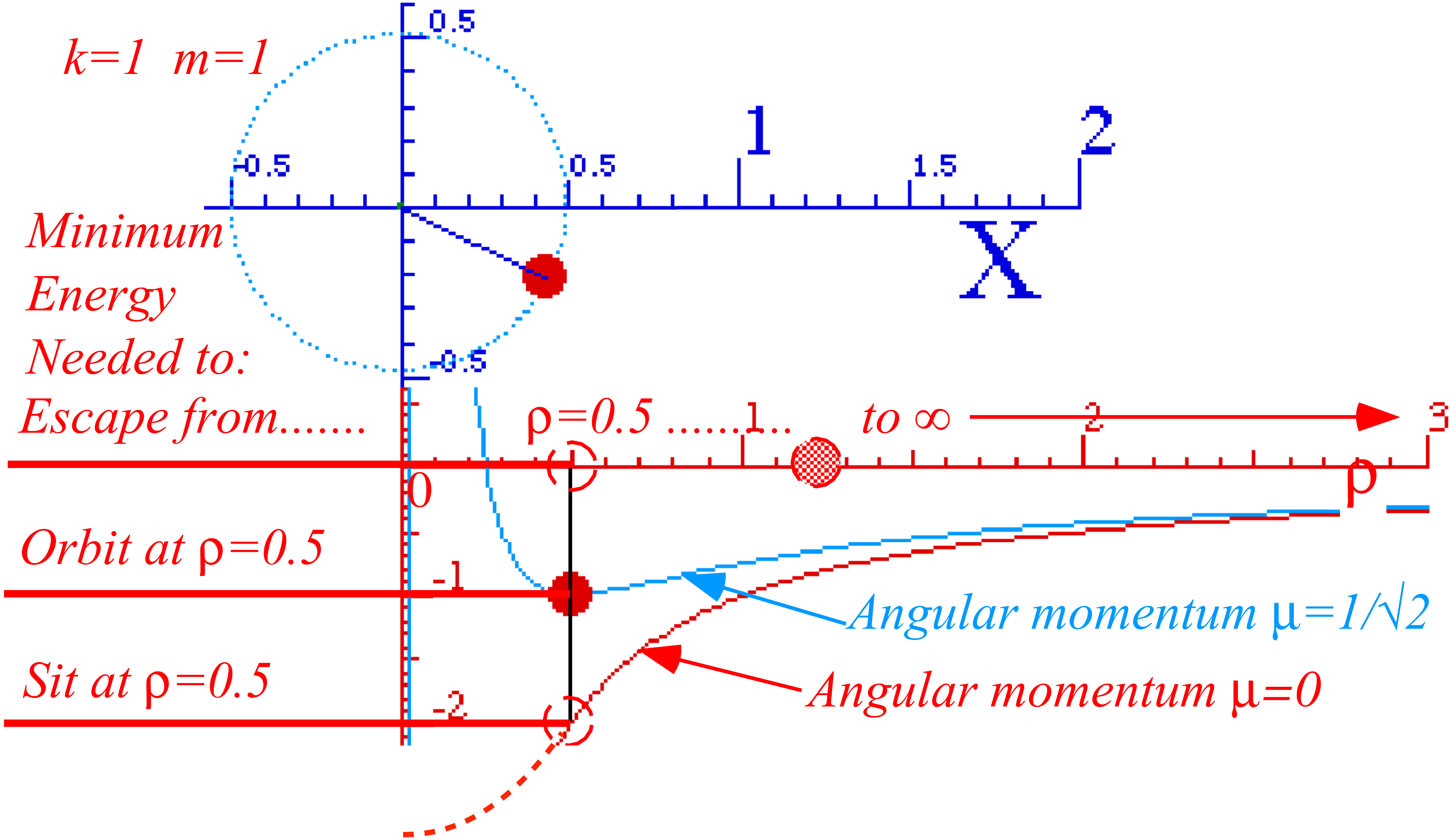
Angular momentum:
$\ell = \sqrt{|km\lambda|} = b\sqrt{(2m|E|)}$



Perigee is faster turning point
ρ_-

Apogee is slower turning point
ρ_+

$$E = \frac{p_r^{\,2}}{2M} + \overbrace{\underbrace{\frac{\ell^2}{2Mr^2}}_{\text{“centifugal-barrier” PE}} - \overbrace{\frac{k}{r}}^{\text{“real” PE}}}^{\text{“effective” PE}}$$



$k=1 \quad m=1$

Minimum Energy Needed to:

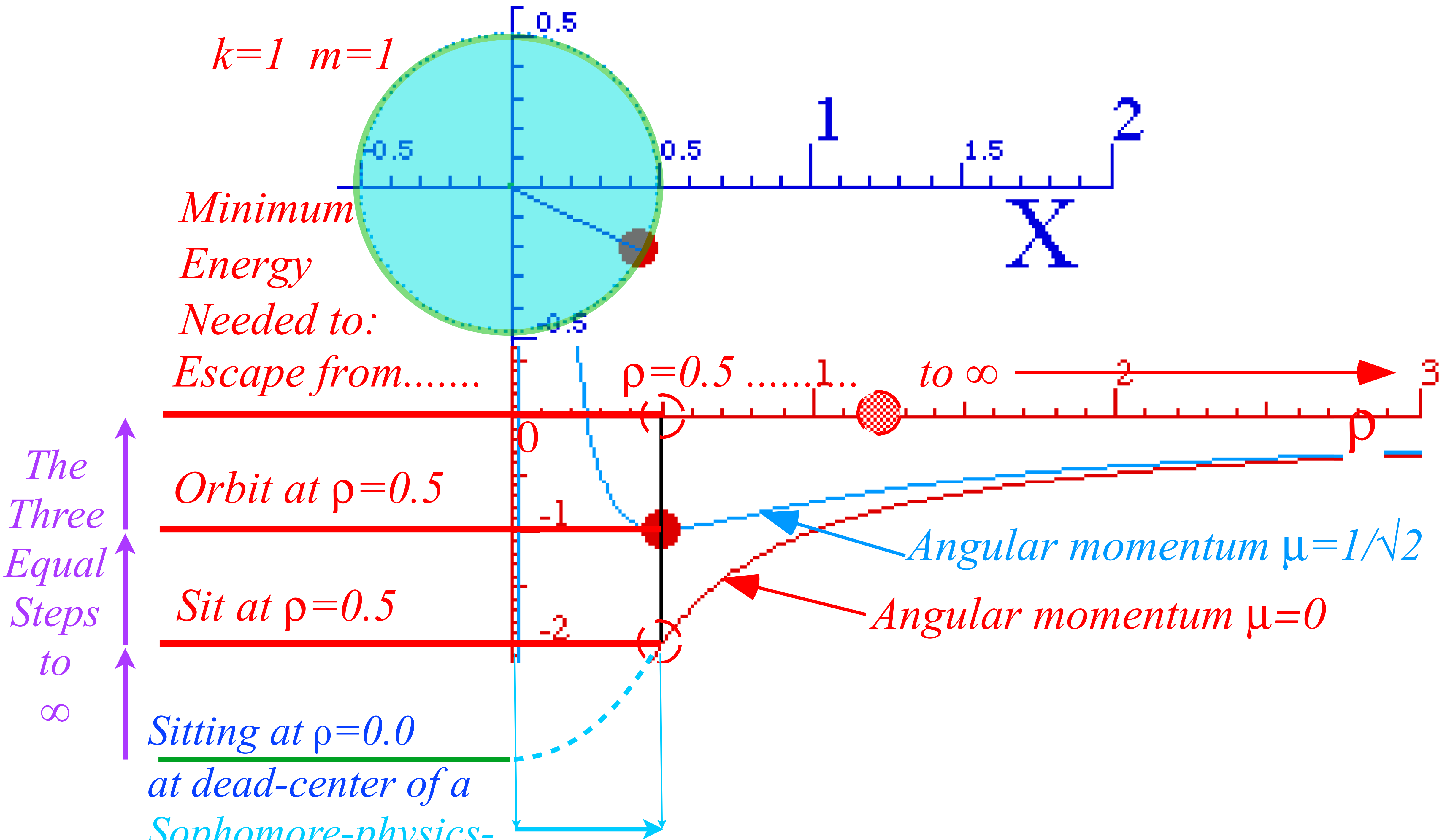
Escape from....... $\rho=0.5$ ......1.. to ∞

Orbit at $\rho=0.5$

Sit at $\rho=0.5$

Angular momentum $\mu=1/\sqrt{2}$

Angular momentum $\mu=0$



k=1 m=1

Minimum Energy Needed to:

Escape from....... ρ=0.5 ......1.. to ∞

The Three Equal Steps to ∞

Orbit at ρ=0.5

Sit at ρ=0.5

Sitting at ρ=0.0 at dead-center of a Sophomore-physics-Earth of radius $R_{\oplus}$=0.5

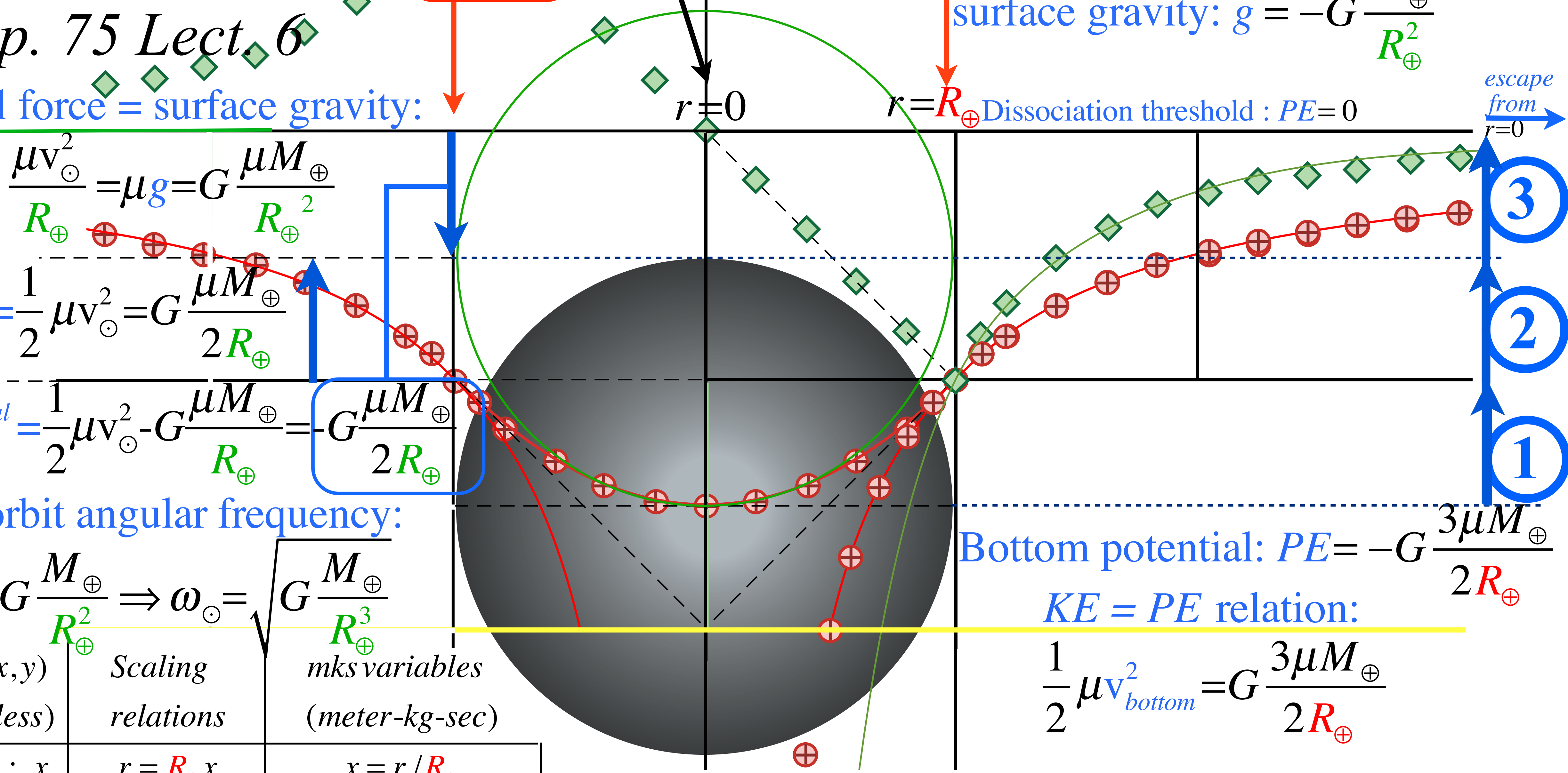
Angular momentum μ=1/√2

Angular momentum μ=0

From p. 74 Lect. 6 on next page

Sophomore-physics-Earth inside and out: "3-steps out of (or into) Hell" ...and surface orbit at $r=R_\oplus$

From p. 75 Lect. 6



surface gravity: $g=-G\dfrac{M_\oplus}{R_\oplus^2}$

$r=R_\oplus$ Dissociation threshold : $PE=0$

Centifugal force = surface gravity:

$$\frac{\mu v_\odot^2}{R_\oplus}=\mu g=G\frac{\mu M_\oplus}{R_\oplus{}^2}$$

Orbit KE$=\dfrac{1}{2}\mu v_\odot^2=G\dfrac{\mu M_\oplus}{2R_\oplus}$

Orbit $E_\odot^{Total}=\dfrac{1}{2}\mu v_\odot^2-G\dfrac{\mu M_\oplus}{R_\oplus}=\boxed{-G\dfrac{\mu M_\oplus}{2R_\oplus}}$

$(r=R_\oplus)$-orbit angular frequency:

$$\omega_\odot^2 R_\oplus=G\frac{M_\oplus}{R_\oplus^2}\Rightarrow\omega_\odot=\sqrt{G\frac{M_\oplus}{R_\oplus^3}}$$

Bottom potential: $PE=-G\dfrac{3\mu M_\oplus}{2R_\oplus}$

KE = PE relation:

$$\frac{1}{2}\mu v_{bottom}^2=G\frac{3\mu M_\oplus}{2R_\oplus}$$

Geometric (x,y) (Dimensionless)	Scaling relations	mks variables (meter-kg-sec)		
space coord.: x	$r=R_\oplus x$	$x=r/R_\oplus$		
PE for $\lvert x\rvert\geq 1$: $y^{PE}=\dfrac{-1}{x}$	$PE^{mks}(r)$ $=\dfrac{GM\mu}{R_\oplus}y^{PE}$	$PE^{mks}(r)=-\dfrac{GM\mu}{r}$ $=-\dfrac{GM\mu}{R_\oplus}\dfrac{1}{x}$	PE for $\lvert x\rvert<1$: $y^{PE}=\dfrac{x^2}{2}-\dfrac{3}{2}$	$PE^{mks}(r)=\dfrac{GM\mu}{R_\oplus}\left(\dfrac{r^2}{2R_\oplus^2}-\dfrac{3}{2}\right)$
Force for $\lvert x\rvert\geq 1$: $y^{Force}=\dfrac{-1}{x^2}$	$F^{mks}(r)$ $=\dfrac{GM\mu}{R_\oplus^2}y^{Force}$	$F^{mks}(r)=-\dfrac{GM\mu}{r^2}$ $=-\dfrac{GM\mu}{R_\oplus^2}\dfrac{1}{x^2}$	Force for $\lvert x\rvert<1$: $y^{Force}=-x$	$F^{mks}(r)=-\dfrac{GM\mu}{R_\oplus^3}r$

$(r=0)$-escape-velocity

$$v_{bottom}=\sqrt{3G\frac{M_\oplus}{R_\oplus}}$$

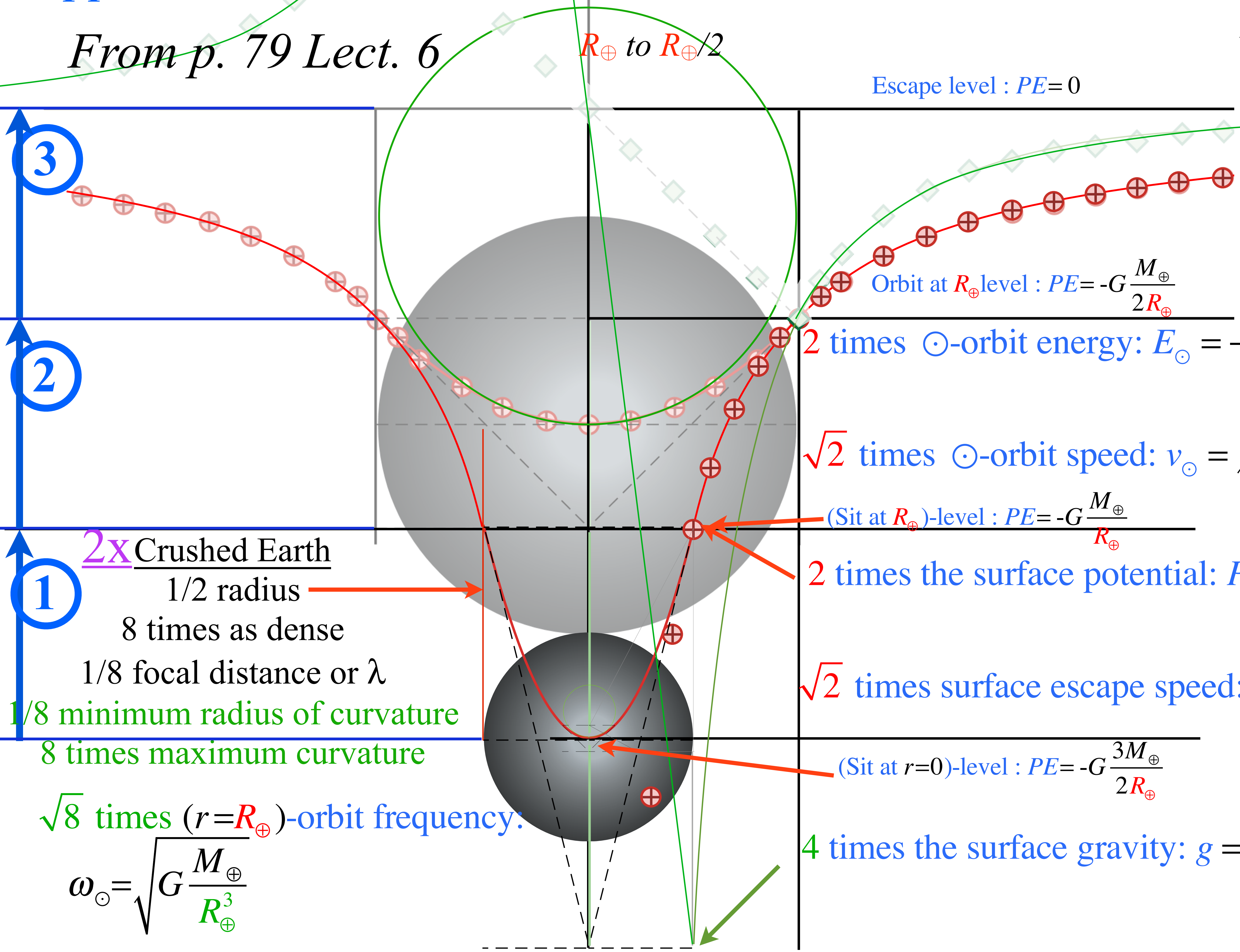
$(r=R_\oplus)$-escape velocity:

$$v_{escape}=\sqrt{2G\frac{M_\oplus}{R_\oplus}}$$

Sophomore-physics-Earth inside and out: “3-steps to Hell”

Suppose Earth radius crushed to 1/2: ($R_\oplus$=6.4·10^6m crushed to $R_\oplus$/2=3.2·10^6m)

From p. 79 Lect. 6



$R_\oplus$ to $R_\oplus/2$

All formulas identical to ones derived on p.15 to 27. Imagine reducing $R_\oplus$ to $R_\oplus/2$

Escape level : $PE= 0$

Orbit at $R_\oplus$ level : $PE= -G\dfrac{M_\oplus}{2R_\oplus}$

2 times ⊙-orbit energy: $E_\odot = -G\dfrac{M_\oplus}{2R_\oplus}$

$\sqrt{2}$ times ⊙-orbit speed: $v_\odot = \sqrt{G\dfrac{M_\oplus}{R_\oplus}}$

(Sit at $R_\oplus$)-level : $PE= -G\dfrac{M_\oplus}{R_\oplus}$

2 times the surface potential: $PE= -G\dfrac{M_\oplus}{R_\oplus}$

$\sqrt{2}$ times surface escape speed: $v_e = \sqrt{G\dfrac{2M_\oplus}{R_\oplus}}$

(Sit at r=0)-level : $PE= -G\dfrac{3M_\oplus}{2R_\oplus}$

4 times the surface gravity: $g = -G\dfrac{M_\oplus}{R_\oplus^2}$

2x Crushed Earth

1/2 radius

8 times as dense

1/8 focal distance or λ

1/8 minimum radius of curvature

8 times maximum curvature

$\sqrt{8}$ times (r=$R_\oplus$)-orbit frequency:

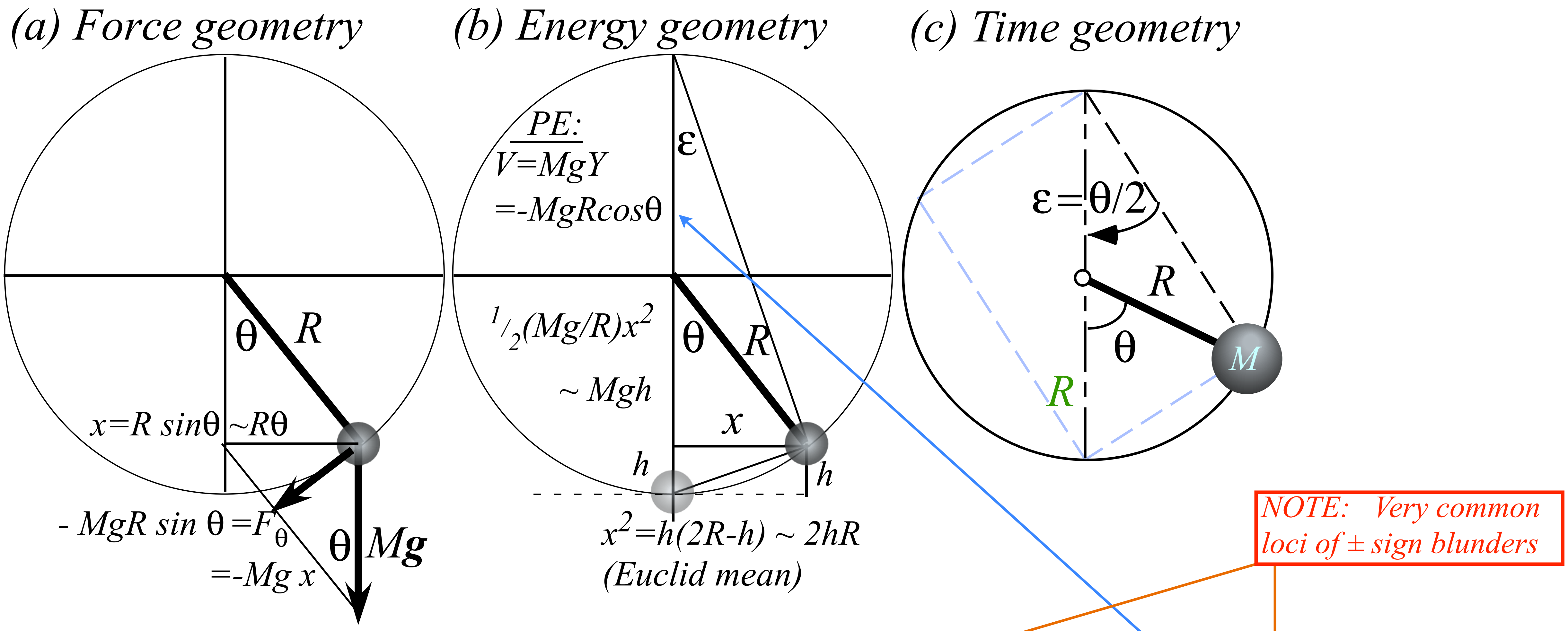
$$\omega_\odot = \sqrt{G\frac{M_\oplus}{R_\oplus^3}}$$

Examples of Hamiltonian mechanics in phase plots

➡ *1D Pendulum and phase plot* *(Web Simulations: Pendulum, Cycloidulum, JerkIt (Vertically Driven Pendulum))*

1D-HO phase-space control *(Classic Simulation of "Catcher in the Eye", Web Simulation:JerkIt)*

1D Pendulum and phase plot

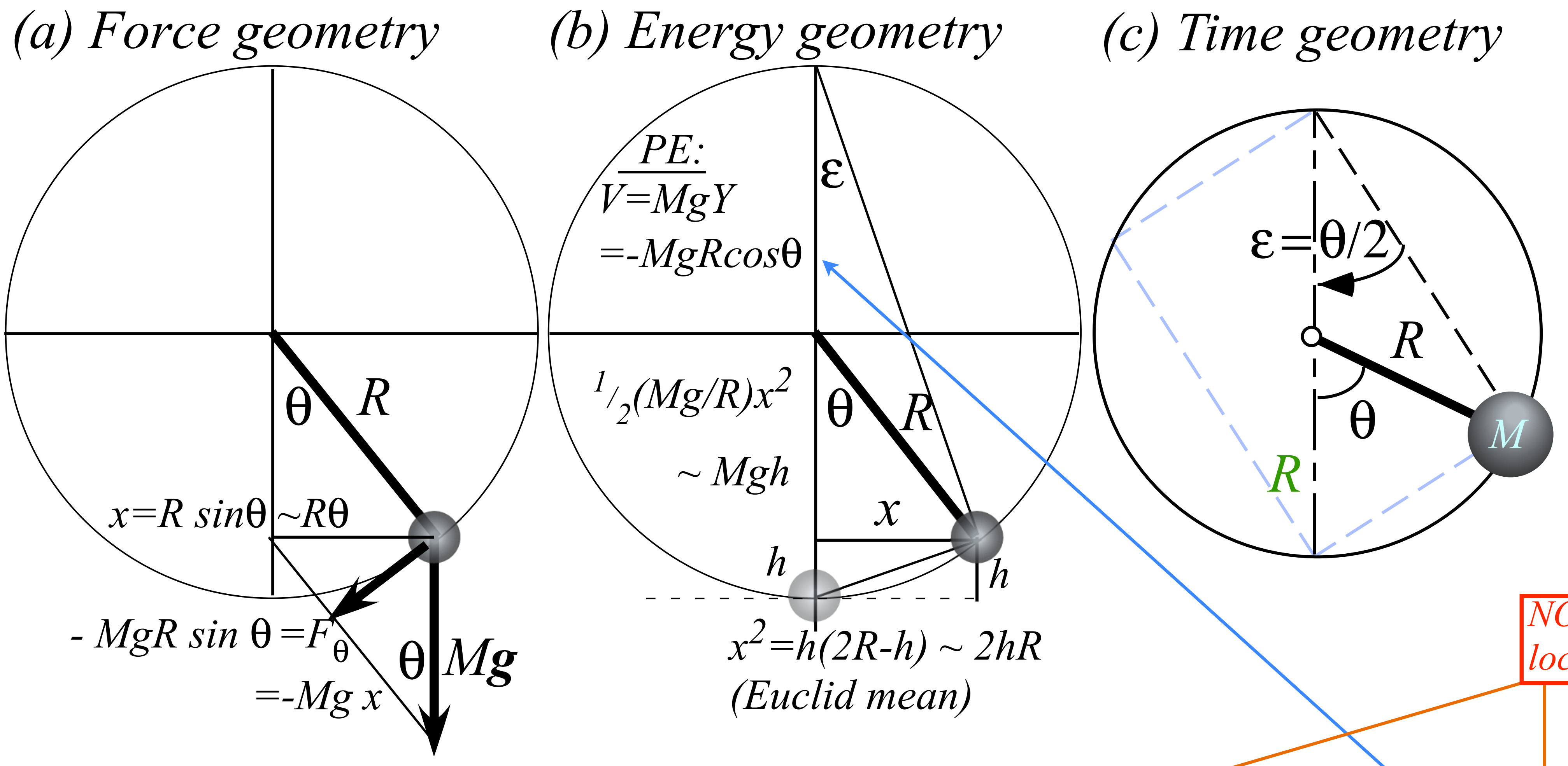


NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot



(a) Force geometry (b) Energy geometry (c) Time geometry

NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

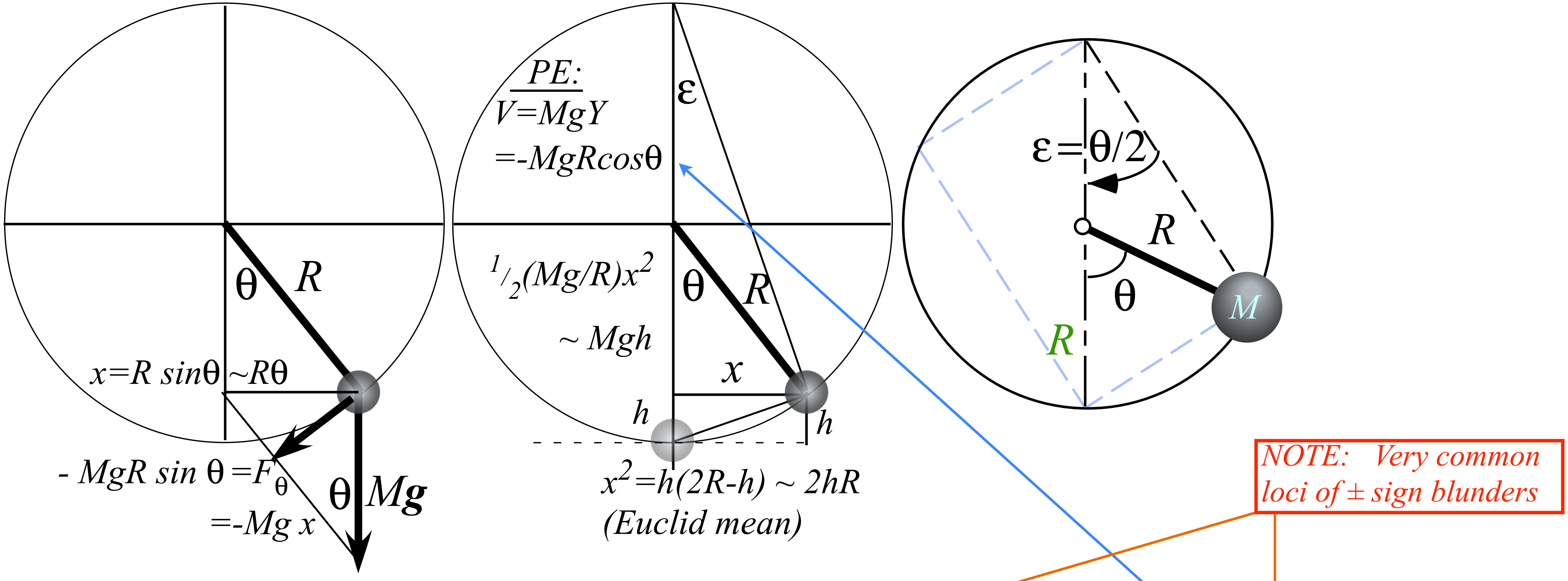
$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta \;=E \quad =const.$$

1D Pendulum and phase plot

(a) Force geometry (b) Energy geometry (c) Time geometry



NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

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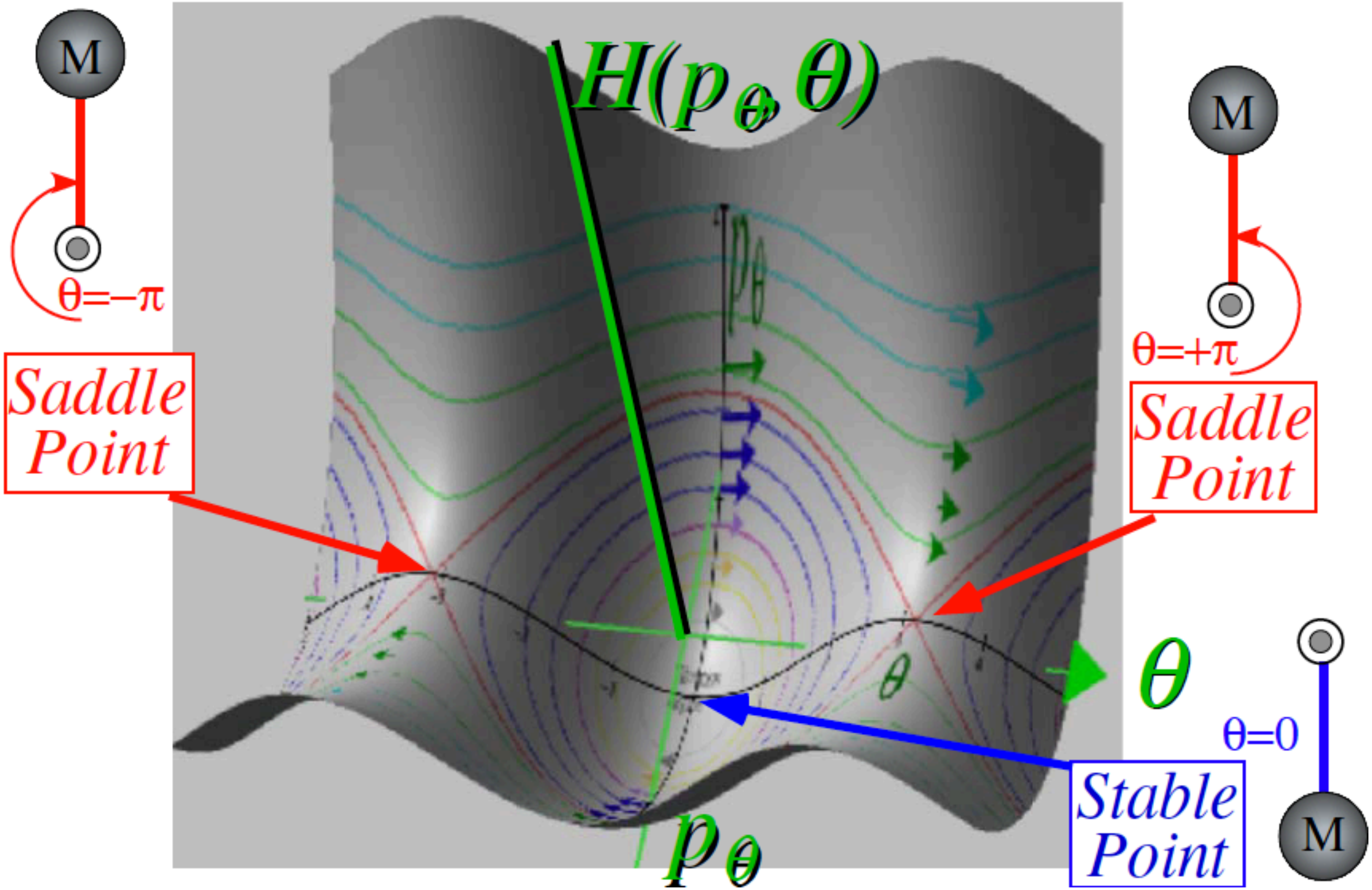
$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ =E\quad =\text{const.}$$

implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



Saddle Point

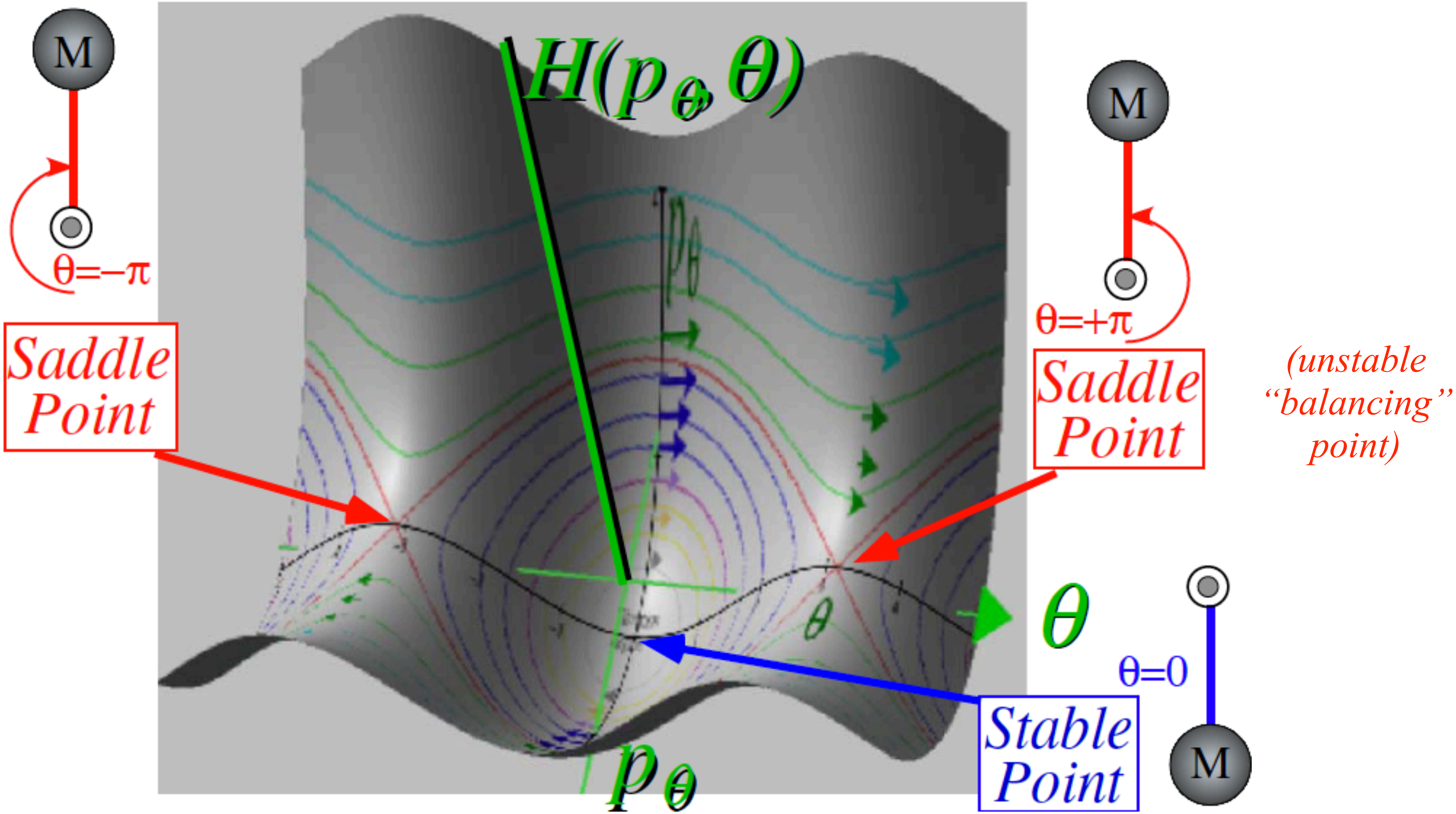
Saddle Point

(unstable "balancing" point)

Stable Point

Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}\, p_\theta{}^2 - MgR\cos\theta \ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix} = \mathbf{e_H}\times\left(-\nabla H\right) = (\text{H-axis})\times(\text{fall line}),\ \text{where:}\begin{cases}(\text{H-axis})=\mathbf{e_H}=\mathbf{e_q}\times\mathbf{e_p}\\ (\text{fall line})=-\nabla H\end{cases}$$

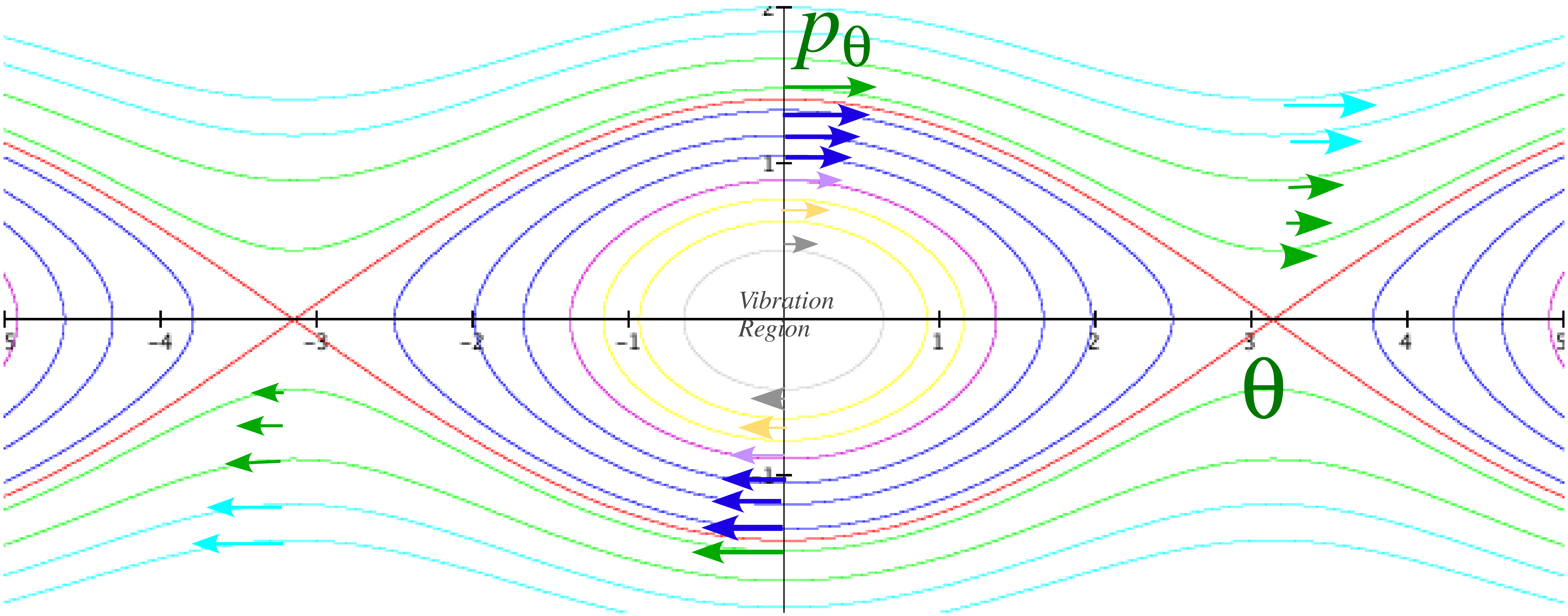


Fig. 2.7.2 Phase portrait or topography map for simple pendulum

(Unit 2 Chapter 7 Fig. 2)

Examples of Hamiltonian mechanics in phase plots

- 1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vert Driven Pendulum))
  - → Circular pendulum dynamics and elliptic functions
  - Cycloid pendulum dynamics and "sawtooth" functions
- 1D-HO phase-space control (Old Mac OS & Web Simulations of "Catcher in the Eye")

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H= KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$

Let $E=MgY=-MgR\cos\theta_0$ be potential energy where $KE=0$ or $p_\theta=0$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

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$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I\left(E + MgR\cos\theta\right)} / I \quad \text{where: } I = MR^2$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H= KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const.$$

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Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta^{\ 2} + U(\theta) = \frac{1}{2I}p_\theta^{\ 2} - MgR\cos\theta \ = E \quad =\text{const.}$$

implies: $p_\theta = \sqrt{2I(E + MgR\cos\theta)}$

Let $E=MgY=-MgR\cos\theta_0$ *be potential energy where* $KE=0$ *or* $p_\theta=0$

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E + MgR\cos\theta)} / I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E + MgR\cos\theta)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$*:*

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = \left(\text{Travel time 0 to } \theta_0\right) = \tau_{1/4}$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ *where potential energy is* $U(\theta) = -MgR\cos\theta$

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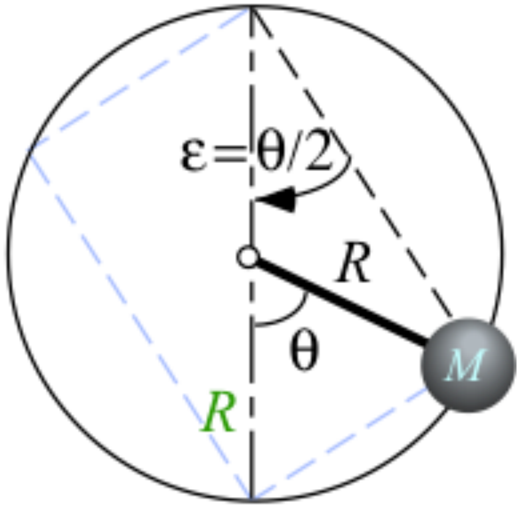

Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\varepsilon,$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

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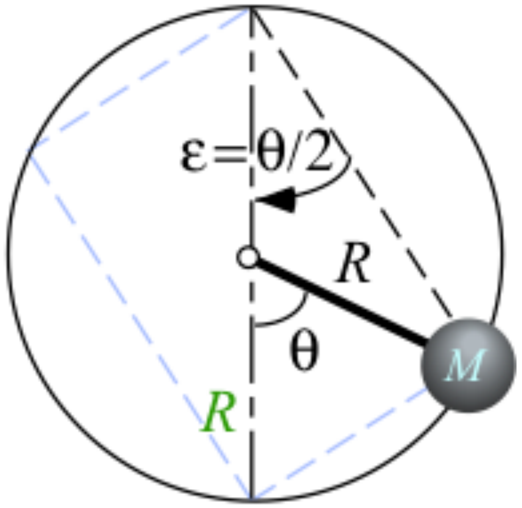
Quadrature integral gives quarter-period $\tau_{1/4}$*:*

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$



Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\sin^2\varepsilon, \qquad \cos\theta - \cos\theta_0 = 2\sin^2\varepsilon_0 - 2\sin^2\varepsilon$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta = E \quad = const. \qquad \text{implies: } p_\theta = \sqrt{2I(E + MgR\cos\theta)}$$

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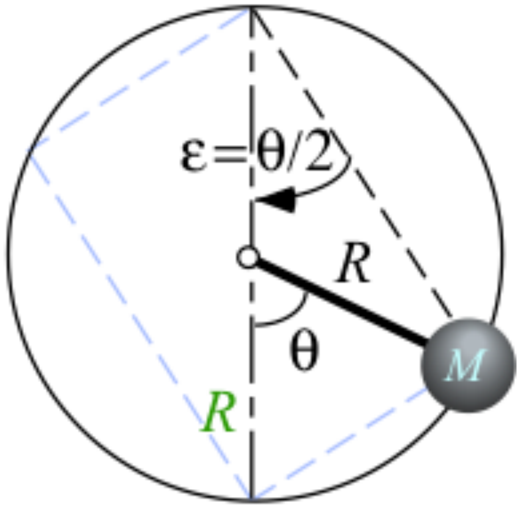
$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$



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$$\tau_{1/4} = \sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0 - \sin^2\varepsilon}}$$

Circular pendulum dynamics and elliptic functions

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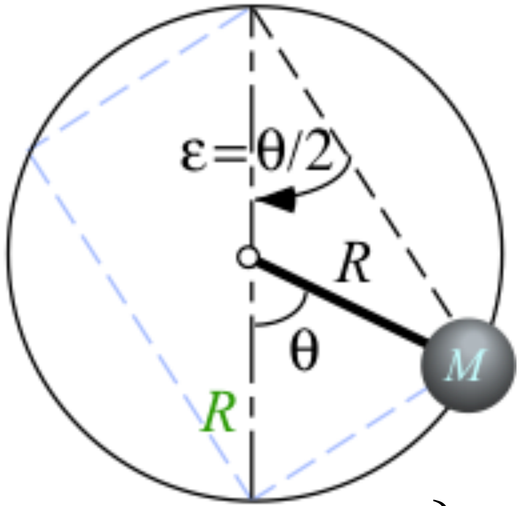

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Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

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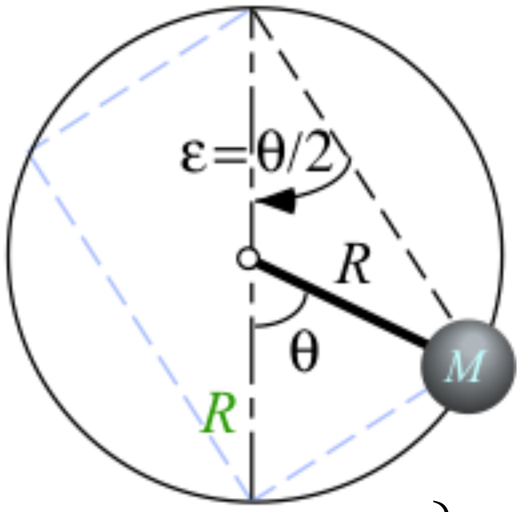

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$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{k\,d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\text{ , where:}\left\{\begin{array}{c}1/k=\sin\varepsilon_0=\sin\dfrac{\theta_0}{2}\\ I=MR^2\end{array}\right\}$$

The integral is an *elliptic integral of the first kind:* $F(k,\varepsilon_0)=am^{-1}$ or the "inverse amu" function.

$$F(k,\varepsilon_0)\equiv\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\equiv am^{-1}(k,\varepsilon_0)$$

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \ = E \quad = const. \qquad implies: \ p_\theta = \sqrt{2I(E+MgR\cos\theta)}$$

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$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{d\theta}{dt} = p_\theta / I = \sqrt{2I(E+MgR\cos\theta)}\,/\,I \quad \text{where: } I = MR^2 \quad \text{or : } dt = \frac{d\theta}{\sqrt{2(E+MgR\cos\theta)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$*:*

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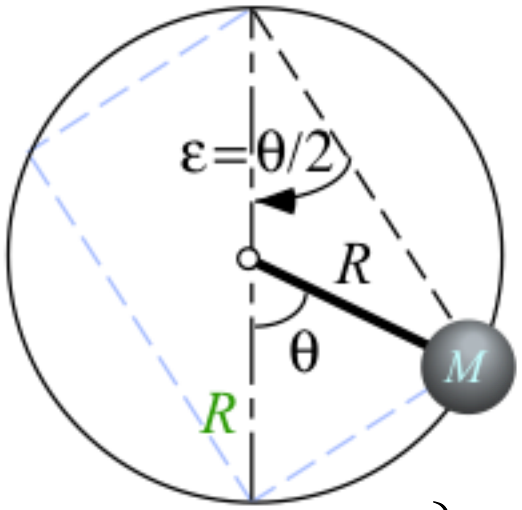

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For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions

Hamiltonian function H= KE + PE = T +U where potential energy is $U(\theta) = -MgR\cos\theta$

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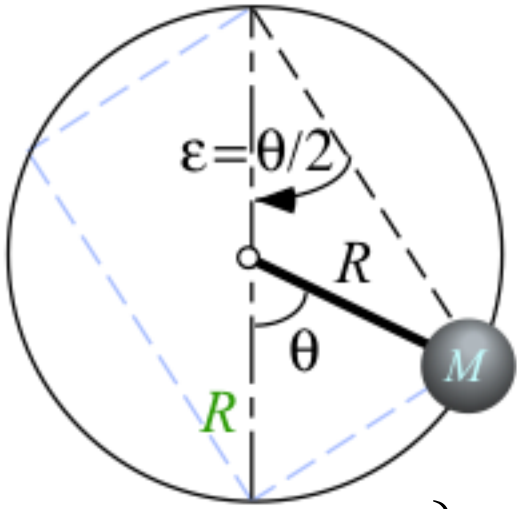
Quadrature integral gives quarter-period $\tau_{1/4}$*:*

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int_0^{\theta_0} dt = (\text{Travel time 0 to } \theta_0) = \tau_{1/4}$$



Uses a half-angle coordinate $\varepsilon=\theta/2$

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The integral is an *elliptic integral of the first kind: F(k,ε$_0$)=am^{-1}* or the "inverse amu" function.

$$F(k,\varepsilon_0) \equiv \int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}} \equiv am^{-1}(k,\varepsilon_0) \qquad \tau_{1/4} = \sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\varepsilon_0{}^2-\varepsilon^2}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon}{\varepsilon_0}\Big|_0^{\varepsilon_0} = \sqrt{\frac{R}{g}}\frac{\pi}{2} = \tau\frac{2\pi}{4}$$

low $\varepsilon \ll 1$: $t = \sqrt{\frac{R}{g}}\int_0^{\varepsilon(t)}\frac{d\varepsilon}{\sqrt{\varepsilon_0^2-\varepsilon^2}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon}{\varepsilon_0}\Big|_0^{\varepsilon(t)} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon(t)}{\varepsilon_0}$ For low amplitude $\varepsilon \ll 1$: $\sin\varepsilon_0 \simeq \varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions

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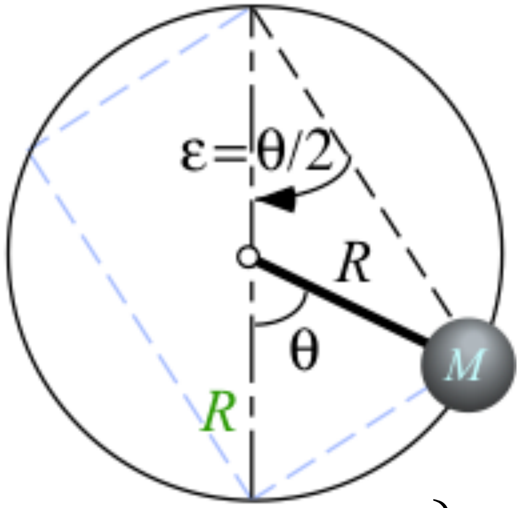
$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\frac{d\theta}{dt}=p_\theta/I=\sqrt{2I\left(E+MgR\cos\theta\right)}/I \quad \text{where: } I=MR^2 \quad \text{or : } dt=\frac{d\theta}{\sqrt{2\left(E+MgR\cos\theta\right)/I}}$$

Quadrature integral gives quarter-period $\tau_{1/4}$:

$$\sqrt{\frac{I}{2MgR}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\int_0^{\theta_0}dt=\left(\text{Travel time 0 to }\theta_0\right)=\tau_{1/4}$$



Uses a half-angle coordinate $\varepsilon=\theta/2$

$$\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\sin^2\varepsilon, \qquad \cos\theta-\cos\theta_0=2\sin^2\varepsilon_0-2\sin^2\varepsilon$$

$$\tau_{1/4}=\sqrt{\frac{I}{MgR}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0-\sin^2\varepsilon}}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{kd\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\text{, where:}\left\{\begin{array}{c}1/k=\sin\varepsilon_0=\sin\dfrac{\theta_0}{2}\\ I=MR^2\end{array}\right\}$$

The integral is an *elliptic integral of the first kind:* $F(k,\varepsilon_0)=am^{-1}$ or the "inverse amu" function.

$$F(k,\varepsilon_0)\equiv\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{1-k^2\sin^2\varepsilon}}\equiv am^{-1}(k,\varepsilon_0)$$

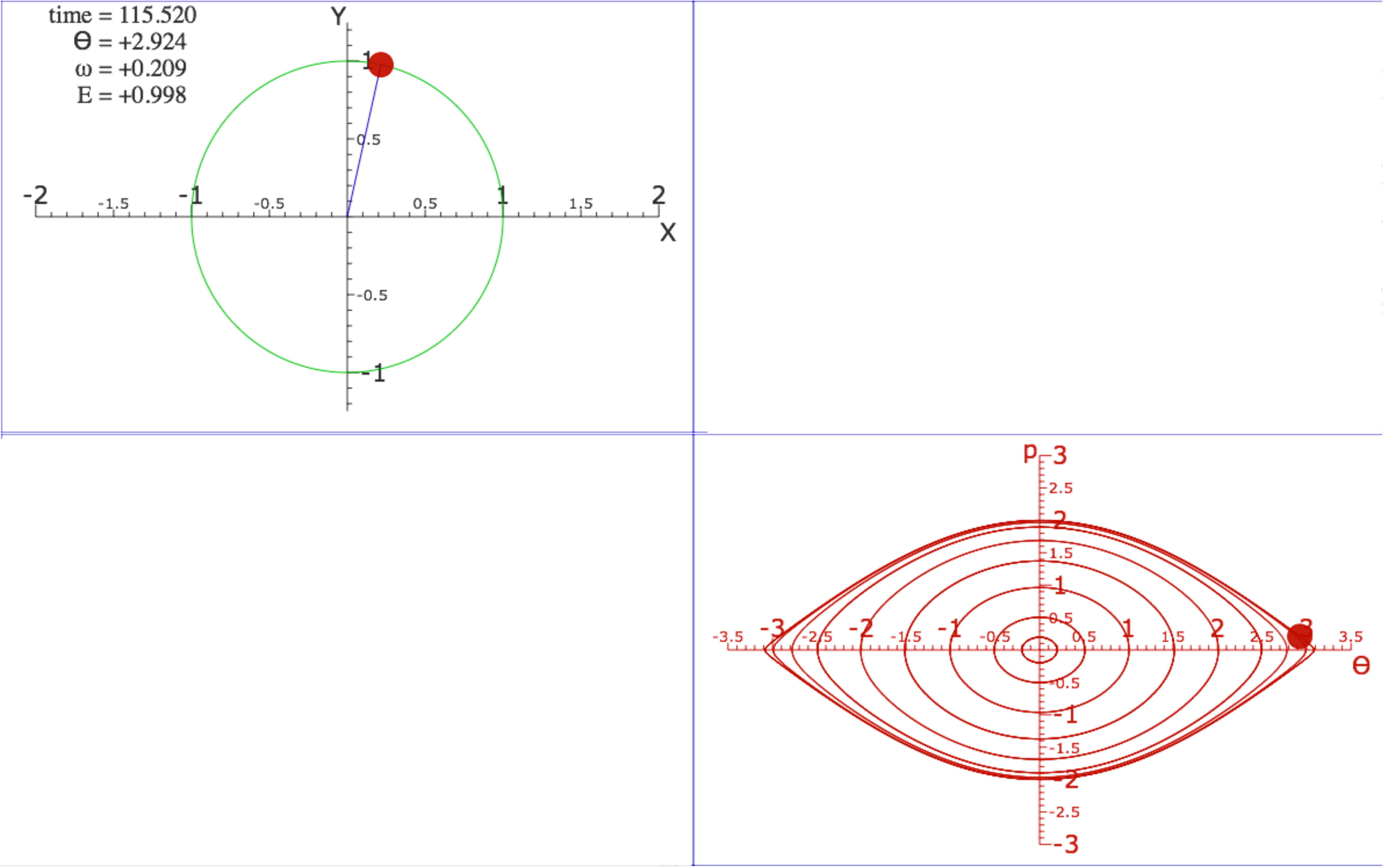
$$\tau_{1/4}=\sqrt{\frac{R}{g}}\int_0^{\varepsilon_0}\frac{d\varepsilon}{\sqrt{\varepsilon_0{}^2-\varepsilon^2}}=\sqrt{\frac{R}{g}}\sin^{-1}\frac{\varepsilon}{\varepsilon_0}\Big|_0^{\varepsilon_0}=\sqrt{\frac{R}{g}}\frac{\pi}{2}=\tau\frac{2\pi}{4}$$

..reduces to sine...

$$\varepsilon(t)=\varepsilon_0\sin\sqrt{\frac{g}{R}}t=\varepsilon_0\sin\omega\, t\ ,\ \text{ where: }\omega=\sqrt{\frac{g}{R}}$$

For low amplitude $\varepsilon\ll 1$: $\sin\varepsilon_0\simeq\varepsilon_0$ reduces $\tau_{1/4}$ to $\tau\frac{2\pi}{4}$

Circular pendulum dynamics and elliptic functions



(Simulations of pendulum)

(See also: Simulation of cycloidally constrained pendulum)

Examples of Hamiltonian mechanics in phase plots

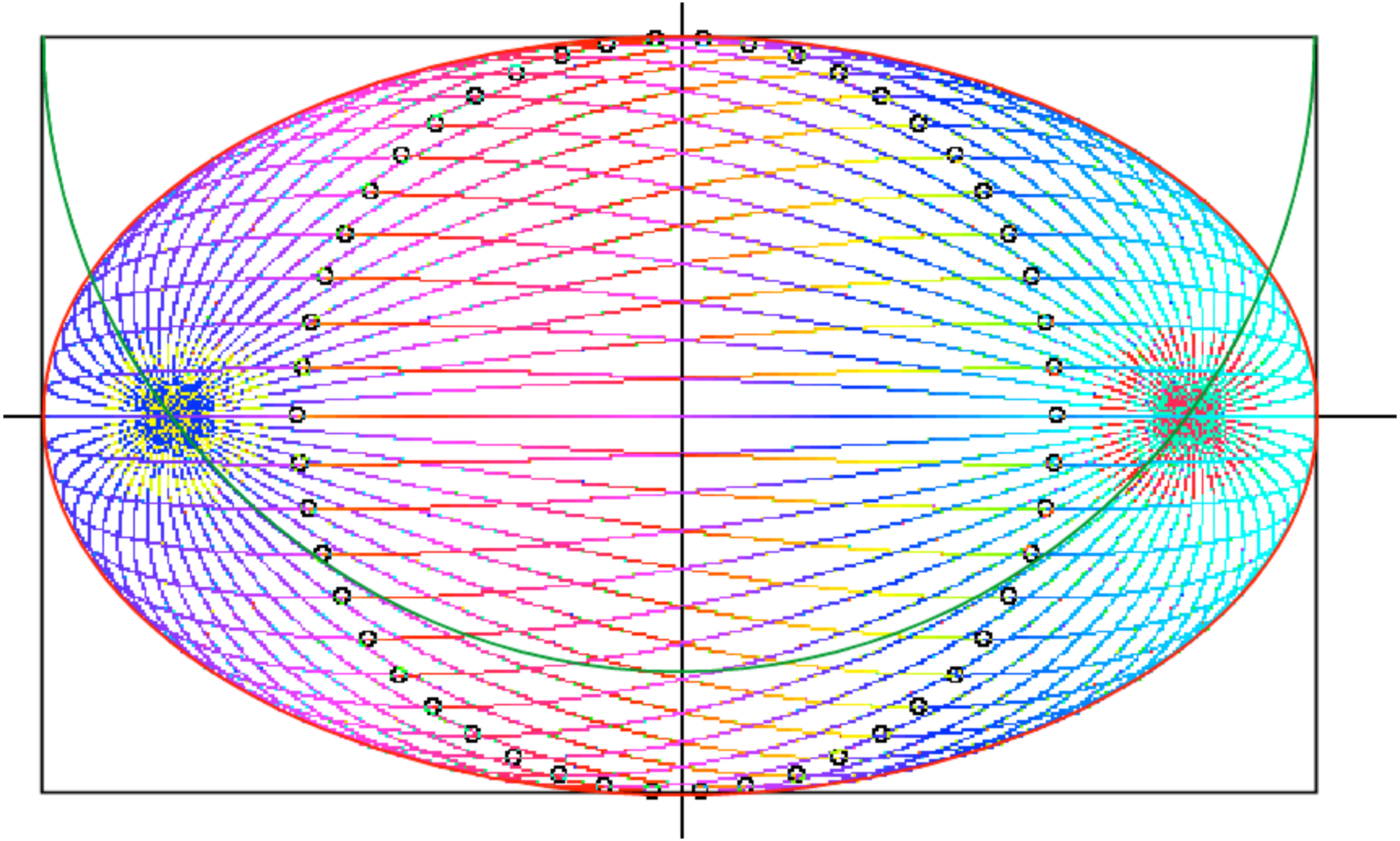
1D Pendulum and phase plot (Web Simulations: Pendulum, Cycloidulum, JerkIt (Vertically Driven Pendulum))

→ 1D-HO phase-space control (Classic Simulation of "Catcher in the Eye", Web Simulation:JerkIt)

Parabolic and 2D-IHO elliptic orbital envelopes

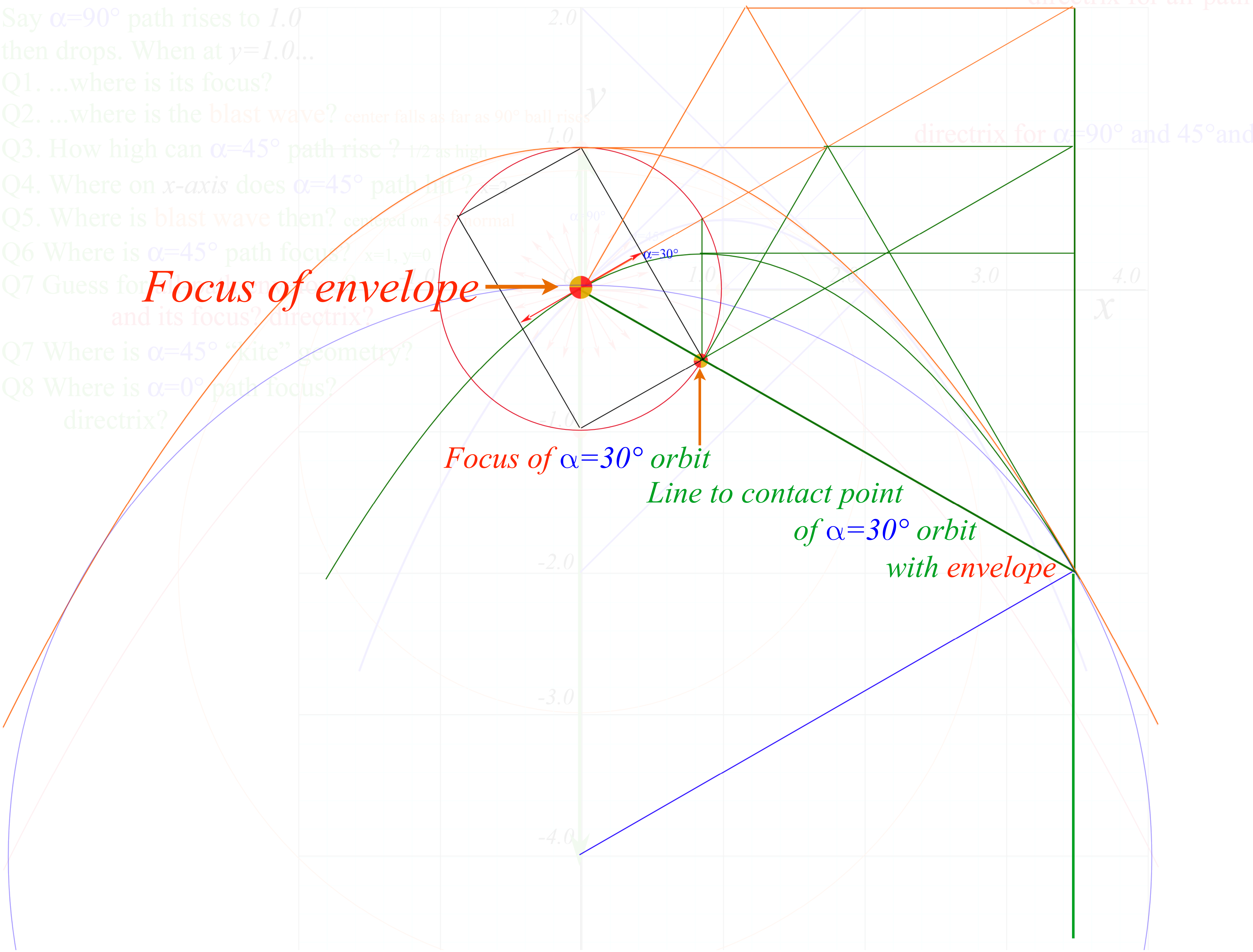
Some clues for future assignment (Mac OS Simulation of “Catcher in the Eye”)

Exploding-starlet elliptical envelope and contacting elliptical trajectories



Say α=90° path rises to *1.0*
then drops. When at *y=1.0*...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90° ball rises
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then? centered on 45° normal
Q6 Where is α=45° path focus? x=1, y=0
Q7 Guess for all-path envelope?
and its focus? directrix?

Q7 Where is α=45° “kite” geometry?
Q8 Where is α=0° path focus?
directrix?



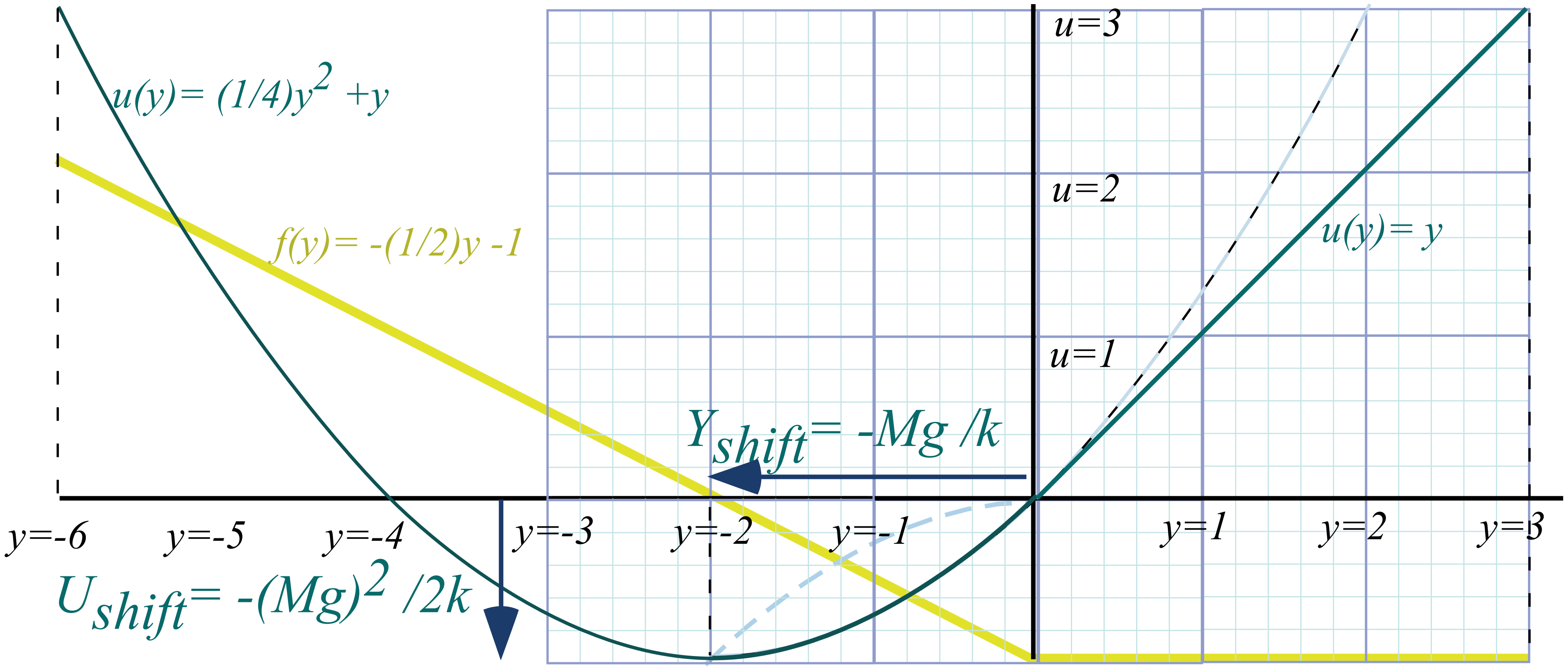
Focus of envelope

Focus of α=30° orbit

Line to contact point
of α=30° orbit
with envelope

Lecture 10 ends here
Fri. 9.23.2016

$F(Y)=-kY-Mg$ $U(Y)=(1/2)kY^2+Mg\ Y$



Unit 1
Fig. 7.4

Web Simulation of atomic classical (or semi-classical) dynamics using varying phase control